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A Study of Asymptotic Solutions of Second Order Linear Differential Equations

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A STUDY OF ASYMPTOTIC SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Consider the second order linear ordinary differential equation with polynomial coefficients of the form:

$$(1.1) \qquad y'' - P(x)y = 0 \qquad \left('' = \frac{d^2}{dx^2} \right)$$

where $P(x)$ is the polynomial in x :

$$(1.2) \qquad P(x) = x^m + a_1 x^{m-1} + \cdots + a_{m-1}x + a_m .$$

It was shown by P. F. Hsieh and Y. Sibuya [1], in 1966, that since the only singular point of (1.1) is at $x = \infty$, a solution of (1.1) is an entire function of $(x, a_1, \ldots, a_m)$ if its initial values are entire functions of $(a_1, \ldots, a_m)$. It was also shown that since $x = \infty$ is an irregular singular point, a solution of (1.1) can be determined by prescribing asymptotic conditions as x tends to infinity in a sector S, if S and the asymptotic conditions are suitably given.

In this paper, we shall consider the following two second order linear differential equations with polynomial coefficients:

$$(1.3) \qquad x^2y'' + (x^2 + \lambda)y = 0$$

$$(1.4) \qquad x^2y'' + (x^3 + \lambda)y = 0$$

where λ is a complex parameter. Notice that $x = 0$ is a regular singular point and $x = \infty$ is an irregular singular point of both (1.3) and (1.4), while $x = \infty$ is the only singular point of (1.1).

The result for (1.3) is stated in §2 and proved in §3 ~ 6, while the result for (1.4) is stated in §7 and proved in §8 ~ 10.

<u>Remark 1</u>: Replacement of u by $x^{-1/2}y$ in Bessel's equation

$$(1.5) \qquad x^2u'' + xu' + (x^2 - \nu^2)u = 0$$

leads to the equation

$$(1.6) \qquad x^2y'' + (x^2 + \frac{1}{4} - \nu^2)y = 0 \quad .$$

Therefore (1.3) is equivalent to Bessel's equation with $\nu = \pm(\frac{1}{4} - \lambda)^{1/2}$.

A Bessel function $J_\nu(x)$ is a solution of (1.5) which is known to have the following asymptotic expansion [2]:

$$J_\nu(x) \cong \left(\frac{2}{\pi x}\right)^{1/2}\left\{\cos\zeta \sum_{s=0}^{\infty} \frac{(-1)^s A_{2s}(\nu)}{x^{2s}} - \sin\zeta \sum_{s=0}^{\infty} \frac{(-1)^s A_{2s+1}(\nu)}{x^{2s+1}}\right\}$$

as x tends to infinity in $|\arg x| \leq \pi - \delta$, uniformly with respect to ν in $|\nu| < r$, $(0 < r < +\infty)$, where

$$\zeta = x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi$$

$$A_s(\nu) = \frac{(4\nu^2 - 1)(4\nu^2 - 3^2)\cdots\{4\nu^2 - (2s-1)^2\}}{s!\, 8^3}$$

and the branch of $x^{1/2}$ is determined by:

$$x^{1/2} = \exp\{\frac{1}{2}\ln|x| + \frac{1}{2} i \arg x\} .$$

Therefore a solution of (1.3) has the form $y = x^{1/2} J_\nu(x)$.

2. ASYMPTOTIC SOLUTION OF $x^2 y'' + (x^2 + \lambda) y = 0$

As the first result of this paper, we shall prove the following theorem:

Theorem 1: The second order linear differential equation (1.3) has a solution $y(x, \lambda)$ that satisfies the following conditions:

(i) $y(x, \lambda)$ is holomorphic with respect to $(x, \lambda) \in S \times D$, where S is a sector in x-plane and D is an open disc in λ-plane defined by:

$$S: \quad -2\pi + \delta \le \arg x \le \pi - \delta, \quad |x| \ge M .$$

$$D: \quad |\lambda| < r, \quad 0 < r < +\infty$$

(ii) $y(x, \lambda)$ and $y'(x, \lambda)$ admit respectively the asymptotic representations:

$$y(x, \lambda) \simeq e^{ix}\left\{1 + \sum_{n=1}^{\infty} \alpha_n x^{-n}\right\} \tag{2.1}$$

$$(2.2) \qquad y'(x, \lambda) \cong ie^{ix} \left\{1 + \sum_{n=1}^{\infty} \beta_n x^{-n}\right\}$$

<u>uniformly with respect to</u> $\lambda \in D$, <u>as</u> x <u>tends to infinity in</u> S, <u>where</u> δ, r <u>and</u> M <u>are positive constants and</u> α_n <u>and</u> β_n <u>are polynomials in</u> λ.

<u>Remark 2</u>: If the asymptotic representations of $y(x, \lambda)$ and $y'(x, \lambda)$ in a sector different from S is desired, it can be constructed as follows:

Let us change the independent variable x by $\hat{x} = e^{i\theta}x$. Then equation (1.3) becomes

$$(2.3) \qquad \hat{x}^2 \frac{d^2y}{d\hat{x}^2} + e^{-2i\theta}(\hat{x}^2 + e^{2i\theta}\lambda)y = 0 .$$

Therefore, if we choose θ so that $e^{-2i\theta} = 1$, the function $y(\hat{x}, e^{2i\theta}\lambda)$ is a solution of (2.3). Hence if we put $\theta = \pi$ and $y_1(x, \lambda) = y(\hat{x}, e^{2i\theta}\lambda) = y(e^{i\pi}x, \lambda)$, then $y_1(x, \lambda)$ is a solution of (1.3) which admits the asymptotic representation $Y(-x, \lambda)$ as x tends to infinity in any closed sector which is contained in the sector S', where S' is defined by:

$$S': -\pi + \delta \le \arg x \le 2\pi - \delta, \qquad |x| \ge M$$

and $Y(x, \lambda)$ is the right-hand member of (2.1).

3. PROOF OF THEOREM 1: PART I.

3.1. Preliminary Transformations.

Let us write (1.3) as the following system of equations

(3.1) $$\frac{dY}{dx} = A(x)Y \ ,$$

where

$$Y = \begin{bmatrix} y \\ y' \end{bmatrix}$$

and

$$A(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\lambda & 0 \end{bmatrix} x^{-2} \equiv A_o + A_2 x^{-2} \ .$$

Since the eigenvalues of A_o are i and $-i$, in order to have the Jordan canonical form of A_o as the leading term in $A(x)$, let us put $Y = TW$, where

$$T = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \ .$$

Then (3.1) becomes

(3.2) $$\frac{dW}{dx} = B(x)W \ ,$$

where

(3.3) $$B(x) = \begin{bmatrix} i\left(1 + \frac{\lambda}{2x^2}\right) & -\frac{\lambda}{2x^2} \\ -\frac{\lambda}{2x^2} & -i\left(1 + \frac{\lambda}{2x^2}\right) \end{bmatrix} \ .$$

We shall now derive a nonlinear first order differential equation associated with the system (3.2).

Put

$$B(x) = \begin{bmatrix} \alpha_1(x) & \beta_1(x) \\ \beta_2(x) & \alpha_2(x) \end{bmatrix}$$

where

$$\alpha_1(x) = i + \frac{\lambda}{2} i x^{-2}$$

$$\alpha_2(x) = -i - \frac{\lambda}{2} i x^{-2}$$

$$\beta_1(x) = -\frac{\lambda}{2} x^{-2}$$

$$\beta_2(x) = -\frac{\lambda}{2} x^{-2}$$

The quantities α_1, α_2, β_1 and β_2 are polynomials in x^{-1} and we have $\alpha_1 = i + O(x^{-2})$, $\alpha_2 = -i + O(x^{-2})$, $\beta_1 = O(x^{-2})$ and $\beta_2 = O(x^{-2})$ as x tends to infinity, uniformly with respect to λ such that $|\lambda| < \infty$.

Now let us put the following expression into system (3.2):

$$(3.4) \qquad W = \begin{pmatrix} 1 \\ p(x) \end{pmatrix} \exp\left\{\int^x \gamma(\eta)\, d\eta\right\}$$

Then we obtain the following relations:

$$(3.5) \qquad \gamma(x) = \alpha_1(x) + \beta_1(x)p(x)$$

$$\frac{dp}{dx} = \beta_2(x) + \{\alpha_2(x) - \alpha_1(x)\}p(x) - \beta_1(x)p^2(x)$$

$$(3.6) \qquad = \frac{-\lambda}{2x^2} - 2i\left(1 + \frac{\lambda}{2x^2}\right)p(x) + \frac{\lambda}{2x^2}p^2(x) .$$

If we determine $p(x)$ from the nonlinear differential equation (3.6) and then $\gamma(x)$ by (3.5), the quantity (3.4) is a solution of (3.2). Therefore

$$(3.7) \qquad Y = T\begin{pmatrix}1\\p\end{pmatrix} \exp\left\{\int^x \gamma(\eta)\,d\eta\right\}$$

will give us a solution of (1.3)

3.2 Formal Solution of (3.6)

Lemma 1. The nonlinear first order differential equation

$$(3.6) \qquad \frac{dp}{dx} = \frac{-\lambda}{2x^2} - 2i\left(1 + \frac{\lambda}{2x^2}\right)p(x) + \frac{\lambda}{2x^2}p^2(x)$$

has a unique formal solution

$$(3.8) \qquad \hat{p}(x) = \sum_{n=1}^{\infty} P_n x^{-n} ,$$

where $P_1 = 0$, $P_2 = \frac{\lambda}{4}i$, $P_3 = \frac{\lambda}{4}$ and

$$(3.9) \qquad P_n = \frac{-i}{2}\{(n-1)P_{n-1} - \lambda i P_{n-2} + \frac{\lambda}{2}\Sigma P_j P_k\}$$

for $n \geq 4$

where the sum is taken over $j, k \geq 1$ and $j + k = n - 2$.

Proof: Let $p = \sum_{n=1}^{\infty} P_n x^{-n}$ and substitute into (3.6). Then we obtain

$$\sum_{n=1}^{\infty} (-n) P_n x^{-n-1}$$

$$= \frac{-\lambda}{2x^2} - 2i\left(1 + \frac{\lambda}{2x^2}\right) \sum_{n=1}^{\infty} P_n x^{-n} + \frac{\lambda}{2x^2} \sum_{n=2}^{\infty} \left(\sum P_j P_k\right) x^{-n},$$

where $\sum P_j P_k$ is taken over $j, k \geq 1$ and $j + k = n$. Equating the coefficients of terms with equal exponent, we have

$$0 = -2i P_1$$

$$-P_1 = -\frac{\lambda}{2} - 2i P_2$$

$$-2P_2 = -2i P_3 - \lambda i P_1$$

$$-nP_n = -2i P_{n+1} - \lambda i P_{n-1} + \frac{\lambda}{2} \sum_{j+k=n-1} P_j P_k$$

for $n \geq 3$.

