Random Factors and Isofactors in Graphs and Digraphs

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RANDOM FACTORS AND ISOFACTORS
IN GRAPHS AND DIGRAPHS

John Frederick Fink, Ph.D.
Western Michigan University, 1982

A factor is a spanning sub(di)graph of a (di)graph. Factors that are generated by an algorithm that incorporates an element of randomness are often called random factors. An isofactor is essentially a factor $G$ that is either empty or for which there exists a connected regular (di)graph $H$ having a nontrivial $G$-factorization. Several topics, each concerning random factors or isofactors, are investigated in this dissertation. An historical introduction to these topics is given in Chapter I.

In Chapter II we define an antipath and say that a digraph is randomly antitraceable if every nonspanning antipath can be extended (at its terminus) to a longer antipath and, therefore, to a hamiltonian antipath. We characterize randomly antitraceable digraphs.

Chapter III is devoted to randomly bitraceable graphs. A graph $G$ is randomly bitraceable if there is a bifactorization $G = R \oplus B$ such that any nonhamiltonian path whose edges alternate between $R$ and $B$ can be extended to a longer alternating path. We determine which randomly traceable graphs are randomly bitraceable and show that such graphs have unique bitraceable bifactorizations.

In Chapter IV we define randomly near-traceable graphs as a generalization of randomly traceable graphs; these are graphs that admit a special type of efficient depth-first search. We show that
all complete multipartite graphs are randomly near-traceable. We also
prove that the radius of a randomly near-traceable graph is 1 or 2.

In Chapter V we offer entirely graph-theoretic proofs that every
graph and digraph is an isofactor. We show that for every nonempty
graph $G$ there is a connected regular $G$-factorable graph $H$ with
$\chi(H) = \chi(G)$. We also show that for every asymmetric digraph $D$ there
is an asymmetric, regular, connected $D$-factorable digraph. With $r_o(G)$
denoting the minimum degree of regularity among all regular connected
$G$-factorable graphs, we show that, for $T$ a tree, $\Delta(T) \leq r_o(T) \leq$
$\Delta(T) + 1$ are sharp bounds. This and other parameters are evaluated
for several important classes of graphs.
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John Frederick Fink
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CHAPTER I

AN HISTORICAL INTRODUCTION

In 1759 Leonard Euler published a solution \[8\] to the Knight's Tour Problem — the problem being to find a tour of the 8 x 8 chessboard in which a knight, by legal moves, visits each square of the board exactly once, and then returns directly to his initial position. Formulated in the modern language of graph theory — Euler found a hamiltonian cycle in the order 64 graph that indicates the admissible moves of a knight on a chessboard. Since a hamiltonian cycle is a special type of factor, it might be said that Leonard Euler solved the first "factor-type" problem in graph theory.

Traditionally, a factor in a (di)graph \( G \) is defined to be a spanning sub(di)graph of \( G \). Since Euler's paper, the literature on problems related to various types of factors has become extensive. Perhaps the most commonly (and thoroughly) studied factors are 1-factors, hamiltonian cycles, hamiltonian paths, and spanning trees.

A 1-factor \( F \) in a graph \( G \) is a 1-regular spanning subgraph of \( G \), that is, each vertex has degree 1 in \( F \), and \( F \) contains each vertex of \( G \). Tutte [26] proved that a graph \( G \) has a 1-factor if and only if for every proper subset \( S \) of the vertex set \( V(G) \), the number of components of odd order in \( G-S \) does not exceed \( |S| \). Tutte [27] later generalized this result and characterized those graphs that, for a given degree sequence, contain a factor with that degree sequence. Other authors have characterized graphs that admit various types of "random factors."
The edges of a 1-factor are pairwise nonadjacent and therefore constitute an independent set of edges called a **perfect matching**. Sumner [23] defined a connected graph $G$ to be **randomly matchable** if every independent set of edges in $G$ can be extended to a perfect matching. Thus, if $G$ is randomly matchable, it is possible to select one edge of $G$ at random, then randomly select another edge of $G$ not adjacent with the first, proceed to select a third not adjacent with either of the first two, and so on, until we ultimately obtain an independent set of edges that induces a 1-factor in $G$ — a "random 1-factor." Sumner proved that the complete graph $K_{2n}$ and the $n$-regular complete bipartite graph $K(n,n)$ are the only randomly matchable graphs.

Chartrand and Kronk [5] investigated a class of graphs in which another type of factor, "random paths", can be randomly generated. They called a graph $G$ **randomly traceable** if it is possible to extend every nonspanning path (with a given initial vertex) in $G$ to a hamiltonian path (with the same first vertex). They went on to characterize all such graphs. In a later paper, Chartrand, Kronk and Lick [6] extended the foregoing concept to define and characterize randomly traceable digraphs. In Chapters II and III we will investigate variations on the theme of random factors.

As we shall see in Chapter IV, a notion that is closely linked to the topic of random factors in graphs is the process known as a depth-first search of a graph. The depth-first search procedure was first developed by Gaston Tarry [25] in 1895 as a method by which one may always escape from a labyrinth. Since then, the depth-first
search procedure has been applied, in various forms, to a variety of problems, most notably in computer science and graph theory. Of interest to us is the well-known fact that a depth-first search of a connected graph $G$ generates a spanning tree in $G$. We take note of the fact that the generation of this tree is accomplished with a large element of randomness; hence, the spanning tree so obtained may be considered a random factor of yet another type. We will investigate a class of graphs in which such a random factor may be generated rather efficiently.

The other topic involving factors that we discuss in this dissertation deals with the concept of isomorphic factorizations.

The first attributable result on isomorphic factorizations of graphs was obtained by König in 1918. In general, if there is a partition of the edge set of a given graph $G$ so that the subgraphs (factors) induced by the subsets in the partition are pairwise isomorphic, then we say that this partition determines an isomorphic factorization of $G$. König [15] showed that there is an isomorphic factorization of every regular bipartite graph into factors that are 1-factors; in other words, he showed that every regular bipartite graph is 1-factorable. It is also true that the complete graph $K_{2n}$ is 1-factorable for each positive integer $n$.

The complete graph $K_p$ is perhaps the most frequently studied graph from the standpoint of isomorphic factorizations. In addition to being 1-factorable, $K_{2n}$ has an isomorphic factorization consisting of $n$ hamiltonian paths. The odd order complete graph $K_{2n+1}$ admits an isomorphic factorization whose factors are hamiltonian cycles. Ringel
[12] conjectured in 1963 that for every tree $T$ of order $n+1$, there is an isomorphic factorization of $K_{2n+1}$ in which each factor is isomorphic to $T$; this conjecture, still unresolved, became the basis for Kotzig's stronger, and now famous, Graceful Tree Conjecture, (communicated by Rosa [19]).

The Graceful Tree Conjecture is just one of several general isomorphic factorization questions. Harary, Robinson and Wormald [12] asked and determined which integers $t$ satisfy the property that $K^*_p$ has as isomorphic factorization comprising $t$ factors. The same authors solved similar problems concerning isomorphic factorizations of the complete symmetric digraph $K^*_p$ in [14] and isomorphic factorizations of the regular complete bipartite graph in [13].

As documented in a survey paper by Bermond and Sotteau [3:], much of the work on isomorphic factorizations has been motivated by the study of Balanced incomplete block designs. Such was the motivation for R.M. Wilson when he proved in [28] that for every nonempty graph $G$ and sufficiently large integers $p$ (satisfying certain congruence relations), the complete graph $K^*_p$ is $G$-factorable — that is, there is an isomorphic factorization of $K^*_p$ into factors that are isomorphic to $G$. If, for a given nonempty graph $G$, there exists a connected, regular, and nontrivially $G$-factorable graph $H$, we say that $G$ is an isofactor. Thus, as a corollary to Wilson's theorem, every nonempty graph is an isofactor. In Chapter V we offer alternate, simpler proofs of this fact that, in fact, prove stronger results. We also investigate several parameters related to isofactors.
We will define all pertinent nonelementary terms as needed. All terms not defined in the text follow the meaning employed in Behzad, Chartrand and Lesniak-Foster [1].
A semipath is an asymmetric digraph whose underlying graph is a path. A semipath which contains no (directed) path of length 2 is termed an antipath. A hamiltonian antipath in a digraph D is an antipath which contains every vertex of D; thus, it is an antipath which is a factor of D. By analogy to the concepts of traceability in graphs and digraphs, a digraph D is said to be antitraceable if it contains a hamiltonian antipath.

Grünbaum [11] showed that with the exceptions of one tournament of order 3, one of order 5, and one of order 7, every tournament is antitraceable. Rosenfeld [20] strengthened this result by proving that in any tournament \( T_p \) of order \( p = 10 \) or \( p \geq 12 \), every vertex is the initial vertex of a hamiltonian antipath.

A graph or digraph in which each vertex is the initial vertex of a hamiltonian path is called homogeneously traceable. Such graphs and digraphs have been studied by Skupiën [22], Chartrand, Gould and Kapoor [4], Simoes-Periería and Zanfirescu [21], and Gould [9]. Chartrand and Kronk [5] investigated a class of graphs which includes the homogeneously traceable graphs. They called a graph G randomly traceable if for each vertex \( v \) of \( G \) every path beginning at \( v \) could be extended to a hamiltonian path beginning at \( v \).
Randomly traceable digraphs were defined in a similar manner and were characterized by Chartrand, Kronk and Lick [6].

With the foregoing in mind, we say that a digraph $D$ is randomly antitraceable if for each vertex $v$ of $D$, every antipath beginning at $v$ can be extended to a hamiltonian antipath beginning at $v$. In this chapter, randomly antitraceable digraphs will be structurally characterized.

Section 2.2

A Characterization

Central in the investigation of randomly antitraceable digraphs is the following characterization of randomly traceable graphs by Chartrand and Kronk [5].

Theorem 2A

A graph $G$ of order $p$ is randomly traceable if and only if $G$ is the cycle $C_p$, the complete graph $K_p$, or the regular complete bipartite graph $K(p/2,p/2)$, the last being possible only if $p$ is even.

This result will be used to establish our first theorem, and later to obtain other results.

Before stating and proving Theorem 2.1, we define a few terms. If $u$ and $v$ are vertices of a digraph, then an antipath that begins at $u$ and ends at $v$ is called a $u-v$ antipath. If both $(u,v)$ and $(v,u)$ are arcs in a digraph, then these two arcs
constitute a symmetric pair of arcs. A digraph is a symmetric digraph if all of its arcs occur in symmetric pairs. The complete symmetric digraph of order \( p \), denoted \( K_p^* \), is the symmetric digraph whose underlying graph is the complete graph \( K_p \). The symmetric digraph whose underlying graph is the cycle \( C_p \) is called the symmetric cycle of order \( p \) and is denoted \( C_p^* \). With these definitions, we now characterize the randomly antitraceable digraphs of odd order.

**Theorem 2.1**

A digraph \( D \) of odd order \( p \) is randomly antitraceable if and only if \( D \) is the symmetric cycle \( C_p^* \) or the complete symmetric digraph \( K_p^* \).

**Proof:** Let \( D \) be a randomly antitraceable digraph of odd order \( p \), and let \((u_1, u_2)\) be an arc of \( D \). By hypothesis, we may extend the antipath \( u_1, (u_1, u_2), u_2 \) to a hamiltonian antipath

\[
P_1: u_1, (u_1, u_2), u_2, (u_3, u_2), u_3, \ldots, u_{p-1}, (u_p, u_{p-1}), u_p.
\]

Also, the \( u_2 - u_p \) subantipath \( P_1' \) of \( P_1 \) may be extended to a \( u_2 - u_1 \) hamiltonian antipath \( P_2 \). Since the final arc of \( P_1' \) is \( (u_p, u_{p-1}) \), the last arc of \( P_2 \) must be \( (u_p, u_1) \). Now, again by hypothesis, the \( u_3 - u_1 \) subantipath \( P_2' \) of \( P_2 \) may be extended to a \( u_3 - u_2 \) hamiltonian antipath \( P_3 \). Since the last arc of \( P_2' \) is \( (u_p, u_1) \), the final arc of \( P_3 \) must be \( (u_2, u_1) \).

Since \( D \) contains the symmetric pair of arcs \( (u_1, u_2) \) and \( (u_2, u_1) \) and since \( (u_1, u_2) \) was chosen arbitrarily, we conclude...
that all arcs of $D$ occur in symmetric pairs. Since $D$ is randomly antitraceable, the underlying graph $G$ of $D$ must be a randomly traceable graph of odd order $p$. Hence, by Theorem A, the graph $G$ is the odd cycle $C_p$ or the complete graph $K_p$. This implies that $D$ is the symmetric cycle $C_p^*$ or the complete symmetric digraph $K_p^*$. It is easily verified that both $C_p^*$ and $K_p^*$ are randomly antitraceable. 

Having characterized randomly antitraceable digraphs of odd order, we turn our attention to randomly antitraceable digraphs of even order.

An asymmetric digraph is a digraph containing no symmetric pair of arcs. A semicycle is an asymmetric digraph whose underlying graph is a cycle. An anticycle is a semicycle which contains at most one (directed) path of length 2. A $p$-anticycle, denoted $A_p^*$, is an anticycle of order $p$. We note that a $p$-anticycle contains no (directed) path of length 2 if $p$ is even, and precisely one if $p$ is odd. Clearly $A_{2n}^*$ is randomly antitraceable.

Let $\tilde{K}(n,n)$ be the digraph whose vertex set can be partitioned as $U \cup V$, where $U = \{u_i | 1 \leq i \leq n\}$ and $V = \{v_i | 1 \leq i \leq n\}$, and whose arc set is $\{(u_i, v_j) | 1 \leq i, j \leq n\}$. Thus, $\tilde{K}(n,n)$ is the asymmetric digraph obtained by uniformly orienting the edges of $K(n,n)$ from one vertex partite set to the other. By noting that every semipath in $\tilde{K}(n,n)$ is an antipath, and employing Theorem 2A, we see that $\tilde{K}(n,n)$ is randomly antitraceable.
We illustrate the digraphs $A_5$, $A_6$, and $K(3,3)$ in Figure 2.1.

![digraphs A5, A6, and K(3,3)]

Figure 2.1

Let $D$ be a digraph (with vertex set $V(D)$ and arc set $E(D)$) and let $F_1$ and $F_2$ be digraphs with $V(D) = V(F_1) = V(F_2)$. If $E(D) = E(F_1) \cup E(F_2)$ and $E(F_1) \cap E(F_2) = \emptyset$, then we write $D = F_1 \oplus F_2$ and say that $D$ is the arc sum of $F_1$ and $F_2$.

Figure 2.2 shows the arc sum of a 6-anticycle and a digraph isomorphic to $K(3,3)$.

![arc sum of a 6-anticycle and K(3,3)]

Figure 2.2

We now present a characterization of randomly antitraceable bipartite digraphs.
Theorem 2.2

A bipartite digraph $D$ is randomly antitraceable if and only if

(i) $D \not\cong A_{2n}$, (ii) $D \not\cong \overline{K}(n,n)$, or (iii) $D = F_1 \oplus F_2$ where $F_1, F_2 \in \{A_{2n}, \overline{K}(n,n)\}$, and for each vertex $v$ of $D$, either $id_{F_1} v = od_{F_1} v = 0$ or $od_{F_1} v = id_{F_2} v = 0$.

Proof: We noted earlier that $A_{2n}$ and $\overline{K}(n,n)$ are randomly antitraceable. Suppose $D$ satisfies (iii). Let $(u,v)$ be the first arc of an antipath $P$ in $D$. Without loss of generality, assume that $(u,v) \in E(F_1)$. Then $od_{F_1} u \neq 0$ and $id_{F_2} v \neq 0$, so $id_{F_1} u = od_{F_2} u = 0$ and $od_{F_1} v = id_{F_2} v = 0$. Thus, if $P$ has length at least 2, the arc following $(u,v)$ on $P$ must also lie in $F_1$. Similarly, any other arcs of $P$ must lie in $F_1$. We conclude that any antipath in $D$ lies entirely in either $F_1$ or $F_2$. Since both $F_1$ and $F_2$ are randomly antitraceable, $D$ is also.

Now, assume that $D$ is randomly antitraceable and that $D \not\cong A_{2n}$ and $D \not\cong \overline{K}(n,n)$. Let $U$ and $V$ be the partite sets of $D$. Let $F_1$ and $F_2$ be the subdigraphs of $D$ having vertex sets $V(F_1) = V(F_2) = V(D)$ and arc sets $E(F_1) = \{(u,v)|u \in U, v \in V\}$ and $E(F_2) = \{(v,u)|u \in U, v \in V\}$. Then, $D = F_1 \oplus F_2$ and for each vertex $w$ of $D$, either $id_{F_1} w = od_{F_2} w = 0$ or $od_{F_1} w = id_{F_2} w = 0$. Furthermore, any antipath of $D$ lies entirely within either $F_1$ or $F_2$. Thus, $F_1$ and $F_2$ are randomly antitraceable. Since any semipath in $F_1$ or $F_2$ is an antipath, each of the underlying bipartite graphs of $F_1$ and $F_2$ must be randomly traceable and hence one of $C_{2n}$ and $K(n,n)$. Thus, $F_1, F_2 \in \{A_{2n}, \overline{K}(n,n)\}$. ■
A **hamiltonian anticycle** in a digraph $D$ is an anticycle which contains each vertex of $D$. A digraph $D$ is said to be **randomly antihamiltonian** if for each vertex $v$ of $D$, any antipath beginning at $v$ can be extended to a hamiltonian anticycle.

