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Graph Multiplicities

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GRAPH MULTIPLICITIES

by

David Burns

A Project Report
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Specialist in Arts Degree

Western Michigan University
Kalamazoo, Michigan
December 1976
ACKNOWLEDGEMENTS

In writing this project, I have benefited from the encouragement, patient advice, and constructive criticism of Professor Gary Chartrand. In addition to thanking Professor Chartrand, I would like to express my gratitude to Professor Arthur T. White for introducing me to Graph Theory and to both Professors White and S. F. Kapoor for serving on my committee. Finally, I would like to thank Professor Seymour Schuster of Carleton College for several valuable suggestions and ideas.

David Burns
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BURNS, David
GRAPH MULTIPLICITIES.

Western Michigan University, Sp.A., 1976
Mathematics

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CHAPTER I

INTRODUCTION

Every graph \( G \) is a subgraph of the complete graph of the same order. However, if \( G \) has order \( p \), there may very well be several (mutually edge-disjoint) copies of \( G \) in \( K_p \). This suggests the following question.

Given a nonempty graph \( G \) of order \( p \) what is the largest integer \( m \) such that the complete graph \( K_p \) contains \( m \) edge-disjoint spanning subgraphs each isomorphic with \( G \)? We call this integer \( m \) the multiplicity of the graph \( G \). The goal of this project is to investigate this concept.

We devote the remainder of this chapter to definitions and notation in graph theory necessary for that investigation. Other terminology and symbols used may be found in Behzad and Chartrand [1].

Let \( G \) and \( H \) be graphs such that \( E(G) \cap E(H) = \emptyset \). If \( V(G) \cap V(H) = \emptyset \) then the union of \( G \) and \( H \), denoted \( G \cup H \), is that graph having vertex set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \). If \( V(G) = V(H) \) then the edge sum of \( G \) and \( H \), denoted \( G \oplus H \), has vertex set \( V(G) \) and edge set \( E(G) \cup E(H) \). If \( n \) is a positive integer then \( nG \) denotes the edge sum of \( n \) graphs each isomorphic with \( G \).
If $G_1$ and $G_2$ are graphs such that all the vertices and edges of $G_1$ are in $G_2$, then $G_1$ is a subgraph of $G_2$ and we write $G_1 \subseteq G_2$. If $U$ is a nonempty set of vertices of $G$, the induced subgraph $\langle U \rangle$ is the maximal subgraph of $G$ with vertex set $U$. If $W$ is a set of edges of $G$, the induced subgraph $\langle W \rangle$ is the graph whose vertex set consists of those vertices of $G$ incident with at least one edge of $W$ and whose edge set is $W$.

The maximum degree (minimum degree) of $G$, denoted $\Delta(G)$ (respectively $\delta(G)$), is the largest (smallest) integer in the set $\{\deg_G v : v \in V(G)\}$. The complete $n$-partite graph $K(p_1, p_2, \ldots, p_n)$, where $n \geq 2$, is the graph whose vertex set can be partitioned into $n$ subsets $V_1, V_2, \ldots, V_n$ such that $|V_i| = p_i \geq 1$ for $1 \leq i \leq n$ and whose edge set contains $uv$ if and only if there exist distinct subscripts $i$ and $j$ with $u \in V_i$ and $v \in V_j$.

If $v$ is a vertex of $G$, then $G - v$ is that subgraph of $G$ consisting of all vertices of $G$ except $v$ and all edges of $G$ not incident with $v$. If $e$ is an edge of $G$, then $G - e$ is that spanning subgraph of $G$ containing all the edges of $G$ except $e$. We denote by $P_m$ the path of order $m$. 

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For a real number $a$, the symbol $\lfloor a \rfloor$ denotes the greatest integer not exceeding $a$ and $\lceil a \rceil = -\lfloor -a \rfloor$ is the smallest integer not less than $a$. 
CHAPTER II

GRAPH MULTIPLEITIES

In this chapter we formalize the notion of graph multiplicity, compute the multiplicity of several graphs and classes of graphs, establish bounds for this parameter, and introduce some related concepts.

Section 2.1

The Multiplicity of a Graph

Definition 2.1. The multiplicity $n(G)$ of a nonempty graph $G$ of order $p$ is the largest positive integer $n$ such that $K_p = nG \oplus R$.

For example, $n(C_5) = 2$ since $K_5$ may be expressed as the edge sum of two edge-disjoint spanning subgraphs, each isomorphic with $C_5$. It is equally easy to establish that $n(K_2 \cup K_3) = 1$ since $K_2 \cup K_3$ contains an odd cycle and therefore is not a subgraph of its bipartite complement. Our first theorem establishes the existence of $n(G)$ for all nonempty graphs $G$ and, at the same time, provides an upper bound for this parameter.
Theorem 2.2. If $G$ is a graph of order $p$ and size $q > 0$, then $n(G)$ exists and

$$1 \leq n(G) \leq \left\lfloor \frac{p(p-1)}{2q} \right\rfloor.$$  

Proof: Since $G$ is a nonempty subgraph of $K_p$, the multiplicity of $G$ is defined and $n(G) \geq 1$. Because $G$ has size $q$ and $K_p$ has size $\frac{p(p-1)}{2}$, there are at most $\left\lfloor \frac{p(p-1)}{2q} \right\rfloor$ edge-disjoint subgraphs of $K_p$ each isomorphic with $G$. $\blacksquare$

The upper bound of Theorem 2.2 is sharp since it is attained by the complete graphs $K_p$, for $p \geq 2$, and by $K_2 \cup (p-2)K_1$, where $p \geq 2$. There are numerous examples of graphs whose multiplicities are less than this upper bound. Each star $K(1,n)$, where $n \geq 3$, has multiplicity one, for example.

Section 2.2

The Multiplicities of Some Classes of Graphs

The first two results (see [7, p. 89]) will aid in determining the multiplicities of the cycles $C_m$ for $m \geq 3$.

Lemma 2.3. If $p$ is an odd integer, then $K_p$ is the edge sum of $(p-1)/2$ spanning cycles.
Lemma 2.4. If $p$ is an even integer, then $K_p$ is the edge sum of a 1-factor and $(p - 2)/2$ spanning cycles.

Theorem 2.5. $n(C_m) = \left\lfloor \frac{(m - 1)}{2} \right\rfloor$ for all $m \geq 3$.

Proof: Let $m (\geq 3)$ be an integer. If $m$ is odd, then $K_m = \frac{(m - 1)}{2} C_m$ by Lemma 2.3. Hence $n(C_m) = \frac{(m - 1)}{2} = \left\lfloor \frac{(m - 1)}{2} \right\rfloor$. If $m$ is even, then there exists $\frac{(m - 2)}{2}$ edge-disjoint subgraphs in $K_m$, each isomorphic with $C_m$, by Lemma 2.4. Thus $n(C_m) \geq \frac{(m - 2)}{2} = \left\lfloor \frac{(m - 1)}{2} \right\rfloor$. Since $C_m$ has size $m$, we know $n(C_m) \leq \left\lfloor \frac{m(m - 1)}{2m} \right\rfloor = \left\lfloor \frac{(m - 1)}{2} \right\rfloor$ by Theorem 2.2. Hence $n(C_m) = \left\lfloor \frac{(m - 1)}{2} \right\rfloor$ which completes the proof. ■

The next result will allow us to use Theorem 2.5 to help establish the multiplicities of the paths $P_m$ of order $m$.

Proposition 2.6. If $G$ is a nonempty spanning subgraph of $H$, then $n(G) \geq n(H)$.

Proof: Suppose $|V(G)| = |V(H)| = p$. We know that $K_p$ contains $n(H)$ copies of $H$. Each such subgraph $H$ contains $G$ as a spanning subgraph. Hence $K_p$ contains at least $n(H)$ edge-disjoint subgraphs, each isomorphic with $G$ which implies that $n(G) \geq n(H)$. ■
We next determine the multiplicities of the paths $P_m$ of order $m$. When $m$ is even this is an immediate consequence of a result by Beineke [2]. We choose, however, to establish this same result using Theorem 2.5, Proposition 2.6, and a well-known 1-factorization of $K_m$.

**Theorem 2.7.** $n(P_m) = \left\lceil \frac{m}{2} \right\rceil$ for all $m \geq 2$.

**Proof:** If $m = 2$, then $P_m = K_2$ and therefore

$$n(P_m) = n(K_2) = 1 = \left\lceil \frac{m}{2} \right\rceil.$$  

If $m \geq 3$ then $P_m$ is a spanning subgraph of $C_m$. Hence, by Theorem 2.5 and Proposition 2.6, $n(P_m) \geq n(C_m) = \left\lceil \frac{(m-1)}{2} \right\rceil$.

Now $P_m$ has size $m-1$; hence $n(P_m) \leq \left\lfloor \frac{m(m-1)}{2(m-1)} \right\rfloor = \frac{m}{2}$ by Theorem 2.2. Thus

$$\left\lceil \frac{(m-1)}{2} \right\rceil \leq n(P_m) \leq \left\lfloor \frac{m}{2} \right\rfloor.$$  

When $m$ is odd, $\left\lceil \frac{(m-1)}{2} \right\rceil = \left\lceil \frac{m}{2} \right\rceil$ and our conclusion is established.

