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An Experimental Study Comparing the Effects of Small Group-Discovery Learning and Conventional Instruction on Student Achievement and Attitudes in Calculus

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AN EXPERIMENTAL STUDY COMPARING THE EFFECTS OF SMALL GROUP-DISCOVERY LEARNING AND CONVENTIONAL INSTRUCTION ON STUDENT ACHIEVEMENT AND ATTITUDES IN CALCULUS

by

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CHAPTER I

INTRODUCTION

Rationale

While it is generally recognized that one of the major pedagogical goals of mathematics instruction should be the active participation of students in the learning process, this goal is seldom or only incidentally achieved in most undergraduate courses. That increased attention should be paid to this goal in collegiate instruction has been urged by Polya (1963), who argues that

> ... student participation is vital to the learning process and that merely by reading books or listening to lectures or looking at moving pictures without adding some action of your own mind you can hardly learn anything and certainly you can not learn much. (p. 607)

The same view has been expressed by Piranian (1975) in more colorful terms.

> I do not know how to present mathematical ideas so effectively that students can take possession of them simply by sitting at my feet and smelling my socks ... after grazing in my lush pastures, the students must ruminate; they must dedicate substantial time to the chewing of the cud. (p. 475)

Although increasing attention is being paid to the necessity of student involvement in the learning process, there is diversity among mathematics educators as to how this can be best accomplished. At one end of the spectrum is Piranian. He claims that student acquisition of mathematical ideas depends upon the enthusiasm, inspiration and class preparation of the instructor, as well as upon staggered homework problems carefully selected by the instructor so as to provide several opportunities for clarifying the basic principles and demonstrating the necessary techniques.

In contrast to the above is the hypothesis that "for efficient learning, the learner should discover by himself as large a fraction of the material to be learnt as feasible under the given circumstances" (Polya, p. 608). This view is supported by Moise (1965), who asserts that "mathematics is capable of being learned as an activity, and that knowledge which is acquired in this way has a power which is out of all proportion to its quantity" (p. 409).

In the same spirit, Halmos (1975) advocates using the modified Moore method which maximizes the activity of the student in the development of mathematical concepts and minimizes the role of the instructor. He presents the theorems to be proved during the first lecture. During the term he gives definitions, supplies some counterexamples, tells about the history of the topic and points out connections with other parts of mathematics. In the meantime the students prove the theorems, and present them to the class for their criticisms. How successful Halmos is as a teacher is indicated by the fact that his students, in contrast to students from other classes, are recognized for their greater mathematical maturity and greater inclination and ability to ask penetrating questions!

Though considerable research has been conducted to assess the effectiveness of discovery teaching and learning, the results of these studies suggest that the effectiveness of this approach is still an open question (Begle, 1969). Moreover, as can be seen in Chapter II, most of this research has been limited to mathematical instruction at the elementary and secondary levels. Furthermore, Cronback (1966) in his review of the experiments on discovery learning, pointed out that there are a multitude of different approaches that have been applied under the rubric of "discovery".

Therefore, for purposes of this experiment, the researcher has chosen to characterize discovery learning by the following attributes:

(1) The students acquire mathematical insights through self-activity.

(2) The instructor initiates the investigations through exploratory exercises.

(3) Concrete experience precedes abstraction.

(4) The students formulate definitions and hypotheses.

(5) The students prove the hypotheses or give counterexamples.

(6) The teacher provides limited guidance in the form of hints, suggestions, leading questions.

Finally, though there appears to be an increasing trend toward individualization of introductory courses, as evidenced in recent volumes of the American Mathematical Monthly and the Two-Year College Mathematics Journal; there is, on the other hand, a growing body of research which strongly suggests that learning is a social process in which the learner becomes a partner in a shared activity (cf., MacPherson, 1972; Steiner, 1972). Hence, the admonition given by MacPherson that "we had better pay close attention to learning in groups before attempting to impose educational practices on the assumption that learning may be exclusively treated as an individual matter" (p. 478). Moise (1975) also alludes to this when he criticizes the use of 'modules' which, he says, fail to provide the student with the most stimulating challenges that he can react to successfully, and convey to him a false impression of intellectual development. Buck's (1962) reminder of over a decade ago again appears appropriate:

> Let me remind you that student interactions are also important in learning, and that at the professional level, much mathematical research springs from discussion between mathematicians. Moreover, a test of understanding is often the ability to communicate it to others; and this act itself is often the final and most crucial step in the learning process. (p. 563)

In accord with the above, Phillips (1966) claims that discussion groups provide a method for students to assimilate facts and test their ability to apply them; for only when the student is left to his own devices and allowed the freedom to make errors can he develop his own style of discovery of learning, an impossibility if students are passive and inert recipients of information dispensed by an erudite professor.

Stein (1972) argues that students are disenchanted with the lecture method and thus lectures should be frequently replaced with active small group learning. Most of his students are "captured" students (students who are suddenly forced to study a subject which they have avoided for years). He had found that even though the pace is much slower when he reverts to small groups, the advantages are sufficient to warrant the change. Stein relates that

> ... This method gives students a chance to talk mathematics with each other and to get immediate feedback on their work; it enables

> teachers to see poor drawings of students, the strange quantities that were made variables, and the difficulty in using geometry to relate variables; it places students in a position to expose weaknesses without embarrassment; it provides an opportunity for students to get to know each other; and, since a "captured" student tends to be mathematically withdrawn, this experience tends to build his confidence and draw him out. (p. 1028)

The integration of the above views of learning into a small group-discovery approach to calculus instruction was first formulated and tested by Davidson (1970). Though Davidson provided both a theoretical base and existence proof for this instructional strategy, the selectivity of the subjects in the experimental class (twelve volunteers with A or B grades in high school mathematics and at least a mild interest in the subject) left open the question of the efficacy of such an approach.

The primary purpose of this study then was to answer the question: What is the effect of the use of a small group-discovery method of teaching as compared with the conventional lecture-discussion method upon concept attainment, skill acquisition, and attitudes toward mathematics in an introductory course in calculus?

Hypotheses

This investigation is concerned with testing the following null hypotheses:

H_1: There is no significant difference in overall achievement between students in the experimental group and the students in the control group.

H_2: There is no significant difference in conceptual understanding between students in the experimental group and the students in the control group.

H_3: There is no significant difference in skill acquisition between the experimental and the control group.

H_4: There is no significant difference in attitude towards mathematics between the experimental and the control group.

H_5: There is no significant interaction between ACT mathematics scores and overall achievement.

H_6: There is no significant interaction between ACT composite scores and overall achievement.

H_7: There is no significant interaction between ACT mathematics and composite scores and overall achievement.

CHAPTER II

REVIEW OF THE RELATED RESEARCH

This chapter is devoted to a review of the research on discovery learning and small group instruction in mathematics at the collegiate level. As will be seen, very little research has been reported on these two alternatives to conventional instruction.

In an early study, Cummins (1960) asked the questions: "Is it really possible to conduct a planned course in beginning calculus at the faster-paced university level and to utilize as much as possible discoveries of students themselves? Will such an approach increase understanding?" (p. 163) In an attempt to find the answers, he conducted an experiment to test the hypothesis that a student-experience discovery approach to calculus in the university is as effective as in the secondary school under similar conditions.

His research was undertaken for three terms with three experimental classes including a small pilot group taught by Cummins and two traditional classes instructed by experienced professors. During the fall term the pilot group was used for preliminary studies and comparisions. In the winter quarter the formal experimental work began with experimental and control sections determined by the way the students scheduled themselves. In the spring term the students for the experimental and control sections were randomly selected from one large section. The same textbook was used for all the sections, although the students of the experimental groups relied on study guides with the textbook serving as a source book. The students in the experimental groups using the study guide sheets either independently or with the help of class discussion were led to formulate hypotheses and definitions, and to discover and prove theorems. At the end of the respective terms a test written by Cummins and another by the instructor of the control group were given to all the students participating in the experiment, and comparisions were made. The results of this research indicated that this method of teaching was especially effective in promoting a deeper understanding of the calculus and that this gain was not at the sacrifice of proficiency in manipulations and applications (significant at the .01 level).

The first investigation on small group instruction was reported by Turner et al. (1966) and involved students from Fundamentals of Mathematics and College Algebra classes. Students from both classes were assigned randomly to an experimental or a control section. The control sections were instructed using a lecture-discussion for each class day (4 days per week for Fundamentals of Mathematics and 5 days per week for College Algebra). Students in the Fundamentals of Mathematics experimental section were taught by a lecture-discussion method for 2 days; those in College Algebra for 3 days. The experimental sections were randomly placed in smaller groups to receive one of three treatments during the remaining two class meetings. The treatment for Group I consisted of having students under the supervision of the regular instructor work together in groups of three students with one student in each group acting as leader. The students in Group II were placed in subgroups of five or six students with an undergraduate mathematics student who was to use a variety of methods of instruction in charge. A graduate assistant using a variety of methods of instruction taught Group III.

A similar experiment was conducted involving students from College Algebra, and Analytic Geometry and Calculus I classes during the next term. Although the control sections continued as in the first experiment, a change regarding the experimental College Algebra class was made. This class again met for three days for a lecture-discussion. On the remaining two days the class was divided into small groups of twelve to twenty students with a senior mathematics major acting as a teaching assistant. These groups were further divided into subgroups of three students working together. The regular instructor was available if needed. The same text and the same tests were used for all comparable experimental and control sections.

A simple randomized design was used to test the null hypothesis of equal means for the control and experimental group. The "F" test statistic in all classes indicated no significant differences in achievement at the .05 level. But on the basis of the analysis and within the limitations of the study, the following conclusions were drawn: teachers participating were able to cover the material in the experimental sections using three of five days for lecture-discussion; teachers of the four-hour courses were rushed in trying to cover material in two days; the use of small groups and the use of the mathematics majors as teaching assistants allowed for individual help and for active student participation; it was an invaluable opportunity for teaching experience for the senior mathematics majors.

In a similar experiment conducted by Eisenberg and Browne (1973), two large sections of pre-calculus were divided into groups of 20 to 25 students, each under the supervision of a member of a mathematics education methods course. Two days a week the large sections met for a lecture, but the attendance was not compulsory. The smaller groups were further subdivided into groups of three or four students and met on the other two days. Attendance was compulsory for these small group sessions which were modeled after those described by Davidson (1970). In addition to favorable comments by the students, the achievement level as indicated by test scores was reported to be significantly affected. Using the majority of the test questions from the same item pool as in previous semesters, the class average on the tests was 10% higher than in previous semesters. In addition, the failure rate was essentially half of the failure rate of previous semesters.

Davidson (1970) did research in calculus instruction combining the small group method (Turner et al. 1966) with discovery learning (Cummins, 1960). The control group consisted of four subsections each of which was taught by a graduate assistant using the lecture-discussion method. Apparently, no attempt was made to maintain equivalency of content presented to the experimental and control sections. The twelve members of the experimental group were volunteers who were either freshmen or sophomores, had no or very little previous knowledge of calculus, studied high school mathematics through trigonometry, had grades of A or B in their high school mathematics courses, and were at least mildly interested in mathematics. The approach to the subject matter was one of guided discovery in which topics were introduced in the form of questions for investigation by the students who, under the guidance of the researcher, formulated definitions, stated and proved theorems, constructed examples and counterexamples, and developed techniques for solving various classes of problems.

The democratic atmosphere pervaded both in the relationship of teacher to students and within the groups. In place of a textbook daily notes containing the students' accomplishments were prepared. In order to use interest in mathematics as a major source of motivation and to provide a non-threatening atmosphere, emphasis upon grades was reduced by using an A - B grading scale and by elimination of formal in-class examinations. Furthermore, the changes regarding group membership were determined by the students themselves.

The teacher's role was one of a moderator of the group discussion. In addition, at the beginning of each class he provided a brief prospectus of the day's topic; he also kept track of the progress of the work groups, offered suggestions or technical information as needed, encouraged the individual members, give hints, proffered constructive praise and criticism, checked the solutions to the problems, and assisted the groups in working cooperatively with each other.