We now state and prove a lemma which is fundamental in the investigation of randomly antitraceable digraphs of even order.

**Lemma 2.1**

If $D$ is a randomly antitraceable digraph of order $p \geq 3$, then $D$ is randomly antihamiltonian.

**Proof:** Let $u_1$ be a vertex of a randomly antitraceable digraph $D$, and let $P$ be any hamiltonian antipath beginning at $u_1$. In addition, let $u_2$ denote the second vertex of $P$ and $u_p$ the last vertex of $P$. If $Q$ denotes the $u_2 - u_p$ subantipath of $P$, then by hypothesis there is an arc $a$ in $D$ which joins $u_p$ and $u_1$, and such that $Q$ followed by $a$ and $u_1$ is a hamiltonian antipath. Now, the antipath $P$ and the arc $a$ determine a hamiltonian anticycle in $D$. Since $P$ was an arbitrary hamiltonian antipath, we conclude that $D$ is randomly antihamiltonian. 

Proceeding further, we state the following result of Dirac and Thomassen [7].

**Theorem 2B**

A hamiltonian graph $G$ of order $p \geq 3$ has the property that each hamiltonian path in $G$ can be extended to a hamiltonian cycle.
if and only if $G$ is the cycle $C_p$, the complete graph $K_p$, or the complete bipartite graph $K(p/2, p/2)$, the last being possible only if $p$ is even.

This theorem about graphs facilitates the proof of the first of the following two lemmas concerning the digraph $K(n, n)$.

**Lemma 2.2**

If $D$ is a randomly antitraceable digraph of order $2n$ that contains an even anticycle of length less than $2n$, then $K(n, n)$ is a spanning subdigraph of $D$.

**Proof:** Consider a smallest even anticycle

$$C_1: u_1, (u_1, v_1), v_1, (u_2, v_1), u_2, \ldots, u_m, (u_m, v_m), v_m, (u_1, v_m), u_1$$

of length $2m (< 2n)$ in $D$. We note that $2 \leq m < n$. By Lemma 2.1, $D$ is randomly antihamiltonian, so we may extend the $u_1 - v_m$ antipath which spans $C_1$ to a hamiltonian anticycle

$$C_2: u_1, (u_1, v_1), v_1, \ldots, (u_m, v_m), v_m, (u_m+1, v_m), u_{m+1},$$

$$(u_{m+1}, v_{m+1}), v_{m+1}, \ldots, u_n, (u_n, v_n), v_n, (u_1, v_n), u_1.$$

Now, let $U = \{u_1, u_2, \ldots, u_n\}$, $V = \{v_1, v_2, \ldots, v_n\}$, and let $G$ be the (undirected) bipartite graph with vertex set $U \cup V$ such that $u_i$ is adjacent with $v_j$ if and only if $(u_i, v_j)$ is an arc in $D$.

Since every path or cycle in $G$ corresponds to an antipath or anticycle in $D$, it follows from Lemma 2.1 that every hamiltonian path of
G is contained in a hamiltonian cycle. Since G is neither a complete graph nor a cycle, it follows from Theorem B that G is isomorphic to K(n,n). Considering the definition of G, we then see that K(n,n) is a spanning subdigraph of D. ■

Lemma 2.3

If D is a randomly antitraceable digraph of order 2n that contains K(n,n) as a subdigraph, then D is bipartite or D is the complete symmetric digraph K^*_n.

Proof: Suppose that D is not bipartite. Since K(n,n) is a spanning subdigraph of D, the vertex set of D may be expressed as V(D) = U U V where U = \{u_i \mid 1 \leq i \leq n\}, V = \{v_i \mid 1 \leq i \leq n\}, and the sets of arcs \{(u_i,v_j) \mid 1 \leq i, j \leq n\} is contained in E(D), the arc set of D. Since D is not bipartite, there is an arc in either \langle U \rangle or \langle V \rangle, the subdigraphs induced by U and V respectively.

Assume that there is an arc (u_i, u_j) in \langle U \rangle, and let k, \ell \in \{1, 2, \ldots, n\} with k \neq \ell. Observe that D contains an antipath which begins as u_j, (u_i, u_j), u_i, then alternates between V and U, ends at v_k, and excludes only the vertex v_k. Since D is randomly antitraceable, this antipath is extendable, so (v_k, v_\ell) \in E(D). Since k and \ell were arbitrary, \langle V \rangle is isomorphic to the complete symmetric digraph K^*_n. A similar argument, with the initial assumption of an arc existing in \langle V \rangle, leads to the conclusion that U \cong K^*_n. Thus, in any case, \langle U \rangle \cong \langle V \rangle \cong K^*_n.
To see that \( D \cong K_n^* \), it remains to show that \( (v_i, u_j) \in E(D) \) for \( 1 \leq i, j \leq n \). We will consider two cases depending on the parity of \( n \).

If \( n \) is even, choose an antipath in \( \langle U \rangle \) which contains each vertex of \( U \) except \( u_j \), and which may be extended with arcs directed from \( U \) to \( V \). Add such an arc which is not incident to \( v_i \), and extend this with an antipath spanning \( \langle V \rangle \) and ending at \( v_i \). Since \( n \) is even, the last arc of this antipath is directed out from \( v_i \). Thus, since \( u_j \) is the only vertex of \( D \) not on this antipath, \( (v_i, u_j) \in E(D) \).

If \( n \) is odd, fix \( k \neq j \), and \( i \neq i \), \( 1 \leq k, \ell \leq n \). Choose an antipath in \( \langle U \rangle \) which contains each vertex of \( U \) except \( u_j \) and \( u_k \), and which may be extended with arcs directed from \( U \) to \( V \). Add such an arc incident to neither \( v_i \) nor \( v_2 \), and extend this with an antipath in \( \langle V \rangle \) which contains each vertex of \( V \) except \( v_i \) and \( v_k \). Call this antipath \( P \), and let \( v_m \) denote its last vertex. Since \( n \) is odd, \( P \) may be extended to a longer antipath \( Q \) by adding \( (u_k, v_m), (u_k, v_2), (v_i, v_2), (v_i, v_i) \). Now since \( u_j \) is the only vertex of \( D \) not on \( Q \), it follows that \( (v_i, u_j) \in E(D) \).

Since \( i \) and \( j \) are arbitrary, \( \{(v_i, u_j) | 1 \leq i, j \leq n \} \subseteq E(D) \), so \( D \cong K_n^* \).

We may now characterize the randomly antitraceable digraphs of even order.
Theorem 2.3

A digraph $D$ of even order $2n \geq 4$ is randomly antitraceable if and only if it is isomorphic to one of the digraphs $K^*_n$, $A_{2n}$, $\bar{K}(n,n)$, and $F_1 \otimes F_2$ where $F_1, F_2 \in \{A_{2n}, \bar{K}(n,n)\}$ and where for each vertex $v$ of $D$ either $id_{F_1}v = od_{F_2}v = 0$ or $od_{F_1}v = id_{F_2}v = 0$.

Proof: The sufficiency follows from Theorem 2.2 together with the observation that $K^*_n$ is randomly antitraceable. For the necessity we need only show, by Theorem 2.2, that if $D$ is randomly antitraceable and not bipartite, then $D \cong K^*_n$.

Assume then that $D$ is randomly antitraceable and not bipartite. Since $D$ contains a hamiltonian anticycle and is not bipartite, $D$ must contain an anticycle of odd length. Let $A$ be a smallest odd anticycle in $D$ and label its vertices so that it has the form

$$u_1, (u_1,v_1), v_1, (u_2,v_2), u_2, \ldots, v_{k-1},$$

$$\quad (u_k,v_{k-1}), u_k, (u_k,u_1), u_1.$$

Extend the $u_1 - u_k$ antipath which spans $A$ to the hamiltonian antipath

$$P_k: u_1, (u_1,v_1), v_1, \ldots, (u_k,v_{k-1}), u_k, (u_k,v_k),$$

$$\quad v_k, (u_{k+1},v_k), u_{k+1}, \ldots, u_n, (u_n,v_n), v_n.$$

Let $Q_k$ be the $v_n - u_1$ antipath in $D$ consisting of the $v_n - u_k$ antipath in $P_k$ followed by $(u_k,u_1)$ and $u_1$. 

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Now, note that if either \((u_i, u_i)\) or \((v_j, v_j)\) is in \(D\), where \(1 \leq i < j \leq k - 1\), then one of

\[
A_u: \ u_i, (u_i, v_i), v_i, (u_{i+1}, v_{i+1}), u_{i+1}, \ldots, v_{j-1},
\]

\[
(u_j, v_{j-1}), u_j, (u_j, u_1), u_1
\]

and

\[
A_v: \ v_1, (u_{i+1}, v_{i+1}), u_{i+1}, (u_{i+1}, v_{i+1+1}), v_{i+1}, \ldots,
\]

\[
u_j, (u_j, v_j), v_j, (v_i, v_j), v_i
\]

is in \(D\) and is an odd anticycle whose length is less than that of \(A\). Hence, \(D\) contains neither \((u_j, u_1)\) nor \((v_i, v_j)\) for \(1 \leq i < j \leq k - 1\).

From the above argument and the fact that \(Q_1\) is nonhamiltonian, we deduce that \(u_1\) must be adjacent from some \(v_j\) with \(1 \leq j \leq k - 1\). If \((v_j, u_1)\) is in \(D\) for \(2 \leq j \leq k - 1\), then the subdigraph of \(D\) comprising the \(v_j - u_k\) subantipath of \(P_1\) and \((u_k, u_1), u_1, (v_j, u_1), v_j\) is an odd anticycle whose length is less than that of \(A\). Thus, \(u_1\) must be adjacent from \(v_1\) and the arc \((v_1, u_1)\) must lie on every antipath that is an extension of \(Q_1\). If the antipath

\[
Q_1, (v_1, u_1), v_1
\]

is not hamiltonian, then \(v_1\) must be adjacent to some \(u_j\) with \(2 \leq j \leq k - 1\). If the arc \((v_1, u_j)\) is in \(D\), with \(3 \leq j \leq k - 1\), then the \(u_j - u_k\) subantipath of \(P_1\) followed by

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determines an odd anticycle

Suppose now that the arcs \((v_1, u_1), (v_1, u_2), \ldots, (v_{i-1}, u_{i-1}), (v_{i-1}, u_i)\) are known to be in \(D\) for some \(i \leq k - 1\). If \(i = k - 1\), then \(v_{k-1}\) is the only vertex of \(D\) that is not on the antipath

\[
Q_i, (v_1, u_1), v_1, (v_1, u_2), \ldots, (v_{k-2}, u_{k-2}), v_{k-2}, (v_{k-2}, u_{k-1}), u_{k-1};
\]

hence \((v_{k-1}, u_{k-1})\) must be an arc of \(D\). If \(j < k - 1\) and the arc \((v_j, u_j)\) is in \(D\), \(i < j \leq k - 1\), then the \(v_j - u_k\) subantipath of \(P_1\) followed by \((u_k, u_1), u_1, (v_1, u_1), v_1, (v_1, u_2), u_2, \ldots, v_{i-1}, (v_{i-1}, u_i), u_i, (v_j, u_j), v_j\) is an odd anticycle which is smaller than \(A\) — a contradiction. Hence, \((v_i, u_i)\) must be in \(D\). Similarly, if the successive arcs \((v_1, u_1), (v_1, u_2), \ldots, (v_{i-1}, u_{i-1}), (v_i, u_i)\) are in \(D\), with \(i < k - 1\), we conclude that \((v_i, u_{i+1})\) is in \(D\).

By successively applying the above arguments, we see that the only hamiltonian antipath to which \(Q_1\) can be extended is

\[
R: Q_1, (v_1, u_1), v_1, (v_1, u_2), u_2, \ldots, (v_{k-1}, u_{k-1}), v_{k-1}.
\]

We employ those arcs of \(R\) which do not lie on \(Q_1\) in what follows.

Extend the \(v_1 - u_1\) antipath that spans the anticycle \(A\) to a hamiltonian antipath
Using arcs of $P_2$ and the arcs of $R$ which are not on $Q_1$, construct the antipath

$$P_2: v_1, (u_2, v_1), u_2, \ldots, v_{k-1}, (u_k, v_{k-1}), u_k,$$

$$(u_k, u_1), u_1, (x_1, u_1), x_1, (x_1, x_2), x_2,$$

$$(x_3, x_2), x_3, \ldots, x_{2n-2k}, (x_{2n-2k+1}, x_{2n-2k}), x_{2n-2k+1}.$$ 

Since $u_k$ is the only vertex of $D$ not on $Q_2$, we conclude that $(v_{k-1}, u_k)$ is an arc of $D$.

Now, extend the antipath

$$u_2, (u_2, v_2), v_2, \ldots, v_{k-1}, (u_k, v_{k-1}), u_k, (u_k, u_1), u_1, (v_1, u_1), v_1$$

to a hamiltonian antipath

$$P_3: u_2, (u_2, v_2), v_2, \ldots, u_1, (v_1, u_1), v_1, (v_1, y_1), y_1, (y_2, y_1), y_2, (y_2, y_3), y_3, \ldots, y_{2n-2k},$$

$$(y_{2n-2k}, y_{2n-2k+1}), y_{2n-2k+1}.$$ 

By using arcs of $P_3$ and $R$, together with the arc $(v_{k-1}, u_k)$ whose existence was demonstrated above, we construct the antipath
Since $Q_3$ contains every vertex of $D$ except $u_1$, the arc $(u_1, u_k)$ must be in $D$.

Now, we have seen that $(v_{k-1}, u_k)$ and $(u_1, u_k)$ are arcs of $D$. The fact that both of the hamiltonian antipaths $P_1$ and $Q_1$ can be extended to hamiltonian anticycles implies that $(u_1, v_n)$ and $(v_{k-1}, v_n)$ are arcs of $D$. With these arcs, we have the 4-anticycle $u_1, (u_1, v_n), v_n, (v_{k-1}, v_n), v_{k-1}, (v_{k-1}, u_k), u_k, (u_1, u_k), u_1$ in $D$.

If $2n > 4$, it follows from Lemmas 2.2 and 2.3 that $D = K_{2n}^*$. If $2n = 4$, then the foregoing arguments and the initial labeling of the vertices of $A$ imply that $\langle\{u_1, v_{k-1}, u_k\}\rangle = \langle u_1, v_1, u_2 \rangle = K_3^*$. Using this observation, one can easily verify that $D = K_4^*$. ■

We close this chapter with the following theorem which, by combining Theorems 2.1 and 2.3, characterizes all randomly antitraceable digraphs.

**Theorem 2.4**

A digraph $D$ of order $p \geq 3$ is randomly antitraceable if and only if

(a) when $p$ is odd, either $D \not\cong C_p^*$ or $D \cong K_p^*$.
(b) when $p$ is even, $D$ is isomorphic to one of the digraphs $K_p^*$, $A_p$, $\tilde{K}(p/2,p/2)$, and $F_1 \oplus F_2$ where $F_1, F_2 \in \{A_p, \tilde{K}(p/2,p/2)\}$ and for each vertex $v$ of $D$ either $\text{id}_F v = \text{od}_F v = 0$ or $\text{od}_F v = \text{id}_F v = 0$. 

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CHAPTER III

RANDOMLY BITRACEABLE GRAPHS

If a graph $G$ is expressed as the edge sum of two factors $R$ and $B$ of $G$, then $G = R \oplus B$ is called a bifactorization of $G$. We refer to the edges of $R$ as red edges, and those of $B$ as blue edges. A path (cycle) in such a bifactorization is an alternating path (cycle) if for every pair of adjacent edges in the path (cycle), one edge is red and the other is blue. Alternating paths and cycles have been studied by Grossman and Häggkvist [10], Petersen [16], and Zahn [29].

An alternating path (cycle) which contains every vertex of the graph $G$ is called a hamiltonian alternating path (cycle). A bifactorization $G = R \oplus B$ is said to be randomly bitraceable if for each vertex $v$ of $G$, every nonhamiltonian alternating path beginning at $v$ can be extended to a longer alternating path with initial vertex $v$. A graph $G$ for which a randomly bitraceable bifactorization exists is also said to be randomly bitraceable. Figure 3.1 shows a randomly bitraceable bifactorization of $K(4,4)$.

![Figure 3.1](image)

--- red

--- blue

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Observe that a bifactorization of a graph is randomly bitraceable if and only if every nonhamiltonian alternating path (with a given initial vertex) can be extended to a hamiltonian alternating path (with the same initial vertex). Consequently, every randomly bitraceable graph is traceable. We note however that not every randomly bitraceable graph is randomly traceable; for example, Figure 3.2 illustrates a randomly bitraceable bifactorization of a graph which, by Theorem 2A, is not randomly traceable.

In addition to the fact that not every randomly bitraceable graph is randomly traceable, we shall see that not every randomly traceable graph is randomly bitraceable. In fact, we will determine precisely which randomly traceable graphs are randomly bitraceable. We shall make extensive use of the following lemma.

