Imbedding Cayley Graphs

Brian L. Garman

Western Michigan University

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IMBEDDING CAYLEY GRAPHS

by

Brian L. Garman

A Project Report
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Brian L. Garman
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In this project we determine the genus of certain Cayley graphs. In particular, the genus of the regular complete 4-partite graph $K_{n,n,n,n}$ of order $4n$ is equal to $(n-1)^2$ for $n \equiv 2 \pmod{4}$; this partially verifies G. Ringel's 1969 conjecture that $\gamma(K_{n,n,n,n}) = (n-1)^2$. 

IMBEDDING CAYLEY GRAPHS

Brian L. Garman, Sp.A.

Western Michigan University, 1974
I. INTRODUCTION

The theory of quotient graphs and quotient manifolds, introduced by Gustin [4], developed by Youngs [14], and unified by Jacques [5], is a beautiful and powerful theory for studying the genus parameter of graphs.

In 1969, G. Ringel [8] conjectured that the genus of the regular complete 4-partite graph $K_{n,n,n,n}$ of order $4n$ is equal to $(n-1)^2$. Using the theory of quotient graphs and quotient manifolds, we show Ringel's conjecture to be true for the cases $n \equiv 2 \pmod{4}$.

In Chapter II, we give basic definitions and terms as well as some known results relevant to general imbedding problems.

We review, in Chapter III, the theory of quotient graphs and quotient manifolds giving examples of previously known quotient structures.

Finally, in Chapter IV, we apply the theory of quotient graphs and quotient manifolds to yield many new genus formulas, the most important result being that the genus of $K_{n,n,n,n}$ equals $(n-1)^2$ for the cases $n \equiv 2 \pmod{4}$.
II. TOPOLOGICAL GRAPH THEORY

In this chapter, we present a few definitions and known results that we will need throughout our discussion. Other terms and definitions not given may be found in Behzad and Chartrand [1]. In addition, many definitions, especially topological terminology, may be found in White [11].

A graph $G$ consists of a finite non-empty set $V(G)$ of vertices together with a set $E(G)$ of unordered pairs of distinct vertices, called edges. The cardinality of the vertex set of $G$ is called the order of $G$ and is denoted $|V(G)|$. The cardinality of the edge set of $G$ is denoted $|E(G)|$. A graph with $p$ vertices and $q$ edges is called a $(p, q)$ graph. The union $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. (We assume $V(G_1) \cap V(G_2) = \emptyset$). We denote $G \cup G$ by $2G$ and $(n-1)G \cup G$ by $nG$, $n \geq 3$. The join $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{[u,v] \mid u \in V(G_1), v \in V(G_2)\}$. (Again we assume $V(G_1) \cap V(G_2) = \emptyset$).

Let $G = K_{m,m,\ldots,m} = K_n(m)$ be defined by $\overline{G} = nK_m$. $G$ is called the regular complete $n$-partite graph of order $nm$.

By a surface we mean a compact orientable 2-manifold. It is known that every surface $M$ is homeomorphic to a
sphere with \( \gamma \) \((\geq 0)\) handles. In this case, we write
\( M = S_{\gamma} \) and say that \( M \) has genus \( \gamma \). A graph is said
to be **imbedded** in a surface \( M \) if it is "drawn" in \( M \)
so that edges intersect only at their common vertices.
If a graph \( G \) is imbedded in a surface \( M \), we call the
components of \( M - G \) **regions** of the imbedding. We refer
to the edges and vertices contiguous with a region \( R \)
as the **boundary** of \( R \). We say that a region of an
imbedding of a graph \( G \) in a surface \( M \) is a **2-cell**
if it is homeomorphic to the open unit disk. If every
region of an imbedding is a 2-cell, we call the imbedding a **2-cell imbedding**. The **genus** \( \gamma(G) \) of a graph
\( G \) is the minimum genus among all surfaces in which \( G \)
can be imbedded; furthermore, an imbedding of \( G \) in a
surface \( M = S_{\gamma(G)} \) is said to be a **minimal imbedding**.
We also note that any imbedding in which every region
boundary is a triangle (3-cycle) is minimal.

In Chapter IV, we will use a procedure called
Edmonds' permutation technique (See Edmonds[2] and
Youngs [13]). This technique is an effective tool for
studying imbedding problems. The method gives a concise
algebraic description to each geometric 2-cell imbedding. We describe Edmonds' permutation technique as
follows.

Denote the vertex set of a connected graph \( G \) by
\( V(G) = \{1, 2, \ldots, n\} \). For each \( i \in V(G) \), let
\( V(i) = \{ k \in V(G) \mid [i,k] \in E(G) \} \). Let \( p_1 : V(i) \to V(i) \) be a cyclic permutation of length \( |V(i)| \). Then any 2-cell imbedding of \( G \) in a surface \( S_\gamma \) of fixed orientation determines the \( p_i \)'s uniquely; conversely, every choice of \( (p_1, p_2, \ldots, p_n) \) gives a 2-cell imbedding of \( G \) in a uniquely determined surface \( S_\gamma \) such that the adjacencies at \( v_i \) are given (in the order imposed by the fixed orientation on \( S_\gamma \)) by \( p_i \), \( 1 \leq i \leq n \). Furthermore, given \( (p_1, p_2, \ldots, p_n) \) there is an algorithm producing the determined imbedding. The algorithm is as follows: Let \( D^* = \{ (a,b) \mid [a,b] \in E(G) \} \) and define \( P^* : D^* \to D^* \) by \( P^*(a,b) = (b, p_d(a)) \). Then \( P^* \) is a permutation on the set \( D^* \) of directed edges of \( G \) (where each edge of \( G \) is associated with two oppositely-directed directed edges), and the orbits under \( P^* \) determine the (2-cell) regions of the corresponding imbedding.

We list here some known results pertinent to imbedding problems. Let \( r \) denote the number of regions of an imbedding of a graph \( G \) in a surface \( M \).

**Proposition 2.1.** (The Euler polyhedral formula) If \( G \) is a connected \((p,q)\) graph imbedded in the sphere \( S_0 \), then \( p - q + r = 2 \).

**Proposition 2.2.** (The generalized Euler polyhedral
formula) Let $G$ be a (connected) $(p,q)$ graph with a 2-cell imbedding in a surface $S^2$. Then $p - q + r = 2 - 2\gamma$.

**Proposition 2.3.** If $G$ is a connected $(p,q)$ graph, where $p \geq 3$, then $\gamma(G) \geq q/6 - p/2 + 1$.

**Proposition 2.4.** (Ringel and Youngs [9]) Let $K_n$ denote the complete graph of order $n$. Then $\gamma(K_n) = \lfloor (n-3)(n-4)/12 \rfloor$, for $n \geq 3$.

**Proposition 2.5.** (Ringel [7]) Let $K_{n,n}$ denote the regular complete bipartite graph of order $2n$. Then $\gamma(K_{n,n}) = \lfloor (n-2)^2/4 \rfloor$, for $n \geq 2$.

**Proposition 2.6.** (Ringel and Youngs [10]) Let $K_{n,n,n}$ denote the regular complete tripartite graph of order $3n$. Then $\gamma(K_{n,n,n}) = (n-1)(n-2)/2$.