On a comparative final examination, the students in the experimental group scored slightly better, but not significantly better, than the students in the control sections. An open-ended questionnaire showed that the small-group discovery method had either positive or non-negative effects upon each student's interest in mathematics and estimate of his problem-solving skill. There were numerous indications that more intensive investigations of the small group-discovery method of instruction should be undertaken.

A preliminary study (Hirsch, 1974) of the efficacy of this method in pre-calculus mathematics was conducted in the Fall of 1973 at Western Michigan University. Students in two classes of intermediate algebra taught by the researcher were assigned in groups of four to students from a secondary mathematics methods class. These prospective teachers were responsible for teaching the groups the content of one unit in the course. Hence, in this case, all instruction occurred in small group settings with group leaders guiding the students to a discovery of the appropriate content. The achievement level of the students in the small group learning project was then compared to that of the students in the large lecture-recitation sections. Although the difference in achievement level was nonsignificat ($p < .18$), the difference favored the students in the small group learning project. The student reaction to the format was highly favorable.

In summary, the studies on discovery and/or small group learning in collegiate instruction in mathematics have been few in number and were characterized by encouraging but not clear-cut results. Davidson's investigation left open the question of the efficacy of the small group-discovery approach to calculus instruction because of the selectivity of the experimental group and lack of experimental controls. The present study was then an attempt to provide additional empirical data regarding the viability of this method.

CHAPTER III

PROCEDURE

Introduction

This research project was intended to be a replication and refinement of an earlier experiment conducted by Davidson (1970) at the University of Wisconsin. Originally, two classes were assigned to the researcher; the expected control class met at 8:00 A.M., and the experimental class met at 1:00 P.M. However, Ary (1972) strongly recommends that all extraneous variables, including time of day, be held constant, and that all subjects in the various groups be treated exactly alike except for their exposure to the independent variable, namely, the treatment. Thus, conjecturing that time might indeed prove to be a selection factor, an experienced professor was asked to let his class, which met at 2:00 P.M., serve as a second control group. Data concerning the differential dropping of this course was collected for the three sections, and the results are summarized in Tables 1, 2 and 3 (p. 17).

Using the χ^2 test with Yates' correction factor for continuity, the difference between the proportion of students who dropped the class at 8:00 A.M. was signi-

TABLE 1

Observed Frequency of Change in Class Membership Between 8:00 A.M. and 1:00 P.M. Sections

	8:00	1:00	
drop	21	7	28
remain	18	23	41
	39	30	69

TABLE 2

Observed Frequency of Change in Class Membership Between 8:00 A.M. and 2:00 P.M. Sections

	8:00	2:00	
drop	21	4	25
remain	18	33	51
	39	37	76

TABLE 3

Observed Frequency of Change in Class Membership Between 1:00 P.M. and 2:00 P.M. Sections

	1:00	2:00	
drop	7	4	11
remain	23	33	56
	30	37	67

ficantly different from that of the 1:00 P.M. class ($\chi^2 = 5.34$, $p < .05$) and that of the 2:00 P.M. class ($\chi^2 = 14.04$, $p < .001$). However, when comparing the two afternoon classes, there was not a significant difference ($\chi^2 = 1.09$, $p = .30$), between the proportions of students dropping the course.

As a result of the above analyses, it was decided that the research would be conducted using the 1:00 P.M. class as the experimental group and the 2:00 P.M. class as the control group. Also it became apparent that the sample population was not comparable to that used by Davidson.

Description of the Experiment

The subjects for the study were 46 students enrolled in two afternoon sections of the introductory calculus course at Western Michigan University. As shown by Table 4 (p. 19), the subjects included both males and females of freshman, sophomore, junior, senior and graduate status.

One section (N = 21) was designated to use the experimental materials, and the other section (N = 25) served as a control group. As can be seen by Table 5 (p. 20) and Table 6 (p. 21), the background of the students varied from three years of high school mathematics

TABLE 4

Summary of Class Ranks and Sex of the Students in Both Sections

Class	Number in Control Group	Number in Experimental Group
Freshmen	14	13
Sophomores	3	7
Juniors	5	
Seniors	2	1
Graduate Students	1	
Total	25	21
Number of females	4	3
Number of males	21	18

or its equivalent through some previous calculus experience. Forty percent of the control group and 57.4 percent of the experimental group had completed 4 years of high school mathematics or its equivalent. Also 24% of the control group had previous contact with the calculus either several years ago, for a period of a few weeks, or were failures in the preceding terms. Only one student in the experimental section had attempted calculus previously. In the control group 32% had studied some college mathematics other than calculus while 33.3% of

TABLE 5

Mathematical Background of the Control Group

Student	High School: No. of years						College Courses: Algebra	Trigonometry	Finite Math.	Tech. Courses	Some Calculus
	1	2	3	3½	4	6*					
1					x						
2			x								
3		x						x			
4			x								x
5			x								x
6					x						x
7			x								
8			x				x				x
9			x								x
10			x								
11		x					x				
12					x				x		
13			x					x			
14				x							
15				x					x		x
16						x					
17					x					x	
18						x					
19	x						x	x			
20					x						
21					x				x		
22					x		x	x			
23				x							
24			x								
25					x						

*foreign students

TABLE 6

Mathematical Background of the Experimental Group

Student	High School						College Courses				
	No. of years						Algebra	Trigonometry	Finite Math.	Tech. Courses	Some Calculus
	1	2	3	3½	4	6*					
1						x					
2					x						
3			x								
4			x					x			
5					x						
6					x						
7					x		x				
8					x						
9					x		x	x			
10			x								
11			x				x				
12					x						
13				x					x		
14					x		x				x
15					x						
16					x						
17			x								
18			x				x				
19				x							
20					x						
21			x					x			

*foreign students

the experimental group had taken mathematics courses other than calculus in college.

An examination of Table 7 shows that, although this course was a curriculum requirement for all these students, only one was a mathematics major.

The pre-calculus competence of the subjects in both groups was measured prior to instruction with a test consisting of 35 multiple-choice items selected from the Cooperative Mathematics Tests (ETS) Algebra II, Algebra III, and Analytic Geometry. This test may be found in Appendix A. Using the Kuder-Richardson Formula 20, a reliability estimate of .86 was calculated for the pooled items on the basis of this sample. The mean indices of discrimination and difficulty were both .45.

At this time the students were also given the Aiken-Dreger Revised Mathematics Attitude Scale (Aiken, 1963) to assess their attitudes toward mathematics on entering the course (see Appendix B for complete scale). However, one adjustment was made in this scale. Statement 13 originally read, "I approach mathematics with a feeling of hesitation, resulting from a fear of not being able to do math." It was changed to read, "I approach calculus with ..." This instrument contains 20 questions, each with 5 possible responses: strongly agree, agree, undecided, disagree, strongly disagree. Responses are

TABLE 7

Summary of Areas of Study
for Each Section

Major	Number in Control Group	Number in Experimental Group
Accounting	2	
Biology	2	5
Computer Systems Engineering	1	
Chemistry	1	1
Drafting and Design Engineering	1	
Electrical Engineering Technology	2	5
Finance	1	
General Business		1
General Curriculum	2	1
Industrial Engineering Technology		2
Mathematics		1
Mechanical Engineering Technology	4	2
Medical Technology	1	
Paper Engineering	1	
Pre-Engineering	4	2
Psychology	2	1
Permission to Take Classes	1	
Total	25	21

scored from 0 to 4, 4 being the response most positive towards mathematics. Thus, a score of 80 is most positive, a score of 40 is considered neutral.

The mean scores and standard deviations on the above two pretests are reported for each group in Table 8.

TABLE 8

Pretest Scores

	Experimental Group		Control Group	
	Mean	S.D.	Mean	S.D.
Pre-Calculus Achievement	19.95	5.86	20.72	7.65
Attitude	53.71	12.28	51.64	14.95

The groups were then given instruction in two different modes and utilizing two different kinds of instructional materials for the duration of one semester. Both groups met for 50 minutes, 4 times per week.

The first decision to be made regarding the experimental treatment involved the question of group size. "In general, members of the group should be able to speak directly to each other with minimum effort in order for it to qualify as 'small'." (Phillips, 1966, p. 41).

Later Phillips states:

> One of the more fundamental notions about group discussion is that the process becomes unwieldly when the members can no longer sit around a table facing each other. As the size of the group increases it becomes more difficult to communicate, for voice level has to be raised and it is harder to look directly at the other members. If the group can be kept to a size where members can talk in normal conversational tones without expending much effort in trying to locate a respondent, the efficiency of the group is improved.
>
> It is very difficult for conflict to remain below the surface in such a group. It comes to light quickly and can be dealt with rather than remaining hidden and acting as a marginal or sublimal influence on individual behavior. It is harder for subgroups to split off from the face-to-face group. (p. 42)

Slater (1958) suggests that both too many and too few members in a group have adverse effects on motivation. In his research of groups containing 2 to 7 college students, he found that members of 5-man groups were most satisfied with their size; that members of groups of sizes 2-4 indicated their groups were too small to achieve maximum success; and that members of groups of sizes 6-7 expressed dissatisfaction because of lack of cooperation, insufficient opportunity to speak one's mind, poor use of time, and mediocre accomplishment.

With respect to dyads, Davidson (1970) succintly pointed out that, because any class of reasonable size would contain many dyads, periods of non-productivity, including deadlocks, might persist for too long a time since the teacher could only spend a small amount of time with each dyad to give needed hints and to mediate differences of opinions. Other disadvantages include the possibility of a member being left alone due to absence of the other member, and the atmosphere not being conducive to effective problem solving because of tenseness and restraint in the relationship of one member to the other.

On the other hand the triad is a unit in which coalitions can be generated, one individual may provoke or mediate disagreements between the others, and the continuance of the group does not depend on the willingness of every member to participate (Steiner, 1972). Since any member can be outvoted by the other two, the will of the group need not reflect the preferences of all its members. Thus the possibility exists that the mathematical ability of the majority could determine the success or failure of the group in problem solving. Consideration that, as with dyads, the size of the class made it prohibitive to split the class into triads and that, if a member were absent, the 3-member group would then become a dyad also influenced the decision not to use triads.

Empirical evidence does not clearly suggest whether groups of size 4 or 5 were more desirable. However, Slater had provided evidence that members of groups of size 2, 3, or 4 were "too tense, passive, tactful, constrained", but these effects decreased with increasing size of the group. Groups of size 4 could subdivide into groups of size 2 so there would not be a majority decision, and, in this situation, the possibility of a deadlock existed. But, the presence of the teacher could help to resolve these deadlocks.

In contrast, Hare (1962) reported that a group of size 5 combines the characteristics that a strict deadlock is not possible; it tends to split into a majority of three and a minority of two so that no individual is isolated; and it permits a member to withdraw easily from an awkward position.

Further consideration of groups of size 4 suggested that they might facilitate more active participation than would groups of size 5. Moreover, if the class was divided into groups of five, this would force one group to consist of 6 members. Thus the decision was made to divide the class into 4 subgroups containing four members and a fifth group consisting of 5 members.

Group membership initially was determined by the instructor on the basis of the results of the mathematics pretest with a view toward distributing pre-calculus competence evenly throughout each of the groups. Group membership was modified at the beginning of each subsequent unit except the final one, using test results, perceived leadership potential and compatibility as guides. The students themselves decided that the group membership was to remain the same for the final unit.

The classroom activity was a social process, taking place in the small groups, with their learning directed by guide sheets based upon materials originally developed by Davidson. However, these materials were revised and supplemented where necessary to be consistent with the content of the course as determined by the mathematics department at Western Michigan University and outlined in Table 9 (p. 29). To ensure that the same pace was maintained by the experimental group as the other sections of this course, identical tests were given by the instructor to her conventionally taught 8:00 A.M. class and the experimental section, but there was variability as to which class took the test first.

