A Generalized Dirac Equation with Spin I

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A GENERALIZED DIRAC EQUATION
WITH SPIN I

by

Tilahun Eneyew

A Thesis
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in partial fulfillment
of the
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Tilahun Entewew
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>4</td>
</tr>
<tr>
<td>III</td>
<td>7</td>
</tr>
<tr>
<td>IV</td>
<td>14</td>
</tr>
<tr>
<td>V</td>
<td>21</td>
</tr>
<tr>
<td>VI</td>
<td>31</td>
</tr>
<tr>
<td>VII</td>
<td>55</td>
</tr>
<tr>
<td>VIII</td>
<td>84</td>
</tr>
</tbody>
</table>

CHAPTER I INTRODUCTION

CHAPTER II MAXWELL'S EQUATIONS FOR THE ELECTROMAGNETIC FIELD

CHAPTER III THE MEANING OF THE \( \vec{M} \) VECTOR MATRIX

CHAPTER IV SPIN 1 (VECTOR) FIELD

CHAPTER V THE SIGMA MATRIX

Construction of a Sigma Matrix for the Vector Field

Generalization of the \( \Sigma(\omega) \)-Matrix to Any Spin Field

CHAPTER VI THE SIGMA MATRIX AND MAXWELL'S EQUATIONS

Maxwell's Equations in Terms of \( \Sigma(\omega) \) Matrix

Generalization of Maxwell's Equations to a Massive Field Equation

The Generalized Dirac Equation in the Presence of a Source

Scalar and Vector Potentials of a Massive Field

CHAPTER VII SPECIAL APPLICATION OF THE GENERALIZED DIRAC EQUATION

The Vector Boson Equation

A Free Particle Solution of the Vector Boson Equation

The Vector Boson Equation in a Central Field

CHAPTER VIII THE "DIRAC MATRICES" OF THE VECTOR BOSON EQUATION

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>IX</td>
<td>THE GAMMA MATRICES AND LORENTZ TRANSFORMATION</td>
</tr>
<tr>
<td>X</td>
<td>CONCLUSION</td>
</tr>
<tr>
<td></td>
<td>BIBLIOGRAPHY</td>
</tr>
<tr>
<td>Properties of the &quot;Dirac Matrices&quot; for Spin 1</td>
<td>84</td>
</tr>
<tr>
<td>The Vector Boson Equation in a Covariant Form</td>
<td>91</td>
</tr>
</tbody>
</table>
FIGURES

FIGURE 1: ROTATION OF $\mathbf{r}$ ABOUT AN AXIS $\mathbf{n}$ BY ANGLE $\phi$.

PAGE 8
I. INTRODUCTION

The Klein-Gordon equation and the Dirac equation describe particles of spin 0 and $\frac{1}{2}$ respectively. In this paper the idea of the Dirac equation is extended to particles of non-zero rest mass with higher spin. To do this, we first start with the classical Maxwell equations and express them in terms of a spin matrix which can then be extended to give a generalized Pauli spin matrix.

The first two chapters are devoted to finding a spin matrix of the vector (electromagnetic) field. From the vector equations which give Ampere's Law and Faraday's Law, a vector matrix $\overrightarrow{M}$ is constructed. In Chapter III the physical meaning of $\overrightarrow{M}$ is investigated in terms of the rotation of a vector function. It turns out that $\overrightarrow{M}$ is, in fact, related to the spin $\overrightarrow{S}$ of the vector field. In Chapter IV the properties of $\overrightarrow{S}$ are discussed in detail. $\overrightarrow{S}$ has three components $S_x$, $S_y$ and $S_z$. By using the properties of $S_z$, a unitary transformation is developed so as to write the components of $\overrightarrow{S}$ in a form that is analogous to the components of the spin of a spinor field. Using this same unitary transformation, we can write the vector operators (gradient, divergence and curl) as matrix opera-

tors. Then one can write down Maxwell's equations using $\overline{S}$ and the matrix forms of the vector operators.

In Chapter V we develop a $4 \times 4$ sigma matrix. This sigma matrix is constructed from $\overline{S}$ and the matrix forms of the vector operators. The sigma matrix is then extended to any spin I to give a generalized Pauli spin matrix. In Chapter VI the $4 \times 4$ sigma matrix is used to express Maxwell's equations in an elegant way. Then a comparison of Maxwell's equations in terms of the sigma matrix with the Dirac equations automatically leads to an extension of the latter to a higher spin field equation by making use of the generalized Pauli spin matrix. In fact, there are two ways of making the extension, as discussed in section two of Chapter VI. But only one of them seems to be physically correct. The equations which scalar and vector potentials of a massive field should satisfy are discussed in section four of Chapter VI.

In Chapters VII and VIII the vector boson equation is discussed as a particular example of the generalized Dirac equation. A free particle solution of the vector boson equation is obtained explicitly, and a formal treatment of a vector boson in a central field is briefly discussed. As in the case of the Dirac equation, the vector boson equation can be elegantly written using gamma matrices. The construction of the gamma matrices and their interesting properties are studied in Chapter VII. Using these
matrices the vector boson equation could be written in a covariant form. Finally, the transformation properties of functions which are bilinear in the wave function of the vector boson and its adjoint are investigated quite briefly.
II. MAXWELL'S EQUATIONS FOR THE ELECTROMAGNETIC FIELD

The Maxwell equations of the electromagnetic field are given in Gaussian units as:

\[ \nabla \cdot \mathbf{E} = 4 \pi \rho, \quad 1 \text{ (Coulomb's Law)} \]

\[ \nabla \cdot \mathbf{B} = 0, \quad 2 \text{ (Gauss's Law-No magnetic charge)} \]

\[ \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\frac{4 \pi}{c} \mathbf{J}, \quad 3 \text{ (Ampere's Law)} \]

\[ \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \quad 4 \text{ (Faraday's Law)} \]

Where \( \mathbf{E} \) is the electric field, \( \mathbf{B} \) is the magnetic field, \( \rho \) is the charge density and \( \mathbf{J} \) is the current density.

The last two are linear differential equations in space and time, and they can be written in matrix form as follows:

First \( \mathbf{E} = E_x \mathbf{i} + E_y \mathbf{j} + E_z \mathbf{k} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \)

\[ \mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \]

We also have:

\[ \nabla \times \mathbf{E} = \begin{pmatrix} \frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y \\ \frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z \\ \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x \end{pmatrix} \begin{pmatrix} 0 & -\frac{2}{\partial x} & \frac{2}{\partial y} \\ \frac{2}{\partial y} & 0 & -\frac{2}{\partial z} \\ -\frac{2}{\partial z} & \frac{2}{\partial x} & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \]
Similarly $\nabla \times \mathbf{B} = \begin{pmatrix} 0 & -\frac{2}{\delta z} & \frac{2}{\delta y} \\ \frac{2}{\delta z} & 0 & -\frac{2}{\delta x} \\ -\frac{2}{\delta y} & \frac{2}{\delta x} & 0 \end{pmatrix}$

Thus we can now write equations 3 and 4 in the matrix forms.

\[
\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + \begin{pmatrix} 0 & -\frac{2}{\delta z} & \frac{2}{\delta y} \\ \frac{2}{\delta z} & 0 & -\frac{2}{\delta x} \\ -\frac{2}{\delta y} & \frac{2}{\delta x} & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0
\]

Consider the matrix:

\[
\begin{pmatrix} 0 & -\frac{2}{\delta z} & \frac{2}{\delta y} \\ \frac{2}{\delta z} & 0 & -\frac{2}{\delta x} \\ -\frac{2}{\delta y} & \frac{2}{\delta x} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \frac{2}{\delta x}
\]

\[
+ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \frac{2}{\delta y} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{2}{\delta x}.
\]

If we now define:

\[
M_H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
then we can define a vector matrix \( \mathbf{M} = M_x \mathbf{i} + M_y \mathbf{j} + M_z \mathbf{k} \) such that
\[
\begin{pmatrix}
0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} \\
-\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0
\end{pmatrix}
= \mathbf{M} \cdot \nabla
\]

Now, using \( \mathbf{M} \) we can write Maxwell's equations as:
\[
\nabla \cdot \mathbf{E} = 4\pi \rho, \\
\nabla \cdot \mathbf{B} = 0, \\
\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - (\mathbf{M} \cdot \nabla) \mathbf{B} = -\frac{4\pi}{c} \mathbf{J}, \\
\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{M} \cdot \nabla) \mathbf{E} = 0.
\]

We shall next investigate the physical meaning of the vector matrix.
III. THE MEANING OF THE $\mathbf{M}$-VECTOR MATRIX

The rotation of a vector function $A(\mathbf{r})$ in three-dimensional space can be represented by a rotation operator $\mathbf{R}(\phi)$. This rotation operator $\mathbf{R}(\phi)$ can be written in terms of the angle of rotation $\phi$ and the axis of rotation $\mathbf{n}$. We also know from quantum mechanics that in the case of infinitesimal rotations the rotation operator can be represented in terms of the total angular momentum of the system and the angle of rotation. In this section we shall investigate the possibility of using the vector matrix $\mathbf{M}$ in the representation of $\mathbf{R}(\phi)$. 

Consider the rotation of a position vector $\mathbf{r}$ about $\mathbf{n}$ by an angle $\phi$ (see Figure 1). Such a rotation can be given by:

$$\mathbf{r}' = \mathbf{R}(\phi \mathbf{n}) \mathbf{r}$$

where $\mathbf{R}(\phi \mathbf{n})$ is a rotational matrix. In a matrix form the above equation can be written as:

$$
\begin{pmatrix}
X' \\
Y' \\
Z'
\end{pmatrix} =
\begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}
$$

Using a transformation from the $(\mathbf{\ell}, \mathbf{j}, \mathbf{k})$ to the $(\mathbf{\ell}', \mathbf{j}', \mathbf{k}')$ system one can find the rotational matrix $\mathbf{R}(\phi \mathbf{n})$ to be:
Figure 1. Rotation of $\mathbf{r}$ about an axis $\mathbf{n}$ by an angle $\phi$. 

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\[
\tilde{R}(\phi_n) = (1 - \cos \phi) \begin{pmatrix}
\eta_x \\
\eta_y \\
\eta_z
\end{pmatrix} + \begin{pmatrix}
\cos \phi, -\eta_z \sin \phi, \eta_y \sin \phi \\
\eta_z \sin \phi, \cos \phi, -\eta_x \sin \phi \\
-\eta_y \sin \phi, \eta_x \sin \phi, \cos \phi
\end{pmatrix}
\]

Consider the second term on the right side of the above matrix equation:

\[
\begin{pmatrix}
\cos \phi - \eta_z \sin \phi & \eta_y \sin \phi \\
\eta_y \sin \phi & \cos \phi - \eta_x \sin \phi \\
-\eta_y \sin \phi & \eta_x \sin \phi & \cos \phi
\end{pmatrix}
= \begin{pmatrix}
0 & -\eta_z & \eta_y \\
\eta_z & 0 & -\eta_x \\
-\eta_y & \eta_x & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} \eta_x + \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix} \eta_y + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \eta_z
\]

\[
= \mathbf{M} \cdot \eta
\]

If we consider an infinitesimal rotation, then \(\cos \phi \approx 1\) and \(\sin \phi \approx \phi\).

Then, \(\tilde{R}(\phi_n) = \mathbf{T} + \phi (\mathbf{M} \cdot \eta)\).

We can also define an inverse rotational operator such that

\[
\mathbf{R} = \tilde{R}^{-1}(\phi_n) \tilde{R}^{-1}.
\]
but \( \widehat{R}(\phi) = \widehat{R}(\phi_{\infty}) = \widehat{R} \cdot \phi(\vec{m} \cdot \vec{n}) \) for an infinitesimal rotation.

Consider a vector function \( A(\vec{r}) \). Under infinitesimal rotation of \( A(\vec{r}) \) we get:

\[
A_R(\vec{r}_R) = \widehat{R}(\phi_{\infty}) A(\vec{r}) = \widehat{R}(\phi_{\infty}) A(\widehat{R}^{-1}(\phi_{\infty}) \vec{r}_R). \tag{1}
\]

Similarly we get:

\[
A_R(\vec{r}) = \widehat{R}(\phi_{\infty}) A(\widehat{R}^{-1}(\phi_{\infty}) \vec{r}). \tag{2}
\]

From (2) we have:

\[
A_R(\vec{r}) = \widehat{R}(\phi_{\infty}) A(\vec{r} - \phi(\vec{m} \cdot \vec{n}) \vec{r})
= \left[ \vec{r} + \vec{r} \phi(\vec{m} \cdot \vec{n}) \right] A(\vec{r} - \phi(\vec{m} \cdot \vec{n}) \vec{r})
= \vec{r} A(\vec{r} - \phi(\vec{m} \cdot \vec{n}) \vec{r}) + \phi(\vec{m} \cdot \vec{n}) A(\vec{r} - \phi(\vec{m} \cdot \vec{n}) \vec{r})
A(\vec{r} - \phi(\vec{m} \cdot \vec{n}) \vec{r}) = A(x - \phi(n_x z - n_z y), y - \phi(n_y x - n_x z),
\]
\[
= A(x', y', z'),
\]

where \( x' = x - \phi(n_x z - n_z y), \ y' = y - \phi(n_y x - n_x z), \ z' = z - \phi(n_y x - n_x y) \).

If we make a Taylor expansion of this function about a point \((x, y, z)\) we get:

\[
A(x', y', z') = A(x, y, z) + \left[ \frac{\partial A(x', y', z')}{\partial x'} \right] (x' - x) + \left[ \frac{\partial A(x', y', z')}{\partial y'} \right] (y' - y)
\]

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This reduces to:

\[ A(x', y', z') = A(x, y, z) - \frac{\partial A(x, y, z)}{\partial x} \phi(n_z - n_y) \]

\[ - \frac{\partial A(x, y, z)}{\partial y} \phi(n_z x - n_x z) - \frac{\partial A(x, y, z)}{\partial z} \phi(n_y - n_x y) \]

\[ + \frac{\partial^2 A(x, y, z)}{\partial x^2} \phi^2(n_z^2 - n_y^2) \]

Since we are considering infinitesimal rotations, \( \phi^2 \) is very small, and we can neglect second order terms.

Hence

\[ A(\vec{r} - \phi(\vec{n}, \vec{n}) \vec{r}) \approx A(x, y, z) - \frac{\partial A(x, y, z)}{\partial x} \phi(n_z - n_y) \]

\[ - \frac{\partial}{\partial y} A(x, y, z) \phi(n_z x - n_x z) \]

\[ - \frac{\partial}{\partial z} A(x, y, z) \phi(n_y - n_x y) \]

Rearranging terms we get:

\[ A(\vec{r} - \phi(\vec{n}, \vec{n}) \vec{r}) \approx A(x, y, z) - n_x \phi\left(\frac{y}{\frac{\partial}{\partial y}} - \frac{x}{\frac{\partial}{\partial z}}\right) A(x, y, z) \]

\[ - n_y \phi\left(\frac{\partial}{\partial x} - \frac{y}{\frac{\partial}{\partial z}}\right) A(x, y, z) \]

\[ - n_z \phi\left(\frac{x}{\frac{\partial}{\partial x}} - \frac{y}{\frac{\partial}{\partial y}}\right) A(x, y, z) \]
But from classical mechanics the orbital angular momentum is given by \( \mathbf{L} = \frac{\hbar}{i} \mathbf{r} \times \nabla \), classically \( \mathbf{k} = 1 \).

We can identify:

\[
\begin{align*}
    \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial y^2} &= i \mathbf{L} \cdot \mathbf{x}, \\
    \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial z^2} &= i \mathbf{L} \cdot \mathbf{y}, \\
    \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial x^2} &= i \mathbf{L} \cdot \mathbf{z}.
\end{align*}
\]

Thus we now have:

\[
A(\mathbf{r} - \phi(\mathbf{m} \cdot \mathbf{n}) \mathbf{r}) = A(\mathbf{r}) - i \phi(\mathbf{n} \cdot \mathbf{L} + \mathbf{m} \cdot \mathbf{L}) A(\mathbf{r}),
\]

Substituting this into our original equation we get:

\[
A_R(\mathbf{r}) = \mathbb{H} A(\mathbf{r}) - i \phi(\mathbf{L} \cdot \mathbf{n}) A(\mathbf{r})
\]

\[
+ \phi(\mathbf{m} \cdot \mathbf{n}) [A(\mathbf{r}) - i \phi(\mathbf{L} \cdot \mathbf{n}) A(\mathbf{r})]
\]

\[
= \mathbb{H} A(\mathbf{r}) - i \phi(\mathbf{L} \cdot \mathbf{n}) A(\mathbf{r}) + \phi(\mathbf{m} \cdot \mathbf{n}) A(\mathbf{r})
\]

\[
- i \phi^2 (\mathbf{L} \cdot \mathbf{n})(\mathbf{m} \cdot \mathbf{n}) A(\mathbf{r})
\]

Neglecting the second order term in \( \phi^2 \) we get:

\[
A_R(\mathbf{r}) \simeq A(\mathbf{r}) + \left[ - i \phi(\mathbf{L} \cdot \mathbf{n}) + (\mathbf{m} \cdot \mathbf{n}) \phi \right] A(\mathbf{r})
\]

\[
= \left[ \mathbb{H} - i \phi(\mathbf{L} \cdot \mathbf{n}) + \phi(\mathbf{m} \cdot \mathbf{n}) \right] A(\mathbf{r})
\]

But from quantum mechanics we know that if \( \mathbf{j} \) is the total
angular momentum of a system, its component along any axis of rotation $\overrightarrow{\mathbf{n}}$ is related to the operator for infinitesimal rotation about that axis by the relation:

$$\hat{R}(\phi \overrightarrow{\mathbf{n}}) = e^{i(\vec{J} \cdot \overrightarrow{\mathbf{n}})\phi}$$

Using this representation, $\hat{A}(\overrightarrow{\mathbf{r}})$ under rotation by $\phi$ about $\overrightarrow{\mathbf{n}}$ gives:

$$\hat{A}(\overrightarrow{\mathbf{r}}) = e^{-i(\vec{J} \cdot \overrightarrow{\mathbf{n}})\phi} \hat{A}(\overrightarrow{\mathbf{r}})$$

$$= \left[ \overrightarrow{\mathbf{n}} - i \phi (\vec{J} - \vec{s}) \cdot \overrightarrow{\mathbf{n}} \right] \hat{A}(\overrightarrow{\mathbf{r}}) \quad \text{(for infinitesimal rotation)}$$

$$= \left[ \overrightarrow{\mathbf{n}} - i \phi \left( \vec{L} + \vec{s} \cdot \overrightarrow{\mathbf{n}} \right) \right] \hat{A}(\overrightarrow{\mathbf{r}})$$

$$= \left[ \overrightarrow{\mathbf{n}} - i \phi \left( \vec{L} \cdot \overrightarrow{\mathbf{n}} + \vec{s} \cdot \overrightarrow{\mathbf{n}}^2 \right) \right] \hat{A}(\overrightarrow{\mathbf{r}}) \quad \text{(4)}$$

Now, comparing (3) and (4) we see that $\overrightarrow{\mathbf{M}} = -i \overrightarrow{\mathbf{s}}$. Thus the $\overrightarrow{\mathbf{M}}$-vector matrix is related to the spin of the vector field\(^1\).

\(^1\)This is taken from a lecture given by Dr. M. Soga in a physics seminar at the Physics Department, Western Michigan University.
IV. SPIN 1 (VECTOR) FIELD

Let \( A(\mathbf{r}) \) be a vector field associated with a physical system. This field can be the electromagnetic field or the wave function of a particle of spin 1. By studying how \( A(\mathbf{r}) \) transforms under rotation, one can explicitly determine the total angular momentum associated with the field and in particular the form of the intrinsic angular momentum \( \mathbf{S} \). In Chapter III we have seen that \( \mathbf{S} = i \mathbf{M} \). Therefore, in the \((x, y, z)\) representation components of \( \mathbf{S} \) are given by:

\[
S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

We see that \( S^2 = S_x^2 + S_y^2 + S_z^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \)

Hence the eigenvalues of \( S^2 \) are \( S(S+1) \) which implies \( S = 1 \). \( S_z \) has three eigenvalues \( 1, 0, -1 \). Suppose \( \chi_1, \chi_0, \chi_{-1} \) are eigenvectors of \( S_z \) associated with the eigenvalues \( 1, 0 \) and \(-1\) respectively, i.e.,

\[1\text{Messiah, Quantum Mechanics Vol., 2, p. 249-250.}\]
$S_0 \chi_1 = 1 \chi_1, \quad S_2 \chi_0 = 0 \chi_0 \quad \text{and} \quad S_3 \chi_1 = -1 \chi_1$.

We can write $\chi_1, \chi_0$ and $\chi_1$ as:

$$\chi_i = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad \chi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \chi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

which are orthonormal column vectors.

$$\chi_i' = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \quad \chi_0' = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_1' = \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}$$

Let $\mathbf{U}$ be a unitary transformation such that

$$\chi_i' = \mathbf{U} \chi_i, \quad \chi_0' = \mathbf{U} \chi_0, \quad \text{and} \quad \chi_1' = \mathbf{U} \chi_1$$.