When $m$ is even it is sufficient to show there exist $\frac{m}{2}$ edge-disjoint spanning paths in $K_m = K_{2s}$, since $n(P_m) \leq \frac{m}{2}$ by Theorem 2.2. To that end, let $V(K_{2s}) = \{v, v_1, \ldots, v_{2s-1}\}$. Arrange the vertices $v_1, v_2, \ldots, v_{2s-1}$ in a regular $(2s-1)$-gon and place $v$ in the center. In the following graphs, represent edges with straight lines. Define $2s - 1$ 1-factors.
$F_1$ of $K_{2s}$ where $1 \leq i \leq 2s - 1$ by $E(F_i) = \{ vv_i \}$ together with all edges of $K_{2s}$ perpendicular to $vv_i$ in the $(2s-1)$-gon. Then $K_{2s} = F_1 \oplus F_2 \oplus \ldots \oplus F_{2s-1}$. This 1-factorization of $K_{2s}$ is illustrated in Figure 2.1 for $s = 3$.

Figure 2.1

Let $e_i$ be that edge of $F_i$ joining the two vertices $v_{i+1}$ and $v_{i-1}$ where subscripts are taken modulo $2s - 1$. Consider the 1-factors $F_1, F_3, F_5, \ldots, F_{2s-3}$. Add 1 to the subscript of every second 1-factor in this list starting at $F_3$ to form a new list

$$S: F_1, F_4, F_5, F_8, F_9, \ldots, F_k$$

where $k = 2s - 2$ if $s$ is odd and $k = 2s - 3$ if $s$ is even. For example $S$ is $F_1, F_4, F_5, F_8, F_9$ when $s = 6$ and $S$ is $F_1, F_4, F_5, F_8, F_9, F_{12}$ when $s = 7$.

Next we form a set $T$ consisting of $s - 1$ edges which are independent in $K_{2s}$ and where one edge is chosen from each of the 1-factors in the list $S$. If $s$ is even each 1-factor $F_i$ in $S$ contributes the edge $e_i$ to $T$. If $s$ is odd every 1-factor $F_i$ in
S, except $F_{2s-2}$, contributes the edge $e_1$ to $T$ and $F_{2s-2}$ contributes $vv_{2s-2}$ to $T$. The edges of $T$ are illustrated for $s = 6$ and $s = 7$ in Figure 2.2.

![Figure 2.2](image)

Now $K_{2s}$ may be expressed as the edge sum of $s - 1$ edge-disjoint spanning cycles and a 1-factor as follows:

$$K_{2s} = (F_1 \cup F_2) \oplus (F_3 \cup F_4) \oplus \ldots \oplus (F_{2s-3} \cup F_{2s-2}) \oplus (F_{2s-1}).$$

Notice now that $T$ contains $s - 1$ edges which are independent in $K_{2s}$ and where each cycle above contributes exactly one edge to $T$. Remove the set $T$ from the cycles above and allocate these $s - 1$ edges to the 1-factor $F_{2s-1}$. Each former cycle is now a spanning path of $K_{2s}$, and the augmented 1-factor $F_{2s-1}$ is also a spanning path of $K_{2s}$. Hence there exist $s$ edge-disjoint spanning paths in $K_{2s}$ which completes the proof. ■
The next result generalizes an example presented earlier in this chapter.

**Theorem 2.8.** If \( a \geq 3 \) and \( b \geq 1 \), then
\[
n(K_a \cup K_b) = 1.
\]

**Proof:** Since \( a \geq 3 \), necessarily \( G = K_a \cup K_b \) contains an odd cycle. The complement of \( G \) is bipartite and therefore contains no odd cycles. Thus \( G \) is not a subgraph of \( \overline{G} \) and hence \( n(G) = 1 \). ■

Section 2.3

Graphs with Multiplicity 1

A graph has multiplicity 1 if and only if it is not a subgraph of its complement. Several sufficient conditions for a graph to have multiplicity 1 may be established using this equivalence.

**Proposition 2.9.** If \( G \) is a \((p,q)\) graph where
\[
q > \left\lceil \frac{p(p-1)}{4} \right\rceil,
\]
then \( n(G) = 1 \).

**Proof:** Since \( q > \left\lceil \frac{p(p-1)}{4} \right\rceil \), it follows that
\[
q > \frac{p(p-1)}{4}.
\]
Therefore \( G \) has larger size than its complement and thus \( G \) is not a subgraph of \( \overline{G} \). ■

We may use this result to produce the following observation.
Proposition 2.10. If G is a graph of order p such that \( \delta(G) > (p - 1)/2 \) then \( n(G) = 1 \).

Proof: Since \( \delta(G) > (p - 1)/2 \), the graph G has size greater than \( p(p - 1)/4 \). The result follows from Proposition 2.9. ■

Only the degrees of a majority of the vertices of a graph need be properly bounded to conclude that the graph has multiplicity 1.

Proposition 2.11. If G is a graph of order p and G has more than \( p/2 \) vertices of degree at least \( p/2 \), then \( n(G) = 1 \).

Proof: If G is a graph satisfying the hypotheses and if G has multiplicity at least 2, then there exists a vertex v in \( K_p \) incident with \( p/2 \) edges from each of two copies of G. But then \( \deg_{K_p} v \geq p \), an impossibility. ■

Relationships involving \( \Delta(G) \) also allow us to conclude \( n(G) = 1 \).

Proposition 2.12. If G is a graph such that \( \Delta(G) > \Delta(G) \) then \( n(G) = 1 \).

Proof: If \( \Delta(G) > \Delta(G) \) then G is not a subgraph of \( \overline{G} \) and hence \( n(G) = 1 \). ■
Proposition 2.13. If $G$ is a graph of order $p$ such that $\Delta(G) = p - 1$, then $n(G) = 1$.

Proof: Since $\Delta(G) = p - 1$, $\Delta(G) \leq p - 2$. The result follows from Proposition 2.12. \[ \Box \]

Next we use Proposition 2.12 to determine the multiplicities of a well-known collection of graphs.

Theorem 2.14. If $G$ is the complete $m$-partite graph $K(a_1, a_2, \ldots, a_m)$, then $n(G) = 1$.

Proof: Since each vertex in $G$ is adjacent to all vertices in partite sets other than its own,

$$\Delta(G) = \max_{i=1,m} \left\{ \left( \sum_{j=1}^{m} a_j \right) - a_i \right\}.$$  

Now $\overline{G} = K_{a_1} \cup K_{a_2} \cup \ldots \cup K_{a_m}$ and so $\Delta(\overline{G}) = \max_{i=1,m} \{ a_i - 1 \}$. Therefore, since $m$ is at least 2, $\Delta(G) > \Delta(\overline{G})$. Hence $n(G) = 1$ by Proposition 2.12. \[ \Box \]

Some of the results in this section are particular cases of a more general remark. A graph-theoretic parameter $f$ is said to be hereditary (with respect to spanning subgraphs) if whenever $G$ is a spanning subgraph of $H$ it follows that $f(G) \leq f(H)$. Chromatic number, genus, connectivity, maximum degree, and size are examples of hereditary parameters.
Proposition 2.15. If \( f \) is a hereditary graph-theoretic parameter and if \( G \) is a graph such that \( f(G) > f(G') \), then \( n(G) = 1 \).

Proof: Since \( f \) is hereditary, \( f(G) > f(G') \) implies that \( G \) is not a subgraph of \( G' \). ■

Section 2.4

Some Upper Bounds for Multiplicity

Theorem 2.2 provides an upper bound for the multiplicity of a graph based on order and size. Other properties of a graph also limit its multiplicity.

Theorem 2.16. Let \( G \) be a nonempty graph of order \( p \) where \( \delta(G) \geq 1 \). If \( G \) contains a vertex of degree \( p - k \) for some positive integer \( k \), then \( n(G) \leq k \).

Proof: Let \( G \) be a graph with the specified properties and assume that \( n(G) \geq k + 1 \). Then there exist at least \( k + 1 \) copies of \( G \) in \( K_p \). Select one such subgraph \( G \) and identify a vertex of degree \( p - k \) (in that subgraph). This vertex \( v \) appears in at least \( k \) other copies of \( G \) in \( K_p \), and has degree at least one in each copy. Hence the degree of \( v \) in \( K_p \) is at least \( (p - k) + k = p \), an impossibility. ■
The bound presented in Theorem 2.16 is sharp since it is attained by the complete graph $\mathcal{G}_p$, for $p \geq 2$.

**Theorem 2.17.** If $G$ is a nonempty graph of order $p$ where $\delta(G) = s \geq 1$ and $\Delta(G) = m$, then $n(G) \leq \left\lfloor \frac{(p - m)}{s} \right\rfloor$.

**Proof:** Let $G$ be a graph with the specified properties and assume that $n(G) \geq \left\lfloor \frac{(p - m)}{s} \right\rfloor + 1$. Therefore there exist at least $\left\lfloor \frac{(p - m)}{s} \right\rfloor + 1$ copies of $G$ in $K_p$. Choose one such subgraph $G$ and identify a vertex $v$ of degree $m$ in that subgraph. Then vertex $v$ appears in at least $\left\lfloor \frac{(p - m)}{s} \right\rfloor$ other edge-disjoint copies of $G$ in $K_p$. Since $v$ has degree at least $s$ in each copy, it follows that

$$\deg_{K_p} v \geq m + \left\lfloor \frac{(p - m)}{s} \right\rfloor s \geq m + \left( \frac{p - m}{s} \right) s = p,$$

an impossibility. ■

The bound in Theorem 2.17 is sharp since it is attained by the family of cycles $C_m$, for $m \geq 3$.

If $T$ is a tree of order $p$ ($p \geq 2$) then $T$ has size $p - 1$. Hence $n(T) \leq \lceil p/2 \rceil$ by Theorem 2.2.

Also, since $\delta(T) = 1$, we conclude that $n(T) \leq p - \Delta(T)$ by Theorem 2.17. We may combine these two observations to produce the following statement.
Corollary 2.18. If \( T \) is a tree of order \( p \geq 2 \), then
\[ 1 \leq n(T) \leq \min\left\{ \left \lfloor \frac{p}{2} \right \rfloor, p - \Delta(T) \right\} \].