**Lemma 3.1**

If \( G = R \oplus B \) is a randomly bitraceable bifactorization of a graph \( G \), then \( G = R \oplus B \) contains no 4-cycle having exactly three edges of the same color (red or blue).
Proof: Assume, to the contrary, that $G = R \oplus B$ is a randomly bi-traceable bifactorization which contains a 4-cycle $u_1, u_2, u_3, u_4, u_1$ having exactly three edges of the same color. Without loss of generality, we assume that edges $u_1u_2$, $u_1u_4$, and $u_3u_4$ are blue while $u_2u_3$ is red (see Figure 3.3).

![Figure 3.3](image)

By hypothesis we may extend the alternating path $u_1, u_2, u_3, u_4$ to a hamiltonian alternating path

$u_1, u_2, u_3, u_4, u_5, \ldots, u_{p-1}, u_p$.

Since $G = R \oplus B$ is a randomly bi-traceable bifactorization, the alternating path

$P: u_p, u_{p-1}, \ldots, u_5, u_4, u_1$,

which contains all vertices of $G$ except $u_2$ and $u_3$, can be extended to a hamiltonian alternating path $P'$. Since $u_2$ and $u_3$ are the only vertices of $G$ not on $P$, the last edge of $P'$ must be $u_2u_3$, a red edge. But, since the last edge of $P$, namely $u_1u_4$, is blue, the last edge of $P'$ must be blue. This produces a
Another useful result is the following.

**Lemma 3.2**

Every nontrivial randomly bitraceable graph has even order.

**Proof:** Assume, to the contrary, that there is a graph $G$ of odd order $p = 2k + 1$, $k \geq 1$, which has a randomly bitraceable bifactorization $G = R \oplus B$. Let $u_1, u_2, \ldots, u_{2k}, u_{2k+1}$ be a hamiltonian alternating path in $G$ and assume, without loss of generality, that for $1 \leq j \leq k$, edges $u_{2j-1}u_{2j}$ are blue and edges $u_{2j}u_{2j+1}$ are red.

Since $u_1$ is the only vertex of $G$ not on the alternating path

$$u_2, u_3, \ldots, u_{2k}, u_{2k+1},$$

and since the edge $u_{2k}u_{2k+1}$ is red, we see that $u_1u_{2k+1}$ is in $G$ and is blue. Similarly, by examining the alternating path

$$u_3, u_4, \ldots, u_{2k}, u_{2k+1}, u_1,$$

we see that the edge $u_1u_2$ must be red. This contradicts the assumption that $u_1u_2$ is blue and completes the proof. ■

Before stating the next lemma, we introduce additional terminology. A bifactorization $G = R \oplus B$ of a graph $G$ is said to be *randomly bihamiltonian* if for each vertex $v$, every alternating path beginning at $v$ can be extended to a hamiltonian alternating cycle (with initial vertex $v$). A graph $G$ which has a randomly bihamil-
tonian bifactorization is itself said to be randomly bihamiltonian. The next lemma establishes the equivalence of the concepts "randomly bihamiltonian" and "randomly bitraceable."

Lemma 3.3

A graph $G$ of order $p \geq 4$ is randomly bitraceable if and only if it is randomly bihamiltonian.

Proof: It is an immediate consequence of the definitions that if $G$ is randomly bihamiltonian, then $G$ is randomly bitraceable.

Suppose that $G$ is randomly bitraceable. By Lemma 3.2, $G$ has even order, say $p = 2k$, where $k \geq 2$. Let $G = R \oplus B$ be a randomly bitraceable bifactorization of $G$, and let $u_1, u_2, \ldots, u_{2k}$ be a Hamiltonian alternating path in $G = R \oplus B$. Without loss of generality, assume that edges $u_{2j-1}u_{2j}$ are blue $(1 \leq j \leq k)$, and edges $u_{2j}u_{2j+1}$ are red $(1 \leq j \leq k-1)$. Since $u_1$ is the only vertex of $G$ not on the alternating path $u_2, u_3, \ldots, u_{2k}$, we see that $u_1u_{2k}$ is an edge of $G$ and, in fact, a red edge. Thus,

$$C: u_1, u_2, \ldots, u_{2k}, u_1$$

is an extension of the given Hamiltonian alternating path to a Hamiltonian alternating cycle. Since for each vertex $v$ of $G$ any alternating path beginning at $v$ can be extended to a Hamiltonian alternating path beginning at $v$, and we have seen that any Hamiltonian alternating path can be extended to a Hamiltonian alternating cycle, we conclude that $G = R \oplus B$ is randomly bihamiltonian. ■
From Lemma 3.3 it follows that every randomly bitraceable bi-factorization of a graph contains a hamiltonian alternating cycle. This observation will prove useful in establishing several results in this chapter. One such result is the following lemma which asserts that the randomly traceable graphs $K_n$ and $K(n,n)$ are not always randomly bitraceable.

**Lemma 3.4**

For every odd integer $n \geq 3$, the graphs $K_{2n}$ and $K(n,n)$ are not randomly bitraceable.

**Proof:** Let $n = 2m + 1$, $(m$ a positive integer), and let $G$ denote either $K_{2n}$ or $K(n,n)$. Assume that $G = R \@ B$ is a randomly bitraceable bi-factorization of $G$, and let

$$u_1, u_2, \ldots, u_{4m+2}, u_1$$

be a hamiltonian alternating cycle in $G = R \@ B$. Without loss of generality, assume that the edge $u_ju_{j+1}$ is blue if $j$ is odd and red if $j$ is even; $1 \leq j \leq 4m + 2$, subscripts modulo $4m + 2$.

Whether $G = K_{2n}$ or $G = K(n,n)$, it is easily seen that all edges of the form $u_ju_{j+3}$ are in $G$ for $1 \leq j \leq 4m + 2$, subscripts modulo $4m + 2$. Moreover, if $u_{j+1}u_{j+2}$ is red then so too is $u_ju_{j+3}$, and if $u_{j+1}u_{j+2}$ is blue then $u_ju_{j+3}$ is blue; otherwise

$$u_j, u_{j+1}, u_{j+2}, u_{j+3}, u_j$$

is a 4-cycle having exactly three edges of the same color, contrary to
the statement of Lemma 3.1. In particular, for $k = 1, 2, \ldots, m$, all edges of the form $u_{4k-3}u_{4k}$ are red while those of the form $u_{4k-2}u_{4k+1}$ are blue. With this knowledge, we see that

$$P: u_1, u_4, u_3, u_2, u_5, u_8, u_7, u_6, \ldots,$$

$$u_{4k-3}, u_{4k}, u_{4k-1}, u_{4k-2}, \ldots,$$

$$u_{4m-3}, u_{4m}, u_{4m-1}, u_{4m-2}, u_{4m+1}$$

is an alternating path which contains every vertex of $G$ except $u_{4m+2}$. Thus, since $G = R \oplus B$ is a randomly bitraceable bifactorization and since $u_{4m-2}u_{4m+1}$, the last edge of $P$, is blue, the edge $u_{4m+1}u_{4m+2}$ must be red. This contradicts the fact that $u_{4m+1}u_{4m+2}$ is blue. Hence $G$ is not randomly bitraceable. \[ \blacksquare \]

With the foregoing lemmas, it is possible to establish two theorems which state precisely when the graphs $K(n,n)$ and $K_p$ are randomly bitraceable.

**Theorem 3.1**

The complete bipartite graph $K(n,n)$, $n \geq 2$, is randomly bitraceable if and only if $n$ is even.

**Proof:** If $K(n,n)$ is randomly bitraceable and $n \geq 2$, then it follows from Lemma 3.4 that $n$ is even.

For the converse, suppose $n$ is even and $n = 2m$. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the partite sets of $K(n,n)$ and define a bifactorization $K(n,n) = R \oplus B$ by prescribing
that $u_i v_j$ be red if and only if $i$ and $j$ have the same parity. We shall show that this bifactorization is randomly bitraceable.

Let $P: x_1, x_2, \ldots, x_k$ be an alternating path in $K(n,n) = R \oplus B$ that cannot be extended to a longer alternating path beginning at $x_1$. By the symmetry of $K(n,n)$, we may, without loss of generality, assume that $x_k \in V$, and say $x_k = v_k$. We now show that $P$ is a hamiltonian alternating path.

First, suppose that $x_{k-1} x_k$ is red. Since $P$ cannot be extended to a longer alternating path with initial vertex $x_1$, the $m$ vertices in the set

$$U_o = \{u_i \mid i \text{ and } j \text{ are of opposite parity}\}$$

are on $P$. The parity relationships imposed by the definition of the bifactorization $R \oplus B$ together with the observation that $x_j \in U$ if and only if $k - j$ is odd imply that in fact

$$U_o = \{x_{k-4t+1} \mid t = 1, 2, \ldots, m\}.$$  

Since $k \leq 4m = 2n$ and $k - 4m + 1 > 1$, we see that $k = 4m$.

Hence $P$ is a hamiltonian alternating path.

If then, $x_{k-1} x_k$ is blue and $P$ cannot be extended to a longer alternating path with initial vertex $x_1$, the $m$ vertices in the set

$$U_s = \{u_i \mid i \text{ and } j \text{ are of the same parity}\}$$

are on $P$. Furthermore, the parity relationships determined by the
bifactorization and the fact that $x_j \in U$ if and only if $k - j$ is odd dictate that

$$U_s = \{x_{k-4t+1} | t = 1, 2, \ldots, m\}.$$ 

As before, the inequalities $k \leq 4m$ and $k - 4m + 1 \geq 1$ imply that $k = 4m$. Hence $P$ is a hamiltonian alternating path, and we conclude that $K(n,n) = R \oplus B$ is randomly bitraceable. ■

**Theorem 3.2**

The complete graph $K_p$, $p \geq 3$ is randomly bitraceable if and only if $p \equiv 0 \pmod{4}$.

**Proof:** If $K_p$, where $p \geq 3$, is randomly bitraceable, then by Lemma 3.2, $p$ is even, say $p = 2n$. By Lemma 3.4, $K_{2n}$ ($n \geq 2$) is not randomly bitraceable if $n$ is odd. Hence $n$ is even and $p \equiv 0 \pmod{4}$.

For the converse, suppose $p = 4n$, where $n$ is a positive integer, and partition the vertex set of $K_p$ into two subsets, each of cardinality $2n$. Define a bifactorization $K_p = R \oplus B$ as follows: let $R$ be the $2n$-regular complete bipartite graph having partite sets $U$ and $V$, and let $B$ be the (union of two complete graphs, one with vertex set $U$ and the other with vertex set $V$, that constitute the) complement of $R$. We wish to show that $K_p = R \oplus B$ is a randomly bitraceable bifactorization of $K_p$.

Let $P: w_1, w_2, \ldots, w_m$ be an alternating path in $K_p = R \oplus B$ which cannot be extended to a longer alternating path having initial
vertex \( w_1 \). We shall consider two cases, depending on whether
\( w_{m-1}w_m \) is red or blue. In these cases we will use the observation
that if \( w_jw_{j+1} \) and \( w_{j+2}w_{j+3} \) are blue edges on \( P \) then one of
these edges joins two vertices of \( U \) and the other joins two vertices
of \( V \); that is, the "consecutive" blue edges alternately belong to the
subgraph induced by \( U \) and the subgraph induced by \( V \).

**Case 1:** Suppose the last edge \( w_{m-1}w_m \) of \( P \) is blue. By the
symmetry within the factors \( R \) and \( B \) in the bifactorization, we may
without loss of generality assume that \( w_{m-1} \) and \( w_m \) belong to \( V \).
Since \( P \) cannot be extended to a longer alternating path having initial
vertex \( w_1 \), all vertices joined by a red edge to \( w_m \), that is all
vertices of \( U \), lie on \( P \). Since \( w_{m-1}w_m \) joins two vertices of \( V \),
we may apply the earlier observation about consecutive edges on \( P \)
to see that

\[ U = \{w_{m-4k+1}w_{m-4k+2} | k = 1, 2, ..., n\}. \]

Since \( m < 4n \) and the subscript \( m - 4n + 1 \geq 1 \), we conclude that
\( m = 4n \) and that \( P \) is a hamiltonian alternating path.

**Case 2:** Suppose the last edge \( w_{m-1}w_m \) of \( P \) is red and as­
sume, without loss of generality, that \( w_m \in U \). Since \( P \) cannot be
extended to a longer alternating path whose first vertex is \( w_1 \), all
vertices joined by a blue edge to \( w_m \), that is all vertices of \( U \),
lie on \( P \). By again applying the foregoing remark concerning con­
secutive blue edges on \( P \), and the fact that \( w_m \in U \), we see that
Moreover, since \( m \leq 4n \) and \( m - 4n + 1 \geq 1 \), we see that \( m = 4n \) and \( P \) is a hamiltonian alternating path.

We conclude that \( K_p = R \oplus B \) is randomly bitraceable. ◼

As stated in Theorem 2A, the randomly traceable graphs are \( K_p \), \( K(p/2,p/2) \), and \( C_p \). It is easily seen that \( C_p \) is randomly bitraceable if and only if \( p \) is even. Thus, as a corollary to Theorems 3.1 and 3.2, we have the following.

**Corollary 3.1**

A graph \( G \) of order \( p \geq 4 \) is randomly traceable and randomly bitraceable if and only if \( p \) is even and \( G \) is the cycle \( C_p \), or \( p \equiv 0 \) (mod 4) and \( G \) is isomorphic to either the complete graph \( K_p \) or the complete bipartite graph \( K(p/2,p/2) \).

It is easy to see that if \( p \) is even then there is a unique randomly bitraceable bifactorization of \( C_p \). Perhaps more surprising are the following two theorems about \( K(n,n) \) and \( K_p \).

**Theorem 3.3**

If \( K(n,n) \) is randomly bitraceable, then it has a unique randomly bitraceable bifactorization.
Proof: If $K(n,n)$ is randomly bitraceable and $n \geq 2$, then $n$ is even by Lemma 3.4. Let

$$C: w_1, w_2, \ldots, w_{2n}, w_1$$

be a hamiltonian alternating cycle in a randomly bitraceable bifactorization $K(n,n) = R \oplus B$. Without loss of generality, assume that $w_iw_{i+1}$ is red if and only if $i$ is odd; $i = 1, 2, \ldots, 2n$, (subscripts modulo $2n$). Note that for $i = 1, 2, \ldots, 2n$ and $k = 1, 2, \ldots, n$, the edge $w_iw_{i+2k-1}$ is in $K(n,n)$; in fact, each edge in $K(n,n)$ can be expressed in this form. We shall show that the color of each of these edges is uniquely determined by the colors of the edges on $C$.

Suppose $i$ is odd, where $1 \leq i < 2n$. Then, as stated earlier $w_iw_{i+1}$ is red; that is, if $k = 1$, then edge $w_iw_{i+2k-1}$ is red. Since

$$w'_i, w'_{i+1}, w'_{i+2}, w'_{i+3}, w'_i$$

is a 4-cycle in which $w_iw_{i+1}$ and $w_{i+2}w_{i+3}$ are red while $w_{i+1}w_{i+2}$ is blue, Lemma 3.1 implies that $w_iw_{i+3}$ is blue; that is, if $k = 2$, then $w_iw_{i+2k-1}$ is blue. Proceeding inductively, with the hypothesis that $w_iw_{i+2(k-1)-1}$ is blue if $k$ is odd and red if $k$ is even, we note the colors of the edges in the 4-cycle

$$w'_i, w'_{i+2(k-1)-1}, w'_{i+2(k-1)}, w'_{i+2k-1}, w'_i$$

and invoke Lemma 3.1 to infer that (when $i$ is odd) $w_iw_{i+2k-1}$ is red if and only if $k$ is odd ($k = 1, 2, \ldots, n$). A similar analysis
shows that if $i$ is even, then $w_iw_{i+2k-1}$ is blue if and only if $k$ is odd ($k = 1, 2, \ldots, n$).

Since the bifactorization $R \oplus B$ and the alternating Hamiltonian cycle $C$ were arbitrary, and since the colors of the edges on $C$ determine the color of each other edge in $K(n,n)$, we conclude that $K(n,n)$ has a unique randomly bitraceable bifactorization. ■

**Theorem 3.4**

If $K_p$ is randomly bitraceable, then it has a unique randomly bitraceable bifactorization.

**Proof:** If $K_p$ is randomly bitraceable and $p \neq 2$, then $p \equiv 0 \pmod{4}$ by Theorem 3.2. Suppose then that $p = 2n$ where $n$ is even. Let

$$C: w_1, w_2, \ldots, w_{2n}, w_1$$

be a Hamiltonian alternating cycle in a randomly bitraceable bifactorization $K_p = R \oplus B$ and, without loss of generality, assume that the edge $w_iw_{i+1}$ is red if and only if $i$ is odd; $i = 1, 2, \ldots, 2n$ (subscripts modulo $2n$).

The arguments used in the proof of Theorem 3.3 show that

(i) if $i$ is odd, then $w_iw_{i+2k-1}$ is red if and only if $k$ is odd ($k = 1, 2, \ldots, n$),

and

(ii) if $i$ is even, then $w_iw_{i+2k-1}$ is blue if and only if $k$ is odd ($k = 1, 2, \ldots, n$).
Thus, it remains to investigate the edges of the form $w_i w_{i+2k}$ for $i = 1, 2, ..., 2n$ and $k = 1, 2, ..., n$ (subscripts modulo $2n$).