Then, the elements of $\mathbf{U}$ are obtained as follows:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$$

$$-\frac{1}{\sqrt{2}} u_{11} - i \frac{1}{\sqrt{2}} u_{12} = 1,$$

$$-\frac{1}{\sqrt{2}} u_{21} - i \frac{1}{\sqrt{2}} u_{22} = 0 \Rightarrow u_{21} = i u_{22},$$

$$-\frac{1}{\sqrt{2}} u_{31} - i \frac{1}{\sqrt{2}} u_{32} = 0 \Rightarrow u_{31} = -i u_{32}.$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u_{13} = 0, \quad u_{23} = 1, \quad u_{33} = 0,$$

from the first matrix equation above. From the second matrix equation we get:
\[
\frac{1}{\gamma} \left( \begin{array}{cc}
-\frac{1}{\gamma} & \frac{1}{\gamma} \\
\frac{1}{\gamma} & -\frac{1}{\gamma}
\end{array} \right) \left( \begin{array}{c}
0 \\
1
\end{array} \right) = 0, \quad \Rightarrow \left( \begin{array}{c}
0 \\
\gamma
\end{array} \right) = \left( \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} \right) \left( \begin{array}{c}
0 \\
\gamma
\end{array} \right),
\]
\[
\frac{1}{\gamma} \left( \begin{array}{cc}
-\frac{1}{\gamma} & \frac{1}{\gamma} \\
\frac{1}{\gamma} & -\frac{1}{\gamma}
\end{array} \right) \left( \begin{array}{c}
1 \\
1
\end{array} \right) = 1, \quad \Rightarrow \left( \begin{array}{c}
0 \\
1
\end{array} \right) = \left( \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} \right) \left( \begin{array}{c}
0 \\
1
\end{array} \right)
\]
\[
\frac{1}{\gamma} \left( \begin{array}{cc}
-\frac{1}{\gamma} & \frac{1}{\gamma} \\
\frac{1}{\gamma} & -\frac{1}{\gamma}
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) = \left( \begin{array}{c}
1 \\
0
\end{array} \right), \quad \Rightarrow \left( \begin{array}{c}
1 \\
0
\end{array} \right) = \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right)
\]
\[
\frac{1}{\gamma} \left( \begin{array}{cc}
-\frac{1}{\gamma} & \frac{1}{\gamma} \\
\frac{1}{\gamma} & -\frac{1}{\gamma}
\end{array} \right) \left( \begin{array}{c}
0 \\
1
\end{array} \right) = \left( \begin{array}{c}
0 \\
1
\end{array} \right), \quad \Rightarrow \left( \begin{array}{c}
0 \\
1
\end{array} \right) = \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{c}
0 \\
1
\end{array} \right)
\]

Hence \( \hat{U} = \left( \begin{array}{ccc}
\frac{1}{\gamma} & 0 & 0 \\
0 & \frac{1}{\gamma} & 0 \\
0 & 0 & 0
\end{array} \right) \)

The inverse of \( \hat{U} \) is given by:

\[
\hat{U}^{-1} = (\hat{U}^*)^* = \hat{U}^* = \left( \begin{array}{ccc}
\frac{1}{\gamma} & 0 & \frac{1}{\gamma} \\
0 & \frac{1}{\gamma} & -\frac{1}{\gamma} \\
\frac{1}{\gamma} & -\frac{1}{\gamma} & 0
\end{array} \right).
\]

By using this unitary transformation we can write \( S_x \)

\( S_x \) and \( S_y \) in a way that is familiar in the case of \( S = \frac{1}{\gamma} \):

\[
S_x' = \hat{U} S_x \hat{U}^{-1}
\]

\[
= \left( \begin{array}{ccc}
\frac{1}{\gamma} & 0 & \frac{1}{\gamma} \\
0 & \frac{1}{\gamma} & -\frac{1}{\gamma} \\
\frac{1}{\gamma} & -\frac{1}{\gamma} & 0
\end{array} \right) \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0
\end{array} \right) \left( \begin{array}{ccc}
\frac{1}{\gamma} & 0 & \frac{1}{\gamma} \\
0 & \frac{1}{\gamma} & -\frac{1}{\gamma} \\
\frac{1}{\gamma} & -\frac{1}{\gamma} & 0
\end{array} \right)
\]

\[
= \left( \begin{array}{ccc}
0 & \frac{1}{\gamma} & 0 \\
\frac{1}{\gamma} & 0 & \frac{1}{\gamma} \\
0 & \frac{1}{\gamma} & 0
\end{array} \right).
\]
Note that $S_2$, $S_4$ and $S_3$ contain forms which are similar to the corresponding $S_2$, $S_4$ and $S_3$ for the spin $\frac{1}{2}$ case. Besides we see that:

$S_2 \chi'_1 = \chi'_1$, $S_4 \chi'_0 = 0 \chi'_0$, $S_3 \chi'_1 = -1 \chi'_1$.

We can now rewrite Maxwell's equations as:

\begin{align*}
\nabla \cdot \vec{E} &= 4\pi \rho, \quad (1) \\
\nabla \cdot \vec{B} &= 0, \quad (2) \\
\kappa \frac{\partial}{\partial t} \vec{E} + i(\vec{S} \cdot \nabla) \vec{B} &= -\frac{4\pi}{c} \vec{J}, \quad (3) \\
\kappa \frac{\partial}{\partial t} \vec{B} - i(\vec{S} \cdot \nabla) \vec{E} &= 0. \quad (4)
\end{align*}

By operation on $\vec{B}$ and $\vec{E}$ with the unitary transformation we developed above, we get:

\[
\tilde{U} \vec{E} = \tilde{U} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} (E_x - iE_y) \\ E_x \\ \frac{1}{2} (E_x + iE_y) \end{pmatrix} = \begin{pmatrix} E_i \\ E_0 \\ E_{-i} \end{pmatrix},
\]
\[ \tilde{U} \tilde{B} = \tilde{U} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} (B_x - iB_y) \\ B_x \\ \frac{1}{\sqrt{2}} (B_x + iB_y) \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}. \]

The three fundamental vector operators (gradient, divergence and curl) can now be written in matrix form using the unitary transformation developed above:

Gradient Operator \( \nabla \)

Let \( f \) be a scalar function such that:

\[ \nabla f = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \]

Multiplying by \( \tilde{U} \) from the left we get:

\[ \tilde{U} \nabla f = \tilde{U} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} E_1 \\ E_0 \end{pmatrix} \]

If we first find the product \( \tilde{U} \nabla \) and then operate on \( f \) we get:

\[ \begin{pmatrix} -\frac{1}{\sqrt{2}} (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \\ \frac{\partial}{\partial x} \\ \frac{1}{\sqrt{2}} (\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \end{pmatrix} f = \begin{pmatrix} E_1 \\ E_0 \end{pmatrix} \]

\[ \begin{pmatrix} -\frac{1}{\sqrt{2}} (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \\ \frac{\partial}{\partial x} \\ \frac{1}{\sqrt{2}} (\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \end{pmatrix} = \begin{pmatrix} -\frac{12}{12} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} \frac{12}{12} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \frac{\partial}{\partial y} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \frac{\partial}{\partial z}. \]

If we define \( G_x = \begin{pmatrix} -\frac{12}{12} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, G_y = \begin{pmatrix} \frac{12}{12} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \) and \( G_z = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \).
then $\nabla \equiv \vec{G} \cdot \nabla$.

The Divergence Operator:

Consider:

$$\nabla \cdot \vec{B} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

$$= \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\tau} & 0 & \frac{1}{\tau} \\ -\frac{1}{\tau} & 0 & -\frac{1}{\tau} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\tau} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \frac{1}{\tau} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

But $\begin{pmatrix} \frac{1}{\tau} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \frac{1}{\tau} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau}, 0, \frac{1}{\tau} \end{pmatrix} \frac{\partial}{\partial x}$

$+ \begin{pmatrix} -\frac{1}{\tau}, 0, \frac{1}{\tau} \end{pmatrix} \frac{\partial}{\partial y} + \begin{pmatrix} 0, 1, 0 \end{pmatrix} \frac{\partial}{\partial z}$.

Defining $\vec{D}_x = (-\frac{1}{\tau}, 0, \frac{1}{\tau}), \vec{D}_y = (-\frac{1}{\tau}, 0, -\frac{1}{\tau})$, and $\vec{D}_z = (0, 1, 0)$,

then we have $\begin{pmatrix} -\frac{1}{\tau} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \frac{1}{\tau} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \end{pmatrix} \equiv \vec{D} \cdot \nabla$.

So, the divergence operator can simply be substituted by its matrix form $\vec{D} \cdot \nabla$.

In a similar way we can write the curl operator in a matrix form. In fact, by looking at the Maxwell equations on page 17 and comparing them to the original equations on
one can see that the curl operator is related to
the spin matrix $\mathbf{S}$ and is given by $\text{curl} (\mathbf{V}) = -i (\mathbf{S} \cdot \mathbf{V})$.
This relation can easily be verified from the definition of
curl and the explicit form of the matrices $S_x$, $S_y$, and $S_z$.

Suppose $\mathbf{A}(\mathbf{r})$ is a vector function.

Then $\nabla \times \mathbf{A}(\mathbf{r}) = \begin{pmatrix} i & 0 & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{A}_x & \mathbf{A}_y & \mathbf{A}_z \end{pmatrix} = \mathbf{i} \left( \frac{\partial}{\partial y} \mathbf{A}_x - \frac{\partial}{\partial z} \mathbf{A}_y \right) - \mathbf{j} \left( \frac{\partial}{\partial z} \mathbf{A}_x - \frac{\partial}{\partial x} \mathbf{A}_z \right) + \mathbf{k} \left( \frac{\partial}{\partial x} \mathbf{A}_y - \frac{\partial}{\partial y} \mathbf{A}_x \right).$

$-i (\mathbf{S} \cdot \mathbf{V}) \mathbf{A}(\mathbf{r}) = -i \begin{pmatrix} S_x \frac{\partial}{\partial x} + S_y \frac{\partial}{\partial y} + S_z \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$

$= -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} A_x + \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} \end{pmatrix} A_y + \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} A_z$

$= -i \begin{pmatrix} 0 & i \frac{\partial}{\partial y} & i \frac{\partial}{\partial z} \\ -i \frac{\partial}{\partial z} & 0 & -i \frac{\partial}{\partial x} \\ i \frac{\partial}{\partial x} & -i \frac{\partial}{\partial y} & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$

The matrix product on the right side gives a $3 \times 1$ matrix
with components:

$\begin{pmatrix} \frac{\partial}{\partial y} A_x - \frac{\partial}{\partial z} A_y \\ \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \\ \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \end{pmatrix}.$

We see that these components of the $3 \times 1$ matrix are
exactly the same as the components of curl $\mathbf{A}(\mathbf{r})$. 

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V. THE SIGMA MATRIX

Construction of a Sigma Matrix for a Vector Field

We shall now write Maxwell's equations in terms of \( \tilde{\Sigma}, \tilde{D}, \) and \( \tilde{E} \) and develop a sigma matrix for these field equations.

\[
(\tilde{D} \cdot \nabla) \tilde{E} = 4\pi \rho, \quad (1)
\]

\[
(\tilde{D} \cdot \nabla) \tilde{B} = 0, \quad (2)
\]

\[
\kappa \frac{\partial \tilde{E}}{\partial t} + i(\tilde{\Sigma} \cdot \nabla) \tilde{B} = -\frac{4\pi}{\kappa} \tilde{J}, \quad (3)
\]

\[
\kappa \frac{\partial \tilde{B}}{\partial t} - i(\tilde{\Sigma} \cdot \nabla) \tilde{E} = 0 \quad (4)
\]

From quantum mechanics the energy and momentum operators are given by:

\[
E \leftrightarrow -\frac{\hbar}{2\kappa} \frac{\partial}{\partial t}, \quad \mathbf{p} \leftrightarrow \hbar \mathbf{\nabla}.
\]

Making these substitutions in equations 1 to 4 we get:

\[
(\tilde{D} \cdot \mathbf{p}) \tilde{E} = 4\pi \hbar \kappa \rho, \quad (5)
\]

\[
(\tilde{D} \cdot \mathbf{p}) \tilde{B} = 0, \quad (6)
\]

\[
(\hbar/c) \tilde{E} - i(\tilde{\Sigma} \cdot \mathbf{p}) \tilde{B} = 4\pi \hbar \kappa \mathbf{J}, \quad (7)
\]

\[
(\hbar/c) \tilde{B} + i(\tilde{\Sigma} \cdot \mathbf{p}) \tilde{E} = 0. \quad (8)
\]

Now we want to define a sigma matrix \( \tilde{\Xi} \) for the vector (electromagnetic) field such that \( \tilde{\Xi} \) satisfies the same conditions the Pauli matrices satisfy. Such a matrix can be constructed from \( \tilde{\Sigma}, \tilde{D} \) and \( \tilde{E} \). It has been shown that the following two matrices:

\[
\tilde{\Xi}^1 = \begin{pmatrix} \tilde{\Sigma} & -\mathbf{\nabla} \\ -\tilde{D} & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\Xi}^2 = \begin{pmatrix} \tilde{\Sigma} & \mathbf{\nabla} \\ \tilde{D} & 0 \end{pmatrix}
\]

are good candidates for the sigma matrix of the electro-
magnetic field\(^1\). Both \(\widetilde{\Sigma}^1\) and \(\widetilde{\Sigma}^2\) satisfy the requirements:

(1) \(\Sigma_n^2(I) = \Sigma_x^2(I) = \Sigma_m^2(I) = \widetilde{\Sigma}\)

and (2) \(\Sigma_n \Sigma_x = -\Sigma_x \Sigma_n = \lambda \Sigma_m\), where \(n, x, m\) are cyclic permutations of \(x, y\) and \(z\).

We can easily show that both \(\widetilde{\Sigma}^1\) and \(\widetilde{\Sigma}^2\) do really satisfy the above requirements. We shall consider only \(\widetilde{\Sigma}^1\) here.

\[
\Sigma_x^1 = \begin{pmatrix}
S_x & -C_x \\
-C_x & S_x
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0
\end{pmatrix}
\]

\[
\Sigma_y^1 = \begin{pmatrix}
S_y & -C_y \\
-C_y & S_y
\end{pmatrix} = \begin{pmatrix}
0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0
\end{pmatrix}
\]

\[
\Sigma_z^1 = \begin{pmatrix}
S_z & -C_z \\
-C_z & S_z
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\]

\[
\Sigma_x^{12} = \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\(^1\)This has been shown by Dr. M. Soga in a lecture at a physics seminar in the Physics Department, Western Michigan University.
Thus $\Sigma_x^2 = \widetilde{I}_{4 \times 4}$. Similar we can show $\Sigma_y^2 = \widetilde{I}_{4 \times 4}$ and $\Sigma_z^2 = \widetilde{I}_{4 \times 4}$.

\[
\Sigma_x^l \Sigma_y^l = \begin{pmatrix}
0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 \\
0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & -i \\
0 & -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
-\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}
= \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & -i & 0 \\
0 & -i & 0 & 0
\end{pmatrix}
= i \Sigma_z^l,
\]

\[
\Sigma_y^l \Sigma_x^l = \begin{pmatrix}
0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\
0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}
\begin{pmatrix}
0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 \\
0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}
= \begin{pmatrix}
-i & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
0 & 0 & i & 0
\end{pmatrix}
= -i \Sigma_x^l.
\]

Hence $\Sigma_x^l \Sigma_y^l = -\Sigma_y^l \Sigma_x^l = i \Sigma_z^l$.

Similarly we have:

\[
\Sigma_y^l \Sigma_z^l = \begin{pmatrix}
0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\
0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}
\begin{pmatrix}
0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 \\
0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}
= i \Sigma_y^l,
\]

\[
\Sigma_z^l \Sigma_y^l = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 \\
0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}
= \begin{pmatrix}
0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 \\
0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}
= -i \Sigma_z^l.
\]

Hence $\Sigma_y^l \Sigma_z^l = -\Sigma_z^l \Sigma_y^l = i \Sigma_x^l$.
Thus we have the following anti-commutation relation for the $\Sigma_0$ matrix.

$$\Sigma_1^{' \dagger} \Sigma_x^{' \dagger} + \Sigma_y^{' \dagger} \Sigma_x = 0,$$

$$\Sigma_y^{' \dagger} \Sigma_1^{' \dagger} + \Sigma_z^{' \dagger} \Sigma_1 = 0,$$

$$\Sigma_z^{' \dagger} \Sigma_y^{' \dagger} + \Sigma_x^{' \dagger} \Sigma_z = 0.$$

Hence we have the following anti-commutation relation for the $\Sigma(1)$ matrix.

$$\Sigma_y^{' \dagger} \Sigma_y = -\Sigma_x^{' \dagger} \Sigma_x = \mathcal{I} \Sigma_y.$$

Thus we have the following anti-commutation relation for the $\Sigma(1)$ matrix.

$$\Sigma_1^{' \dagger} \Sigma_x^{' \dagger} + \Sigma_y^{' \dagger} \Sigma_x = 0,$$

$$\Sigma_y^{' \dagger} \Sigma_1^{' \dagger} + \Sigma_z^{' \dagger} \Sigma_1 = 0,$$

$$\Sigma_z^{' \dagger} \Sigma_y^{' \dagger} + \Sigma_x^{' \dagger} \Sigma_z = 0.$$

We also have:

$$\Sigma_1^{' \dagger} \Sigma_y^{' \dagger} - \Sigma_y^{' \dagger} \Sigma_1 = 2i \Sigma_2,$$

$$\Sigma_y^{' \dagger} \Sigma_1^{' \dagger} - \Sigma_1^{' \dagger} \Sigma_y = 2i \Sigma_x,$$

$$\Sigma_z^{' \dagger} \Sigma_1^{' \dagger} - \Sigma_1^{' \dagger} \Sigma_z = 2i \Sigma_y.$$

Hence $\Sigma(1)$ has the same properties as the Pauli spin matrices. In fact, using a unitary transformation $\Sigma_x^{' \dagger} \Sigma_1^{' \dagger}$ and $\Sigma_1^{' \dagger}$ can be written in the familiar forms of the Pauli spin matrices. That is, we can find a unitary transformation $\hat{U}$ such that:

$$\hat{U}^{-1} \Sigma_x^{' \dagger} \hat{U} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$
From the first of the above matrix equations we get:

\[ \tilde{U} \Sigma_x \tilde{U} = \tilde{U} \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix} \]

\[ \Sigma_x \tilde{U} = \tilde{U} \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix} \]

(9)

And from the third matrix equation we have:

\[ \tilde{U} \Sigma_z \tilde{U} = \tilde{U} \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix} \]

\[ \Sigma_z \tilde{U} = \tilde{U} \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix} \]

(10)

Solving 9 and 10 algebraically and using the fact that \( \tilde{U}^+ \tilde{U} = \Pi \) (since \( \tilde{U}^+ = \tilde{U}^* \) for a unitary transformation) we can explicitly determine \( \tilde{U} \). The correct unitary transformation turns out to be:

\[
\tilde{U} = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
-\frac{1}{2} \sin \theta & -\frac{1}{2} \cos \theta & \frac{1}{2} \cos \theta & -\frac{1}{2} \sin \theta \\
0 & 0 & -\sin \theta & -\cos \theta \\
\frac{1}{2} e^{i\theta} \sin \theta & \frac{1}{2} e^{i\theta} \cos \theta & \frac{1}{2} e^{i\theta} \cos \theta & -\frac{1}{2} e^{i\theta} \sin \theta
\end{pmatrix}
\]

We can easily check that this \( \tilde{U} \) is the desired transformation by computing \( \tilde{U}^{-1} \Sigma_z \tilde{U} \).
\[
\begin{pmatrix}
\cos \theta, -\frac{1}{2} \sin \theta, 0, \frac{1}{2} e^{i \theta} \\
-\sin \theta, -\frac{1}{2} \cos \theta, 0, \frac{1}{2} e^{i \theta} \\
0, \frac{1}{2} \cos \theta, -\sin \theta, \frac{1}{2} e^{i \theta} \\
0, -\frac{1}{2} \sin \theta, -\cos \theta, -\frac{1}{2} e^{i \theta}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -e^{i \theta} \\
0 & 0 & -1 & 0 \\
0 & -e^{i \theta} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\cos \theta, -\frac{1}{2} \sin \theta, 0, \frac{1}{2} e^{i \theta} \\
-\frac{1}{2} \sin \theta, -\frac{1}{2} \cos \theta, 0, \frac{1}{2} e^{i \theta} \\
0, \frac{1}{2} \cos \theta, -\sin \theta, \frac{1}{2} e^{i \theta} \\
0, -\frac{1}{2} \sin \theta, -\cos \theta, -\frac{1}{2} e^{i \theta}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\cos \theta, -\frac{1}{2} \sin \theta, 0, \frac{1}{2} e^{i \theta} \\
-\sin \theta, -\frac{1}{2} \cos \theta, 0, \frac{1}{2} e^{i \theta} \\
0, \frac{1}{2} \cos \theta, -\sin \theta, \frac{1}{2} e^{i \theta} \\
0, -\frac{1}{2} \sin \theta, -\cos \theta, -\frac{1}{2} e^{i \theta}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
Generalization of the $\mathbf{Z}$- Matrix to Any Spin Field

Any sigma matrix should satisfy the two conditions that the Pauli sigma matrices satisfy, i.e., $\Sigma^2_n(I) = I$ and $\Sigma_n \Sigma_n = -\Sigma_n \Sigma_n = i \Sigma_n$.

Using this requirement as a guide, one can develop a sigma matrix for any spin field. Let us closely look again at the $\Sigma^2$ matrix for the vector field. We can consider either $\Sigma^1$ or $\Sigma^2$.

If we try to write for any spin $I$

$$\Sigma(I) = \begin{pmatrix} S(I) & G(I) \\ D(I) & D(I) \end{pmatrix}$$

as a generalized Pauli spin matrix, it at first seems acceptable since it is easily reduced to the sigma matrix of the vector field. But, in this general form the sigma matrix fails to satisfy the two requirements mentioned above. A thorough investigation of possible forms of the generalized sigma matrix gives:

$$\Sigma(I) = \frac{1}{I} \begin{pmatrix} S(I) & G(I) \\ D(I) & -S(I) \end{pmatrix}.$$ 

This generalized Pauli spin matrix satisfies both requirements. This can easily be verified for the case of $I = \frac{1}{2}$ and $I = \frac{3}{2}$. 