These relations determine P_n $(n \geq 1)$ uniquely, namely $P_1 = 0$, $P_2 = \frac{\lambda}{4} i$, $P_3 = \frac{\lambda}{4}$ and

$$P_n = \frac{-i}{2}\left\{(n-1)P_{n-1} - \lambda i\ P_{n-2} + \frac{\lambda}{2}\sum_{j+k=n-2} P_j\ P_k\right\}$$

$$\text{for} \quad n \geq 4 \quad .$$

By Lemma 1 and (3.4), we obtain a formal solution $w_1(x, \lambda)$ of (3.2):

$$w_1(x, \lambda) = \left\{1 - \frac{\lambda}{4}x^{-2} + \frac{\lambda}{4}ix^{-3} + \frac{1}{8}(3\lambda + \lambda^2)x^{-4} + \cdots\right\}$$

$$\times \exp\left\{ix - \frac{\lambda}{2}ix^{-1} + \frac{\lambda^2}{24}ix^{-3} + \frac{\lambda^2}{32}x^{-4} + \cdots\right\}$$

3.3 <u>Analyticity of a solution of (3.6)</u>

In order to find the analytic meaning of the formal solution $\hat{P}(x)$ in (3.8), we need the following:

Lemma 2: <u>There exists a unique solution</u> $p(x)$ <u>of Eq. (3.6) that satisfies the following conditions</u>:

i) <u>For each positive constant</u> r, <u>there exists a positive constant</u> N_r <u>such that</u> $p(x)$ <u>is holomorphic with respect to</u> (x, λ) <u>in the domain determined by</u>:

$$|\arg(2i) + \arg x| \leq \frac{3\pi}{2} - \delta \quad , \quad |x| > N_r$$

(3.10)

$$|\lambda| < r, \quad 0 < r < \infty$$

ii) $p(x)$ *is asymptotic to the formal solution* $\hat{p}(x)$ *uniformly on each compact set in the* λ*-space as* x *tends to infinity in the sector* $-2\pi + \delta \le \arg x \le \pi - \delta$, *where* δ *is a fixed sufficiently small positive number.*

The proof of this lemma will be provided in §5.

4. PROOF OF THEOREM 1: PART II

In this section we shall complete the proof of Theorem 1 by the use of Lemma 1 and Lemma 2.

Applying Lemma 2, using the fact that $\arg 2i = \frac{\pi}{2}$, we have a unique solution $p(x)$ of Eq. (3.6) holomorphic with respect to (x, λ) in the domain $S \times D$, where

$$S: \quad -2\pi + \delta \le \arg x \le \pi + \delta, \qquad |x| \ge M$$

$$D: \quad |\lambda| < r \quad 0 < r < \infty .$$

By Eq. (3.5), we have

$$\gamma(x) = i + \frac{\lambda}{2} i x^{-2} - \frac{\lambda}{2} x^{-2} p(x)$$

which is holomorphic with respect to $(x, \lambda) \in S \times D$. Hence we have

$$\exp\left\{\int^{x} \gamma(\eta)\,d\eta\right\}$$

$$(4.1) \qquad = \exp\left[\int^{x}\left\{i + \frac{\lambda}{2}\, i\eta^{-2} - \frac{\lambda}{2}\eta^{-2}p(\eta)\right\} d\eta\right]$$

$$\cong \exp\left[\int^{x}\left\{i + \frac{\lambda}{2} i\eta^{-2} - \frac{\lambda}{2}\eta^{-2}\,\hat{p}(\eta)\right\} d\eta\right]$$

uniformly with respect to $\lambda \in D$ as x tends to infinity in S. Inserting the series obtained by Lemma 1 into the expression in (4.1) and integrating, we obtain

$$\exp\left\{\int^{x} \gamma(\eta)\,d\eta\right\}$$

$$\cong \exp\left\{ix - \frac{\lambda}{2} i\, x^{-1} + \frac{\lambda^2}{24}\, ix^{-3} + \frac{\lambda^2}{32}\, x^{-4} + \cdots\right\}.$$

By (3.7) we have

$$y(x\,,\lambda) = \{1 + ip(x)\}\, \exp\left\{\int^{x} \gamma(\eta)\,d\eta\right\}$$

$$\cong e^{ix}\left\{1 - \frac{i}{2}\lambda x^{-1} - \frac{1}{8}(2\lambda + \lambda^2)x^{-2} + \frac{i}{24}(6\lambda + \lambda^2 - \lambda^3)x^{-3} + \cdots\right\}$$

$$= e^{ix}\left\{1 + \sum_{n=1}^{\infty} \alpha_n x^{-n}\right\}$$

uniformly with respect to $\lambda \in D$ as x tends to infinity in S, where α_n are polynomials of λ.

5. PROOF OF LEMMA 2: PART I

In this section we shall prove Lemma 2 in four steps:

i) Construct a function $\hat{P}_r(x, \lambda)$ holomorphic with respect to $(x, \lambda) \in S \times D$ and asymptotic to the formal solution $\hat{P}(x)$ of Eq. (3.6) uniformly with respect to $\lambda \in D$ as $x \to \infty$ in S.

ii) Find a function $q_r(x, \lambda)$ holomorphic with respect to $(x, \lambda) \in S \times D$ such that $q_r(x, \lambda) + \hat{P}_r(x, \lambda)$ is a solution of Eq. (3.6), by constructing an integral equation.

iii) Show that $q_r(x, \lambda) \simeq 0$ uniformly with respect to $\lambda \in D$ as x tends to infinity in S.

iv) Show that $q_r(x, \lambda)$ is actually independent of r. Let

$$p(x) = q_r(x, \lambda) + \hat{P}_r(x, \lambda)$$

Then $p(x)$ is the desired solution of Eq. (3.6).

5.1 Construction of $\hat{P}_r(x)$.

In order to construct $\hat{P}_r(x)$, the following theorem is applied:

Theorem 2: (Borel-Ritt Theorem) [5]. *Corresponding to every formal power series*

$$\sum_{n=0}^{\infty} A_n x^{-n}$$

<u>and to every sector</u> $S: \theta_1 \le \arg x \le \theta_2$ $(\theta_1, \theta_2:$ <u>constants</u>), <u>there exists a function</u> $f(x)$ <u>holomorphic in</u> S <u>for</u> $|x| \ge x_0$ $(x_0$ <u>an arbitrary constant</u>) <u>such that</u>

$$f(x) \simeq \sum_{n=0}^{\infty} A_n x^{-n}$$

<u>as</u> x <u>tends to infinity in</u> S .

We shall apply this theorem with the formal solution $\hat{P}(x) = \sum_{n=1}^{\infty} P_n x^{-n}$ of Eq. (3.6). Notice that all coefficients P_n are independent of x and polynomials in λ. Therefore for each positive constant r, P_n are holomorphic with respect to λ such that $|\lambda| < r$. Let δ be a sufficiently small positive number and Ω be an arbitrary fixed positive number. Let us define a sector S_δ in x-plane by:

$$|\arg(2i) + \arg x| \le \frac{3\pi}{2} - \delta , \quad |x| \ge \Omega \ .$$

The reason for choosing such a sector will be explained in Section 5.2. Let D_r be a domain in λ-plane such that $|\lambda| < r$ $(0 < r < \infty)$.

Now, Borel-Ritt theorem asserts that corresponding to $\hat{P}(x) = \sum_{n=1}^{\infty} P_n x^{-n}$ and to the domain $S_\delta \times D_r$, there exists a function $\hat{P}_r(x)$ so that $\hat{P}_r(x)$ is holomorphic with respect to $(x, \lambda) \in S_\delta \times D_r$ and $\hat{P}_r(x)$ and $d\hat{P}_r(x)/dx$ admit the uniform asymptotic expansions

$$\hat{P}_r(x) \cong \sum_{n=1}^{\infty} P_n x^{-n} \quad , \quad \frac{d\hat{P}_r(x)}{dx} \cong \frac{d\hat{P}(x)}{dx}$$

for $\lambda \in D_r$ as x tend to infinity in the sector S_δ.

5.2. Construction of an Integral Equation.

Put $p = q + \hat{P}_r(x)$ in Eq. (3.6). Then the differential equation (3.6) is reduced to

$$(5.1) \qquad \frac{dq}{dx} = \mu_r(x) + \psi_r(x)q + \nu_r(x)q^2$$

where

$$\mu_r(x) = \frac{-\lambda}{2x^2} - 2i\left(1 + \frac{\lambda}{2x^2}\right)\hat{P}_r(x) + \frac{\lambda}{2x^2}\hat{P}_r^2(x) - \frac{d\hat{P}_r(x)}{dx}$$

$$\psi_r(x) = -2i\left(1 + \frac{\lambda}{2x^2}\right) + \frac{\lambda}{x^2}\hat{P}_r(x)$$

$$\nu_r(x) = \frac{\lambda}{2x^2}$$

Notice that these three quantities are holomorphic in $S_\delta \times D_r$, and we have

$$(5.2) \qquad \mu_r(x) \cong 0$$

$$(5.3) \qquad \psi_r(x) = -2i + \mathcal{O}(x^{-2})$$

$$(5.4) \qquad \nu_r(x) = \mathcal{O}(x^{-2})$$

uniformly for $\lambda \in D_r$ as x tends to infinity in S_δ. The asymptotic relation (5.2) is derived from the fact that the asymptotic expansion of $\hat{P}_r(x)$ is a formal solution of the differential equation (3.6). Let us put

$$\psi_r(x) = -2i + \phi_r(x) \quad .$$

Then the relation (5.3) implies that

$$(5.5) \qquad \phi_r(x) = \mathcal{O}(x^{-2})$$

uniformly for $\lambda \in D_r$ as x tends to infinity in S_δ.