First consider the edges $w_i w_{i+2}$. Symmetry considerations allow us to assume, without loss of generality, that the edge $w_1 w_3$ is red. Proceeding inductively, with the hypothesis that $w_i w_{i+1}$ is red $(i \geq 2)$, we examine the $4$-cycle 

$$w_{i-1}, w_i, w_{i+2}, w_{i+1}, w_{i-1}$$

noting that $w_i w_1$ and $w_i w_{i+2}$ share the same color (red or blue) while $w_{i-1} w_{i+1}$ is red, and deduce from Lemma 3.1 that $w_i w_{i+2}$ must be red for each $i = 1, 2, ..., 2n$. Restated, if $k = 1$, then $w_i w_{i+2k}$ is red for $i = 1, 2, ..., 2n$.

Now, on the $4$-cycle

$$w_i, w_{i+2(k-1)}, w_{i+2k-1}, w_{i+2k}, w_i$$

one of the edges $w_{i+2(k-1)} w_{i+2k-1}$ and $w_{i+2k-1} w_{i+2k}$ is red and the other is blue. Hence, by Lemma 3.1, the edges $w_i w_{i+2(k-1)}$ and $w_i w_{i+2k}$ must have opposite colors. Thus, for each $i = 1, 2, ..., 2n$, we may show inductively that $w_i w_{i+2k}$ is red if and only if $k$ is odd $(k = 1, 2, ..., n)$.

We have now seen that the color of each edge in $K_p$ is essentially determined by the colors of the edges of $C$. Since $R \bowtie B$ and $C$ were arbitrary, we conclude that $K_p$, where $p \equiv 0 \ (\text{mod} \ 4)$, has a unique randomly bitraceable bifactorization.

We have now seen that, when they are randomly bitraceable, the graphs $C_p$, $K_p$ and $K(p/2, p/2)$ have unique randomly bitraceable
bifactORIZATIONS. It is also easily verified that the graph of Figure 3.2 has a unique randomly bitraceable bifactorization. In light of these results and other examples investigated by the author, we close this chapter with the following conjecture.

**Conjecture 3.1**

Every randomly bitraceable graph has a unique randomly bitraceable bifactorization.
CHAPTER IV

RANDOMLY NEAR-TRACEABLE GRAPHS

Section 4.1

Introduction

In this chapter we will define and investigate a class of graphs we refer to as randomly near-traceable graphs. The development of this topic will be based on the concept of a depth-first search of a connected graph. We will see that the definition of a randomly traceable graph can also be stated in terms of the depth-first search procedure and, as a consequence, that every randomly traceable graph is randomly near-traceable.

A depth-first search of a connected graph \( G \) is a step-by-step method for generating a walk that visits (i.e. includes) each vertex of \( G \). At a given step in a depth-first search of \( G \), the vertex which is currently being visited is designated the active vertex.

To begin a depth-first search of a connected graph \( G \), we randomly select a first vertex to visit — this is the first active vertex and the first vertex of our walk. Next we select, at random, a vertex adjacent to our first active vertex and visit it — this becomes the new active vertex and the second vertex in our walk. In general, if \( v_a \) denotes the current active vertex in our search, and if the walk generated so far is not a spanning walk, we proceed as follows. If there are unvisited vertices adjacent with \( v_a \), select one at random, visit it, designate it the new active vertex, and append it to our walk.
If each vertex adjacent with $v_a$ has been visited, we backtrack to (i.e. revisit) the vertex that was the active vertex immediately before $v_a$ was first visited, designate this the current active vertex and add it to our walk. We repeat the foregoing general procedure (using the new active vertex) until each vertex of $G$ has been visited. As soon as each vertex has been visited, the depth-first search terminates.

A walk generated by a (not necessarily completed) depth-first search of a graph is called a depth-first search walk, or, more briefly, a DFS walk. We see now that a graph $G$ is randomly traceable if and only if $G$ is connected and every DFS walk in $G$ is a path. Thus, every completed depth-first search of a randomly traceable graph yields a hamiltonian path in $G$. Reformulated, a connected graph $G$ is randomly traceable if and only if every depth-first search of $G$ is completed without backtracking (i.e. revisiting a vertex).

If no depth-first search walk $W: w_1, w_2, \ldots, w_n$ in a connected graph $G$ contains consecutive vertices $w_k$ and $w_{k+1}$ both of which appear on the subwalk $w_1, w_2, \ldots, w_{k-1}$ of $W$, then $G$ is said to be randomly near-traceable. Thus, in a depth-first search of a randomly near-traceable graph, whenever we backtrack to a previously visited vertex, that vertex is adjacent to at least one unvisited vertex. To illustrate this concept, we will demonstrate that, of the two graphs in Figure 4.1, only $G_1$ is randomly near-traceable.
To show that \( G_1 \) is randomly near-traceable, it suffices by symmetry to examine only those DFS walks which begin at \( u_1 \) and proceed next to either \( u_2 \) or \( u_6 \). It is easily seen that every DFS walk which begins as \( u_1, u_6 \) is a Hamiltonian path. Hence, we now consider only those DFS walks which begins as \( u_1, u_2 \). The two DFS walks which begin as \( u_1, u_2, u_3 \) are Hamiltonian paths. Thus, we now consider those DFS walks which begin as \( u_1, u_2, u_5 \); these are

\[
W_1: u_1, u_2, u_5, u_6, u_4, u_3, \\
W_2: u_1, u_2, u_3, u_4, u_6, u_4, u_3.
\]

and

\[
W_3: u_1, u_2, u_5, u_4, u_6, u_4, u_3.
\]

Among these walks, backtracking occurs only in \( W_2 \) and \( W_3 \). In each of these two walks, we backtrack to \( u_4 \) and find an unvisited vertex (\( u_6 \) or \( u_3 \) respectively) adjacent to it and continue our depth-first search by visiting that vertex. We conclude that \( G_1 \) is randomly near-traceable.
To see that $G_2$ is not randomly near-traceable, we consider the nonspanning DFS walk

$$v_1, v_4, v_5, v_2, v_3$$

in $G_2$. Since each vertex adjacent to $v_3$ is on this walk, it is necessary now to backtrack to $v_2$. However, it is now the case that every vertex adjacent with the previously visited vertex $v_2$ has already been visited. Thus, $G_2$ is not randomly near-traceable.

By definition, whether or not a given graph is randomly near-traceable depends on the structural characteristics of each depth-first search of the graph. As we shall see from the next lemma, a depth-first search walk in a randomly near-traceable graph has a very explicit structure.

Lemma 4.1

A graph $G$ is randomly near-traceable if and only if every spanning depth-first search walk $W: w_1, w_2, \ldots, w_n$ in $G$ is either a hamiltonian path or satisfies the condition that for some integer $k$, with $1 \leq k \leq n-3$, the subwalk $w_1, w_2, \ldots, w_{k+1}$ is a path, $w_k = w_{k+2} = w_{k+4} = \ldots = w_{n-1}$, and $\{w_{k+1}, w_{k+3}, w_{k+5}, \ldots, w_n\}$ is an independent set of vertices.

Proof. Suppose that each depth-first search walk in $G$ is either a hamiltonian path or satisfies the condition stated above. Then, on no DFS walk $W: w_1, w_2, \ldots, w_n$ do there appear consecutive vertices $w_k$ and $w_{k+1}$ which are both visited on the subwalk $w_1, w_2, \ldots, w_{k-1}$. Hence, by definition, $G$ is randomly near-traceable.
For the converse, suppose that \( W: w_1, w_2, \ldots, w_n \) is a spanning DFS walk in a randomly near-traceable graph \( G \), and that \( W \) is not a hamiltonian path. Let \( k \) be the largest integer for which

\( P: w_1, w_2, \ldots, w_{k+1} \) is a path. Since \( P \) cannot be extended to a longer path beginning at \( w_1 \), each vertex of \( G \) which is adjacent to \( w_{k+1} \) must be on \( P \). Also, since \( G \) is randomly near-traceable, it follows that \( w_k = w_{k+2} \) and that the vertex \( w_{k+3} \) is not on \( P \).

Note that if \( i < k \), then \( w_i \) is visited only once on \( W \). To see this, assume to the contrary that for some \( i < k \), vertex \( w_i \) is revisited on \( W \). Let \( \ell \) be chosen so that \( w_{\ell} \) represents the second occurrence of \( w_i \) on \( W \). Since \( P \) is a path, and since \( w_{\ell-2} \) and \( w_{\ell+3} \) is not on \( P \), it follows that \( \ell > k + 3 \).

Since \( G \) is randomly near-traceable, it follows from definition that the vertex labelled \( w_{\ell-1} \) does not occur on \( w_1, w_2, \ldots, w_{\ell-2} \).

Thus, on the initial visit to \( w_{\ell-1} \) in the depth-first search corresponding to \( W \), it was necessary to backtrack to the vertex preceding \( w_{\ell-1} \) on \( W \), namely \( w_{\ell-2} \). Hence \( w_{\ell-2} = w_{\ell} = w_{\ell-1} \). This, however, is a contradiction since \( i < \ell - 2 < \ell \) implies that \( w_{\ell} \) does not represent the second occurrence of \( w_i \) on \( W \).

Now, since

\[
\begin{align*}
  w_1, w_2, \ldots, w_k, w_{k+3}, w_{k+4}, \ldots, w_{n-1}, w_n
\end{align*}
\]

is a DFS walk containing every vertex of \( G \) except \( w_{k+1} \) and since \( w_{k+1} \) is not adjacent to \( w_n \), it follows that \( w_{k+1} \) must be adjacent with \( w_{n-1} \). Since \( w_{k+1} \) is adjacent only to vertices on
P and since none of the vertices \( w_1, w_2, \ldots, w_{k-1} \) is revisited on \( W \), we see that \( w_{n-1} = w_k \).

Since \( G \) is randomly near-traceable and since \( w_{n-1} \) does not represent the first occurrence of \( w_k \) on \( W \), it follows that \( w_{n-2} \) does not appear on the subwalk \( w_1, w_2, \ldots, w_{n-3} \). Thus, \( w_{n-3} = w_{n-1} = w_k \). By continuing to argue in the above manner, we conclude that \( (n-1) - k \) is an even number and that
\[
\cdots = w_{k+2} = w_k.
\]

Since \( w_k = w_{k+2} = w_{k+4} = \cdots = w_{n-1} \), we see that \( w_{k+2j-1} \) is not adjacent to \( w_{k+2j-1} \) for any integers \( j \) and \( \ell \) where \( 1 \leq j < \ell \leq (n-k+1)/2 \). Hence \( \{w_{k+1}, w_{k+3}, w_{k+5}, \ldots, w_n\} \) is an independent set of vertices. ■

If \( G, W \) and \( k \) are as in the statement of Lemma 4.1, then we will usually denote the BFS walk \( W \) by
\[
W: w_1, w_2, \ldots, w_k \rightarrow (w_{k+1}, w_{k+3}, \ldots, w_n).
\]

Usually we will label the \( i \)-th newly visited vertex in \( W \) as \( u_i \). Hence, if \( G \) has order \( p \), then
\[
W: u_1, u_2, \ldots, u_k \rightarrow (u_{k+1}, u_{k+2}, \ldots, u_p)
\]
denotes the DFS walk
\[
W: u_1, u_2, \ldots, u_{k-1}, u_k, u_{k+1}, u_k, u_{k+2}, u_k, u_{k+3}, \ldots, u_{p-1}, u_k, u_p.
\]
The tree induced by a spanning DFS walk $W$ in a randomly near-traceable graph $G$ will, by Lemma 4.1, necessarily have one of the forms illustrated in Figure 4.2.

Figure 4.2
Also from Lemma 4.1, it follows that the DFS walk in a randomly near-traceable graph which results from an incomplete depth-first search in which at least one backtrack step has occurred has the form

\[ u_1, u_2, \ldots, u_k, u_{k+1}, u_k, u_{k+2}, \ldots, u_k, u_{k+m}; \]

this will be denoted by

\[ u_1, u_2, \ldots, u_k + (u_{k+1}, \ldots, u_{k+m}); \]

Section 4.2

n-Partite Randomly Near-Traceable Graphs

Since there is no backtracking in a depth-first search of a randomly traceable graph, it follows that every randomly traceable graph is randomly near-traceable. Thus, by Theorem 2A every cycle, complete graph, and regular complete bipartite graph is randomly near-traceable. The complete graphs and regular complete bipartite graphs are therefore examples of randomly near-traceable complete n-partite graphs (for appropriate values of n). The following theorem asserts that in fact every complete n-partite graph is randomly near-traceable.

**Theorem 4.1**

Every complete n-partite graph is randomly near-traceable (n ≥ 2).

**Proof:** Let \( W: w_1, w_2, \ldots, w_n \) be a spanning depth-first search walk in a complete n-partite graph \( G \), where \( n ≥ 2 \). Suppose that \( W \) is not a hamiltonian path, and let \( k \) be the least integer such that
vertex \( v_k \) is visited more than once on \( W \). Then, since \( W \) is a DFS walk, the subwalk

\[
W_0: w_1, w_2, \ldots, w_k, v_{k+1}
\]

is a path which cannot be extended to a longer path beginning at \( w_1 \). Thus \( w_k = w_{k+2} \) and each vertex of \( G \) that is not on \( W_0 \) is not adjacent to \( w_{k+1} \); hence each vertex not on \( W_0 \) belongs to the partite set of \( G \) that contains \( w_{k+1} \). Since \( G \) is a complete \( n \)-partite graph, this implies that each vertex not on \( W_0 \) is adjacent to \( w_k \) and that \( w_k = w_{k+2} = w_{k+4} = \ldots = w_{t-1} \) while \( w_{k+1}, w_{k+3}, w_{k+5}, \ldots, w_t \) are distinct. Since \( W \) was an arbitrary DFS walk in \( G \), we conclude that \( G \) is randomly near-traceable.

Thus, randomly traceable graphs and complete \( n \)-partite graphs are randomly near-traceable. These are however not the **only** such graphs. For example, the graph \( G_1 \) of Figure 4.1 and the graph obtained by joining one pair of nonadjacent vertices in \( K(m,m) \), where \( m \geq 3 \), are examples of tripartite randomly near-traceable graphs. Thus Theorems 2A and 4.1 do not provide **all** examples of randomly near-traceable tripartite graphs. As the next theorem shows, however, they **do** include all bipartite randomly near-traceable graphs.

**Theorem 4.2**

A bipartite graph \( G \) is randomly near-traceable if and only if \( G \) is a cycle or a complete bipartite graph.
Proof: Let $G$ be a randomly near-traceable bipartite graph. Let $U$ and $V$ be the partite sets of $G$ and suppose that $|U| \leq |V|$. Furthermore, suppose that $G$ is neither a cycle nor a regular complete bipartite graph; hence, by Theorem 2A, the graph $G$ is not randomly traceable.

Since $G$ is not randomly traceable, there is a spanning DFS walk $W$ of $G$ having the form

$$W: x_1, x_2, \ldots, x_k \rightarrow (x_{k+1}, x_{k+2}, \ldots, x_p)$$

where $p$ is the order of $G$. [Note that if $k = 1$, then $G = K(l, p-1)$; hence we shall assume that $k \geq 2$.] Since $p-k \geq 2$, and since $G$ is bipartite with $|U| \leq |V|$, we conclude that $|U| < |V|$ and that $x_{k+1}, x_{k+2}, \ldots, x_p \in V$. Moreover, it follows that $x_{k-j} \in V$ if and only if $j$ is odd. It is this condition that leads us to consider two cases.

Case 1: Suppose $k = 2n$. We relabel the vertices as follows:

for $1 \leq i \leq n$ we set $v_i = x_{2i-1}$ and $u_i = x_{2i}$; for $j = 1, 2, \ldots, p-k$ we set $v_{n+j} = x_{k+j}$. Thus $W$ has the form

$$W: v_1, u_1, v_2, u_2, \ldots, v_n, u_n \rightarrow (v_{n+1}, v_{n+2}, \ldots, v_{n+p-k})$$

The tree induced by $W$ is illustrated in Figure 4.3.

![Figure 4.3](image-url)
Observe now that \( U = \{u_1, u_2, \ldots, u_n\} \) and \( V = \{v_1, v_2, \ldots, v_{n+p-k}\} \). With this fact and the nonspanning DFS walk

\[ u_1, v_2, u_2, v_3, \ldots, v_n, u_n \]

we see that \( u_1 v_1 \) is an edge of \( G \). Using this edge to construct the nonspanning DFS walk

\[ v_1, u_n, v_n, u_{n-1}, \ldots, v_2, u_1 \]

we see that \( u_1 v_{n+j} \) is an edge of \( G \) for \( 1 \leq j \leq p-k \).

We now proceed inductively to show that for \( i = 1, 2, \ldots, n-1 \) the edges \( u_i v_1 \) and \( u_i v_{n+j} \) where \( 1 \leq j \leq p-k \) are in \( G \). Assuming that \( u_{i-1} v_1 \) and all edges \( u_{i-1} v_{n+j} \) are in \( G \), we consider the nonspanning DFS walks

\[ v_i, u_{i-1}, v_{i-1}, u_{i-2}, \ldots, v_2, u_1, v_{n+k}, u_n, v_n, u_{n-1}, v_{n-1}, \ldots, v_{i+1}, u_i \] (for \( i = 1, 2 \))

and see that each of the edges \( u_i v_1 \) and \( u_i v_{n+j} \) where \( 1 \leq j \leq p-k \) must be in \( G \). Thus, for \( i = 1, 2, \ldots, n-1 \) the edges \( u_i v_1 \) and \( u_i v_{n+j} \) are in \( G \), for all \( j \) where \( 1 \leq j \leq p-k \). In particular, \( v_1 \), the first vertex of \( W \) is adjacent to each vertex in \( U \). By repeating the foregoing arguments for each spanning DFS walk of the form

\[ v_j, u_j, v_{j+1}, u_{j+1}, \ldots, v_n, u_n, v_1, u_1, v_2, u_2, \ldots, v_{j-1}, u_{j-1} + (v_{n+1}, v_{n+2}, \ldots, v_{n+p-k}) \]
where \( j = 2, 3, \ldots, n \) we see that each vertex \( v_j \) is adjacent to every vertex in \( U \). Thus \( G \) is isomorphic to the complete bipartite graph \( K(n, n+p-k) \).