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For $\mathbf{I} = \frac{1}{2}$, $\mathbf{S}(\frac{1}{2}) = \frac{\hbar}{\sqrt{2}} \mathbf{S}(\frac{1}{2}) = 2 \mathbf{S}(\frac{1}{2})$

and $\mathbf{S} = 2 \mathbf{S}$,

the usual relation for Pauli matrices. Here $\mathbf{S}$ is the spin of a Fermion.

For $\mathbf{I} = 1$,

$$\mathbf{S}(I) = \begin{pmatrix} \mathbf{S}(I) & \mathbf{G}(I) \\ \mathbf{D}(I) & 0 \end{pmatrix}$$

which has been discussed earlier as the spin matrix for a vector field.

It is very useful to write down explicitly the properties of $\mathbf{S}(I)$ for later reference:

$$\Sigma_n^2(I) = \frac{1}{I^2} \begin{pmatrix} \mathbf{S}_n(I) & \mathbf{G}_n(I) \\ \mathbf{D}_n(I) & -\mathbf{S}_n(I) \end{pmatrix} \begin{pmatrix} \mathbf{S}_n(I) & \mathbf{G}_n(I) \\ \mathbf{D}_n(I) & -\mathbf{S}_n(I) \end{pmatrix}$$

$$= \frac{1}{I^2} \begin{pmatrix} \mathbf{S}_n^2(I) + \mathbf{G}_n(I) \mathbf{D}_n(I) & \mathbf{S}_n(I) \mathbf{G}_n(I) - \mathbf{G}_n(I) \mathbf{S}_n(I) \\ \mathbf{D}_n(I) \mathbf{S}_n(I) - \mathbf{D}_n(I) \mathbf{S}_n(I) & \mathbf{D}_n(I) \mathbf{G}_n(I) + \mathbf{S}_n^2(I) \end{pmatrix}$$

This property of $\mathbf{S}(I)$ gives us:

$$\mathbf{S}_n^2(I) + \mathbf{G}_n(I) \mathbf{D}_n(I) = I^2 \mathbf{P}(I),$$

$$\mathbf{S}_n(I) \mathbf{G}_n(I) - \mathbf{G}_n(I) \mathbf{S}_n(I) = 0,$$

$$\mathbf{D}_n(I) \mathbf{S}_n(I) - \mathbf{S}_n(I) \mathbf{D}_n(I) = 0,$$

$$\mathbf{D}_n(I) \mathbf{G}_n(I) + \mathbf{S}_n^2(I) = I^2 \mathbf{P}(I),$$
From $\Sigma_n(I)\Sigma_e(I) = -\Sigma_e(I)\Sigma_n(I) = i\Sigma_m(I)$ we get:

$$\begin{pmatrix}
\frac{1}{I^2} \left( S_n(I) G_n(I) - G_n(I) D_n(I) \right) \\
\frac{1}{I^2} \left( S_e(I) G_e(I) - G_e(I) D_e(I) \right)
\end{pmatrix} = -\frac{1}{I^2} \begin{pmatrix}
S_e(I) G_e(I) - G_e(I) S_e(I-I) \\
D_e(I) S_e(I-I) - S_e(I-I) D_e(I)
\end{pmatrix}$$

$$= \frac{i}{I} \begin{pmatrix}
S_m(I) G_m(I) \\
D_m(I) - S_m(I-I)
\end{pmatrix}.$$
matrices should also be taken into consideration to
determine them. $S(I)$ and $S(I-1)$ can be easily determined
from the relations:
\[
\langle I M' | S_+(I) | I M \rangle = \sqrt{(I-M)(I+M-1)} \delta_{M',M+1}
\]
\[
\langle I M' | S_-(I) | I M \rangle = \sqrt{(I+M)(I-M+1)} \delta_{M',M-1}
\]
\[
\langle I M' | S_0(I) | I M \rangle = M \delta_{M',M}
\]

Once we find $S_+(I)$, $S_-(I)$ and $S_0(I)$ we can easily
find $S_{\chi}(I)$, $S_{\psi}(I)$ and $S_2(I)$ from:

\[
S_{\chi}(I) = \frac{1}{2} \left( S_+(I) + S_-(I) \right)
\]
\[
S_{\psi}(I) = \frac{1}{2} \left( S_+(I) - S_-(I) \right)
\]
\[
S_2(I) = S_0(I)
\]

Similarly $S_{\chi}(I-1)$, $S_{\psi}(I-1)$ and $S_2(I-1)$ can be
obtained.

Thus, having developed a spin matrix for any spin
field, we shall next attempt to write Maxwell's field
equations in terms of $S(I)$. 

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VI. THE SIGMA MATRIX AND MAXWELL'S EQUATIONS

Maxwell's Equations in Terms of the \( \Sigma \)-Matrix

The \( \Sigma (i) \) matrix developed for the electromagnetic field can now be used to write the Maxwell equations. First we couple equations 6 with 7 and 5 with 8 on page ... If we write the coupled equations in a matrix form, we can easily see how to introduce the \( \Sigma (i) \)-matrix into the Maxwell equations. The coupled equations are:

\[
\begin{align*}
\mathbf{E} & = \mathbf{E} + \frac{4\pi k}{} \mathbf{J} \\
\mathbf{D} & = (\mathbf{D} \cdot \mathbf{P}) \mathbf{B} \\
\mathbf{B} & = -i \mathbf{C} (\mathbf{D} \cdot \mathbf{P}) \mathbf{E} \\
\mathbf{H} & = (\mathbf{D} \cdot \mathbf{P}) \mathbf{E} - \frac{4\pi k}{} \mathbf{P} \\
\mathbf{E} & = -i \mathbf{C} (\mathbf{D} \cdot \mathbf{P}) \mathbf{E} + 4\pi k \mathbf{P} \\
\mathbf{D} & = (\mathbf{D} \cdot \mathbf{P}) \mathbf{E} - \frac{4\pi k}{} \mathbf{P} \\
\mathbf{H} & = -i \mathbf{C} (\mathbf{D} \cdot \mathbf{P}) \mathbf{E} + 4\pi k \mathbf{P}
\end{align*}
\]

Writing them in matrix form we get:

\[
\begin{align*}
\mathbf{E} \left( \begin{array}{c} E_1 \\ 0 \end{array} \right) & = \mathbf{C} \left( \begin{array}{cc} \mathbf{D} & 0 \\ \mathbf{D} & 0 \end{array} \right) \cdot \mathbf{P} \left( \begin{array}{c} B_1 \\ 0 \end{array} \right) + \frac{4\pi k}{} \mathbf{J} \\
\mathbf{D} \left( \begin{array}{c} D_1 \\ 0 \end{array} \right) & = \mathbf{C} \left( \begin{array}{cc} \mathbf{D} & 0 \\ \mathbf{D} & 0 \end{array} \right) \cdot \mathbf{P} \left( \begin{array}{c} E_1 \\ 0 \end{array} \right) + \frac{4\pi k}{} \mathbf{P}
\end{align*}
\]
If we arbitrarily introduce $E_s$ and $B_s$ as fourth components of $\overline{E}$ and $\overline{B}$ respectively, then we can write equations 5 and 6 in a general form as:

$$\mathcal{E}\left(\begin{vmatrix} \overline{E} \\ E_s \end{vmatrix}\right) = i c \mathcal{C}\left(\begin{vmatrix} \overline{E} \\ E_s \end{vmatrix}\right) \cdot \mathcal{P}\left(\begin{vmatrix} \overline{B} \\ B_s \end{vmatrix}\right) + \frac{4\pi\kappa}{\lambda} \left(\begin{vmatrix} \overline{J} \\ 0 \end{vmatrix}\right)$$  (7)

$$\mathcal{E}\left(\begin{vmatrix} \overline{B} \\ B_s \end{vmatrix}\right) = -i c \mathcal{C}\left(\begin{vmatrix} \overline{E} \\ E_s \end{vmatrix}\right) \cdot \mathcal{P}\left(\begin{vmatrix} \overline{E} \\ E_s \end{vmatrix}\right) + \frac{4\pi\kappa}{\lambda} \left(\begin{vmatrix} 0 \\ c\rho \end{vmatrix}\right)$$  (8)

We can now write equations 1 to 4 above in a general form:

$$\mathcal{E}\overline{E} = i c \left\{ (\overline{\mathcal{D}} \cdot \overline{\mathcal{P}}) \overline{B} + (\overline{\mathcal{G}} \cdot \overline{\mathcal{P}}) B_s \right\} + \frac{4\pi\kappa}{\lambda} \overline{J} ,$$

$$\mathcal{E}E_s = i c (\overline{\mathcal{D}} \cdot \overline{\mathcal{P}}) \overline{B} ,$$

$$\mathcal{E}\overline{B} = -i c \left\{ (\overline{\mathcal{G}} \cdot \overline{\mathcal{P}}) \overline{E} + (\overline{\mathcal{G}} \cdot \overline{\mathcal{P}}) E_s \right\} ,$$

$$\mathcal{E}B_s = -i c \left\{ (\overline{\mathcal{D}} \cdot \overline{\mathcal{P}}) E_s \right\} + \frac{4\pi\kappa}{\lambda} \rho .$$

When $E_s = B_s = 0$, these equations reduce the original Maxwell equations 1 to 4. Since $\mathcal{\Xi}^{2} = \begin{vmatrix} \overline{\mathcal{G}} \\ \overline{\mathcal{D}} \\ 0 \end{vmatrix}$, we can re-write equations 7 and 8 as:

$$\mathcal{E}\left(\begin{vmatrix} \overline{E} \\ E_s \end{vmatrix}\right) = i c \left\{ (\mathcal{\Xi}^{2} \cdot \overline{\mathcal{P}}) \overline{B} \right\} + \frac{4\pi\kappa}{\lambda} \left(\begin{vmatrix} \overline{J} \\ 0 \end{vmatrix}\right) ,$$

$$\mathcal{E}\left(\begin{vmatrix} \overline{B} \\ B_s \end{vmatrix}\right) = -i c (\mathcal{\Xi}^{2} \cdot \overline{\mathcal{P}}) \left(\begin{vmatrix} \overline{E} \\ E_s \end{vmatrix}\right) + \frac{4\pi\kappa}{\lambda} \left(\begin{vmatrix} 0 \\ c\rho \end{vmatrix}\right) .$$
In fact, we could have used $\Sigma = \begin{pmatrix} \mathcal{S} & -\mathcal{C} \\ -\mathcal{D} & 0 \end{pmatrix}$ and obtained a set of equations which would reduce to Maxwell's equation when $E_s = B_s = 0$. Dropping the superscript on $\Sigma$ we write the above equation as:

$$
\begin{pmatrix} \frac{E}{C} \\ E_s \end{pmatrix} = (\Sigma \cdot \vec{p}) i \begin{pmatrix} B_s \\ E_s \end{pmatrix} + \frac{4 \pi \mu_0}{\epsilon} \begin{pmatrix} J \\ 0 \end{pmatrix}
$$

(9)

$$
\begin{pmatrix} \frac{E}{C} \\ E_s \end{pmatrix} = (\Sigma \cdot \vec{p}) E_s - \frac{4 \pi \mu_0}{\epsilon} \begin{pmatrix} 0 \\ c \rho \end{pmatrix}
$$

(10)

We can now generalize Maxwell's equations to any spin $I$ using the generalized $\Sigma$ matrix.

$$
\begin{pmatrix} \frac{E}{C} \\ E_s(I) \end{pmatrix} = (\Sigma(I) \cdot \vec{p}) i \begin{pmatrix} B_s(I) \\ B_s(I) \end{pmatrix} + \frac{4 \pi \mu_0}{\epsilon} \begin{pmatrix} J(I) \\ 0 \end{pmatrix}
$$

(11)

$$
\begin{pmatrix} \frac{E}{C} \\ E_s(I) \end{pmatrix} = (\Sigma(I) \cdot \vec{p}) E_s(I) - \frac{4 \pi \mu_0}{\epsilon} \begin{pmatrix} 0 \\ c \rho(I) \end{pmatrix}
$$

(12)

where:

$$
\Sigma(I) = \frac{1}{I} \begin{pmatrix} \mathcal{S}(I) & \mathcal{G}(I) \\ \mathcal{D}(I) & -\mathcal{S}(I) \end{pmatrix}
$$
Generalization of Maxwell's Equations to a Massive Field Equation

To generalize the Maxwell equations into a massive field equation, we first closely study the form of the generalized Maxwell equations and compare them with the Dirac equation. The generalized Maxwell equations are given by:

\[
\begin{align*}
\left( \frac{\varepsilon}{c} \right) \begin{pmatrix} E(I) \\ E_s(I) \end{pmatrix} - \left( \bar{\alpha}(I) \cdot \bar{p} \right) i \begin{pmatrix} B \\ B_s(I) \end{pmatrix} &= \frac{4\pi i^2}{c^2} \begin{pmatrix} J(I) \\ 0 \end{pmatrix} \\
\left( \frac{\varepsilon}{c} \right) i \begin{pmatrix} B(I) \\ B_s(I) \end{pmatrix} - \left( \bar{\alpha}(I) \cdot \bar{p} \right) i \begin{pmatrix} E(I) \\ E_s(I) \end{pmatrix} &= -\frac{4\pi i}{c^2} \begin{pmatrix} 0 \\ (\bar{p} I_{11} \bar{p}) \end{pmatrix}
\end{align*}
\]  

(1)

(2)

The Dirac equation for a free particle is given by:

\[ H_D \psi = [\frac{\gamma}{c} (\bar{\alpha} \cdot \bar{p}) + \beta m_0 c^2] \psi, \]

where \( H_D = \gamma (\bar{\alpha} \cdot \bar{p}) + \beta m_0 c^2 \) is the Dirac Hamiltonian which satisfies \( H_D^2 = -c^2 \frac{\partial^2}{\partial t^2} + m_0^2 c^4 \bar{\alpha} \) and \( \beta \) are represented by:

\[ \bar{\alpha} = \begin{pmatrix} 0 & \sigma_y \\ \sigma_x & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix} \quad \text{or} \quad \beta = i \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix} \]

\( \bar{\alpha} \) has components:

\[ \alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \quad \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \quad \text{and} \quad \alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \]
where $\sigma_x$, $\sigma_y$ and $\sigma_z$ are the familiar Pauli spin matrices.

The Dirac equation can now be written as:

$$\mathbf{E} \begin{pmatrix} \bar{\mathbf{P}} & 0 \\ 0 & \bar{\mathbf{P}} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \left[ \mathbf{C} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} + m_0 c^2 \bar{\mathbf{P}} \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} \right] \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

where $\psi_+$ and $\psi_-$ are each two component eigen spinors.

Then:

$$\mathbf{E} \psi_+ = \mathbf{C} \left( \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_+ \end{pmatrix} \psi_+ \right) + m_0 c^2 \psi_+ ,$$
$$\mathbf{E} \psi_- = \mathbf{C} \left( \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_+ \end{pmatrix} \psi_- \right) - m_0 c^2 \psi_- ,$$

$$\mathbf{E}/c - m_0 c \psi_+ = \mathbf{C} \left( \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_+ \end{pmatrix} \psi_- \right) ,$$
$$\mathbf{E}/c + m_0 c \psi_- = \mathbf{C} \left( \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_+ \end{pmatrix} \psi_+ \right) .$$

(3)

If we use $\mathbf{p} = \mathbf{c} \begin{pmatrix} \sigma_y \\ 0 \end{pmatrix}$, we get:

$$\mathbf{E} \begin{pmatrix} \bar{\mathbf{P}} & 0 \\ 0 & \bar{\mathbf{P}} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \left[ \mathbf{C} \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_+ \end{pmatrix} + i m_0 c^2 \bar{\mathbf{P}} \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} \right] \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

$$\mathbf{E} \psi_+ = \left[ \mathbf{C} \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_+ \end{pmatrix} + i m_0 c^2 \right] \psi_+ ,$$
$$\mathbf{E} \psi_- = \left[ \mathbf{C} \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_+ \end{pmatrix} - i m_0 c^2 \right] \psi_+ ,$$

$$\mathbf{E}/c \psi_+ = \left[ \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_+ \end{pmatrix} + i m_0 c \right] \psi_+ ,$$
$$\mathbf{E}/c \psi_- = \left[ \begin{pmatrix} \sigma_+ & 0 \\ 0 & \sigma_+ \end{pmatrix} - i m_0 c \right] \psi_+ .$$

(5)

To make a comparison between the generalized Maxwell equation and Dirac's equation we define:
\( \psi_i = \begin{pmatrix} E(i) \\ E_s(i) \end{pmatrix}, \quad \psi_i = \begin{pmatrix} B(i) \\ B_s(i) \end{pmatrix}. \)

Then equations 1 and 2 can be written as:

\[
\left( \frac{e}{c} \right) \psi_+ = \left( \Sigma(i) \cdot \vec{p} \right) \psi_+ + 4\pi i \frac{k}{c} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7}
\]

\[
\left( \frac{e}{c} \right) \psi_- = \left( \Sigma(i) \cdot \vec{p} \right) \psi_- - 4\pi i \frac{k}{c} \begin{pmatrix} 0 \\ C \varphi(i) \end{pmatrix} \tag{8}
\]

Now we require 7 and 8 to be written in a general form such that for the special case of \( I = \frac{1}{2} \) they reduce to 3 and 4 or 5 and 6. For a free particle we don't have a current source or a charge source. Hence, the source terms in 7 and 8 can be neglected for a free particle. Then, comparing the rest of the terms in equations 7 and 8 to 3 and 4 or 5 and 6, we get two possible ways to generalize Maxwell's equation into a massive field equation.

\[
I \cdot \left( \frac{e}{c} - m_0 c \right) \psi_+ = \left( \Sigma(i) \cdot \vec{p} \right) \psi_- \tag{9}
\]

\[
\left( \frac{e}{c} + m_0 c \right) \psi_- = \left( \Sigma(i) \cdot \vec{p} \right) \psi_+ \tag{10}
\]

or

\[
II \cdot \left( \frac{e}{c} \right) \psi_+ = \left( \Sigma(i) \cdot \vec{p} \right) + i m_0 c \] \psi_- = \left( \frac{e}{c} \right) \psi_- = \left( \Sigma(i) \cdot \vec{p} \right) - i m_0 c \] \psi_+ \tag{11}
\]

Note that for the special case of \( I = \frac{1}{2} \), \( \Sigma = \vec{g} \), 9 and 10
reduce to the Dirac equation 3 and 4. Similarly 11 and 12 reduce to 5 and 6. Which of these two sets of equations, i.e., 9 and 10 or 11 and 12, we should use depends on which set satisfies the physical requirements imposed on them. We shall discuss this in detail later.

