A T-Matrix Analysis for the Scattering Cross Section

Michael J. Linville

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A T-MATRIX ANALYSIS
FOR THE SCATTERING CROSS SECTION

by
Michael J. Linville

A Thesis
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
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Michael J. Linville
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I. INTRODUCTION

One of the prominent features of nature is "motion". If this motion is a relative motion of two objects it is possible to consider a collision occurring between them. It is likely that during the collision process these objects will be deflected from their original path. Through a systematic analysis of the collision it should be possible to determine some information about the forces of interaction between the objects. Possibly information about the internal structure of the objects could be derived from the investigation. If we regress momentarily, the marble ring was where many experimentalists actually did their first scattering experiments. If the concern be with the collisions of objects in the context of the human senses, the concern can also be with the collision of particles or systems with atomic or sub-atomic dimensions. By the nature of atomic particles or systems there is little recourse other than scattering to obtain detailed information. This detail involves very small separations of the order $10^{-15}$ meter; the primary probe for such dimensions is a flux of atomic particles with relatively high kinetic energy and
De Broglie wavelength that is small, at least compared to the scatterer. Bohr established the theoretical structure of the nucleus through the analysis of data from the scattering of nucleons by nuclei.

The physical quantity called the differential scattering cross section characterizes the angular distribution of the scattered particles. Physically it is defined as the ratio of the number of particles \(dN_{\text{scat}}\) scattered per unit time into the solid angle \(d\Omega\) to the flux density of the incident particles.

It is necessary to distinguish between elastic and inelastic scattering. In elastic scattering the internal state of both the scatterer and the scattered system remains unchanged. In inelastic scattering the internal states of one or both are affected. Initially we will discuss elastic scattering because of its relative simplicity since internal structure can be ignored.

The elastic scattering cross section can be expressed in terms of the physical quantity, the so-called "phase shift". We may recall that the only effect a central potential \(U(r)\) has on an incident wave is to change its phase with respect to the phase of the same wave if the potential were absent.
Solving the Schrödinger equation for the case of the centrally symmetric potential will eventually yield the expression for the total cross section

$$\sigma = \frac{2}{\pi} \frac{4\pi}{R} (2l+1) \sum \xi^2 \xi_{\ell}.$$  

The phase shifts \( \xi_{\ell} \) then links the formal theory, Schrödinger's equation, with the physically measurable total cross section. This bridging between the theory and the physically measurable will be generalized to the inelastic case. Instead of the phase shifts of the elastic case, the analogous quantity will be the Transition "T" matrix in the inelastic case. An expression for the inelastic scattering cross section will be sought that has the geometry of the problem distinguished from the physical factors. These purely physical aspects of the cross section will be represented by the T matrix.

Chapter two will see the analysis of the elastic scattering, of a spinless projectile from a spinless target, by means of Green's Function techniques. The T matrix will be developed for this case. The next step in the progression will be to the theoretical analysis of a spinless projectile from a target with spin \( \mathbf{I} \). As a natural extension this will also yield a T matrix by means of the Green's Function technique. The third case to be considered
will be the scattering of a projectile with spin \( i_0 \) from a scatterer with spin \( I_0 \). The most general case to be analyzed in terms of the technique will be the inelastic scattering of a spin \( L \), isospin \( T \) projectile from a target with spin \( I \), isospin \( T \).

Chapter three will apply the expression for the scattering amplitude to some practical low-energy nuclear physics; protons with energy less than 12 MeV scattered inelastically by a target nucleus with initial spin-parity \( 0^+ \) and final spin-parity \( 2^+ \). This is the case most encountered with the Western Michigan University Tandem-Vande Graf accelerator.

Chapter four will show some examples of how to extract physical quantities, \( T \) matrices, from experimental data. The relations may not give unique solutions, but definitely will provide information about the \( T \) matrix combined with additional data, say polarization, asymmetry experiments; these results may yield unique solutions for a group of \( T \) matrices.
II. THEORY

A. Potential Scattering

The scattering of spin $I_0$, isospin $T_0$, particles by a spin $I_0$, isospin $T_0$ target will be investigated. To help clarify the theoretical investigation less complicated scattering problems will be considered first. The least complicated is the potential scattering of a spinless projectile from a spinless target. The two-particle Hamiltonian is

$$H = \frac{1}{2m_1} \vec{p}_1^2 + \frac{1}{2m_2} \vec{p}_2^2 + V(r)$$

(1)

for a projectile of mass $m_1$ incident on a target of mass $m_2$ with the interaction potential $V(r)$. The potential depends on the relative displacement $r$, defined as $r \equiv (|\vec{r}_1 - \vec{r}_2|)$, where $\vec{r}_1$ is the position of the projectile and $\vec{r}_2$ is the position of the target in the laboratory frame. Using the commutation properties of the operators for momentum and position and using the definitions $\mu \equiv m_1 m_2 / (m_1 + m_2)$ and $M \equiv m_1 + m_2$ the Hamiltonian takes the form

$$H = \frac{1}{2M} \vec{p}_{\text{cm}}^2 + \frac{1}{2\mu} \vec{p}_{\text{rel}}^2 + V(r)$$

(2)

where $\vec{p}_{\text{cm}}$ is the center-of-mass momentum operator and
\( \vec{p}_{rel} \) is the relative momentum operator of the system. Since the center-of-mass motion is a plane wave, only the relative motion will be considered. The energy of relative motion is \( E_{rel} = \frac{1}{2} \mu \vec{k}^2 \), where \( \vec{R} \) is the wave number vector of the projectile. Thus the Schrodinger equation in the relative frame is

\[
\left[ T + V(r) \right] \psi^+_r(\vec{r}) = E \psi^+_E(\vec{r})
\]

(3)

where the kinetic energy \( (\frac{1}{2} \mu \vec{p}_{rel}^2 \) operator is

\[
T = -\frac{\hbar^2}{2\mu} \nabla^2 = -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{2\mu r^2} \hat{\mathbf{L}}^2
\]

(4)

noting that \( \hat{\mathbf{L}} \) is the orbital angular momentum operator and \( \omega = (\theta, \phi) \) are angles defined in Figure 1.

Since \( \hat{\mathbf{L}}^2 \) commutes with any function of \( r \), particularly \( \hat{\mathbf{H}} = T + V(r) \), it is possible to separate the Schrodinger equation into angular and radial coordinates. All solutions of the Schrodinger equation are obtained as linear combinations of the product of spherical harmonics and radial solutions.

The spherical harmonics \( Y_{lm}^{(\theta, \phi)} \) simultaneously satisfy

\[
\hat{\mathbf{L}}^2 Y_{lm}^{(\theta, \phi)} = \ell(\ell+1) \hbar^2 Y_{lm}^{(\theta, \phi)}
\]

\[
\hat{L}_z Y_{lm}^{(\theta, \phi)} = m \hbar Y_{lm}^{(\theta, \phi)}
\]

(5)
Figure (1). Coordinate system defining $\theta$ and $\phi$.
where $l$ is a non-negative integer and $m$ has some integral value $-l, \ldots, l$.

Since the eigenfunctions with angular dependence are completely known, it remains then to determine the eigenfunctions of the radial equation. We write the Schrödinger equation as an inhomogeneous partial differential equation

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 - \frac{1}{kr^2} \left( \frac{1}{r^2} \right) + \frac{1}{k^2} \right] \Psi^+(\mathbf{r}) = \frac{2m}{k^2} \sqrt{V(r)} \Psi^+(\mathbf{r})$$  \hspace{1cm} (6)

When $V(r)=0$ the radial portion of equation 6 has the form of Bessel's differential equation which has as solutions $J_l(kr)$. These Bessel functions in combination with the spherical harmonics form a complete set of eigenfunctions for $H(V(r)=0)$. Since $e^{ik \cdot \mathbf{r}}$ also forms a complete set over the free particle Hamiltonian, the two forms must be equivalent, which results in the useful expansion

$$e^{ik \cdot \mathbf{r}} = 4\pi \sum_k \sum_m l^l J_l(kr) \gamma_{km}^\ell( \alpha_k ) \gamma_{lm}^\ell( \alpha_l )$$  \hspace{1cm} (7)

of the plane wave. There remains to find the eigenfunctions of the inhomogeneous radial equation. The method of Green's function will be used to that end.

The resulting Green's function method is

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 - \frac{1}{kr^2} \left( \frac{1}{r^2} \right) + \frac{1}{k^2} \right] G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$$  \hspace{1cm} (8)
To solve this, expand $G(r,r')$ linearly in terms of an as yet unknown radial function $g_l(r)$ and $\gamma_l^{(2)}$.

$$G(r,r') = \sum_l \frac{g_l(r)}{r} \frac{\gamma_l^{(2)}(r')}{r'} Y_{lm} Y_{l'm} \tag{9}$$

The radial equation

$$[\frac{d^2}{dr^2} - \frac{1}{r^2} l(l+1) + k^2] g_l(r) = -\delta(r-r') \frac{\partial^2}{\partial r \partial r'} \tag{10}$$

results from separating the orthogonal functions of equation 9. The Bessel function $j_l(kr)$ is the regular solution at $r=0$ and the Bessel function $h_l^{(2)}(kr)$ is the regular solution at $r=\infty$. Matching these at $r=r'$ yields

$$g_l(r) = c \frac{j_l(kr)}{r} h_l^{(2)}(kr) \, . \tag{11}$$

After integrating equation 10 from $r-\delta$ to $r+\delta$ with $G_l(r,r')$ expressed in terms of the Bessel functions and using