**Case 2:** Suppose \( k = 2n + 1 \). Relabel the vertices of \( W \) as follows: for \( 1 \leq i \leq n \), set \( u_i = x_{2i-1} \) and \( v_i = x_{2i} \); set \( u_{n+1} = x_k \) and for \( j = 1, 2, \ldots, p-k \) set \( v_{n+j} = x_{k+j} \). Then \( U = \{u_1, u_2, \ldots, u_{n+1}\} \), \( V = \{v_1, v_2, \ldots, v_{n+p-k}\} \) and \( W \) has the form

\[
W: u_1, v_1, u_2, v_2, \ldots, u_n, v_n, u_{n+1} + (v_{n+1}, v_{n+2}, \ldots, v_{n+p-k})
\]

The tree induced by \( W \) is shown in Figure 4.4.

![Figure 4.4](image)

Since \( u_1 \) is not adjacent to \( u_{n+1} \) the two DFS walks

\[
v_1, u_2, v_2, u_3, \ldots, v_n, u_{n+1}, v_{n+1}
\]

and

\[
v_1, u_2, v_2, u_3, \ldots, v_n, u_{n+1}, v_{n+2}
\]

imply that \( u_1v_{n+j} \) is an edge of \( G \) for each \( j = 1, 2, \ldots, p-k \).

Similarly, for each \( i = 2, 3, \ldots, n \), the nonspanning DFS walks
$v_{i-1}, u_{i-1}, v_{i-2}, u_{i-2}, \ldots, v_1, u_1, v_{n+1}, u_{n+1}, v_n, u_n, \ldots, v_i, u_i$

and

$v_{i-1}, u_{i-1}, v_{i-2}, u_{i-2}, \ldots, v_1, u_1, v_{n+2}, u_{n+1}, v_n, u_n, \ldots, v_i, u_i$

imply that $u_j v_{n+j}$ is an edge of $G$ for each $j = 1, 2, \ldots, p-k$. In particular, we have, for each fixed $j = 1, 2, \ldots, p-k$, that $v_{n+j}$ is adjacent to each vertex in $U$.

The nonspanning DFS walk

$u_1, v_{n+1}, u_2, v_{n+2}, u_{n+1}, v_n, u_n, v_{n-1}, u_{n-1}, \ldots, u_j, v_2$

implies that $v_1 u_j$ is an edge of $G$. In general, for $4 \leq j \leq n$, the nonspanning DFS walk

$u_1, v_{n+1}, u_2, v_{n+2}, u_3, v_3, \ldots, v_{j-2}, u_{j-1}, v_{n+2}, u_{n+1}, v_n, u_n, v_{n-1}, u_{n-1}, \ldots, u_j, v_{j-1}$

implies that $v_1 u_j$ is an edge of $G$. Thus, $v_1$ is adjacent to each vertex in $U$.

By applying arguments similar to the foregoing arguments to the spanning DFS walk
we see that \( v_1 \) is adjacent to each vertex of \( U \) for each \( i = 1, 2, \ldots, n \). Thus, \( G \) is isomorphic to the complete bipartite graph \( K(n+1,n+p-k) \).

For the converse, we see from Theorem 2A that any cycle is randomly near-traceable, and from Theorem 4.1 that any complete bipartite graph is randomly near traceable. ■

Section 4.3

Radius and Diameter of Randomly Near-Traceable Graphs

If \( u \) and \( v \) are any two vertices of a connected graph \( G \), then the \textbf{distance} from \( u \) to \( v \), denoted \( d(u,v) \), is the length of a shortest \( u-v \) path in \( G \). The \textbf{diameter} of \( G \), denoted \( \text{diam } G \), is the maximum distance between any two vertices in \( G \). The \textbf{radius} of \( G \) is denoted \( \text{rad } G \) and defined by

\[
\text{rad}(G) = \min_{u \in V(G)} \left\{ \max_{v \in V(G)} d(u,v) \right\}
\]

Thus, if \( \text{rad } G = r \), then there is at least one vertex \( u \) in \( G \) such that \( d(u,v) \leq r \) for every \( v \in V(G) \).
Each randomly near-traceable graph (except cycles) that we have discussed so far has radius at most 2. As the next theorem shows, this is not coincidental.

**Theorem 4.3**

If $G$ is a randomly near-traceable graph that is not a cycle, then $\text{rad } G \leq 2$.

**Proof:** If $G$ is also randomly traceable, then, by Theorem 2A, either $G$ is a complete graph or a regular complete bipartite graph. In either instance $\text{rad } G \leq 2$. Thus, we henceforth assume that $G$ is not randomly traceable.

Suppose also that $G$ has order $p$ and that $\text{rad } G \neq 1$; that is $\Delta(G) \leq p-2$. Then, by Lemma 4.1, there is a depth-first search of $G$ which yields a spanning walk of the form

$$W: u_1, u_2, \ldots, u_k \rightarrow (u_{k+1}, u_{k+2}, \ldots, u_p).$$

The spanning tree induced by $W$ is indicated in Figure 4.5.

![Figure 4.5](image_url)

Clearly, if $2 \leq k \leq 4$ then $d(u_{k-1}, u_j) \leq 2$ for $j = 1, 2, \ldots, p$, so that $\text{rad } G \leq 2$. Suppose then that $k \geq 5$. We shall show that
We consider two cases.

**Case 1:** Assume that $u_1 u_k$ is an edge of $G$. Then $d(u_{k+1}, u_i) \leq 2$ and $d(u_{k+1}, u_{k-1}) \leq 2$ for $i = 1, 2, \ldots, p-k$. Also, if for each $j = 1, 2, \ldots, k-2$, the vertex $u_{k+i}$ is adjacent to one of $u_j$ and $u_{j+1}$ then $d(u_{k+i}, u_j) \leq 2$. Assume then, to the contrary, that for some $j$ in the range $1 \leq j \leq k-2$, and some $i = 1, 2, \ldots, p-k$ neither $u_j u_{k+i}$ nor $u_{j+1} u_{k+i}$ is an edge of $G$. Now, however, the DFS walk

$$u_{j+2}, u_{j+3}, \ldots, u_k, u_1, u_2, \ldots, u_j, u_{j+1}$$

cannot be continued to include every vertex in the independent set

$$\{u_{k+1}, u_{k+2}, \ldots, u_p\}$$

without backtracking to a vertex preceding $u_j$. This is a contradiction. Hence, for every $j = 1, 2, \ldots, p$ and every $i = 1, 2, \ldots, p-k$, we have $d(u_{k+i}, u_j) \leq 2$.

**Case 2:** Suppose that $u_1 u_k$ is not an edge of $G$. Then the DFS walk

$$u_2, u_3, \ldots, u_{k-1}, u_k, u_{k+i}$$

for any $i$ in the range $1 \leq i \leq p-k$, together with the fact that

$$\{u_{k+1}, u_{k+2}, \ldots, u_p\}$$

is an independent set, implies that $u_1$ is adjacent to each vertex in

$$\{u_{k+1}, u_{k+2}, \ldots, u_p\}.$$ [See Figure 4.6].

---

Figure 4.6

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Now, if $2 \leq j \leq k-2$, then the DFS walk

$$W_1: u_{j+2}, u_{j+3}, \ldots, u_k, u_{k+1}, u_1, u_2, \ldots, u_j, u_{j+1}$$

and

$$W_2: u_{j+2}, u_{j+3}, \ldots, u_k, u_{k+2}, u_1, u_2, \ldots, u_j, u_{j+1}$$

together with the fact that $\{u_{k+1}, u_{k+2}, \ldots, u_p\}$ is an independent set, imply that either $u_j$ or $u_{j+1}$ is adjacent to each of the vertices $u_{k+1}, u_{k+2}, \ldots, u_p$. Thus, for each $j = 1, 2, \ldots, p$ and each $i = 1, 2, \ldots, p-k$, we have $d(u_{k+i}, u_j) \leq 2$.

From Cases 1 and 2 we conclude that $\text{rad } G \leq 2$. ■

Since it is always the case that $\text{diam } G \leq 2 \text{ rad } G$, we have the following corollary to Theorem 4.3.

**Corollary 4.1**

If $G$ is a randomly near-traceable graph which is not a cycle, then $\text{diam } G \leq 4$.

We see then that if $G$ is a randomly near-traceable graph which is not a cycle, the distance between any pair of vertices in $G$ is at most 4. Each example of a randomly near-traceable graph (which is not a cycle) that we have investigated herein has diameter 1 or 2. In light of this observation, we close this chapter with the following.
Conjecture 4.1

If $G$ is a randomly near-traceable graph that is not a cycle, then $\text{diam } G \leq 2$. 
CHAPTER V

ISOFACTORS

Section 5.1

Graph Isofactors

If $H$ is a graph and $H_1, H_2, \ldots, H_n$ ($n \geq 2$) are nonempty, pairwise edge-disjoint subgraphs of $H$ satisfying the property that

$$E(H) = \bigcup_{i=1}^{n} E(H_i),$$

we say that $H$ is the edge sum of the factors $H_1, H_2, \ldots, H_n$ and write

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_n.$$  

If there is a graph $G$ that is isomorphic to each of the factors $H_1, H_2, \ldots, H_n$, then the expression

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

is said to be an isomorphic factorization or $G$-factorization of $H$, and $H$ is said to be $G$-factorable.

It is well-known that if $p$ is odd and $p \geq 5$, then the complete graph $K_p$ can be expressed as the edge sum of $(p-1)/2$ hamiltonian cycles; that is, when $p$ ($\geq 5$) is odd, $K_p$ is $C_p$-factorable. If $p$ is even and $p \geq 4$, then $K_p$ is expressible both as the edge sum of $p/2$ hamiltonian paths, and as the edge sum of $(p-1)$ 1-regular spanning subgraphs called 1-factors. Harary,
Robinson and Wormald [12] showed that there is an isomorphic factorization of $K_p$ into $t$ (isomorphic) subgraphs whenever $t$ divides $p(p-1)/2$. A discussion of other results on isomorphic factorizations of $K_p$ can be found in [3] by Bermond and Sotteau.

The complete graph is only one of the regular graphs for which isomorphic factorizations have been investigated. König [15] showed that every $r$-regular bipartite graph ($r \geq 2$) is the edge sum of $r$ 1-factors. Harary, Robinson and Wormald [13] showed that there is an isomorphic factorization of $K(n,n)$ into $t$ subgraphs whenever $t$ divides $n^2$; they also showed that if $n$ is even, there is an isomorphic factorization of the regular complete tripartite graph $K(n,n,n)$ into $t$ subgraphs when $t = 2,4$.

Most often, the topic of isomorphic factorizations has been approached by asking whether a given connected regular graph $H$ has a $G$-factorization for some $G$. Wilson [28] examined the subject from a different point of view and, using a combination of arguments involving finite fields, pairwise balanced designs, and finite vector spaces, proved the following "asymptotic" result.

**Theorem 5A**

If $G$ is a nonempty graph of size $q$, then the complete graph $K_p$ is $G$-factorable for all sufficiently large $p$ satisfying

(i) $p(p-1)/2 \equiv 0 \pmod{q}$

and

(ii) $(p-1) \equiv 0 \pmod{d}$

where $d = \gcd(\deg_G v \mid v \in V(G))$. 

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We say that a graph $G$ is an isofactor if it is either empty, or there exists a regular connected graph $H$ that is $G$-factorable. As a consequence of Theorem 5A we have the following.

**Theorem 5.1**

Every graph is an isofactor.

In what follows, we will prove a result that is stronger than Theorem 5.1. We will first define several new terms and establish a useful lemma.

An $n$-coloring of a graph $G$ is an assignment of $n$ colors (elements of some set) to the vertices of $G$ so that no two adjacent vertices receive the same color. The smallest integer $n$ for which an $n$-coloring of $G$ exists is called the chromatic number of $G$ and denoted $\chi(G)$. If $\chi(G) = n$, then $G$ is said to be $n$-chromatic.

It follows by definition that a $\chi(G)$-coloring of $G$ partitions the vertex set of $G$ into $\chi(G)$ independent sets, each such set containing all vertices of one assigned color; these independent sets are called color classes.

The cartesian product $G_1 \times G_2$ of two graphs $G_1$ and $G_2$ has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$; two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if and only if either

$$u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)$$

or

$$u_2 = v_2 \text{ and } u_1v_1 \in E(G_1).$$
We may now state and prove a result of Behzad and Mahmoodian [2].

Lemma 5A

If \( G_1 \) and \( G_2 \) are graphs, then the chromatic number of the cartesian product \( G_1 \times G_2 \) is

\[
\chi(G_1 \times G_2) = \max \{\chi(G_1),\chi(G_2)\}
\]

Proof: Let \( m = \chi(G_1) \) and \( n = \chi(G_2) \), and assume, without loss of generality, that \( m \geq n \). Let \( U_1, U_2, \ldots, U_m \) be the color classes in an \( m \)-coloring of \( G_1 \), and let \( V_1, V_2, \ldots, V_n \) be the color classes of an \( n \)-coloring of \( G_2 \). Then

\[
W_i = \bigcup_{j=1}^{n} (U_{i+j-1} \times V_j)
\]

(subscripts modulo \( m \)) is an independent set of vertices in \( G_1 \times G_2 \), for \( i = 1, 2, \ldots, m \). Since \( V(G_1 \times G_2) \) is the (disjoint) union of the sets \( W_1, W_2, \ldots, W_m \), we see that \( \chi(G_1 \times G_2) \leq m \). Since \( G_1 \times G_2 \) contains at least one subgraph isomorphic to \( G_1 \), it follows that \( \chi(G_1 \times G_2) \geq m \). Thus, \( \chi(G_1 \times G_2) = m \). ■

Before stating and proving the main result of this section, we introduce some notation. The floor function \( \lfloor x \rfloor \) of a real number \( x \) is the greatest integer less than or equal to \( x \). The notation \( \lceil x \rceil \) denotes the value of the ceiling function of \( x \) and is defined to be the least integer greater than or equal to \( x \).
Theorem 5.2

For every nonempty graph $G$, there exists a connected regular $G$-factorable graph $H$ such that $\chi(H) = \chi(G)$.

Proof: Suppose first that $G$ is a connected nontrivial graph that is not regular and has chromatic number $\chi(G) = n$. Define

$$m = \text{lcm}\{\text{deg}_G v | v \in V(G)\}$$

and let $V_1, V_2, \ldots, V_n$ be the color classes in an $n$-coloring of $G$. Define

$$m_i = \text{lcm}\{m/\text{deg}_G v | v \in V_i\}$$

for $i = 1, 2, \ldots, n$, set

$$t = \prod_{i=1}^{n} m_i,$$

and let $G_1, G_2, \ldots, G_t$ be $t$ distinct copies of $G$. If $u$ is a vertex of some $G_j$, and $u$ corresponds to a vertex of $G$ that belongs to color class $V_i$, then we will say that $u$ has color $i$; hence, the $n$-coloring of $G$ induces an $n$-coloring of $G_j$, for $j = 1, 2, \ldots, t$. Finally, for each vertex $v$ of $G$, define

$$r_v = m/\text{deg}_G v.$$

In what follows, we shall describe a step-by-step procedure in which certain vertices of the graphs $G_1, G_2, \ldots, G_t$ will be identified in order to ultimately obtain a connected, $m$-regular, $n$-chromatic graph $H$ having $G$-factorization.
\[ H = G_1 \oplus G_2 \oplus \cdots \oplus G_t. \]

**Step 1:** We first consider the graphs \( G_1, G_2, \ldots, G_m \). Let \( v \in V_1 \), and identify the vertices that correspond to \( v \) in the graphs \( G_1, G_2, \ldots, G_m \); perform the same vertex identification procedure with each set of graphs

\[
\{G_{r_v+1}, G_{r_v+2}, \ldots, G_{2r_v}\},
\{G_{2r_v+1}, G_{2r_v+2}, \ldots, G_{3r_v}\}, \ldots,
\{G_{m-r_v+1}, G_{m-r_v+2}, \ldots, G_m\}.
\]

By repeating this process for each \( v \in V_1 \), we obtain a connected graph which has the \( G \)-factorization

\[ G_1 \oplus G_2 \oplus \cdots \oplus G_m. \]

Also, since all vertices identified in this step have color 1, no multiple edges have been formed, and the \( n \)-colorings of \( G_1, G_2, \ldots, G_m \) have induced an \( n \)-coloring of this new graph (in which, in fact, each vertex is assigned the same colors as before). Each vertex \( u \) of color 1 in this graph corresponds (in some \( G_i \)) to a vertex \( v \) in \( G \) and has degree equal to

\[ r_v \cdot \deg_{G,v} = m. \]

By applying the foregoing procedure to each set of graphs
we obtain a total of \( \frac{t}{m} \) connected n-chromatic graphs, the \( i \)-th having the G-factorization

\[
G^{(i-1)m+1} \oplus G^{(i-1)m+2} \oplus \cdots \oplus G^{im}
\]

for \( i = 1, 2, \ldots, \frac{t}{m} \). Each of these graphs has an n-coloring defined by the n-colorings of the subgraphs \( G_j \), and every vertex of color 1 has degree \( m \). Thus, by taking the union of these \( \frac{t}{m} \) graphs, we obtain a graph \( H_1 \) with \( \frac{t}{m} \) components and G-factorization

\[
H_1 = G_1 \oplus G_2 \oplus \cdots \oplus G_t.
\]

Moreover, \( H_1 \) is n-chromatic and has an n-coloring defined by the n-colorings of \( G_1, G_2, \ldots, G_t \); that is, vertex \( u \) has color \( i \) in \( H_1 \) if and only if \( u \) has color \( i \) in the initial n-coloring of each (sub)graph \( G_j \) containing \( u \). Also, each vertex of color 1 in \( H_1 \) has degree \( m \).