Let

$$(5.6) \qquad q(x) = \int_\infty^x \{\mu_r(\eta) + \phi_r(\eta)q(\eta) + \nu_r(\eta)q^2(\eta)\} \exp\{-2i(x-\eta)\}d\eta ,$$

where the path of integration is a straight line, to be determined in S_δ, given by

$$(5.7) \qquad \eta = x + te^{i\theta}, \quad \theta\text{:constant}, \quad 0 \le t < \infty$$

such that along this path $\exp\{-2i(x-\eta)\}$ converges as $\eta \to \infty$. Then by differentiating both sides of (5.6), we obtain

$$\frac{dq}{dx} = -2i\, e^{-2ix}\int_{\infty}^{x}\{\mu_r(\eta) + \phi_r(\eta)q(\eta) + \nu_r(\eta)q^2(\eta)\}e^{2i\eta}d\eta$$

$$+ e^{-2ix}\{\mu_r(x) + \phi_r(x)q(x) + \nu_r(x)q^2(x)\}e^{2ix}$$

$$= -2i\, q(x) + \mu_r(x) + \phi_r(x)q(x) + \nu_r(x)q^2(x)$$

$$= \mu_r(x) + \{-2i + \phi_r(x)\}q(x) + \nu_r(x)q^2(x)$$

$$= \mu_r(x) + \psi_r(x)q(x) + \nu_r(x)q^2(x)$$

Therefore a solution of the integral equation (5.6) satisfies the differential equation (5.1). A question arises here. Is it possible to have a straight line path in S_δ, from any point x in S_δ to infinity so that $\exp\{-2i(x-\eta)\}$ converges as $\eta \to \infty$ on this line? The answer is affirmative. Suppose we chose a path so that $\mathrm{Re}(i\eta)$ tends to negative infinity as η tends to infinity on this path, then $\exp\{-2i(x-\eta)\}$ tends to 0. Since

$$\mathrm{Re}(i\eta) = -t \sin\theta$$

on the path (5.7), this can be accomplished by choosing θ such that $\frac{\delta}{2} \le \theta \le \pi - \frac{\delta}{2}$. It is readily seen from the figure below that such a straight line path is contained entirely in S_δ .

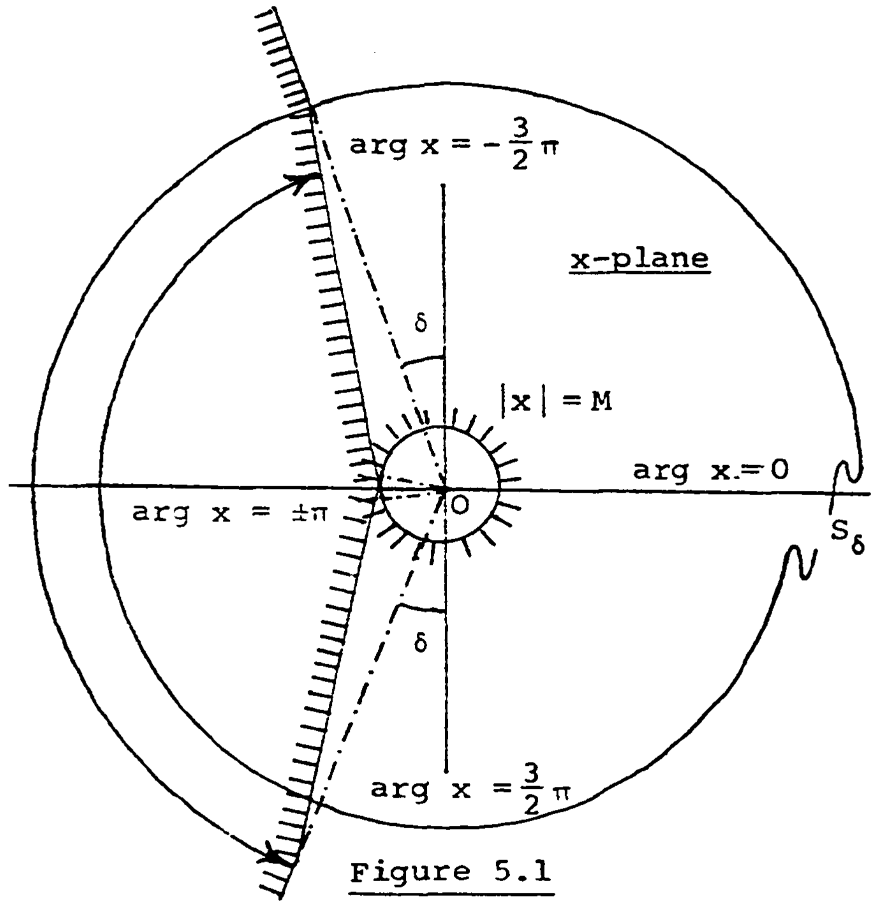

Figure 5.1

Thus it is established that a solution of integral equation (5.6) satisfies the differential equation (5.1) by choosing the path of integration to be

$$\eta = x + te^{i\theta}, \quad \frac{\delta}{2} \le \theta \le \pi - \frac{\delta}{2}, \quad 0 \le t < \infty$$

$$\text{and} \quad x \in S_\delta .$$

Now we shall prove the existence of a holomorphic solution of the integral equation (5.6) by the method of successive approximations. There are several key inequalities used in the process. Let us now state these inequalities. Proofs will be provided in the later sections.

Lemma 3: [3 and 4] Let positive constants δ and ρ be given, where δ is sufficiently small, while ρ is arbitrary. Then there exist positive constants $M_{\delta\rho}$ and L_0 such that we have

$$(5.8) \qquad \int_{\infty}^{s} |\sigma|^{-\rho} |e^{s-\sigma}| \, |d\sigma| \le L_0 |s|^{-\rho}$$

for all s in the sector, $S_{\delta M_{\delta\rho}}$ defined by

$$(5.9) \qquad |\arg x| \le \frac{3\pi}{2} - \delta , \qquad |x| \ge M_{\delta\rho} ,$$

where L_0 is independent of ρ, and the path of integration is the straight line

$$\sigma = s + te^{i\theta}, \quad 0 \le t < \infty, \frac{\delta}{2} \le \theta \le \pi - \frac{\delta}{2} .$$

Lemma 4: The relations (5.2), (5.5) and (5.4) imply that there exists a positive constant L such that

$$(5.10) \qquad |\mu_r(x)| \le L \, |x|^{-2}$$

$$(5.11) \qquad |\phi_r(x)| \le L \, |x|^{-1}$$

$$(5.12) \qquad |v_r(x)| \le L \, |x|^{-1}$$

<u>in the domain</u> $S_\delta \times D_r$.

<u>Lemma 5</u>: <u>Let</u> $f(m) = m^{-1/2}$ <u>for</u> $m > 0$. <u>Then</u> <u>since</u> $\lim_{m\to\infty} f(m) = 0$, <u>there exist positive constants</u> K <u>and</u> M' <u>such that</u>

$$(5.13) \quad \frac{1}{2} L_o L f(M') \{1 + K + f(M')K^2\} \le K$$

$$(5.14) \quad \frac{1}{2} L_o L f(M') \{1 + 2f(M')K\} \le \frac{1}{2}$$

Now let us choose a positive constant M such that

$$(5.15) \qquad M \ge \max\{M' , M_{\delta\rho}\} \quad \text{where} \quad \rho = \frac{1}{2}$$

and let us define the successive approximations in $S_{\delta M}$ by:

$$(5.16) \qquad \begin{aligned} q_o(x) &\equiv 0 \\ q_n(x) &= \int_\infty^x \{\mu_r(\eta) + \phi_r(\eta) q_{n-1}(\eta) + \nu_r(\eta) q_{n-1}^2(\eta)\} \\ &\qquad \times \exp\{-2i(x - \eta)\} d\eta \end{aligned}$$

for $n = 1 , 2 , \ldots$.

Then it can be shown, by induction, with the aid of Lemma 3, 4 and 5, that

$$(5.17) \qquad |q_n(x)| \le K\, |x|^{-1} , \quad |q_{n+1}(x) - q_n(x)| \le \frac{1}{2^n}$$

in the domain $S_{\delta M} \times D_r$. As a matter of fact, suppose that (5.17) holds for $n = j - 1$, then for $n = j$ let $x = -is$ and $\eta = -i\sigma$ in (5.16). By the use of Lemma 3,4 and 5, we have

$$|q_j(x)|$$

$$\leq \int_{\infty}^{s} \{|\mu_r(\eta)| + |\phi_r(\eta)| \cdot |q_{j-1}(\eta)| + |\nu_r(\eta)| |q_{j-1}^2(\eta)|\} |e^{s-\sigma}| |d\sigma|$$

$$\leq \int_{\infty}^{s} \{L|\eta|^{-2} + L|\eta|^{-1} \cdot K|\eta|^{-1} + L|\eta|^{-1} \cdot K^2|\eta|^{-2}\} |e^{s-\sigma}| |d\sigma|$$

$$= L\int_{\infty}^{s} |\sigma|^{-2}\{1 + K + K^2|\sigma|^{-1}\} \cdot |e^{s-\sigma}| |d\sigma|$$

$$= L(1+K)\int_{\infty}^{s} |\sigma|^{-2} |e^{s-\sigma}| |d\sigma| + LK^2 \int_{\infty}^{s} |\sigma|^{-3} |e^{s-\sigma}| |d\sigma|$$

$$\leq L(1+K)L_o|s|^{-2} + LK^2L_o|s|^{-3}$$

$$= LL_o|s|^{-2}(1 + K + K^2|s|^{-1})$$

$$\leq |x|^{-1}LL_of(M')\{1 + K + K^2f(M')\} \leq |x|^{-1} \cdot K .$$

$$|q_{j+1}(x) - q_j(x)|$$

$$\leq \int_{\infty}^{s} \{ |\phi_r(\eta)| |q_j(\eta) - q_{j-1}(\eta)| + |\nu_r(\eta)| |q_j^2(\eta) - q_{j-1}^2(\eta)| \} |e^{s-\sigma}| |d\sigma|$$

$$\leq \int_{\infty}^{s} \{ L|\eta|^{-1} \frac{1}{2^j} + L|\eta|^{-1} \frac{1}{2^j} |q_j(\eta) + q_{j-1}(\eta)| \} |e^{s-\sigma}| |d\sigma|$$

$$\leq \frac{L}{2^j} \int_{\infty}^{s} |\eta|^{-1} \{ 1 + |q_j(\eta) + q_{j-1}(\eta)| \} |e^{s-\sigma}| |d\sigma|$$

$$\leq \frac{L}{2^j} \int_{\infty}^{s} |\eta|^{-1} (1 + 2K|\eta|^{-1}) |e^{s-\sigma}| |d\sigma|$$

$$\leq \frac{L}{2^j} (L_o|s|^{-1} + 2KL_o|s|^{-2})$$

$$\leq \frac{LL_o}{2^j} \{ f(M') + 2Kf^2(M') \}$$

$$= \frac{1}{2^j} LL_o f(M') \{ 1 + 2Kf(M') \}$$

$$\leq \frac{1}{2^j} \cdot \frac{1}{2} = \frac{1}{2^{j+1}}$$

Therefore the sequence $\{q_n(x)\}$ converges uniformly in $S_{\delta M} \times D_r$.