Within each group there was an atmosphere of cooperation, rather than competition. Early in the course, the teacher found it necessary to encourage the students to use the blackboard. Since the groups were heterogeneous with respect to mathematical ability and/or mathematical experience, there was a tendency for one person to dominate the group. This was overcome by suggesting that the students take turns writing the solutions on the board. Emphasis was upon group solutions and understanding of

TABLE 9

Outline of Course for Calculus I

I. Plane Analytic Geometry
    A. Review of basic definitions and formulas
    B. Division of a line segment
    C. Distance from a point to a line
    D. The circle
II. Functions
    A. Review of basic definitions
    B. Algebra of functions
    C. Introduction to limits
        1. Intuitive definition of limit
        2. Algebra of limits
III. Derivative
    A. Definition of slope of the tangent to a point on a curve
    B. Definition of derivative
    C. Relationship between velocity and slope of a tangent at a point
    D. Formulas
        1. Sum and Difference
        2. Product
        3. Quotient
        4. Chain rule
        5. Power rule

E. Implicit differentiation

F. Multiple derivatives

G. Curve sketching

1. Intercepts and asymptotes

2. Symmetry

3. Relative and absolute maxima and minima

a. First derivative test

b. Second derivative test and inflection points

4. Endpoint extrema

H. Applications

1. Related rates

2. Differential

G. Antiderivatives

IV. Integral

A. Area under a curve

1. Approximation using inscribed or circumscribed rectangles

2. Computations of areas as limits

B. Definite integral

C. Fundamental theorem of calculus

D. Integral as an antiderivative

E. Area between two curves

F. Approximate integration

1. Trapezoidal rule

2. Simpson's rule

G. Conic Sections

1. Translation

---

problems. But in problems involving skills, frequently the groups split into subgroups and then compared solutions.

In this setting, the students with some guidance from the instructor formulated definitions, constructed examples and counterexamples, discovered and proved many of the theorems of calculus, and developed techniques for solving various classes of problems. The emphasis was placed upon discovery of new ideas rather than upon rigor and formal proof. Typically units were introduced by open-ended questions such as: "What would you mean by a tangent to a curve?" or "How would you find the area under a given curve?" As a unit progressed or as the material became more deductive in nature, the amount of guidance which varied from topic to topic and from group to group, usually increased. As can be seen from the following examples, different methods were used to lead the students down the road to a new discovery.

In the development of the derivative; velocity and slope of the tangent to a curve were used to introduce the concept of the limit. These guide sheets may be found in Appendix C. Generally, the students found it very difficult making the "jump" from the concrete to the abstract with respect to velocity, but once this idea was perceived, they needed little guidance to comprehend the concept of the slope of a curve.

Using the definition of derivative, the students, with the exception of one group, were able to develop the formulas for the derivative of the sum and differences of two functions (see Appendix D) as well as the derivative of the reciprocal of a function and the power rule (see Appendix E).

In order to deduce the formula for the derivative of the product of two functions, a table was constructed, a conjecture was formulated, and this conjecture was proved or disproved (see Appendix E for the student guide sheets). Although most of the students had little trouble determining this rule, four out of the five groups were unable to develop deductively the chain rule without the direction of the instructor.

Through the use of progressive exercises (see Appendix F), the majority of the students successfully proved the theorem that if $f(x)$ is continuous, non-negative and always increasing or decreasing, the

$$\frac{d}{dx} \int_0^x f(t)dt = f(x) .$$

At the same time, ideas were being planted in preparation for further study. For example, one of the problems discussed by the work groups concerning derivatives was: Let $f(x)$ be a function whose graph in the first quadrant is a line which lies entirely above the x-axis. Let $A(x)$ be the function whose value at x_o is the area of the region bounded by the lines $x = 0$ and $x = x_o$, the x-axis, and the graph of $y = f(x)$. Show that $A'(x) = f(x_o)$.

The spirit of discovery was also carried into the homework, for among the problems can be found not only applications of the ideas discussed during class time but also extensions of these ideas as well as counter-examples (see Appendix D).

Throughout the course, the emphasis in the experimental group was on learning mathematics by doing mathematics. The teacher spent most of the period with the small work groups keeping track of the progress of the groups, answering or asking questions, making corrections and suggestions, giving hints, providing encouragement and seeing that the groups were functioning smoothly. Usually no more than five or ten minutes of class time was spent in introducing new concepts or definitions and setting the stage for the day's activities. However, sometimes it was necessary, if there was a common difficulty, as with the development of the chain rule or implicit differentiation, to bring the class together for a general class discussion under the guidance of the instructor. Since there is a tendency for the students in this atmosphere to move more slowly than in a traditional class, a few times a special session was scheduled for a particular group in order to enable these students to "catch up" with the other groups.

As is demonstrated in Appendices C - F, the sequencing of the material in the guide sheets proceeded from the more concrete to the more abstract. Also the skills developed in a traditional calculus course were nurtured in this course as well. However, in this approach the skills were formed under conditions requiring thought and maintained in the process of discovering solutions to new problems.

The control group was taught by an experienced professor using a lecture-discussion approach based upon the adopted textbook (Riddle, 1974) for the course.

Upon completion of the experiment, subjects in both treatment groups were re-administered the attitude scale and were given a comprehensive test consisting of 36 multiple-choice items (odd numbered items involving primarily manipulative skills and even numbered items measuring understanding of concepts) selected from the Cooperative Mathematics Tests (ETS) Calculus, Parts I and II and Vervoort's (1970) Calculus Test. This test may be found in Appendix G . A reliability estimate of .80 was calculated for the pooled items using the Kuder-Richardson Formula 20. The mean index of difficulty was .48 and the mean index of discrimination was .37.

Experimental subjects also responded in an open-ended manner to the following questions regarding their learning environment:

(1) What are your likes about the manner in which this class was handled?

(2) What are your dislikes about the manner in which this class was handled?

(3) Would you like to take another course taught in this manner?

CHAPTER IV

ANALYSIS OF RESULTS

Analysis of covariance with student pretest scores as the covariate, was applied to the data about the two treatments on the total posttest, skills subtest, and concepts subtest. Post-treatment attitude scores were also adjusted using analysis of covariance with pre-treatment attitude scores as the covariate. Prior to each analysis, a test (Lindquist, 1953, pp. 330-331) was made of the assumption of homogeneity of regression and in each case homogeneous regression was clearly tenable. The results of these tests are summarized in Tables 21-24 respectively which may be found in Appendix H.

The results of the analysis of treatment effect on overall achievement, skill acquisition, and concept attainment are reported in Tables 10 through 12 respectively and are further summarized in Table 13 (pp. 37-38).

The null hypotheses of no significant differences with respect to overall achievement and skill acquisition were rejected beyond the .02 and .01 levels respectively. In each case, the differences favored the experimental group.

The null hypothesis concerning differences in concept attainment was not rejected. However, the observed

TABLE 10

Analysis of Covariance for Total Posttest

Source	df	ss	ms	F	p
Treatments	1	134.60	134.60	6.75	.013
Within	43	857.78	19.95		
Total	44	992.38			

TABLE 11

Analysis of Covariance for Skills Subtest

Source	df	ss	ms	F	p
Treatments	1	81.13	81.13	8.35	.006
Within	43	417.93	9.72		
Total	44	499.06			

TABLE 12

Analysis of Covariance for Concepts Subtest

Source	df	ss	ms	F	p
Treatments	1	6.73	6.73	1.19	.281
Within	43	242.55	5.64		
Total	44	249.28			

TABLE 13

Posttest Scores

	Experimental Group			Control Group		
	Mean	S.D.	Adj. Mean	Mean	S.D.	Adj. Mean
Overall achievement	20.43	5.10	20.65	17.40	6.24	17.21
Skill acquisition	12.90	2.98	13.02	10.44	4.03	10.35
Concept attainment	7.52	3.11	7.64	6.96	2.88	6.87
Attitude	54.48	12.08	53.66	50.56	15.82	51.24

difference again favored the experimental group. A further analysis of the results indicated a significant interaction ($p < .02$) between pre-calculus achievement and method in the case of those items dealing with differentiation (see Table 25 of Appendix I for test of homogeneity of regression). Figure 1 shows the regression slopes for scores on the conceptual items involving differentiation. The regression slope rises more sharply for the experimental (E) group than for the control (C) group. The point of intersection is (14.08, 2.19). This suggests that pupils who scored above 14 on the pretest had a better conceptual understanding of differentiation if their instruction involved small group-discovery learning, whereas those pupils who scored 14 or below did better under a conventional lecture format.

In contrast, a test of homogeneity of regression (see Appendix I, Table 26) of those items dealing with integration on pre-calculus achievement indicated no significant interaction. Hence, an analysis of covariance with the pretest as covariate was carried out. The results which are reported in Table 14 imply that no significant difference exists between treatments ($p < .83$). However,

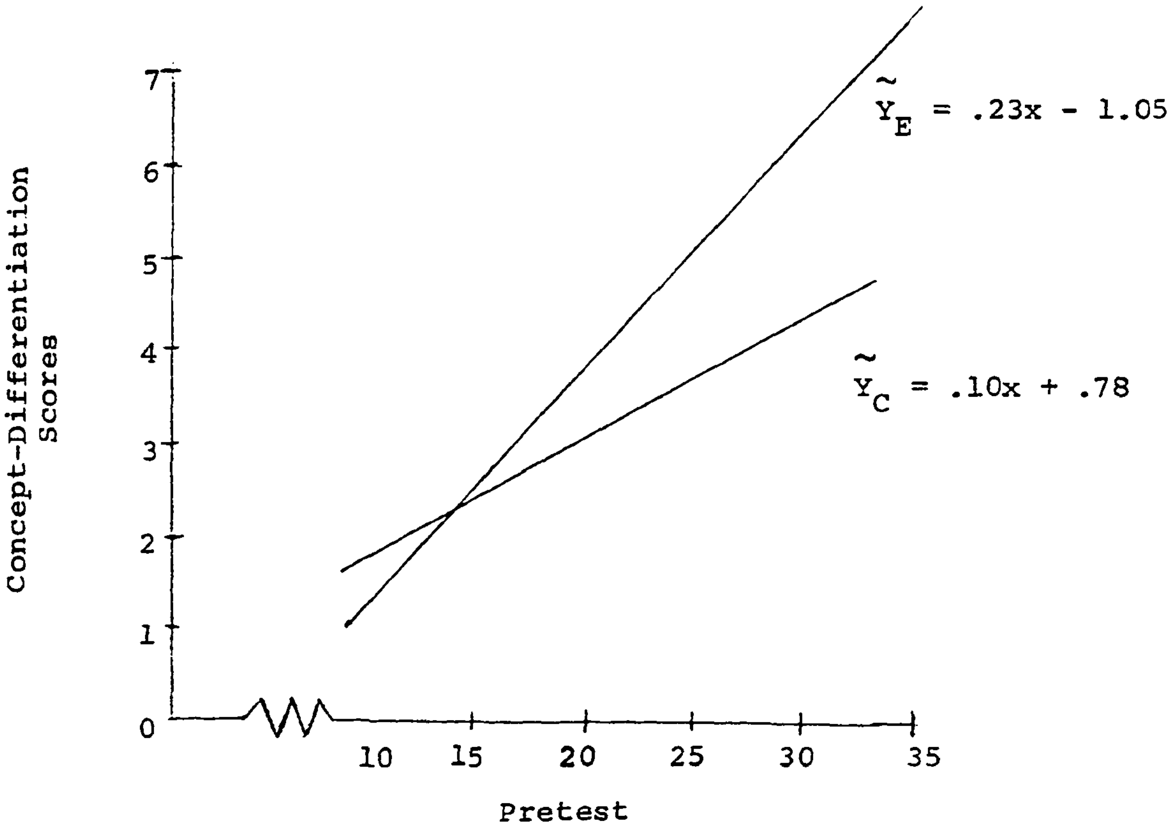


Fig. 1 Interaction of Regression Slopes

TABLE 14

Analysis of Covariance for Integration Items in Concepts Subtest

Source	df	ss	ms	F	p
Treatments	1	0.12	0.12	0.05	.83
Within	43	111.29	2.59		
Total	44	111.41			

the adjusted mean for the experimental group on the integration items was 2.86 and the control group had an adjusted mean of 2.76.