**Step 2:** First consider the (sub)graphs \( G_1, G_2, \ldots, G_{\frac{t}{m}m_2} \).

In Step 1 these were used to form \( m_2 \) n-chromatic components of \( H_1 \) that, respectively, have the G-factorizations
Let \( v \in V_2 \) and identify the vertices that correspond to \( v \) in the (sub)graphs

\[
G_1, G_{m_1+1}, G_{2m_1+1}, \ldots, G_{r_m+m_1+1};
\]

apply the same identification procedure to each set of \( r_v \) (sub)graphs

\[
\{G_{r_v+m_1+1}, G_{r_v+m_1+1}^*, \ldots, G_{2r_v+m_1+1}^*\},
\]

\[
\{G_{2r_v+m_1+1}, G_{2r_v+m_1+1}^*, \ldots, G_{3r_v+m_1+1}^*\},
\]

\[
\ldots,
\]

\[
\{G_{m_1+m_1-j, G_{m_1+m_1-j}^*, \ldots, G_{m_1+m_1-j}^*}\}.
\]

Perform the above procedure for each set of graphs

\[
\{G_j, G_{m_1+j}, \ldots, G_{r_v+m_1+j}\}
\]

\[
\{G_{r_v+m_1+j}, G_{r_v+m_1+j}^*, \ldots, G_{2r_v+m_1+j}^*\},
\]

\[
\{G_{2r_v+m_1+j}, G_{2r_v+m_1+j}^*, \ldots, G_{3r_v+m_1+j}^*\},
\]

\[
\ldots,
\]

\[
\{G_{m_1+m_1-j, G_{m_1+m_1-j}^*, \ldots, G_{m_1+m_1-j}^*}\}.
\]
where \( j = 1, 2, \ldots, m_1 \). By repeating the foregoing for each vertex \( v \in V_2 \), we create a connected graph having the \( \text{G-factorization} \)

\[
G \oplus G_2 \oplus \cdots \oplus G_{m_1 m_2}
\]

Since each pair of vertices identified in this step had color 2 in \( H_1 \) and belonged to different components of \( H_1 \), no multiple edges have been formed, and an \( n \)-coloring of this new graph has been determined by the \( n \)-coloring of \( H_1 \) (where vertex \( u \) has color \( i \) in this graph if and only if the color of \( u \), or the vertices identified to form \( u \), is \( i \) in \( H_1 \)). Furthermore, each vertex in this graph that corresponds (in some \( G_j \)) to a vertex \( v \) of either \( V_1 \) or \( V_2 \) has degree

\[
\deg_G v = m;
\]

thus, each vertex of color 1 or 2 has degree \( m \). Now, by repeating the foregoing procedure for each collection of (sub)graphs

\[
\{G_{km_1 m_2 + 1}, G_{km_1 m_2 + 2}, \ldots, G_{(k+1)m_1 m_2}\},
\]

where \( k = 0, 1, \ldots, t/(m_1 m_2) - 1 \), we obtain \( t/(m_1 m_2) \) connected \( n \)-chromatic graphs, the \( k \)-th of which has the \( \text{G-factorization} \)

\[
G_{km_1 m_2 + 1} \oplus G_{km_1 m_2 + 2} \oplus \cdots \oplus G_{(k+1)m_1 m_2}.
\]

By forming the union of these \( t/(m_1 m_2) \) graphs, we obtain an \( n \)-chromatic graph \( H_2 \) with \( t/(m_1 m_2) \) components and \( \text{G-factorization} \)

\[
H_2 = G_1 \oplus G_2 \oplus \cdots \oplus G_t.
\]
Also, $H_2$ has an $n$-coloring imposed (ultimately) by the original $n$-colorings of the (sub)graphs $G_1, G_2, \ldots, G_t$: a vertex $u$ has color $i$ in $H_2$ if and only if it has color $i$ in the original $n$-coloring of some $G_j$ containing $u$. Moreover, each vertex of color 1 or 2 in $H_2$ has degree $m$.

If $n = 2$, then $H_2$ is the desired $m$-regular, connected, $n$-chromatic, $G$-factorable graph. If $n > 2$, we proceed to Step 3.

We next describe the general Step $s + 1$ in the construction of the desired $m$-regular, connected, $n$-chromatic, $G$-factorable graph. Step $s + 1$ is performed after the completion of Steps 1, 2, ..., $s$, where $1 \leq s < n$. Our hypothesis is that at the conclusion of Step $s$, we have constructed an $n$-chromatic $G$-factorable graph $H_s$ having $t/(m_1m_2\ldots m_s)$ components and $G$-factorization

$$H_s = G_1 \oplus G_2 \oplus \cdots \oplus G_t;$$

the $k$-th component of $H_s$ has $G$-factorization

$$G_k(m_1m_2\ldots m_s)+1 \oplus G_k(m_1m_2\ldots m_s)+2 \oplus \cdots \oplus G_{k+1}(m_1m_2\ldots m_s)$$

for $k = 0, 1, \ldots, t/(m_1m_2\ldots m_s) - 1$. Also, we assume that $H_s$ has been given an $n$-coloring in which vertex $u$ has color $i$ if and only if $u$ had color $i$ in each of the original $n$-colorings given to those (sub)graphs $G_j$ which contain $u$. Furthermore, we hypothesize that each vertex having color 1, 2, ..., $s-1$, or $s$ has degree $m$ in $H_s$. 

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Step $s + 1$: Consider first the (sub)graphs $G_1, G_2, \ldots,$ $G_{(m_1 m_2 \ldots m_{s+1})}$. These (sub)graphs were used, in Steps 1 through $s$, to form the $m_{s+1}$ components of $H_s$ that have, respectively, the $G$-factorizations

$$G_1 \oplus G_2 \oplus \cdots \oplus G_{(m_1 m_2 \ldots m_s)} ,$$

$$G_{(m_1 m_2 \ldots m_s)+1} \oplus G_{(m_1 m_2 \ldots m_s)+2} \oplus \cdots \oplus G_{2(m_1 m_2 \ldots m_s)} ,$$

$$\vdots$$

$$G_{(m_{s+1}-1)(m_1 m_2 \ldots m_s)+1} \oplus G_{(m_{s+1}-1)(m_1 m_2 \ldots m_s)+2} \oplus \cdots \oplus G_{(m_1 m_2 \ldots m_{s+1})} .$$

Let $v \in V_{s+1}$ and identify the vertices that correspond to $v$ in the $r_v$ (sub)graphs

$$G_1, G_{(m_1 m_2 \ldots m_s)+1}, G_{2(m_1 m_2 \ldots m_s)+1},$$

$$\cdots, G_{(r_v-1)(m_1 m_2 \ldots m_s)+1} ;$$

do the same with each set of $r_v$ (sub)graphs

$$\{G_{r_v(m_1 m_2 \ldots m_s)+1}, G_{(r_v+1)(m_1 m_2 \ldots m_s)+1} ,$$

$$\cdots, G_{(2r_v-1)(m_1 m_2 \ldots m_s)+1}\} ;$$

$$\{G_{2r_v(m_1 m_2 \ldots m_s)+1}, G_{(2r_v+1)(m_1 m_2 \ldots m_s)+1} ,$$

$$\cdots, G_{(3r_v-1)(m_1 m_2 \ldots m_s)+1}\} .$$
Apply the foregoing vertex identification procedure to each set of $r_v (\text{sub})$graphs

$$\{G_{(r_v-1)(m_1 m_2 \ldots m_s)} + j\}, \ldots, G_{(r_v)(m_1 m_2 \ldots m_s)} + j\},$$

$$\{G_{r_v(m_1 m_2 \ldots m_s)} + j, G_{(r_v+1)(m_1 m_2 \ldots m_s)} + j\},$$

$$\ldots, G_{(2r_v-1)(m_1 m_2 \ldots m_s)} + j\},$$

$$\ldots, G_{(m_{s+1}-r_v)(m_1 m_2 \ldots m_s)} + j\}, G_{(m_{s+1}-r_v+1)(m_1 m_2 \ldots m_s)} + j\},$$

$$\ldots, G_{(m_{s+1}-1)(m_1 m_2 \ldots m_s)} + j\},$$

for $j = 1, 2, \ldots, (m_1 m_2 \ldots m_s)$. By repeating the preceding process for each $v \in V_{s+1}$, we produce a connected graph having the $G$-factorization

$$G_1 \oplus G_2 \oplus \ldots \oplus G_{(m_1 m_2 \ldots m_{s+1})}.$$

Since $V_{s+1}$ is an independent set, and no two vertices identified in this procedure belonged to the same component of $H_s$, no multiple edges have been formed. Since every two vertices identified in this step had color $s + 1$, this new graph is $n$-chromatic and has an $n$-coloring defined by the initial $n$-colorings of the subgraphs $G_j$. Also, every vertex of color $1, 2, \ldots, s$ or $s + 1$ in this graph has degree
Therefore, by applying the foregoing procedure to each set of (sub)graphs

\[ \{G_k(m_1 m_2 \ldots m_{s+1})+1', G_k(m_1 m_2 \ldots m_{s+1})+2', \ldots, G_{k+1}(m_1 m_2 \ldots m_{s+1})\} \]

where \( k = 0, 1, \ldots, t/(m_1 m_2 \ldots m_{s+1})-1 \), and forming the union of the resulting graphs, we obtain an \( n \)-chromatic graph \( H_{s+1} \) with \( t/(m_1 m_2 \ldots m_{s+1}) \) components and \( G \)-factorization

\[ H_{s+1} = G_1 \oplus G_2 \oplus \cdots \oplus G_t ; \]

the \( k \)-th component of \( H_{s+1} \) has the \( G \)-factorization

\[ G_k(m_1 m_2 \ldots m_{s+1})+1 \oplus G_k(m_1 m_2 \ldots m_{s+1})+2 \oplus \cdots \oplus G_{k+1}(m_1 m_2 \ldots m_{s+1}) \]

where \( k = 0, 1, \ldots, t/(m_1 m_2 \ldots m_{s+1})-1 \), and \( H_{s+1} \) has an \( n \)-coloring defined by the initial \( n \)-colorings of the subgraphs \( G_1, G_2, \ldots, G_t \). Furthermore, each vertex having color 1, 2, \ldots, \( s \), or \( s + 1 \) in \( H_{s+1} \) has degree \( m \).

We repeat the general procedure described above until we have completed Step \( n \) and constructed the graph \( H_n \). This will be the desired \( m \)-regular, \( n \)-chromatic graph with \( t/(m_1 m_2 \ldots m_n) = 1 \) component and \( G \)-factorization

\[ H_n = G_1 \oplus G_2 \oplus \cdots \oplus G_t . \]

Thus, for every nonempty connected graph \( G \) that is not regular, there exists a connected regular \( G \)-factorable graph \( H \) such that \( \chi(H) = \chi(G) \).
Now, suppose that $G$ is a nonempty connected $r$-regular graph and let $V(G) = \{v_1, v_2, \ldots, v_p\}$. If $H$ denotes the cartesian product $G \times G$, then $H$ is $2r$-regular and has vertex set $V(H) = \{(v_i, v_j) | 1 \leq i, j \leq p\}$. Moreover, if we define

$$G_{i,j} = \{(v_i, v_j) | j = 1, 2, \ldots, p\}$$

for $i = 1, 2, \ldots, p$, and

$$G_{j} = \{(v_i, v_j) | i = 1, 2, \ldots, p\}$$

for $j = 1, 2, \ldots, p$, then $G_{i,j} = G_{j} = G$ for all $i$ and $j$. Furthermore,

$$H = G_{1} \oplus G_{2} \oplus \cdots \oplus G_{p} \oplus G_{1} \oplus G_{2} \oplus \cdots \oplus G_{p}$$

so that $H$ is $G$-factorable. Also, by Lemma 5A, we see that $X(H) = X(G)$. Thus, for every nonempty connected regular graph $G$, there is a connected, regular $G$-factorable graph $H$ such that $X(H) = X(G)$.

We have now verified our claim for all nonempty connected graphs. Thus, we next turn our attention to disconnected graphs.

Suppose that $G$ is a disconnected graph in which each component is nontrivial. Let $G_a$ and $G_b$ be two copies of $G$, and let $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$ be, respectively, the components of $G_a$ and $G_b$. Since each component of $G$ is nontrivial $\chi(G) \geq 2$ and $\chi(A_i), \chi(B_i) \geq 2$ for $i = 1, 2, \ldots, k$. Thus, using the colors $1, 2, \ldots, \chi(G)$, we may define $\chi(G)$-colorings of $G_a$ and $G_b$ in such a way that each of the components of $G_a$ and $G_b$ contains at
least one vertex of color 1 and at least one vertex of color 2.

Given these n-colorings, select vertices \( u^i, v^i \in V(A^i) \) and \( w^i, x^i \in V(B^i) \) such that \( u^i \) and \( w^i \) have color 1, and \( v^i \) and \( x^i \) have color 2, for \( i = 1, 2, \ldots, k \). Construct a new graph \( G^+ \) from \( G_a \) and \( G_b \) by identifying \( v^i \) with \( x^i \), and \( w^i \) with \( u^{i+1} \) for \( i = 1, 2, \ldots, k \) with subscripts modulo \( k \). Then \( G^+ = G_a \oplus G_b \) and \( G^+ \) is connected. Also, since every two vertices that were identified to form \( G^+ \) had the same color, we see that \( \chi(G^+) = \chi(G) \). From the earlier part of this proof, we know that there is a connected regular graph \( H \) that is \( G^+ \)-factorable and satisfies \( \chi(H) = \chi(G^+) \). Since \( G^+ \) is \( G \)-factorable and \( \chi(G^+) = \chi(G) \), we see that \( H \) is \( G \)-factorable and \( \chi(H) = \chi(G) \). Thus our claim is true for all nonempty graphs that have no trivial components.

Finally, suppose that \( G \) is disconnected and has both trivial and nontrivial components. Let \( p \) be the order of \( G \) and let \( k \) denote the number of isolated vertices in \( G \). Set \( m = \lceil p/(p-k) \rceil \) and let \( F \) be the union of \( m \) copies of \( (E(G)) \). Then \( F \) has no trivial components and is the edge-sum of \( m \) subgraphs isomorphic to \( G \); each such subgraph comprises one of the copies of \( (E(G)) \) used to form \( F \), and \( k \) other vertices of \( F \). Clearly \( \chi(F) = \chi(G) \). Now, by our earlier analysis, there is a connected regular \( F \)-factorable graph \( H \) with \( \chi(H) = \chi(F) = \chi(G) \). Since \( F \) is \( G \)-factorable, so too is \( H \). This completes the proof.  

Theorem 5.2 now provides an independent, simpler (than the proof of Theorem 5A), and completely graph-theoretic proof of Theorem 5.1.
We restate this as a corollary to Theorem 5.2.

**Corollary 5.1**

Every graph is an isofactor.

**Section 5.2**

**Digraph Isofactors**

If \( F \) is a digraph and \( F_1, F_2, \ldots, F_n \) (\( n \geq 2 \)) are nonempty pairwise arc-disjoint subdigraphs of \( F \) satisfying the property that

\[
E(F) = \bigcup_{i=1}^{n} E(F_i),
\]

we say that \( F \) is the arc sum of the factors \( F_1, F_2, \ldots, F_n \) and write

\[
F = F_1 \oplus F_2 \oplus \ldots \oplus F_n.
\]

If there is a digraph \( D \) that is isomorphic to each of the factors \( F_1, F_2, \ldots, F_n \), then the expression

\[
F = F_1 \oplus F_2 \oplus \ldots \oplus F_n
\]

is called an isomorphic factorization or \( D \)-factorization of \( F \), and \( F \) is said to be \( D \)-factorable.