Section 2.5

The Multiplicity of the Complement

The task of determining the multiplicity of a graph \( G \) is equivalent to the problem of finding the maximum number of edge-disjoint subgraphs of \( \overline{G} \) each isomorphic with \( G \). It seems natural, with the graph \( \overline{G} \) at hand, to inquire about its multiplicity and to compare the two numbers \( n(G) \) and \( n(\overline{G}) \). The first result shows that both these numbers cannot be large.

Theorem 2.19. If \( k \) is a positive integer then there exists a graph \( G \) such that \( \min(n(G), n(\overline{G})) = k \) if and only if \( k \leq 2 \).

Proof: The graphs \( G_1 = K(1,3) \) and \( G_2 = P_4 \) have the property that \( \min(n(G_k), n(\overline{G}_k)) = k \) for \( k = 1 \) and 2.
If, on the other hand, \( G \) is a graph such that \( \min(n(G), n(\overline{G})) = k \), where \( k \geq 3 \), then \( n(G) \) is at least 3 which implies that \( \overline{G} \) contains at least two edge-disjoint subgraphs isomorphic with \( G \). Hence
\[ \overline{q} \geq 2q \] where \( \overline{q} = |E(\overline{G})| \) and \( q = |E(G)| \). However \( n(\overline{G}) \) is also at least 3. Thus we have \( q \geq 2\overline{q} \) by a similar argument. These two inequalities are irreconcilable. \( \blacksquare \)
Theorem 2.20. If $k$ is a positive integer, then there exists a graph $G_k$ such that
\[ \max\{n(G_k), n(G_k)\} = k. \]

Proof: The graph $G_1 = K(1,3)$ establishes the result for $k = 1$. For $k \geq 2$ and $k$ even, let $G_k = C_{2k+2}$. By Theorem 2.5, $n(C_{2k+2}) = \left(\frac{(2k + 1)}{2}\right) = k$. Now $C_{2k+2}$ has $2k^2 + k - 1$ edges. Since $k \geq 2$, $|E(C_{2k+2})| > |E(C_{2k+2})|$. Hence $n(C_{2k+2}) = 1$ and $\max\{n(G_k), n(G_k)\} = k$ as required.

For $k \geq 2$ and $k$ odd, let $G_k = P_{2k+1}$. By Theorem 2.7, $n(P_{2k+1}) = \left(\frac{(2k + 1)}{2}\right) = k$. Again, since $k \geq 2$, $P_{2k+1}$ has more edges than $P_{2k+1}$. Hence $n(P_{2k+1}) = 1$ and $\max\{n(G_k), n(G_k)\} = k$. ■

The graphs discussed in the preceding proof may be used again to establish the following result.

Theorem 2.21. If $k$ is a positive integer, then there exist graphs $G$, $H$, and $I$ such that
\begin{enumerate}
  \item $n(G) \cdot n(G) = k$
  \item $n(H)/n(H) = k$
  \item $n(I) - n(I) = k$.
\end{enumerate}
Moreover, if $k \geq 2$, there exists a graph $J$ such that $n(J) + n(J) = k$.
Proof: For all positive integers $k$, a graph $G_k$ was defined in the proof of Theorem 2.20 such that $n(G_k) = k$ and $n(G_k^*) = 1$. Hence we may take $G = H = G_k$ and the corresponding conclusions are true.

Now set $I = G_{k+1}$. Then $n(I) - n(I^*) = (k + 1) - 1 = k$.

Finally, for $k \geq 2$, set $J = G_{k-1}$. Here $n(J) + n(J^*) = (k - 1) + 1 = k$. ■

Section 2.6

Minimal Graphs

For a graph $G$ of size at least 2 consider the collection of spanning subgraphs $G - e$ where $e$ is an edge of $G$. Now $n(G-e) \geq n(G)$, by Proposition 2.6. We now examine graphs where this inequality is always strict.

Definition 2.22. A graph $G$ is minimal (with respect to multiplicity) if $n(G-e) > n(G)$ for all edges $e$ in $E(G)$. A graph is $m$-minimal (with respect to multiplicity) if it is minimal and has multiplicity $m$.

Proposition 2.23. For each positive integer $k$, there exists a $k$-minimal graph.

Proof: Let $k$ be a positive integer, and consider $C_{2k+2}$. We know $n(C_{2k+2}) = [(2k+1)/2] = k$ by Theorem
2.5. If \( e \) is an edge of \( C_{2k+2} \), then 
\[ n(C_{2k+2} - e) = n(P_{2k+2}) = [(2k+2)/2] = k + 1 \] by Theorem 2.7. Hence \( C_{2k+2} \) is \( k \)-minimal. 

**Proposition 2.24.** For every positive integer \( k \), there exists a minimal graph \( G \) such that \( n(G-e) - n(G) > k \) for all edges \( e \) in \( G \).

**Proof:** For a fixed positive integer \( k \), select a positive integer \( p \) such that \( \frac{p(p-1)}{4} > k \). Note that \( p \) must be at least 3. Let \( G = P_3 \cup (p-3)K_1 \). \( n(G) \leq \left\lceil \frac{p(p-1)}{4} \right\rceil \) by Theorem 2.2. For all edges \( e \) in \( G \), \( n(G-e) = \frac{p(p-1)}{2} \) since \( G-e \) has exactly one edge. Since \( p \geq 3 \) the graph \( G \) is minimal. Now 
\[ n(G-e) - n(G) \geq \frac{p(p-1)}{2} - \left\lceil \frac{p(p-1)}{4} \right\rceil \geq \frac{p(p-1)}{4} > k. \]

The next result provides a bound for the number \( n(G-e) \).

**Proposition 2.25.** If \( G \) is a \((p,q)\) graph where \( q \geq 2 \) and \( G \) has multiplicity \( m \) then 
\[ n(G-e) \leq \left\lceil \frac{p(p-1) - mq}{q-1} \right\rceil \] for all edges \( e \) in \( G \).

**Proof:** Since \( G \) has multiplicity \( m \), there exist \( m \) copies of \( G \) in \( K_p \). Hence \( mq \leq \frac{p(p-1)}{2} \). Select an edge \( e \) in \( G \) and let \( n(G-e) = r \). Hence there
exist \( r \) copies of \( G-e \) in \( K_p \). Since \( G-e \) has size \( q-1 \), it follows that \( r(q-1) \leq \frac{p(p-1)}{2} \).

Hence \( r(q-1) + mq \leq p(p-1) \) which yields

\[
r \leq \left[ \frac{p(p-1) - mq}{q-1} \right].
\]
CHAPTER III
DIVISORS AND REMAINDERS

Let $G$ be a graph of order $p$ and multiplicity $m$. Then $K_p = mG \oplus R$. In this chapter we shift our attention from the given graph $G$ and its multiplicity to the last graph in the above sum, $R$. Since the graph $G$ must be nonempty for its multiplicity to be defined, $R$ cannot be complete. There are other graphs which can never appear as $R$ in the above equation. These graphs are characterized in Section 3.2. First we examine, in a general setting, the situation when $R$ is empty.

Section 3.1

Divisors of Graphs

If $R = \overline{K_p}$ in the graphical equation $K_p = mG \oplus R$, then we may write $K_p = mG$. It will be helpful to give a name to this relationship.

Definition 3.1 A spanning subgraph $G$ of a nonempty graph $H$ is a divisor of $H$ if $H = mG$ for some positive integer $m$.

An interesting theorem involving this concept is due to Harary, Robinson, and Wormald [8].
Theorem 3.2. The complete graph $K_p$ has a divisor of size $q$ if and only if $q$ divides $\binom{p}{2}$.

Notice now that every $(p,q)$ graph $G$, where $q \geq 2$, has at least two distinct divisors, namely $K_2 \cup (p - 2)K_1$ and $G$ itself. We distinguish two subclasses of graphs based on this observation and borrow terms for them from Number Theory [6, p. 9].

Definition 3.3. A graph $G$ is prime if it has exactly two distinct divisors. A graph $G$ of size at least two which is not prime is called composite.

Since the size of any divisor of a graph $G$ must divide the size of $G$ it is clear that every graph of prime size is prime. The converse is false; $K_3 \cup K_2$ is a prime graph of composite size. We next establish some sufficient conditions for a graph to be composite.

Theorem 3.4. If $G$ is an eulerian graph of even size, then $G$ is composite.

Proof: Let $C: v_0, v_1, \ldots, v_q = v_0$ be an eulerian circuit of $G$ where $q = |E(G)|$. Since $q$ is even and there is no eulerian graph of size 2, it follows that $q$ is at least 4 and $E(G)$ can be partitioned into $q/2$ subsets, each containing two adjacent edges of $G$, namely:
Let $P_1 = \{v_0v_1, v_1v_2\}$

$P_2 = \{v_2v_3, v_3v_4\}$

$\vdots$

$P_{q/2} = \{v_{q-2}v_{q-1}, v_{q-1}v_q\}$.

We form $q/2$ spanning subraphs $G_i$ of $G$ where $E(G_i) = P_i$ for $1 \leq i \leq q/2$. For each $i$

$G_i = P_3 \cup (p - 3)K_1$; hence $G = q/2(P_3 \cup (p - 3)K_1)$.

Since $q$ is at least 4, $G \neq P_3 \cup (p - 3)K_1$. Therefore $G$ has a divisor other than $K_2 \cup (p-2)K_1$, and $G$ itself which implies that $G$ is composite. ■

**Corollary 3.5.** If $G$ is a connected $(p,q)$ graph containing an eulerian trail where $p \geq 4$ and $q$ is even, then $G$ is composite.