TABLE 15

Analysis of Covariance for Attitude Scores

Source	df	ss	ms	F	p
Treatments	1	66.22	66.22	.63	.433
Within	43	4539.99	105.58		
Total	44	4606.21			

The results of the analysis of treatment effect on attitudes toward mathematics are reported in Table 15 and summarized further in Table 13. No significant difference between treatments was found. The observed difference, however, favored the experimental group.

The ACT mathematics and composite scores for 15 students from each of the control and experimental groups were obtained from the Records Office at Western Michigan University. These scores were not available for foreign or transfer students. An analysis of Tables 16 through 18 indicates that in each case equality of slopes of the regression lines is tenable. Hence, the null hypotheses of no significant interactions between the ACT mathematics scores and overall achievement ($p < .19$), the ACT composite scores and overall achievement ($p < .91$), the ACT composite and mathematics scores, and overall achievement ($p < .28$) are not rejected.

TABLE 16

Results of Test for Homogeneity of Regression of Overall Calculus Achievement on ACT Mathematics Scores

Source	df	ss	ms	F	p
Equality of slopes	1	31.24	31.24	1.83	.19
Error	26	442.92	17.04		

Results of the open-ended questionnaire indicated that, in general, experimental subjects perception of their

TABLE 17

Results of Test for Homogeneity of Regression of Overall Calculus Achievement on ACT Composite Scores

Source	df	ss	ms	F	p
Equality of slopes	1	0.31	0.31	.01	.91
Error	26	562.18	21.62		

TABLE 18

Results of Test for Homogeneity of Regression of Overall Calculus Achievement on ACT Mathematics and Composite Scores

Source	df	ss	ms	F	p
Equality of slopes	2	44.86	22.43	1.34	.28
Error	24	402.47	16.77		

learning environment was highly favorable. Their comments are summarized in Tables 19 and 20.

Eighteen students responded that they would like to take another mathematics course taught in this manner; two students did not respond to this question; and one student was undecided about taking another course with a

TABLE 19

A Summary of Positive Student Reactions to Experimental Format

Student Comments	Frequency
Course was well-organized	6
Learned more in this class than in any previous math class	6
Like opportunity to work with others	4
Class was enjoyable - not dull	4
Stimulated interest in the material	3
Class time went fast	3
Development of mathematical concepts by students increased understanding	2
Aids in retention of material	1
Made one feel at ease with fellow students	1
Provided opportunity for greater student participation	1
Encouraged the development of friendships	1
Helps the teacher to understand the students' problems	1
Gives the teacher new insights into the teaching of calculus	1

similar format. The "undecided" student indicated that the pace was too fast for him and he would have to retake this course.

TABLE 20

A Summary of Negative Student Reactions to Experimental Format

Student Comments	Frequency
Teacher unable to get around efficiently to each of the groups	2
Need an assistant because of size of class	2
More explanation by the instructor needed	2
Need a source book to supplement the guide sheets	2
Rushed at times to complete units	1

As can be gleaned from Table 20, two students felt that the instructor should have an assistant, perhaps a senior mathematics major. It should be noted that the size of the class necessitated the use of two classrooms in order to have sufficient blackboard space and to keep the groups separated so that they could work independently. This, at times, made it difficult for the instructor to keep track of the progress of each group. Thus, as reported in Table 20 by two students, the teacher on several occasions was unable to get around efficiently to each of the groups.

It should be noted that most of the above negative comments can be attributed to the same two individuals.

CHAPTER V

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary

This paper is a description of an experiment in the teaching of an introductory course in calculus using a small group-discovery method and the results thereof. To determine the viability of this method of instruction, this study was carried out for one term.

The experiment was designed originally as a replication of one carried out by Davidson (1970) at the University of Wisconsin. However, a careful analysis of the Davidson study indicated a failure to control several concomitant variables. Consequently, this investigation was designed to be a refinement and extension of Davidson's earlier work.

Materials to be used by the experimental group were obtained from Davidson and revised in order to meet the course requirements (Table 9, p. 29) of the mathematics department at Western Michigan University.

Prior to instruction a pretest was given in order to measure the pre-calculus competency of the subjects in both the control and experimental groups. Students in both groups were also given the Aiken-Dreger Revised

Mathematics Attitude Scale to assess their attitudes toward mathematics on entering the course.

The 21 students in the experimental group were divided into 4 groups of four students and one group of five students. In an atmosphere of cooperation and through the use of dittoed notes the students were led to discover concepts, and to formulate basic definitions and theorems as they proceeded from the more concrete to the more abstract. In this setting they discussed open-ended questions, worked with exercises which were systematically chosen to guide the students to particular results, or sought solutions through deductive reasoning. Proofs were not intended to be rigorous. The instructor spent most of the class periods with the small groups, giving hints or constructive criticism, asking leading questions, and checking solutions.

The 25 students in the control group were taught by an experienced professor using the lecture-textbook method. The textbook was one chosen by the mathematics department.

At the end of the term subjects in both treatment groups were re-administered the attitude scale and were given a comprehensive test consisting of 36 multiple-choice items (odd numbered items involving primarily manipulative skills and even numbered items measuring understanding of concepts).

The results of the tests of Hypotheses 1-7 are summarized below.

Hypothesis 1: There is no significant difference in overall achievement between students in the experimental group and the students in the control group.

This hypothesis was rejected at the .02 level of significance. The adjusted mean for the experimental group was 20.65 and the adjusted mean for the control group was 17.21.

Hypothesis 2: There is no significant difference in conceptual understanding between students in the experimental group and students in the control group.

Since $p < .281$, this hypothesis was accepted. The adjusted means for the experimental and control groups were 7.64 and 6.87 respectively.

Hypothesis 3: There is no significant difference in skill acquisition between the experimental and the control groups.

Hypothesis 3 was rejected beyond the .01 level of significance. In this case the adjusted means were found to be 13.02 and 10.35 for the experimental and control groups respectively.

Hypothesis 4: There is no significant difference in attitude towards mathematics between the experimental and the control group.

Although the adjusted mean for the experimental group was 53.66 and the adjusted mean for the control group was 51.24, this hypothesis was accepted at the .433

level of significance.

Hypothesis 5: There is no significant interaction between ACT mathematics scores and overall achievement.

With $p < .19$, homogeneity of regression is tenable and, therefore, this hypothesis was accepted.

Hypothesis 6: There is no significant interaction between ACT composite scores and overall achievement.

Equality of slopes of the regression lines was clearly tenable since $p < .91$. Thus, this hypothesis was accepted.

Hypothesis 7: There is no significant interaction between ACT mathematics and composite scores and overall achievement.

A significance level of .28 resulted in the conclusion that homogeneity of regression was maintainable and, therefore, this hypothesis was to be accepted.

Conclusions

In general, subject to the conditions of this investigation, the results of the present study indicate that the use of small group-discovery learning in an introductory calculus course can be an effective means of improving student achievement, particularly with respect to the acquisition of manipulative skills. The results regarding concept attainment are not quite as encouraging; and, in fact, are somewhat perplexing. One would have expected that the active participation of the students in building up the calculus would have resulted in a deeper understanding of the fundamental theory and logical relations among its parts. Though students in the small group-discovery class performed better on the concept related test items, their performance was not significantly better than that of the control group. However, an item analysis yielded a mean index of difficulty of .61 for the concepts subtest as compared with .36 for the skills subtest. This suggests that the concepts subtest was too difficult for the subjects involved and this may in turn have resulted in random errors of measurement which reduced the possibility of finding a more significant difference between the treatments.

The results in the affective domain indicated that most students had a favorable attitude toward the small group-discovery method of instruction. Eighteen of the twenty-one students in the experimental class indicated a desire to take another course taught in this manner. The effect of this method on improvement of attitudes toward mathematics, however, was only slight. On the other hand, attitudes toward mathematics in the control class actually declined.

An analysis of the results of the tests involving the ACT scores indicate that the ACT mathematics and/or composite scores can not be used to predict whether the students will perform better under the small-group discovery method or traditional lecture-textbook method.

Recommendations

Finally, in the investigator's opinion, the experiences in this study and the results suggest the following recommendations for future research and development:

(1) A replication of this experiment involving a revision of the conceptual items on the posttest should be conducted.

(2) An investigation of the effects of small group-discovery learning upon problem-solving skills and long-term retention should be undertaken.

(3) A follow-up study of the differential success of the two groups in their future mathematics courses should be carried out.

(4) The question of whether certain personological aptitudes can predict the success or failure of the subjects in the two treatment groups should be investigated.

(5) A modification of this study to include students in the secondary mathematics methods course as assistants seems warranted.

(6) An exploratory study of the feasibility of

developing small group-activity oriented materials for use in various pre-calculus courses seems desirable.

APPENDIX A

THE PRETEST

MATH 122 PRE-TEST

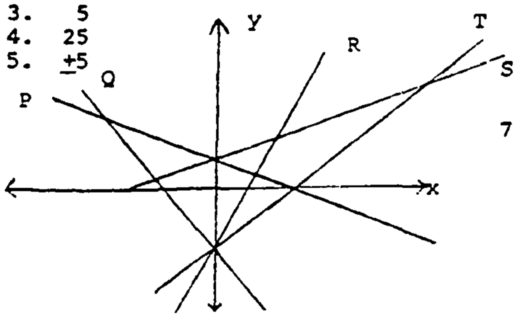
Do NOT open this booklet until you are told to do so. This test is a multiple choice test covering material from your high school mathematics courses. It will be scored as the number of items correct. When you receive an answer sheet, enter your name, the date, and your Social Security Number.

You will have 45 minutes to answer this test. Do not write in this test booklet. Do all your scratch work on the paper provided. Use a soft (No. 2) lead pencil to record your answers. Mark only one answer for each question.

Do not begin until you are told to do so.

1. $\sqrt[3]{-125}$ = (?)

   1. -25
   2. -5
   3. 5
   4. 25
   5. ±5



2. Which of the five lettered lines has (have) positive slope?

   1. T only
   2. P and Q only
   3. R and S only
   4. P, Q, and R
   5. R, S, and T

3. $(2x^2)(3x^3)$ = (?)

   1. $5x^5$
   2. $6x^6$
   3. $5x^6$
   4. $6x^5$
   5. None of these

4. Solve the equation $x^2 - 7x + 12 = 0$ for x.

   1. x = -3 and x = -4
   2. x = -3 and x = 4
   3. x = 3 and x = 4
   4. x = 2 and x = -6
   5. x = 2 and x = 6

5. What is the distance between the points (6,7) and (3,9)?

   1. $\sqrt{13}$
   2. $\sqrt{5}$
   3. 5
   4. $\sqrt{148}$
   5. 13

6. If 5/n - 3/n = 1/4, then n = (?)

   1. 1/8
   2. 1/2
   3. 2
   4. 4
   5. 8

7. 5^0 = (?)

   1. -1/5
   2. 0
   3. 1/5
   4. 1
   5. 5

8. What is the slope of the line 6y = 9 - 3x?

   1. -2
   2. -1/2
   3. 1/2
   4. 2
   5. 3

9. What are all the numbers which satisfy the equation $x^2 - 5x - 14 = 0$ and also satisfy the inequality $x > 5$?

   1. -2 only
   2. 7 only
   3. -2 and 7
   4. $x > 5$
   5. $x \geq 7$

10. $2\sqrt{12} + 3\sqrt{48} - 5\sqrt{27}$ = (?)