It seems natural to study $\gamma(K_{n,n,n})$ next; in Chapter IV we address ourselves to this problem, as well as several related problems. The material of Chapter III will be necessary to this purpose.
III. QUOTIENT GRAPHS AND QUOTIENT MANIFOLDS

We discuss, in this chapter, the general theory of quotient graphs and quotient manifolds. The material that we present is based on work by Jacques [5].

The following insights into Jacques' work are due to White [See 11 and 12]. Given a finite group \( \Gamma \) with a set \( \Delta \) of generators for \( \Gamma \), the Cayley color graph \( C_{\Delta}(\Gamma) \) has vertex set \( \Gamma \), with \((g, g')\) a directed edge—labeled with generator \( \delta_i \)—if and only if \( g' = g\delta_i \). We assume that if \( \delta_i \in \Delta \), then \( \delta_i^{-1} \notin \Delta \) unless \( \delta_i \) has order 2. In this latter case, the two directed edges \((g, g\delta_i)\) and \((g\delta_i, g)\) are represented as a single undirected edge \([g, g\delta_i]\), labeled with \( \delta_i \). The graph obtained by deleting all labels (colors) and arrows (directions) from the edges of \( C_{\Delta}(\Gamma) \) is called the Cayley graph \( G_{\Delta}(\Gamma) \).

Consider a Cayley graph \( G_{\Delta}(\Gamma) \) 2-cell imbedded in a surface \( M \). We study the imbedding of the Cayley color graph \( C_{\Delta}(\Gamma) \) thus determined. Let \( \Delta^{-1} = \{ \delta_i^{-1} \mid \delta_i \in \Delta \} \) and form \( \Delta^* = \Delta \cup \Delta^{-1} \). The elements of \( \Delta^* \) are called currents. The imbedding is characterized (by Edmonds' technique) by giving, at each \( g \in \Gamma \), the cyclic permutation \( \sigma_g \) of \( g\Delta^* \) determined by the orientation imposed by \( M \). For example, see Figure 3.1.

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Let \( \Omega \) be a subgroup of \( \Gamma \) such that if \( \Omega g = \Omega g' \), then \( g^* = g'^* \), where \( g^* \) denotes the cyclic permutation of \( \Delta^* \) induced by the action of \( g \) on \( h^* \); such a subgroup always exists since we can take \( \Omega = \{ e \} \), where \( e \) is the identity of \( \Gamma \). It is to our advantage, however, to choose \( \Omega \) as large as possible. In the terminology of Jacques, \( \Gamma \) and \( \Omega \) determine a quotient constellation \( C' \) for the constellation \( C = (C_\Delta(\Gamma) \text{ in } M) \). The quotient constellation is an imbedding of the Schreier coset graph (See White [11]) for \( \Omega \) in \( \Gamma \), the imbedding being determined by the collection \( \{ g^* \} \), taken over any set \( \{ h \} \) of right coset representatives of \( \Omega \) in \( \Gamma \).

The reduced constellation \( (C')^* \) is the dual of the quotient constellation. This is a 2-cell imbedding of a pseudograph \( K \) (with each edge directed and labeled
with the current of its dual edge; see Figure 3.2) in a compact orientable 2-manifold, called by Youngs [14] the quotient graph and quotient manifold respectively, for $\mathcal{C}_\Delta(\Gamma)$ and $\Omega$. Youngs obtains this structure by a different process: he first takes the dual of $\mathcal{C}_\Delta(\Gamma)$ in $\mathcal{M}$ and then "mods out" regions with identically labeled boundaries in accordance with the subgroup $\Omega$. Jacques' approach is consistent with that of Gustin [4] and has the advantage of applying also to irregular imbeddings (i.e., $r \neq r_k$, for each $k$).

Define (after Jacques) a brin to be an ordered pair $(c, \delta^*)$ where $c$ is a vertex in $\mathcal{C}$, $\mathcal{C}'$, or $(\mathcal{C}')^*$ and $\delta^* \in \Delta^*$. The following two theorems characterize a quotient graph in its quotient manifold.

**Theorem 3.1.** A quotient graph in its quotient manifold, denoted $\mathcal{M}(\Gamma/\Omega)$, satisfies the following five properties:
1) each brin carries a current from $\Delta^*$;

2) two opposing brins $x = (c, \delta^*)$ and $x^{-1} = (c', \delta^{*-1})$ (See Figure 3.3)
carry inverse currents (if $x = x^{-1}$, the current must be of order 2; in this case, the brin appears as in Figure 3.4);

Figure 3.3

Figure 3.4
3) the regions are in one-to-one correspondence with the right cosets of \( \Omega \) in \( \Gamma \) (The index of \( \Omega \) in \( \Gamma \) is called the index of the imbedding.);

4) the currents appearing in a region boundary are in one-to-one correspondence with \( \Delta^* \);

5) if a brin \( x \) appears in the boundary of a region associated with \( \Omega g \) and its opposite brin \( x^{-1} \) in the boundary of a region associated with \( \Omega g' \), then the current by \( x \) is in the set \( g^{-1}\Omega g' \).

For imbedding problems, what is important is the converse:

**Theorem 3.2.** A quotient graph in its quotient manifold \( M(\Gamma/\Omega) \) for \( C_\Delta(\Gamma) \) and \( \Omega \) (i.e., a pseudograph 2-cell imbedded, with directed edges and regions labeled with elements of \( \Delta^* \) and right cosets of \( \Omega \) in \( \Gamma \), respectively) satisfying the five properties of Theorem 3.1 determines a 2-cell imbedding \( C \) of \( C_\Delta(\Gamma) \) in \( M \) such that \( (C')^* = M(\Gamma/\Omega) \).

A third theorem involving \( M(\Gamma/\Omega) \) gives us information about the number of regions of length \( n \) (for each \( n \geq 3 \)) in the imbedding of \( C_\Delta(\Gamma) \) in \( M \). Before giving
this theorem, the following definition is needed. Let \( \xi \) be a vertex of \( M(\Gamma/\Omega) \), with \( \pi \) the product of currents (from \( \Delta^* \)) directed away from \( \xi \), in the order given by the orientation. We call the order of \( \pi \) in \( \Gamma \) the valence of \( \xi \). If \( \Gamma \) is non-abelian, we note that \( \pi \) may not be uniquely determined; nevertheless, if \( \pi \) and \( \pi' \) are two products at vertex \( \xi \), then \( \pi \) and \( \pi' \) are conjugate elements of \( \Gamma \) and thus have the same order so that the valence of \( \xi \) is, indeed, well-defined. We now give Theorem 3.3.

**Theorem 3.3.** A vertex of degree \( k \) and valence \( v \) in \( M(\Gamma/\Omega) \) determines \( |\Omega|/v \) regions of length \( kv \) in the imbedding of \( C_\Delta(\Gamma) \) in \( M \).

In order to apply the above theory to the imbedding of a graph \( G \), we are faced with the task of first finding a group \( \Gamma \) and a generating set \( \Delta \) giving \( G = G_\Delta(\Gamma) \). (This, of course, may not be possible.) Second, we must find an imbedding of a graph (perhaps a pseudo-graph) \( K \) so that the currents (elements of \( \Delta^* \)) are assignable to the edges of \( K \) in such a way that \( K \) satisfies the five properties of Theorem 3.1. Knowing this, we are guaranteed a 2-cell imbedding of \( G \) by Theorem 3.2.