A digraph \( D \) is \( r \)-regular (or simply regular) if every vertex of \( D \) has both indegree and outdegree equal to \( r \). The most frequently investigated regular digraph, from the standpoint of isomorphic factorizations, is the complete symmetric digraph \( K_p^* \) (see Bermond and Sotteau [3]). Harary, Robinson, and Wormald [14] have shown that if \( t \) divides \( p(p-1) \), then \( K_p^* \) has an isomorphic factorization into \( t \) copies of...
some digraph; they also showed that the n-regular complete symmetric bipartite digraph $K^*(n,n)$ has an isomorphic factorization into $t$ factors if and only if $t$ is odd and divides $2n^2$. Wilson [28], using algebraic techniques, showed the following

Theorem 5B

If $D$ is a nonempty digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_k\}$ and having size $q$, then the complete symmetric digraph $K^*_p$ is $D$-factorable for all sufficiently large integers $p$ satisfying

(i) $p(p-1) \equiv 0 \pmod{q}$

and

(ii) $(p-1) \equiv 0 \pmod{g_D}$

where $g_D$ is the least positive integer $g$ for which the vector equation

$$\sum_{i=1}^{k} x_i(idv_i, odv_i) = (g, g)$$

has an integral solution $x_1, x_2, \ldots, x_k$.

A digraph $D$ is said to be an isofactor if either it is empty, or there exists a connected regular digraph $F$ that is $D$-factorable. As a consequence of Theorem 5B we have the following.

Theorem 5.3

Every digraph is an isofactor.

Before stating and proving our next theorem, a result that is stronger than Theorem 5.3, it is necessary to introduce some notation. If $n$ is a positive integer, and $t$ is an integer, then the expression $[t]_n$
denotes the number in the set \{1,2,...,n\} that is congruent to \(t\) modulo \(n\).

**Theorem 5.4**

For every nonempty digraph \(D\) there exists a connected regular digraph \(F\) having a \(D\)-factorization satisfying the property that every symmetric pair of arcs of \(F\) belongs to some factor.

**Proof**: Suppose initially that \(D\) is connected, has vertex set \(V(D) = \{v_1,v_2,...,v_p\}\), and has \(q\) arcs. Let \(D_1, D_2, ..., D_p\) be \(p\) distinct digraphs, each isomorphic to \(D\), and define \(F_0\) to be the union of these digraphs. Let \((i;j)\) denote the vertex of \(D_i\) that corresponds to the vertex \(v_j\) in \(D\).

In what follows, we will describe a \(p\)-step procedure for constructing a \(q\)-regular connected digraph \(F\) having the \(D\)-factorization

\[ F = D_1 \oplus D_2 \oplus ... \oplus D_p ; \]

this \(D\)-factorization will satisfy the property that every symmetric pair of arcs in \(F\) belongs to some factor \(D_j\). At the \(s\)-th step in our procedure \((1 \leq s \leq p)\) we will identify the \(p\) vertices in the set

\[ \{(ip^s + kp^{s-1} + [k+j]_{s-1}; [k+j+s-1]_p) | k = 0,1,...,p-1\} \]

to form a new vertex, for each \(i = 0, 1, ..., p^{n-1}\) and \(j = 1, 2, ..., p^{s-1}\). Before describing this procedure in more detail, we define the set

\[ A = \bigcup_{s} \bigcup_{i} \bigcup_{j} \{(ip^s + kp^{s-1} + [k+j]_{s-1}; [k+j+s-1]_p) | k = 0,1,...,p-1\} \]
Let \((t;n) \in V(D_{s})\) where \(1 \leq t \leq p^{s}\). We will show that \((t;n) \in A\).

Define

\[ s = [n-t]_{p} + 1 \quad (= [n - [t]_{p}]_{p} + 1) \]

if \([t]_{p} \neq n\) and \(s = 1\) otherwise. Choose

\[ i \in \{0, 1, \ldots, p^{s-1}\}, \]

\[ j \in \{1, 2, \ldots, p^{s-1}\}, \]

and

\[ k \in \{0, 1, \ldots, p - 1\} \]

so that

\[ t = ip^{s} + kp^{s-1} + \left[k+j\right]_{p}^{s-1}. \]

Note that if \(s = 1\), then \(j = 1\) and \([t]_{p} = k + 1 = [k+j]_{p}\). If \(s \geq 2\), then

\[ [t]_{p} = \left[k+j\right]_{p}^{s-1} = [k+j]_{p}. \]
Therefore,

\[ [k+j+s-1]_p = ([k+j]_p + [n - [t]_p]_p)_p \]

\[ = ([t]_p + n - [t]_p)_p \]

\[ = n. \]

Hence,

\[(t;n) = (ip^s + kp^{s-1} + [k+j]_{s-1}; [k+j+s-1]_p)\]

and we conclude that \( V(D_t) \subseteq A \). Thus,

\[ A = \bigcup_{Z=1}^{p^p} V(D_Z) \]

and each vertex of \( \bigcup V(D_Z) \) belongs to exactly one of the \( p^p \) p-element sets

\[ \{(ip^s + kp^{s-1} + [k+j]_{s-1}; [k+j+s-1]_p)|k = 0, 1, ..., p-1}\} \]

Hence, each vertex of \( \bigcup V(D_Z) \) will be identified with exactly \( p-1 \) other vertices in our construction. We now describe the p-step construction of \( F \) in more detail.

**Step 1**: As indicated before, we construct a new digraph \( F_1 \) from \( F_0 \) by identifying the \( p \) vertices in the set

\[ \{(ip+k+1;k+1)|k = 0, 1, ..., p-1\} \]

to form a new vertex for each \( i = 0, 1, ..., p^{p-1} - 1 \); each such new vertex has indegree equal to
\[ \sum_{k=0}^{p-1} \text{id}(ip+k+1; k+1) = \sum_{k=0}^{p-1} \text{id}v_{k+1} = q, \]

and, by similar reasoning, has outdegree of \( q \) as well. Since no two vertices from the same component of \( F_0 \) were identified in this step, and since exactly one vertex from each subdigraph \( D_k \) was identified with other vertices in this step, neither multiple arcs nor new symmetric pairs of arcs were introduced in forming \( F_1 \) from \( F_0 \). We remark that \( F_1 \) has \( p \)-components, the \( i \)-th of which has the D-factorization

\[ D_{ip+1} \oplus D_{ip+2} \oplus \cdots \oplus D_{(i+1)p}, \]

where \( 0 \leq i \leq p^{p-1} - 1 \). Thus

\[ F_1 = D_1 \oplus D_2 \oplus \cdots \oplus D_p; \]

since, as discussed above, no new symmetric pairs of arcs were formed in this step, each symmetric pair of arcs in \( F_1 \) belongs to some factor in this D-factorization.

We now describe the general \( s \)-th step in our construction.

**Step \( s \) \((s > 2)\):** This step is performed under the (inductive) assumption that \( F_{s-1} \) has \( p^{p-s+1} \) components, the \( i \)-th of which has the D-factorization

\[ D_{ip^{s-1}+1} \oplus D_{ip^{s-1}+2} \oplus \cdots \oplus D_{(i+1)p^{s-1}}, \]

for \( i = 0, 1, \ldots, p^{p-s+1} - 1 \); furthermore, each symmetric pair of arcs in \( F_{s-1} \) belongs to some factor in the D-factorization.
We also hypothesize that each vertex of $F_{s-1}$ that was formed by identifying vertices in one of steps 1 through $s-1$ has both indegree and outdegree equal to $q$.

As indicated earlier, in this step we create new vertices and form $F_s$ from $F_{s-1}$ by identifying the $p$ vertices in each set

$$\{(ip^s + kp^{s-1} + [k+j])_{p^{s-1}}; [k+j+s-1]_p^q; k = 0, 1, \ldots, p-1\}$$

for $i = 0, 1, \ldots, p^{s-s} - 1$ and $j = 1, 2, \ldots, p^{s-1}$. Note that none of these sets contains either two vertices from the same component of $F_{s-1}$, or two vertices from the same factor $D_t$ of $F_{s-1}$, where $1 \leq t \leq p^s$. Furthermore, if $u$ belongs to one of these sets and $v$ to another, then $u$ and $v$ do not belong to the same factor $D_t$ in $F_{s-1}$; that is, if

$$ip^s + kp^{s-1} + [k+j]_{p^{s-1}} = np^s + np^{s-1} + [n+m]_{p^{s-1}},$$

(where $0 \leq k, n \leq p-1$ while $1 \leq j, m \leq p^{s-1}$ and $0 \leq i, l \leq p^{s-s} - 1$) then $i = l$, $k = n$, and $j = m$. In fact, for $i$ fixed, $(0 \leq i \leq p^{s-s} - 1)$, the union of the $p^{s-1}$ $p$-element sets

$$\{(ip^s + kp^{s-1} + [k+j])_{p^{s-1}}; [k+j+s-1]_p^q; k = 0, 1, \ldots, p-1\}$$

over $j = 1, 2, \ldots, p^{s-1}$ contains exactly one vertex from each of the $p^s$ factors

$$D_{ip^s+1}, D_{ip^s+2}, \ldots, D_{(i+1)p^s}.$$
Thus, with these observations, we see that neither multiple arcs nor new symmetric pairs of arcs were introduced into $F_s$ by identifying vertices in this step. Note that $F_s$ has $p^{p-s}$ components, the $i$-th having the D-factorization

$$D_{ip^{p+1}} \Phi D_{ip^{p+2}} \Phi \cdots \Phi D_{(i+1)p^s},$$

and thus,

$$F_s = D_1 \Phi D_2 \Phi \cdots \Phi D_{p^p};$$

from the foregoing remark, each symmetric pair of arcs in $F_s$ must belong to some factor $D_i$. Also, like the vertices formed in steps 1 through $s-1$, each new vertex created by identifying vertices in this step has indegree

$$\sum_{k=0}^{p-1} \text{id}(ip^{s+1} + [k+j]_{s-1}; [k+j+s-1]_p)$$

and, by similar reasoning, has outdegree equal to $q$ as well.

Note that if $s = p$, then $F_s$ has one component (i.e., is connected), is $q$-regular, has D-factorization

$$F_s = D_1 \Phi D_2 \Phi \cdots \Phi D_{p^p},$$

and satisfies the property that every symmetric pair of arcs in $F_s$...
belongs to some factor $D^p$. Thus, $F_p$ is the desired digraph $F$.

Suppose now that $D$ is a disconnected digraph in which each component is nontrivial. Let $D_a$ and $D_b$ be two copies of $D$, and let $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$ be, respectively, the components of $D_a$ and $D_b$. Choose pairs of distinct vertices $v_1, v_2 \in V(A_i)$ and $w_1, w_2 \in V(B_i)$ for each $i = 1, 2, \ldots, k$. Form a new digraph $D^+$ from $D_a$ and $D_b$ by identifying $v_i$ with $w_i$, and $x_i$ with $u_i$ for $i = 1, 2, \ldots, k$ with subscripts modulo $k$. Then $D^+$ is connected, has the $D$-factorization

$$D^+ = D_a \oplus D_b,$$

and satisfies the property that each symmetric pair of arcs in $D^+$ belongs to either $D_a$ or $D_b$. Since $D^+$ is connected and nonempty, it follows from the first part of this proof that there is a regular connected digraph $F$ having a $D^+$-factorization satisfying the property that every symmetric pair of arcs in $F$ belongs to some factor. Since $D^+$ is $D$-factorable in such a way that every symmetric pair of arcs belongs to some factor, so too is $F$. Thus, our claim is true for every digraph that has no trivial component.

Suppose finally that $D$ is a disconnected digraph that has both trivial and nontrivial components. Let $p$ denote the order of $D$ and let $k$ be the number of isolated vertices in $D$. Put $m = \lceil p/(p-k) \rceil$ and let $H$ be the union of $m$ digraphs, each isomorphic to $\langle E(D) \rangle$. Then $H$ has no trivial components and is the arc sum of $m$ subdigraphs that are isomorphic to $D$, each such subdigraph consists of one of the copies of $\langle E(D) \rangle$ used to form $H$. 

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and \( k \) vertices of \( H \) that do not lie on that copy of \( \langle E(D) \rangle \).

Since \( H \) has no trivial component, we may, by earlier results conclude that there exists a regular connected digraph \( F \) that has a \( D \)-factorization that satisfies the condition that each symmetric pair of arcs in \( F \) belongs to some factor. This completes the proof. \( \blacksquare \)

We have the following immediate corollary to Theorem 5.4.

**Corollary 5.2**

For every nonempty asymmetric digraph \( D \), there exists a connected, regular, asymmetric, \( D \)-factorable digraph.

Theorem 5.4 also provides an independent, simpler (than the proof of Theorem 5B), and entirely graph-theoretic proof for Theorem 5.3. Thus, we restate Theorem 5.3 as a corollary.

**Corollary 5.3**

Every digraph is an isofactor.

**Section 5.3**

**Isofactor Parameters**

The proofs of Theorems 5.2 and 5.4 (if we consider a graph to be a symmetric digraph) both provide examples of \( G \)-factorable graphs for every nonempty graph \( G \). For a given nonempty graph \( G \), the proof of Theorem 5.2 actually constructs a \( G \)-factorable graph having the minimum chromatic number among all \( G \)-factorable graphs. In general,
neither proof provides a $G$-factorable graph having minimum order, minimum degree of regularity, or the minimum number of factors in its $G$-factorizations. With this in mind, we define several parameters.

For a nonempty graph $G$, define $r_o(G)$ to be the smallest integer $r$ for which there exists an $r$-regular, connected, $G$-factorable graph. Define $p_o(G)$ to be the minimum order among all regular, connected $G$-factorable graphs. Let $f_o(G)$ be the least integer $t$ such that there exists a regular, connected graph with a $G$-factorization comprising $t$ factors. If the graph $G$ is clear from context, we sometimes write $r_o$, $p_o$, and $f_o$ rather than $r_o(G)$, $p_o(G)$, and $f_o(G)$. In the three propositions that follow, we determine the exact values of these parameters for some well-known classes of graphs.

Proposition 5.1

For the complete graph $K_n$, where $n \geq 2$,

$$r_o(K_n) = 2n - 2,$$

$$p_o(K_n) = \frac{(n+1)n}{2},$$

and

$$f_o(K_n) = n + 1.$$

Proof: Let $H$ be a connected $r$-regular $K_n$-factorable graph, and let

$$H = H_1 \oplus H_2 \oplus \ldots \oplus H_k$$

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be a $K_n$-factorization of $H$. Since $H$ is connected and regular, every vertex $v$ of $H^1$ belongs to another subgraph $H^i$, where $i \neq 1$. Then

$$r = \deg_H v \leq \deg_{H^1} v + \deg_{H^i} v = 2(n-1).$$

and we conclude that $r_o(K_n) \geq 2n - 2$.

Since $r \geq 2n - 2$, each vertex of $H$ belongs to at least two of the subgraphs $H_1, H_2, \ldots, H_k$. Since each $H_i$ is complete, no two of the $n$ vertices in $H^1$ belong to the same $H_j$ for $j \neq 1$; thus we see that $k \geq n + 1$ and $f_o(K_n) \geq n + 1$.

Let $v_1, v_2, \ldots, v_n$ denote the vertices in $H^1$. Since no two of these vertices belong to the same $H_j$ when $j \neq 1$, we may, without loss of generality, assume that $v_i$ belongs to $H_{i+1}$ for $i = 1, 2, \ldots, n$. Since each $H_i$, for $i = 1, 2, \ldots, n + 1$, is complete, any two of these subgraphs share at most one vertex. Thus, the number of vertices in $V(H_1) \cup V(H_2) \cup \ldots \cup V(H_{n+1})$ is at least $(n+1)n/2$; hence the order of $H$ is at least $(n+1)n/2$ and therefore $p_o(K_n) \geq (n+1)n/2$.

We have seen that $r_o(K_n) \geq 2n - 2$, while $p_o(K_n) \geq (n+1)n/2$, and $f_o(K_n) \geq n + 1$. We next show that the reverse inequalities also are true and, therefore, that equality holds in these expressions.

Let $H$ denote the line graph of $K_{n+1}$, and suppose that $K_{n+1}$ has vertex set \{v_1, v_2, \ldots, v_{n+1}\}. Let $H_i$ denote the subgraph of $H$ that is induced by the $n$ vertices which correspond to the edges of $K_{n+1}$ incident with $v_i$, for $i = 1, 2, \ldots, n+1$. 

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Then $H_i = K_n$ for each $i = 1, 2, \ldots, n+1$, and

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_{n+1}$$

is a $K_n$-factorization of $H$; thus $f_o(K_n) \leq n + 1$. Also, $H$ is $(2n-2)$-regular and has order $(n+1)n/2$. Thus, $r_o(K_n) \leq 2n - 2$ and $p_o(K_n) \leq (n+1)n/2$. □

**Proposition 5.2**

For the cycle $C_n$, where $n \geq 3$,

$$r_o(C_n) = 4,$$

$$p_o(C_n) = \begin{cases} 6 & \text{if } n = 3 \text{ or } n = 4 \\ n & \text{if } n \geq 5, \end{cases}$$

and

$$f_o(C_n) = \begin{cases} 4 & \text{if } n = 3 \\ 3 & \text{if } n = 4 \\ 2 & \text{if } n \geq 5. \end{cases}$$

**Proof:** For $n = 3$, the result follows from Proposition 5.1. Since $C_n$ is 2-regular, every vertex in a connected, regular, $C_n$-factorable graph $H$ must belong to at least two of the subgraphs in each $C_n$-factorization of $H$; hence $r_o(C_n) \geq 4$. Trivially, $p_o(C_n) \geq n$ and $f_o(C_n) \geq 2$. Since there is no regular graph of order 4 or 5 whose size is at least 8 and divisible by 4, we see that $p_o(C_4) \geq 6$; with $p_o(C_4) \geq 6$ it is easily seen that $f_o(C_4) \geq 3$. The graph of the octahedron is a connected, 4-regular, graph of order 6 that has a $C_4$-factorization into 3 factors; thus we conclude that
For $n \geq 5$ it is well-known that $K_n$ contains two (or more) edge-disjoint hamiltonian cycles. Thus, there is a connected 4-regular graph of order $n$ that has an isomorphic factorization into 2 copies of $C_n$. We conclude that $r_o(C_n) = 4$, $p_o(C_n) = n$ and $f_o(C_n) = 2$ when $n \geq 5$. 