    1. $\sqrt{3}$
    2. $2\sqrt{3}$
    3. $3\sqrt{3}$
    4. $4\sqrt{3}$
    5. None of these

11. If x = 3/4, then x^{-2} = (?)

    1. -16/9
    2. -6/8
    3. -9/16
    4. 8/6
    5. 16/9

12. Which of the following, when added to m + n, yields n - m?

1. 2m
2. -2m
3. 2m - 2n
4. m - n
5. n - m

13. The equation of the line through (2,1) and (-4,3) is

1. $x - y - 1 = 0$
2. $x + y - 1 = 0$
3. $x - y + 7 = 0$
4. $x + 3y + 5 = 0$
5. $x + 3y - 5 = 0$

14. Which of the following is an equation for a line perpendicular to the line $2x + 5y = 16$?

1. $2x + 5y = -1/16$
2. $2x - 5y = 16$
3. $5x + 2y = -16$
4. $5x - 2y = 16$
5. $3x + 4y = 16$

15. If x and y are real numbers, what is the domain of the function defined by $y = \frac{x}{\sqrt{9 - x^2}}$?

1. All x except x = 3
2. All x except x = 3 and x = -3
3. $x < -3$ and $x > 3$
4. $-3 < x < 3$
5. $x < 3$

16. If $2x + y = 7$ and $x - 4y = 4$, then y = (?)

1. 15/9
2. -1/9
3. 7/16
4. 11/9
5. 7

17. $(3 + 4i)(2 - i) = (?)$ where $i = \sqrt{-1}$

1. 10 + 11i
2. 2 + 5i
3. 10 + 5i
4. 2 + 11i
5. None of these

18. $\frac{1}{y} + \frac{2}{3 + y} = (?)$

1. $\frac{3}{3 + 2y}$
2. $\frac{3y + 3}{3 + 2y}$
3. $\frac{y + 1}{y + y^2}$
4. $\frac{y + 3}{3y + y^2}$
5. $\frac{3y + 3}{3y + y^2}$

19. If $f(x) = 2x^3 - 3x^2 - x + 2$, then $f(-1) = (?)$

1. -4
2. -2
3. 0
4. 2
5. 4

20. The lowest common denominator of the fractions $\frac{1}{s^2 + s - 12}$ and $\frac{1}{s^2 - 5s + 6}$ is

1. $(s - 2)(s - 3)(s + 4)$
2. $(s - 2)(s - 3)^2(s + 4)$
3. $(s - 2)(s - 3)(s + 3)(s - 4)$
4. $(s - 2)(s + 3)(s - 4)$
5. $(s + 2)(s - 3)(s + 3)(s - 4)$

21. If $x^3 - 8x^2 = 0$, then x = (?)

1. 0 only
2. 2 only
3. 8 only
4. 0 or 2
5. 0 or 8

22. If $x^3 - 2x^2 - 3$ is multiplied by $2x^2 - 5$, what is the coefficient of x^2 in the product?

1. -16
2. -4
3. 4
4. 0
5. 16

23. $(\sqrt{3} - 6)(5\sqrt{3} - 2) = (?)$

1. $27 - 28\sqrt{3}$
2. $27 - 32\sqrt{3}$
3. $5\sqrt{6} - 32\sqrt{3} + 12$
4. $28\sqrt{3} + 27$
5. $32\sqrt{3} + 27$

24. The two solutions of the equation $2x^2 - x - 4 = 0$ are

1. opposite in sign
2. both positive and equal
3. both negative and equal
4. both positive but unequal
5. both negative but unequal

25. If the line through (0,3) and (3,7) is parallel to the line through (1,5) and (4,k), then k = (?)

1. 8
2. 9
3. 10
4. 11
5. 12

26. When simplified, $\dfrac{t^6}{t^4 + t^2} = (?)$

1. t^2
2. $\dfrac{t^4}{t^2 + 1}$
3. 1
4. $\dfrac{t^3}{t^2 + 1}$
5. $t^2 + t^4$

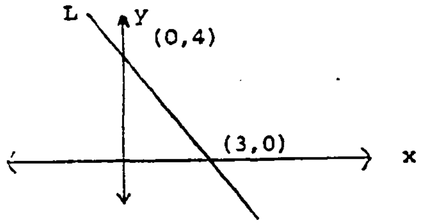
27. Which of the following expressions is equivalent to $x(x-a) + a(x+a)$?

1. $(x + a)^2(x - a)$
2. $(x + a)(x - a)^2$
3. $(x + a)^3$
4. $(x + a)^2$
5. $x^2 + a^2$

28. $|x - 2| < 4$ if and only if

1. $x < 2$
2. $x < 6$
3. $-2 < x < 6$
4. $x > -2$
5. $x < -2$ or $x > 6$

29.



What is the equation of the line L which crosses the X-axis and the Y-axis at the points indicated in the figure above?

1. $3y + 4x = 0$
2. $4y + 3x = 0$
3. $x + y = 12$
4. $3y + 4x = 12$
5. $4y + 3x = 12$

30. The graphs of two linear equations are distinct and do not intersect. How many solutions do the two equations have in common?

1. None
2. One
3. Two
4. An unlimited number
5. It cannot be determined from the information given

31. If an integer x is divided by another integer y, the quotient is 24. If the sum of the two integers is 75, then x = (?)

1. 3
2. 8
3. 25
4. 48
5. 72

32. If $x - 7$ is a factor of $x^2 - 3x + p$, what is the value of p?

1. -28
2. -21
3. -10
4. 21
5. 28

33. $\frac{3 + 2i}{i} = (?)$

1. 1
2. 5
3. 5i
4. -2 + 3i
5. 2 - 3i

34. If the graph of the equation $cy = dx^2 - 4$ passes through the points (2,0) and (-4,3), then c = (?)

1. 4
2. 0
3. -9/2
4. -20/3
5. -20

35. Find the center C and radius R of the circle $(x + 2)^2 + (y - 3)^2 = 49$

1. C = (-2,3) and R = 6
2. C = (2,-3) and R = 6
3. C = (-2,3) and R = 7
4. C = (2,-3) and R = 7
5. C = (0,0) and R = 7

APPENDIX B

THE MATHEMATICS ATTITUDE SCALE

MATHEMATICS ATTITUDE SCALE

When you receive an answer sheet, enter your name, the date, and your Social Security Number.

Each of the statements on this opinionaire expresses a feeling which a particular person has toward mathematics. You are to express, on a five-point scale, the extent of agreement between the feeling expressed in each statement and your own personal feeling. The following statements are to be rated on a scale 1-5 with the interpretation:

Strongly Disagree	Disagree	Undecided	Agree	Strongly Agree
1	2	3	4	5

---

1. I am always under a terrible strain in a math class.
2. I do not like mathematics, and it scares me to have to take it.
3. Mathematics is very interesting to me, and I enjoy math courses.
4. Mathematics is fascinating and fun.

5. Mathematics makes me feel secure, and at the same time it is stimulating.
6. My mind goes blank, and I am unable to think clearly when working math.
7. I feel a sense of insecurity when attempting mathematics.
8. Mathematics makes me feel uncomfortable, restless, irritable, and impatient.
9. The feeling that I have toward mathematics is a good feeling.
10. Mathematics makes me feel as though I'm lost in a jungle of numbers and can't find my way out.
11. Mathematics is something which I enjoy a great deal.
12. When I hear the word math, I have a feeling of dislike.
13. I approach calculus with a feeling of hesitation, resulting from a fear of not being able to do math.
14. I really like mathematics.
15. Mathematics is a course in school which I have always enjoyed studying.
16. It makes me nervous to even think about having to do a math problem.
17. I have never liked math, and it is my most dreaded subject.

18. I am happier in a math class than in any other class.

19. I feel at ease in mathematics, and I like it very much.

20. I feel a definite positive reaction to mathematics; it's enjoyable.

APPENDIX C

VELOCITY AND SLOPES OF CURVES

CHAPTER II

SECTION 1. DERIVATIVES

In this chapter we study one of the two major problems of calculus, finding the line tangent to a given curve. The concept involved - rate of change - occurs frequently in mechanics, economics, biology, etc., and this universality is perhaps the major reason that calculus is so useful.

Historically the problem of tangents was studied much later than the other major problem, the problem of areas. Archimedes (287?-212 B.C.), for example, could compute areas of circles, regions bounded by parabolas, etc., but the determination of tangents required the invention of analytic geometry simultaneously by Descartes (1596-1650) and Fermat (1601-1665) in the 17th century.

PROBLEM

The Box Problem: An open box is to be made from a 4 ft by 5 ft piece of cardboard by cutting out squares of equal size from the four corners and bending up the flaps to form sides. Using only algebra, geometry, and analytic geometry, find approximately the dimensions of the box of largest volume which can be made in this way. What do you need to know in order to find the exact dimensions?

PROBLEMS

1. (a) An auto started from rest and during the second between $t = 1$ sec and $t = 2$ sec it went 30 feet. What was the average velocity between $t = 1$ and $t = 2$ sec?

   (b) During the first half-second between $t = 1$ and $t = 2$ the auto went 10 feet. What was the average velocity between $t = 1$ and $t = 1\frac{1}{2}$ sec?

   (c) During the first fourth-second between $t = 1$ and $t = 2$ the auto went 4 feet. What was the average velocity between $t = 1$ and $t = 1\frac{1}{4}$ sec?

   (d) Which of these is the "best approximation" to the velocity at the end of the first second $(t = 1)$?

2. Suppose that an object is dropped from a high building and that the distance travelled during the first t seconds is given by $s(t) = 16t^2$.

   (a) Find the average velocity between $t = 3$ and $t = 5$.

   (b) Find the average velocity between $t = 3$ and $t = 4$.

(c) Find the average velocity between $t = 3$ and $t = 3.1$.

(d) Find the average velocity between $t = 3$ and $t = 3 + h$, $h \neq 0$. (your answer should depend on h).

(e) What number would you assign as the <u>instantaneous velocity</u> at $t = 3$?

3. In Problem 2 the time intervals "squeezed down" on $t = 3$ from the right. Would your answer in (e) have been different if you had "squeezed down" on $t = 3$ from the left?

4. Sketch a graph of the function $s(t)$ in Problem 2 and give a geometric interpretation to each of parts (a)-(d). To what geometric idea does part (e) seem to be related?

HOMEWORK

1. Each of the following formulas gives distance $s(t)$ traveled by an object in time t. Find the average velocity between $t = 1$ and $t = 5$, between $t = 1$ and $t = 2$, and between $t = 1$ and $t = 1 + h$, $h \neq 0$.

(a) $s(t) = 2t - 5$

(b) $s(t) = 2t^2 - 1$

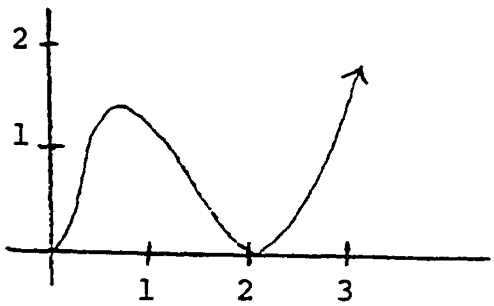
(c) $s(t) = t^2 - 6t$

3. If $s(t)$ is the distance traveled by an object in time t, define the instantaneous velocity when $t = 2$.

4. If $s(t)$ is the distance traveled by an object in time t, define the instantaneous velocity when $t = t_o$.

5. Suppose that a car travels a distance of 60 miles in 90 minutes. Is it possible to exceed the speed limit of 60 miles per hour? Use a graph to give a geometric interpretation to your answer.

6. The graph of a certain distance function is given in the figure at the right. Sketch a graph of the corresponding velocity function.



HOMEWORK

1. If a marble rolls down a certain straight groove, the distance covered (from the time of release) is $s(t) = 2t^2$. The motion stops at time 10.

   (a) Find a formula, in terms of t and h, for the average velocity between times t and $t + h$, where both t and $t + h$ are between 0 and 10, and $h \neq 0$.

2. For each of the distance functions in Problem 1, what number would you assign as the instantaneous velocity at $t = 1$?

3. What can you say about the motion of a car whose average velocity is always 0 in any time interval?

4. What can you say about the average velocity of an object which remains at rest?

SECTION 2. VELOCITY

PROBLEMS

1. An object is dropped from a high building and the distance traveled during the first t seconds is given by $s(t) = 16t^2$.

   (a) Find the average velocity between $t = a$ and $t = a + h, h \neq 0$.

   (b) What number would you assign as the instantaneous velocity at $t = a$?

   (c) Find a formula for the velocity $v(t)$ at the end of the first t seconds.

2. The distance traveled by an object in time t is $s(t)$.

   (a) Find the distance traveled between times t_o and $t_o + h, h \neq 0$.

   (b) Find the average velocity between times t_o and $t_o + h, h \neq 0$.

   (c) Explain how you would find the instantaneous velocity at time t_o.

   (d) Use a graph to interpret parts (a) - (c) geometrically.

   (e) How is the velocity at time t_o related to the graph of $y = s(t)$ at t_o?