Put

$$q_r(x, \lambda) = \lim_{n\to\infty} q_n(x) .$$

Then $q_r(x, \lambda)$ is a holomorphic solution of the integral equation (5.6) such that $|q_r(x, \lambda)| \leq K|x|^{-1}$ in $S_{\delta M} \times D_r$.

5.3 <u>Asymptotic Property of</u> $q_r(x, \lambda)$.

Now, we shall prove, by induction, that

$$q_r(x, \lambda) \cong 0$$

uniformly for $\lambda \in D_r$ as x tends to infinity in $S_{\delta M}$.

Suppose $q_n(x, r) \cong 0$ $(n = 0, 1, 2, \ldots, j)$ uniformly for $\lambda \in D_r$ as x tends to infinity in $S_{\delta M}$. Then

$$\lim_{x\to\infty} x^m q_n(x, r) = 0 \quad \text{for} \quad m = 0, 1, 2, \ldots \quad \text{and}$$
$$n = 0, 1, 2, \ldots, j .$$

Therefore for each nonnegative integer m, there exists a positive constant C such that

$$|q_n(x, r)| \le C|x|^{-m-1} \quad (n = 0, 1, 2, \ldots, j).$$

Also by the relation (5.2), we have

$$|\mu_r(x)| \le L|x|^{-m-2} \quad \text{for} \quad m = 0, 1, 2, \ldots \quad .$$

Hence by (5.16) for $n = j + 1$, we have

$$|x^m q_n(x , r)| \le |x|^m \int_\infty^x$$

$$\left\{L|\eta|^{-m-2} + L|\eta|^{-1}C|\eta|^{-m-1} + L|\eta|^{-1}C^2|\eta|^{-2(m+1)}\right\}|e^{-2i(x-\eta)}||d\eta|$$

Put $\eta = -i\sigma$ and $x = -is$ as before, then we have

$$|x^m q_n(x , r)|$$

$$\le |s|^m\left\{\int_\infty^s L|\sigma|^{-m-2}(1+C)|e^{s-\sigma}||d\sigma| + \int_\infty^s LC^2|\sigma|^{-2m-3}|e^{s-\sigma}||d\sigma|\right\}$$

$$\le |s|^m LL_o(1+C)|s|^{-m-2} + |s|^m LL_oC^2|s|^{-2m-3} \qquad \text{by Lemma 3}$$

$$= LL_o|s|^{-2}(1+C+C^2|s|^{-m-1})$$

$$\le |x|^{-1}LL_of(M')\{1+C+C^2f(M')\} \le |x|^{-1} \cdot C$$

Therefore $x^m q_n(x , r)$ tends to 0 uniformly for $\lambda \in D_r$ as x tends to infinity in $S_{\delta M}$ for all nonnegative integer m. (i.e. $q_n(x , r) \cong 0$ for all $n = 0 , 1 , 2 , \ldots$). Hence we have

$$q_r(x , \lambda) \cong 0$$

uniformly for $\lambda \in D_r$ as x tends to infinity in $S_{\delta M}$.

5.4 <u>Proof of Lemma 2</u>.

Thus far we have shown that

$$\text{(5.18)} \qquad p(x , r) \equiv q_r(x , \lambda) + \hat{P}_r(x , \lambda)$$

is a holomorphic solution of (3.6) with respect to $(x, \lambda) \in S_{\delta M} \times D_r$. In this section we shall prove that $p(x, r)$ is actually independent of r, which implies the uniqueness of the solution.

In order to prove the independence of $p(x, r)$ from r, let us consider $p(x, r_1)$, and $p(x, r_2)$, which are defined in $S_{\delta M_1} \times D_{r_1}$ and $S_{\delta M_2} \times D_{r_2}$, respectively. Let $M = \max(M_1, M_2)$ and $r = \min(r_1, r_2)$. Then $S_{\delta M} \times D_r = (S_{\delta M_2} \times D_{r_1}) \cap (S_{\delta M_2} \times D_{r_2})$, and $q_{r_1}(x,\lambda)$ and $q_{r_2}(x,\lambda)$ are asymptotic to 0 uniformly for $\lambda \in D_r$ as x tends to infinity in $S_{\delta M}$.

Let $u(x) = p(x, r_1) - p(x, r_2)$. Then we have

$$\frac{du}{dx} = \frac{d}{dx} p(x, r_1) - \frac{d}{dx} p(x, r_2),$$

where

$$\frac{d}{dx} p(x, r_1) = \beta_2 + (\alpha_2 - \alpha_1) p(x, r_1) - \beta_1 p^2(x, r_1),$$

by (3.6).

Similarly

$$\frac{d}{dx} p(x, r_2) = \beta_2 + (\alpha_2 - \alpha_1) p(x, r_2) - \beta_1 p^2(x, r_2).$$

By (3.6), we have

$$\frac{du}{dx} = \left[(\alpha_2 - \alpha_1) - \beta_1 \{p(x, r_1) + p(x, r_2)\}\right] u \equiv J(x) u \tag{5.19}$$

where

$$J(x) = \alpha_2 - \alpha_1 - \beta_1\{p(x, r_1) + p(x, r_2)\} .$$

Thus

$$|J(x) + 2i| = |O(x^{-1}) + O(x^{-1})\{p(x, r_1) + p(x, r_2)\}| \le K|x|^{-1}$$

for $x \in S_{\delta M}$, where K is a positive constant.

Let x_o be an arbitrary point in $S_{\delta M}$. Then (5.19) has a solution of the form

$$(5.20) \qquad u(x) = u(x_o) \exp\left\{\int_{x_o}^{x} J(\eta)\,d\eta\right\}$$

In order to have $u(x) \cong 0$ as x tends to infinity in $S_{\delta M}$, we must have $u(x_o) = 0$. Since x_o was chosen arbitrarily, this implies that $u(x) \equiv 0$ for $(x, \lambda) \in S_{\delta M} \times D_r$, i.e. $p(x, r_1) \equiv p(x, r_2)$ for $(x, \lambda) \in S_{\delta M} \times D_r$. Therefore $p(x, r)$ is actually independent of r.

This completes the proof of Lemma 2.

6. PROOF OF LEMMA 3.

This lemma is due to P. F. Hsieh and Y. Sibuya and the complete proof is given in [3 and 4]. In this section we shall prove the lemma as it is applied to our case.

For a fixed point s in $S_{\delta M}$ and a fixed θ such that $\frac{\delta}{2} \le \theta \le \pi - \frac{\delta}{2}$, let α be the angle between the two straight lines

(6.1) $\quad \sigma = s + te^{i\theta} \quad 0 \le t < \infty, \quad \frac{\delta}{2} \le \theta \le \pi - \frac{\delta}{2}$

(6.2) $\quad \arg \sigma = \arg s$

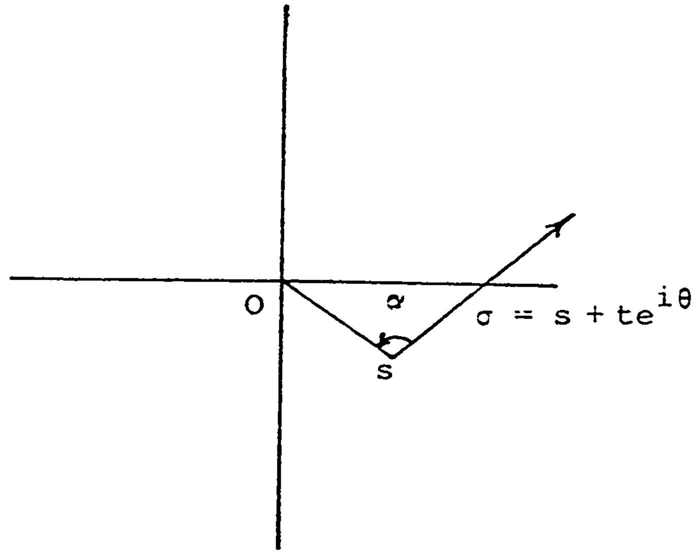

Figure 6.1

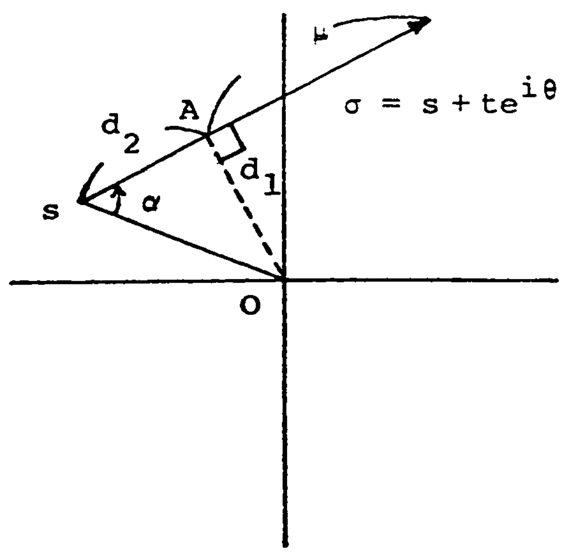

Figure 6.2

Case I: When $\alpha \geq \frac{\pi}{2}$ (Figure 6.1).

Since $\alpha \geq \frac{\pi}{2}$ implies $|\sigma| \geq |s|$, $|\sigma|^{-\rho} \leq |s|^{-\rho}$.

Also $s - \sigma = -te^{i\theta}$ implies $|e^{s-\sigma}| = e^{-t\cos\theta}$. Therefore we have

$$\int_{\infty}^{s} |\sigma|^{-\rho} |e^{s-\sigma}|\,|d\sigma| \leq |s|^{-\rho} \int_0^{\infty} e^{-t\cos\theta}\,dt$$

$$= |s|^{-\rho} \frac{1}{\cos\theta}$$

$$\leq L_o |s|^{-\rho}$$

where

$$L_o = \max_{|\theta| \leq \frac{\pi}{2} - \frac{\delta}{2}} \left\{\frac{1}{\cos\theta}\right\}$$

Case II: When $0 \leq \alpha < \frac{\pi}{2}$ (Figure 6.2).