If we now define:

$$\vec{\alpha}' = \begin{pmatrix} 0 & \Sigma \\ -\Sigma & 0 \end{pmatrix}, \quad \vec{\beta}' = \begin{pmatrix} \Pi & 0 \\ 0 & -\Pi \end{pmatrix},$$

then 9 and 10 could be written as:

$$\left[ (\vec{\alpha}' \cdot \vec{p}) + \vec{\beta}'m_0c \right] \psi = \xi_c \psi, \text{ where } \psi = \begin{pmatrix} \psi^- \\ \psi^+ \end{pmatrix}.$$

Similarly we have:

$$\left[ (\vec{\alpha}' \cdot \vec{p}) + \vec{\beta}''m_0c \right] \psi = \xi \psi, \quad \vec{\beta}'' = i \begin{pmatrix} 0 & \Pi \\ -\Pi & 0 \end{pmatrix}$$

If we multiply 9 by $\left( \frac{\xi}{\xi_c} + m_0c \right)$ we get:

$$\left( \frac{\xi}{\xi_c} - m_0c \right) \left( \frac{\xi}{\xi_c} + m_0c \right) \psi = (\vec{\alpha}(\vec{p}) \cdot \vec{p}) \left( \frac{\xi}{\xi_c} + m_0c \right) \psi$$

$$\left[ (\frac{\xi}{\xi_c})^2 - m_0^2c^2 \right] \psi_+ = (\vec{\alpha}(\vec{p}) \cdot \vec{p})^2 \psi_+$$

$$\left[ (\frac{\xi}{\xi_c})^2 - m_0^2c^2 \right] \psi_+ = 0$$

Similarly multiplying 10 by $\left( \frac{\xi}{\xi_c} - m_0c \right)$ we get:

$$\left[ (\frac{\xi}{\xi_c})^2 - m_0^2c^2 \right] \psi_- = 0$$
The Generalized Dirac Equation in the Presence of a Source

In the presence of a current or charge source, the generalized Dirac equation can be written as:

\[ \left( \frac{\gamma}{c} - m_0 c \right) \psi_+ = (\Sigma(I) \cdot \overline{P}) \psi_+ + \frac{4\pi k}{ic} \begin{pmatrix} J^{(I)} \\ 0 \end{pmatrix} \tag{1} \]

\[ \left( \frac{\gamma}{c} + m_0 c \right) \psi_- = (\Sigma(I) \cdot \overline{P}) \psi_- - \frac{4\pi k}{ic} \begin{pmatrix} 0 \\ C\rho(E) \end{pmatrix} \tag{2} \]

OR

\[ \left( \frac{\gamma}{c} \right) \psi_+ = \begin{pmatrix} (\Sigma(I) \cdot \overline{P}) + im_0 c \end{pmatrix} \psi_+ + \frac{4\pi k}{ic} \begin{pmatrix} J^{(I)} \\ 0 \end{pmatrix} \tag{3} \]

\[ \left( \frac{\gamma}{c} \right) \psi_- = \begin{pmatrix} (\Sigma(I) \cdot \overline{P}) - im_0 c \end{pmatrix} \psi_- + \frac{4\pi k}{ic} \begin{pmatrix} 0 \\ C\rho(E) \end{pmatrix} \tag{4} \]

From 1 and 2 we get:

\[ \left( \frac{\gamma}{c} - m_0 c \right) \left( \frac{\gamma}{c} + m_0 c \right) \psi_+ = (\Sigma(I) \cdot \overline{P}) \left( \frac{\gamma}{c} + m_0 c \right) \psi_+ + \left( \frac{\gamma}{c} + m_0 c \right) \frac{4\pi k}{ic} \begin{pmatrix} J^{(I)} \\ 0 \end{pmatrix}, \]

\[ \left[ \left( \frac{\gamma}{c} \right)^2 - m_0^2 c^2 \right] \psi_+ = (\Sigma(I) \cdot \overline{P})^2 \psi_+ - (\Sigma(I) \cdot \overline{P}) \frac{4\pi k}{ic} \begin{pmatrix} J^{(I)} \\ 0 \end{pmatrix} + \left( \frac{\gamma}{c} + m_0 c \right) \frac{4\pi k}{ic} \begin{pmatrix} C\rho(E) \end{pmatrix} \]

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\[
\left( \frac{\varepsilon}{c} \right)^2 - p^2 - m_0^2 c^2 \right) \psi = \frac{4 \pi i}{c} \left[ \left( \frac{\varepsilon}{c} + m_0 c \right) J^{(I)} \right]_0 - \left( \Sigma^{(I)} \cdot \bar{p} \right) \left( \frac{0}{c_p(I)} \right)
\]

Similarly we get:
\[
\left( \frac{\varepsilon}{c} \right)^2 - p^2 - m_0^2 c^2 \right) \psi = \frac{4 \pi i}{c} \left[ \left( \frac{\varepsilon}{c} + m_0 c \right) J^{(I)} \right]_0 - \left( \frac{\varepsilon - m_0 c}{c} \right) 
\times \left( \frac{0}{c_p(I)} \right)
\]

From equations 3 and 4 we get:
\[
\left( \frac{\varepsilon}{c} \right)^2 \psi = \left[ \left( \Sigma^{(I)} \cdot \bar{p} + i m_0 c \right) \right] \frac{\varepsilon}{c} \psi + \frac{4 \pi i}{c} \frac{\varepsilon}{c} J^{(I)}
\]

\[
= \left[ \left( \Sigma^{(I)} \cdot \bar{p} \right)^2 + m_0^2 c^2 \right] \psi - \left[ \Sigma^{(I)} \cdot \bar{p} + i m_0 c \right] \frac{4 \pi i}{c} \frac{0}{c_p(I)}
\]

\[
+ \frac{4 \pi i}{c} \frac{\varepsilon}{c} \left( J^{(I)} \right)_0
\]

\[
\left( \frac{\varepsilon}{c} \right)^2 - p^2 - m_0^2 c^2 \right) \psi = \frac{4 \pi i}{c} \left[ \left( \frac{\varepsilon}{c} \right)^2 \left( J^{(I)} \right) \right]_0 - \left[ \Sigma^{(I)} \cdot \bar{p} + i m_0 c \right] \frac{0}{c_p(I)}
\]

Similarly we get:
\[
\left( \frac{\varepsilon}{c} \right)^2 - p^2 - m_0^2 c^2 \right) \psi = \frac{4 \pi i}{c} \left[ \left( \Sigma^{(I)} \cdot \bar{p} - i m_0 c \right) \left( J^{(I)} \right) \right]_0 - \frac{\varepsilon}{c} \frac{0}{c_p(I)}
\]
Let us now investigate which sets of equations, i.e., 5 and 6 or 7 and 8, give physically meaningful results. If we set \( E_s = B_s = 0 \), then equations 5 and 6 give:

\[
\left[\frac{\mathcal{E}^2}{c^2} - p^2/m_0 c^2\right] \mathcal{E}(I) = \frac{4\pi k}{x_c} \left[\frac{\mathcal{E} + m_0 c}{c}\right] J(I) - \frac{i}{I} \left( \mathcal{C}(I) \cdot \mathcal{P} \right) \mathcal{C} \rho(I-I) \]

(9)

\[
0 = \frac{i}{I} \left( \mathcal{S}(I-I) \cdot \mathcal{P} \right) \mathcal{C} \rho(I-I) \tag{10}
\]

\[
\left[\frac{\mathcal{E}^2}{c^2} - p^2/m_0 c^2\right] \mathcal{B}(I) = \frac{4\pi k}{x_c} \left[\frac{1}{I} \left( \mathcal{S}(I) \cdot \mathcal{P} \right) J(I)\right] \tag{11}
\]

\[
0 = \frac{1}{I} \left( \mathcal{D}(I) \cdot \mathcal{P} \right) J(I) - \left( \frac{\mathcal{E}}{c} - m_0 c \right) \mathcal{C} \rho(I-I) \tag{12}
\]

To understand the physical significance of equations 9 through 12 we consider the case \( I = 1 \) which gives:

\[
\left[\frac{\mathcal{E}^2}{c^2} - p^2/m_0 c^2\right] \mathcal{E} = \frac{4\pi k}{x_c} \left[\frac{\mathcal{E} + m_0 c}{c}\right] J - \left( \mathcal{C} \cdot \mathcal{P} \right) \mathcal{C} \rho \tag{13}
\]
\[
\left[\left(\frac{\varepsilon}{c}\right)^2 - \frac{p^2 - m_0^2 c^2}{c^2}\right] \varepsilon \mathbf{B} = \frac{4\pi}{c} \mathbf{\varepsilon} \mathbf{\cdot F}(\mathbf{\vec{S}}) \mathbf{\cdot \vec{J}}
\]

\[\nabla \cdot \mathbf{D} = (\varepsilon - m_0 c^2)\mathbf{J} = 0. \quad (15)\]

Equation 10 vanishes for \( I = 1 \) since \( \mathbf{S}(0) \equiv 0 \). Equations 13 and 14 give a relationship between the electric and magnetic fields and their sources. Equation 15 could be interpreted as a continuity equation for a massive field. When \( m_0 = 0 \) it reduces to the familiar continuity equation.

Hence:

\[
(\nabla \cdot \mathbf{p}) \mathbf{J} - (\frac{\varepsilon}{c} - m_0 c) \mathbf{J} = 0,
\]

\[
(\nabla \cdot \mathbf{p}) - \varepsilon \mathbf{J} = m_0 c^2 \mathbf{J},
\]

and if \( m_0 = 0 \), \( (\nabla \cdot \mathbf{p}) \mathbf{J} - \varepsilon \mathbf{J} = 0 \),

\[
\frac{k}{\varepsilon} (\nabla \cdot \mathbf{p}) \mathbf{J} + \frac{1}{\varepsilon} \frac{\partial}{\partial t} \mathbf{J} = 0,
\]

\[
\nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \mathbf{J} = 0,
\]

which is the usual continuity equation.

When \( E_s = B_s = 0 \), 7 and 8 give:
\[ \left[ \left( \frac{E}{c} \right)^2 - \mathbf{p}^2 \right] E(I) = \frac{4 \pi \hbar}{i c} \left[ \frac{E}{c} (J(I) - \frac{1}{i} (S'(I) \cdot \mathbf{p}) c \mathbf{p} \right] \]

\[ \left[ \left( \frac{E}{c} \right)^2 - \mathbf{p}^2 \right] i B(I) = \frac{4 \pi \hbar}{i c} \left[ \frac{1}{i} (S(I) \cdot \mathbf{p}) J(I) - i m_0 c J(I) \right] \]

These two equations give:

\[ \left[ \left( \frac{E}{c} \right)^2 - \mathbf{p}^2 \right] E(I) = \frac{4 \pi \hbar}{i c} \left[ \frac{E}{c} (J(I) - \frac{1}{i} (S'(I) \cdot \mathbf{p}) c \mathbf{p} \right] \]

\[ 0 = \frac{1}{i} (S(I) \cdot \mathbf{p}) - i m_0 c c \mathbf{p} \]

\[ \left[ \left( \frac{E}{c} \right)^2 - \mathbf{p}^2 \right] i B(I) = \frac{4 \pi \hbar}{i c} \left[ \frac{1}{i} (S(I) \cdot \mathbf{p}) J(I) - i m_0 c J(I) \right] \]

\[ = \frac{4 \pi \hbar}{i c} \left[ \frac{1}{i} (S(I) \cdot \mathbf{p}) - i m_0 c J(I) \right] \]
\[ 0 = \frac{1}{\varpi} (\mathcal{D}(\varpi) \cdot \vec{P}) \mathcal{D}(\varpi) - \frac{\varpi}{c^2} \mathcal{C} \mathcal{P}(\varpi) \cdot \vec{P} \cdot \mathcal{C} \mathcal{P}(\varpi). \]  

(19)

In the case of \( \varpi = 1 \) we get:

\[ \left[ \frac{\varpi^2 - m_o^2 c^2}{c^2} \right] \vec{E} = \frac{4 \pi \hbar}{i c} \left[ \frac{\varpi}{c^2} \mathcal{J} - (\mathcal{C} \cdot \vec{P}) \mathcal{C} \mathcal{P}(\varpi) \right] \]  

(20)

\[ 0 = - (i m_o c) \mathcal{C} \mathcal{P}, \]  

(21)

\[ \left[ \frac{\varpi^2 - m_o^2 c^2}{c^2} \right] \vec{B} = \frac{4 \pi \hbar}{i c} \left[ (\mathcal{C} \cdot \vec{P}) - i m_o c^2 \mathcal{J} \right], \]  

(22)

\[ 0 = (\mathcal{C} \cdot \vec{P}) \mathcal{J} - \mathcal{E} \mathcal{P}. \]  

(23)

Like 13 and 14, equations 20 and 22 relate the \( \vec{E} \) and \( \vec{B} \) fields to their respective sources. But equation 21 implies either \( m_o = 0 \), or \( \mathcal{P} = 0 \). This contradicts the fact that the equations are supposed to describe the kinematics of massive particles with any spin. So, this equation fails to satisfy this requirement. Besides, equation 23 is the familiar continuity equation for a massless field. It is a special case of equation 15.

Thus, if \( \varpi = 0 \), then the massive field equations given by 1 and 2 provide a more meaningful relation between physical quantities than do equations 3 and 4.
On the other hand, if \( E_s \neq 0, B_s \neq 0 \), then we get from equations 5 and 6:

\[
\left( \frac{E}{c} \right)^2 - p^2 m_0^2 c^2 \right) \frac{E_i}{i} = \frac{4\pi \hbar}{i c} \left[ \frac{E}{c} + m_0 c \right] J_i - \frac{i}{\lambda} (G_i \cdot \vec{p}) C P_i \tag{24}
\]

\[
\left( \frac{E}{c} \right)^2 - p^2 m_0^2 c^2 \right) E_s(i) = \frac{4\pi \hbar}{i c} \left[ \frac{1}{\lambda} (S_i \cdot \vec{p}) C P_i \right] \tag{25}
\]

\[
\left( \frac{E}{c} \right)^2 - p^2 m_0^2 c^2 \right) i B_s(i) = \frac{4\pi \hbar}{i c} \left[ \frac{1}{\lambda} (S_i \cdot \vec{p}) J_i \right] \tag{26}
\]

\[
\left( \frac{E}{c} \right)^2 - p^2 m_0^2 c^2 \right) i B_s(i) = \frac{4\pi \hbar}{i c} \left[ \frac{1}{\lambda} (D_i \cdot \vec{p}) C P_i \right] \tag{27}
\]

Similarly from equations 7 and 8 we have:

\[
\left( \frac{E}{c} \right)^2 - p^2 m_0^2 c^2 \right) E_i = \frac{4\pi \hbar}{i c} \left[ \frac{E}{c} J_i - \frac{i}{\lambda} (G_i \cdot \vec{p}) C P_i \right] \tag{28}
\]

\[
\left( \frac{E}{c} \right)^2 - p^2 m_0^2 c^2 \right) E_s(i) = \frac{4\pi \hbar}{i c} \left[ \frac{1}{\lambda} (S_i \cdot \vec{p}) C P_i \right] \tag{29}
\]

\[
\left( \frac{E}{c} \right)^2 - p^2 m_0^2 c^2 \right) i B_s(i) = \frac{4\pi \hbar}{i c} \left[ \frac{1}{\lambda} (S_i \cdot \vec{p}) J_i - \frac{i}{\lambda} m_0 c \right] \tag{30}
\]

\[
\left( \frac{E}{c} \right)^2 - p^2 m_0^2 c^2 \right) i B_s(i) = \frac{4\pi \hbar}{i c} \left[ \frac{1}{\lambda} (D_i \cdot \vec{p}) J_i - \frac{i}{\lambda} E C P_i \right] \tag{31}
\]
Thus if $E_S \neq 0$ and $B_S \neq 0$, then it is hard to choose 1 and 2 or 3 and 4 as the correct equations for a massive field. To begin with if $E_S$ and $B_S$ are non-zero, what do they physically represent? To avoid such a conceptual problem, we shall set $E_S$ and $B_S$ to zero. As we have seen on page 41, 1 and 2 are the correct equations when $E_S = B_S = 0$. Hence, hereafter we shall develop all other equations for bosons or other particles using 1 and 2.

Scalar and Vector Potentials of a Massive Field

The massive field has scalar and vector potentials. The equations which the scalar and vector potentials should satisfy can be obtained by first defining generalized $E(I)$ and $B(I)$ fields. The $E(I)$ and $B(I)$ fields should be reducible to $\vec{E}$ and $\vec{B}$ of the electromagnetic field when we set $I = 1$, $m_0 = 0$.

First consider the massive field equations:

$$\left[\left(\frac{\varepsilon}{c}\right)^2 p^2 - m_0^2 c^2\right]E(I) = \frac{4\pi \hbar}{i c} \left[\frac{\varepsilon}{c} + m_0 c\right]j(I) - \frac{i}{2} (\vec{G}(I) \cdot \vec{p}) c P(I)$$

(1)

$$0 = \frac{i}{2} (\vec{s}(I) \cdot \vec{p}) c P(I)$$

(2)
\[ \left( \frac{E}{c} \right)^2 - p^2 = m_0^2 c^2 \left\{ \int B(1) = \frac{4\pi}{c} \left[ \frac{1}{l} \left( S(1) \cdot \bar{P} \right) J(1) \right], \right. \]  

\[ 0 = \frac{l}{l} \left( D(1) \cdot \bar{P} \right) J(1) - \left( \frac{E}{c} - m_0 c \right) C P(I) \cdot \bar{P} \right). \]

We now want to find equations which the scalar and vector potentials satisfy. First, from the quantized Maxwell equations we have:

\[ E = \frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \nabla \phi = \frac{A}{\lambda} \left[ \frac{E}{c} \vec{A} - \left( \vec{G} \cdot \bar{P} \right) \phi \right], \]

\[ B = \nabla \times \vec{A} = \frac{1}{\lambda} \left( \vec{S} \cdot \bar{P} \right) \vec{A}. \]

By closely studying the forms of the right hand side terms of equations 1 to 3 above and by a parallel analogy to the quantized Maxwell equations we define \( E(I) \) and \( B(I) \) for massive fields as follows:

\[ E(I) = \frac{\lambda}{\lambda} \left[ \left( \frac{E}{c} + m_0 c \right) A(I) - \frac{1}{l} \left( G(I) \cdot \bar{P} \right) \phi(I) \right], \]

\[ B(I) = \frac{1}{\lambda} \frac{1}{l} \left( S(I) \cdot \bar{P} \right) A(I). \]

By substituting these identities into equations 1 and 2 on
page 38 we obtain:

\[
\begin{align*}
\left( \frac{\varepsilon}{c} - m_0 c \right) \begin{pmatrix} E(I) \\ 0 \end{pmatrix} &= \left( \Sigma(I) \cdot \bar{P} \right) \begin{pmatrix} B(I) \\ 0 \end{pmatrix} + \frac{4 \pi \hbar}{i c} \begin{pmatrix} \psi(I) \\ 0 \end{pmatrix} , \\
\left( \frac{\varepsilon}{c} + m_0 c \right) i \begin{pmatrix} B(I) \\ 0 \end{pmatrix} &= \left( \Sigma(I) \cdot \bar{P} \right) \begin{pmatrix} E(I) \\ 0 \end{pmatrix} - \frac{4 \pi \hbar}{i c} \begin{pmatrix} 0 \\ C \psi(I) \end{pmatrix} .
\end{align*}
\]

Substituting the explicit form of \( \Sigma(I) \) we get:

\[
\begin{align*}
\left( \frac{\varepsilon}{c} - m_0 c \right) \begin{pmatrix} E(I) \\ 0 \end{pmatrix} &= -\frac{1}{i} \begin{pmatrix} S(I) & G(I) \end{pmatrix} \cdot \bar{P} \begin{pmatrix} B(I) \\ 0 \end{pmatrix} = \frac{4 \pi \hbar}{i c} \begin{pmatrix} \psi(I) \\ 0 \end{pmatrix} , \\
\left( \frac{\varepsilon}{c} + m_0 c \right) i \begin{pmatrix} B(I) \\ 0 \end{pmatrix} &= -\frac{1}{i} \begin{pmatrix} S(I) & G(I) \end{pmatrix} \cdot \bar{P} \begin{pmatrix} E(I) \\ 0 \end{pmatrix} = -\frac{4 \pi \hbar}{i c} \begin{pmatrix} 0 \\ C \psi(I) \end{pmatrix} .
\end{align*}
\]

These two equations give:

\[
\begin{align*}
\left( \frac{\varepsilon}{c} - m_0 c \right) E(I) - \frac{1}{i} (S(I) \cdot \bar{P}) i B(I) &= \frac{4 \pi \hbar}{i c} \psi(I) , \\
-\frac{1}{i} (D(I) \cdot \bar{P}) i B(I) &= 0 , \\
\left( \frac{\varepsilon}{c} + m_0 c \right) i B(I) - \frac{1}{i} (S(I) \cdot \bar{P}) E(I) &= 0 , \\
\frac{1}{i} (D(I) \cdot \bar{P}) E(I) &= \frac{4 \pi \hbar}{i c} C \psi(I) .
\end{align*}
\]
Substituting the identities for $E(I)$ and $B(I)$ into 5 and 7 we get:

$$
\frac{i}{\hbar} \left( \frac{e}{c} - m_0 c \right) \left[ \left( \frac{e}{c} + m_0 c \right) A(I) - \frac{1}{i} \left( G(I) \cdot \vec{P} \right) \phi(I) \right]
$$

$$
- \frac{i}{\hbar} \frac{1}{I^2} (\vec{S}(I) \cdot \vec{P})^2 A(I) = \frac{4\pi k}{i\hbar} \phi(I)
$$

$$(\vec{S}(I) \cdot \vec{P})^2 = (\vec{S}(I) \cdot \vec{n})^2 p^2$$

$\vec{n}$ is a unit vector in the direction of $\vec{P}$.

$$(\vec{S}(I) \cdot \vec{n})^2 = S_x^2 I_x^2 + S_y^2 I_y^2 + S_z^2 I_z^2$$

$$+ (S_x S_y + S_y S_x) I_x I_y$$

$$+ (S_y S_z + S_z S_y) I_y I_z$$

$$+ (S_z S_x + S_x S_z) I_z I_x$$.