**Proof:** Let $T: v_0, v_1, \ldots, v_q$ be an eulerian trail in $G$. Since $G$ has order at least 4 and is connected, $q$ is not 2; thus $q$ is at least 4. Now the argument employed to prove Theorem 3.4 may again be used to show that $G$ has $P_3 \cup (p - 3)K_1$ as a divisor and that $G \neq P_3 \cup (p - 3)K_1$. Hence $G$ is composite. ■

The following result, due to Chartrand, Polimeni, and Stewart [5] will be useful.
Theorem 3.6. If $G$ is a connected graph with $2n$ odd vertices ($n \geq 1$), then $E(G)$ may be partitioned into $n$ subsets each of which induces a trail in $G$ and where at most one of these trails has odd length.

Theorem 3.7. Let $G$ be a connected $(p,q)$ graph where $p$ is at least 4 and $q$ is even. Then $G$ is composite.

Proof: If $G$ has no odd vertices then $G$ is eulerian and we may conclude that $G$ is composite by Theorem 3.4. The only remaining possibility is that $G$ contains $2n$ odd vertices for some $n \geq 1$. By Theorem 3.6, $E(G)$ may be partitioned into $n$ subsets each of which induces a trail in $G$ and where at most one of these trails has odd length. If exactly one trail had odd length, then $G$ would have odd size, a contradiction. Hence all $n$ trails have even length. Therefore the edge set of each trail may be partitioned into subsets each containing two edges which are adjacent on the trail. Hence $E(G)$ may be partitioned into $q/2$ subsets each containing two edges which are adjacent in $G$. Therefore, as before, $G = q/2(P_3 \cup (p-3)K_1)$. Since $p$ is at least 4, $G$ is connected, and $q$ is even, we know $q \geq 4$ and hence $G \neq P_3 \cup (p-3)K_1$. Hence $G$ has a divisor other than $K_2 \cup (p-2)K_1$ and $G$ itself which implies that $G$ is composite. ■
Using Theorem 3.7, we may write one corollary dropping the requirement that the given graph be connected.

Corollary 3.8. If $G$ is a graph of size at least 4 and every component of $G$ has even size, then $G$ is composite.

Proof: The edge set of any component of $G$ having order at least 4 can be partitioned into subsets each containing exactly two edges which are adjacent in $G$, as was shown in the proof of Theorem 3.7. Every component of $G$ having order less that 4 must either be isomorphic with $P_3$ or be an isolated vertex since all components of $G$ have even size. Hence $E(G)$ may be partitioned into subsets each containing two edges which are adjacent in $G$. Since $G \neq P_3 \cup (p - 3)K_1$, the graph $G$ has a divisor $(P_3 \cup (p - 3)K_1)$ other than $K_2 \cup (p - 2)K_1$ and $G$ itself, so that $G$ is composite.

We present one result concluding that $G$ is composite without the requirement that $G$ have even size.

Theorem 3.9. If $G$ is a connected bipartite $(p,q)$ graph ($p \geq 5$) with at most two odd vertices and if $q = 3n$ for some integer $n$, then $G$ is composite.
Proof: Since $G$ is connected, nontrivial, and contains at most two odd vertices, $G$ is either eulerian or contains an eulerian trial. In either case $G$ contains a trail $T: v_0, v_1', \ldots, v_q$ containing all edges of $G$. Partition $E(G)$ into subsets $P_i$ $(1 \leq i \leq n)$ as follows:

$$P_1 = \{v_0v_1', v_1'v_2, v_2v_3\}$$
$$P_2 = \{v_3v_4', v_4'v_5, v_5v_6\}$$
$$\vdots$$
$$P_n = \{v_{q-3}v_{q-2}', v_{q-2}'v_{q-1}, v_{q-1}'v_q\}.$$

For all $i$, the graph $\langle P_i \rangle = P_4$ since $G$ is bipartite. Form $n$ spanning subgraphs $G_i$ of $G$ where $E(G_i) = P_i$ for $1 \leq i \leq n$. For each $i$, $G_i = P_4 \cup (p - 4)K_1$. Hence $G = n(P_4 \cup (p - 4)K_1)$. Because $p$ is at least 5 and $G$ is connected, $n$ is at least 2 and hence $G \neq P_4 \cup (p - 4)K_1$. Thus $G$ contains a divisor other than $K_2 \cup (p - 2)K_1$ and $G$ itself which implies that $G$ is composite. ■

Section 3.2

Graph Remainders

In the graphical equation $K_p = mG \oplus R$, the graph $R$ contains those edges of $K_p$ which remain after $m$ edge-disjoint copies of the graph $G$ have been deleted. This observation suggests the following definition.
Definition 3.10. A graph \( G \) of order \( p \) is a **remainder** if there exists a graph \( H \) of order \( p \) and multiplicity \( m \) such that \( K_p = mH \oplus G \).

We begin with several characterizations of remainders.

Theorem 3.11. The following statements are equivalent.

1. \( G \) is a remainder.
2. There exists a divisor of \( G \) which is not a subgraph of \( G \).
3. \( \overline{G} \) is not a subgraph of \( G \).
4. \( \overline{G} \) has multiplicity 1.

**Proof:** It is easy to see that statement 3 implies statement 4 and that statement 4 implies statement 1. We first show that statement 2 follows from statement 1.

If \( G \) is a remainder having order \( p \), then there exists a graph \( H \) of order \( p \) and multiplicity \( m \) such that \( K_p = mH \oplus G \). Hence \( \overline{G} = mH \) which implies that \( H \) is a divisor of \( \overline{G} \). If \( H \) is a subgraph of \( G \), then there exists at least \( m + 1 \) edge-disjoint isomorphic copies of \( H \) in \( K_p \) (\( m \) copies in \( \overline{G} \), at least one copy in \( G \)). This contradicts our assumption that \( H \) has multiplicity \( m \). Hence \( H \) is a divisor of \( \overline{G} \) which is not a subgraph of \( G \).
The last argument necessary to complete the proof is that statement 3 is a consequence of statement 2 which we now show in its contrapositive form.

Assume that $\overline{G}$ is a subgraph of $G$ and let $H$ be an arbitrary divisor of $\overline{G}$. We must show that $H$ is a subgraph of $G$. Since $H$ is a divisor of $\overline{G}$, clearly $H$ is a subgraph of $\overline{G}$. We have assumed that $\overline{G}$ is a subgraph of $G$. Hence $H$ is a subgraph of $G$ which completes the proof. ■

Theorem 3.11 has several consequences. Recall (see Berge [3, p. 164] for example) that the connectivity $K(G)$ of a graph $G$ is the minimum number of vertices whose removal from $G$ results in a disconnected or trivial graph.

**Corollary 3.12.** If $K(\overline{G}) > K(G)$, then $G$ is a remainder.

**Proof:** Clearly, if $H$ and $J$ are two graphs such that $H$ is a spanning subgraph of $J$, then $K(H) \leq K(J)$. Hence if $K(\overline{G}) > K(G)$ we may conclude that $\overline{G}$ is not a subgraph of $G$ which implies, by Theorem 3.11, that $G$ is a remainder. ■

**Corollary 3.13.** If $G$ is disconnected, then $G$ is a remainder.
Proof: If $G$ is disconnected then $K(G) = 0$. Also, we know $\overline{G}$ is connected. Since $\overline{G}$ is nontrivial, $K(\overline{G}) \geq 1$. Thus $K(\overline{G}) > K(G)$ and the result follows from Corollary 3.12.

The next result is an immediate consequence of Theorem 3.11.

Corollary 3.14. No self-complementary graph is a remainder.

The question of which graphs are remainders seems intimately tied with the relationship between a graph and its complement. Some graphs $G$ have the property that both $G$ and $\overline{G}$ are remainders.

Theorem 3.15. If $G$ is not self-complementary, then $G$ and $\overline{G}$ are remainders if and only if $n(G) = n(\overline{G})$.

Proof: If both $G$ and $\overline{G}$ are remainders then, by Theorem 3.11, $G \not\subseteq \overline{G}$ and $\overline{G} \not\subseteq G$. Hence $n(G) = n(\overline{G}) = 1$. If, on the other hand, $n(G) = n(\overline{G}) = k$ then $k = 1$ or $2$ by Theorem 2.19. If $n(G) = n(\overline{G}) = 2$ then $G \subseteq \overline{G}$ and $\overline{G} \subseteq G$, which implies that $G$ is self-complementary, a contradiction. Hence $n(G) = n(\overline{G}) = 1$, which implies that $G \not\subseteq \overline{G}$ and $\overline{G} \not\subseteq G$. But, by Theorem 3.11, this implies that both $G$ and $\overline{G}$ are remainders. ■
Theorem 3.16. If $G$ is an acyclic graph with order at least 5 then $G$ is a remainder.

Proof: If $G$ is disconnected, the result follows from Corollary 3.13. The only remaining possibility is that $G$ is a tree of order, say, $p$. Hence $G$ has size $p - 1$ which is less than $p(p - 1)/4$ since $p$ is at least 5. Hence $\overline{G}$ is not a subgraph of $G$ which yields, by Theorem 3.11, the conclusion that $G$ is a remainder. ■

Corollary 3.17. If $T$ is a tree, then $T$ is a remainder if and only if $T$ is none of the graphs: $K_1, K_2, K(1,2), P_4$.

Proof: Each of the trees $K_1, K_2, K(1,2), P_4$ contains its complement as a subgraph. Hence, by Theorem 3.11, none of these trees is a remainder. If $T$ is a tree other than $K_1, K_2, K(1,2), P_4$, then either $T = K(1,3)$ or $T$ has order at least 5. The graph $K(1,3)$ is not a subgraph of $K(1,3)$. Hence $K(1,3)$ is a remainder. If $T$ has order at least 5, then $T$ is a remainder by Theorem 3.16. ■

We include the following more general result involving the concept of hereditary graph-theoretic parameters.
Proposition 3.18. If \( f \) is a hereditary graph-theoretic parameter and if \( G \) is a graph such that \( f(\bar{G}) > f(G) \), then \( G \) is a remainder.