To determine the genus of graph \( G \) we must also show that the imbedding of \( G \) is minimal. Assuming \( G \)

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has 3-cycles, it suffices to show that every region in the imbedding of $G$ is a triangle.

We can easily deduce if this imbedding of $G$ is triangular from Theorem 3.3; namely, by ascertaining if $k v = 3$ for each vertex $v$. Note that if $k v = 3$, then either $k = 3$ and $v = 1$ or $k = 1$ and $v = 3$. In the case that $k = 3$ and $v = 1$, we say that $v$ satisfies Kirchoff's Current Law (KCL), that is, that the product of the currents directed away from $v$ is the identity element of $\Gamma$. Any cubic quotient structure for a graph $G = G_\Delta(\Gamma)$ satisfying the KCL at each vertex necessarily produces a minimal imbedding for $G$; this thus gives the genus of $G$.

Given a cubic quotient graph $K$ in its quotient manifold $S_\gamma$ with $\gamma \geq 1$, it is often convenient to simplify the presentation of $K$ by drawing it in the plane with the following device describing the local vertex permutations: a solid dot—representing a vertex—has its incident edges ordered clockwise; a hollow dot, counterclockwise.

We now give two examples illustrating the power and beauty of the theory of quotient graphs and quotient manifolds. The first example illustrates the most ideal quotient structure that we could hope for: an index one imbedding of a cubic graph satisfying the KCL at each vertex.
Figure 3.5 is a representation of $K_{3,3}$ (with currents assigned) in the plane giving $(C')^*$ for $G_{\Delta}(\Gamma)$ where $\Gamma = \Omega = \mathbb{Z}_{24}$ and $\Delta = \{1, 2, 3, 5, 6, 7, 9, 10, 11\}$. It should be noted that $S_2$ is actually the quotient manifold and that the quotient graph $K_{3,3}$ is drawn according to the aforementioned scheme.

![Figure 3.5](image)

The five properties of Theorem 3.1 can easily be verified; moreover, the KCL is satisfied at each vertex. Theorem 3.3 guarantees that the imbedding of $G_{\Delta}(\Gamma)$ is minimal; thus the genus of $G_{\Delta}(\Gamma)$ is determined. We show in Chapter IV that $G_{\Delta}(\Gamma) = K_{6,6,6,6}$. Using Euler's formula with $p = 24$, $q = 216$, and $r = 144$, we see that $\gamma(K_{6,6,6,6}) = 5^2 = (6-1)^2$, verifying Ringel's conjecture for $n = 6$.

As a second example, Figure 3.6 is a quotient graph actually drawn in its quotient manifold, the sphere.
(Compare with example one.) This quotient structure is an index two imbedding giving $G_\Delta(\mathfrak{T}) = K_{2,2,2}$ where $\mathfrak{T} = \mathbb{Z}_6$, $\Omega = \{0, 2, 4\}$, and $\Delta = \{1, 2\}$. (The reader should verify that the five properties of Theorem 3.1 hold.)

Figure 3.6

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IV. ON THE GENUS OF CAYLEY GRAPHS

In this chapter, we give an asymptotic genus formula for the graph $K_m(2) = K_{2,2,2,\ldots,2}$, the regular complete $m$-partite graph of order $2m$, where $m$ is a power of three; we show that many new genus formulas can be obtained from certain known quotient structures; and we prove the main result of this project: $\gamma(K_{n,n,n,n}) = (n-1)^2$ for $n \equiv 2 \pmod{4}$, where $K_{n,n,n,n}$ is the regular complete $4$-partite graph of order $4n$.

The graphs $K_m(2)$ are called the octahedral graphs, deriving their name from the bottom case $K_{2,2,2}$ which is the one-skeleton of the octahedron. The Euler formula shows that $\gamma(K_m(2)) \geq (m-1) \cdot (m-3)/3$, with equality possible only if $m \equiv 0$ or $1 \pmod{3}$. For $m = 1, 3, \text{or } 4$, equality is well-known. Gross and Alpert [3] have recently shown that equality holds for $m \equiv 4 \pmod{6}$; White [12] has shown that equality holds for $m = 6$.

Once we establish that Figure 4.1 (labeled properly) is a quotient structure for the graphs $K_{3^n(2)}$ for $n = 1, 2, \ldots$, we will show that $\gamma(K_m(2)) \sim (m-1) \cdot (m-3)/3$ for $m$ a power of $3$; this latter result will be stated in Theorem 4.1. The symbol "\sim" means "is asymptotic to" and we write $f(x) \sim g(x)$ for functions $f$ and $g$, if $\lim_{x \to \infty} f(x)/g(x) = 1$. To facilitate the proof of Theorem 15.
4.1, we first state and prove several properties about the graph in Figure 4.1.

Let $\Gamma = \mathbb{Z}_2 \times (\mathbb{Z}_3)^n$, the direct product of the cyclic group of order two with the elementary abelian group of order $3^n$, where $n = 1, 2, 3, \ldots$. Define

$\Delta = \{ \delta = (a, x_1, x_2, \ldots, x_n) | \delta \in \Gamma, a = 0 \text{ or } 1, \text{ the first non-zero } x_i = 1, \text{ but not all } x_i = 0 \}$. Then $\Delta^* = \Gamma - \{ (0,0,0,\ldots,0), (1,0,0,\ldots,0) \}$ so that a simple counting argument gives $|\Delta| = (2 \cdot 3^n - 2)/2 = 3^n - 1$.

First we show that $G_\Delta(\Gamma)$, the underlying graph for $C(\Gamma)$, is $K_{3n}$, where $K_{3n}$ denotes the regular complete $3^n$-partite graph $K_{2,2,\ldots,2}$ of order $2 \cdot 3^n$.

Since $V(G_\Delta(\Gamma)) = \Gamma$, $|V(G_\Delta(\Gamma))| = 2 \cdot 3^n = |V(K_{3n})|$.

Next we note that for each $n \geq 1$, $\delta = (1,0,0,0,\ldots,0)$ \( \not\in \Delta \) and $\delta^2 = (0,0,\ldots,0)$. Let $\overline{\Delta} = \{ \delta \}$; it is clear that $G_{\overline{\Delta}}(\Gamma) = 3^n \cdot K_2$. It now follows that

$G_\Delta(\Gamma) = \overline{G_{\overline{\Delta}}(\Gamma)} = \overline{3^n \cdot K_2} = K_{3n(2)}$.