Let A be the intersection of $\sigma = s + te^{i\theta}$ and the ray $\arg\sigma = \theta + \frac{\pi}{2}$. Let $d_1 = |OA|$ and $d_2 = |SA|$. Then

$$|\sigma|^2 = d_1^2 + (t - d_2)^2$$

Put $t = \mu + d_2$ where $-d_2 \leq \mu < \infty$. Then we have

$$\int_{\infty}^{s} |\sigma|^{-\rho} |e^{s-\sigma}| \, |d\sigma| = \int_{-d_2}^{\infty} (d_1^2 + \mu^2)^{-\rho/2} e^{-(\mu + d_2)\cos\theta} \, d\mu$$

(6.1)

$$= e^{-d_2 \cos\theta} \int_{-d_2}^{\infty} (d_1^2 + \mu^2)^{-\rho/2} e^{-\mu\cos\theta} \, d\mu$$

Put

(6.2) $$G(\tau) = (d_1^2 + \tau^2)^{\rho/2} e^{\tau\cos\theta} \int_{\tau}^{\infty} (d_1^2 + \tau^2)^{-\rho/2} e^{-\mu\cos\theta} \, d\mu \, .$$

For $\tau \geq 0$, μ varies over the interval $[\tau, \infty)$.

Thus

$$(d_1^2 + \mu^2)^{-\rho/2} \leq (d_1^2 + \tau^2)^{-\rho/2}$$

Hence

(6.3) $$G(\tau) \leq e^{\tau\cos\theta} \int_{\tau}^{\infty} e^{-\mu\cos\theta} \, d\mu = \frac{1}{\cos\theta}$$

For $\tau < 0$,

(6.4) $$\frac{d}{d\tau} G(\tau) = \frac{\tau\rho}{d_1^2 + \tau^2} G(\tau) + \cos\theta \cdot G(\tau) - 1$$

Now, it can be shown that

(6.5) $$\left|\frac{\tau}{d_1^2 + \tau^2}\right| \leq \frac{1}{2d_1} \qquad \text{for all } \tau \, .$$

Since $s \in S_{\delta M}$, $d_1 \geq M$.

Hence

$$\left|\frac{\tau}{d_1^2+\tau^2}\right| \le \frac{1}{2d_1} \le \frac{1}{2M}$$

Thus if $M_{\hat{\delta}\rho}$ is large, by making $\frac{\tau\rho}{d_1^2+\tau^2} \ge -\frac{1}{2}\cos\theta$

we have

(6.6) $$\frac{d}{d\tau}G(\tau) \ge \frac{1}{2}\cos\theta\, G(\tau) - 1 .$$

Therefore, by (6.4) and (6.6) we have

(6.7) $$\frac{d}{d\tau}G(\tau) = \frac{d}{d\tau}[G(\tau) - \frac{2}{\cos\theta}] \ge \frac{1}{2}\cos\theta\, G(\tau) - 1$$
$$= \frac{1}{2}\cos\theta\{G(\tau) - \frac{2}{\cos\theta}\},$$

which implies

(6.8) $$\frac{d}{d\tau}\left[e^{-\frac{\tau}{2}\cos\theta}\{G(\tau) - \frac{2}{\cos\theta}\}\right] \ge 0 .$$

Integrating (6.8) from $-d_2$ to 0, we obtain the following inequality

(6.9) $$G(0) - \frac{2}{\cos\theta} \ge e^{\frac{1}{2}d_2\cos\theta}\{G(-d_2) - \frac{2}{\cos\theta}\} .$$

But by(6.3),

$$G(\tau) \le \frac{1}{\cos\theta} < \frac{2}{\cos\theta} \qquad \text{for } \tau \ge 0 ,$$

in particular

$$G(0) \le \frac{2}{\cos\theta} .$$

Therefore by (6.9) we have

$$G(-d_2) < \frac{2}{\cos\theta} \quad . \tag{6.10}$$

Using (6.1), (6.2), (6.10) and the fact that $d_1^2 + d_2^2 = |s|^2$, we obtain

$$\int_{\infty}^{s} |\sigma|^{-\rho} |e^{s-\sigma}| |d\sigma| = G(-d_2) \cdot (d_1^2 + d_2^2)^{-\rho/2}$$

$$\le |s|^{-\rho} \frac{2}{\cos\theta} \le L_0 |s|^{-\rho}$$

where

$$L_0 = \max_{|\theta| \le \frac{\pi}{2} - \frac{\delta}{2}} \left\{\frac{2}{\cos\theta}\right\} .$$

This completes the proof of Lemma 3 .

7. ASYMPTOTIC SOLUTION OF $x^2y'' + (x^3 + \lambda)y = 0$.

In this section, we shall state the second result of this paper. The proof will be given in the following sections.

<u>Theorem 3. The second order linear differential equation</u> (1.4) <u>has a solution</u> $y(x, \lambda)$ <u>that satisfies the following:</u>

(i) $y(x, \lambda)$ <u>is holomorphic with respect to</u> $(x, \lambda) \in S \times D$, <u>where</u> S <u>is a sector in X-plane and</u> D <u>is an open disc in λ-plane defined by</u>:

$$S: \frac{-2\pi}{3} + \frac{2\delta}{3} \leq \arg x \leq \frac{4\pi}{3} - \frac{2\delta}{3}, \quad |x| \geq M .$$

$$D: |\lambda| < r, \quad 0 < r < +\infty .$$

(ii) $y(x, \lambda)$ <u>and</u> $y'(x, \lambda)$ <u>admit respectively the asymptotic representations</u>:

$$(7.1)\ y(x, \lambda) \cong x^{-1/4} \exp\left\{\frac{2i}{3} x^{3/2}\right\} \left\{1 + \sum_{n=1}^{\infty} \alpha_n x^{-3n/2}\right\}$$

$$(7.2)\ y'(x, \lambda) \cong ix^{1/4} \exp\left\{\frac{2i}{3} x^{3/2}\right\} \left\{1 + \sum_{n=1}^{\infty} \beta_n x^{-3n/2}\right\}$$

<u>uniformly with respect to</u> $\lambda \in D$ <u>as</u> x <u>tends to infinity in</u> S, <u>where all</u> α_n <u>and</u> β_n <u>are polynomials in</u> λ <u>and</u> $x^a = \exp\{a(\ln|x| + i \arg x)\}$ <u>for any constant</u> a.

<u>The quantities</u> δ, r <u>and</u> M <u>are positive constants</u>.

<u>Remark 3</u>: For the construction of the asymptotic representation of the solution $y(x, \lambda)$ of (1.4) as x tends to infinity in a sector different from the sector S, let us change the independent variable x by

$$\hat{x} = e^{i\theta} x .$$

Then equation (1.4) becomes

$$(7.3) \qquad \hat{x}^2 \frac{d^2y}{d\hat{x}^2} + e^{-3i\theta}(\hat{x}^3 + e^{3i\theta}\lambda)y = 0 .$$

Therefore, if we choose θ so that $e^{-3i\theta} = 1$, the fucntion $y(\hat{x}, e^{3i\theta}\lambda)$ is a solution of (7.3). Hence if we put

$$\theta_K = \frac{2K\pi}{3} \qquad (K = 0, 1, 2) \quad ,$$

then $y_K(x, \lambda) = y(\hat{x}, e^{3i\theta_K}\lambda) = y(e^{i\theta_K}x, \lambda)$ are solutions of (1.4) which admit the asymptotic representations $Y(e^{i\theta_K}x, \lambda)$ as x tends to infinity in any closed sector contained in the sector S_K, where S_K is given by

$$\frac{-2\pi}{3} + \frac{2\delta}{3} - \frac{2K\pi}{3} \le \arg x \le \frac{4\pi}{3} - \frac{2K\pi}{3} - \frac{2\delta}{3} \quad (K = 0, 1, 2)$$

and $Y(x, \lambda)$ is the right-hand member of (7.1).
That is

$$y_0(x, \lambda) \cong Y(x, \lambda) \text{ as } x\to\infty \text{ in } \frac{-2\pi}{3} + \frac{2\delta}{3} \le \arg x \le \frac{4\pi}{3} - \frac{2\delta}{3}$$

$$y_1(x, \lambda) \cong Y(e^{\frac{2\pi}{3}i}x, \lambda) \text{ as } x\to\infty \text{ in } \frac{-4\pi}{3} + \frac{2\delta}{3} \le \arg x \le \frac{2\pi}{3} - \frac{2\delta}{3}$$

$$y_2(x, \lambda) \cong Y(e^{\frac{4\pi}{3}i}x, \lambda) \text{ as } x\to\infty \text{ in } -2\pi + \frac{2\delta}{3} \le \arg x \le -\frac{2\delta}{3} .$$

<u>Remark 4</u>: In [1], the asymptotic solutions of second order linear ordinary differential equations with polynomial coefficients were discussed. If (1.4) is

divided by x^2, it becomes

$$y'' + \left(x + \frac{\lambda}{x^2}\right)y = 0 .$$

Therefore in the neighborhood of $x = \infty$, the coefficient $x + \frac{\lambda}{x^2}$ can be approximated by the polynomial $P(x) = x$.

For this reason, the procedure used in the proof of this theorem will be similar to the one used in [1].

8. PROOF OF THEOREM 3: PART I .

8.1. Preliminary Transformations.

Let us write equation (1.4) as the following system of equations.

$$\frac{dY}{dx} = A(x)Y , \tag{8.1}$$

where

$$Y = \begin{bmatrix} y \\ y' \end{bmatrix}$$

and

$$A(x) = \begin{bmatrix} 0 & 1 \\ -x - \frac{\lambda}{x^2} & 0 \end{bmatrix} .$$

Put

$$x = \xi^2 , \quad Y = \begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix} U \quad .$$

Then system (8.1) is reduced to

$$(8.2) \qquad \frac{dU}{d\xi} = \left\{ \xi^2 \sum_{n=0}^{6} A_n \, \xi^{-n} \right\} U \, ,$$

where

$$A_o = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 0 \\ -2\lambda & 0 \end{bmatrix}$$

and $A_1 = A_2 = A_4 = A_5 =$ Zero matrix .