1. General properties of Green's function
2. General properties of continuous functions
3. $j_\alpha(z) \frac{d}{dz} - \frac{1}{2} \frac{d^2}{dz^2} = - \frac{1}{2}$

where $\alpha$ could be complex, we allow $\beta \to 0$ which results in the determination of $c$, namely $c=ik$. The explicit form of the Green's function expansion, equation (9) is then

$$G(r,r') = ik \sum_l \frac{j_l(kr)}{r} \frac{h_l^{(2)}(kr)}{r} Y_{lm}^{(2)} Y_{l'm}^{(2)} \tag{12}$$
Analogous to the expansion of the plane wave in terms of the Bessel function and spherical harmonics, we expand formally $\Phi^+_e(\vec{r})$ in terms of an as yet indeterminate eigenfunction of the radial problem

$$\Phi^+_e(\vec{r}) = 4\pi \sum_{l,m} \sum_{\ell,M} u_l(kr) \gamma^\ell_{\ell M} \gamma_l^{(m)}$$

The integral equation

$$\Phi^+_e(\vec{r}) = e^{ik\cdot\vec{r}} - \int G(r-r') \psi(r') \psi^+_e(\vec{r}) d\vec{r}'$$

can be used to explicitly determine $u_l(kr)$ by substituting the expansions for the plane wave and the outgoing wave. Upon separating and factoring we find that

$$u_l(kr) = j_l(kr) - ik \int j_l(kr_0) \rho_{11}^{(l)}(kr)$$

$$\times \int u_l(r') u_l(kr') r'^2 d\vec{r}'$$

where we have used the interaction potential, defined as

$$\mathcal{U}_{k'}(r') \delta_{k''} \delta_{m'} \delta_{m''} = \frac{2\mu}{\hbar^2} \mathcal{V}(r') \sum_{l'M'} \mathcal{Y}_{l'M'}^* \mathcal{Y}_{l'M}$$

A simplification of equation (14) is possible by considering the form that ignores the details near the scattering center, since from physical considerations $r$ is much greater than $r''$ (distance of detector from the scattering center is very large compared to
scattering center dimensions). We consider the large-\( \mathcal{V} \) form of \( h^{(n)}_\mathcal{V}(kr) \):

\[
h^{(n)}_\mathcal{V}(kr) \xrightarrow{r \to \infty} \frac{1}{ikr} e^{i(kr-\mathcal{V}r/2)}
\]

then

\[
\mathcal{U}_\mathcal{V}(kr) = J_\mathcal{V}(kr) + \frac{1}{ikr} e^{i(kr-\mathcal{V}r/2)} T_\mathcal{V}
\]

where \( T_\mathcal{V} \) is defined as

\[
T_\mathcal{V} = -k \int J_\mathcal{V}(kr') \mathcal{U}_\mathcal{V}(kr') \mathcal{U}_\mathcal{V}(kr '') r''^2 dr''
\]

The transition matrix \( T_\mathcal{V} \) describes the scattering of a partial \( l \) incident wave by the potential.

The explicit radial expression for \( \mathcal{U}_\mathcal{V}(kr) \) can now be substituted into equation (8) yielding

\[
\Psi_{\mathcal{V}}(r) = e^{-ikr} + \frac{1}{r} e^{ikr} \int \mathcal{U}_\mathcal{V}(r') dr'
\]

recalling the plane wave expansion and defining the scattering amplitude

\[
\mathcal{f}(\mathcal{N}, \mathcal{R}) = 4\pi \sum_{\mathcal{L}_\mathcal{R}} T_{\mathcal{L}_\mathcal{R}} \mathcal{Y}_{\mathcal{L}_\mathcal{R}}(\mathcal{R}) \mathcal{Y}_{\mathcal{L}_\mathcal{R}}(\mathcal{R})
\]

The scattering amplitude \( \mathcal{f}(\mathcal{N}) \) describes the probability amplitude of a particle scattered from an incident direction \( \mathcal{N}_\mathcal{R} = (\epsilon_{\mathcal{R}}, \phi_{\mathcal{R}}) \) to the arbitrary direction \( \mathcal{N}_\mathcal{L} = (\epsilon, \phi) \). This result is completely independent of the coordinate system chosen.
B. Spinless Projectile,  
Spin $I_o$ Target Inelastic Scattering

The next case to be considered introduces another term in the Hamiltonian, an interaction that depends on the internal variables of the target and the relative distance of the projectile from it. The projectile is spinless, but the target has spin $I_o$ initially and has excited states $I^n$. Schrodinger's equation for this case is

$$\left[T + H_{st}(\vec{q}) + V(\vec{r}, \vec{q})\right] \Psi^+(\vec{r}, \vec{q}) = E \Psi^+(\vec{r}, \vec{q}) \quad \text{(20)}$$

The spin dependence of the total wave function is known and satisfies

$$H_{st}(\vec{q}) \chi_n(\vec{q}) = E_{n} \chi_n(\vec{q}) \quad \text{(21)}$$

where $H_{st}(\vec{q})$ is the spin Hamiltonian of the target and $\chi_n(\vec{q})$ is chosen as an eigenfunction of $I^z$ and $I^2$, the square of the spin operator and its $z$ component. The internal spin coordinates of the target are represented by $\vec{q}$. The energy relation is $E_{n} = (\frac{2m}{\hbar^2}) \vec{k}_n^2$ where $\vec{k}_n$ is the wave number of the projectile.

The radial eigenfunctions become more complicated with the introduction of spin, but the technique of
finding the radial eigenfunction remains similar to the previous pure potential scattering case. Thus, express equation (20) inhomogeneously and then separate variables. The radial equation has solutions which with the spherical harmonics will generate the Green's function

$$G(\vec{r}, \vec{r}', \vec{r}'') = -\frac{2\mu}{\hbar^2} \sum_{l} \sum_{m} \sum_{l'k} \sum_{m'} \left( k_{l,m} f_{l,k}(k_{l,m}) h_{l'}^{(l)}(k_{l',m'}) \right) \times Y_{l,m}^{*(\vec{r})} Y_{l,m}^{(\vec{r}')\vec{r}''} (\vec{r}'')$$

(22)

Eigenfunctions of the total angular momentum $\vec{J} = \vec{L} + \vec{\ell}$ will be generated from a linear combination of the eigenfunctions of $\vec{L}$ and $\vec{\ell}$. The coefficients of the expansion will be the Clebsch-Gordon coefficients with values that are dictated by the angular momentum operator. Equation (22) becomes

$$G(\vec{r}, \vec{r}', \vec{r}'') = -\frac{2\mu}{\hbar^2} \sum_{l} \sum_{m} \sum_{l'} \sum_{m'} \left( k_{l,m} f_{l,k}(k_{l,m}) h_{l'}^{(l)}(k_{l',m'}) \right) \times \left[ Y_{l,m}^{\vec{J}^{\vec{r}'} \vec{J}^{\vec{r}''}} \right]$$

(23)

upon forming the set of eigenfunctions $Y_{l,m}^{\vec{J}^{\vec{r}'} \vec{J}^{\vec{r}''}}$ of the total angular momentum by standard techniques discussed in textbooks on quantum mechanics. Using these same general techniques of adding angular momenta enables the initial wave

$$\exp{i\vec{k}_{\vec{r}} \cdot \vec{r}} \chi_{l,m}^{\vec{J}}(\vec{r}) = 4\pi \sum_{l} \sum_{m} \left( f_{l,k}(k_{l,m}) \chi_{l,m}^{\vec{J}}(\vec{r}) \right)$$

(24)
to be expanded in terms of eigenfunctions of \( J \)

\[
= 4\pi \sum_{l} \sum_{m} \sum_{J} \sum_{M} i^{l} \frac{d_{l}(k_{n}r)}{U_{l}} \psi_{l}^{*}(\Omega_{kn}) \times (l m l M | J M) \chi_{l}^{l M} \delta_{l M} \delta_{l M} \tag{25}
\]

which can be written in the form

\[
= 4\pi \sum_{l} \sum_{m} \sum_{J} \sum_{M} i^{l} \frac{d_{l}(k_{n}r)}{U_{l}} \psi_{l}^{*}(\Omega_{kn}) \times (l m l M | J M) \chi_{l}^{l M} \delta_{l M} \delta_{l M} \tag{26}
\]

which will be more useful later.

At this point assuming there is a complete set of eigenfunctions of the non-homogeneous radial equation formally expand \( \Psi(r) \)

\[
\Psi(r) = 4\pi \sum_{l} \sum_{m} \sum_{J} \sum_{M} i^{l} \frac{d_{l}(k_{n}r)}{U_{l}} \psi_{l}^{*}(\Omega_{kn}) \times (l m l M | J M) \chi_{l}^{l M} \delta_{l M} \delta_{l M} \tag{27}
\]

The factor \( (l m l M | J M) \) in the expansion will allow a later simplification. The integral equation with these expansions becomes

\[
= 4\pi \sum_{l} \sum_{m} \sum_{J} \sum_{M} i^{l} \frac{d_{l}(k_{n}r)}{U_{l}} \psi_{l}^{*}(\Omega_{kn}) \times (l m l M | J M) \chi_{l}^{l M} \delta_{l M} \delta_{l M} \tag{28}
\]
in which the potential is defined as

\[
J \psi_j p k \left( \frac{1}{r} \right) = \sum \gamma^j \gamma^k \psi_j p k \left( \frac{1}{r} \right)
\]

Factoring the common terms in equation (28) leaves

\[
\mathcal{U}^{J}(k_{n}r) = J_{\ell}(k_{n}r) \sum_{l' l} \Psi^{J'}_{l' l} \left( \frac{1}{r} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right)
\]

As in the spinless case consider the asymptotic region \( r \gg r' \). then

\[
\mathcal{U}^{J}(k_{n}r) = J_{\ell}(k_{n}r) \sum_{l' l} \Psi^{J'}_{l' l} \left( \frac{1}{r} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right)
\]