Relative to the graph in Figure 4.1 (See Figure 4.2 for cases $n = 2$ and 3), it will be convenient to introduce the following terminology: each edge incident with a vertex of degree one will be called a spur. Spurs contained inside the circle will be called inner-spurs; spurs contained outside the circle will be called outer-spurs. Edges on the circle will be referred to as circle-edges.
Figure 4.1
Case $n = 2$:

![Diagram for Case $n = 2$]

Case $n = 3$:

![Diagram for Case $n = 3$]

Figure 4.2

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We define the graph $G$ in Figure 4.1 (See Figure 4.2 for special cases $n = 2$ and 3) to have order $p = |\Delta|$ and to have $|\Delta|/2$ inner-spurs alternately sequenced with $|\Delta|/2$ outer-spurs. Necessarily, the number of circle-edges equals $|\Delta|$. Both the inner-spurs and outer-spurs are directed with an arrow pointing away from the center of the circle. The two circle-edges adjacent to an inner-spur will both be directed with an arrow toward their common vertex. With each current $\delta = (z_1, z_2, \ldots, z_{n+1})$, we associate the unique ternary numeral $\delta' = z_1z_2\ldots z_{n+1}$; and instead of labeling a given directed edge of $G$ with current $\delta$, we label it with $\delta'$, the associated ternary numeral. Let $\Delta_0 = \{0x_1x_2\ldots x_n \mid (0,x_1^\prime,x_2^\prime\ldots x_n) \in \Delta\}$ and $\Delta_1 = \{1x_1\ldots x_n \mid 0x_1\ldots x_n \in \Delta_0\}$. Clearly, $|\Delta_0| = |\Delta_1| = |\Delta|/2$; furthermore, $\Delta_0 \cup \Delta_1 = \{ax_1\ldots x_n \mid (a, x_1, \ldots, x_n) \in \Delta\}$ and $\Delta_0 \cap \Delta_1 = \emptyset$. Pick any inner-spur and label it $00\ldots 01$ ($n$ zeros). Continue, in a counter-clockwise direction, labeling each successive inner-spur with the next largest ternary numeral from $\Delta_0$. (Note that each of the $(3^n-1)/2$ distinct elements of $\Delta_0$ is assigned to exactly one of the $(3^n-1)/2$ inner-spurs.) For each inner-spur assigned with ternary numeral $0x_1x_2\ldots x_n$, label the two unique adjacent circle-edges with the ternary numeral $1x_1x_2\ldots x_n$. (Note that each of the $3^n-1$ circle-edges has a uniquely determined ternary numeral assigned to it; furthermore, each of the
$\frac{3^n - 1}{2}$ distinct ternary numerals from $\Delta_1$ has been assigned to two uniquely determined circle-edges.) At every vertex of degree three incident with an outer-spur, label the outer-spur with the ternary numeral associated with the current that makes the KCL hold at this vertex. (Note that each such label necessarily has the form $0x_1x_2...x_n$, but the first non-zero $x_1$ need not be a one.)

Now we show the graph $G$ of Figure 4.1, labeled as described above, has the following four properties:

**Property A.** For $n \geq 2$, none of the $n$ ternary numerals of the form $0aa...abb..., where $a = 0$ and $b = 1$, the number of $a$'s is $k \geq 0$, the number of $b$'s is $m \geq 1$, and $k + m = n$, appears as a label of an outer-spur.

**Proof.** Translating back to currents, each outer-spur label arises from the solution of an equation of the form:

\[
\begin{align*}
* (0, 0, ..., 0) &= (1, x_1, x_2, ..., x_n) + \\
&+ (1, y_1, y_2, ..., y_n) + \\
&+ (0, z_1, z_2, ..., z_n),
\end{align*}
\]

where $lx_1...x_n$ and $ly_1...y_n$ are the labels of the adjacent circle-edges and $0z_1...z_n$ is the label assigned to the outer-spur. Thus we show that $0z_1...z_n$ cannot have the form $0aa...abb..., that is,
equation * has no acceptable solution with \((0, z_1, \ldots, z_n)\) = \((0, a, a, \ldots, b, b, \ldots, b)\). First, suppose \((0, z_1, \ldots, z_n)\) = \((0, 1, 1, \ldots, 1)\). Then \(0 \equiv x_i + y_i + 1 \pmod{3}\) for \(1 \leq i \leq n\). Since neither \(x_1 = 2\) nor \(y_i = 2\) (by the way we have labeled circle-edges), necessarily \(x_1 = y_1 = 1\). Thus \(11x_2 \ldots x_n\) and \(11y_2 \ldots y_n\) are consecutive ternary numerals, and without loss of generality we assume
\(11x_2 \ldots x_n + 1 = 11y_2 \ldots y_n\). Consider \(x_n\); if \(x_n = 0\), then \(y_n\) must satisfy both \(0 = 0 + y_n + 1\) and \(0 + 1 = y_n\), an impossibility; if \(x_n = 1\), then \(y_n\) must satisfy both \(0 = 1 + y_n + 1\) and \(1 + 1 = y_n\), again an impossibility; thus \(x_n = 2\) and \(y_n = 0\). Repeating the argument consecutively for \(x_{n-1}, x_{n-2}, \ldots, x_2\), we find that \(11x_1 x_2 \ldots x_n = 0122 \ldots 2\) and \(11y_1 y_2 \ldots y_n = 0100 \ldots 0\); but this again is impossible since \(012 \ldots 2 + 1 \neq 010 \ldots 0\).

Second, suppose \(0z_1 \ldots z_n\) \(\neq 011 \ldots 1\) but that \(0z_1 \ldots z_n\) has the specified form. Let \(z_1, \ldots, z_k\) be zero (necessarily, \(k > 0\)). Then \(0 \equiv x_1 + y_1 + 0 \pmod{3}\); and since (as in the first case) neither \(x_1 = 2\) nor \(y_1 = 2\), so necessarily \(x_1 = y_1 = 0\). Similarly, \(x_2 = y_2 = 0, \ldots, x_k = y_k = 0\). Now consider \(z_{k+1} = 1\) (\(z_{k+1} \neq 2\) by assumption). We see that \(0 \equiv x_{k+1} + y_{k+1} + 1 \pmod{3}\) implying that \(x_{k+1} = y_{k+1} = 1\) since again neither \(x_{k+1} = 2\) nor \(y_{k+1} = 2\). Thus \(10 \ldots 01x_{k+2} \ldots x_n\) and \(10 \ldots 01y_{k+2} \ldots y_n\) are consecutive ternary numerals,
and without loss of generality we, again, assume that
\[10 \ldots 01x_{k+2} \ldots x_n + 1 = 10 \ldots 0ly_{k+2} \ldots y_n.\]
Since each digit in \(lx_1 \ldots x_n\) preceding the digit \(x_{k+2}\) is
equal to the corresponding digit in \(ly_1 \ldots y_n\) and since
\(lx_1 \ldots x_n\) and \(ly_1 \ldots y_n\) are consecutive ternary
numerals, necessarily \(x_n \neq 2\) and \(x_{k+j} = y_{k+j}\) for
\(2 \leq j \leq n-k-1\). But \(0 \equiv x_{k+j} + y_{k+j} + 1 \pmod{3}\),
implying that \(x_{k+j} = y_{k+j} = 1\) for \(2 \leq j \leq n-k-1\).
Thus \(lx_1 \ldots x_n = 10 \ldots 011 \ldots lx_n\) and \(ly_1 \ldots y_n =
10 \ldots 011 \ldots ly_n\). If \(x_n = 0\) or \(1\), then \(y_n =
1\) or \(2\), respectively; this contradicts \(0 \equiv x_n + y_n +
1 \pmod{3}\). Hence Property A is established.

**Property B.** Among the labels assigned to outer-spurs,
there are exactly \(n\) ternary numerals of the form
\(0z_1z_2 \ldots z_n\) whose first non-zero \(z_i = 2\).