Proof: Since \( f \) is hereditary, \( f(\bar{G}) > f(G) \) implies that \( \bar{G} \) is not a subgraph of \( G \). Hence, by Theorem 3.11, \( G \) is a remainder. ■
CHAPTER IV

THE MULTIPLICITIES OF GRAPHS WITH SMALL SIZE

It is clear that graphs whose sizes are small relative to their orders tend to have large multiplicities. We have seen that the star $K(1,p-1)$, having size $p-1$ and order $p$, has multiplicity 1. Therefore, for each $q$ with $p - 1 \leq q \leq \binom{p}{2}$, there exists a $(p,q)$ graph of multiplicity 1; namely select any $(p,q)$ graph containing $K(1,p-1)$ as a (spanning) subgraph. In this chapter we show that the bound $p - 1$ cannot be reduced; furthermore, we investigate the bound $p - 1$ itself.

Section 4.1

On the Multiplicity of $(p,p-2)$ Graphs

The following result of H. J. Straight (unpublished) will be useful. We include its proof for completeness.

**Lemma 4.1.** Every tree which is not a star is a subgraph of its complement.

**Proof:** We employ induction on $p$, the order of the tree. Note that the lemma holds for $p = 1$. If $T$ is a nonempty tree which is not a star, then the order
of \( T \) is at least 4. Figure 4.1 shows the three trees of order 4 or 5 which are not stars where the edges of each tree are indicated with solid lines. In each case dashed lines are used to indicate the tree as a subgraph of its complement.

![Fig 4.1](image)

Figure 4.1

Thus the lemma follows for \( p \leq 5 \). Assume the result holds for all trees which are not stars and which have orders less than \( k \) (where \( k \) is at least 6) and let \( T \) be a tree of order \( k \) which is not a star.

**Case 1.** Assume that \( T \) contains a vertex \( v \) of degree \( k - 2 \). Let the vertices of \( T \) adjacent with \( v \) be denoted \( v_1, v_2, ..., v_{k-2} \). Let \( w \) be the remaining vertex of \( T \). Clearly \( w \) is not adjacent with \( v \). Since \( T \) is both acyclic and connected, it follows that \( w \) is adjacent to exactly one vertex of the set \( \{v_1, v_2, ..., v_{k-2}\} \). Without loss of generality, assume \( wv_{k-2} \) is an edge in \( T \). Then, in \( \overline{T} \), the vertex \( w \) is adjacent with the \( k - 2 \) vertices.
v_1, v_2, \ldots, v_{k-3}$. Since $v_1v_{k-2}$ is an edge in $\overline{T}$, we conclude that $T \subseteq \overline{T}$.

**Case 2.** Assume that $\Delta(T)$ is at most $k - 3$. In this case, $T$ contains two end-vertices $x$ and $y$ such that $d_{\overline{T}}(x,y)$ is at least 3 and such that $T - \{x,y\}$ is not a star. Since $x$ and $y$ are end-vertices of $T$, the subgraph $T - \{x,y\}$ is a tree and we may conclude by the inductive assumption that $T - \{x,y\}$ is a subgraph of $\overline{T - \{x,y\}}$. Let $u$ and $v$ be the vertices of $T$ adjacent with $x$ and $y$ respectively.

Since $d_{\overline{T}}(x,y) > 2$, vertices $u$ and $v$ are distinct. Because $T - \{x,y\}$ is isomorphic with a subgraph of $\overline{T - \{x,y\}}$, there exist distinct vertices $u'$ and $v'$ in $\overline{T - \{x,y\}}$ corresponding (under the isomorphism) to $u$ and $v$ respectively in $T - \{x,y\}$. If $u' \neq u$ and $v' \neq v$, then $xu'$ and $yv'$ are edges of $\overline{T}$, and hence $T$ is a subgraph of $\overline{T}$. The only remaining possibility is that $u' = u$ or $v' = v$. If $u' = u$, we may conclude that $v' \neq u$ and $u' \neq v$, since $u' \neq v'$ and $u \neq v$. If $v' = v$, we may again conclude that $v' \neq u$ and $u' \neq v$ by the same reasoning. Hence if $u' = u$ or $v' = v$ then $v' \neq u$ and $u' \neq v$ which means that $xv'$ and $yu'$ are edges in $\overline{T}$ and thus $T$ is a subgraph of $\overline{T}$. ■
With the aid of Lemma 4.1, the main result of this section may be presented. Part of the argument in Subcase 2a of the following proof was suggested by S. Schuster.

**Theorem 4.2.** Every graph of order $p \geq 2$ and size $p - 2$ is a subgraph of its complement.

**Proof:** We proceed by induction on $p$. The theorem is easily seen to be true for $2 \leq p \leq 5$ by inspecting the eight graphs in question. Assume that every $(p, p-2)$ graph is contained in its complement for $2 \leq p < k$, where $k$ is at least 6, and let $G$ be a $(k, k-2)$ graph. If $G$ is acyclic, then $G$ has exactly two components each of which is a tree. Since $k \geq 4$, an edge can be added to $G$ to form a tree $G'$ which is not a star. By Lemma 4.1, it follows that $G' \subseteq \overline{G'}$. Hence $G \subseteq G' \subseteq \overline{G'} \subseteq \overline{G}$ and our conclusion is established. For the remainder of the proof we assume that $G$ is cyclic.

**Case 1.** Assume that $G$ contains two nonadjacent end-vertices $x$ and $y$, not both adjacent to some single third vertex. Consider $G - \{x, y\}$, a $(k-2, k-4)$ graph. By the inductive assumption, $G - \{x, y\}$ is a subgraph of $\overline{G - \{x, y\}}$. Let $u$ and $v$ be the distinct vertices of $G$ adjacent with $x$ and $y$ respectively.
Since $G - \{x,y\} \subseteq \overline{G - \{x,y\}}$, there exist distinct vertices $u'$ and $v'$ in $\overline{G - \{x,y\}}$ corresponding to $u$ and $v$ respectively in $G - \{x,y\}$. Arguing as in Case 2 of the proof of Lemma 4.1, we first consider the possibility that $u' \neq u$ and $v' \neq v$. Here $xu'$ and $yv'$ are edges in $\overline{G}$ and therefore $G$ is a subgraph of $\overline{G}$. The only remaining possibility is that $u' = u$ or $v' = v$. By the same reasoning as used in Case 2 of the preceding proof, we may conclude that $v' \neq u$ and $u' \neq v$, and hence $xv'$ and $yu'$ are edges in $\overline{G}$. This implies that $G$ is a subgraph of $\overline{G}$.

**Case 2.** Assume that $G$ does not contain two end-vertices as specified in Case 1. Then the number of nontrivial acyclic components of $G$ is at most 1. This follows because if $G$ contains two or more nontrivial acyclic components, then an end-vertex chosen from each of two of these components may play the role of $x$ and $y$ in Case 1, contradicting our assumption here that no such pair of end-vertices exist in $G$.

We distinguish two subcases, depending on whether the number of nontrivial acyclic components of $G$ is zero or one.

**Subcase 2a.** Assume that $G$ contains exactly one nontrivial acyclic component. Remove this component from
G, leaving a \((p, p-1)\) graph \(H\) for some \(p < k\). Each component of \(H\) is either cyclic or is an isolated vertex. Let \(m\) denote the number of vertices of \(H\) (and hence of \(G\)) which belong to cyclic components of \(H\) and let \(m'\) denote the number of isolated vertices of \(H\) (and hence of \(G\)). Then \(m + m' = p\). Let vertex \(v\) have minimum degree among all vertices of \(H\) (and of \(G\)) which belong to cyclic components of \(H\). Let \(\deg_{H} v = \deg_{G} v = n\). If \(n = 1\), then vertex \(v\) together with an end-vertex in the nontrivial acyclic component of \(G\) would produce the situation in Case 1, a contradiction. Hence \(n \geq 2\).

The cyclic components of \(H\) contain \(m\) vertices, each having degree at least \(n\). Hence \(H\) contains at least \(mn/2\) edges. Since \(H\) has size \(p - 1\) we have \(p - 1 \geq mn/2\). Substituting \(m + m'\) for \(p\) yields \(m + m' - 1 \geq mn/2\).

Now

\[
m' \geq \frac{mn}{2} - m + 1
\]

\[
= m\left(\frac{n}{2} - 1\right) + 1.
\]

Since \(m \geq 3\) and \(n \geq 2\),

\[
m' \geq 3\left(\frac{n}{2} - 1\right) + 1
\]

\[
= \frac{3n}{2} - 2
\]

\[
\geq n - 1.
\]
This says that \( H \) (and \( G \)) contains a vertex \( v \), of degree \( n \) and a collection of \( n - 1 \) isolated vertices, say \( v_1, v_2, \ldots, v_{n-1} \). Consider the graph \( G - \{v, v_1, v_2, \ldots, v_{n-1}\} \). This graph has order \( r \) and size \( r - 2 \), for some \( r \) satisfying \( 2 \leq r < k \).