The transition matrix is defined as

\[
T^{J}_{l l'} = - \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right) \frac{2 \lambda}{\ell} \frac{2 \lambda}{\ell} \frac{i}{\ell} \left( k_{l} \right)
\]

and describes the transition from a state \( I, I_{J} \) to \( I', I_{J} \). The asymptotic radial expression becomes
The scattering amplitude is expressed as

\[ \mathcal{U}^J(k\nu r) = \sum_{lm} \omega_{lm} \mathcal{U}^J(\nu, r) \sum_{I} \mathcal{D}_{I_m} \mathcal{D}_{I_m^*} + \frac{e^{ik\nu r}}{k\nu r} \mathcal{J}^{J'}_{I_m I_n^*} \cdot \]  

(33)

Substitute equation (33) in equation (27)

\[ \mathcal{U}^J(\nu, r) = 4\pi \sum_{l_1 l_2} \mathcal{D}_{I_m} \mathcal{D}_{I_m^*} i^l \mathcal{D}_{I_m} \mathcal{D}_{I_m^*} \mathcal{Y}^*_{l_m I_m} \]

\[ \times (\nu, r) \sum_{l_1 l_2} \mathcal{D}_{I_m} \mathcal{D}_{I_m^*} \mathcal{Y}^*_{l_m I_m} \]

\[ \times i^{l'} \frac{e^{ik\nu r}}{k\nu r} \mathcal{J}^{J'}_{I_m I_n^*} \mathcal{Y}^*_{l_n I_n} \]

\[ \times \sum_{l' m'} \mathcal{D}_{I_m} \mathcal{D}_{I_m^*} \mathcal{Y}^*_{l_m I_m} \mathcal{Y}_{l_m I_m^*} \]

\[ \mathcal{Y}^*_{l_m I_m} \mathcal{Y}_{l_m I_m^*} \]  

(34)

The scattering amplitude is expressed as

\[ \sum_{lm} \mathcal{J}^{J'}_{I_m I_n^*} \mathcal{Y}^*_{l_m I_m} \mathcal{Y}_{l_m I_m^*} = \mathcal{J}^{J'}_{I_m I_n^*} \mathcal{Y}^*_{l_m I_m} \mathcal{Y}_{l_m I_m^*} \]  

using

\[ \sum_{l' m'} \mathcal{D}_{I_m} \mathcal{D}_{I_m^*} \mathcal{Y}^*_{l_m I_m} \mathcal{Y}_{l_m I_m^*} = \mathcal{J}^{J'}_{I_m I_n^*} \mathcal{Y}^*_{l_m I_m} \mathcal{Y}_{l_m I_m^*} \]

The scattering amplitude describes the transition from a spinless wave incident with direction \( \omega_{\nu, r} \) on a target in the spin state \( I_m \) with \( \frac{1}{2} \) component \( M_n \), to the outgoing wave in direction \( \omega \) leaving the target in the spin state \( I_n \) with spin \( \frac{1}{2} \) component \( M_n \).
C. Spin \( I_0 \) Projectile,  
Spin \( I_n \) Target Inelastic Scattering

The projectile and target both have initial spin \( I_0 \) and \( I_n \) respectively. After the inelastic collision the projectile is in the excited spin state \( I_n \) and the target is in the spin state \( I_n \). The spin wave function is known and satisfies

\[
\begin{align*}
\mathcal{H}_{sp}(\vec{n}) \chi_{\iota_n}(\vec{n}) &= \mathcal{E}_n \chi_{\iota_n}(\vec{n}) \\
\mathcal{H}_{sp}(\vec{\eta}) \chi_{\iota_n}(\vec{\eta}) &= \mathcal{E}_n \chi_{\iota_n}(\vec{\eta})
\end{align*}
\]  

(35)

where \( \mathcal{H}_{sp}(\vec{n}) \) is the projectile spin Hamiltonian and \( \chi_{\iota_n}(\vec{n}) \) is chosen as an eigenfunction of \( \vec{I}^2 \) and \( I_y \) as in the last section. The internal spin coordinates of the projectile are represented by \( \vec{n} \). The energy relation of the system is \( E_n - E_m - E_n = (\mathcal{E}_m + \mathcal{E}_n) + \mathcal{E}_n \). The target spin Hamiltonian from the last section remains intact.

As a direct extension of previous Green's functions we have the expression

\[
\begin{align*}
\chi_{\iota_n}^*(\vec{\eta}) \chi_{\iota_n}(\vec{\eta}) \chi_{\iota_n}(\vec{\eta}) \chi_{\iota_n}(\vec{\eta}) \\
\chi_{\iota_n}^*(\vec{\eta}) \chi_{\iota_n}(\vec{n}) \chi_{\iota_n}(\vec{n}) \chi_{\iota_n}(\vec{n})
\end{align*}
\]  

(36)
Using the vector addition properties

\[ \mathbf{Y}(\mathbf{L}_1) \mathbf{Y}(\mathbf{L}_2) = \sum_{\mathbf{L}_3} \left( \mathbf{L}_1 \mathbf{L}_2 \mathbf{L}_3 \right) \frac{\mathbf{Y}^{\mathbf{L}_3}(\theta, \phi)}{2^j J} \]

(37a)

and the property

\[ \sum_{\mathbf{L}_1} \sum_{\mathbf{L}_2} \sum_{\mathbf{L}_3} \sum_{\mathbf{L}_4} \left( \mathbf{L}_1 \mathbf{L}_2 \mathbf{L}_3 \mathbf{L}_4 \right) X^{\mathbf{L}_1}(\xi, \eta) X^{\mathbf{L}_2}(\xi, \eta) = \delta_{\mathbf{L}_1, \mathbf{L}_2} \]

(37b)

where \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \) are any two general angular momenta, equation (36) becomes

\[ \mathcal{G}(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4) = \frac{-3}{8} \sum_{\mathbf{L}_1} \sum_{\mathbf{L}_2} \sum_{\mathbf{L}_3} \sum_{\mathbf{L}_4} \frac{i k_n f(k_n)}{k^2} \]

(38)

Expanding the initial wave with eigenfunctions of \( \mathbf{L}, \bar{\mathbf{L}} \) and applying general vector addition properties the initial wave expansion becomes

\[ e^{i k_n \cdot \xi} \mathbf{Y}^{\mathbf{L}}(\xi, \eta) = 4 \pi \sum_{\mathbf{L}_1} \sum_{\mathbf{L}_2} \sum_{\mathbf{L}_3} \sum_{\mathbf{L}_4} \frac{i k_n f(k_n)}{k^2} \]

(39)

Analogously expand \( \Psi^+(\mathbf{L}) \) formally in terms of the radial eigenfunctions and the vector spherical harmonics

\[ \Psi^+(\mathbf{L}) = 4 \pi \sum_{\mathbf{L}_1} \sum_{\mathbf{L}_2} \sum_{\mathbf{L}_3} \sum_{\mathbf{L}_4} \frac{i k_n f(k_n)}{k^2} \]

(40)
Substitute the expansions, equations (38), (39), (40) in the integral equation in which

\[
\mathcal{J} \quad \mathcal{K} \quad \mathcal{L} \quad \mathcal{M} \quad \mathcal{N} \quad \mathcal{O} \quad \mathcal{P} \quad \mathcal{Q} \quad \mathcal{R} \quad \mathcal{S} \quad \mathcal{T} \quad \mathcal{U} \quad \mathcal{V} \quad \mathcal{W} \quad \mathcal{X} \quad \mathcal{Y} \quad \mathcal{Z}
\]

is defined as the interaction potential associated with the temporary state \( |I \rangle \otimes |\mu \rangle \otimes |J' \rangle \) and the final state \( |I \rangle \otimes |\mu \rangle \otimes |J \rangle \). Factoring the common terms of angle and spin dependence and recalling that the expression will only be used asymptotically, that is for \( r \) much larger than \( r' \), then the asymptotically regular Bessel function is removed from the integral, resulting in

\[
U_J(kr) \cdot \sum \prod \delta \delta = J_0(kr) \sum \prod \delta \delta \]

If the definition

\[
T_J \quad \mathcal{I} \quad \mathcal{J} \quad \mathcal{K} \quad \mathcal{L} \quad \mathcal{M} \quad \mathcal{N} \quad \mathcal{O} \quad \mathcal{P} \quad \mathcal{Q} \quad \mathcal{R} \quad \mathcal{S} \quad \mathcal{T} \quad \mathcal{U} \quad \mathcal{V} \quad \mathcal{W} \quad \mathcal{X} \quad \mathcal{Y} \quad \mathcal{Z}
\]
is used then

$$U(kr)_{1(l',m')I'I} = J_{e}(kr)_{1(l',m')I'I} + \frac{i}{kr} e^{-ikr} T_{1(l',m')I'I}$$

(43)