But

$$S_x^2(I) = I_x^2 - G_x(I) D_x(I),$$

$$S_y^2(I) = I_y^2 - G_y(I) D_y(I),$$

$$S_z^2(I) = I_z^2 - G_z(I) D_z(I).$$

and
\[ S_x S_y + S_y S_x = -(G_x D_y + G_y D_x), \]
\[ S_y S_z + S_z S_y = -(G_y D_z + G_z D_y), \]
\[ S_z S_x + S_x S_z = -(G_z D_x + G_x D_z). \]

Hence
\[
(S(i) \cdot \vec{n})^2 = I^2 (\eta_x^2 + \eta_y^2 + \eta_z^2) - G_x D_x \eta_x^2
- G_y D_y \eta_y^2 - G_z D_z \eta_z^2
-(G_x D_y + G_y D_x) \eta_x \eta_y
-(G_y D_z + G_z D_y) \eta_y \eta_z
-(G_z D_x + G_x D_z) \eta_x \eta_z
\]
\[
= I^2 - \begin{pmatrix}
G_x \eta_x \\
G_y \eta_y \\
G_z \eta_z
\end{pmatrix} \begin{pmatrix}
D_x \eta_x, D_y \eta_y, D_z \eta_z
\end{pmatrix}
\]
\[
= I^2 - (G \cdot \vec{n})(\vec{D} \cdot \vec{n})
\]

Hence
\[
(S(i) \cdot \vec{p})^2 = p^2 (S(i) \cdot \vec{n})^2 = p^2 [I^2 (G \cdot \vec{n})(\vec{D} \cdot \vec{n})]
\]
\[
= p^2 I^2 - (G \cdot \vec{p} \vec{n}) (\vec{D} \cdot \vec{p} \vec{n})
\]
\[
= p^2 I^2 (G \cdot \vec{p}) (\vec{D} \cdot \vec{p})
\]

Thus we have:
\[
\frac{d}{dt} \left[ \frac{(E)^2}{c^2} - m_0^2 c^2 \right] A(i) - \frac{1}{c} (E(i) \cdot \vec{p})(E(i) - m_0 c) \Phi(i-1)
\]
\[
- \frac{d}{dt} \left[ p^2 I^2 - (G(i) \cdot \vec{p})(D(i) \cdot \vec{p}) \right] A(i) = \frac{4\pi}{c} \delta^{(4)}(i)
\]
If we let 
\[
\left(\frac{\varepsilon}{\kappa} - m_0 c\right) \phi(I-I) - \frac{i}{\kappa} \left(D(I) \cdot \overline{\rho}\right) A(I) = 0,
\]
then
\[
\frac{i}{\kappa} \left(\frac{\varepsilon}{\kappa} - m_0^2 c^2\right) A(I) = \frac{4\pi}{\kappa} j(I),
\]
\[
\left[\left(\frac{\varepsilon}{\kappa}\right)^2 - m_0^2 c^2\right] A(I) = -4\pi j(I).
\]
From 6 we have:
\[
-\frac{i}{\kappa} \left(D(I) \cdot \overline{\rho}\right) - \frac{1}{i\kappa} \left(S(I) \cdot \overline{\rho}\right) A(I) = 0,
\]
\[
\frac{i}{\kappa} \left(D(I) \cdot \overline{\rho}\right) (S(I) \cdot \overline{\rho}) A(I) = 0,
\]
since \(\text{div} \ \text{curl} \ A(I) = 0\).
From 7 we have:
\[
\left(\frac{\varepsilon}{\kappa} + m_0 c\right) \frac{i}{\kappa} \cdot \left(S(I) \cdot \overline{\rho}\right) A(I) - \frac{1}{\kappa} (S(I) \cdot \overline{\rho}) \left\{ \frac{i}{\kappa} \left(\frac{\varepsilon}{\kappa} + m_0 c\right) A(I)
\right\} = 0,
\]
\[
\frac{i}{\kappa} \left(\frac{\varepsilon}{\kappa} + m_0 c\right) (S(I) \cdot \overline{\rho}) A(I) - \frac{1}{\kappa} \left(\frac{\varepsilon}{\kappa} + m_0 c\right) (S(I) \cdot \overline{\rho}) A(I)
\]
\[
+ \frac{1}{i\kappa} \left(\frac{\varepsilon}{\kappa} + m_0 c\right) (S(I) \cdot \overline{\rho}) (G(I) \cdot \overline{\rho}) \phi(I-I) = 0,
\]
Since
\[
\frac{1}{i\kappa} \left(\frac{\varepsilon}{\kappa} + m_0 c\right) (S(I) \cdot \overline{\rho}) (G(I) \cdot \overline{\rho}) \phi(I-I) = \frac{1}{i\kappa} \text{Curl} \ \text{grad} \ \phi(I-I) = 0,
\]
The above result is obviously true.
Finally from equation 8 we have:
\[
\frac{i}{\kappa} \left(D(I) \cdot \overline{\rho}\right) \frac{i}{\kappa} \left[\left(\frac{\varepsilon}{\kappa} + m_0 c\right) A(I) - \frac{1}{\kappa} (G(I) \cdot \overline{\rho}) \phi(I-I)\right] = \frac{4\pi}{\kappa} c \phi(I-I)
\]
\[ i \frac{\hbar}{\mathcal{K}} \left( \frac{\mathcal{E}}{\mathcal{C}} + \mathcal{M}_0 \mathcal{C} \right) (D(I) \cdot \mathcal{P}) A(I) - \frac{1}{I^2} (D(I) \cdot \mathcal{P}) (G(I) \cdot \mathcal{P}) \phi(I) \]

\[ = \frac{4 \pi \kappa}{\mathcal{C}} \mathcal{P}(I-1), \]

however \( \frac{1}{I} (D(I) \cdot \mathcal{P}) A(I) = (\mathcal{E}_C - \mathcal{M}_0 \mathcal{C}) \phi(I-1), \)

and \( (D(I) \cdot \mathcal{P})(G(I) \cdot \mathcal{P}) = (D(I) \cdot \mathcal{P})(G(I) \cdot \mathcal{P}) \mathcal{P}^2 \)

\[ (D(I) \cdot \mathcal{P})(G(I) \cdot \mathcal{P}) = (D_x G_x) \mathcal{H}_x^2 + (D_y G_y) \mathcal{H}_y^2 + (D_z G_z) \mathcal{H}_z^2 \]

\[ + (D_x G_y + D_y G_x) \mathcal{H}_x \mathcal{H}_y \]

\[ + (D_y G_z + D_z G_y) \mathcal{H}_y \mathcal{H}_z \]

\[ + (D_z G_x + D_x G_z) \mathcal{H}_z \mathcal{H}_x, \]

\[ D_x G_x = I^2 \mathcal{P}(I-1) - \mathcal{S}_x^2 (I-1), \]

\[ D_y G_y = I^2 \mathcal{P}(I-1) - \mathcal{S}_y^2 (I-1), \]

\[ D_z G_z = I^2 \mathcal{P}(I-1) - \mathcal{S}_z^2 (I-1), \]

and \( D_x G_y + D_y G_x = -\left( \mathcal{S}_x(I-1) \mathcal{S}_y(I-1) + \mathcal{S}_y(I-1) \mathcal{S}_x(I-1) \right), \)

\[ D_y G_z + D_z G_y = -\left( \mathcal{S}_y(I-1) \mathcal{S}_z(I-1) + \mathcal{S}_z(I-1) \mathcal{S}_y(I-1) \right), \]

\[ D_z G_x + D_x G_z = -\left( \mathcal{S}_z(I-1) \mathcal{S}_x(I-1) + \mathcal{S}_x(I-1) \mathcal{S}_z(I-1) \right). \]
\[
(I(I) \cdot \vec{n}) (G(I) \cdot \vec{n}) = I^2 \mathcal{M}(I) (\eta_x^2 + \eta_y^2 + \eta_z^2 - S_x^2(I) \eta_x^2
\]
\[
- S_y^2(I) \eta_y^2 - S_z^2(I) \eta_z^2
\]
\[
- \left( S_x(I) S_y(I) + S_y(I) S_x(I) \right) \eta_x \eta_y
\]
\[
- \left( S_y(I) S_z(I) + S_z(I) S_y(I) \right) \eta_y \eta_z
\]
\[
- \left( S_z(I) S_x(I) + S_x(I) S_z(I) \right) \eta_z \eta_x
\]
\[
= I^2 \mathcal{M}(I) - \begin{pmatrix} S_x(I) \eta_x, S_y(I) \eta_y, S_z(I) \eta_z \end{pmatrix}
\]
\[
= I^2 \mathcal{M}(I) - (S'(I) \cdot \vec{n}) (S'(I) \cdot \vec{n})
\]

Hence \( (J'(I) \cdot \vec{p}) (G(I) \cdot \vec{p}) = P^2 I^2 \mathcal{M}(I) - (S'(I) \cdot \vec{p}) (S'(I) \cdot \vec{p}) \)
\[
= P^2 I^2 \mathcal{M}(I) - (S'(I) \cdot \vec{p})^2
\]

Thus, substituting these results in the above equation we get
\[
\frac{d}{dt} \left[ \frac{\varepsilon}{\varepsilon_C} + m_0 C (\varepsilon - m_0 C) \phi(I) \right] - \frac{1}{\varepsilon_C} \left[ P^2 I^2 \mathcal{M}(I) - (S'(I) \cdot \vec{p})^2 \right] \phi(I)
\]
\[
= \frac{4\pi}{\varepsilon_C} \frac{\mu}{\varepsilon} \vec{p}(I)\cdot \vec{n}
\]
\[
= \frac{4\pi}{\varepsilon_C} \frac{\mu}{\varepsilon} \vec{p}(I) \cdot \vec{n}
\]

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We see that if \( I = 1 \), the last term on the left automatically disappears since \( S(0) = 0 \). Besides, \( \Phi \) is one component (scalar) for a vector field and hence \( (S(I-I) \cdot \vec{p}) \), which is a curl operator, cannot operate on a scalar. Now we impose the physical condition:

\[
\cdot (S(I-I) \cdot \vec{p}) \Phi(I-I) = 0,
\]

which implies \( \Phi(I-I) \) is an irrotational field. This is physically reasonable since \( \Phi(I-I) \) is related to a static charge source of \( \rho(I-I) \). So, using the above condition we obtain:

\[
\left[ (\xi^2 - \rho^2 - m_0^2 c^2) \right] \Phi(I-I) = -4\pi\hbar^2 \rho(I-I) \cdot
\]

Hence the equations which the scalar and vector potentials of a massive field should satisfy are:

\[
\left[ (\xi^2 - \rho^2 - m_0^2 c^2) \right] A(I) = -\frac{4\pi\hbar^2}{c} J(I),
\]

\[
\left[ (\xi^2 - \rho^2 - m_0^2 c^2) \right] \Phi(I-I) = -4\pi\hbar^2 \Phi(I-I),
\]

\[
(\xi^2 - m_0 c) \Phi(I-I) - \frac{1}{I} (D(I) \cdot \vec{p}) A(I) = 0.
\]

From these equations one can find \( \Phi(I-I) \) and \( A(I) \) for any massive field with a given spin. For spin I the equations reduce to wave equations of a vector field:

\[
\left[ (\xi^2 - \rho^2 - m_0^2 c^2) \right] \overline{A}(\vec{r}) = -\frac{4\pi\hbar^2}{c} J,
\]

\[
\left[ (\xi^2 - \rho^2 - m_0^2 c^2) \right] \Phi(\vec{r}) = -4\pi\hbar^2 \rho,
\]

\[
(\xi^2 - m_0 c) \Phi(\vec{r}) - (\overline{D} \cdot \vec{p}) A(\vec{r}) = 0.
\]
The solutions of these equations could give the scalar and vector potentials of a massive vector field.
VII. SPECIAL APPLICATION OF THE GENERALIZED DIRAC EQUATION

The Vector Boson Equation

In this section we will discuss the vector boson equation as a special case of the generalized Dirac equation. The vector boson equation can be written in terms of the sigma matrix $\sigma (i)$ developed for the vector (electromagnetic) field. The Hamiltonian for a vector boson can be written as:

$$H_B = C (\vec{x} \cdot \vec{p}) + \beta m_0 c^2,$$

where

$$\vec{\alpha} = \begin{pmatrix} \theta & \xi (1) \\ \bar{\xi} (1) & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \vec{n} & 0 \\ 0 & -\vec{n} \end{pmatrix}.$$

Each identity matrix is $4 \times 4$.

Then the vector boson field equation can be written as:

$$i \hbar \frac{\partial}{\partial t} \psi (r, t) = \left[ C (\vec{\alpha} \cdot \vec{p}) + \beta m_0 c^2 \right] \psi (r, t),$$

$$= H_B \psi (r, t).$$

We know that the total angular momentum $\vec{J} = \vec{L} + \frac{\hbar}{2} \vec{S}$ is a constant of the motion for the Dirac equation. For the vector boson equation we expect the total angular momentum $\vec{J} = \vec{L} + \hbar \vec{S}$ to be conserved and hence it is a constant of motion for the system. We can easily verify this assertion as follows:
\[
[H, \bar{\bar{J}}] = [H, \bar{\bar{L}} + \bar{\bar{L}} \bar{\bar{S}}] = [H, \bar{\bar{L}}] + [H, \bar{\bar{L}} \bar{\bar{S}}]
\]

Let us consider only \( \bar{\bar{S}} = \bar{\bar{L}} + \bar{\bar{S}} \) part first.

\[
[H, \bar{\bar{L}} + \bar{\bar{L}} \bar{\bar{S}}] = [H, \bar{\bar{L}}] + [H, \bar{\bar{L}} \bar{\bar{S}}]
\]

\[
[H, \bar{\bar{L}}] = [\mathbf{C}(\bar{\bar{x}} \cdot \bar{\bar{p}}) + \beta m_0 c^2, \bar{\bar{L}}]
\]

\[
= [\mathbf{C}(\bar{\bar{x}} \cdot \bar{\bar{p}}), \bar{\bar{L}}] + [\beta m_0 c^2, \bar{\bar{L}}]
\]

\[
[\beta m_0 c^2, \bar{\bar{L}}] = m_0 c^2 [\beta, \bar{\bar{L}}] = 0
\]

\[
[\mathbf{C}(\bar{\bar{x}} \cdot \bar{\bar{p}}), \bar{\bar{L}}] = \left[ \mathbf{C}(\bar{\bar{x}} P_x + \bar{\bar{x}} P_y + \bar{\bar{x}} P_z), Y P_y - Y P_z \right],
\]

\[
= \mathbf{C} \left( \bar{\bar{x}} [P_x, y P_z] + \bar{\bar{x}} [P_y, y P_z] + \bar{\bar{x}} [P_z, y P_z] \right)
\]

\[
- \mathbf{C} \left( \bar{\bar{x}} [P_x, 2 P_y + \bar{\bar{x}} [P_y, 2 P_y] + \bar{\bar{x}} [P_z, 2 P_z] \right)
\]

\[
= \mathbf{C} \left( \bar{\bar{x}} (P_x y P_z - y P_x P_z) - \bar{\bar{x}} (P_x P_z - P_z P_x) \right).
\]

But \( P_x = -i \hbar \frac{\partial}{\partial y} \) and \( P_z = -i \hbar \frac{\partial}{\partial y} \) so that:

\[
[\mathbf{C}(\bar{\bar{x}} \cdot \bar{\bar{p}}), \bar{\bar{L}}] = (-i \hbar)^2 \mathbf{C} \bar{\bar{x}} \left( \frac{\partial}{\partial y} \left( y \frac{\partial}{\partial y} \right) - y \frac{\partial^2}{\partial y^2} \right)
\]

\[
= (-i \hbar)^2 \mathbf{C} \bar{\bar{x}} \left( \frac{\partial}{\partial y} \left( y \frac{\partial}{\partial y} \right) - \frac{\partial^2}{\partial y^2} \right)
\]

\[
= (-i \hbar)^2 \mathbf{C} \bar{\bar{x}} \left( \frac{\partial}{\partial y} + y \frac{\partial^2}{\partial y^2} - y \frac{\partial^2}{\partial y^2} \right)\]

\[
= (-i \hbar)^2 \mathbf{C} \bar{\bar{x}} \left( \frac{\partial}{\partial y} + y \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y^2} \right)
\]

\[
= (-i \hbar)^2 \mathbf{C} \bar{\bar{x}} \frac{\partial}{\partial y} - (-i \hbar)^2 \mathbf{C} \bar{\bar{x}} \frac{\partial}{\partial y}
\]

\[
= -i \hbar \mathbf{C} \bar{\bar{x}} (-i \hbar \frac{\partial}{\partial y}) - (-i \hbar) \mathbf{C} \bar{\bar{x}} (-i \hbar \frac{\partial}{\partial y})
\]
\[
\begin{align*}
&= -i \not\! \not\! \! \not C \bar{x}_y \not\! \! \not P_3 - (\not\! \not C \not\! \! \not P_4) \\
&= -i \not\! \not C (\bar{x}_y \not\! \! \not P_3 - \bar{x}_y \not\! \! \not P_4) .
\end{align*}
\]
\[
\begin{align*}
[H, \not\! \! \not C S_x] &= [C (\bar{x}_y \not\! \! \not P_3 + \beta m_0 c^2, \not\! \! \not C S_x] , \\
&= [C (\bar{x}_y \not\! \! \not P_3, \not\! \! \not C S_x] + [\beta m_0 c^2, \not\! \! \not C S_x] , \\
&= C \not\! \! \not C [\bar{x}_y \not\! \! \not P_3 + \bar{x}_y \not\! \! \not P_4 + \bar{x}_y \! \not\! \! \not P_3 , S_x] , \\
\end{align*}
\]
\[
\begin{align*}
\bar{x}_x \not\! \! \not P_x , S_x \not\! \! \not S_x] &= \bar{x}_x \not\! \! \not P_x + \bar{x}_x [P_x , S_x] , \\
&= (\bar{x}_x S_x - S_x \bar{x}_x) P_x .
\end{align*}
\]
Similarly we have
\[
\begin{align*}
\bar{x}_y \not\! \! \not P_y , S_x \not\! \! \not S_x] &= (\bar{x}_y S_x - S_x \bar{x}_y) P_y \\
\end{align*}
\]
and
\[
\begin{align*}
\bar{x}_y \not\! \! \not P_3 , S_x \not\! \! \not S_x] &= (\bar{x}_y S_x - S_x \bar{x}_y) P_3 .
\end{align*}
\]
\[
\bar{x}_x, \bar{x}_y \text{ and } \bar{x}_z \text{ are } 4 \times 4 \text{ matrices while } S_x \text{ is a } 3 \times 3 \text{ matrix. But, since we have expanded the vector field into a four component field by introducing } E_s \text{ and } B_s , \text{ we can expand } S_x \text{ into a } 4 \times 4 \text{ matrix by writing in general:}
\]
\[
S'(I) = \begin{pmatrix}
S(I) & 0 \\
0 & S(I) - I
\end{pmatrix}
\]
For spin I this gives:
\[
S'^{I}(I) = \begin{pmatrix}
S(I) & 0 \\
0 & 0
\end{pmatrix} .
\]
Thus we have

\[ S'_x(1) = \begin{pmatrix}
0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \]

Now we can easily multiply the matrices

\[ \Sigma_x S'_x, \Sigma_y S'_y \text{ and } \Sigma_z S'_z. \]

\[ \Sigma_x S'_x = \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0
\end{pmatrix} \begin{pmatrix}
0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 1 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \]

\[ S'_x \Sigma_x = \begin{pmatrix}
0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 1 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \]

\[ \Sigma_x S'_x - S'_x \Sigma_x = 0. \]
\[
S_x^{\prime} \Sigma_y = \begin{pmatrix}
0 & \frac{1}{\tau} & 0 & 0 \\
\frac{1}{\tau} & 0 & \frac{1}{\tau} & 0 \\
0 & \frac{1}{\tau} & 0 & \frac{1}{\tau} \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & \frac{1}{\tau} & 0 & 0 \\
\frac{1}{\tau} & 0 & \frac{1}{\tau} & 0 \\
0 & \frac{1}{\tau} & 0 & \frac{1}{\tau} \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\tau} & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\Sigma_y S_x^{\prime} - S_x^{\prime} \Sigma_y = \begin{pmatrix}
-i & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & 0 & i & 0
\end{pmatrix} = -i \Sigma_y
\]

\[
\Sigma_2 S_x^{\prime} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\tau} & 0 & 0 \\
0 & 0 & \frac{1}{\tau} & 0 \\
0 & 0 & 0 & \frac{1}{\tau}
\end{pmatrix} \begin{pmatrix}
0 & \frac{1}{\tau} & 0 & 0 \\
\frac{1}{\tau} & 0 & \frac{1}{\tau} & 0 \\
0 & \frac{1}{\tau} & 0 & \frac{1}{\tau} \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{\tau} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
S_x^{\prime} \Sigma_2 = \begin{pmatrix}
0 & \frac{1}{\tau} & 0 & 0 \\
\frac{1}{\tau} & 0 & \frac{1}{\tau} & 0 \\
0 & \frac{1}{\tau} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\tau} & 0 & 0 \\
0 & 0 & \frac{1}{\tau} & 0 \\
0 & 0 & 0 & \frac{1}{\tau}
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{\tau} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
S_x \Sigma_y = \begin{pmatrix}
0 & \frac{1}{\tau} & 0 & 0 \\
\frac{1}{\tau} & 0 & \frac{1}{\tau} & 0 \\
0 & \frac{1}{\tau} & 0 & \frac{1}{\tau} \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\tau} & 0 & 0 \\
0 & 0 & \frac{1}{\tau} & 0 \\
0 & 0 & 0 & \frac{1}{\tau}
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{\tau} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

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\[
\sum_{\gamma} s_\gamma^\prime s_\gamma = \left(\begin{array}{cccc}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
-\frac{1}{2} & 0 & -\frac{1}{2} & 0
\end{array}\right) \cdot \xi \xi_\gamma.
\]

Hence:
\[
\mathcal{C}^+ \{ [\sum_{\gamma} p_\gamma, s_\gamma] + [\sum_{\gamma} p_\gamma, s_\gamma] + [\sum_{\gamma} p_\gamma, s_\gamma] \} = -i \mathcal{C}^+ (\sum_{\gamma} p_\gamma - \sum_{\gamma} p_z).
\]

Hence:
\[
[\mathcal{H}, \sum_{\gamma} \xi \xi_\gamma] = -i \mathcal{C}^+ (\sum_{\gamma} p_\gamma - \sum_{\gamma} p_z) + i \mathcal{C}^+
(\sum_{\gamma} p_\gamma - \sum_{\gamma} p_z) = 0.
\]

Similarly we can show that:
\[
[\mathcal{H}, \sum_{\gamma} \xi \xi_\gamma] = [\mathcal{H}, \sum_{\gamma} \xi \xi_\gamma] = 0
\]

Hence:
\[
[\mathcal{H}, \sum_{\gamma} \xi \xi_\gamma] = 0.
\]

Hence \( \mathcal{H} \) and \( \sum_{\gamma} \) commute.

Thus, \( \tilde{\mathcal{J}} = \tilde{\mathcal{L}} + \hbar \tilde{\mathcal{S}} \) is definitely a constant of the motion for the vector boson equation.