Hence \( G - \{v, v_1, v_2, \ldots, v_{n-1}\} \) is a subgraph of \( G - \{v, v_1, v_2, \ldots, v_{n-1}\} \) by the inductive assumption. Therefore each vertex \( u \) of \( G - \{v, v_1, v_2, \ldots, v_{n-1}\} \) corresponds to exactly one vertex \( u' \) of \( G - \{v, v_1, v_2, \ldots, v_{n-1}\} \). Since \( n \) is at least 2, there is at least one isolated vertex \( v_1 \) in the set \( \{v_1, v_2, \ldots, v_{n-1}\} \). Define a map \( \hat{\phi}: V(G) \to V(\overline{G}) \) as follows: \( \hat{\phi}(v) = v_1, \hat{\phi}(v_1) = v, \hat{\phi}(v_i) = v_i \) for all \( i \) satisfying \( 2 \leq i \leq n - 1 \), and \( \hat{\phi}(u) = u' \) for all vertices \( u \) belonging to \( G - \{v, v_1, \ldots, v_{n-1}\} \). Since \( \hat{\phi}(v) = v_1 \) is isolated in \( G \), \( v_1 \) has all the required adjacencies in \( \overline{G} \). Therefore \( G \) is a subgraph of \( \overline{G} \) as indicated by the map \( \hat{\phi} \).

Subcase 2b. Assume that \( G \) contains no nontrivial acyclic components. Here \( G \) consists of cyclic components together with isolated vertices. Suppose \( G \) contains \( s \) end-vertices, say \( v_1, v_2, \ldots, v_s \). If \( s = 0 \), select a vertex \( v \) of minimum degree among the vertices of \( G \) which have degree at least 2. Say \( \deg_G v = n \geq 2 \). Let \( m \) denote the number of vertices...
in $G$ of degree at least 2, and let $m'$ denote the number of isolated vertices of $G$. Then $G$ contains $m$ vertices, each of degree at least $n$. Therefore $G$ contains at least $mn/2$ edges. Since $G$ is a $(k,k-2)$ graph, $m + m' = k$ and $k - 2 \geq mn/2$. Hence
\[ m + m' - 2 \geq \frac{mn}{2} \]
which yields
\[ m' \geq \frac{mn}{2} - m + 2 \]
\[ = m\left(\frac{n}{2} - 1\right) + 2. \]
Since $m \geq 3$ and $n \geq 2$,
\[ m' \geq 3\left(\frac{n}{2} - 1\right) + 2 \]
\[ = \frac{3n}{2} - 1 \]
\[ > n - 1. \]
Therefore $G$ contains a vertex of degree $n$ and at least $n - 1$ isolated vertices where $n$ is at least 2, and the argument of Subcase 2a may be employed to conclude that $G$ is a subgraph of $G$.

Next we consider the possibility that $s > 0$. Here the conditions of Case 2 require that all the end-vertices $v_1, v_2, \ldots, v_s$ belong to one cyclic component of $G$, and that they are all adjacent to a single vertex, say $v_{s+1}$, in that component. If the degree of $v_{s+1}$ is at least 2 in the graph $G - \{v_1, v_2, \ldots, v_s\}$, then choose a vertex $v$ of
minimum degree among the non-isolated vertices of
G - \{v_1, v_2, ..., v_s\} other than v_{s+1}. Say
deg_{G-\{v_1,v_2,...,v_s\}} = deg_v = m \geq 2. Since each of
the vertices v_1, v_2, ..., v_s is of degree 1, the
graph G - \{v_1, v_2, ..., v_s\} is a (p,p-2) graph for
some p < k. Let m' be the number of isolated ver-
tices of G - \{v_1, v_2, ..., v_s\} (and of G). Let r
be the number of non-isolated vertices of
G - \{v_1, v_2, ..., v_s\} other than v_{s+1}. Each of these
r vertices has degree at least m in
G - \{v_1, v_2, ..., v_s\} and in G. Since G is cyclic,
r is at least 2. Finally G - \{v_1, v_2, ..., v_s\}
contains vertex v_{s+1} which has degree at least two.
Hence m' + r + 1 = p.

Now G - \{v_1, v_2, ..., v_s\} contains r vertices
of degree at least m and one vertex v_{s+1} of degree
at least 2. Therefore G - \{v_1, v_2, ..., v_s\}
contains at least \((2 + rm)/2\) edges. Thus
\[ p - 2 \geq (2 + rm)/2 \] or equivalently
\[ 2p - 4 \geq 2 + rm. \]
Hence
\[ 2(m' + r + 1) - 4 \geq 2 + rm \]
which yields
\[ 2m' \geq 4 + r(m - 2). \]
Since m and r are at least 2,
\[ 2m' \geq 4 + 2(m - 2) = 2m. \]
Therefore $G$ contains a vertex of degree $m \geq 2$ and at least $m - 1$ isolated vertices, and the argument of Subcase 2a may be employed to conclude that $G$ is a subgraph of $\overline{G}$.

The only remaining possibility is that $s > 0$ and that the degree of vertex $v_{s+1}$ in $G - \{v_1, v_2, \ldots, v_s\}$ is one. Figure 4.2 illustrates this situation.

![Figure 4.2](image_url)

Let $v_{s+2}$ be the single vertex of $G - \{v_1, v_2, \ldots, v_s\}$ adjacent to $v_{s+1}$ and consider the degree of $v_{s+2}$ in the graph $G - \{v_1, v_2, \ldots, v_s, v_{s+1}\}$. If this degree is 1, let $v_{s+3}$ be the single vertex in $G - \{v_1, v_2, \ldots, v_{s+1}\}$ adjacent to $v_{s+2}$ and consider

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the degree of \( v_{s+3} \) in the graph \( G - \{v_1, v_2, \ldots, v_{s+2}\} \). Continue in this fashion. Since these vertices all belong to one cyclic component of \( G \) whose only end-vertices are \( v_1, v_2, \ldots, v_s \), there must exist a least index \( r \) such that all non-isolated vertices of \( G - \{v_1, v_2, \ldots, v_r\} \) have degree at least 2. In forming the graph \( G - \{v_1, v_2, \ldots, v_r\} \), we have removed \( r \) vertices and \( r \) edges from \( G \). Select a vertex \( v \) of minimum degree among the non-isolated vertices of \( G - \{v_1, v_2, \ldots, v_r\} \). Let 
\[
\deg_{G - \{v_1, v_2, \ldots, v_r\}} v = n \geq 2.
\]
Let \( G - \{v_1, v_2, \ldots, v_r\} \) have \( m' \) isolated vertices and \( m \) vertices of degree at least \( n \). Hence \( G - \{v_1, v_2, \ldots, v_r\} \) has at least \( mn/2 \) edges.

Now reconsider the graph \( G \). It follows that \( G \) contains \( m' \) isolated vertices, at least \( mn/2 + r \) edges, and \( m + r \) vertices belonging to cyclic components of \( G \) where \( m \) is at least 3. Hence 
\[
m + r + m' = k \quad \text{and} \quad k - 2 \geq mn/2 + r.
\]
Substituting \( m + r + m' \) for \( k \) yields 
\[
m + r + m' - 2 \geq \dfrac{mn}{2} + r.
\]
Hence 
\[
m + m' - 2 \geq \dfrac{mn}{2}
\]
and
\[
m' \geq \frac{mn}{2} - m + 2 \\
= m\left(\frac{n}{2} - 1\right) + 2.
\]
Since \( m \geq 3 \) and \( n \geq 2 \),
\[
m' \geq 3\left(\frac{n}{2} - 1\right) + 2 \\
= \frac{3n}{2} - 1 \\
> n - 1.
\]
Hence \( G \) contains at least \( n \) isolated vertices.

Recall that \( G - \{v_1, v_2, \ldots, v_r\} \) contains a vertex \( v \) of degree \( n \). If \( v \) is adjacent to \( v_r \) in \( G \), then \( \deg_G(v) = n + 1 \). If \( v \) is not adjacent to \( v_r \) in \( G \), then \( \deg_G(v) = n \). In either case there are at least \( \deg_G(v) - 1 \) isolated vertices in the graph \( G \) and the argument of Subcase 2a may be employed to show that \( G \) is a subgraph of \( \overline{G} \). This completes the proof. ■

Immediate consequences of this result are now given in the following three corollaries.

**Corollary 4.3.** If \( G \) is a \((p,q)\) graph where \( p \geq 3 \) and \( 0 < q \leq p - 2 \), then \( n(G) \geq 2 \).

**Proof:** Construct a \((p,q')\) supergraph \( H \) of \( G \) by adding sufficiently many edges to \( G \) so that \( q' = p - 2 \). By Theorem 4.2, \( H \subset \overline{H} \). Hence \( G \subset H \subset \overline{H} \subset \overline{G} \). Since
q > 0 , the multiplicity of G is defined and since $G \subseteq \overline{G}$, $n(G) \geq 2$. ■

**Corollary 4.4.** If G is a (p,q) graph where $p \geq 2$ and $q \leq p - 2$, then G is a subgraph of $\overline{G}$.

**Proof:** Let H be the (p,p-2) supergraph of G defined in the proof of Corollary 4.3. By Theorem 4.2, $H \subseteq \overline{H}$. Therefore, $G \subseteq H \subseteq \overline{H} \subseteq \overline{G}$. ■

**Corollary 4.5.** If G is a (p,q) graph where $p \geq 2$ and $q > (p^2 - 3p + 2)/2$, then G is not a remainder.

**Proof:** Since $q > \frac{p^2 - 3p + 2}{2} = \binom{p}{2} - (p - 1)$, the complement of G has fewer than $p - 1$ edges. By Corollary 4.4, $\overline{G} \subseteq G$ and hence G is not a remainder by Theorem 3.11. ■

We are now able to show that the bound $p - 1$ discussed in the introduction to this chapter cannot be reduced.

**Corollary 4.6.** Given a positive integer $p (\geq 2)$, the least integer q such that there exists a (p,q) graph having multiplicity 1 is $p - 1$.