This $T$ matrix describes the transition between the initial and final states. If the explicit radial expression equation (43) is introduced into the formal expansion of the total wave function, the resulting asymptotically valid wave function

$$\Psi^+(r) = e^{ikr} \chi(l''m'') \chi(l'm'm)$$

$$+ 4\pi \sum_{m'} \sum_{m''} \sum_{I'I} \sum_{I'I'} \sum_{m'''} \sum_{m'''} \sum_{J''J''} \frac{1}{kr} e^{-ikr}$$

$$\times i^{-l''} T_{1(l',m')I'I} \sum_{l''m''} \chi_{l''m''}(l'm'm/I'I')$$

$$\times (J''m''I'''M''/JM) \sum_{m'm''} \sum_{M''} \chi_{m'm''M''}(l'm'm/I'I')$$

$$\times (J'M'M'I'M'/JM) \chi_{l'n'n''}(l'm'm/I'I') \chi_{m'm'I'I'}$$

(44)

results if the initial wave expansion and the properties

$$\sum_{m'} \sum_{m''} \sum_{m'''} \sum_{m''''} \sum_{I'I'} \sum_{I'I''} \chi_{l'm'm''m'''}(l'm'm''m''') \chi_{l'I'I'I''I''I''} = \sum_{J'M'M'I'M'/JM} \chi_{l'n'n''m'm''}(l'n'n''m'm'') \chi_{m'm'I'I'}$$

where $l$, $I$, and $J$ are representative angular momenta, are used.
The scattering amplitude is, then, expressed as

\[
f(\Omega_{m, n} \Omega) = 4\pi \sum_{J_1, \ldots, J_N} (-1)^{J_1 + \ldots + J_N} \frac{1}{4} \sum_{I_1, \ldots, I_N} \frac{1}{Z_1 \ldots Z_N} \frac{1}{Z_1' \ldots Z_N'}
\]

\[
x i^{J_1' - J_1} \mathcal{D}_{m,j}^{l_0} \mathcal{D}_{m,j}^{l_1} \ldots \mathcal{D}_{m,j}^{l_N} \mathcal{D}_{m,j}^{l_1'} \ldots \mathcal{D}_{m,j}^{l_N'} \mathcal{D}_{m,j}^{l_0'} \, (l_{m_0} m_0 | j_{M_j}) \, (l_{M_0} m_0 | J_{M_j}) \,
\]

\[
\times (f_{m_1} m_1 m_0 | J_{M_j}) \, (l_{m_1} m_1 m_0 | J_{M_j}) \, (l_{m_1'} m_1' m_0 | J_{M_j}) \, \mathcal{Y}_{2 \alpha}(\Omega)
\]  \hspace{1cm} (45)

This scattering amplitude describes the transition from a wave characterized with spin \( \zeta_0 \) and direction \( \vartheta_0 \) incident on a target in the spin state \( \Omega_0 \), to an outgoing wave in the spin state \( \zeta_m \) with the residual target in the spin state \( \Omega_m \).
D. Spin \( \downarrow \) Isospin \( \uparrow \) Projectile, Spin \( \uparrow \) Isospin \( \downarrow \) Target Inelastic Scattering

The scattering formalism of this section will be the most general case in this thesis. The isospin states of the projectile with initial spin \( I_p \) and the target with initial spin \( I_t \) are respectively \( I_p \) and \( I_t \). After scattering, the target is in the spin state \( I_t \) and isospin state \( T_t \); the projectile is in the spin state \( I_p \) and isospin state \( T_p \). The spin wave function remains intact from the previous analysis. The isospin wave function satisfies

\[
H_{T_p}(\overline{T}) \chi_{\overline{T}}(\overline{T}) = \varepsilon_{\alpha} \chi_{\overline{T}}(\overline{T}) \tag{46a}
\]

where \( H_{T_p}(\overline{T}) \) is the projectile's isospin Hamiltonian. \( \chi_{\overline{T}}(\overline{T}) \) is chosen as an eigenfunction of \( T^z \) and \( T^2 \). The internal isospin coordinates of the projectile are represented by \( \overline{T} \). The isospin is represented by \( T \) and its \( z \) component by \( W \). A similar notation is followed for the target

\[
H_{T_t}(\overline{V}) \chi_{\overline{V}}(\overline{V}) = \varepsilon_{\alpha} \chi_{\overline{V}}(\overline{V}) \tag{46b}
\]

The isospin eigenfunctions are formally analogous to the spin eigenfunctions.
Using the properties of the Clebsch-Gordon coefficients for addition of angular momenta the Green's function is

\[
G(r, r') = -\frac{\alpha^{r^2}}{\hbar} \sum_{\ell m} \sum_{\ell' m'} \sum_{\ell'' m''} \sum_{\ell''' m'''} i k_n
\]

\[
\times J_{\ell}(kr) J_{\ell'}(kr') \sum_{\ell m} U_{\ell m}(\alpha_{r^2}) U_{\ell m}(\alpha_{r'^2})
\times U_{\ell m}^{*}(\alpha_{T^2}) U_{\ell m}^{*}(\alpha_{T'^2})
\times U_{\ell m}^{*}(\alpha_{T''2}) U_{\ell m}^{*}(\alpha_{T''2})
\times U_{\ell m}^{*}(\alpha_{T'''2}) U_{\ell m}^{*}(\alpha_{T'''2})
\]

(47)

and the initial wave is

\[
\psi_0(r) = \sum_{\ell m} \sum_{\ell' m'} \sum_{\ell'' m''} \sum_{\ell''' m'''} \sum_{\ell''' m'''\ell'' m''} \sum_{\ell''' m'''\ell'' m''}
\times J_{\ell}(kr) J_{\ell'}(kr') \sum_{\ell m} U_{\ell m}(\alpha_{r^2}) U_{\ell m}(\alpha_{r'^2})
\times U_{\ell m}^{*}(\alpha_{T^2}) U_{\ell m}^{*}(\alpha_{T'^2})
\times U_{\ell m}^{*}(\alpha_{T''2}) U_{\ell m}^{*}(\alpha_{T''2})
\times U_{\ell m}^{*}(\alpha_{T'''2}) U_{\ell m}^{*}(\alpha_{T'''2})
\times U_{\ell m}^{*}(\alpha_{T'''2}) U_{\ell m}^{*}(\alpha_{T'''2})
\]

Expand the total wave function formally in terms of \( \psi_0(r) \), the radial eigenfunction is

\[
\psi_T(r) = \sum_{\ell m} \sum_{\ell' m'} \sum_{\ell'' m''} \sum_{\ell''' m'''} \sum_{\ell''' m'''\ell'' m''} \sum_{\ell''' m'''\ell'' m''}
\times J_{\ell}(kr) J_{\ell'}(kr') \sum_{\ell m} U_{\ell m}(\alpha_{r^2}) U_{\ell m}(\alpha_{r'^2})
\times U_{\ell m}^{*}(\alpha_{T^2}) U_{\ell m}^{*}(\alpha_{T'^2})
\times U_{\ell m}^{*}(\alpha_{T''2}) U_{\ell m}^{*}(\alpha_{T''2})
\times U_{\ell m}^{*}(\alpha_{T'''2}) U_{\ell m}^{*}(\alpha_{T'''2})
\times U_{\ell m}^{*}(\alpha_{T'''2}) U_{\ell m}^{*}(\alpha_{T'''2})
\]

(48)

The explicit form of the radial wave function...
will be determined by substitution of equations (47) and (48) into the Integral equation. Using the definition of the interaction equation (41)

\[ \int \frac{d^3 p' d^3 q' d^3 p d^3 q}{f (n') f (n) f (n') f (n)} \]

where the interaction is between an initial state characterized by the projectiles spin and isospin \( \mathbf{s}_i, \mathbf{t}_i \) and the targets spin and isospin \( \mathbf{I}_i', \mathbf{T}_i' \), and a final state characterized by the projectiles spin and isospin \( \mathbf{s}_f, \mathbf{t}_f \) and the targets spin and isospin \( \mathbf{I}_f, \mathbf{T}_f \). The initial states refer to a virtual compound state consisting of the target and projectile combined in an excited system that decays into an outgoing wave and target in excited states \( \mathbf{s}_i, \mathbf{t}_i; \mathbf{I}_i, \mathbf{T}_i \)

Upon eliminating the common terms in the expanded form of the integral equation, and using the interaction the asymptotic form of the radial wave function is explicitly determined

\[ U_{j_i j_f}^{(I_i T_i; I_f T_f)}(k, \mathbf{r}) = J_{2j_i j_f}^{(I_i T_i; I_f T_f)}(k, \mathbf{r}) \mathcal{S}_{j_i j_f}^{(I_i T_i; I_f T_f)}(k, \mathbf{r}) \mathcal{S}_{j_i j_f}^{(I_i T_i; I_f T_f)}(k, \mathbf{r}) \]  

(49)
The transition matrix is then defined in the preceding equation as

\[
T_{\ell' \ell m'}^{\ell m} = -\frac{2\mu}{k^2} \sum_{\ell''} \sum_{m''} \sum_{\ell'''} \sum_{m'''} i k_n
\]

and describes the transition from the state \((\ell, \ell_m, \ell_n, \ell_o, T_o)\) to \((\ell', \ell_m, \ell_n, \ell_o, T_n)\).

Rewriting equation (49) using the preceding definition yields the expression

\[
\psi_n(k r) = \sum_{\ell m} \sum_{\ell''} \sum_{m''} \sum_{\ell'''} \sum_{m'''} i k_n \int \psi_\ell^{(T)}(r') V_{\ell''m''; \ell'''m'''}(T_n) \, \frac{e^{-i k r r'}}{r} \, dr'.
\]

If this explicit expression for the radial eigenfunction is then used in the formal expansion for the total wave function the resulting equation satisfies the asymptotic requirements. After some manipulation parallel to earlier sections, the wave function becomes

\[
\psi_n(k r) e^{i k N} = e^{i k N} \sum_{\ell m} \sum_{\ell''} \sum_{m''} \sum_{\ell'''} \sum_{m'''} e^{i k r r'} \frac{\psi_\ell^{(T_n)}(r)}{r} \frac{e^{-i k r r'}}{r} \, dr'.
\]
and expressing \( \mathcal{U}^{+}(\vec{r}, \vec{r}', \tilde{r}) \) in terms of eigenfunctions of \( \vec{J} \) and \( \vec{T} \)

\[
\mathcal{U}^{+}(\vec{r}, \vec{r}', \tilde{r}) = e^{i\tilde{b}_{m}^{\dagger} r} \frac{\chi^*_{J_{1} M_{1}}(\vec{r}) \chi^*_{J'_{1} M'_{1}}(\vec{r}) \chi^*_{T_{1} W_{1}}(\tilde{r}) \chi_{T'_{1} W'_{1}}(\tilde{r})}{\sqrt{2(2J+1)(2J'+1)}}
\]