Also by analogy to \( \tilde{\mathcal{J}} = \tilde{\mathcal{L}} + \frac{1}{2} \mathcal{S}(\mathcal{V}) \) we suspect that \( \tilde{\mathcal{J}}' = \tilde{\mathcal{L}} + \frac{1}{2} \mathcal{S}(\mathcal{V}) \) could be a constant of the motion for the vector boson equation.
\[ [H, \overline{\mathfrak{J}}'] = [C(\mathbf{\hat{r}} \cdot \mathbf{P}) + \beta m_0 c^2, \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}(\omega)] = [C(\mathbf{\hat{r}} \cdot \mathbf{P}), \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}(\omega)] + [\beta m_0 c^2, \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}(\omega)] \]

Consider first \( \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}} \).

\[ [H, \overline{\mathfrak{L}}_{x}] = [C(\mathbf{\hat{r}} \cdot \mathbf{P}), \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}}] + [\beta m_0 c^2, \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}}] = [C(\mathbf{\hat{r}} \cdot \mathbf{P}), \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}}] \]

\[ [C(\mathbf{\hat{r}} \cdot \mathbf{P}), \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}}] = -i \hbar C(\mathbf{\hat{z}}_y P_y - \mathbf{\hat{z}}_3 P_3), \quad \text{(from page 57)} \]

\[ [C(\mathbf{\hat{r}} \cdot \mathbf{P}), \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}}] = \frac{\hbar}{2} \left[ \mathbf{L}_{x}, \mathbf{L}_{x} \right] \mathbf{P}_{x} + \mathbf{L}_{y} \mathbf{P}_{y} + \mathbf{L}_{z} \mathbf{P}_{z}, \quad \right] \mathbf{\mathcal{E}}_{\mathbf{x}} \right] \]

\[ \mathbf{L}_{x} \mathbf{L}_{x} \mathbf{P}_{x} + \mathbf{L}_{y} \mathbf{P}_{y} + \mathbf{L}_{z} \mathbf{P}_{z} + \mathbf{\mathcal{E}}_{\mathbf{x}} \mathbf{P}_{y} \]

\[ = -i \hbar C(\mathbf{\hat{z}}_y P_y - \mathbf{\hat{z}}_3 P_3). \]

Thus, \( [H, \overline{\mathfrak{J}}'] = -i \hbar C(\mathbf{\hat{z}}_y P_y - \mathbf{\hat{z}}_3 P_3) = i \hbar C(\mathbf{\hat{z}}_3 P_3 - \mathbf{\hat{z}}_y P_y) \)

Thus \( [H, \overline{\mathfrak{J}}'] = 0 \) showing that \( \overline{\mathfrak{J}}' \) is also a conserved quantity and is thus a constant of the motion for the system.

Thus there are two constants of motion for the vector boson equation. One is the total angular momentum \( \overline{\mathfrak{J}} = \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}} \)

and the other is \( \overline{\mathfrak{J}}' = \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}}. \) In the case of the Dirac equation \( \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}} \) is the spin angular momentum \( \mathbf{S}(\gamma). \) But, here \( \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}} \) is not the same as the spin angular momentum \( \mathbf{S}(\gamma). \) So what is the physical meaning of \( \overline{\mathfrak{J}}' = \mathbf{L} + \frac{1}{2} \mathbf{\mathcal{E}}_{\mathbf{x}} \)?
Right now there is no clear explanation of what \( \mathcal{F} = \hat{L} + \frac{1}{2} \mathcal{J}^2 \) physically represents.

In the next section we shall discuss the solution of the vector boson equation for a free particle (boson).

A Free Particle Solution of the Vector Boson Equation

A free particle solution of the vector boson equation

\[
\mathcal{H}_B \Psi_{j m} = \mathcal{E} \Psi_{j m}
\]

will be discussed now. We have already seen that \( \mathcal{H}_B = c (\mathbf{\hat{r}} \cdot \mathbf{p}) + \beta mc^2 \) the Hamiltonian for a free vector boson, commutes with \( \mathcal{J}_r \) and \( \mathcal{J}_z \). It can also be shown that

\[
[\mathcal{J}_r^2, \mathcal{J}_z] = 0 \quad \text{and} \quad [\mathcal{J}_z^2, \mathcal{H}_B] = 0.
\]

Thus, we can now construct a simultaneous eigenfunction of \( \mathcal{J}_r^2, \mathcal{J}_z \) and \( \mathcal{H}_B \). Such an eigenfunction can be written in terms of spherical harmonics and Clebsh-Gordon coefficients.

\[
\Psi_{j m} = \alpha \sum\limits_{\mu} \left( \begin{array}{c}
J+1, m-\mu, l, l l 3 \end{array} J, m \right) f_{j+1}(r) Y_{j+1, m_{-\mu}} (\mathbf{\hat{r}})
+ \beta \sum\limits_{\mu} \left( \begin{array}{c}
J, m-\mu, l, l l 3 \end{array} J, m \right) f_j (r) Y_{j, m_{-\mu}} (\mathbf{\hat{r}})
+ \gamma \sum\limits_{\mu} \left( \begin{array}{c}
J-1, m-\mu, l, l l 3 \end{array} J, m \right) f_{j-1} (r) Y_{j-1, m_{-\mu}} (\mathbf{\hat{r}})
\]

Where \( \mu = 1, 0, -1 \) and \( \alpha, \beta, \gamma \) are arbitrary constants.

\[
\Psi_{j m} = \left\{ \begin{array}{c}
\alpha \left( \begin{array}{c}
J+1, m-1, l, l l 3 \end{array} J, m \right) f_{j+1}(r) Y_{j+1, m_{-1}} (\mathbf{\hat{r}})
+ \beta \left( \begin{array}{c}
J, m-1, l, l l 3 \end{array} J, m \right) f_j (r) Y_{j, m_{-1}} (\mathbf{\hat{r}})
+ \gamma \left( \begin{array}{c}
J-1, m-1, l, l l 3 \end{array} J, m \right) f_{j-1} (r) Y_{j-1, m_{-1}} (\mathbf{\hat{r}})
\right\}
\]

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\[ \begin{align*}
+ \{ &\alpha(j+1, m, l, 0|m) f_{j+1}(r) Y_{j+1,m}(\omega) \\
+ &\beta(j, m, l, 0|m) f_{j}(r) Y_{j,m}(\omega) \\
+ &\delta(j-1, m, l, 0|m) f_{j-1}(r) Y_{j-1,m}(\omega) \} \\
+ \{ &\alpha(j+1, m+1, l-1|m) f_{j+1}(r) Y_{j+1,m+1}(\omega) \\
+ &\beta(j, m+1, l-1|m) f_{j}(r) Y_{j,m+1}(\omega) \\
+ &\delta(j-1, m+1, l-1|m) f_{j-1}(r) Y_{j-1,m+1}(\omega) \} \\
(j+1, m-1, 1, 1|m) &= \sqrt{\frac{(j-m+1)(j-m+2)}{2(j+1)(2j+3)}} = C_1 \\
(j, m-1, l, 1|m) &= -\sqrt{\frac{(j+m)(j-m+1)}{2j(j+1)}} = C_2 \\
(j-1, m-1, 1, 1|m) &= \sqrt{\frac{(j+m-1)(j+m)}{2j(2j-1)}} = C_3 \\
(j+1, m, 1, 0|m) &= -\sqrt{\frac{(j-m+1)(j+m+1)}{(j+1)(2j+3)}} = C_4 \\
(j, m, l, 0|m) &= \frac{M}{\sqrt{j(j+1)}} = C_5
\end{align*} \]
\[
(j, m) = \sqrt{\frac{(j-m)(j+m)}{j(j+1)}} = C_6
\]
\[
(j+1, m+1, -1/m) = \sqrt{\frac{(j+1)(j+m+1)}{2j(2j+1)}} = C_7
\]
\[
(j, m+1, 1, -1/m) = \sqrt{\frac{(j-m)(j+m+1)}{2j(2j-1)}} = C_8
\]
\[
(j-1, m+1, 1, -1/m) = \sqrt{\frac{(j-m)(j-m-1)}{2j(2j-1)}} = C_9
\]

We now have:

\[
\left[ \mathcal{C} (\vec{x} \cdot \vec{p}) + \beta m_0 c^2 \right] \Psi_{jm} = \mathcal{E} \Psi_{jm},
\]

\[
\begin{bmatrix}
0 & \vec{x} \\
\vec{x} & 0
\end{bmatrix} \cdot \vec{p} + m_0 c \begin{bmatrix}
\vec{u} & 0 \\
0 & -\vec{u}
\end{bmatrix} \Psi_{jm} = \frac{\mathcal{E}}{c} \Psi_{jm},
\]

\[
\begin{bmatrix}
0 & \vec{x} \\
\vec{x} & 0
\end{bmatrix} \cdot \vec{p} + m_0 c \begin{bmatrix}
\vec{u} & 0 \\
0 & -\vec{u}
\end{bmatrix} \begin{bmatrix}
\Psi_{jm}^+ \\
\Psi_{jm}^-
\end{bmatrix} = \frac{\mathcal{E}}{c} \begin{bmatrix}
\Psi_{jm}^+ \\
\Psi_{jm}^-
\end{bmatrix}
\]

where \( \Psi_{jm}^+ \) and \( \Psi_{jm}^- \) are eigenfunctions corresponding to positive energy eigenvalue and negative energy eigenvalue respectively. Since \( \vec{x} \) and \( \beta \) are 4 x 4 matrices, \( \Psi_{jm}^+ \) and \( \Psi_{jm}^- \) should each have four components.
\( \Psi_{j_{m}}^{+} = \begin{pmatrix} \varphi_{1}^{j_{m}} \\ \varphi_{0}^{j_{m}} \\ \varphi_{-1}^{j_{m}} \\ \varphi_{-2}^{j_{m}} \end{pmatrix}, \quad \Psi_{j_{m}}^{-} = \begin{pmatrix} \chi_{1}^{j_{m}} \\ \chi_{0}^{j_{m}} \\ \chi_{-1}^{j_{m}} \\ \chi_{-2}^{j_{m}} \end{pmatrix} \)

where

\( \varphi_{1}^{j_{m}} = \alpha C_{4} f_{j_{1}} (r) Y_{j+1, m} (\psi) + \beta C_{5} f_{j_{1}} (r) Y_{j, m} (\psi) + \gamma C_{6} f_{j_{1}} (r) Y_{j-1, m} (\psi) \)

\( \varphi_{0}^{j_{m}} = \alpha C_{4} f_{j_{1}} (r) Y_{j+1, m} (\psi) + \beta C_{5} f_{j_{1}} (r) Y_{j, m} (\psi) + \gamma C_{6} f_{j_{1}} (r) Y_{j-1, m} (\psi) \)

\( \varphi_{-1}^{j_{m}} = \alpha C_{4} f_{j_{1}} (r) Y_{j+1, m-1} (\psi) + \beta C_{5} f_{j_{1}} (r) Y_{j, m-1} (\psi) + \gamma C_{6} f_{j_{1}} (r) Y_{j-1, m-1} (\psi) \)

\( \varphi_{-2}^{j_{m}} \) is simply a scalar field arbitrarily introduced as the fourth component of \( \Psi_{j_{m}}^{+} \).

Similarly:

\( \chi_{1}^{j_{m}} = \alpha C_{4} g_{j+1} (r) Y_{j+1, m-1} (\psi) + \beta C_{5} g_{j} (r) Y_{j, m-1} (\psi) + \gamma C_{6} g_{j-1} (r) Y_{j-1, m-1} (\psi) \)

\( \chi_{0}^{j_{m}} = \alpha C_{4} g_{j+1} (r) Y_{j+1, m} (\psi) + \beta C_{5} g_{j} (r) Y_{j, m} (\psi) + \gamma C_{6} g_{j-1} (r) Y_{j-1, m} (\psi) \)
\begin{align*}
\chi_{j_1}^{j_0} &= \alpha C_1 \, g_{j_1}^*(r) \, \psi_{j_1, m_{j_1}^0}^{(c_1)} + \beta C_2 g_j(r) \psi_{j, m_{j_1}^0}^{(c_2)} \\
&\quad + \gamma C_3 g_{j_{-1}}(r) \psi_{j_{-1}, m_{j_1}^0}^{(c_2)}.
\end{align*}

\chi_{j_1}^{j_0} \quad \text{is a scalar field.}

From the eigenvalue equation we get two coupled equations:

\begin{align*}
(\hat{\Sigma} \cdot \vec{P}) \psi_{j_1 m_1}^{-} + \sqrt{m_0 c} \psi_{j_1 m_1}^{+} &= \frac{E}{c} \psi_{j_1 m_1}^{+}, \\
(\hat{\Sigma} \cdot \vec{P}) \psi_{j_1 m_1}^{+} - \sqrt{m_0 c} \psi_{j_1 m_1}^{-} &= \frac{E}{c} \psi_{j_1 m_1}^{-}, \\
(\hat{\Sigma} \cdot \vec{P}) \psi_{j_1 m_1}^{-} &= \left(\frac{E}{c} - m_0 c\right) \psi_{j_1 m_1}^{+}, \\
(\hat{\Sigma} \cdot \vec{P}) \psi_{j_1 m_1}^{+} &= \left(\frac{E}{c} + m_0 c\right) \psi_{j_1 m_1}^{-}.
\end{align*}

Equation (1) becomes:

By solving either (1) or (2) we can find a relation between the radial functions \( f(r) \) and \( g(r) \).

We shall solve (1).

\[
(\hat{\Sigma} \cdot \vec{P}) = \hat{\Sigma}_x P_x + \hat{\Sigma}_y P_y + \hat{\Sigma}_z P_z,
\]

\[
= \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0
\end{pmatrix} P_x + \begin{pmatrix}
0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0
\end{pmatrix} P_y + \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} P_z.
\]

Equation (1) becomes:
Equations (1) and (2) can be written as:

\[
\frac{1}{2} P_x (\chi_0^{jm} + \chi_5^{jm}) - \frac{1}{2} P_y (\chi_0^{jm} + \chi_5^{jm}) + P_3 \chi_1^{jm} = \left( \frac{\epsilon}{c^2} - m_0 c \right) \Phi_1^{jm},
\]

(1)

\[
\frac{1}{2} P_x (\chi_1^{jm} + \chi_{-1}^{jm}) + \frac{1}{2} P_y (\chi_1^{jm} - \chi_{-1}^{jm}) - P_3 \chi_5^{jm} = \left( \frac{\epsilon}{c^2} - m_0 c \right) \Phi_0^{jm},
\]

(2)

\[
\frac{1}{2} P_x (\chi_0^{jm} - \chi_5^{jm}) + \frac{1}{2} P_y (\chi_0^{jm} - \chi_5^{jm}) - P_3 \chi_{-1}^{jm} = \left( \frac{\epsilon}{c^2} - m_0 c \right) \Phi_{-1}^{jm},
\]

(3)

\[
\frac{1}{2} P_x (\chi_{1}^{jm} - \chi_{-1}^{jm}) + \frac{1}{2} P_y (\chi_{1}^{jm} + \chi_{-1}^{jm}) - P_3 \chi_{-5}^{jm} = \left( \frac{\epsilon}{c^2} - m_0 c \right) \Phi_{-5}^{jm}.
\]

(4)

By adding (2) and (4) we get:

\[
\frac{1}{2} (P_x + i P_y) \chi_1^{jm} - P_3 (\chi_0^{jm} + \chi_5^{jm}) = \left( \frac{\epsilon}{c^2} - m_0 c \right) (\Phi_0^{jm} + \Phi_5^{jm}).
\]

(5)
But \( P_x = -i \hbar \frac{\partial}{\partial x} \), \( P_y = -i \hbar \frac{\partial}{\partial y} \), \( P_z = -i \hbar \frac{\partial}{\partial z} \), so that \( \frac{1}{\sqrt{2}} (P_x + i P_y) = -i \hbar \left( \frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} \right) \).

But \( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} = \left( \frac{8\pi}{5} \right)^{1/2} \left[ -Y_{11} \frac{\partial}{\partial r} + \frac{1}{\alpha r} Y_{11} L + \frac{1}{\alpha r} Y_{11} L_0 \right] \),

\( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} = \left( \frac{8\pi}{5} \right)^{1/2} \left[ Y_{11} \frac{\partial}{\partial r} - \frac{1}{\alpha r} Y_{11} L + \frac{1}{\alpha r} Y_{11} L_0 \right] \),

and \( \frac{\partial}{\partial z} = \left( \frac{4\pi}{5} \right)^{1/2} \left[ Y_{10} \frac{\partial}{\partial r} - \frac{1}{\alpha r} Y_{10} L + \frac{1}{\alpha r} Y_{10} L_0 \right] \).

Making these substitutions in (5), (6) and (7) we get:

\[ -i \hbar \left( \frac{4\pi}{3} \right)^{1/2} \left[ -Y_{11} \frac{\partial}{\partial r} + \frac{1}{\alpha r} Y_{10} L - \frac{i}{\alpha r} Y_{11} L_0 \right] (\chi_0^m + \chi_s^m) \]

\[ -i \hbar \left( \frac{4\pi}{3} \right)^{1/2} \left[ -Y_{11} \frac{\partial}{\partial r} + \frac{1}{\alpha r} Y_{10} L - \frac{i}{\alpha r} Y_{11} L_0 \right] \chi_s^m \]

\[ = (\frac{\varepsilon}{\hbar} - m_0 c) \phi_s^m \]

(8)

\[ -i \hbar \left( \frac{4\pi}{3} \right)^{1/2} \left[ Y_{10} \frac{\partial}{\partial r} - \frac{i}{\alpha r} Y_{10} L - \frac{i}{\alpha r} Y_{10} L_0 \right] \chi_l^m \]

\[ + i \hbar \left( \frac{4\pi}{3} \right)^{1/2} \left[ Y_{10} \frac{\partial}{\partial r} - \frac{i}{\alpha r} Y_{10} L - \frac{i}{\alpha r} Y_{10} L_0 \right] \chi_s^m \]

\[ = (\frac{\varepsilon}{\hbar} - m_0 c) \phi_l^m \]

(9)

\[ -i \hbar \left( \frac{16\pi}{3} \right)^{1/2} \left[ -Y_{11} \frac{\partial}{\partial r} + \frac{1}{\alpha r} Y_{11} L + \frac{1}{\alpha r} Y_{11} L_0 \right] \chi_l^m \]

\[ + i \hbar \left( \frac{4\pi}{3} \right)^{1/2} \left[ Y_{10} \frac{\partial}{\partial r} - \frac{i}{\alpha r} Y_{11} L + \frac{i}{\alpha r} Y_{10} L_0 \right] (\chi_0^m + \chi_s^m) \]

\[ = (\frac{\varepsilon}{\hbar} - m_0 c) (\phi_0^m + \phi_s^m) \]

(10)

Similarly, going through the same procedure as above, we get from equation (2) on page 66:
By solving equation (8) we can derive a relationship between the radial functions $f(r)$ and $g(r)$. We shall also get a similar relationship from equation (11) above. Then the two results could be coupled together to find an explicit solution of the radial functions.
Consider the first term of equation 8:

\[-\ell^2 \left(\frac{4\pi^2}{3}\right)^{\frac{3}{2}} \int \gamma_{j+1} \frac{\partial}{\partial r} \gamma_{j-1} \frac{\partial}{\partial r} \gamma_{j+1} \frac{\partial}{\partial r} \gamma_{j-1} \lambda_0 \left( \chi_{j+1}^{\prime\prime} \chi_j^{\prime\prime} \right)\]

\[
\gamma_{j+1} \frac{\partial}{\partial r} \chi_j^{\prime\prime} := \gamma_{j+1} \frac{\partial}{\partial r} \left( \alpha C_4 g_{j+1}(r) \gamma_{j+1} \chi_j^{(2)} + \beta C_5 g_j(r) \gamma_j \chi_j^{(2)} + \delta C_6 g_{j-1}(r) \gamma_{j-1} \chi_j^{(2)} \right)
\]

\[
= \alpha C_4 \frac{dg_{j+1}(r)}{dr} \left[ \frac{3 (j-m+2)(j-m+3)}{8\pi (2j+3)(2j+5)} \right]^{\frac{3}{2}} \gamma_{j+2} \chi_j^{(2)}
\]

\[- \left( \frac{3 (j+m) (j+m+1)}{4\pi (2j+1)(2j+3)} \right)^{\frac{3}{2}} \gamma_j \chi_j^{(2)}
\]

\[
+ \beta C_5 \frac{dg_j(r)}{dr} \left[ \frac{3 (j-m+1)(j-m+2)}{8\pi (2j+1)(2j+3)} \right]^{\frac{3}{2}} \gamma_{j+1} \chi_{j-1}^{(2)}
\]

\[- \left( \frac{3 (j+m) (j+m-1)}{8\pi (2j-1)(2j+1)} \right)^{\frac{3}{2}} \gamma_j \chi_{j-1}^{(2)}
\]

\[
+ \delta C_6 \frac{dg_{j-1}(r)}{dr} \left[ \frac{3 (j-m)(j-m+1)}{8\pi (2j-1)(2j+1)} \right]^{\frac{3}{2}} \gamma_j \chi_{j-1}^{(2)}
\]

\[- \left( \frac{3 (j+m-1)(j+m-2)}{8\pi (2j-1)(2j-3)} \right)^{\frac{3}{2}} \gamma_j \chi_{j-2}^{(2)}
\]
$$\begin{align*}
- \frac{1}{2\pi r} Y_{l_0} L - \chi_{0}^{L} &= - \frac{1}{2\pi r} Y_{l_0} \left[ g_{j+1}(r) L - Y_{j+1, m}(e) \right] \\
&\quad + \beta C_5 g_j(r) L - Y_{j, m}(e) + \gamma C_6 g_{j-1}(r) L - Y_{j-1, m}(e) \\
&= - \frac{\alpha C_4 g_{j+1}(r) \left[(J+1)(J-1)ight]^{1/2}}{\gamma} \left[ \frac{3(J+1)(J-1)}{8\pi(2J+1)(2J+3)} \right]^{1/2} \frac{1}{\gamma} Y_{j, m}(e) \\
&\quad + \left( \frac{3(J+1)(J-1)}{8\pi(2J+1)(2J+3)} \right)^{1/2} \frac{1}{\gamma} Y_{j, m-1}(e) \\
&\quad - \frac{\beta C_5}{r} g_j(r) \left[(J-m)(J+m-1)\right]^{1/2} \left[ \frac{3(J-m)(J+m-1)}{8\pi(2J-1)(2J+1)} \right]^{1/2} \frac{1}{\gamma} Y_{j, m-1}(e) \\
&\quad + \left( \frac{3(J-m)(J+m-1)}{8\pi(2J-3)(2J-1)} \right)^{1/2} \frac{1}{\gamma} Y_{j-1, m}(e) \\
&\quad - \frac{\gamma C_6}{r} g_{j-1}(r) \left[(J-m)(J+m-2)\right]^{1/2} \left[ \frac{3(J-m)(J+m-2)}{8\pi(2J-1)(2J+1)} \right]^{1/2} \frac{1}{\gamma} Y_{j-1, m}(e) \\
&= \frac{1}{2\pi r} Y_{l_1} L - \chi_{0}^{L} = \frac{m}{r} \left[ \alpha C_4 g_{j+1}(r) Y_{l_1}(e) Y_{j+1, m}(e) \\
&\quad + \beta C_5 g_j(r) Y_{l_1}(e) Y_{j, m}(e) + \gamma C_6 g_{j-1}(r) Y_{l_1}(e) Y_{j-1, m}(e) \right] \\
&\quad + \frac{1}{2\pi r} Y_{l_0} L - \chi_{0}^{L} = \frac{m}{r} \left[ \alpha C_4 g_{j+1}(r) Y_{l_1}(e) Y_{j+1, m}(e) \\
&\quad + \beta C_5 g_j(r) Y_{l_1}(e) Y_{j, m}(e) + \gamma C_6 g_{j-1}(r) Y_{l_1}(e) Y_{j-1, m}(e) \right]
\end{align*}$$
\[
\begin{align*}
Y_{j-1} \frac{\partial}{\partial r} \chi^\text{jm}_s &= \frac{d \Phi(r)}{dr} \left[ \left( \frac{3 (j-m+1) (j-m+2)}{8 \pi (2j+3)(2j+5)} \right)^{1/2} y_{j+1,m-1}^{(2j+3)} \right. \\
&\quad - \left. \left( \frac{3 (j+m) (j+m+1)}{8 \pi (2j+1)(2j+3)} \right)^{1/2} y_{j,m-1}^{(2j+1)} \right]
\end{align*}
\]
\[
-\frac{1}{2\pi r} \chi_{ij}^{j_{1n}} = \frac{h(r) (j+m)(j-m+1)}{r} \left[ \frac{3(j-m+1)(j+m)}{8\pi(2j+1)(2j+3)} \right]^{\frac{1}{2}} \\
\]

\[
y_{j+1,m-1}(\omega) + \left( \frac{3(j-m+1)(j+m-1)}{8\pi(2j-1)(2j+1)} \right)^{\frac{1}{2}} y_{j-1,m-1}(\omega) \\
\]

\[
- \left( \frac{3(j+m)(j+m-1)}{8\pi(2j-1)(2j+1)} \right)^{\frac{1}{2}} y_{j-1,m-1}(\omega) \\
\]