**Proof:** The result is obvious for $p = 2$. When $p$ is at least 3, the star graph $K(1,p-1)$ has size $p - 1$ and multiplicity 1. Any nonempty graph of order $p$
with fewer than \( p - 1 \) edges has multiplicity at least 2 by Corollary 4.3. ■

Section 4.2

On the Multiplicity of \((p,p-1)\) Graphs

Theorem 4.2 asserts that every \((p,p-2)\) graph, \( p \geq 2 \), is a subgraph of its complement. This result is best possible in the sense that not all \((p,p-1)\) graphs are subgraphs of their complements. For example, neither \( K(1,p-1) \) nor \( K_3 \cup K(1,p-4) \), with \( p \geq 8 \), are subgraphs of their complements. The following theorem is useful for the task of characterizing those \((p,p-1)\) graphs of even order which are subgraphs of their complements.

**Theorem 4.7.** Let \( G \) be a \((p,p-1)\) graph, where \( p \) is at least 6, satisfying the following three conditions:

1. exactly one component of \( G \) is a star,
2. all components of \( G \) other than the star are cycles,
3. \( G \) is neither a star nor \( K_3 \cup K(1,n) \), where \( n \geq 4 \).

Then \( G \) is a subgraph of \( \overline{G} \).

**Proof:** We employ induction on \( m \), the number of cycles in \( G \). Note that \( m \geq 1 \) since \( G \) is not a star.
If \( m = 1 \), then \( G = C_r \cup K(l,n) \) for some \( r \geq 3 \) and \( n \geq 1 \). If \( r = 3 \), then \( n \leq 3 \) since \( G \) is not \( K_3 \cup K(l,n) \) for \( n \geq 4 \). Also, if \( r = 3 \), then \( n \geq 2 \) since \( G \) has order at least 6. Hence, for \( r = 3 \), \( G = K_3 \cup K(l,2) \) or \( G = K_3 \cup K(l,3) \). In either case, \( G \subset \overline{G} \) as indicated in Diagrams 1 and 2, respectively, of Figure 4.3 where the edges of the graph \( G \) are indicated with solid lines and where dashed lines indicate edges present in \( \overline{G} \). If \( r = 4 \) and \( n = 1 \), Diagram 3 shows that \( G \subset \overline{G} \). If \( r = 4 \) and \( n = 2 \), Diagram 4 shows that \( G \subset \overline{G} \) where the solid vertex represents the presence of dashed edges between itself and all other vertices in that diagram which are not already incident with dashed edges. For \( r = 4 \) and \( n \geq 3 \), simply add \( n - 2 \) vertices to Diagram 4, each adjacent to vertex \( v \). With the same understanding about the meaning of the solid vertex, this modified Diagram 4 shows \( G \subset \overline{G} \). If \( r \geq 5 \) and \( n = 1 \), \( G = C_r \cup K_2 \) as indicated in Diagram 5. Select adjacent vertices \( v_1 \) and \( v_2 \) belonging to \( C_r \) and join them in \( \overline{G} \) to vertex \( u \), one of the two end-vertices. Connect \( v_1 \) and \( v_2 \) by a path of length \( r - 2 \) where all vertices on the path belong to \( C_r \) and all edges belong to \( \overline{G} \). Since \( r \geq 5 \) such a path always exists. This procedure is indicated only for
\( r = 5 \) in the diagram. Allow the remaining solid vertex of \( C_r \) to have the same meaning as before, and we may conclude that \( G \subseteq \overline{G} \). If \( r \geq 5 \) and \( n \geq 2 \), add \( n - 1 \) vertices to Diagram 5, each adjacent to vertex \( w \), and this modified diagram shows \( G \subseteq \overline{G} \). Thus the result holds for \( m = 1 \).

Assume that \( G \subseteq \overline{G} \) for any \((k,k-1)\) graph \( G \) (where \( k \geq 6 \)) consisting of exactly one star and \( m \) other components each of which is a cycle, \( 1 \leq m \leq \ell \) with \( \ell \geq 2 \) and where \( G \) is neither a star nor \( K_3 \cup K(l,n) \) for \( n = 4 \).

Let \( H \) be a \((p,p-1)\) graph, where \( p \geq 6 \), and where \( H \) consists of exactly one star and \( \ell \) other components each of which is a cycle, and where \( H \) is neither a star nor \( K_3 \cup K(l,n) \) for \( n \geq 4 \).

To complete the proof we must show \( H \subseteq \overline{H} \).

First assume that \( H \) contains \( C_r \) as a component, where \( r \geq 5 \). Consider the graph \( H - C_r \). This graph \( H - C_r \) is a \((k,k-1)\) graph for some \( k < p \).

Now \( H - C_r \) is not a star since \( \ell \geq 2 \) and we have removed only one cycle from \( H \). If \( H - C_r = K_3 \cup K(l,n) \) for \( n = 4 \), consider \( H \) as indicated in Diagram 6.

Here we form \( C_r \) in \( \overline{H} \) as in Diagram 5 and form \( K_3 \) in \( \overline{H} \) using the three remaining end-vertices of \( K(l,n) \).

The single remaining vertex of \( C_r \) is solid; hence \( H \subseteq \overline{H} \). If \( H - C_r = K_3 \cup K(l,n) \) for \( n \geq 5 \), add ...
n - 4 vertices to Diagram 6, each adjacent to vertex w in H, and this modified diagram shows H \subset \overline{H}, as before. If H - C_r has order at most 5, then
H = C_r \cup K_3 \cup K_2 since H - C_r contains at least one cycle and exactly one star. This situation is represented by Diagram 7 for the case r = 5. Here we let
the vertices of C_r be sequentially labeled v_1, v_2, ..., v_r and the vertices of K_3 be labeled w_1, w_2, w_3. If the vertices of K_2 are labeled u_1 and u_2, then v_2u_1, u_1w_1, and w_1v_1 induce P_4 in \overline{H}. Next connect v_1 and v_2 by a path of length r - 3 all of whose vertices belong to C_r and all of whose edges are in \overline{H}. Choose this path so that the two vertices of C_r not incident with path edges are not adjacent in H. Such a path exists in C_r for all r \geq 5. These two vertices of C_r, together with w_3, induce K_3 in \overline{H}. Finally, u_2 and w_2 induce K_2 in \overline{H}. Hence H \subset \overline{H}.

The only remaining possibility is that H - C_r is neither a star nor K_3 \cup K(1,n) for n \geq 4 and H - C_r has order at least 6. Since H - C_r has \ell - 1 cycles where \ell \geq 2, we may conclude by the inductive assumption that H - C_r \subset \overline{H - C_r}. Now, since r \geq 5, it follows that C_r \subset \overline{C_r} and thus H \subset \overline{H}.

We now have completed the inductive implication for all graphs H as specified above which contain a cycle.
Cr for r ≥ 5. We now consider graphs H as specified subject to the condition that all cycles of H have orders 3 or 4. We distinguish two cases depending on whether C₄ is a component of H.

Case 1. Assume that H contains C₄ as a component. Then, since l ≥ 2, H has another component which is a cycle, say Cᵣ where r = 3 or 4. Consider H - {C₄, Cᵣ}. If this graph is a star, then

H = C₄ U C₄ U K(l,n) for n ≥ 1 or H = C₄ U K₃ U K(l,n) for n ≥ 1. These two possibilities are illustrated in Diagrams 8 and 9, respectively, for the case n = 1 and can be extended for any positive integral value of n as described earlier. In each case we see H ⊆ H.

If H - {C₄, Cᵣ} is K₃ U K(l,n) for n ≥ 4, then H = C₄ U C₄ U K₃ U K(l,n) with n ≥ 4 or H = C₄ U K₃ U K₃ U K(l,n) with n ≥ 4. In the first case H - {C₄, K₃} = C₄ U K(l,n) with n ≥ 4. Hence H - {C₄, K₃} ⊆ H - {C₄, K₃} by the inductive assumption.

Now C₄ U K₃ ⊆ C₄ U K₃ as indicated in Diagram 10. Hence H ⊆ H. In the second case H ⊆ H as indicated in Diagram 11 for the case n = 4. For the case n ≥ 5 modify the diagram as before by adding n - 4 vertices each adjacent to vertex v.

Now if H - {C₄, Cᵣ} is neither a star nor K₃ U K(l,n) for n ≥ 4, then H - {C₄, Cᵣ} has exactly
one component which is a star and at least one component which is a cycle. Thus if \( H - \{C_4, C_r\} \) has order at most 5, then \( H - \{C_4, C_r\} = K_3 \cup K_2 \). Hence 
\[ H = C_4 \cup C_4 \cup K_3 \cup K_2 \] 
or 
\[ H = C_4 \cup K_3 \cup K_3 \cup K_2 \]. 
In the first case \( H - \{C_4, K_3\} = C_4 \cup K_2 \subset C_4 \cup K_2 \) as indicated in Diagram 3. Now \( C_4 \cup K_3 \subset C_4 \cup K_3 \) as shown in Diagram 10; hence \( H \subset \overline{H} \). In the second case, \( H \subset \overline{H} \) as indicated in Diagram 12.

The only remaining possibility is that \( H - \{C_4, C_r\} \) is neither a star nor \( K_3 \cup K(l,n) \) for \( n \geq 4 \) and has order at least 6. Since \( H - \{C_4, C_r\} \) is a \((k,k-1)\) graph and consists of exactly one star and at least one other component which is a cycle, \( H - \{C_4, C_r\} \subset \overline{H - \{C_4, C_r\}} \) by the inductive assumption. Now 
\[ C_4 \cup C_r \subset C_4 \cup C_r \] 
as shown in Diagram 10 or 13 depending on whether \( r = 3 \) or 4. Hence \( H \subset \overline{H} \).