(52)

\[
+ 4\pi \sum_{J_{1} M_{1} J_{1}' M_{1}'} \sum_{M_{1} M_{1}' T_{1} W_{1}} \sum_{M_{2} M_{2}' T_{2} W_{2}} \frac{1}{\sqrt{2(2J+1)(2J'+1)}} e^{i\tilde{b}_{m}^{\dagger} r} \frac{\chi^*_{J_{1} M_{1}}(\vec{r}) \chi^*_{J'_{1} M'_{1}}(\vec{r}) \chi^*_{T_{1} W_{1}}(\tilde{r}) \chi_{T'_{1} W'_{1}}(\tilde{r})}{\sqrt{2(2J+1)(2J'+1)}}
\]

The scattering amplitude is then expressed as

\[
\mathcal{S}^{\dagger}(\vec{r}, \vec{r}', \tilde{r}) = 4\pi \sum_{J_{1} M_{1} J_{1}' M_{1}'} \sum_{M_{1} M_{1}' T_{1} W_{1}} \sum_{M_{2} M_{2}' T_{2} W_{2}} \frac{1}{\sqrt{2(2J+1)(2J'+1)}} e^{i\tilde{b}_{m}^{\dagger} r} \frac{\chi^*_{J_{1} M_{1}}(\vec{r}) \chi^*_{J'_{1} M'_{1}}(\vec{r}) \chi^*_{T_{1} W_{1}}(\tilde{r}) \chi_{T'_{1} W'_{1}}(\tilde{r})}{\sqrt{2(2J+1)(2J'+1)}}
\]

(53)

This scattering amplitude describes the transition from a wave characterized with spin and isospin...
\[ \omega, t_0 \] and direction \( \Omega_{k_n} = (\theta_{k_n}, \phi_{k_n}) \) incident a target in the spin and isospin states \( I_0, T_0 \) to an outgoing wave in the spin and isospin states \( I_n, T_n \) with the residual target in the spin and isospin states \( I_n, T_n \).
III. APPLICATIONS

The series of the partial wave expansion of the preceding scattering amplitude will converge to a practically acceptable value for small \( l \) if limited in application to low energies. A classical justification is based on the short-range nature of the nuclear forces.

Consider the target to be a hard massive sphere of radius

\[
R_T = 1.2 A^{3/2} \cdot 10^{-15} \text{ meter}
\]

where \( A \) is the mass number, and the projectile, a proton, to be a point mass with

\[
m = 1.67 \cdot 10^{-27} \text{ kg}
\]

The projectile's impact parameter must be less than \( R_T \) or no nuclear interaction will occur.

The impact parameter of a \( 12 \text{ MeV} \) proton (which is available at Western Michigan University tandem Van de Graaff accelerator) is found from the expression

\[
|\vec{r} \times \vec{p}| = \sqrt{l(l+1)} \hbar
\]

\[
r = \sqrt{\frac{l(l+1)}{12mE}} \hbar
\]

When \( l = 3 \)

\[
r = 4.5 \cdot 10^{-15} \text{ meter}
\]

and when \( l = 4 \)

\[
r = 5.8 \cdot 10^{-15} \text{ meter}
\]
If \( A = 64 \) the nuclear radius is

\[
R_T = 5.6 \cdot 10^{-15} \text{ meter}
\]

The \( l = 4 \) proton's impact parameter is greater than the radius of the \( A = 64 \) nucleus. Therefore for these circumstances the \( l \geq 4 \) partial waves can be dropped from the expansion.

The total angular momentum and parity of the projectile-target system is conserved. Consider the scattering process:

\[
\sum_{\omega} \chi(\pi) + \Phi(l, j) \rightarrow \sum_{\omega} \chi^{\pi}(j) \rightarrow \sum_{\omega} \chi(\pi) + \Phi'(l', j')
\]

where \( \chi(\pi) \) represents the target initially with spin, parity \( (I_o, \pi_o) \), \( \Phi(l, j) \) represents the partial wave \( l \), \( j(= l \pm \frac{1}{2}) \) of the incident proton; the intermediate state is represented by \( \sum_{\omega} \chi^{\pi}(j) \) with spin, parity \( (j, \Pi) \); the residual state of the target is represented as \( \sum_{\omega} \chi(\pi) \) with spin, parity \( (I_m, \Pi_f) \), \( \Phi'(l', j') \) represents the partial wave \( l', j'(= l' \pm \frac{1}{2}) \) of the outgoing protons. The spin angular momentum of the compound state must equal the spin orbital angular momentum of the projectile combined with the spin of the target

\[
\bar{J} + \bar{I}_o = \bar{J}
\]  

(54)
The total angular momentum of the projectile is

\[ \overline{J} = \overline{l} + \overline{L} \tag{55} \]

where \( \overline{L} \) is the spin of the proton (magnitude \( \frac{1}{2} \)); thus

\[ (\overline{l} + \overline{L}) + \overline{I}_o = \overline{J} \tag{56} \]

Since parity is conserved the parity of the projectile-target system must equal that of the compound state

\[ \Pi_i (-1)^q = \Pi \tag{57} \]

Similarly, the spin angular momentum of the intermediate compound state must equal the total angular momentum of the final system

\[ \overline{J} = \overline{I}_m + \overline{J}' \tag{58} \]

This results in the expression

\[ \overline{J} = \overline{I}_m + (\overline{l}' + \overline{I}_m) \tag{59} \]

upon separating the total angular momentum of the outgoing proton into its components of orbital and spin angular momentum \( \overline{l}' \) and \( \overline{I}_m \) (magnitude \( \frac{1}{2} \)). The expression for conservation of parity in the transition from the compound state to the final system is

\[ \Pi = (-1)^q \Pi_f \tag{60} \]
Equations (57) and (60) express the equality of parity in the initial and final systems

$$\Pi_i(-1)^{1/2} = (-1)^{L'}\Pi_f$$ \hspace{1cm} (61)

This leads to the conclusion that only odd-numbered outgoing partial waves are associated with odd-numbered initial partial waves, if \( \Pi_i = \Pi_f \); a similar relation holds for the even-numbered partial waves.

These conservation relations are applied to the system described in figure 2. Table 1 summarizes the allowable partial waves, thus the corresponding T matrices that contribute in the partial wave expansion of the scattering amplitude, for a transition from \( 0^+ \) to \( 2^+ \) of the target nucleus.
Figure (2). Schematic energy diagram corresponding to $\pi^+(0^+) + p \rightarrow \pi^+ (2^+) + p'$ through a compound state with spin-parity $J^\pi$. 

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Table (1). Summary of T matrices, incoming partial wave, all possible outgoing waves, that contribute in the partial wave expansion of the scattering amplitude.
<table>
<thead>
<tr>
<th>Even Parity</th>
<th>Odd Parity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{2\ell,0\ell}$</td>
<td>$T_{2\ell,0\ell}$</td>
</tr>
<tr>
<td>$T_{2\ell,2\ell}$</td>
<td>$T_{2\ell,2\ell}$</td>
</tr>
<tr>
<td>$T_{4\ell,2\ell}$</td>
<td>$T_{4\ell,2\ell}$</td>
</tr>
<tr>
<td>$T_{2\ell,2\ell}$</td>
<td>$T_{2\ell,2\ell}$</td>
</tr>
<tr>
<td>$T_{4\ell,2\ell}$</td>
<td>$T_{4\ell,2\ell}$</td>
</tr>
</tbody>
</table>

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Consider next the previously mentioned example where $I^z = 0^+$ and $I^z_n = 2^+$. Figure (3) defines the coordinate system where the xy plane is the scattering plane. If we use the relations established in the last section we find that the scattering amplitude has the form

$$f(a_m, a_n) = \frac{4\pi}{k} \sum_{l} \sum_{j} i^{l-j} T_{lj} \sum_{m} Y_m(a_m)$$

$$\times Y_l(\mu a_n) \langle M', M_0 | J M + M_0 \rangle$$

$$\times \left( \langle \mu a_n | M_0 - M_M \rangle \right)$$

(62)

Define the constant $a_{l', J} M_{l_1} M_0$

$$a_{l', J} M_{l_1} M_0 = 4\pi i^{l'} Y_{l M}^*(a_m) \langle M'_M | M_0 \rangle$$

$$\times \left( \langle \mu a_n | M_0 - M_M \rangle \right)$$

$$\times \left( \langle J M + M_0 | 2 M_0 \rangle \right)$$

which is the coefficient of $\frac{1}{k} T_{lj} Y_{l M_0} Y_{l' M_0}$ in equation (62).