(b) Find a formula which gives the instantaneous velocity as a function of time.

2. Each of the following formulas gives the distance $s(t)$ traveled by an object in t seconds. Find the velocity when $t = 3$.

   (a) $s(t) = 6 - t$

   (b) $s(t) = t^2 + 8$

   (c) $s(t) = t^3$

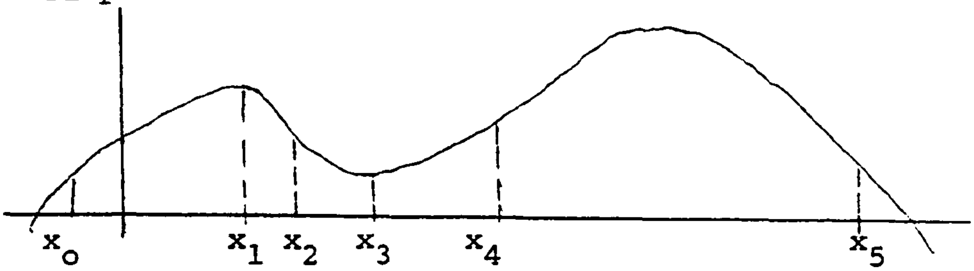
3. Sketch a graph of each of the distance functions in Problem 2, and give a geometric interpretation to the velocity when $t = 3$.

4. For each of the distance functions in Problem 2 find a formula which gives the velocity $v(t)$ at time t.

SECTION 3. SLOPES OF CURVES

PROBLEMS

1. At each of the indicated points sketch in the tangent line and make a rough estimate of its slope.



2. For each of the points in Problem 1, tell whether the function is increasing, decreasing, or neither near that point. How are these answers related to your answers in Problem 1?

3. (a) Let $f(x) = x^2$. Sketch a graph of $f(x)$ and estimate the slope of the line through $(2, 4)$ tangent to the curve.

   (b) Find the slope of the line through $(2, 4)$ and $(3, f(3))$.

   (c) Find the slope of the line through $(2, 4)$ and $(2.1, f(2.1))$.

   (d) Find the slope of the line through $(2, 4)$ and $(2 + h, f(2 + h))$. (Your answer should depend on h).

(e) What number would you assign as the slope of the tangent to the curve $f(x) = x^2$ at the point (2, 4) ?

4. Suppose $s(t)$ is the distance traveled by an object in time t. Explain the relationship between the tangent line at the point $(t_o, s(t_o))$ on the graph of $y = s(t)$ and the velocity at t_o.

5. Let f be a function. Define the slope of the line through the point $(2, f(2))$ tangent to the graph of $y = f(x)$.

6. Define the slope of a line tangent to the curve $y = f(x)$ at the point $(x_o, f(x_o))$.

7. Find the equation of the line through the point (3, 9), tangent to $y = x^2$.

8. Find the slope of the line through (1, 1) tangent to $y = x^3$.

HOMEWORK

1. Compute the slope of the line tangent to the curve at the given point. Sketch each graph and the tangent line.

   (a) $y = f(x) = 3x^2$ at (1, 3)

   (b) $y = f(x) = 3x^2$ at (-1, 3)

   (c) $y = g(x) = 3x^2 + 2$ at (2, 14)

   (d) $y = h(x) = 3x^2 + 1$ at (1, 3)

   (e) $y = h(x) = 3x^2 + 1$ at (0, 1)

2. Find the equation of the line through (2, 6) tangent to the given curve.

(a) $y = 3x^2 + 21$

(b) $y = x^2 - 2x + 14$

(c) $y = x^2 + 6$

APPENDIX D

DEFINITION OF DERIVATIVE, SUMS, DIFFERENCES AND POWERS

SECTION 5. DERIVATIVES

We now combine the ideas of the preceding sections in the following definition:

DEFINITION. The <u>derivative</u> of a function f at x_o is $\lim_{h\to 0} \frac{f(x_o + h) - f(x_o)}{h}$, provided the limit exists. The derivative is denoted by $f'(x_o)$ (read f prime at x_o). (Other notations will be introduced later.)

DEFINITION. A function f is differentiable at x_o if $f'(x_o)$ exists. Note: The particular name "x" is not important. We often write

$$f'(x) = \lim_{n\to 0} \frac{f(x + h) - f(x)}{h} .$$

PROBLEMS

1. Compute the derivatives of the following functions using the definition of derivative. First write down the difference quotient $\frac{f(x + h) - f(x)}{h}$ and then compute its limit as $h \to 0$.

   (a) $f(x) = x$ (d) $f(x) = 7$

   (b) $f(x) = x^2$ (e) $f(x) = 1/x$

   (c) $f(x) = x^3$ (f) $f(x) = 3x^2 - 2x$

2. Suppose that an object travels a distance $s(t) = t^3 - 2t$ in time t.

   (a) Find the average velocity between t_o and $t_o + h$.

   (b) Find the instantaneous velocity at time t_o.

3. (a) Find the equation of the line through the point (1,1) tangent to the curve $y = 3x^2 - 2x$.

   (b) Find the equation of the line through (1,1) and normal to the curve $y = 3x^2 - 2x$. (A normal is perpendicular to the tangent.)

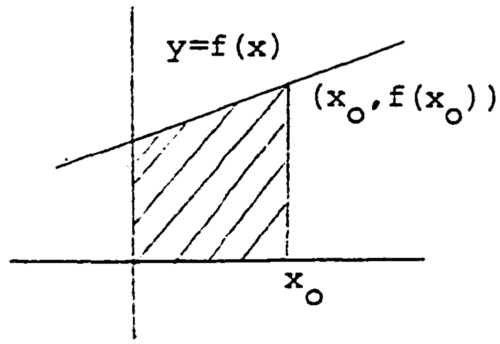
4. Show that $f(x) = |x|$ is not differentiable at $x = 0$. Sketch the graph.

5. Show that $f'(x_o)$ can be written as

$$\lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o}.$$ Give a geometric interpretation

   to this expression.

6. Let $f(x)$ be a function whose graph in the first quadrant is a line which lies entirely above the x-axis. Let $A(x)$ be the function whose value at x_o is the area of the region bounded by the lines $x = 0$ and $x = x_o$, the x-axis, and the graph of $y = f(x)$ Show $A'(x_o) = f(x_o)$.



HOMEWORK

1. For each of the following functions, compute $f'(2)$, if it exists. Sketch a graph of $y = f(x)$ and the tangent line passing through the point $(2, f(2))$.

   (a) $f(x) = 1 - \frac{1}{x}$

   (b) $f(x) = x^2 - 4x + 4$

   (c) $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4x - 4 & \text{if } x \geq 2 \end{cases}$

   (d) $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x \geq 2 \end{cases}$

2. Compute the derivatives of the following functions, wherever they exist.

   (a) $f(x) = \frac{1}{2x + 1}$

   (b) $f(x) = x^4$

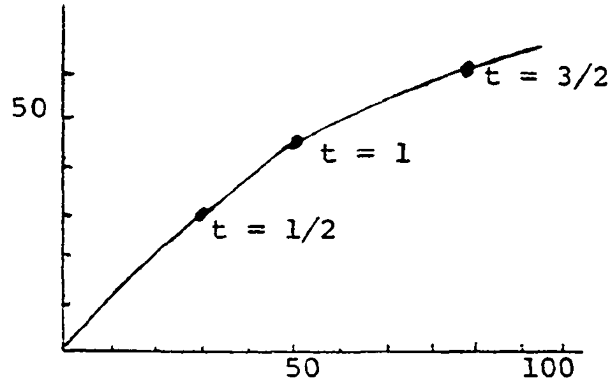
   (c) $f(x) = \frac{1}{x^2}$

   (d) $f(x) = x^2 - 3x + 2$

   (e) $f(x) = \pi x$

   (f) $f(x) = \begin{cases} 2x^2 + 3 & \text{if } x \leq 1 \\ 4x + 1 & \text{if } x > 1 \end{cases}$

   (g) $f(x) = \begin{cases} 2x + 2 & \text{if } x \leq 1 \\ 3x^2 + 1 & \text{if } x > 1 \end{cases}$



3. Suppose that a ball is thrown along a path in the x - y plane so that the horizontal distance is $x(t) = 60t$ and the vertical distance is $y(t) = -16t^2 + 64t$.

3. continued

(a) Find the horizontal and vertical velocities,

(b) If the x-axis represents the ground, find the maximum horizontal distance.

(c) Find the maximum height.

(d) Find the horizontal and vertical acceleration. (Acceleration is defined to be the derivative of velocity.)

4. Find the equations of the lines:

(a) through (1,6), tangent to $y = x^2 + 2x + 3$.

(b) through (1,6), normal to $y = x^2 + 2x + 3$.

(c) through (1,6), tangent to $y = 6/x$.

(d) through (1,6), normal to $y = 6/x$.

SECTION 6, DERIVATIVES OF SUMS, DIFFERENCES, AND POWERS

The aim of the next few sections is to develop some rules that will make it easy to compute derivatives. If you don't feel such rules are necessary, you might try to differentiate $g(x) = \left(\frac{x + 1}{x^2 - 3}\right)^{3/2}$ by using the definition.

DEFINITION. If f and g are functions, then the function

(i) $f + g$ is defined by $(f + g)(x) = f(x) + g(x)$

(ii) $f - g$ is defined by $(f - g)(x) = f(x) - g(x)$

(iii) $f \cdot g$ is defined by $(f \cdot g)(x) = f(x) \cdot g(x)$

(iv) f/g is defined by $(f/g)(x) = f(x)/g(x)$, $g(x) \neq 0$

(v) cf, where c is a constant, is defined by

$(cf)(x) = c \cdot f(x)$. (Of course these equations make sense only when $f(x)$ and $g(x)$ are both defined.

PROBLEMS

1. Let $f(x) = 2x^2 + 1$ and $g(x) = 4x$. Compute:

   (a) $(f + g)(3)$

   (b) $(f - g)(1)$

   (c) $(f \cdot g)(5)$

   (d) $(f/g)(2)$

   (e) $(3f)(-2)$

2. Complete the following statements. Assume that f and g are differentiable and that c is a constant.

(a) $(f + g)' =$ ________________. Restate in words.

(b) $(f - g)' =$ ________________. Restate in words.

(c) $(cf)' =$ ____________________. Restate in words.

3. Prove each of the rules in Problem 2, using the definition of derivative.

4. Differentiate the following functions, using any results obtained in this course.

(a) $f(x) = 3x^3 - 2x^2 - 3x + 1$

(b) $f(x) = 9/2x^2 - 1/(4x) + 1/2\ x$

5. Show that $x - x_o$ is a factor of each of the following and find the other factor:

(a) $x^2 - x_o^2$

(b) $x^3 - x_o^3$

(c) $x^4 - x_o^4$

(d) $x^n - x_o^n$ (n a positive integer)

6. Let $f(x) = x^n$. Find $f'(x)$. (Use the fact that

$$f'(x_o) = \lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o} \Big)$$

If $f(x) = x^n$ and n is a positive integer, then $f'(x) =$

____________________.

7. Differentiate $P(x) = 3x^5 - 1/3\ x^3 - 1/4\ x^2 + x + 1/2$.

8. Compute the derivative of the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o .$$

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o$,

then $P'(x) =$ ______________________.

9. Differentiate:

(a) $f(x) = 7x^8 - 3x^4 - x^2 - 2x + 1$

(b) $h(x) = x^7 - 3/2\, x^4 - 2x^6 - 3x + 7$

Challenge

10. In your proofs in Problem 3 you probably used some properties of limits which have not been proved in class.

(a) State the properties that you used.

(b) Give arguments which make these properties plausible.