**Case 2.** Assume that every cyclic component of \( H \) is \( K_3 \). Hence \( H \) contains \( l + 1 \) components; namely one star and \( l \) triangles where \( l \geq 2 \). If \( l = 2, 3, \) or 4, then \( H \subset \overline{H} \) as indicated in Diagrams 14, 15, and 16, respectively. As before, the diagrams indicate that \( H \) is a subgraph of \( \overline{H} \) only when the star is \( K_2 \). When the star is \( K(l,n) \) for \( n \geq 2 \), add \( n - 1 \) vertices to the appropriate diagram each adjacent to vertex \( v \).
Figure 4.3

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Figure 4.3 continued
Figure 4.3 continued
If \( l \geq 5 \) consider \( H - \{K_3, K_3, K_3\} \). Now \( H - \{K_3, K_3, K_3\} \subset H - \{K_3, K_3, K_3\} \) by the inductive assumption. Since \( K_3 \cup K_3 \cup K_3 \subset K_3 \cup K_3 \cup K_3 \), as indicated in Diagram 17, it follows that \( H \subset \overline{H} \) which completes the proof. ■

Theorem 4.7 may now be used to characterize those \((p, p-1)\) graphs of even order which are subgraphs of their complements. The arguments in Case 2a of the following proof were suggested by S. Schuster.

**Theorem 4.8.** Let \( p (\geq 6) \) be an even integer, and let \( G \) be a \((p, p-1)\) graph. Then \( G \subset \overline{G} \) if and only if \( G \) is neither a star nor \( K_3 \cup K(l,n) \) with \( n \geq 4 \).

**Proof:** It follows readily that \( G \) is not a subgraph of \( \overline{G} \) when \( G \) is a star or when \( G = K_3 \cup K(l,n) \) for \( n \geq 4 \).

It remains to show that \( G \subset \overline{G} \) if \( G \) is a \((p, p-1)\) graph for \( p (\geq 6) \) even and where \( G \) is neither a star nor \( K_3 \cup K(l,n) \) for \( n \geq 4 \). The argument is by induction on the even positive integers \( p \geq 6 \).

If \( p = 6 \), then \( G \neq K_3 \cup K(l,n) \) for \( n \geq 4 \) since the order of that graph is at least 8. To establish the inductive anchor then, it is sufficient to prove that every \((6,5)\) graph with the exception of \( K(l,5) \) is a subgraph of its complement. All 14 such
graphs appear in Figure 4.4 with solid edges and, in each case, the graph appears in the complement with dashed edges.

Let \( k \geq 8 \) be an even integer. Assume all \((p,p-1)\) graphs \( H \) are subgraphs of their complements where \( p \) is even, \( 6 \leq p < k \), and where \( H \) is neither a star nor \( K_3 \cup K(l,n) \) for \( n \geq 4 \).

Let \( G \) be a \((k,k-1)\) graph which is neither a star nor \( K_3 \cup K(l,n) \) for \( n \geq 4 \).

If \( G \) is acyclic, then \( G \subset \bar{G} \) by Lemma 4.1. For the remainder of the proof, we assume \( G \) is cyclic.

**Case 1.** Assume \( G \) contains two non-adjacent end-vertices \( x \) and \( y \) not both adjacent to some single third vertex. Then \( G - \{x,y\} \) is a \((p,p-1)\) graph where \( p \) is even and \( 6 \leq p < k \). If \( G - \{x,y\} \) is a star then \( G \) is acyclic, contradicting our assumption in this portion of the proof. If \( G - \{x,y\} \) is \( K_3 \cup K(l,n) \), for \( n \geq 4 \), then either two other end-vertices \( x' \) and \( y' \) can be chosen in \( G \) with the same properties as \( x \) and \( y \) and such that \( G - \{x',y'\} \) is not \( K_3 \cup K(l,n) \), for \( n \geq 4 \), or \( G \) is one of the two graphs in Figure 4.5.

Each of these two graphs is a subgraph of its complement, as indicated in Figure 4.5. Hence either
Figure 4.4

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our conclusion is established or we make the substitution $x'$ for $x$ and $y'$ for $y$.

Figure 4.5

Therefore we may assume that $G - \{x, y\}$ is neither a star nor $K_3 \cup K(1,n)$, for $n \geq 4$. Now $G - \{x, y\} \subseteq G - \{x, y\}$ by the inductive assumption. Here the argument of Case 1 of the proof for Theorem 4.2 may be employed to conclude that $G \subseteq \overline{G}$.

Case 2. Assume that no pair of end-vertices as required in Case 1 exist. Then there is at most one nontrivial acyclic component in $G$.

Subcase 2a. Assume that $G$ contains no nontrivial acyclic component. Hence every component of $G$ is either cyclic or is an isolated vertex. Clearly, not every component of $G$ is cyclic since $G$ is a $(k,k-1)$ graph. Hence $G$ has an isolated vertex $x$. If $G$ contains a vertex $v$ of degree at least 3 then consider the graph $G - \{v, x\}$. This graph has $k - 2$ vertices and at most $k - 4$ edges, where $k \geq 4$. 

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Hence $G - \{v,x\} \subseteq \overline{G - \{v,x\}}$ by Corollary 4.4. Therefore, each vertex $u$ of $G - \{v,x\}$ corresponds to some vertex $u'$ of $\overline{G - \{v,x\}}$. Define 
\[ \hat{\xi} : V(G) \to V(\overline{G}) \] by $\hat{\xi}(v) = x$, $\hat{\xi}(x) = v$, and $\hat{\xi}(u) = u'$ for all $u$ different from $v$ and $x$.
Hence $G \subseteq \overline{G}$ as indicated by the map $\hat{\xi}$.

The only remaining possibility in Subcase 2a is that every vertex in $G$ has degree at most 2. Here $G$ consists only of cycles and isolated vertices. Hence $G$ has a vertex $v$ of degree 2. Again consider the $(p,p-1)$ graph $G - \{v,x\}$ where $p$ is even and $6 \leq p < k$. Vertex $v$ belongs to a cyclic component of $G$. If $G - \{v,x\}$ is a star, then that star belongs to the same cyclic component of $G$ as $v$ does, since $G$ contains no component which is a star. Moreover, there is only one cyclic component in $G$, namely the one containing $v$. Also, the star must be $K(1,2)$ or $K(1,1)$ since every vertex in $G$ has degree at most 2. These conditions imply that $G$ is either $K_3 \cup K_1$ or $C_4 \cup K_1$. But the order of $G$ is at least 8. Hence $G - \{v,x\}$ is not a star.

If $G - \{v,x\}$ is the graph $K_3 \cup K(1,n)$ for $n \geq 4$, then $G$ contains a vertex of degree 3 or more, contradicting our assumption that $\Delta(G) \leq 2$. Hence $G - \{v,x\} \neq K_3 \cup K(1,n)$, for $n \geq 4$. 

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Now we conclude, using the inductive assumption, that $G - \{v, x\} \subset \overline{G} - \{v, x\}$. Define the map $\xi$ as before and we may conclude that $G \subset \overline{G}$.

**Subcase 2b.** Assume $G$ contains exactly one nontrivial acyclic component. Here we know that this nontrivial acyclic component must be a star because of the assumption for Case 2. $G$ also has cyclic components and may have isolated vertices as well. All cyclic components of $G$ have minimum degree at least 2, otherwise two end vertices exist in $G$ as required in Case 1. If there exists a vertex $v$ of degree at least 3 in the cyclic components of $G$, then $G$ contains an isolated vertex $x$ and $G - \{v, x\} \subset \overline{G} - \{v, x\}$ by Corollary 4.4. Define the map $\xi: V(G) \to V(\overline{G})$ as before, and it follows that $G \subset \overline{G}$.

The only remaining possibility in Subcase 2b is that every cyclic component of $G$ is a cycle. Here $G$ is a $(k, k-1)$ graph which is neither a star nor $K_3 \cup K(1, n)$, with $n \geq 4$, and $G$ contains exactly one component which is a star and all other components are cycles. Since $k \geq 8$, we know $G \subset \overline{G}$ by Theorem 4.7. This completes the proof. $\blacksquare$

If every occurrence of the word "even" in the preceding argument were changed to "odd", then the
inductive implication would still hold. The difficulty when \( p \) is odd is with the "anchor". The theorem is false for \( p = 7 \) since, for example, \( K_3 \cup K_3 \cup K_1 \) is not a subgraph of its complement. The best hope for a similar result dealing with \((p,p-1)\) graphs for \( p \) odd is to establish an odd anchor at \( p = 9 \). Such an anchor would require verification of this conjecture for approximately 350 graphs.

**Corollary 4.9.** If \( p \geq 2 \) is even and if \( G \) is a \((p,p-1)\) graph, then \( n(G) = 1 \) if and only if \( G \) is one of the graphs \( K_2, K_3 \cup K_1, K(1,p-1) \) or \( K_3 \cup K(1,n) \) for \( n \geq 4 \).

**Proof:** The complete graph \( K_2 \) is the only \((2,1)\) graph and \( K_2 \not\subset K_2 \). Among the three \((4,3)\) graphs, \( G \not\subset \overline{G} \) if and only if \( G \) is \( K_3 \cup K_1 \) or \( K(1,3) \). The situation for \( p \geq 6 \) and \( p \) even is the content of Theorem 4.8, with the understanding that \( n(G) = 1 \) if and only if \( G \not\subset \overline{G} \).

The next result follows immediately from Corollary 4.3 and Corollary 4.9.

**Corollary 4.10.** If \( p \geq 2 \) is even and \( G \) is a \((p,q)\) graph where \( 0 < q \leq p - 1 \) then \( n(G) = 1 \) if and only if \( q = p - 1 \) and \( G \) is \( K_2, K_3 \cup K_1, K(1,p-1), \) or \( K_3 \cup K(1,n) \), for \( n \geq 4 \).
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