Bohr's Theorem is

$$P_{\text{init}} e^{i \tau m_j} = P_{\text{fin}} e^{i \tau m_j'}$$

(63)

where $P_{\text{init}}$, $m_j$ and $P_{\text{fin}}$, $m_j'$ are respectively parity, spin projection of the system in its initial and final states. Since $P_{\text{init}} = P_{\text{fin}}$ it follows that
Figure (3). The coordinate system used for this calculation. The $x$ axis is the incident beam direction and the $xy$ plane is the scattering plane.
\[ m_j - m_j' = 0, \pm 2, \pm 4, \ldots \]

\[ m_0 - (m_n + M_n) = 0, \pm 2, \pm 4, \ldots . \]

When \( m_0, m_n \) are parallel and \( m_0 = m_n \)

\[ M_n = 0, \pm 2, \pm 4, \ldots \]

which is the spin-nonflip case. The spin-flip case is represented when \( m_0, m_n \) are anti-parallel and \( m_0 = -m_n \); then

\[ M_n = \pm 1, \pm 3, \ldots \]

From these relations the scattering amplitudes \( f_{m_0 m_n} \) representing spin-nonflip cases are

\[
\begin{align*}
    f_{k_0 z; y_2} &= f_{-k_0 z; -y_2} \\
    f_{k_0; y_2} &= f_{-k_0; -y_2} \\
    f_{k_0-2; y_2} &= f_{-k_0-2; -y_2}
\end{align*}
\]

(64)

and the spin-flip amplitudes are

\[
\begin{align*}
    f_{-k_0-1; y_2} &= f_{k_0+1; -y_2} \\
    f_{-k_0-1; y_2} &= f_{k_0+1; -y_2}
\end{align*}
\]

(65)
Equation (62) evaluated for the spin-nonflip case, say, $f_{kk;1N}^{(2)}$, is
\[
\begin{align*}
\sum_{kk} f_{kk;1N}^{(2)} &= - T_{2_{20}} Y_{2} Y_{2}^{0} - T_{2_{20}} Y_{2} Y_{2}^{0} \\
&+ T_{2_{22}} Y_{2} Y_{2}^{0} - T_{2_{22}} Y_{2} Y_{2}^{0} \\
&+ T_{4_{22}} Y_{2} Y_{2}^{0} - T_{4_{22}} Y_{2} Y_{2}^{0} \\
&+ T_{2_{22}} Y_{2} Y_{2}^{0} - T_{2_{22}} Y_{2} Y_{2}^{0} \\
&+ T_{0_{22}} Y_{0} Y_{0}^{0} + T_{1_{22}} Y_{1} Y_{1}^{0}
\end{align*}
\]
The spherical harmonics for $m \neq 0$ are

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} (-1)^m e^{im\phi} \mathcal{P}_\eta(\cos \theta) ;$$ (67)

for $m < 0$, we may use:

$$Y_{\ell m}(\theta, \phi) = (-1)^m Y_{\ell - m}^{*}(\theta, \phi)$$

and the Legendre polynomials $\mathcal{P}_\eta(\cos \theta)$ are given by

$$\mathcal{P}_\eta(\cos \theta) = \left( \frac{-1}{2^\ell \ell !} \left( \frac{d}{d(\cos \theta)} \right)^{\ell - m} \frac{d^m}{d(\cos \theta)^m} (\cos^2 \theta - 1)^{\ell / 2} \right) .$$

Equation (67) has the form

$$Y_{\ell m}(\theta, \phi) = C_\ell^m e^{im\phi}$$ (68)

defining $\sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} (-1)^m \mathcal{P}_\eta(\cos \theta) \equiv C_\ell^m$ if the scattering angle $\Omega = (\theta = \theta_1, \phi = \phi_0)$, as defined in figure (3), is used.

Substitute for the spherical harmonics in equation (66) using equation (68) and take the product of the scattering amplitude and its complex conjugate.

A term of the product has the general form

$$a_{\ell_1 m_1} a_{\ell_2 m_2} C_{\ell_1} C_{\ell_2} T_{\ell_1 m_1} T_{\ell_2 m_2}^* i(m-m')\theta_p$$

which can be written

$$a_{\ell_1 m_1} a_{\ell_2 m_2} C_{\ell_1} C_{\ell_2} \left\{ \begin{array}{c} \Re \left( T_{\ell_1 m_1} T_{\ell_2 m_2}^* \cos (m-m')\theta_p \right) \\ \Im \left( T_{\ell_1 m_1} T_{\ell_2 m_2}^* \sin (m-m')\theta_p \right) \end{array} \right\} .$$ (69)
This form can be expressed in terms of the Legendre polynomials, as
\[
\sum_{k=0}^{m} \frac{(-1)^k}{k!} \left( \frac{2m+1}{2} \right) (m-k)! \left( \frac{(m-k)!}{m!} \right) \frac{(2k)!}{k!(k-m)!} P_{k}(\cos \theta) \left( T_{\ell+2} \pi_{\ell+2} \right) \frac{d_\ell^{m\ell}}{\xi_{\ell}^{m\ell}} \frac{d_{m\ell}}{\xi_{m\ell}} \frac{d_{m-m'}}{\xi_{m-m'}} P_{m}(\cos \theta).
\]
(7)

noting the change of dummy variable \((m-m') \rightarrow \ell\).

A computer program calculates the coefficient
\[
A_{\ell}(m_{1}m_{1}', \ldots, m_{n}m_{n}') = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell+1)(\ell)!} \left( \frac{2m_{1}+1}{2} \right) \left( \frac{2m_{2}+1}{2} \right) \cdots \left( \frac{2m_{n}+1}{2} \right) \frac{(2\ell)!}{\ell!(\ell-m_{1})!(\ell-m_{2})! \cdots (\ell-m_{n})!} P_{\ell}(\cos \theta).
\]

Table 2 and table 3 list the numerical values of the coefficient \(A_{\ell}\) for the \(I_{\ell} = 0^{+}\) to \(I_{\ell} = 2^{+}\) excitation by \(\ell_{n} = \frac{1}{2}\) projectile.

The scattering matrix \(|f_{y_{1}y_{2}}|^{2}\) has the form
\[
|f(\theta_{p})|^{2} = |f_{\text{even}}(\theta_{p}) + f_{\text{odd}}(\theta_{p})|^{2}.
\]
\[
= |f_{\text{even}}(\theta_{p})|^{2} + |f_{\text{odd}}(\theta_{p})|^{2} + 2 \Re \left[ f_{\text{even}}(\theta_{p})f^{*}(\theta_{p}) \right] + 2 \Re \left[ f_{\text{odd}}(\theta_{p})f^{*}(\theta_{p}) \right].
\]

The \(|f_{\text{even}}(\theta_{p})|^{2}\) and \(|f_{\text{odd}}(\theta_{p})|^{2}\) terms contribute to coefficients of \(P_{\ell, \text{even}}(\cos \theta)\) while the \(\Re \left[ f_{\text{even}}(\theta_{p})f^{*}(\theta_{p}) \right]\) and \(\Re \left[ f_{\text{odd}}(\theta_{p})f^{*}(\theta_{p}) \right]\) terms contribute to coefficients of \(P_{\ell, \text{odd}}(\cos \theta)\).

If we are interested in only even terms of \(P_{\ell, \text{even}}(\cos \theta)\), which is the case in this paper, we need only \(|f_{\text{even}}(\theta_{p})|^{2}\) and \(|f_{\text{odd}}(\theta_{p})|^{2}\).
Table (2). Coefficients $A_n$ for positive parity partial waves. Inelastic coefficients in column 1, 5 and 9; spin-flip coefficients in column 2, 4, 6, 8, 10, and 12; spin-nonflip coefficients in column 3, 5, 7, 9, and 11.
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Table (3). Coefficients $A_{\lambda}$ for negative parity partial waves. Inelastic coefficients in column 1, 5, and 9; spin-flip coefficients in column 2, 4, 6, 8, 10, and 12; spin-nonflip coefficients in column 3, 5, 7, 9, and 11.
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IV. RESULTS

The differential scattering cross section can be expressed in a Legendre polynomial expansion as

$$\frac{d\sigma}{d\Omega} = \sum_L A_L P_L(\cos \Theta) .$$

The experimentally determined differential cross section \( \frac{d\sigma}{d\Omega} \) is best fit to determine the required Legendre polynomial coefficients.

A set of linear equations relating the coefficients \( A_L \) and the transition matrix elements can be established from table (1). In some cases these linear equations can be used to establish a definite relation between the coefficients \( A_L \). Data extracted from phase shifts of elastic scattering experiments can be used to determine the amplitude of the various background terms. The experimental evidence might indicate a negligible amplitude for a particular background element in which case it could be eliminated from the set of equations.

The following transition matrix elements are expected for an \( I^\pi = 0^+ \) to \( I'_n = 2^+ \) process through an intermediate \( \gamma_2^+ \) state:

\[ \Gamma_{02}, \Gamma_{032}, \Gamma_{24}, \Gamma_{26} \]
It is the background terms $T_{3/2,1/2}$ and $T_{1/2,3/2}$ that are susceptible to elimination from the experimental evidence. Table (1) yields the following equations if the process is a pure $T_{d_{x^2}}$ resonance.

$$A_0 = \left| T_{d_{x^2}} \right|^2$$

$$A_2 = \frac{3}{5} \sqrt{2} T_{d_{x^2}} T_{s_{z^2}d_{y^2}} - \frac{12}{5} \sqrt{2} T_{d_{x^2}} T_{s_{x^2}d_{y^2}}$$

$$A_5 = \frac{3}{10} \left| T_{d_{x^2}} \right|^2 + \frac{3}{5} \sqrt{2} T_{d_{x^2}} T_{s_{z^2}d_{y^2}}$$

$$- \frac{2}{5} \sqrt{2} T_{d_{x^2}} T_{s_{x^2}d_{y^2}} + \frac{2}{5} \sqrt{2} T_{d_{x^2}} T_{d_{z^2}}$$

$$A_7 = \frac{3}{5} \sqrt{2} T_{d_{x^2}} T_{s_{x^2}d_{y^2}} - \frac{2}{5} \sqrt{2} T_{d_{x^2}} T_{s_{z^2}d_{y^2}}$$

The set of equations for a pure $T_{d_{x^2}}$ resonance is

$$A_0 = \left| T_{d_{x^2}} \right|^2$$

$$A_2 = \frac{8}{5} \sqrt{3} T_{d_{x^2}} T_{s_{z^2}d_{y^2}} + \frac{2}{5} T_{d_{x^2}} T_{s_{x^2}d_{y^2}}$$

$$A_5 = \frac{1}{5} \left| T_{d_{x^2}} \right|^2 + \frac{2}{5} \sqrt{3} T_{d_{x^2}} T_{s_{z^2}d_{y^2}}$$

$$- \frac{2}{5} \sqrt{2} T_{d_{x^2}} T_{s_{x^2}d_{y^2}} + \frac{2}{5} \sqrt{2} T_{d_{x^2}} T_{d_{z^2}}$$

$$A_7 = \frac{4}{5} \sqrt{3} T_{d_{x^2}} T_{s_{x^2}d_{y^2}} - \frac{4}{5} \sqrt{2} T_{d_{x^2}} T_{s_{z^2}d_{y^2}}.$$