HOMEWORK

1. Differentiate the following functions. Use any result proved in class.

(a) $f(x) = 3x^6 - 4/3\, x^3 - 1/2\, x + 3$

(b) $g(x) = 7x^{15} - 3x + 1/(2x)$

(c) $h(t) = 7t^6 - (3t^2)/2 + 1$

(d) $H(y) = 7y^3 - 3y^2 - 1/3\, y$

(e) $P(w) = -8w^5 + 3/4\, w^2 - 1/w$

(f) $f(z) = 8(z^3 - 3z + 1)$

(g) $g(x) = (x + 1)^2$

1. continued

   (h) $f(x) = x^6 - 3/x^2$

   (i) $f(x) = x^2 - x^3 + 4x^4$

   (j) $f(x) = 4/x^2 - 3/x + 2x$

2. Suppose that in time t a particle travels a distance $s(t) = t^3 - 6t^2 + 6t + 8$.

   (a) Find the velocity $v(t)$ and tell where $v(t) = 0$.

   (b) Find the acceleration $a(t)$ and tell where $a(t) = 0$.

3. Find the equation of the line through (1,5) tangent to:

   (a) $y = 3x^8 + x^5 - 3x + 4$

   (b) $y = 7x^5 - x^3 + x - 2$

4. (a) Let $A(x)$ be the function whose value at x_o is the area of the region bounded by the lines $x = 0$ and $x = x_o$, the x-axis, and the graph of $y = 2x$. Find a formula for $A(x)$.

   (b) Show that $A'(x) = 2x$.

Challenge

5. Suppose that $f(x)$ is differentiable everywhere. Is it always true that $|f(x)|$ is differentiable? Give some examples.

APPENDIX E

DERIVATIVES OF PRODUCTS AND QUOTIENTS

DERIVATIVES OF PRODUCTS AND QUOTIENTS

SECTION 7

The formula for the derivative of the product of two functions eluded Leibniz (1646-1716) for a number of years. He unsuccessfully attempted to prove that the derivative of the product of two functions was the product of their derivatives before conjecturing and proving the correct rule. Once we conjecture and prove the product rule, similar results for quotients will follow easily.

PROBLEMS

1. Find two functions $f(x)$ and $g(x)$ so that $(fg)' \neq f'g'$.
2. To conjecture a formula for the derivative (fg), we can use the following matrix.

f	g	fg
f'	g'	$(fg)'$

For each given pair of functions f, g fill in the remaining four entries in the matrix for that pair.

	f	g	fg	f	g	fg
Function	Example x	x^4	x^5	x	x	
Derivative	1	$4x^4$	$5x^4$			
Function	x^2	x^5		x^3	x^7	
Derivative						
Function	x	$x^2 + x$		x^2	$x^3 + x$	
Derivative						

For each pair of functions, try to express $(fg)'$ as some combination of the other entries in the matrix. State a conjecture: If f and g are differentiable functions

$(fg)' =$

Your answer should involve f' and g' and perhaps other terms. State your answer in words. Test your conjecture with $f(x) = x^{67}$, $g(x) = x^{28}$.

3. (a) Prove the rule obtained in Problem 2.

(b) In your proof you may have used either $\lim_{h \to 0} f(x + h) = f(x)$ or $\lim_{h \to 0} g(x + h) = g(x)$. Justify this by proving the following theorem: If f is differentiable at x_o, then f is continuous at x_o. (Hint Prove $\lim_{x \to 0} (f(x_o + h) - f(x_o)) = 0$.)

(c) Is the converse of the theorem true. If f is continuous at x_o, is f differentiable at x_o? Prove or give a counter-example.

4. Differentiate without multiplying out.

(a) $(3x^5 - 4x + 1)(2x^6 - 3x + 4)$

(b) $(x^5 - 3x^2 - 4)(2x^3 - x + 1)$

(c) $x^2(3x + 1)(4x + 6)$

5. Find a rule for the derivative of a reciprocal: that is, if $f(x) = \frac{1}{g(x)}$, find $f'(x)$ in terms of $g(x)$ and $g'(x)$.

6. Differentiate

(a) $f(x) = \frac{1}{x^6}$

(b) $g(x) = \dfrac{1}{x^3 + 3x + 1}$

7. State and prove a rule for the derivative of a quotient f/g, in terms of f, g, f' and g'. Restate in words.

$(f/g)' =$

8. Differentiate, using any method.

(a) $f(x) = \dfrac{7x^3 - 3x}{x^2 - x + 1}$

(b) $h(y) = \frac{6y^3 - 3y + 1}{y - 1}$

(c) $l(x) = \frac{x + 1}{x^2 + 1}$

9. Let $f(x) = x^n$, where n is a negative integer. Find $f'(x)$. (Hint: Let $n = -k$, where k is a positive integer.)

10. Let $f(x) = x^n$, where $n = 0$. Find $f'(x)$.

11. Complete the statement of the rules.

If $f(x) = x^n$, where n is <u>any integer</u>, than $f'(x) =$

Challenge

12. In your proofs in Problems 3 and 5, you probably used some properties of limits that have not been proved in class

    (a) State the properties that you used.

    (b) Give arguments which make these properties plausible.

HOMEWORK

1. Differentiate. You may use any computational techniques you know.

    (a) $f(x) = 3x^5 - 4x + \frac{1}{2}x^2$

    (b) $f(x) = (3x + 1)(4 - \frac{2}{x^2} + \frac{1}{2x})$

    (c) $f(x) = (7x^6 - 3x)(3x)(2x^6 + 1)$

(d) $h(x) = \frac{(x + 1)(x^2 + 2)}{x + 3}$

(e) $h(x) = \frac{x^2 - 3x}{x^2 + 3x}$

(f) $l(x) = (7x^2 - \frac{3}{x^8})(x^4 - 3x^5)$

(g) $g(t) = (7t^3 - 3t + 1)(t^3 - t^2 + 3)$

(h) $f(t) = (t - 3)(t^2 - 2)$

(i) $h(x) = x(x - 1)(x - 2)$

(j) $p(x) = (x + \frac{1}{x})(x - \frac{1}{x})$

(k) $f(x) = \frac{x^2 + x}{2x - 3}$

2. Find all values of x for which $f'(x) = 0$.

(a) $f(x) = \frac{x}{x^2 + 1}$

(b) $f(x) = \frac{1}{x^2 + 1}$

(c) $f(x) = (x^2 + 1)(x^2)$

Challenge

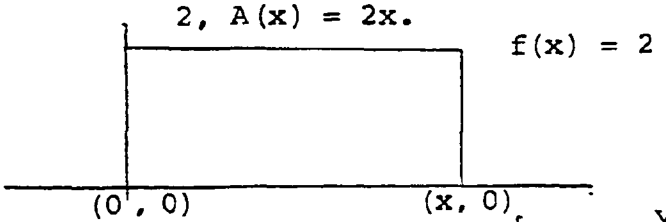
3. Let $P(x)$ be a polynomial. Prove that $P'(x) = 0$ if $P(x)$ is divisible by $(x - x_o)^2$ (i.e., $P(x) = (x - x_o)^2 Q(x)$ for some polynomial $Q(x)$).

APPENDIX F

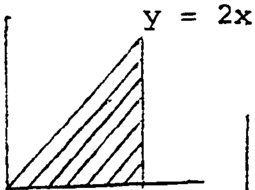
AREA FUNCTIONS

SECTION 3. AREA FUNCTIONS

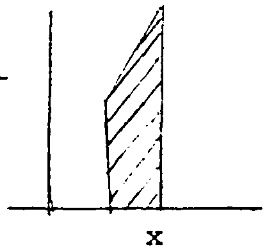
Notation: If $f(x)$ is a non-negative function and $x \geq 0$, let $A(x)$ be the area of the region bounded by the curve $y = f(x)$, the x-axis, and vertical lines through $(0, 0)$ and $(x, 0)$. For example, if $f(x) = 2$, $A(x) = 2x$.



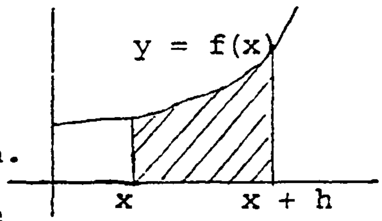
Problems.



1. Let $f(x) = 2x$. Find the area $A(x)$. Also find $\frac{d}{dx} A(x)$.



2. Let $f(x) = 2x$. Find the area $B(x)$ of the shaded region. Also find $\frac{d}{dx} B(x)$



3. Let $f(x)$ be a continuous increasing non-negative function. Let $A(x)$ be the area under the curve from 0 to x.
   a. Express the area of the shaded region in terms of the area function $A(x)$, x, and h.

b. Find two rectangles R_1 and R_2 so that area of $R_1 \leq A(x+H) - A(x) \leq$ area of R_2.

c. Compute $A'(x)$, using the definition of derivative.

$A'(x) =$

d. In parts a, b, and c we assumed $h > 0$. Repeat the same process if $h < 0$.

4. For each of the following functions $f(x)$, compute $A(x)$ and $A'(x)$.

a. $f(x) = x^2$

b. $f(x) = x^3$

c. $f(x) = x^5 + x$

HOMEWORK

1. For each of the following functions, find $A(x)$ and $A'(x)$:

a. $f(x) = 3x^6 + 2x^2$

b. $f(x) = 7x^3 + x + 3$

c. $f(x) = x^2 + x^3$

d. $g(x) = x^{1/2}$

e. $h(x) = 3x^2 + 4$

2. Let $f(x)$ be a continuous non-negative decreasing function. Follow the same procedure as class problem (3) to find $A'(x)$.

APPENDIX G

THE POSTTEST

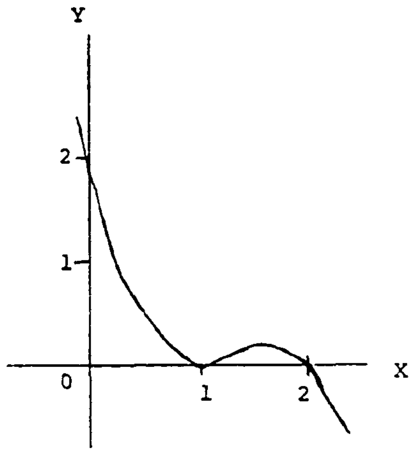
MATH 122 POST TEST

Do NOT open this booklet until you are told to do so. This test is a multiple choice test covering material from your MATH 122 course. It will be scored as the number of items correct. When you receive an answer sheet, enter your name, the date, and your Social Security Number.

You will have 50 minutes to answer this test. Do not write in this test booklet. Do all your scratch work on the paper provided. Use a soft (No. 2) lead pencil to record your answers. Mark only one answer for each question.

Do not begin until you are told to do so.

1. If $f(x) = 2x^3 - 3x + 1$, evaluate $f'(x)$ at $x = 2$.
   1. 9
   2. 11
   3. 13
   4. 18
   5. 21

2. 

   Which of the following can be the equation of the graph shown above?
   1. $y = (1 - x)(x - 2)$
   2. $y = (1 - x)(2 - x)$
   3. $y = (1 - x)^2(2 - x)$
   4. $y = (1 - x)^2(x - 2)$
   5. $y = (1 - x)(2 - x)^2$

3. If $y = \sqrt{0.25}$, then $\frac{dy}{dx} = (?)$
   1. 0
   2. $\frac{1}{4}$
   3. $\frac{1}{2}$
   4. 1
   5. 2

4. At what point on the curve $y = 3x^2 + 2x + 1$ is its slope 8?
   1. $\left(-\frac{5}{3}, 6\right)$
   2. $\left(-\frac{5}{16}, -\frac{69}{16}\right)$
   3. $(1, 6)$
   4. $(1, 9)$
   5. $\left(\frac{4}{3}, 9\right)$

5. $\int_1^2 x^2 dx = (?)$
   1. 2
   2. $\frac{7}{3}$
   3. 3
   4. $\frac{7}{2}$
   5. 7

6. If $y = x^6$, then Δy, the increase in y as x increases from 2 to 2.1, is approximately
   1. 10^{-6}
   2. 0.5
   3. 0.6
   4. 3.0
   5. 19.2

7. What is the equation of the line through (0,1) parallel to the line $3x + 5y = 7$?
   1. $3x + 5y = 5$
   2. $3x + 5y = 3$
   3. $3x - 5y = -5$
   4. $5x - 3y = 7$
   5. $5x - 3y = -3$

8. If the maximum value of $f(x)$ is 2 and if $f'(x) = 2 - 2x$, then $f(x) = (?)$

1. $-x^2 + 2x + 1$
2. $-x^2 + 2x + 2$
3. $-x^2 + 2x$
4. $-x^2 + 2x - 1$
5. $-x^2 + 2x - 2$

9. If $y = \sqrt{x^2 + x + 1}$, then $\frac{dy}{dx} = (?)$

1. $(2x + 1)\sqrt{x^2 + x + 1}$
2. $\frac{(2x + 1)\sqrt{x^2 + x + 1}}{2}$
3. $\frac{1}{2\sqrt{x^2 + x + 1}}$
4. $\frac{2x + 1}{2\sqrt{x^2 + x + 1}}$
5. $\frac{2x + 1}{\sqrt{x^2 + x + 1}}$

10. If $f(x) = x^{20}$ and if $f^{(n)}(x)$ denotes the nth derivative of $f(x)$, what is the smallest n for which $f^{(n)}(x)$ is a constant?