Since the eigenvalues are $2i$ and $-2i$, in order to have the Jordan canonical form of A_o as the leading coefficient of (8.2) let us put $U = Tw$ where

$$T = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} .$$

Then we have

$$(8.3) \qquad \frac{dw}{d\xi} = \xi^2 \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_2 & \alpha_2 \end{bmatrix} w \, ,$$

where

$$(8.4) \qquad \alpha_1(\xi) = 2i - \frac{1}{2}\xi^{-3} + \lambda i \, \xi^{-6} \, ,$$

$$(8.5) \qquad \alpha_2(\xi) = -2i - \frac{1}{2}\xi^{-3} - \lambda i \, \xi^{-6} \, ,$$

$$(8.6) \qquad \beta_1(\xi) = \frac{1}{2} \, \xi^{-3} + \lambda i \, \xi^{-6} \, ,$$

$$(8.7) \qquad \beta_2(\xi) = \frac{1}{2} \, \xi^{-3} - \lambda i \, \xi^{-6} \, ,$$

Note that α_1, α_2, β_1 and β_2 are linear in λ and polynomials in ξ^{-1}, and we have

$$\begin{aligned} \alpha_1 &= 2i + O(\xi^{-3}), \quad \beta_1 = O(\xi^{-3}), \\ \alpha_2 &= -2i + O(\xi^{-3}), \quad \beta_2 = O(\xi^{-3}). \end{aligned} \tag{8.8}$$

Now let us put the following expression in (8.3)

$$w = \begin{bmatrix} 1 \\ p(\xi) \end{bmatrix} \exp\left\{\int^{\xi} \eta^2 \gamma(\eta)\, d\eta\right\}. \tag{8.9}$$

Then we have the following relations:

$$\gamma(\xi) = \alpha_1 + \beta_1 p, \tag{8.10}$$

$$\begin{aligned} \frac{dp}{d\xi} &= \xi^2\{\beta_2 + (\alpha_2 - \alpha_1)p - \beta_1 p^2\} \\ &= \tfrac{1}{2}\xi^{-1} - \lambda i \xi^{-4} - (4i\xi^2 + 2\lambda i \xi^{-4})p - (\tfrac{1}{2}\xi^{-1} + \lambda i \xi^{-4})p^2. \end{aligned} \tag{8.11}$$

If we determine $p(\xi)$ from nonlinear differential equation (8.11) and then $\gamma(\xi)$ by (8.10), the quantity in (8.9) is a solution of (8.3). Therefore

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix} T \begin{bmatrix} 1 \\ p \end{bmatrix} \exp\left\{\int^{\xi} \eta^2 \gamma(\eta)\, d\eta\right\} \tag{8.12}$$

will give us a solution of (1.4).

8.2. Formal Solution of Eq. (8.11)

Lemma 6. The nonlinear differential equation (8.11) *has a unique formal solution*

$$\hat{P}(\xi) = \sum_{n=1}^{\infty} P_n \xi^{-n} \tag{8.13}$$

where

$$P_n = 0 \quad \text{if} \quad n \not\equiv 0 \pmod 3$$

$$P_{3n} = \frac{i}{4}\Big\{2\lambda i\, P_{3(n-2)} - 3(n-1)P_{3(n-1)}$$

$$+ \frac{1}{2}\sum_{j+k=n-1} P_{3j} P_{3K} + \lambda i \sum_{j+K=n-2} P_{3j} P_{3K}\Big\}$$

for $n \geq 3$, *where the summations are over* $j \geq 1$ *and* $k \geq 1$ *such that* $j+k=n-1$ *and* $j+k=n-2$.

Proof: Let $p = \sum_{n=1}^{\infty} P_n \xi^{-n}$ and substitute into (8.11), then we obtain

$$\sum_{n=1}^{\infty}(-n)P_n \xi^{-n-1}$$

$$= \frac{1}{2}\xi^{-1} - \lambda i\, \xi^{-4} - (4i\,\xi^2 + 2\lambda i\,\xi^{-4})\sum_{n=1}^{\infty} P_n \xi^{-n} - \left(\frac{1}{2}\xi^{-1} + \lambda i\,\xi^{-4}\right)$$

$$\cdot\left(\sum_{n=1}^{\infty} P_n \xi^{-n}\right)^2$$

Equating the coefficients of terms with equal exponent, we have

$$0 = -4i\, P_1 \ ,$$

$$0 = -4i\, P_2 \ ,$$

$$0 = \frac{1}{2} - 4i\, P_3 \ ,$$

$$-P_1 = -4i\, P_4 \ ,$$

$$-2P_2 = -4i\, P_5 - \frac{1}{2} \sum_{j+k=2} P_j P_k \ ,$$

$$-3P_3 = -\lambda i - 4i\, P_6 - \frac{1}{2} \sum_{j+k=3} P_j\, P_k \ ,$$

$$-4P_4 = -4i\, P_7 - 2\lambda i\, P_1 - \frac{1}{2} \sum_{j+k=4} P_j\, P_k \ ,$$

$$-5P_5 = -4i\, P_8 - 2\lambda i\, P_2 - \frac{1}{2} \sum_{j+k=5} P_j\, P_k - \lambda i \sum_{j+k=2} P_j\, P_k \ ,$$

$$-6P_6 = -4i\, P_9 - 2\lambda i\, P_3 - \frac{1}{2} \sum_{j+k=6} P_j\, P_k - \lambda i \sum_{j+k=3} P_j\, P_k \ ,$$

$$-nP_n = -4i\, P_{n+3} - 2\lambda i\, P_{n-3} - \frac{1}{2} \sum_{j+k=n} P_j\, P_k - \lambda i \sum_{j+k=n-3} P_j\, P_k \ ,$$

for $n \geq 7$, where the summations are taken over $j \geq 1$ and $k \geq 1$ such that $j + k = n$ and $j + k = n - 3$.

These relations determine P_n, $(n \geq 1)$, uniquely. Furthermore it can be shown that $P_n = 0$ for all n such that $n \neq 3m$ $(m = 1, 2, \ldots)$. Thus we obtain the formal solution

$$\hat{p}(\xi) = \sum_{n=1}^{\infty} P_{3n}\, \xi^{-3n}$$

$$= \frac{-i}{8}\, \xi^{-3} - \frac{3+8\lambda}{32}\, \xi^{-6} + \frac{71+224\lambda}{512}\, i\, \xi^{-9} + \cdots .$$

8.3. Analyticity of a Solution of Eq. (8.11).

In order to find the analytic meaning of the formal solution $\hat{p}(\xi)$ in (8.13), we need the following:

Lemma 7. There exists a unique solution $p(\xi)$ of (8.11) that satisfies the following conditions:

(i) For each positive constant r, there exists a positive constnat N_r such that $p(\xi)$ is holomorphic with respect to (ξ, λ) in the domain determined by:

$$(8.14)\quad \begin{aligned} &|\arg(-4i) + 3 \arg \xi| \leq \frac{3\pi}{2} - \delta, \qquad |\xi| \geq N_r \\ &|\lambda| < r \qquad 0 < r < \infty \end{aligned}$$

(ii) $p(\xi)$ is asymptotic to the formal solution $\hat{p}(\xi)$ uniformly on each compact set in the λ-space as ξ tends to infinity in the sector

$|\arg(-4i) + 3 \arg \xi| \le 3\pi/2 - \delta$, *where* δ *is a fixed sufficiently small positive number.*

The proof of this lemma will be given in Section 10.

9. PROOF OF THEOREM 3: PART II

In this section we shall complete the proof of Theorem 3 by the use of Lemma 6 and Lemma 7.

Applying Lemma 7 using the fact that $\arg(-4i) = -\pi/2$, we have a unique solution $p(\xi)$ of equation (8.11) holomorphic with respect to (ξ, λ) in the domain $S' \times D$, where

$$S': \frac{\delta}{3} - \frac{\pi}{3} \le \arg \xi \le \frac{2\pi}{3} - \frac{\delta}{3}, \quad |\xi| \ge M$$

$$D: \quad |\lambda| < r, \quad 0 < r < \infty$$

By (8.10), we have

$$\gamma(\xi) = 2i - \frac{1}{2}\xi^{-3} + \lambda i \xi^{-6} + (\frac{1}{2}\xi^{-3} + \lambda i \xi^{-6})p(\xi)$$

which is holomorphic with respect to $(\xi, \lambda) \in S' \times D$. Hence we have

$$\exp\left\{\int^{\xi} \eta^2 \gamma(\eta) d\eta\right\}$$

$$(9.1) \quad = \exp\left[\int^{\xi} \eta^2 \{2i - \tfrac{1}{2}\, \eta^{-3} + \lambda i \eta^{-6} + (\tfrac{1}{2}\, \eta^{-3} + \lambda i \eta^{-6})\, p(\eta)\} d\eta\right]$$

$$\cong \exp\left[\int^{\xi} \eta^2 \{2i - \tfrac{1}{2}\, \eta^{-3} + \lambda i\, \eta^{-6} + (\tfrac{1}{2}\, \eta^{-3} + \lambda i\, \eta^{-6})\, \hat{p}(\eta)\} d\eta\right]$$

uniformly with respect to $\lambda \in D$ as ξ tends to infinity in S'. Inserting the series obtained by Lemma 6 into the expression in (9.1) and integrating, we obtain

$$\exp\left\{\int^{\xi} \eta^2\, \gamma(\eta)\, d\eta\right\}$$

$$\cong \exp\left\{\frac{2i}{3}\, \xi^3 - \frac{1}{2} \log\, \xi + \frac{1 - 16\lambda}{48}\, i\, \xi^{-3} + \frac{1}{128}\, \xi^{-6} + \cdots\right\} .$$

By (8.12), for ξ such that $\frac{\delta}{3} - \frac{\pi}{3} \le \arg\, \xi \le \frac{2\pi}{3} - \frac{\delta}{3}$, we have

$$y(\xi, \lambda) = \{1 + p(\xi)\} \exp\left\{\int^{\xi} \eta^2 \gamma(\eta) d\eta\right\}$$

$$\cong \{1 + \hat{p}(\xi)\} \exp\{\tfrac{2i}{3}\, \xi^3\} \cdot \xi^{-1/2}$$

$$\cdot \exp\left\{\xi^{-3}\left(\frac{1 - 16\lambda}{18}\, i + \frac{1}{128}\, \xi^{-3} + \cdots\right)\right\}$$

$$= \xi^{-1/2} \exp\left\{\frac{2i}{3}\, \xi^3\right\} \cdot \left\{1 + \sum_{n=1}^{\infty} \alpha_n\, \xi^{-3n}\right\},$$

where α_n $(n \ge 1)$ are polynomials in λ and the expression

$$1 + \sum_{n=1}^{\infty} \alpha_n\, \xi^{-3n}$$

is obtained from,

$$\{1 + \hat{P}(\xi)\} \exp\left\{\xi^{-3}\left(\frac{1-16\lambda}{18} i + \frac{1}{128}\xi^{-3} + \dots\right)\right\}$$

$$= \left\{1 - \frac{i}{8}\xi^{-3} - \frac{3+8\lambda}{32}\xi^{-6} + \frac{71+224\lambda}{512}\xi^{-9} + \dots\right\}$$

$$\cdot \left\{1 + \xi^{-3}\tau(\xi) + \frac{\xi^{-6}}{2!}\tau^2(\xi) + \dots\right\}$$

where $\tau(\xi) = \frac{1-16\lambda}{18} i + \frac{1}{128}\xi^{-3} + \dots$.

Thus, by (9.2) and $x = \xi^2$, we have the desired asymptotic representation

$$y(x, \lambda) \cong x^{-1/4} \cdot \exp\left\{\frac{2i}{3} x^{3/2}\right\} \left\{1 + \sum_{n=1}^{\infty} \alpha_n x^{-3n/2}\right\} .$$

uniformly with respect to $\lambda \in D$ as x tends to infinity in S .