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The transition matrices expected for a $\frac{3}{2}^+$ resonance in the $I^g = 0^+$ to $I^g = 2^+$ process are

$$T_{\text{res}} = T_{\text{res}}^2, T_{\text{res}}^3, T_{\text{res}}^4, T_{\text{res}}^5, T_{\text{res}}^6, T_{\text{res}}^7, T_{\text{res}}^8, T_{\text{res}}^9.$$

The first three elements are the possible resonant terms. The set of equations for the pure $T_{\text{res}}$ resonance is

$$A_0 = 2 |T_{\text{res}}| \neq 0,$$

$$A_2 = \frac{2}{\sqrt{3}} T_{\text{res}} T_{\text{res}} + \frac{1}{\sqrt{3}} T_{\text{res}} T_{\text{res}} + \frac{1}{\sqrt{3}} T_{\text{res}} T_{\text{res}},$$

$$A_0 = \frac{2}{\sqrt{3}} |T_{\text{res}}| \neq 0,$$

$$A_2 = \frac{2}{\sqrt{3}} T_{\text{res}} T_{\text{res}} + \frac{1}{\sqrt{3}} T_{\text{res}} T_{\text{res}} + \frac{1}{\sqrt{3}} T_{\text{res}} T_{\text{res}},$$

for a pure $T_{\text{res}}$.

$$A_0 = 2 |T_{\text{res}}| \neq 0,$$

$$A_2 = \frac{2}{\sqrt{3}} T_{\text{res}} T_{\text{res}} + \frac{1}{\sqrt{3}} T_{\text{res}} T_{\text{res}} + \frac{1}{\sqrt{3}} T_{\text{res}} T_{\text{res}},$$

$$A_0 = \frac{2}{\sqrt{3}} |T_{\text{res}}| \neq 0,$$

$$A_2 = \frac{2}{\sqrt{3}} T_{\text{res}} T_{\text{res}} + \frac{1}{\sqrt{3}} T_{\text{res}} T_{\text{res}} + \frac{1}{\sqrt{3}} T_{\text{res}} T_{\text{res}},$$

for a pure $T_{\text{res}}$.
for a pure $T\text{d}_g d\frac{3}{2}$,

$$A_0 = \frac{2}{10} |T_{\text{d}_g d\frac{3}{2}}|^2$$

$$A_2 = \frac{8}{\sqrt{15}} |T_{\text{d}_g d\frac{3}{2}}|^2 - \frac{8}{5} \sqrt{\frac{3}{15}} T_{\text{d}_g d\frac{3}{2}} T_{\text{d}_g d\frac{3}{2}} + \frac{8}{3} |T_{\text{d}_g d\frac{3}{2}}|^2 T_{\text{d}_g d\frac{3}{2}}$$

$$A_0 = \frac{12}{70} |T_{\text{d}_g d\frac{3}{2}}|^2 - \frac{8}{5} \sqrt{\frac{3}{15}} T_{\text{d}_g d\frac{3}{2}} T_{\text{d}_g d\frac{3}{2}} + \frac{8}{3} |T_{\text{d}_g d\frac{3}{2}}|^2 T_{\text{d}_g d\frac{3}{2}}$$

$$A_2 = -\frac{2}{\sqrt{15}} |T_{\text{d}_g d\frac{3}{2}}|^2 + \frac{8}{5} \sqrt{\frac{3}{15}} T_{\text{d}_g d\frac{3}{2}} T_{\text{d}_g d\frac{3}{2}}$$

The transition matrices expected for a $\frac{3}{2}^+$ resonance in the $I_0^\pi = 0^+$ to $I_n^\pi = 2^+$ process are

$$T_{\text{d}_g d\frac{3}{2}}, T_{\text{d}_g d\frac{3}{2}}, T_{\text{d}_g d\frac{3}{2}}, T_{\text{d}_g d\frac{3}{2}}, T_{\text{d}_g d\frac{3}{2}}, T_{\text{d}_g d\frac{3}{2}}, T_{\text{d}_g d\frac{3}{2}}.$$

The set of equations for a pure $T_{\text{d}_g d\frac{3}{2}}$ resonance is

$$A_0 = 3 |T_{\text{d}_g d\frac{3}{2}}|^2$$

$$A_2 = -\frac{12}{5} \sqrt{\frac{3}{15}} T_{\text{d}_g d\frac{3}{2}} T_{\text{d}_g d\frac{3}{2}} + \frac{2}{5} T_{\text{d}_g d\frac{3}{2}} T_{\text{d}_g d\frac{3}{2}}$$

$$A_0 = \frac{12}{10} |T_{\text{d}_g d\frac{3}{2}}|^2 - \frac{6}{5} |T_{\text{d}_g d\frac{3}{2}}|^2 T_{\text{d}_g d\frac{3}{2}} - \frac{2}{5} \sqrt{\frac{3}{15}} T_{\text{d}_g d\frac{3}{2}} T_{\text{d}_g d\frac{3}{2}}$$

$$A_2 = -\frac{2}{5} \sqrt{\frac{3}{15}} T_{\text{d}_g d\frac{3}{2}} T_{\text{d}_g d\frac{3}{2}}$$

$$A_0 = -\frac{2}{5} \sqrt{\frac{3}{15}} T_{\text{d}_g d\frac{3}{2}} T_{\text{d}_g d\frac{3}{2}} - \frac{4}{5} \sqrt{\frac{3}{15}} T_{\text{d}_g d\frac{3}{2}} T_{\text{d}_g d\frac{3}{2}}.$$
for a pure \( T_{d^{3}d^{2}} \) resonance,

\[
A_{o} = 3 \left| T_{d^{3}d^{2}} \right|^2
\]

\[
A_{2} = -\frac{12}{35} \left| T_{d^{3}d^{2}} \right|^2 - \frac{24}{5} \sqrt{\frac{3}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} + \frac{48}{7} \sqrt{\frac{2}{7}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}}
\]

\[
- \frac{4}{5} \sqrt{\frac{7}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} - \frac{4}{5} \sqrt{\frac{7}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} \]

\[
A_{4}^{s} = \frac{144}{70} \left| T_{d^{2}d^{2}} \right|^2 + \frac{9}{7} \sqrt{\frac{2}{7}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} - \frac{18}{7} \sqrt{\frac{2}{7}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}}
\]

\[
+ \frac{1}{5} \sqrt{\frac{7}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} + \frac{3}{5} \sqrt{\frac{7}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}}
\]

\[
A_{2}^{s} = -\frac{22}{35} \left| T_{d^{2}d^{2}} \right|^2 - \frac{24}{5} \sqrt{\frac{3}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} + \frac{24}{5} \sqrt{\frac{3}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}}
\]

\[
- \frac{16}{5} \sqrt{\frac{7}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} - \frac{16}{5} \sqrt{\frac{7}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}}
\]

for a pure \( T_{d^{3}d^{2}} \) resonance,

\[
A_{o} = 3 \left| T_{d^{3}d^{2}} \right|^2
\]

\[
A_{2} = \frac{12}{35} \left| T_{d^{3}d^{2}} \right|^2 + \frac{24}{5} \sqrt{\frac{3}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} - \frac{12}{5} \sqrt{\frac{7}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}}
\]

\[
- \frac{24}{5} \sqrt{\frac{3}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} - \frac{36}{5} \sqrt{\frac{3}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}}
\]

\[
A_{4} = -\frac{9}{7} \left| T_{d^{3}d^{2}} \right|^2
\]

\[
A_{0}^{s} = \frac{108}{140} \left| T_{d^{3}d^{2}} \right|^2 + \frac{3}{5} \sqrt{\frac{7}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} + \frac{1}{5} \sqrt{\frac{7}{5}} T_{d^{2}d^{2}} T_{d^{2}d^{2}} T_{d^{2}d^{2}}
\]
The set of equations for a pure $T_{p_1 p_2}, y_2$ resonance is

$$A_0 = \left| T_{p_1 p_2} \right|^2$$

$$A_2 = \frac{2}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4} - \frac{4}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4} - \frac{2}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4}$$

$$A_6 = \frac{2}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4} + \frac{2}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4} + \frac{2}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4}$$

$$A_8 = -\frac{2}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4} T_{p_5 p_6} - \frac{2}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4}$$

$$A_{10} = -\frac{2}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4} T_{p_5 p_6} - \frac{2}{3} \sqrt{3} T_{p_1 p_2} T_{p_3 p_4} + 2 \sqrt{3} T_{p_1 p_2} T_{p_3 p_4} T_{p_5 p_6}$$
The set of equations for a pure $T_{f\pi P_{3/2}}, {\frac{1}{2}}^-$ resonance is

$$A_0 = |T_{f\pi P_{3/2}}|^2$$

$$A_2 = -\frac{2}{5} \sqrt{\frac{2}{7}} T_{f\pi P_{3/2}} T_{f\pi P_{3/2}} P_{3/2} + \frac{3}{5} \sqrt{\frac{2}{7}} T_{f\pi P_{3/2}} T_{f\pi P_{3/2}} P_{3/2}$$

$$+ \frac{3}{5} \sqrt{\frac{2}{7}} T_{f\pi P_{3/2}} T_{f\pi P_{3/2}} P_{3/2} - \frac{4}{5} \sqrt{\frac{2}{7}} T_{f\pi P_{3/2}} T_{f\pi P_{3/2}} P_{3/2}$$