**Proof.** As we remarked in the proof of Property A, each
outer-spur label arises from the solution to an equation
of the form given by equation *.* Let \(lx_1 \ldots x_n\) and
\(ly_1 \ldots y_n\), as before, represent the adjacent circle-
edges and \(0z_1 \ldots z_n\) represent the outer-spur. We
consider two cases:

a) \(lx_1 \ldots x_n = 1122 \ldots 2\) and \(ly_1 \ldots y_n = 10 \ldots 01\)
(or vice versa)
b) \(lx_1 \ldots x_n\) and \(ly_1 \ldots y_n\) are not the labels
in part a.
In case a, 0z₁...zn = 0211...10, accounting for one label of the specified type. In case b, we may assume that lx₁...xn < ly₁...yn. Let t = min {i,j}, where xᵢ is the first non-zero x and yⱼ is the first non-zero y (1 ≤ t ≤ n - 1). As we noted before, neither xᵢ = 2 nor yⱼ = 2; therefore, xᵢ = yⱼ = 1, or xᵢ = 0 and yⱼ = 1. Only xᵢ = 0 and yⱼ = 1 give zᵢ = 2. Moreover, each of the n - 1 labels ly₁...yn with yᵢ = 1 and zero otherwise, for 1 ≤ i ≤ n - 1, gives rise to an outer-spur label of the specified type. It is easy to see that these n - 1 labels are 00...022, 0...0221, 0...02211, ..., 02211...1. Since these are distinct and all are different from the label 0211...10 arising in case a, there are exactly n labels of the form 0z₁...zn whose first non-zero zᵢ = 2.

Property C. No two outer-spurs bear the same label.

Proof. By Property B and its proof, we need only consider labels 0z₁...zn whose first non-zero zᵢ = 1. Suppose two distinct pairs—lx₁...xn and ly₁...yn, lα₁...αᵣ and lβ₁...βᵣ—give rise to the same label 0z₁...zn. We may assume lx₁...xn < ly₁...yn and lα₁...αᵣ < lβ₁...βᵣ. In fact, lx₁...xn + 1 = ly₁...yn and lα₁...αᵣ + 1 = lβ₁...βᵣ (base 3). Let zᵢ be the first non-zero zᵢ; zᵢ = 1 by assumption. For 1 ≤ i < t, zᵢ = 0 and, necessarily,
\[ x_i = y_i = \alpha_i = \beta_i = 0. \] Now \( z_t = 1 \) implies

\[ x_t = \alpha_t = y_t = \beta_t = 1. \] Surely \( x_n + 1 \equiv y_n \pmod{3} \)

and \( \alpha_n + 1 \equiv \beta_n \pmod{3} \); furthermore, \( x_i + y_i + z_i = \alpha_i + \beta_i + z_i \equiv 0 \pmod{3} \) for \( 1 \leq i \leq n \). We now consider the three cases for \( z_n \): a) \( z_n = 0 \); b) \( z_n = 2 \); c) \( z_n = 1 \). If \( z_n = 0 \), then \( x_n = \alpha_n = 1 \) and 

\[ y_n = \beta_n = 2; \] moreover, both \( x_i = y_i \) and \( \alpha_i = \beta_i \) for \( t \leq i \leq n-1 \). Since \( x_i + y_i + z_i = \alpha_i + \beta_i + z_i \pmod{3} \), it follows that \( 2\alpha_i = 2x_i \pmod{3} \), and so \( \alpha_i = x_i \) for \( t \leq i \leq n-1 \). Similarly, \( \beta_i = y_i \) for \( t \leq i \leq n-1 \). Thus \( l \alpha_1 \ldots \alpha_n = l \alpha_1 \ldots \alpha_n \) and \( l \beta_1 \ldots \beta_n = 1 \alpha_1 \ldots \beta_n \), a contradiction. If \( z_n = 2 \), then \( x_n = \alpha_n = 0 \) and \( \beta_n = y_n = 1 \); thus, \( x_i = y_i \) and \( \alpha_i = \beta_i \) for \( t \leq i \leq n-1 \). This leads to the same contradiction that arose for the case \( z_n = 0 \). Finally, \( i ) z_n = 1 \), then \( x_n = \alpha_n = 2 \) and \( y_n = \beta_n = 0 \).

We now apply the above analysis to \( z_{n-1} \). If \( z_{n-1} = 0 \) or \( 2 \), we are done; if not, \( x_{n-1} = \alpha_{n-1} = 2 \) and \( y_{n-1} = \beta_{n-1} = 0 \), and we proceed to \( z_{n-2} \). Since \( x_t = y_t = 1 \), this procedure must end, and we again reach a contradiction. Property C follows.

**Property D.** Every one of the \( n \) labels described in Property B, except one, gives rise to the inverse of a current already labeled on an outer-spur.

**Proof.** The inverse currents of the prescribed type are
(0, ..., 0,1,1), (0, ..., 1,1,2), (0, ..., 1,1,2,2), ..., (0,1,1,2, ..., 2), and (0,1,2, ..., 2,0). By Property A 00 ... 011 does not appear as a label. Now, by Properties A and C, each of the labels 00 ... 112, 00 ... 01122, ..., 0112 ... 2, and 0122 ... 20 does appear as a label.

We are now in a position to use Properties A, B, C, and D to help prove our first theorem.

**Theorem 4.1.** $\gamma(K_m(2)) \sim (m-1) \cdot (m-3)/3$ where $m = 3^n$ and $n = 1, 2, 3, ...$

**Proof.** We prove this theorem by relabeling the graph in Figure 4.1 and showing that the resulting structure is an index-two imbedding of a quotient graph in its quotient manifold, the sphere. This will give an imbedding of $K_{3^n(2)}$ which almost certainly will be non-minimal; however, enough of the regions in the imbedding will be triangular to guarantee the desired asymptotic result. To this end, we proceed.

For each of the $n$ outer-spurs (See Figure 4.1) labeled with $0z_1 ... z_n$ where the first non-zero $z_i = 2$, except the outer-spur labeled with $0 ... 022$, we arbitrarily relabel the spur with an element chosen from $A = \{ 0 ... 01 \} \cup \{ 0 ... 0111, 0 ... 01111, ..., 011 ... 1 \}$ so that each element of $A$ is assigned exactly once.
We now verify that the relabeled graph (lower labels) in Figure 4.1 with \( \Gamma = \mathbb{Z}_2 \times (\mathbb{Z}_3)^n \) , \( \Delta = \{ (a, x_1, \ldots, x_n) \in \Gamma \mid a = 0 \text{ or } l \text{, the first non-zero } x_i = 1 \text{, but not all } x_i = 0 \} \) , and \( \Omega = \{ (0, x_1, \ldots, x_n) \in \Gamma \mid \text{the first non-zero } x_i = 1 \text{, but not all } x_i = 0 \} \) satisfy the five basic properties of a quotient graph in its quotient manifold. Property one is readily apparent since \( \Delta^* = \Delta \cup \Delta^{-1} = \Gamma - \{ (0, \ldots, 0), (1, 0, \ldots, 0) \} \). By convention, an edge carrying current \( \delta \) going with the arrow carries current \( \delta^{-1} \) going against the arrow; this makes property two immediate. The order of \( \Omega \) is easily seen to be \( 3^n \), one half of the order of \( \Gamma \); thus the index of \( \Omega \) in \( \Gamma \) is two. Since Figure 4.1 gives the actual imbedding of \( M(\Gamma/\Omega) \) (in the sphere), the two regions are surely in one-to-one correspondence with the right cosets of \( \Omega \) in \( \Gamma \); property three follows. Recall that \( \Delta_0 = \{ 0x_1x_2\ldots x_n \mid (0, x_1, \ldots, x_n) \in \Delta \} \) and that \( |\Delta_0| = |\Delta|/2 \). As noted before, there is a one-to-one correspondence with the inner-spurs and the elements of \( \Delta_0 \). Appealing to Properties A, C, and D (and the proof of D), and our relabeling scheme for outer-spurs, it is clear that the outer-spur labels are also in a one-to-one correspondence with the elements of \( \Delta_0 \). As we noted before, each element of \( \Delta_1 (\Delta_1 = \{ 1x_1 \ldots x_n \mid \}) \)
0x_1 \ldots x_n \in \Delta_0 \} \) appears exactly twice as a circle-edge label: once with the circle-edge directed clockwise; once, counterclockwise. Since in tracing out either region boundary we traverse each of the \( 3^n - 1 \) circle-edges exactly once, it is clear that we have accounted for \( 3^n - 1 \) distinct elements of \( \Delta^* \), namely, the currents and the inverses associated with the elements of \( \Delta_l \).