1. 11
2. 19
3. 20
4. 21
5. 22

11. If $f'(x) = x^3 + 4x$ and $f(0) = 4$, then $f(x) = (?)$

1. $3x^2 + 4$
2. $x^3 + 4x + 4$
3. $x^4 + 4x^2 + 4$
4. $\frac{1}{4}x^4 + 2x^2 - 4$
5. $\frac{1}{4}x^4 + 2x^2 + 4$

12. If the line $y = 3x + k$ is tangent to the parabola $y = x^2 - x + 9$, then $k = (?)$

1. -1
2. 1
3. 3
4. 5
5. 9

13. What is the area of the region bounded by the curve $y = x^3$, the X-axis, and the lines $x = 1$ and $x = 2$?

1. $\frac{7}{3}$
2. 3
3. $\frac{7}{2}$
4. $\frac{15}{4}$
5. 4

14. If $f(x) = (x - a)^3 G(x)$ where $G(x)$ is differentiable, find $f'(a)$.

1. 0
2. $G'(a)$
3. $-a^3 G'(a)$
4. $3a^2 G(a) - a^3 G'(a)$
5. $3(G'(a))^2$

15. What is the slope of the line normal to the curve $y = x^2 + 2x$ at the point $(2, 8)$?

1. $-\frac{1}{4}$
2. $-\frac{1}{6}$
3. $\frac{1}{8}$
4. $\frac{1}{6}$
5. $\frac{1}{4}$

16. If $G(t) = \int_2^t f(x)dx$, then $G'(t) = (?)$

1. $f'(t) - f'(2)$
2. $f'(t)$
3. $f(t) - f(2)$
4. $f(t)$
5. $G(t) - G(2)$

17. $\int \sqrt{x - 4}\, dx = (?)$

1. $\frac{2}{3}x^{\frac{3}{2}} - 2x + C$
2. $\frac{1}{2}(x - 4)^{-\frac{1}{2}} + C$
3. $\frac{2}{3}(x - 4)^{\frac{3}{2}} + C$
4. $\frac{1}{2}(x - 4)^{\frac{3}{2}} + C$
5. $\frac{3}{2}(x - 4)^{\frac{3}{2}} + C$

18. The lower sum L_n of the function $f(x) = x^2$ with respect to a partition of the interval $[0, 4]$ into 4 equal subintervals is

1. 6
2. 10
3. 14
4. 30
5. none of the above

19. Evaluate: $\lim_{x \to \infty} \frac{2x^2 - 5x + 2}{x^2 + 5x + 6}$

1. $-\frac{1}{12}$
2. $\frac{1}{3}$
3. 1
4. 2
5. The limit does not exist.

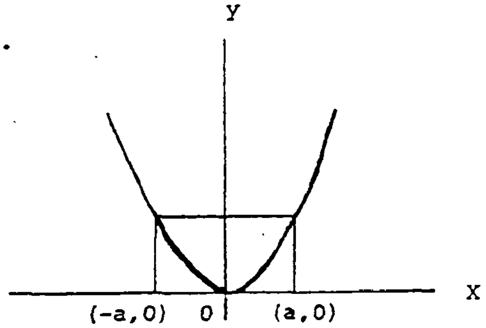
20. Which of the following statements is always CORRECT?

1. relative maximum ≤ relative minimum
2. relative minimum ≤ relative maximum
3. absolute maximum ≤ relative maximum
4. relative maximum ≤ absolute minimum
5. none of the above are always correct.

21. For what values of x is the graph of $y = x^3 - 3x + 2$ concave upward?

1. $x > 0$
2. $x < -2$
3. $-2 < x < -1$
4. $-1 < x < 0$
5. All values

22.



The figure above shows a rectangle with two of its vertices at (a,0) and (-a,0) and the other two on the parabola $y = x^2$. What fraction of the area of the rectangle lies below the parabola?

1. $\frac{1}{6}$
2. $\frac{1}{4}$
3. $\frac{1}{3}$
4. $\frac{2}{5}$
5. $\frac{1}{2}$

23. What are all values of x for which the function $f(x) = x^2 - 4x + 3$ is increasing?

1. $x < 2$
2. $x > 2$
3. $1 < x < 3$
4. $x > 1$
5. $x < 1,\ x > 3$

24. If g is the antiderivative of f, where f is continuous, then

1. $\int_b^a f(x)dx = g(b) - g(a)$
2. $\int_b^a f(x)dx = g(a) - g(b)$

24 continued

3. $\int_b^a g(x)dx = f(b) - f(a)$
4. $\int_b^a g(x)dx = f(a) - f(b)$
5. $f' = g$

25. $\int x\sqrt{2x^2 + 1}\, dx = (?)$

1. $\frac{1}{6}(2x^2 + 1)^{\frac{3}{2}} + C$
2. $\frac{1}{4}(2x^2 + 1)^{\frac{3}{2}} + C$
3. $\frac{2}{3}(2x^2 + 1)^{\frac{3}{2}} + C$
4. $\frac{x}{3}(2x^2 + 1)^{\frac{3}{2}} + C$
5. $\frac{2x^2}{\sqrt{2x^2 + 1}} + C$

26. Which of the following statements is <u>NOT</u> CORRECT?

1. if $\lim_{x\to 1} f(x) = A$, then $\lim_{x\to 1}[f(x) - A] = 0$
2. If $f(x) \le k$ for all x in (0,1), then $\lim_{x\to 1^\ell} f(x) \le k$
3. $\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = 2$
4. if $\lim_{x\to 1} g(x) \ne 0$, then $\lim_{x\to 1} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x)}$
5. $\lim_{x\to 1} f(x^2) = \lim_{x\to 1} [f(x)]^2$

27. Find y' when $y = 1/\sqrt{x}$

1. $\frac{-x^{-3/2}}{2}$
2. $\frac{-x^{-1/2}}{2}$
3. $\frac{x^{1/2}}{2}$
4. $\frac{-x^{1/2}}{2}$
5. none of the above

28. If f and g are differentiable functions of x such that $\frac{df}{dx} = \frac{dg}{dx}$ for all x, then

1. $\int f(x)dx = \int g(x)dx$
2. $\int f(x)dx + \int g(x)dx = 0$
3. $f(x) = cg(x)$
4. $f(x) = g(x + c)$
5. $f(x) = g(x) + c$

29.. What is the slope of the curve $y = x^3 - 3x^2 - 9x + 20$ at its inflection point?

1. -12
2. -7
3. 0
4. 1
5. 9

30. Which of the following statements is CORRECT?

1. the derivative of $y = f(x)$ is defined as the ratio of dy and dx.
2. if f is a differentiable function on $[a,b]$ and t an element of $[a,b]$ such that $f'(t) = 0$, then $f(t)$ is either the maximum or the minimum value of the function on the interval

30. continued

3. if $f(x)$ assumes all values between $f(a)$ and $f(b)$ then f is continuous on (a,b)
4. if f and g are differentiable functions such that $f' = g'$ then f is linear implies that g is linear.
5. the maximum value of the function $f(x) = \frac{1}{x^2 - 2a}$ occurs when $x = a$

31. if $3xy^3 = 8$, find $\frac{dy}{dx}$.

1. $\frac{8}{3y^3}$
2. $\frac{1}{9x^2}$
3. $3y^3$
4. $-\frac{3x}{y}$
5. $-\frac{y}{3x}$

32. If f is an integrable function of x, and if a, b, and c are constants such that $a < b < c$, which of the following is (are) true?

I. $\int_b^a f(x)dx = \int_a^b f(x)dx$

II. $\int_{a/3}^{b/3} f(3x)dx = \frac{1}{3}\int_a^b f(x)dx$

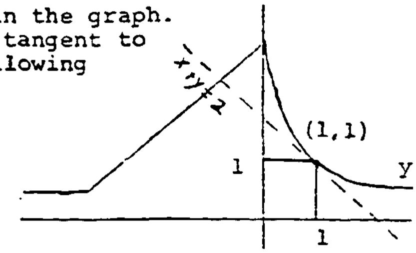
III. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

1. I only
2. I and II only
3. I and III only
4. II and III only
5. I, II, and III

33. What is the minimum value of the function $y = x^2 - 5x + 4$ on the interval $1 \le x \le 2$?

1. $-\frac{5}{4}$
2. -2
3. 0
4. 1
5. 2

34. Given the function $y = f(x)$ as shown in the graph. Let $x + y = 2$ be the equation of the tangent to the curve at $(1, 1)$. Which of the following statements about the graph is CORRECT when $x = 1$?



1. $y' \ge 0$
2. $y'' < 0$
3. $y'' \ge 0$
4. whether $y' < 0$ can not be determined without additional information.
5. whether $y'' < 0$ can not be determined without additional information.

35. The graph of which of the following functions has the X-axis as an asymptote?

1. $y = x$
2. $y = \frac{x - 1}{x}$
3. $y = \frac{x + 1}{x}$
4. $y = \frac{x}{x - 1}$
5. $y = \frac{1}{x - 1}$

36. If $f(x) \ge 0$, continuous, and increasing on the interval $a \le x \le b$, which of the following is NOT true of A, the area of the region bounded by $f(x)$, the X-axis, and the lines $x = a$ and $x = b$?

1. $A = \int_a^b f(x)\,dx$
2. $f(a)(b - a) < A < f(b)(b - a)$
3. $A > 0$
4. $A = \frac{f(b) - f(a)}{b - a}$
5. If $a < c < b$, $A = \int_a^c f(x)dx + \int_c^b f(x)dx$

APPENDIX H

TESTS FOR HOMOGENEITY OF REGRESSION
RELATIVE TO HYPOTHESES 1-4

TABLE 21

Results of Test for Homogeneity of Regression of Overall Calculus Achievement on Pretest Scores

Source	df	ss	ms	F	p
Equality of slopes	1	4.70	4.70	.23	.63
Error	42	853.08	20.31		

TABLE 22

Results of Test for Homogeneity of Regression of Skill Acquisition on Pretest Scores

Source	df	ss	ms	F	p
Equality of slopes	1	.002	.002	.000	1.00
Error	42	417.93	9.95		

TABLE 23

Results of Test for Homogeneity of Regression of Concepts Attainment on Pretest Scores

Source	df	ss	ms	F	p
Equality of slopes	1	4.53	4.53	.80	.38
Error	42	238.02	5.67		

TABLE 24

Results of Test for Homogeneity of Regression of Post-Calculus Mathematical Attitudes on Pre-calculus Attitude Test Scores

Source	df	ss	ms	F	p
Equality of slopes	1	.01	.01	.00	1.00
Error	42	4539.98	108.09		

APPENDIX I

TESTS FOR HOMOGENEITY
OF REGRESSION RELATIVE TO CONCEPT ITEMS
INVOLVING DIFFERENTATION AND INTEGRATION

TABLE 25

Results of Test for Homogeneity of Regression of Concept-differentiation Scores on Pretest Scores

Source	df	ss	ms	F	p
Equality of slopes	1	8.59	8.59	5.81	.02
Error	42	62.10	1.48		

TABLE 26

Results of Test for Homogeneity of Regression of Concept-integration Scores on Pretest Scores

Source	df	ss	ms	F	p
Equality of slopes	1	0.77	0.77	.29	.59
Error	42	110.52	2.63		

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