10. PROOF OF LEMMA 7

In this section we shall prove Lemma 7 in steps similar to the ones in Section 5.

By Theorem 2, corresponding to the formal solution $\hat{P}(\xi) = \sum_{n=1}^{\infty} P_{3n}\xi^{-3n}$ of (8.11) and to the secter

$$S_\Omega: \frac{\delta}{3} - \frac{\pi}{3} \le \arg \xi \le \frac{2\pi}{3} - \frac{\delta}{3}, \quad |\xi| \ge \Omega .$$

there exists a function $\hat{p}_r(\xi)$ holomorphic with respect to $(\xi, \lambda) \in S_\Omega \times D_r$ such that $\hat{p}_r(\xi)$ and $d\,\hat{p}_r(\xi)/d\xi$ admit the uniform asymptotic expansions

$$\hat{p}_r(\xi) \cong \sum_{n=1}^{\infty} p_{3n}\, \xi^{-3n}, \quad \frac{d\,\hat{p}_r(\xi)}{d\xi} \cong \frac{d\,\hat{p}(\xi)}{d\xi}$$

for $\lambda \in D_r$ as ξ tends to infinity in S_Ω.

Put $p = q + \hat{p}_r(\xi)$ in (8.11). Then the differential equation (8.11) is reduced to

$$\frac{dq}{d\xi} = \xi^2\{\mu_r(\xi) + \psi_r(\xi)q + \nu_r(\xi)q^2\} \tag{10.1}$$

where

$$\mu_r(\xi) = \frac{1}{2}\,\xi^{-3} - \lambda i\,\xi^{-6} - (4i + 2\lambda i\,\xi^{-6})\hat{p}_r(\xi) - \left(\frac{1}{2}\,\xi^{-3} + \lambda i\,\xi^{-6}\right)\hat{p}_r^2(\xi) - \frac{d\,\hat{p}_r(\xi)}{d\xi}$$

$$\psi_r(\xi) = -(4i + 2\lambda i\,\xi^{-6}) - (\xi^{-3} + 2\lambda i\,\xi^{-6})\hat{p}_r(\xi)$$

$$\nu_r(\xi) = -\left(\frac{1}{2}\,\xi^{-3} + \lambda i\,\xi^{-6}\right).$$

Notice that these three quantities are holomorphic in $S_\Omega \times D_r$ and

$$\mu_r(\xi) \cong 0 \tag{10.2}$$

$$(10.3) \qquad \psi_r(\xi) = -4i + O(\xi^{-3})$$

$$(10.4) \qquad \nu_r(\xi) = O(\xi^{-3})$$

as ξ tends to infinity in S_Ω, uniformly for $\lambda \in D_r$. The asymptotic relation in (10.2) is derived from the fact that $\hat{P}_r(\xi)$ is asymptotic to the formal solution of (8.11). In (10.3), let us put

$$\psi_r(\xi) = -4i + \phi_r(\xi) \quad .$$

Then the relation (10.3) implies that

$$(10.5) \qquad \phi_r(\xi) = O(\xi^{-3})$$

uniformly for $\lambda \in D_r$ as ξ tends to infinity in S_Ω.

Let

$$(10.6) \quad q(\xi) = \int_\infty^\xi \eta^2 \left\{ \mu_r(\eta) + \phi_r(\eta) q(\eta) + \nu_r(\eta) q^2(\eta) \right\}$$

$$\cdot \exp\left\{ \frac{-4i}{3} (\xi^3 - \eta^3) \right\} d\eta$$

where the path of integration is a straight line in S_Ω of the form

$$(10.7) \qquad \eta = \xi + te^{i\theta_o} \qquad 0 \le t < \infty$$

where θ_o is a suitably determined constant so that the entire path (10.7) lies in the secter S_Ω and

$\exp\left\{ \frac{-4i}{3} (\xi^3 - \eta^3) \right\}$ converges uniformly as η tends to infinity along the line.

Differentiating both sides of (10.6), we obtain

$$\frac{dq}{d\xi} = \xi^2 \left\{ \mu_r(\xi) + \phi_r(\xi) q(\xi) + \nu_r(\xi) q^2(\xi) \right\} - 4i\ \xi^2 q(\xi)$$

$$= \xi^2 \left\{ \mu_r(\xi) + \psi_r(\xi) q(\xi) + \nu_r(\xi) q^2(\xi) \right\} .$$

Therefore a solution of the integral equation (10.6) satisfies the differential equation (10.1). The validity of the equivalence of (10.5) and (10.6) depends on the path of integration. In order to determine such a path that maintains the validity, let us put

(10.8) $$s = \frac{-4i}{3} \xi^3$$

(10.9) $$\sigma = \frac{-4i}{3} \eta^3 .$$

Then (10.6) becomes

(10.10) $$q(\xi) = -\frac{1}{4i} \int_{\infty}^{s} \left\{ \mu_r(\eta) + \phi_r(\eta) q(\eta) + \nu_r(\eta) q^2(\eta) \right\} e^{s-\sigma} d\sigma .$$

Let $\sigma = s + te^{i\theta}$ be the path of integration, where $0 \leq t < \infty$ and θ is appropriately chosen constant so that $\lim_{\sigma \to \infty} |e^{s-\sigma}|$ exists and the corresponding path (10.7) is contained entirely in S_Ω of ξ-plane .

This can be achieved by choosing θ such that $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, since $\mathrm{Re}(s - \sigma) = -t \cos \theta$. The following figures show that this is possible.

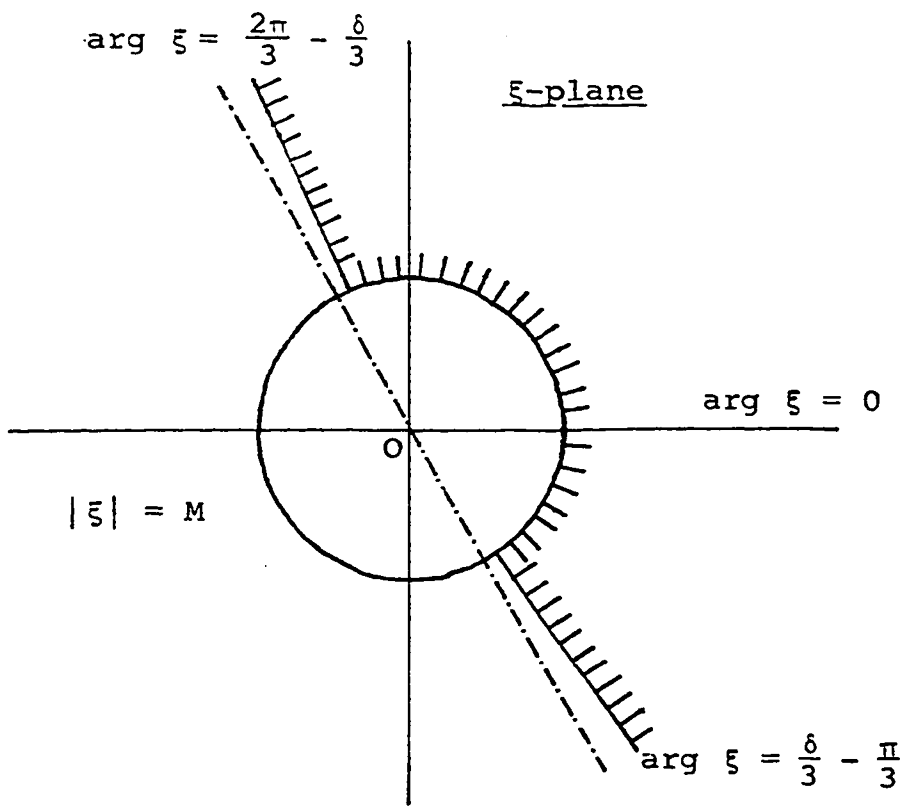

Figure 10.1

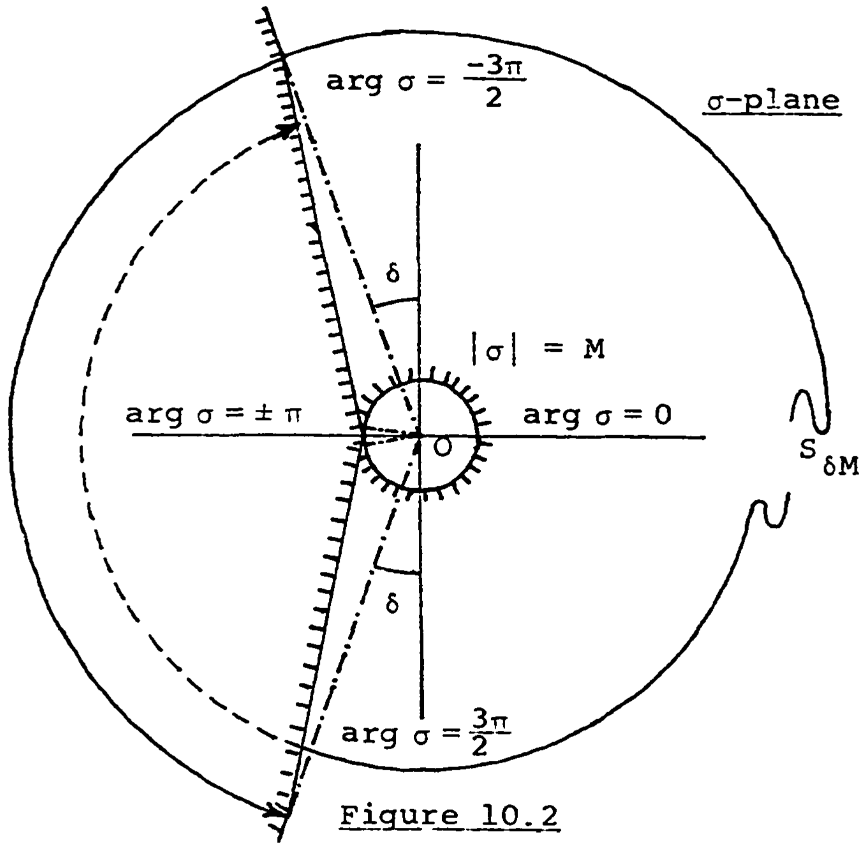

Figure 10.2

Figure 10.1 shows the sector S_Ω in ξ-plane and Figure 10.2 shows the corresponding sector in σ-plane. Complete detail of construction of the sector in Figure 10.2 can be found in [1].