For the inner region, the other \( 3^n - 1 \) elements of \( \Delta^* \) are accounted for by traversing each inner-spur in both directions; for the outer region, we traverse the outer-spurs. It should now be clear that the currents of \( \Delta^* \) are in a one-to-one correspondence with the brins of each of the two region boundaries; property four is established.

To verify property five, we consider two cases. For notational convenience, let \( e = (0, 0, \ldots, 0) \) and \( \delta = (1, 0, \ldots, 0) \). First, suppose \( x \) is a brin associated with any one of the spurs. Then both \( x \) and \( x^{-1} \) appear in the boundary of the region associated with \( \Omega e \), or both \( x \) and \( x^{-1} \) appear in the boundary of the region associated with \( \Omega \delta \). The current carried by \( x \) has the form \( (0, x_1, x_2, \ldots, x_n) \); since both \( e^{-1} \Omega e = \Omega \) and \( \delta^{-1} \Omega \delta = \Omega \), property five follows for this case. Second, suppose \( x \) is a brin associated with any one of the circle-edges. Therefore \( x \) appears in the boundary of one of the two regions while \( x^{-1} \) appears in the boundary of the other region. The current carried by \( x \) (or \( x^{-1} \))
has the form \((1, x_1, \ldots, x_n)\); since both \(e^{-1}\Omega \delta = \Omega \delta\) and \(\delta^{-1}\Omega e = \Omega \delta\), property five is satisfied in this case. Hence in both cases the fifth property holds. This proves that \(M(\Gamma/\Omega)\) is, indeed, a quotient graph in its quotient manifold; however, we now show that it is not the case that the KCL is satisfied at each vertex of degree three. This probably means that the guaranteed imbedding of \(G_\Delta(\Gamma) = K_{3^n(2)}\) is not minimal.

It is clear from the way \(M(\Gamma/\Omega)\) is now labeled that every vertex of degree three satisfies the KCL, except those \(n - 1\) vertices incident with the relabeled outer-sprays. The valence of each of these \(n - 1\) vertices is easily seen to be three. Moreover, each vertex of degree one has valence three also. We now apply Theorem 3.3. Each of the \((3^n - 1) - (n - 1) = 3^n - n\) vertices for which the KCL holds gives rise to \(|\Omega|/\nu = 3^n/1 = 3^n\) triangular regions in the imbedding of \(G_\Delta(\Gamma) = K_{3^n(2)}\); each of the \(3^n - 1\) vertices of degree one gives rise to \(|\Omega|/\nu = 3^n/3 = 3^{n-1}\) more triangular regions in this imbedding, and each of the \(n - 1\) vertices of degree three for which the KCL does not hold gives rise to \(|\Omega|/\nu = 3^n/3 = 3^{n-1}\) regions of length \(\nu_\nu = 3 \cdot 3 = 9\). Hence there are \((3^n - n) \cdot 3^n + (3^n - 1) \cdot 3^{n-1} = 3^{2n} + 3^{2n-1} - n \cdot 3^n - 3^{n-1}\) 3-sided regions and \((n - 1) \cdot 3^{n-1}\) 9-sided regions. Appealing to the Euler formula with \(\rho = 2 \cdot 3^n\),
q = 2 \cdot 3^n (3^n - 1), and \( r = 3^{2n} + 3^{2n-1} - n \cdot 3^n + (n-2) \cdot 3^{n-1} \),
we have

\[
\gamma = 1 + \frac{1}{2} (q - p - r)
\]

\[
= 1 + \frac{1}{2} (2 \cdot 3^n \cdot (3^n - 1) - 2 \cdot 3^n - (3^{2n} + 3^{2n-1} - n \cdot 3^n + (n-2) \cdot 3^{n-1}))
\]

\[
= 1 + 3^{2n} - 3^n - 3^n - \frac{1}{2} [3^{2n-1} (3+1) - 3^{n-1} (3n - (n-2))]
\]

\[
= 1 + 3^{2n} - 2 \cdot 3^{2n-1} - 2 \cdot 3^n + 3^{n-1} (n+1)
\]

\[
= 1 + 3^{2n-1} + (n-5) \cdot 3^{n-1}
\]

\[
= \frac{3^{2n} + (n-5) \cdot 3^n + 3}{3}
\]

We have seen before that \( \gamma(K_m(2)) \geq \frac{(m-1) \cdot (m-3)}{3} \).

Substituting \( m = 3^n \), we get \( \gamma(K_{3^n(2)}) \geq \frac{(3^n-1) \cdot (3^n-3)}{3} \).

\[
= \frac{3^{2n} - 4 \cdot 3^n + 3}{3}.
\]

Thus

\[
\frac{3^{2n} - 4 \cdot 3^n + 3}{3} \leq \gamma(K_{3^n(2)}) \leq \frac{3^{2n} + (n-5) \cdot 3^n + 3}{3}.
\]

Dividing by \( \frac{3^{2n} - 4 \cdot 3^n + 3}{3} \) gives

\[
1 \leq \frac{\gamma(K_{3^n(2)})}{\frac{3^{2n} - 4 \cdot 3^n + 3}{3}} \leq \frac{1 + \frac{n-5}{3^n} + \frac{1}{3^{2n-1}}}{1 - \frac{4}{3^n} + \frac{1}{3^{2n-1}}}.
\]
Therefore

\[ \lim_{n \to \infty} \frac{\gamma(K \mathbb{Z}/3^n(2))}{\mathbb{Z}/3^{2n-4} \cdot 3^n + 3} = \lim_{n \to \infty} \frac{\gamma(K \mathbb{Z}/3^n(2))}{(3^n-1) \cdot (3^n-3)} = 1. \]

This completes the proof.