From the second term of equation 8 we obtain:

\[
\gamma_{10} \frac{2}{\pi r} \chi_{i}^{j_{1n}} = \alpha C_1 \frac{d g_{j+1}(r)}{dr} y_{10}(\omega) y_{j+1,m-1}(\omega) + \beta C_2 \frac{d g_{j}(r)}{dr} \\
\]

\[
y_{10}(\omega) y_{j+1,m-1}(\omega) + \kappa C_3 \frac{d g_{j}(r)}{dr} y_{10}(\omega) y_{j+1,m-1}(\omega) \\
= \alpha C_1 \frac{d g_{j+1}(r)}{dr} \left[ \frac{(3(j-m+1)(j+m+1))^{\frac{1}{2}} y_{j+2,m-1}(\omega)}{4\pi(2j+3)(2j+5)} \right] + \beta C_2 \frac{d g_{j}(r)}{dr} \\
+ \left( \frac{3(j-m+2)(j+m)}{4\pi(2j+1)(2j+3)} \right)^{\frac{1}{2}} y_{j+1,m-1}(\omega) \\
\]

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\[ + \beta C_2 \frac{dg_{2j}}{dr} \left[ \left( \frac{3 (j-m+2)(j+m)}{4 \pi (2j+1)(2j+3)} \right)^{\frac{1}{2}} \right] Y_{j+1, m-1}^{(2j)} \]

\[ + \left( \frac{3 (j-m+1)(j+m-1)}{4 \pi (2j-1)(2j+1)} \right)^{\frac{1}{2}} Y_{j-1, m-1}^{(2j)} \]

\[ + \gamma C_3 \frac{dg_{j-1}}{dr} \left[ \left( \frac{3 (j-m+1)(j+m-1)}{4 \pi (2j-1)(2j+1)} \right)^{\frac{1}{2}} \right] Y_{j, m-1}^{(2j)} \]

\[ + \left( \frac{3 (j-m)(j+m-2)}{4 \pi (2j-3)(2j-1)} \right)^{\frac{1}{2}} Y_{j-2, m-1}^{(2j)} \]

\[ - \frac{\sqrt{2}}{\pi r} \Delta_{1-1} L_{j-1} + \chi_{j}^{(j)} = - \frac{\sqrt{2}}{\pi r} Y_{j-1} \left[ \alpha C_1 g_{j+1}(r) L_{j+1, m-1}^{(2j)} + \beta C_2 g_{j}(r) L_{j, m-1}^{(2j)} + \gamma C_3 g_{j-1}(r) L_{j-1, m-1}^{(2j)} \right] \]

\[ = - \frac{\alpha C_1 g_{j+1}(r)}{\pi r} \left( \frac{(j-m+2)(j+m+1)}{4 \pi (2j+1)(2j+3)} \right)^{\frac{1}{2}} \left[ \left( \frac{3 (j-m+2)(j+m+1)}{4 \pi (2j+1)(2j+3)} \right)^{\frac{1}{2}} \right] \]

\[ Y_{j+2, m-1}^{(2j)} - \left( \frac{3 (j+m)(j+m+1)}{2 \pi (2j+1)(2j+3)} \right)^{\frac{1}{2}} Y_{j, m-1}^{(2j)} \]

\[ - \frac{\beta C_2 g_{j}(r)}{\pi r} \left( \frac{(j+m)(j-m+1)}{4 \pi (2j+1)(2j+3)} \right)^{\frac{1}{2}} \left[ \left( \frac{3 (j-m+1)(j-m+2)}{4 \pi (2j-1)(2j+1)} \right)^{\frac{1}{2}} \right] \]

\[ Y_{j+1, m-1}^{(2j)} - \left( \frac{3 (j+m)(j+m-1)}{4 \pi (2j-1)(2j+1)} \right)^{\frac{1}{2}} Y_{j-1, m-1}^{(2j)} \]
\[-\frac{1}{\lambda} Y_0 L_0 \chi_j^m = - \frac{(m-1)}{R} \left[ \alpha c_1 g_{j+1}(r) Y_0 (\omega) Y_{j+1,m-1} \right. \\
+ \beta c_2 g_{j+k}(r) Y_0 (\omega) Y_{j+k,m-1} + \kappa c_3 g_{j-1}(r) Y_{0} (\omega) Y_{j-1,m-1} \right] \\
= - \frac{\alpha c_1 g_{j+1}(r)}{R} (m-1) \left[ \frac{3 (j-m+3) (j+m+1)}{4 \pi (2j+3)(2j+5)} \right]^{1/2} Y_{j+2,m-1} \\
+ \left( \frac{3(j-m+2)(j+m)}{4 \pi (2j+1)(2j+3)} \right)^{1/2} Y_{j,m-1} \\
- \frac{\beta c_2 g_{j+k}(r)}{R} (m-1) \left[ \frac{3(j-m+2)(j+m+1)}{4 \pi (2j+1)(2j+3)} \right]^{1/2} Y_{j+k,m-1} \\
+ \left( \frac{3(j-m+1)(j+m-1)}{4 \pi (2j-1)(2j+1)} \right)^{1/2} Y_{j-1,m-1} \\
- \frac{\kappa c_3 g_{j-1}(r)}{R} (m-1) \left[ \frac{3(j-m+1)(j+m-1)}{4 \pi (2j-1)(2j+1)} \right]^{1/2} Y_{j-1,m-1} \]
We can now collect like terms together. First, since $J^m$ is an arbitrary scalar field, we can set it to be zero to make calculations simple. When all terms are collected together, the terms with $\chi_{J+2, m-1}$ and $\chi_{J-2, m-1}$ drop out. Thus equation 8 reduces to:

\[
\begin{align*}
& + \left( \frac{3(j-m)(j+m-2)}{4\pi(2j-1)(2j+1)} \right)^{1/2} \chi_{J-2, m-1} \\
\end{align*}
\]

\[
\begin{align*}
& - i \frac{(j-m-1)(j-m+2)}{2j(2j+1)(2j+3)} \left[ \frac{dg_j(r)}{dr} - \frac{j+1}{r} g_j(r) \right] \chi_{J+2, m-1} \\
& - i \frac{\alpha'(j+m)(j-m+1)}{2(j+1)(2j+1)} \left[ \frac{dg_{j+1}(r)}{dr} + \frac{j+2}{r} g_{j+1}(r) \right] \chi_{J, m-1} \\
& + \frac{\alpha'(j+m)(j-m+1)}{2j(2j+1)} \left[ \frac{dg_{j-1}(r)}{dr} - \frac{j-1}{r} g_{j-1}(r) \right] \chi_{J, m-1} \\
& + \frac{i a'(j+m)(j+m-1)}{2j(j+1)(2j-1)(2j+1)} \left[ \frac{dg_{j+2}(r)}{dr} + \frac{j+1}{r} g_{j+2}(r) \right] \chi_{J+2, m-1} \\
& = \left( \frac{\alpha'(j+m+1)(j-m+2)}{2j(j+1)} \right)^{1/2} \frac{f_j(r)}{\chi_{J+2, m-1}} \\
& - \beta \frac{(j+m)(j-m+1)}{2j(j+1)} \left[ \frac{dg_{j+1}(r)}{dr} + \frac{j+1}{r} g_{j+1}(r) \right] \chi_{J, m-1}
\end{align*}
\]
\[ + \mathcal{I} \left( \frac{(j+m)(j+m-1)}{2j(2j-1)} \right)^{1/2} f_{j+1}(r) \left( j_{-1/2} m_{m-1} \right) \]

Hence, equating like terms, we obtain:

\[ i \hbar \beta \sqrt{\frac{2}{2j+1}} \left[ \frac{dg_j(r)}{dr} - \frac{j}{r} g_j(r) \right] - \alpha (\frac{E}{c} - m_0 c) f_{j+1}(r) = 0 \]

\[ \frac{\partial}{\partial r} - \frac{j+1}{r} \]

\[ i \hbar \alpha \sqrt{\frac{2}{2j+1}} g_j(r) = 0 \quad (1) \]

\[ -i \hbar \alpha \left( \frac{1}{(j+1)(2j+1)} \right)^{1/2} \left[ \frac{d^2 g_{j+1}(r)}{dr^2} + \frac{j+2}{r} g_{j+1}(r) \right] \]

\[ \frac{\partial^2}{\partial r^2} + \frac{j+1}{r} \]

\[ \sqrt{\frac{j+1}{2j+1}} g_{j-1}(r) - \beta (\frac{E}{c} - m_0 c) f_j(r) = 0 \quad (2) \]

\[ i \hbar \beta \left( \frac{j+1}{2j+1} \right)^{1/2} \left[ \frac{dg_j(r)}{dr} + \frac{j+1}{r} g_j(r) \right] = \gamma (\frac{E}{c} - m_0 c) f_{j+1}(r) \]

\[ \frac{\partial}{\partial r} + \frac{j+1}{r} \]

\[ i \hbar \beta \left( \frac{j+1}{2j+1} \right)^{1/2} g_j(r) - \gamma (\frac{E}{c} - m_0 c) f_{j-1}(r) = 0 \quad (3) \]
Equation 11 on page 69 gives us the same type of relations for the radial functions as equation 8 except that \( f(r) \) and \( g(r) \) are interchanged. We get:

\[
\left[ \frac{\partial}{\partial r} - \frac{j}{r} \right] i^+ \beta \sqrt{\frac{j}{2j+1}} f_j(r) - \alpha \left( \frac{\varepsilon}{\alpha} + m_0 c \right) g_j(r) = 0, \quad (1)
\]

\[
\left[ \frac{\partial}{\partial r} + \frac{j+2}{r} \right] \frac{i^+ \alpha}{\sqrt{\frac{j}{2j+1}}} f_j(r) + \frac{\partial}{\partial r} - \frac{j-1}{r} \frac{i^+}{\sqrt{\frac{j}{2j+1}}} f_j(r)
\]

\[-\beta \left( \frac{\varepsilon}{\alpha} + m_0 c \right) g_j(r) = 0, \quad (2)\]

\[
\left[ \frac{\partial}{\partial r} + \frac{j+1}{r} \right] i^+ \beta \sqrt{\frac{j+1}{2j+1}} f_j(r) - \n \left( \frac{\varepsilon}{\alpha} + m_0 c \right) g_j(r) = 0. \quad (3)
\]

From equation 1 on page 77 we have:

\[
f_{j+1}(r) = \left[ \frac{\partial}{\partial r} - \frac{j}{r} \right] i^+ \frac{\beta}{\left( \frac{\varepsilon}{\alpha} - m_0 c \right)} \frac{\sqrt{j}}{\sqrt{2j+1}} g_j(r).
\]

From equation 3 on page 77 we have:

\[
f_{j-1}(r) = \left[ \frac{\partial}{\partial r} + \frac{j+1}{r} \right] i^+ \frac{\beta}{\left( \frac{\varepsilon}{\alpha} - m_0 c \right)} \frac{\sqrt{j+1}}{\sqrt{2j+1}} g_j(r).
\]

Making these substitutions in equation 2 above we get:
\[
\begin{align*}
(\mathbf{i}^2)^2 & \frac{1}{\varepsilon_c - m_0 c} \beta \left( \frac{1}{2j+1} \right) \left( \frac{2}{\partial r} + \frac{j+2}{2 \partial r} \left( \frac{2}{\partial r} - \frac{1}{r} \right) g_j(r) \right) \\
+ (\mathbf{i}^2) & \frac{1}{\varepsilon_c - m_0 c} \beta \left( \frac{j+1}{2j+1} \right) \left( \frac{2}{\partial r} - \frac{r-1}{r} \left( \frac{2}{\partial r} + \frac{j+1}{r} \right) g_j(r) \right) \\
& = \beta \left( \frac{\varepsilon}{c} + m_0 c \right) g_j(r),
\end{align*}
\]

\[-\frac{k^2 \beta}{(\varepsilon_c - m_0 c)} \left\{ \left( \frac{j}{2j+1} \right) \left( \frac{2}{\partial r^2} + \frac{2}{r} \frac{2}{\partial r} - \frac{j(j+1)}{r^2} \right) + \left( \frac{j+1}{2j+1} \right) \right\} \left( \frac{2}{\partial r} + \frac{2}{r} \frac{2}{\partial r} - \frac{j(j+1)}{r^2} \right) g_j(r) = \beta \left( \frac{\varepsilon}{c} + m_0 c \right) g_j(r),\]

\[-\frac{k^2 \beta}{(\varepsilon_c - m_0 c)} \left[ \frac{2}{\partial r^2} + \frac{2}{r} \frac{2}{\partial r} - \frac{j(j+1)}{r^2} \right] g_j(r) = \beta \left( \frac{\varepsilon}{c} + m_0 c \right) g_j(r),\]

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{j(j+1)}{r^2} \right] g_j(r) = -\left( \frac{\varepsilon^2 m_0 c^4}{\hbar^2 c^2} \right) g_j(r).
\]
Let \( k^2 = \frac{\varepsilon^2 - m_0^2 c^4}{\hbar^2 c^2} \),

\[
\mathcal{P} = -i \frac{\mathbf{e}}{r},
\]

\[
\frac{d}{dr} = \frac{d}{d\rho} \frac{d}{dr} = i \frac{k}{\rho},
\]

\[
\frac{d^2}{dr^2} = \frac{d}{dr} \left( \frac{d}{dr} \right) = i \frac{k}{\rho} \frac{d}{dr} = \frac{k}{\rho} \frac{d}{dr} \left( \frac{d}{dr} \right) = k^2 \frac{d^2}{dr^2} - \frac{k^2}{\rho^2},
\]

\[
\rho = \frac{\varphi^2}{k^2},
\]

Thus we get:

\[
\left[ k^2 \frac{d^2}{d\rho^2} + \frac{2k^2}{\rho} \frac{d}{d\rho} - i (j+1) k^2 + k^2 \right] g_j(\varphi) = 0,
\]

\[
\left[ \frac{d}{d\rho^2} + \frac{2}{\rho} \frac{d}{d\rho} - i (j+1) - 1 \right] g_j(\varphi) = 0.
\]

A regular (finite at the origin) solution of this equation is the spherical Bessel function \( J_j(kr) \).

\[
g_j(r) = C J_j(\varphi) = C J_j(kr),
\]

\[
f_{J_{j1}}(r) = \frac{\mathbf{e}}{\varepsilon c^2 - m_0 c^2} \frac{B}{\sqrt{2j+1}} \left( \frac{d}{dr} - \frac{j}{r} \right) C J_j(kr)
\]

\[
= \left( \frac{\mathbf{e} c}{\varepsilon c^2 - m_0 c^2} \right) \frac{B}{\sqrt{2j+1}} \left( \frac{J_j(kr)}{r} - \frac{j}{r} J_j(kr) \right),
\]

where \( J_j'(kr) = \frac{d J_j(kr)}{dr} \).
\[ f_{j-1}(r) = \frac{(\frac{\hbar^2}{2m})^\frac{1}{2}}{\left(\frac{\hbar^2}{2m}\right)^{\frac{1}{2}}} \left(\frac{1}{r} \frac{d}{dr} \right)^j \frac{1}{r} \left[ J_j(kr) + \frac{j+1}{r} J_j(kr) \right]. \]

If we eliminate \( g_{j+1}(r) \) and \( g_{j-1}(r) \) from equation 2 on page 77, we get the radial equation:

\[ \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \left( \frac{j(j+1)}{r^2} \right) \right] f_j(r) = 0, \]

whose solution is:

\[ f_j(r) = C' J_j(kr). \]

Then

\[ g_{j+1}(r) = \frac{(\frac{\hbar^2}{2m})^\frac{1}{2}}{\left(\frac{\hbar^2}{2m}\right)^{\frac{1}{2}}} \frac{1}{r} \left(\frac{1}{r} \frac{d}{dr} \right)^{j+1} \frac{1}{r} \left[ J_j(kr) - \frac{j}{r} J_j(kr) \right], \]

\[ g_{j-1}(r) = \frac{(\frac{\hbar^2}{2m})^\frac{1}{2}}{\left(\frac{\hbar^2}{2m}\right)^{\frac{1}{2}}} \frac{1}{r} \left(\frac{1}{r} \frac{d}{dr} \right)^{j-1} \frac{1}{r} \left[ J_j(kr) + \frac{j+1}{r} J_j(kr) \right]. \]

We see that:
Thus, for a given value of the angular momentum, one can explicitly write the eigenfunction for the Hamiltonian of a free boson.

The boson equation could also be solved in the presence of a central potential by a procedure similar to that above. This is formally done in the next section.

The Vector Boson Equation in a Central Field

In the presence of a central field described by a potential \( V(r) \), the vector boson Hamiltonian is written as:

\[
H_B = C (\vec{\alpha} \cdot \vec{p}) + \beta m_0 c^2 + V(r).
\]

The boson equation will now be:

\[
\begin{bmatrix}
C (\vec{\alpha} \cdot \vec{p}) + \beta m_0 c^2 + V(r)
\end{bmatrix} \psi_m = \varepsilon \psi_m.
\]

\[
\begin{bmatrix}
C \left( \begin{array}{cc}
0 & \Sigma \\
\Sigma & 0
\end{array} \right) \cdot \vec{p} + m_0 c^2 (\begin{array}{cc}
0 & -\beta \\
\beta & 0
\end{array}) + V(r)
\end{bmatrix} \begin{bmatrix}
\psi_m^+ \\
\psi_m^-
\end{bmatrix} = \varepsilon \begin{bmatrix}
\psi_m^+ \\
\psi_m^-
\end{bmatrix}.
\]
\[ C (\vec{\Sigma} \cdot \vec{p}) \psi_{jm}^- + \left\{ m_o c^2 + V(r) \right\} \psi_{jm}^+ = \varepsilon \psi_{jm}^+ , \]
\[ C (\vec{\Sigma} \cdot \vec{p}) \psi_{jm}^+ + \left\{ -m_o c^2 + V(r) \right\} \psi_{jm}^- = \varepsilon \psi_{jm}^- , \]
\[ (\vec{\Sigma} \cdot \vec{p}) \psi_{jm}^- = \frac{1}{c} \left( \varepsilon - m_o c^2 - V(r) \right) \psi_{jm}^+ , \tag{1} \]
\[ (\vec{\Sigma} \cdot \vec{p}) \psi_{jm}^+ = \frac{1}{c} \left( \varepsilon + m_o c^2 - V(r) \right) \psi_{jm}^- . \tag{2} \]
These two equations could be solved to find the explicit forms of the radial functions of the eigenfunction as we did in the case of a free boson in the second section of Chapter \textit{VII}. 

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Properties of the "Dirac Matrices" for Spin 1

In the vector boson equation we have introduced a matrix

\[ \alpha' = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \]

which is analogous to the Dirac matrix:

\[ \alpha = \begin{pmatrix} 0 & \sigma^- \\ \sigma^- & 0 \end{pmatrix} \]

We shall now investigate whether the components of

satisfy the same anti-commutation relations as the com-

ponents of the Dirac matrix . The components of

can be explicitly written as follows:

\[ \alpha'_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \]

\[ \alpha'_x = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \], \quad \alpha'_y = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \]

\[ \alpha'_z = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \]
$$\alpha' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\alpha_1' = \begin{pmatrix}
0 & 0 & 0 & 0 & -\frac{i}{2} & 0 & -\frac{i}{2} \\
0 & 0 & 0 & \frac{i}{2} & 0 & -\frac{i}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{i}{2} & 0 & -\frac{i}{2} \\
0 & \frac{i}{2} & 0 & \frac{i}{2} & 0 & 0 & 0 \\
\frac{i}{2} & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{i}{2} & 0 & -\frac{i}{2} & 0 & 0 & 0 \\
\frac{i}{2} & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 \\
\frac{i}{2} & 0 & \frac{i}{2} & 0 & 0 & 0 & 0
\end{pmatrix}$$
\[ \alpha_2^i = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \]

\[ \beta = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix} \]
Anti-Commutation Relation for the $\alpha'$ Matrices

$$\alpha'_x \alpha'_y + \alpha'_y \alpha'_x = \begin{pmatrix} 0 & \Sigma_x & 0 \\ \Sigma_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Sigma_y & 0 \\ \Sigma_y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$= \begin{pmatrix} \Sigma_x \Sigma_y & 0 & 0 \\ 0 & \Sigma_x \Sigma_y & 0 \\ 0 & 0 & \Sigma_x \Sigma_y \Sigma_x \Sigma_y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Hence $\alpha'_x \alpha'_y + \alpha'_y \alpha'_x = 0$.