The set of equations for a pure $T_{f\pi P_{3/2}}, {\frac{3}{2}}^-$ resonance is

$$A_0 = 2 |T_{f\pi P_{3/2}}|^2$$

$$A_2 = -\frac{2}{5} \sqrt{\frac{2}{7}} T_{f\pi P_{3/2}} T_{f\pi P_{3/2}} P_{3/2} + \frac{3}{5} \sqrt{\frac{2}{7}} T_{f\pi P_{3/2}} T_{f\pi P_{3/2}} P_{3/2}$$

$$+ \frac{3}{5} \sqrt{\frac{2}{7}} T_{f\pi P_{3/2}} T_{f\pi P_{3/2}} P_{3/2} - \frac{4}{5} \sqrt{\frac{2}{7}} T_{f\pi P_{3/2}} T_{f\pi P_{3/2}} P_{3/2}$$
The set of equations for a pure $T_{pV_p J_p}, \frac{1}{2}$ resonance is

$$A_0 = 2 |T_{pV_p J_p}|^2$$

$$A_0^* = \frac{4}{3} |T_{pV_p J_p}|^2 - \frac{4i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p} + \frac{2i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p}$$

$$- \frac{4}{3} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p} + \frac{2i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p}$$

$$- \frac{4}{3} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p}$$

$$A_2 = -\frac{8}{3} |T_{pV_p J_p}|^2 + \frac{4i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p} - \frac{2i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p}$$

$$- \frac{4}{3} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p} + \frac{2i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p}$$

$$A_4 = \frac{2i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p}$$

$$- \frac{2i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p}$$

$$A_6 = \frac{2i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p}$$

$$- \frac{2i}{3} \sqrt{\frac{2}{3}} T_{pV_p J_p} T_{pV_p J_p} T_{pV_p J_p}$$

The set of equations for a pure $T_{fJ_p J_p}, \frac{1}{2}$ resonance is

$$A_0 = 2 |T_{fJ_p J_p}|^2$$

$$A_0^* = \frac{4}{3} |T_{fJ_p J_p}|^2 - \frac{4i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p} + \frac{2i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p}$$

$$- \frac{4}{3} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p} + \frac{2i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p}$$

$$- \frac{4}{3} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p}$$

$$A_2 = -\frac{8}{3} |T_{fJ_p J_p}|^2 + \frac{4i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p} - \frac{2i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p}$$

$$- \frac{4}{3} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p} + \frac{2i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p}$$

$$A_4 = \frac{2i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p}$$

$$- \frac{2i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p}$$

$$A_6 = \frac{2i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p}$$

$$- \frac{2i}{3} \sqrt{\frac{2}{3}} T_{fJ_p J_p} T_{fJ_p J_p} T_{fJ_p J_p}$$

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The set of equations for a pure $T_{p_3p_3}$, $\frac{3}{2}^-$ resonance is

$$A_0 = 2 |T_{p_3p_3}|^2$$

$$A_0^\prime = \frac{9}{35} |T_{p_3p_3}|^2 + \frac{3}{35} T_{p_3p_3}^2 T_{p_3f_3}^2 T_{p_3f_3}^2 + \frac{2}{35} T_{p_3p_3} T_{p_3f_3} T_{p_3f_3}^2$$

$$A_1 = \frac{10}{25} |T_{p_3p_3}|^2 + \frac{144}{25} T_{p_3p_3} T_{p_3f_3} T_{p_3f_3}^2 + \frac{8}{25} T_{p_3p_3} T_{p_3f_3} T_{p_3f_3}^2$$

The set of equations for a pure $T_{p_4f_2}$, $\frac{5}{2}^-$ resonance is

$$A_0 = 3 |T_{p_4f_2}|^2$$

$$A_0^\prime = \frac{9}{35} |T_{p_4f_2}|^2 - \frac{6}{35} T_{p_4f_2} T_{p_4f_2} T_{p_4f_2} - \frac{8}{35} T_{p_4f_2} T_{p_4f_2} T_{p_4f_2}$$

$$A_1 = \frac{12}{25} T_{p_4f_2} T_{p_4f_2} T_{p_4f_2} + \frac{18}{25} T_{p_4f_2} T_{p_4f_2} T_{p_4f_2}$$

$$A_2^\prime = \frac{32}{25} T_{p_4f_2} T_{p_4f_2} T_{p_4f_2}$$

$$A_3 = 12 |T_{p_4f_2}|^2 T_{p_3f_3}^2$$
The set of equations for a pure $T_{\pi\sigma^+}, S_2^-$ resonance is

$$A_0 = 3 |T_{\pi\sigma^+}|^2$$

$$A^3 = \frac{18}{35} |T_{\pi\sigma^+}|^2 - \frac{6}{35} \sqrt{\frac{2}{3}} T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+} + \frac{3}{35} T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+}$$

$$A_2 = -\frac{2}{35} |T_{\pi\sigma^+}|^2 - \frac{12}{35} \sqrt{\frac{2}{3}} T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+} - \frac{2}{35} \sqrt{\frac{2}{3}} T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+}$$

$$A_4 = \frac{2}{49} 16 T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+} + \frac{2}{49} T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+}$$

$$A^3 = \frac{144}{49} |T_{\pi\sigma^+}|^2 + \frac{2}{49} T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+} T_{\pi\sigma^+}.$$

The set of equations for a pure $T_{\sigma\pi^+}, S_2^-$ resonance is

$$A_0 = 3 |T_{\sigma\pi^+}|^2$$

$$A^3 = \frac{2}{35} |T_{\sigma\pi^+}|^2 - \frac{6}{35} \sqrt{2} T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+} + \frac{3}{35} T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+}$$

$$A_2 = -\frac{2}{35} |T_{\sigma\pi^+}|^2 - \frac{12}{35} \sqrt{2} T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+} - \frac{2}{35} \sqrt{2} T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+}$$

$$A_4 = \frac{2}{49} 16 T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+} + \frac{2}{49} T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+}$$

$$A^3 = \frac{144}{49} |T_{\sigma\pi^+}|^2 + \frac{2}{49} T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+} T_{\sigma\pi^+}.$$
The set of equations for a pure $T_{p_3f_1/2}$, $f_2^-$ resonance is

$$A_0 = 3 \left| T_{p_3f_1/2} \right|^2$$

$$A_0^s = \frac{1}{6^2} \left| T_{p_3f_1/2} \right|^2 - \frac{1}{3 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} - \frac{\sqrt{6}}{2} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$

$$A_2 = \frac{1}{6^2} \left| T_{p_3f_1/2} \right|^2 + \frac{1}{3 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} + \frac{\sqrt{6}}{2} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$

$$A_2^s = \frac{1}{6^2} \left| T_{p_3f_1/2} \right|^2 + \frac{1}{3 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} + \frac{\sqrt{6}}{2} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$

$$A_4 = -\frac{9}{7} \left| T_{p_3f_1/2} \right|^2 + \frac{8}{7 \sqrt{5}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} - \frac{16}{7 \sqrt{5}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$

$$A_4^s = -\frac{9}{7} \left| T_{p_3f_1/2} \right|^2 + \frac{8}{7 \sqrt{5}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} - \frac{16}{7 \sqrt{5}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$

The set of equations for a pure $T_{p_3f_1/2}$, $f_2^-$ resonance is

$$A_0 = 4 \left| T_{p_3f_1/2} \right|^2$$

$$A_0^s = \frac{2}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} - \frac{3}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$

$$A_2 = \frac{3}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} - \frac{3}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$

$$A_2^s = \frac{3}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} - \frac{3}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$

$$A_4 = -\frac{3}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} - \frac{3}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$

$$A_4^s = -\frac{3}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2} - \frac{3}{5 \sqrt{2}} T_{p_3f_1/2} T_{p_3f_2} T_{p_3f_2}$$
The set of equations for a pure $T_{\ell_2^\ell_1^2}$ resonance is

$$A_0 = 4 |T_{\ell_2^\ell_1^2}|^2$$

$$A_2 = \frac{\sqrt{2}}{\sqrt{3}} T_{\ell_2^\ell_1^2} T_{\ell_2^\ell_1^2} P_{\ell_1} - \frac{\sqrt{2}}{\sqrt{3}} T_{\ell_2^\ell_1^2} T_{\ell_2^\ell_1^2} P_{\ell_1}$$

$$A_4 = \frac{1}{\sqrt{7}} |T_{\ell_2^\ell_1^2}|^2 - \frac{1}{\sqrt{7}} T_{\ell_2^\ell_1^2} T_{\ell_2^\ell_1^2} P_{\ell_1}$$

The set of equations for a pure $T_{\ell_2^\ell_1^2}$, $\ell_1^2$ resonance is

$$A_0 = 4 |T_{\ell_2^\ell_1^2}|^2$$

$$A_2 = \frac{\sqrt{2}}{\sqrt{3}} T_{\ell_2^\ell_1^2} T_{\ell_2^\ell_1^2} P_{\ell_1} - \frac{\sqrt{2}}{\sqrt{3}} T_{\ell_2^\ell_1^2} T_{\ell_2^\ell_1^2} P_{\ell_1}$$

$$A_4 = \frac{1}{\sqrt{7}} |T_{\ell_2^\ell_1^2}|^2 - \frac{1}{\sqrt{7}} T_{\ell_2^\ell_1^2} T_{\ell_2^\ell_1^2} P_{\ell_1}$$

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The preceding equations give information about the transition matrices in terms of the experimentally determined parameters $A_L$. The coefficients $A_L$ are, as mentioned earlier, determined from spin-flip and spin-nonflip elastic scattering differential cross sections. These relations alone are not sufficient to explicitly determine the transition matrices. Additional elastic scattering cross sections and polarization measurements would provide the means to explicitly determine the transition matrices. For example, these relations with elastic scattering data determine uniquely which partial waves predominantly contribute to the $\frac{1}{2}^-$ resonances of $^{12}C$. 

\[
A_L = -\frac{160}{99} |T_{f,0}f,0|^2
\]

\[
A_S = -\frac{40}{93} |T_{f,0}f,2|^2
\]
REFERENCES