We now examine certain quotient structures that possess a special KCL condition. Let \( \Gamma = \mathbb{Z}_n \) be the cyclic group of order \( n \), let \( \Omega_1 = \Gamma \), and let \( \Delta \) be a set of generators for \( \mathbb{Z}_n \). Furthermore, let \( M(\mathbb{Z}_n / \Omega_1) \) be a cubic quotient structure for the graph \( G_\Delta(\mathbb{Z}_n) \). It follows that if the KCL holds at each vertex of \( M(\mathbb{Z}_n / \Omega_1) \), then each region in the imbedding of \( G_\Delta(\Gamma) \) is triangular. This implies that the imbedding is minimal and yields the genus of \( G_\Delta(\mathbb{Z}_n) \). We said, in Chapter III, that KCL holds at a vertex \( v \) if the sum (product) of currents (directed away) at \( v \) equals the identity element of the group. For \( \Gamma = \mathbb{Z}_n \), this condition can be satisfied in two ways: the sum is identically zero or the sum is a non-zero multiple of \( n \). If the sum at every vertex of \( M(\mathbb{Z}_n / \Omega_1) \) is 0 (not a non-zero multiple of \( n \)), the KCL does not depend on the group \( \mathbb{Z}_n \). Thus, in this case, for every cyclic group \( \mathbb{Z}_p \) with \( \Omega = \mathbb{Z}_p \), where \( p > 2 \cdot \max \{ \delta \} \), then \( M(\mathbb{Z}_n / \Omega_1) \) is also a quotient structure for the graph \( G_\Delta(\mathbb{Z}_p) \). Hence we obtain \( \gamma(G_\Delta(\mathbb{Z}_p)) \).
We give three quotient structures possessing the properties as described in the preceding paragraph. Consequently, each such structure yields a genus formula for an infinite class of graphs. The first two structures were previously known (See the given references). That the last structure is a quotient structure is proved in Theorem 4.3 of this project. In all three examples, that the KCL at each vertex is identically zero is easily verified.

a) Let $\Gamma = \mathbb{Z}_n = \Omega$ and $\Delta = \{1, 2, 3\}$ for $n > 6$.

\[
M(\Gamma/\Omega) : \quad 3 \quad 2 \quad 1
\]

Figure 4.3

This index-one imbedding appears in Youngs [15] and gives $\gamma(G_\Delta(\mathbb{Z}_n)) = 1$ for all $n > 6$.

b) Let $\Gamma = \mathbb{Z}_n = \Omega$ and $\Delta = \{1, 2, 3, \ldots, (6s+3)\}$, where $n > 12s + 6$ and $s = 0, 1, 2, \ldots$.

$M(\Gamma/\Omega)$ was discovered by Youngs [15]. This index-one imbedding gives $\gamma(G_\Delta(\mathbb{Z}_n)) = 1 + n \cdot s$.
for all $n > 12s + 6$ and $s = 0, 1, 2, \ldots$.

c) Let $\Gamma = \mathbb{Z}_n = \Omega$ and $\Delta = \{1, 2, 3, \ldots, 8s+4\} \setminus \{4, 8, \ldots, 8s+4\}$, where $n > 16s + 6$ and $s = 0, 1, 2, \ldots$. $M(\Gamma / \Omega)$ is given in Figure 4.4 of this project (page 33). $\gamma(G^*_\Delta(\mathbb{Z}_n)) = 1 + ns$ for all $n > 16s + 6$ and $s = 0, 1, 2, \ldots$ for this index-one embedding.

We establish the major theorem of this project by showing that Figure 4.4 is a quotient structure. For the graph in Figure 4.4, we understand that a solid dot has its incident edges ordered clockwise; a hollow dot, counterclockwise, as previously formulated in Chapter III.

We now define the graph $G_n$ in Figure 4.4 and describe the assignment of the labels (currents). Let $\Gamma = \mathbb{Z}_{4n} = \Omega = \{0, 1, 2, \ldots, 4n-1\}$ and $\Delta = \{1, 2, 3, 4, \ldots, 8s+4\} \setminus \{4, 8, 12, \ldots, 8s+4\}$, where $n = 4s + 2$, and $s = 0, 1, 2, \ldots$. The order of $G_n$ is $n = 4s + 2$, and the cardinality of the edge-set of $G$ is $3n/2 = 6s + 3$. The vertices of $G_n$ may be pictured as equally spaced around the circumference of a circle. Pick any vertex and represent it with a hollow dot; represent all other vertices with a solid dot. Starting with the hollow dot, order the vertices consecutively, in a clockwise direction, as $1\text{st}$ through $(4s+2)\text{nd}$. Every two diametrically opposite vertices are adjacent. (Recall
Figure 4.4

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that $G_n$ is represented schematically in the plane; the
actual imbedding of $G_n$ is on a surface of genus $1 + s$
—we shall prove this shortly.) For the following discus-
sion it will be convenient to make two definitions: an
edge joining two diametrically opposite vertices will be
called a diameter-edge; an edge joining two consecutive
vertices on the circle will be referred to as a circle-
edge. Next, we define two functions: $D$ and $C$, where $D(K)$
gives the (undirected) label of the diameter-edge incident
with the kth vertex, and $C(k)$ and $C(k+1)$ give the
labels of the two circle-edges incident with the kth
vertex. (Note that the 1st circle-edge is immediately
to the left of the 1st vertex.) The two functions are
described by the following recipes:

$$D(k) = \begin{cases} 
(8s+2) - 8(k-1) & \text{for } 1 \leq k \leq (s+1) \\
6 + 8[k-(s+2)] & \text{for } (s+2) \leq k \leq (2s+1) \\
(8s+2) - 8[k-(2s+2)] & \text{for } (2s+2) \leq k \leq (3s+2) \\
6 + 8[k-(3s+3)] & \text{for } (3s+3) \leq k \leq (4s+2) 
\end{cases}$$

and

$$C(k) = \begin{cases} 
1 + 4(k-1) & \text{for } 1 \leq k \leq (2s+1), \text{ } k \text{ odd} \\
(8s+3) - 4(k-2) & \text{for } 2 \leq k \leq (2s+2), \text{ } k \text{ even} \\
(8s-1) - 4[k-(2s+3)] & \text{for } (2s+3) \leq k \leq (4s+1), \text{ } k \text{ odd} \\
5 + 4[k-(2s+4)] & \text{for } (2s+4) \leq k \leq (4s+2), \text{ } k \text{ even} 
\end{cases}$$

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The diameter-edges labeled with integers congruent to $2 \pmod{8}$ have their arrows directed away from the center of the circle if $k$ is odd, toward the center if $k$ is even. Those diameter-edges labeled with integers congruent to $6 \pmod{8}$ have the arrow directed toward the center if $k$ is odd, away from the center if $k$ is even. The $k$th circle-edge has its arrow pointing counterclockwise for $1 \leq k \leq (2s+2)$ and clockwise for $(2s+3) \leq k \leq (4s+2)$. Appealing to the equivalence classes $\pmod{8}$, it is easy to see that the function $C$ has exactly $4s+2$ distinct values—one for each of the $4s+2$ circle-edges. Similarly, the function $D$ has $4s+2$ values, but only $2s+1$ are distinct since for each $k$, $1 \leq k \leq (2s+1)$, the $k$th and $(k+2s+1)$st vertices (diametrically opposite vertices) are joined by a common diameter-edge. Thus, each of the $2s+1$ diameter-edges has exactly one of the $2s+1$ distinct values of $D$ assigned to it. Moreover, (again appealing to the equivalence classes $\pmod{8}$) no values of $C$ and $D$ are equal so that $\Delta$ is composed precisely of the $6s+3$ distinct values of $C$ and $D$.