Similarly $\alpha'_y \alpha'_z + \alpha'_z \alpha'_y = \alpha'_z \alpha'_x + \alpha'_x \alpha'_z = 0$.

$$\alpha'_x \beta' + \beta' \alpha'_x = \begin{pmatrix} 0 & \Sigma_x & 0 \\ \Sigma_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Sigma_y & 0 \\ \Sigma_y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\Sigma_x \Sigma_y & 0 \\ \Sigma_x \Sigma_y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$
\[
\begin{pmatrix}
0 & \hat{\Pi}(x_2 - x_1) \\
\hat{\Pi}(x_2 - x_1) & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

Hence \( \alpha'_x \beta' + \beta' \alpha'_x = 0 \).

Similarly \( \alpha'_y \beta' + \beta' \alpha'_y = \alpha'_2 \beta' + \beta' \alpha'_2 = 0 \).

Analogous to the standard Dirac \( \gamma \)-matrices we define:

\( \gamma'_1 = -i \beta \alpha'_x = -i \begin{pmatrix} \hat{\Pi} & 0 \\ 0 & -\hat{\Pi} \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \)

\( = -i \begin{pmatrix} \hat{\Pi} x_2 \\ -x_1 \end{pmatrix} \)

\( \gamma'_2 = -i \beta \alpha'_y = -i \begin{pmatrix} 0 & \hat{\Pi} \\ -\hat{\Pi} & 0 \end{pmatrix} \begin{pmatrix} x_4 \\ x_3 \end{pmatrix} \)

\( \gamma'_3 = -i \beta \alpha'_3 = -i \begin{pmatrix} 0 & \hat{\Pi} \\ -\hat{\Pi} & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \)
The $\gamma$ matrices defined above have properties similar to the Dirac $\gamma$ matrices.

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \beta.$$

We can similarly show that $\gamma_2 = \gamma_3 = \gamma_4 = \beta$.

Hence $\gamma_i^2 = \beta$, $i = 1, 2, 3, 4$.

We have seen that $\alpha_i^\prime \alpha_j^\prime + \alpha_j^\prime \alpha_i^\prime = 0$, $i, j = (x, y, z)$.

$$(-i \beta^\prime)^2 \alpha_i^\prime \alpha_j^\prime + (-i \beta^\prime)^2 \alpha_j^\prime \alpha_i^\prime = 0,$$

$$(-i \beta^\prime)(-i \beta^\prime)\alpha_i^\prime \alpha_j^\prime + (-i \beta^\prime)(-i \beta^\prime)\alpha_j^\prime \alpha_i^\prime = 0,$$

Since $\alpha^\prime$s anti-commute with $\beta^\prime$ we get:

$$(-i \beta^\prime)(-1) \alpha_i^\prime (-i \beta^\prime) \alpha_j^\prime + (-i \beta^\prime)(-1) \alpha_j^\prime (-i \beta^\prime) \alpha_i^\prime = 0,$$

$$(-i \beta^\prime \alpha_i^\prime)(-i \beta^\prime \alpha_j^\prime) + (-i \beta^\prime \alpha_j^\prime)(-i \beta^\prime \alpha_i^\prime) = 0,$$

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 0.$$
The properties of the $\gamma$-matrices can thus be written as:

\[
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \Gamma_{\mu \nu} \delta_{\mu \nu}, \quad (\mu, \nu = 1, 2, 3, 4)
\]

\[
\gamma_\mu \gamma^\nu + \gamma^\nu \gamma_\mu = 0, \quad (\text{if } \mu \neq \nu)
\]

\[
\gamma_\mu \gamma^\mu = 2 . \quad (\text{if } \mu = \nu)
\]

It is also usual to define $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$.

\[
\gamma_5 = \begin{pmatrix}
0 - i \Sigma_\tau & 0 - i \Sigma_3 \\
(i \Sigma_\tau) 0 & (i \Sigma_3) 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\Sigma_\tau 0 & 0 - i \Sigma_3 \\
0 \Sigma_\tau 0 & i \Sigma_3 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & i (i \Sigma_3) \Sigma_\tau \\
(i i \Sigma_3) \Sigma_\tau & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & -\vec{\tau} \\
-\vec{\tau} & 0
\end{pmatrix}
\]

It is also possible to show that:

\[
\gamma_5 \gamma_\mu + \gamma_\mu \gamma_5 = 0,
\]

\[
\gamma_5^2 = \vec{\tau}.
\]
The Vector Boson Equation in Covariant Form

We can now write the boson equation in a covariant form using the $\gamma$-matrices. We have:

$$\mp \frac{\partial \psi}{\partial t} = \left[ C (\vec{\alpha} \cdot \vec{P}) + \beta m_0 c^2 \right] \psi$$

$$= \left[ \frac{\kappa}{c} (\vec{\alpha}' \cdot \vec{A}) + \beta m_0 c \right] \psi$$

$$= \left[ \frac{\kappa}{c} (\alpha'_x \frac{\partial}{\partial x} + \alpha'_y \frac{\partial}{\partial y} + \alpha'_z \frac{\partial}{\partial z}) + \beta m_0 c \right] \psi.$$ 

We multiply throughout by $\beta/i c$ to get:

$$\mp \frac{\gamma^\mu}{i c} \frac{\partial \psi}{\partial x^\mu} = \left[ \frac{\kappa}{c} (\alpha'_x \frac{\partial}{\partial x} + \alpha'_y \frac{\partial}{\partial y} + \alpha'_z \frac{\partial}{\partial z}) + \beta m_0 c \right] \psi.$$

Let $x = x_1$, $y = x_2$, $z = x_3$, $ic t = x_4$.

$$- \frac{\gamma^4}{c} \frac{\partial \psi}{\partial x^4} = \left[ \frac{\kappa}{c} (-i \beta \alpha'_x \frac{\partial}{\partial x_1} - i \beta \alpha'_y \frac{\partial}{\partial x_2} - i \beta \alpha'_z \frac{\partial}{\partial x_3} + \beta m_0 c \right] \psi,$$

$$\beta^2 = \alpha_4^2 = 1,$$

$$\left[ \frac{\kappa}{c} (-i \beta \alpha'_x \frac{\partial}{\partial x_1} - i \beta \alpha'_y \frac{\partial}{\partial x_2} - i \beta \alpha'_z \frac{\partial}{\partial x_3} + \beta \frac{\partial}{\partial x_4}) - i \mp m_0 c \right] \psi = 0$$

$$\left[ \frac{\kappa}{2} \left( \frac{\partial}{\partial x_1} + \frac{\alpha_2}{\partial x_2} + \frac{\alpha_3}{\partial x_3} + \frac{\alpha_4}{\partial x_4} \right) - i \mp m_0 c \right] \psi = 0$$

$$\left[ \frac{\kappa}{2} \frac{\partial}{\partial x^\mu} - i \mp m_0 c \right] \psi = 0$$

$$\left( \frac{\kappa}{\mu} \frac{\partial}{\partial x^\mu} + \frac{m_0 c^2}{c} \right) \psi = 0$$
Using Einstein's summation convention we can simply write the last equation as:

\[ \partial_{\mu} \frac{\partial \psi}{\partial x_\mu} + k \psi = 0 \quad , \mu = 1, 2, 3, 4. \]

where \( k = \frac{m_0 c}{\hbar} \) is the inverse of the Compton wave length of the boson.

It is also possible to show that in the presence of an external electromagnetic force, the boson equation has to be:

\[ \partial_{\mu} \left( \frac{\partial}{\partial x_\mu} - i \frac{e}{\hbar c} A_\mu \right) \psi + k \psi = 0, \]

where \( A_\mu \) is the electromagnetic vector potential.
IX. THE GAMMA MATRICES AND THE LORENTZ TRANSFORMATION

Consider two inertial frames of reference $\mathcal{O}$ and $\mathcal{O}'$. The coordinate transformation between the two equivalent systems is given by the Lorentz transformation:

$$\mathbf{x}'_\mu = a_{\mu \nu} x_\nu + b_\mu,$$

where the coefficients $a_{\mu \nu}$ satisfy an orthogonality condition $a_{\mu \nu} a_{\lambda \rho} = \delta_{\mu \lambda} \delta_{\nu \rho}$.

If we assume the $\gamma$ matrices remain unchanged in the primed system, the vector boson equation could be written as:

$$\partial_\mu \frac{\partial}{\partial x'_\mu} \psi'(x'_\mu, t') + k \psi'(x'_\mu, t') = 0$$

(1)

We also assume that the vector boson wave function in the primed system is related to the one in the unprimed by a linear relation of the type:

$$\psi'(x'_\mu, t') = S \psi(x_\mu, t)$$

(2)

where $S$ is an 8 x 8 matrix which depends only on the nature of the Lorentz transformation and is completely independent of the space-time coordinates. Now using $S$ and $\frac{\partial}{\partial x'_\mu} = a_{\mu \nu} \frac{\partial}{\partial x_\nu}$ we can write equation 1 as:

$$\partial_\mu a_{\mu \nu} \frac{\partial}{\partial x_\nu} S \psi(x_\mu, t) + k S \psi(x_\mu, t) = 0.$$
Multiplying by $S^{-1}$ from the left we get:

$$S^{-1} \gamma_{\mu} a_{\mu \nu} \frac{\partial}{\partial x_{\nu}} S \psi(x, t) + k \gamma^0 S \psi(x, t) = 0,$$

$$S^{-1} \gamma_{\mu} S a_{\mu \nu} \frac{\partial}{\partial x_{\nu}} \psi(x, t) + k \psi(x, t) = 0.$$  \hspace{1cm} (3)

We require (3) to be equivalent to the boson equation on page . This requirement is satisfied if we set:

$$S^{-1} \gamma_{\mu} S a_{\mu \nu} = \nu_{\nu},$$

or multiplying both sides by $a_{\lambda \nu}$ and summing over $\nu$ we get:

$$S^{-1} \gamma_{\mu} S \sum_{\nu} a_{\mu \nu} a_{\lambda \nu} = \sum_{\nu} a_{\lambda \nu} \nu_{\nu}.$$  

But

$$\sum_{\nu} a_{\mu \nu} a_{\lambda \nu} = \delta_{\mu \lambda}.$$  

Hence

$$S^{-1} \gamma_{\mu} S \nu = a_{\mu \nu} \nu_{\nu}. \hspace{1cm} (4)$$

We also require that $S$ satisfies the property:

$$S^{-1} = \nu_{4} S^{+} \nu_{4}, \hspace{1cm} S^{+} = \nu_{4} S^{-1} \nu_{4}. \hspace{1cm} (5)$$

If the boson equation is to be covariant under space inversion also, then one can construct the right matrix such that:

$$\psi'(x', t) = S_{p} \psi(x, t),$$

where $S_{p}$ is an 8 x 8 matrix and satisfies the relation:
\begin{align*}
S^\rho_\mu S^\rho_\nu &= -\delta^\rho_\mu, \\
S^\rho_\mu \delta_4^\nu S^\rho_\nu &= \delta_4^\nu.
\end{align*}

Now, using the boson wave function \( \Psi(x_\mu, t) \) and its adjoint \( \overline{\Psi}(x_\mu, t) \), we can study the properties of bilinear functions \( \overline{\Psi}(x_\mu, t) \Gamma \Psi(x_\mu, t) \) under proper Lorentz transformation and space inversion. \( \Gamma \) is a product of the \( \gamma \)-matrices. The adjoint wave function is defined by \( \overline{\Psi}(x_\mu, t) = \Psi^+(x_\mu, t) \gamma^4 \).

Case 1. \( \Gamma = 1 \)

\[
\overline{\Psi}'(x'_\mu, t') \Psi'(x'_\mu, t') = \overline{\Psi}'(x'_\mu, t') \overline{\Psi}(x_\mu, t)
\]

\[
\overline{\Psi}'(x'_\mu, t) = \Psi^+(x'_\mu, t') \gamma^4 \quad (\text{by eq. 8})
\]

\[
= \Psi^+(x_\mu, t) \gamma^4 S^+ \gamma^4 \quad (\text{by eq. 2})
\]

\[
= \Psi^+(x_\mu, t) \gamma^4 S^+ \gamma^4 \quad (\text{since})
\]

\[
= \Psi^+(x_\mu, t) \gamma^4 S^{-1} \quad (\text{by eq. 5})
\]

\[
= \overline{\Psi}(x_\mu, t) \gamma^4 S^{-1} \quad (\text{by eq. 8})
\]

Hence
\[
\overline{\Psi}(x'_\mu, t') \Psi'(x'_\mu, t') = \overline{\Psi}(x_\mu, t) S^{-1} S \Psi(x_\mu, t)
\]

\[
= \overline{\Psi}(x_\mu, t) \Psi(x_\mu, t).
\]

Hence \( \overline{\Psi}(x_\mu, t) \Psi(x_\mu, t) \) is invariant under proper
Lorentz transformation and space inversion.

Case 2  \( \Gamma = x u \).

\[
\overline{\psi}'(x'_\mu, t') \psi'(x'_\mu, t') = \overline{\psi}(x_u, t) s' s \psi(x_u, t)
\]

But \( s' s = a_{\mu \nu} s_{\nu} \). (by eq. 4)

Hence

\[
\overline{\psi}'(x_u, t) a_{\mu} \psi'(x'_\mu, t) = \overline{\psi}(x_u, t) a_{\mu \nu} s_{\nu} \psi(x_u, t)
\]

\[
= a_{\mu \nu} \overline{\psi}(x_u, t) s_{\nu} \psi(x_u, t).
\]

Under space inversion we get:

\[
\overline{\psi}'(x'_\mu, t) a_{\mu} \psi'(x'_\mu, t) = \overline{\psi}(x_u, t) s' s \psi(x_u, t)
\]

\[
= \overline{\psi}(x_u, t) a_{\mu} \psi(x_u, t)
\]

But

\[
\overline{\psi}'(x'_\mu, t') a_{\mu} \psi'(x'_\mu, t') = \overline{\psi}(x_u, t) s' s \psi(x_u, t)
\]

Thus \( \overline{\psi}'(x'_\mu, t') a_{\mu} \psi'(x'_\mu, t') \) transforms as a four component vector.

Using similar techniques one can derive the following summarized properties of bilinear functions under proper orthochronous Lorentz transformation and space inversion.

\[
\overline{\psi}(x_u, t) \psi(x_u, t)
\]

(1)

\[
\overline{\psi}(x_u, t) a_{\mu} \psi(x_u, t)
\]

(2)
\[
\begin{align*}
\overline{\psi(x_u, t)} & \partial_5 \chi_5 \psi(x_u, t) \\
\overline{\psi(x_u, t)} & \Lambda_{\mu \nu} \psi(x_u, t) \\
\overline{\psi(x_u, t)} & \partial_5 \psi(x_u, t)
\end{align*}
\]

where \( \Lambda_{\mu \nu} = \frac{e}{2} \left[ \gamma_\mu, \gamma_\nu \right] \).
The extension of the idea of the Dirac equation to particles of non-zero rest mass with higher spin has been successfully executed by constructing a generalized Pauli spin matrix $\Sigma (I)$ which reduces to the Pauli spin matrix $\hat{\sigma}$ for particles with spin $\frac{1}{2}$. A close study of the classical Maxwell equations, in particular Ampere's Law and Faraday's Law, leads to the construction of a vector matrix $\mathbf{M}$ which is related to the spin $\mathbf{S}$ of the electromagnetic field. Then, the Maxwell equations were expressed in terms of $\mathbf{S}$ and the matrix forms of the vector operators gradient, divergence and curl. Just by introducing two constant scalar fields $E_s$ and $B_s$ as fourth components of the electric and magnetic fields respectively, Maxwell's basic four equations could be written in a matrix form. One can then easily construct a $4 \times 4$ sigma matrix from $\mathbf{S}$, $\mathbf{D}$ and $\mathbf{G}$ where $\mathbf{D}$ and $\mathbf{G}$ are related to the matrix forms of divergence and gradient respectively. Though the vector field has only three components, the sigma matrix of the vector field is $4 \times 4$. The fourth component of sigma gives a divergence condition of the electromagnetic field.

The $4 \times 4$ sigma matrix of the vector field has been extended to any spin field using the two basic requirements that the Pauli spin matrices satisfy, i.e., $\Sigma (I) \Sigma (I) = \mathbf{I}$.
and \((\sigma) \Sigma_n(i) \Sigma_{n}(i) = - \Sigma_{n}(i) \Sigma_{n}(i) = \xi \Sigma_{n} \), where \((n, \ell, m)\) are cyclic permutations of \((x, y, \beta)\). Once a generalized Pauli spin matrix \(\Sigma(i)\) is obtained, Maxwell's equations can then be extended to any massless field with spin \(I\).

But, in this paper the emphasis was on extending the Dirac equation to particles with higher spin. This extension was easily achieved by comparing the generalized Maxwell equations with the Dirac equation. In fact, there are two ways to make the extension depending upon our choice of the representation of \(\beta\) in the Dirac equation. We can choose \(\beta = \begin{pmatrix} \gamma_0 & 0 \\ 0 & -\gamma_0 \end{pmatrix}\) or \(\beta = \begin{pmatrix} 0 & i \gamma_0 \\ -i \gamma_0 & 0 \end{pmatrix}\).

It turns out that only one of the first representations of \(\beta\) gives a more physically meaningful result than the other extension based on the second representation of \(\beta\). This distinction of the two ways of generalizing the Dirac equation is more clearly seen if the components \(E_s(I-I)\) and \(B_s(I-I)\) are set to zero. But, even if \(E_s(I-I)\) and \(B_s(I-I)\) are non-zero, still the first extension of the Dirac equation is seen to be more physically correct at least for the case of a vector boson than the second extension. Thus, only the first extension of the Dirac equation has been fully developed and applied to a specific example in this paper.

As in the case of the electromagnetic field, the massive field has its own scalar and vector potentials. What
types of equations the scalar and vector potentials of a
massive field should satisfy have been investigated in
general and the case of a vector boson in particular. A
solution of the equations is not given but one can easily
write the solutions for a special case, such as the vector
boson in terms of spherical harmonics and radial functions.
Once the scalar and vector potentials of a certain massive
field are explicitly determined, then the fields could
easily be determined.

To confirm the validity of the generalized Dirac
equation, one has to see if it reduces to Dirac's equation
for spin $\frac{1}{2}$ particles and also apply it to a specific ex-
ample so as to check whether it gives a meaningful result
or not. The generalized Dirac equation does automatically
reduce to Dirac's equation when $\tau = \frac{1}{2}$. As a special
application of the generalized equation, the vector boson
equation has been studied. It is very interesting to note
that the vector boson equation has two constants of motion:

1. $\mathcal{F} = \mathcal{L} + \frac{h}{2} S$
2. $\mathcal{F}' = \mathcal{L} + \frac{h}{2} \mathcal{L} \tau$.

Unlike the case of
the Dirac equation where $\frac{h}{2} \mathcal{F} = \mathcal{S}$, here $\frac{h}{2} \mathcal{L} \tau \neq \mathcal{S}$.
So, $\mathcal{F}$ and $\mathcal{F}'$ are completely different. Of course, $\mathcal{F}$
can be identified as the total angular momentum of the
system. But, what $\mathcal{F}'$ physically represents is not yet clear.
It may be possible to study relationships between the two
constants of motion so as to understand the significance of
$\mathcal{F}'$. 

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A free particle solution of the vector boson equation has been obtained. This solution could be applied in the study of scattering of vector bosons. A formal treatment of a vector boson in a central field has been briefly discussed. This could help in studying the behavior of a vector boson in a particular coulomb field.

In Chapter VII gamma matrices of the vector boson equation were developed and their properties studied. It is quite interesting that these gamma matrices have exactly the same properties as the Dirac gamma matrices. A modest attempt has also been made to show the relativistic invariance of the vector boson equation. It is also shown that as one can construct tensor quantities of different rank using a spinor, its adjoint and products of Dirac gamma matrices, we can also do the same thing using the wave function of the vector boson, its adjoint and products of the gamma matrices.

Having achieved the formal extension of the Dirac equation, one can now pursue the study of its application to some more specific examples such as spin $\frac{3}{2}$, etc., particles. Besides, a thorough investigation of the properties of the sigma matrices corresponding to higher spin particles may lead, as in the case of massless particles discussed by Dr. M. Soga and Ralph A. Mudgett\(^1\), to

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\(^1\)Ralph A. Mudgett and M. Soga, *A U(2) Theory of Radiation Fields with spin I* (to be published).
the concept of composite massive fields. That is, a massive field with a higher spin \( I \) may be composed of a spinor and an \( (I-\frac{1}{2}) \) field. This idea of composing fields is exciting since one is forced to ask the question: What is the basic (fundamental) field from which all other fields could be generalized? Since we cannot generate the spinor field with any other field, the spinor field has to be the one which is basic or fundamental. As we have composite fields, is it also possible to have a composite mass, i.e., can one express a certain particle (massive) with a higher spin as a composite of a basic (fundamental) particle? How do we create mass from a massless field?