Proposition 5.3

For the graph $K(l,n)$, where $n \geq 2$,

$$r_o(K(l,n)) = n,$$

$$p_o(K(l,n)) = 2n,$$

and

$$f_o(K(l,n)) = n.$$ 

Proof: Let $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ denote the partite sets of the $n$-regular complete bipartite graph $K(n,n)$, and set $H_i = \langle \{u_i\} \cup V \rangle$ for $i = 1, 2, \ldots, n$. Then $H_n \cong K(1,n)$ and

$$K(n,n) = H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

is a $K(l,n)$-factorization of $K(n,n)$. Thus, $p_o \leq 2n$, $r_o \leq n$, and $f_o \leq n$.

Since $r_o \geq \Delta(K(l,n)) = n$, we see that $r_o = n$.

To see that $f_o \geq n$, assume to the contrary that there is a regular graph $H$ with a $K(l,n)$-factorization.
where $m < n$. Since $H_i$ has order $n + 1$ and $m < n$, some vertex $v$ in $H_i$ is not the central vertex of any (star) $H_i$; $i = 1, 2, \ldots, m$.

But then,

$$\deg_{H_i} v \leq m < n = r^o$$

-- a contradiction. Thus $f_o \geq n$ and we conclude that $f_o = n$.

Now, let $H = H_1 \oplus H_2 \oplus \ldots \oplus H_m$ be a $K(1,n)$-factorization of a connected, $r$-regular, $K(1,n)$-factorable graph $H$ of order $p = p_o$ and size $q$. Assume that $p_o < 2n$. Note that $m \geq n = f_o$. Note also that no vertex $v$ of $H$ is the central vertex of more than one $H_i$; for otherwise $\deg_{H_i} v > 2n > p$, a contradiction. If each vertex of $H$ is the central vertex of some $H_i$, then $q = rp/2 = np$ so that $r = 2n > p$, a contradiction. Thus, some vertex $v$ of $H$ is not the central vertex of any $H_i$. Hence, $r = \deg_{H_i} v \leq m$. But then,

$$q = rp/2 < mn = q,$$

again a contradiction. Thus, $p_o \geq 2n$ and we conclude that $p_o = 2n$. 

We next investigate the value of $r_o(T)$ for an arbitrary tree $T$. We will make extensive use of the following (known) lemma.

**Lemma 5B**

The number of end-vertices in a nontrivial tree $T$ is equal to

$$2 + \sum_{v \in V(T) \setminus \{v : \deg v = 3\}} (\deg v - 2).$$
With the aid of this lemma, we may prove the following theorem.

**Theorem 5.5**

For every nontrivial tree $T$,

$$r(o)(T) = \Delta(T) \text{ if } \Delta(T) \text{ is even,}$$

and

$$\Delta(T) \leq r(o)(T) \leq \Delta(T) + 1 \text{ if } \Delta(T) \text{ is odd.}$$

**Proof:** Certainly $r(o)(T) \geq \Delta(T)$. Now, define

$$r = \begin{cases} 
\Delta(T) & \text{if } \Delta(T) \text{ is even,} \\
\Delta(T) + 1 & \text{if } \Delta(T) \text{ is odd.}
\end{cases}$$

We will construct a connected, $r$-regular, $T$-factorable graph $H$.

First we construct two other $T$-factorable graphs $F$ and $G$.

**Construction of $F$:** Let $U$ and $V$ be the partite sets of (the bipartite graph) $T$, and define

$$A = \{ u \in U \mid 2 \leq \deg_T u \leq r/2 \}$$

and

$$B = \{ v \in V \mid 2 \leq \deg_T v \leq r/2 \}.$$ 

For each vertex $w \in A \cup B$, define

$$n_w = \lfloor r/\deg_T w \rfloor$$

and set
Let $T_1, T_2, \ldots, T_{m_a m_b}$ be distinct trees, each isomorphic to $T$; these trees will be used to construct $F$.

For each vertex $u \in A$, and each $i = 0, 1, \ldots, (m_a m_b/n_u) - 1$, identify the vertices that correspond to $u$ in the $n_u$ trees $T_{in_u+1}, T_{in_u+2}, \ldots, T_{(i+1)n_u}$.

Doing this for all vertices in $A$ yields a graph $F_o$ having $m_b$ components, the $i$-th of which has the $T$-factorization $T_{im_a+1} \oplus T_{im_a+2} \oplus \cdots \oplus T_{(i+1)m_a}$ where $i = 0, 1, \ldots, m_b - 1$. Moreover, since $A$ is an independent set, no multiple edges were introduced into $F_o$.

Now, for each vertex $v \in B$, identify the vertices that correspond to $v$ in the $n_v$ factors in the set $\{T_{(in_v+j)m_a+k} | j = 0, 1, \ldots, n_v-1\}$.

do this for $i = 0, 1, \ldots, m_b/n_v - 1$ and $k = 1, 2, \ldots, m_a$. Since $B$ is an independent set, and since each of the above sets contains exactly
one factor from each component of $F^*_o$, the graph $F$ that we obtain
by this vertex identification procedure has no multiple edges and is
connected. Also,

$$F = T_1 \Theta T_2 \Theta \ldots \Theta T_m$$

Since each vertex $v$ of $F$ either belongs to exactly one factor $T_i$,
or was obtained by identifying corresponding vertices in several $T_i$'s,
there is exactly one vertex of $T$ that corresponds (in every $T_i$
that contains $v$) to $v$; we denote this vertex of $T$ by $v^*$. If $v \in V(F)$ and $v^* \in A \cup B$, then $v$ belongs to $\left\lfloor \frac{r}{\deg_{T_i}v^*} \right\rfloor$
factors $T_i$ and hence

$$\deg_Fv = \left\lfloor \frac{r}{\deg_{T_i}v^*} \right\rfloor \cdot \deg_Tv^* .$$

If we define

$$J = \{v \in V(F) | v^* \in A \cup B \text{ and } \deg_{T_i}v^* \geq 3 \}$$

and

$$K = \{v \in V(F) | \deg_{T_i}v^* > \frac{r}{2} \},$$

then, by Lemma 5B, there are more than

$$\sum_{v \in J} \left\lfloor \frac{r}{\deg_{T_i}v^*} \right\rfloor \cdot (\deg_{T_i}v^* - 2) + \sum_{v \in K} (\deg_{T_i}v^* - 2)$$

vertices of degree 1 in $F$. Also, if $v \in J$, then
If $v \in K$, then
\[
deg^{*}_r - 2 \geq r/2 - 1
\]
\[
\geq r - \deg^{*}_r = r - \deg^{*}_r.
\]
Thus, the number of end-vertices in $F$ exceeds
\[
\sum_{v \in V(F)} (r - \deg^{*}_v). \quad \text{deg}_{\Gamma}v \neq 1
\]
We will use this observation in the construction of a T-factorable graph $G$ each of whose vertices will have degree 1 or $r$.

**Construction of $G$:** If the degree of each vertex of $F$ is either 1 or $r$, let $G = F$. Otherwise, proceed as follows.

Let
\[
m = \max\{r - \deg^{*}_v | v \in V(F) \text{ and } \deg^{*}_v \neq 1\},
\]
let $G_0 = F$, and let $t_0$ denote the number of vertices of degree 1 in $G_0$. We now describe an $m$-step procedure for constructing $G$.

The graph constructed in Step $n$, where $1 \leq n \leq m$, will be denoted $G_n$, and $t_n$ will represent the number of end-vertices in $G_n$. Also,
\[ \text{deg}_n v \] will denote the degree of the vertex \( v \) in \( G_n \), where \( 0 \leq n \leq m \).

We have seen that \( G_0 \) is connected and T-factorable, and that

\[ t_0 > \sum_{v \in V(G_0)} (r-\text{deg}_0 v). \quad \text{deg}_0 v \neq 1 \]

We begin Step \( n \) (for \( 1 \leq n \leq m \)) with the (inductive) hypothesis that \( G_{n-1} \) is connected and T-factorable, and that

\[ t_{n-1} > \sum_{v \in V(G_{n-1})} (r-\text{deg}_{n-1} v). \quad \text{deg}_{n-1} v \neq 1 \]

Step \( n \) (\( 1 \leq n \leq m \)): Let \( F_1, F_2, \) and \( F_3 \) be three distinct graphs, each isomorphic to \( G_{n-1} \). Let

\[ N_i = \{ v \in V(F_i) | 1 < \text{deg}_{F_i} v < r \} \]

for \( i = 1, 2, 3 \), and set \( s = |N_1| = |N_2| = |N_3| \). Since each \( F_i \) has \( t_{n-1} \) end-vertices, and since

\[ t_{n-1} > \sum_{v \in N_i} (r-\text{deg}_{F_i} v) > s \]

for each \( i = 1, 2, 3 \), we may choose \( s \) end-vertices in \( F_{i-1} \) and identify each one of these with exactly one vertex of \( N_i \), thus increasing the degree of each vertex in \( N_i \) by 1 (for \( i = 1, 2, 3 \) with subscripts modulo 3). Since \( s \geq 1 \), and since \( F_1, F_2, \) and \( F_3 \) are connected, the graph \( G_n \) which results from this vertex identification process is connected. Also, it is easily seen
that $G_n$ has no multiple edges. Since $F_1$, $F_2$, and $F_3$ are each T-factorable, so too is $G_n$. Finally we note that $t_n$, the number of end-vertices in $G_n$, satisfies the following:

$$t_n = 3t_{n-1} - 3s$$

$$> 3 \sum_{v \in V(G_n)} (r - \text{deg}_n v) - 3s$$

$$= \sum_{v \in V(G_n)} (r - \text{deg}_n v).$$

Note that upon completing Step $m$, we will have constructed a connected, T-factorable graph $G_m$ in which each vertex has degree 1 or $r$. Thus, we let $G = G_m$.

Let $H_1, H_2, \ldots, H_r$ be $r$ distinct graphs, each isomorphic to $G$. Identify each end-vertex in $H_1$ with the corresponding end-vertices in $H_2, \ldots, H_r$; call the resulting graph $H$. Then, $H$ is connected and $r$-regular. Also, since each of the subgraphs $H_1, H_2, \ldots, H_r$ is T-factorable, $H$ is T-factorable. We conclude that $r_o(T) \leq r$.

If $T$ is a tree whose maximum degree is odd, then, by Theorem 5.5, we know that $\Delta(T) \leq r_o(T) \leq \Delta(T) + 1$. There are trees with odd maximum degree for which $r_o(T) = \Delta(T)$; consider for example $T = K(1,2n+1)$ where $n \geq 1$. In what follows, we will illustrate a tree $T$ whose maximum degree is $\Delta(T) = 2n + 1$, where $n \geq 1$, and we will verify that $r_o(T) = 2n + 2 = \Delta(T) + 1$; hence we will see
that the bounds in the inequality of Theorem 5.5 are sharp.

Let $n$ be a positive integer, and let $T$ be the tree with vertex set

$$V(T) = \{u, v, w\} \cup \{u_i | i = 1, 2, \ldots, 2n-1\}$$

$$\cup \{v_i, w_i | i = 1, 2, \ldots, n\}$$

$$\cup \{v_{ij}, w_{ij} | i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n\}$$

and edge set

$$E(T) = \{uv, uw\} \cup \{uu_i | i = 1, 2, \ldots, 2n-1\}$$

$$\cup \{vv_i, ww_i | i = 1, 2, \ldots, n\}$$

$$\cup \{v_{ij}, w_{ij} | i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n\}$$

We illustrate $T$ in Figure 5.1. Note that $T$ has exactly one vertex of degree $2n + 1$, namely $u$, and $2n + 2$ vertices of degree $n + 1$, namely $v, w, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$. The remaining $2n^2 + 2n - 1$ vertices of $T$ are end-vertices.

To show that $r_o(T) = \Delta(T) + 1 = 2n + 2$, we assume to the contrary that $r_o(T) = \Delta(T) = 2n + 1$. Then there is a connected $(2n+1)$-regular graph $H$ which is $T$-factorable. Let

$$H = T_1 \oplus T_2 \oplus \cdots \oplus T_k$$

be a $T$-factorization of $H$. Each vertex of $H$ that has degree $n + 1$ in some factor $T_i$ must be an end-vertex in exactly $n$ other distinct $T_j$'s, $j \neq i$. Also, since each factor $T_i$ has $2n + 2$ vertices of
Figure 5.1
degree $n + 1$, we see that $H$ has $k(2n+2)$ vertices which correspond to vertices of degree $n + 1$ in the subgraphs $T_1, T_2, \ldots, T_k$. Thus, if we let $t_i$ denote the number of end-vertices in the factor $T_i$, for $i = 1, 2, \ldots, k$, we have

$$\sum_{i=1}^{k} t_i \geq k(2n+2)n.$$ 

However, since $T$ has $2n^2 + 2n - 1$ end-vertices,

$$\sum_{i=1}^{k} t_i = k(2n^2 + 2n - 1),$$

a contradiction to the inequality above. We conclude, by Theorem 5.5, that $r_o(T) = 2n + 2 = \Delta(T) + 1$.

Theorem 5.5 tells us that if the maximum degree of a tree $T$ is odd, then $\Delta(T) \leq r_o(T) \leq \Delta(T) + 1$. If $\Delta(T) = 3$, we can say more.

**Proposition 5.4**

If $T$ is a tree with maximum degree $\Delta(T) = 3$, and if $t_i$ denotes the number of vertices of degree $i$ in $T$, for $i = 1, 2, 3$, then

$$r_o(T) = \begin{cases} 3 & \text{if } t_1 \geq t_2 \\ 4 & \text{if } t_1 < t_2. \end{cases}$$

**Proof:** Suppose first that $t_1 \geq t_2$. Let $T_1, T_2,$ and $T_3$ be three copies of $T$. Since $t_1 \geq t_2$, we can choose $t_2$
end-vertices from $T_i$ and identify each one of these with one vertex of degree 2 in $T_{i+1}$, (for $i = 1, 2, 3$ with subscripts modulo 3) and obtain a connected $T$-factorable graph $F$ in which each vertex has degree 1 or 3. If $F$ is 3-regular, then we see that $r_0(T) = 3$. If $F$ is not 3-regular, let $F_1, F_2,$ and $F_3$ be isomorphic copies of $F$. Form a 3-regular connected graph $G$ by identifying each end-vertex in $F_1$ with the corresponding end-vertices in $F_2$ and $F_3$. Since $F_1, F_2,$ and $F_3$ are $T$-factorable, so too is $G$. Hence, if $t_1 \geq t_2$, then $r_0(T) = 3$.

Now, suppose that $t_1 < t_2$, and assume that $r_0(T) = 3$. Then, there exists a 3-regular, connected, $T$-factorable graph $H$. Let $H = T_1 \oplus T_2 \oplus \ldots \oplus T_k$ be a $T$-factorization of $H$, and let $t_i(T_j)$ denote the number of vertices of degree $i$ in $T_j$, for $i = 1, 2$ and $j = 1, 2, \ldots, k$. Clearly $t_i = t_i(T_j)$ for $i = 1, 2$ and $j = 1, 2, \ldots, k$.

Since $H$ is 3-regular, each vertex of $H$ which has degree 2 in some $T_i$ is an end-vertex in exactly one $T_j$, $j \neq i$; there must be precisely $kt_2$ such vertices in $H$. Thus,

$$\sum_{j=1}^{k} t_i(T_j) \geq kt_2$$

or

$$kt_1 \geq kt_2.$$
This contradicts the hypothesis that $t_1 < t_2$. Hence, $r_o(T) \neq 3$, and by Theorem 5.5, we see that $r_o(T) = 4$. ■

It appears that the determination of good bounds on $p_o(T)$ and $f_o(T)$, for an arbitrary tree $T$, is extremely difficult. A related and well-known problem is the unresolved Graceful Tree Conjecture. A tree $T$ is **gracefully numbered** when each vertex $v$ is assigned a value $\psi(v)$, each edge $uv$ is assigned the value $\psi(uv) = |\psi(u) - \psi(v)|$, and the set equations

$$\{\psi(v) | v \in V(T)\} = \{1, 2, ..., p\},$$

$$\{\psi(uv) | uv \in E(T)\} = \{1, 2, ..., p-1\}$$

hold, where $p$ is the order of $T$. A tree is **graceful** if it admits a graceful numbering. The Graceful Tree Conjecture asserts that every tree is graceful. Rosa [19] showed that if $T$ is a graceful tree of order $p$, then $K_{2p-1}$ is $T$-factorable. Thus, if the Graceful Tree Conjecture is ever verified, it will follow that

$$p \leq p_o(T) \leq 2p - 1$$

and

$$2 \leq f_o(T) \leq 2p - 1$$

for every tree $T$ of order $p$. 

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