By Lemma 3, there exist positive constants M and L_o such that

$$\text{(10.11)} \qquad \int_{\infty}^{s} |\sigma|^{-\rho} |e^{s-\sigma}|\,|d\sigma| \le L_o |s|^{-\rho}$$

for each s in the sector $S_{\delta M}$ in σ-plane given by

$$S_{\delta M}: \quad \delta - \frac{3\pi}{2} \le \arg \sigma \le \frac{3\pi}{2} - \delta , \quad |\sigma| \ge M .$$

where L_o is a positive constant independent of ρ and the path of integration is the straight line $\sigma = s + te^{i\theta}$ ($0 \le t < \infty$ and $|\theta| < \frac{\pi}{2} - \delta$) . Let L be a sufficiently large positive constant such that

$$(10.12) \qquad |\mu_r(\xi)| \le L|\xi|^{-2}$$

$$(10.13) \qquad |\phi_r(\xi)| \le L|\xi|^{-3}$$

$$(10.14) \qquad |\nu_r(\xi)| \le L|\xi|^{-3}$$

in the domain $S_{\delta M} \times D_r$. Such a constant L is assignable by (10.2), (10.3) and (10.4).

Now, in order to find a solution $q(\xi)$ of (10.10), let us define the successive approximations in $S_{\delta M}$ by,

$$(10.15) \qquad \begin{aligned} &q_o(\xi) \equiv 0 \\ &q_n(\xi) = \frac{-1}{4i}\int_\infty^s \{\mu_r(\eta) + \phi_r(\eta)q_{n-1}(\eta) + \nu_r(\eta)q_{n-1}^2(\eta)\}e^{s-\sigma}d\sigma \\ &\qquad\qquad (n \ge 1) \end{aligned}$$

We shall show that the sequence $\{q_n(\xi)\}$ converges uniformly in $S_{\delta M} \times D_r$ and the limit is asymptotic to 0 uniformly for $\lambda \in D_r$ as ξ tends to infinity in $S_{\delta M}$. Note that $q_n(\xi)$ is holomorphic in $S_{\delta M} \times D_r$ for all n.

Let $f(m) = \frac{1}{m}$. Then by Lemma 5, there exists a constant K such that

(10.16) $\frac{1}{2}L_o \, Lf(M)\{1 + K + f(M)K^2\} \le K$

(10.17) $\frac{1}{2}L_o Lf(M)\{1 + 2f(M)K\} \le \frac{1}{2}$

Applying the inequalities (10.11), (10.12), (10.13), (10.14), (10.16) and (10.17) in a similar fashion used in Section 5.2, it can be shown that

$$|q_n(\xi)| \le K|\xi|^{-1}$$

$$|q_{n+1}(\xi) - q_n(\xi)| \le \frac{1}{2^n}$$

in the domain $S_{\delta M} \times D_r$. Let

$$q_r(\xi , \lambda) = \lim_{n\to\infty} q_n(\xi) .$$

Then $q_r(\xi , \lambda)$ is, as the uniform limit of holomorphic functions, holomorphic with respect to $(\xi , \lambda) \in S_{\delta M} \times D_r$ and is a solution of the integral equation (10.6).

Furthermore it can be shown that

(10.18) $$q_r(\xi , \lambda) \cong 0$$

uniformly with respect to $\lambda \in D_r$ as ξ tends to infinity in $S_{\delta M}$. We shall show (10.18) by induction. Suppose $q_n(\xi) \cong 0$ $(n = 0 , 1 , 2 , \ldots , j)$ uniformly for $\lambda \in D_r$ as ξ tends to infinity in $S_{\delta M}$. Then

$$\lim \xi^m q_n(\xi) = 0 \quad (n = 0 , 1 , 2 , \ldots , j)$$

for all nonnegative integer m, where t. . limit is taken as ξ tends to infinity in $S_{\delta M}$. Therefore for each nonnegative integer m, there exists a positive constant C such that

$$|q_n(\xi)| \le C|\xi|^{-m-3} \quad (n = 0, 1, 2, \ldots, j) \quad .$$

Also by relation (10.2), we have

$$|\mu_r(\xi)| \le L|\xi|^{-m-6} \quad \text{for} \quad m = 0, 1, 2, \ldots \quad .$$

Hence by (10.15) for $n = j + 1$, we have

$$|\xi^m q_n(\xi)|$$

$$\le \frac{|\xi|^m}{4} \int_\infty^s \{L|\eta|^{-m-6} + L|\eta|^{-3}C|\eta|^{-m-3} + L|\eta|^{-3}C^2|\eta|^{-2m-6}\} \times |e^{s-\sigma}||d\sigma| \quad .$$

By (10.8) and (10.9), we have $|s| = \frac{4}{3}|\xi|^3$ and $|\sigma| = \frac{4}{3}|\eta|^3$. Thus

$$|\xi^m q_n(\xi)|$$

$$\le \frac{|\xi|^m}{4} \left\{\int_\infty^s L\left(\frac{3}{4}|\sigma|\right)^{\frac{-m-6}{3}} |e^{s-\sigma}||d\sigma| + \int_\infty^s LC\left(\frac{4}{3}\,|\sigma|\right)^{\frac{-m-6}{3}} |e^{s-\sigma}||d\sigma| + \int_\infty^s LC^2\left(\frac{3}{4}|\sigma|\right)^{\frac{-2m-9}{3}} |e^{s-\sigma}||d\sigma|\right\}$$

$$\le \frac{1}{4}\left(\frac{3}{4}|s|\right)^{\frac{m}{3}}\left\{L\, L_o\left(\frac{3}{4}|s|\right)^{\frac{-m-6}{3}} + L\, L_oC\left(\frac{3}{4}|s|\right)^{\frac{-m-6}{3}} + L\, L_oC^2\left(\frac{3}{4}|s|\right)^{\frac{-2m-9}{3}}\right\}$$

$$= \frac{1}{4} L L_o\left\{\left(\frac{3}{4}|s|\right)^{-2} + c\left(\frac{3}{4}|s|\right)^{-2} + c^2\left(\frac{3}{4}|s|\right)^{-m/3 - 3}\right\}$$

$$= \frac{1}{4}\left(\frac{3}{4}|s|\right)^{-1} L L_o\left(\frac{3}{4}|s|\right)^{-1}\left\{1 + c + c^2\left(\frac{3}{4}|s|\right)^{\frac{-m}{3}-1}\right\}$$

$$\leq \frac{1}{4}|\xi|^{-3} c$$

where the last inequality is obtained by Lemma 5, and this implies that

$$\lim_{\substack{\xi\to\infty \\ \xi \in S_{\delta M}}} |\xi^m q_n(\xi)| = 0$$

uniformly for $\lambda \in D_r$, for all nonnegative integer m. Therefore by induction $q_n(\xi) \cong 0$ as ξ tends to infinity in $S_{\delta M}$, uniformly with respect to $\lambda \in D_r$. Hence, as the uniform limit of the sequence $\{q_n(\xi)\}$, $q_r(\xi, \lambda)$ is asymptotic to 0 uniformly for $\lambda \in D_r$ as ξ tends to infinity in $S_{\delta M}$.

Thus far, we have shown that

$$p_r(\xi, \lambda) \equiv q_r(\xi, \lambda) + \hat{p}_r(\xi, \lambda)$$

is a solution of (8.11) holomorphic with respect to $(\xi, \lambda) \in S_{\delta M} \times D_r$, such that $p_r(\xi, \lambda) \cong \hat{p}(\xi)$, the formal solution of (8.11) as ξ tends to infinity in $S_{\delta M}$. To complete the proof of Lemma 7, we must show that $p_r(\xi, \lambda)$ is actually independent of r, which implies the uniqueness of the solution.

Let us consider two solutions $p_{r_1}(\xi, \lambda)$ and $p_{r_2}(\xi, \lambda)$ which are defined in $S_{\delta M_1} \times D_{r_1}$ and $S_{\delta M_2} \times D_{r_2}$, respectively. Let $M = \max(M_1, M_2)$ and $r = \min(r_1, r_2)$. Then

$$S_{\delta M} \times D_r = (S_{\delta M_1} \times D_{r_1}) \cap (S_{\delta M_2} \times D_{r_2})$$

and $q_{r_1}(\xi, \lambda)$ and $q_{r_2}(\xi, \lambda)$ are asymptotic to 0 uniformly for $\lambda \in D_r$ as ξ tends to infinity in $S_{\delta M}$.

Let $u(\xi) = p_{r_1}(\xi, \lambda) - p_{r_2}(\xi, \lambda)$. Then we have

$$\frac{du}{d\xi} = \frac{d\, p_{r_1}}{d\xi} - \frac{d\, p_{r_2}}{d\xi}$$

where

$$\frac{d\, p_{r_j}}{d\xi} = \beta_2 + (\alpha_2 - \alpha_1) p_{r_j}(\xi, \lambda) - \beta_1\, p^2_{r_j}(\xi, \lambda)$$

$$(j = 1, 2)$$

Hence we have

$$\text{(10.19)} \quad \frac{du}{d\xi} = \left[(\alpha_2 - \alpha_1) - \beta_1 \left\{p_{r_1}(\xi, \lambda) + p_{r_2}(\xi, \lambda)\right\}\right] u$$

$$\equiv J(\xi, \lambda) u$$

where

$$J(\xi, \lambda) = \alpha_2 - \alpha_1 - \beta_1\left\{p_{r_1}(\xi, \lambda) + p_{r_2}(\xi, \lambda)\right\}$$

Thus

$$|J(\xi, \lambda) + 4_i|$$

$$= |O(\xi^{-3}) + O(\xi^{-3})\ \{p_{r_1}(\xi, \lambda) + p_{r_2}(\xi, \lambda)\}|$$

$$\le K|\xi|^{-3}$$

for $\xi \in S_{\delta M}$, where K is a positive constant.

Let ξ_o be an arbitrary point in $S_{\delta M}$. Then (10.19) has a solution of the form

$$u(\xi, \lambda) = u(\xi_o, \lambda)\ \exp\left\{\int_{\xi_o}^{\xi} J(\eta, \lambda)d\eta\right\} .$$

In order to have $u(\xi, \lambda) \cong 0$ as ξ tends to infinity in $S_{\delta M}$, we must have $u(\xi_o, \lambda) = 0$. Since ξ_o was chosen arbitrarily, this implies that $u(\xi, \lambda) \equiv 0$ for $(\xi, \lambda) \in S_{\delta M} \times D_r$, that is $p_{r_1}(\xi, \lambda) \equiv p_{r_2}(\xi, \lambda)$ in $S_{\delta M} \times D_r$. Therefore $p_r(\xi, \lambda)$ is actually independent of r.

This completes the proof of Lemma 7.

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