We have stated that Figure 4.4 represents an imbedding of $G_n$ on a surface of genus $1+s$. This imbedding, in fact, is an index-one imbedding. To prove this, our plan of attack will be to give a quotient structure that will yield an imbedding of $G_n$ with three regions and to apply
a lemma of Ringeisen [6] that will reduce the 3-region imbedding to a 1-region imbedding of $G$. As a consequence of this 3-region imbedding of $G_n$, we obtain the genus of $G_n + \overline{K}_3$ (the join of $G_n$ and $\overline{K}_3$); we will state this as a theorem. First, however, we state Ringeisen's lemma.

**Lemma.** (The Edge-Adding Technique) Let $G$ be a connected graph and $v$ and $w$ denote non-adjacent vertices of $G$. Let $T$ be a 2-cell imbedding of $G$ which has vertex $v$ on the boundary of region $F_v$ and vertex $w$ on the boundary of region $F_w$. Let $G'$ be the graph $G$ with edge $[v, w]$ added. Then if $F_v \neq F_w$, $G'$ has a 2-cell imbedding with one less face (region) than $T$.

Second, let $\Gamma_1 = \mathbb{Z}_n = \Omega_1 = \{0, 1, 2, \ldots, n-1\}$ and $\Delta_1 = \{1, n/2\}$, where $n = 4s + 2$ and $s = 0, 1, 2, \ldots$. $G_{\Delta_1}(\Gamma_1)$ is easily seen to be $G_n$; moreover, $M(\Gamma_1 / \Omega_1)$ in Figure 4.5 is readily seen to be a quotient graph imbedded in its quotient manifold, the sphere.

![Figure 4.5](image)

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\(M(\Gamma_1/\Omega_1)\) gives an imbedding of \(G_n = G_{\Delta_1}(\Gamma_1)\) with \(r_n = 3\). To show this, we apply Theorem 3.3. It is clear that the valence of the vertex of degree 1 is \(n\), so this vertex contributes \(|\Omega|/\nu = n/n = 1\) region of length \(k_\nu = 1\cdot n = n\) to the imbedding \(G_{\Delta_1}(\Gamma_1)\).

The sum of the currents directed away from the vertex of degree two is \(1 + n/2\); we give a number theoretic argument to show that the order of this element is \(n/2\) and, hence, the valence at this vertex is \(n/2\). Once this is established, the vertex of degree two will account for \(|\Omega|/\nu = n/(n/2) = 2\) other regions of length \(k_\nu = 2\cdot (n/2) = n\); hence \(r = r_n = 3\). Since \(n = 4s + 2\), \(1 + (n/2) = 2s + 2\). From number theory, \([2s + 2, 4s + 2] = (2s + 2)\cdot (4s + 2)/(2s + 2, 4s + 2)\), where \([2s + 2, 4s + 2]\) and \((2s + 2, 4s + 2)\) denote the least common multiple and greatest common divisor of \(2s + 2\) and \(4s + 2\), respectively. The Euclidean algorithm shows that \((2s + 2, 4s + 2) = 2\); thus \([2s + 2, 4s + 2] = (2s + 2)\cdot (2s + 1)\). It follows that \((n/2) + 1 = 2s + 2\) has order \(n/2 = 2s + 1\).

We now state Theorem 4.2, giving the genus of \(G_n + \overline{K}_3\), the join of \(G_n\) and \(\overline{K}_3\).

**Theorem 4.2.** Let \(G_n\) be defined as in Figure 4.4, where \(n = 4s + 2\) and \(s = 0, 1, 2, 3, \ldots\). Then \(\gamma(G_n + \overline{K}_3) = s\).

**Proof.** The quotient structure in Figure 4.5 gives \(r = r_n = 3\)
for $G_n$ on $S_\gamma$, where $\gamma = s$ (by Euler). Using Edmonds' permutation technique, we can easily express the three region boundaries as follows:

region a: $0, n-1, n-2, \ldots, 3, 2, 1$

region b: $0, 1; \frac{n}{2}+1, \frac{n}{2}+2; 2, 3; \frac{n}{2}+3, \frac{n}{2}+4; \ldots; -2, -1; \frac{n}{2}-1, \frac{n}{2}$

region c: $-1, 0; \frac{n}{2}, \frac{n}{2}+1; 1, 2; \frac{n}{2}+2, \frac{n}{2}+3; \ldots; -3, -2; \frac{n}{2}-2, \frac{n}{2}-1$

We can see from the patterns formed that each $n$-sided region is, in fact, an $n$-cycle. Knowing this, we place a vertex in each of the three regions and then connect each such vertex to every vertex in the corresponding $n$-cycle. This gives a triangulation (all regions being triangular) of $G_n + \bar{K}_3$ on a surface of genus $s$; hence, $\gamma(G_n + \bar{K}_3) = s$ (since any triangulation is a minimal imbedding).

We now come to the major theorem of this project. Ringel's conjecture that $\gamma(K_{n,n,n,n}) = (n-1)^2$ was previously known for $n = 1, 2$, and 4. Recently, Gross showed the conjecture true for $n = 1$ or 5 (mod 6). We now state and prove the conjecture for $n = 2$ (mod 4). (Note that the values of $n = 2$ (mod 4) do not overlap with any of those values established by Gross.)
Theorem 4.3. Let $K_{n,n,n,n}$ be the regular complete 4-partite graph of order $4n$, where $n \equiv 2 \pmod{4}$. Then $\gamma(K_{n,n,n,n}) = (n-1)^2$.

Proof. Let $\Gamma = Z_{4n} = \Omega = \{0, 1, 2, \ldots, 4n-1\}$ and $\Delta = \{1, 2, 3, 4, \ldots, 2n\} \setminus \{4, 8, \ldots, 2n\}$, where $n = 4s + 2$ and $s = 0, 1, 2, \ldots$. First, we show that $G_\Delta(\Gamma) = K_{n,n,n,n}$. Set $\overline{\Delta} = \{4, 8, \ldots, 2n\}$ and observe that $\overline{\Delta}$ partitions $V(G_\Delta(\Gamma))$ into four sets:

- $A_0 = \{0, 4, 8, \ldots, 4n-4\}$,
- $A_1 = \{1, 5, 9, \ldots, 4n-3\}$,
- $A_2 = \{2, 6, 10, \ldots, 4n-2\}$,
- and $A_3 = \{3, 7, 11, \ldots, 4n-1\}$.

The generator $2n$ of $\overline{\Delta}$ has order 2 in $Z_{4n}$ and therefore contributes 1 to the degree of each vertex in $G_\Delta(\Gamma)$; each of the other $(n/2)-1$ generators of $\overline{\Delta}$ has order $\geq 3$ and thus contributes 2 to the degree of each vertex in $G_\Delta(\Gamma)$. It is thus easy to see that each vertex in the four disjoint sets $A_0, A_1, A_2, A_3$ has degree $2 \cdot ((n/2)-1) + 1 \cdot 1 = n-1$. Since the order of each of these sets is n and each set gives rise to a component of $G_\Delta(\Gamma)$, it is clear that each set determines one copy of $K_n$; and so, $G_\Delta(\Gamma) = 4K_n$. 

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(four copies of \( K_n \)). Now we observe that \( G_\Delta (\Gamma) = \bar{G}_\Delta (\Gamma) = \frac{4K_n}{n} = K_{n,n,n,n} \). Second, we verify that

Figure 4.4 is a quotient structure; that is, it satisfies the five basic properties of Theorem 3.1. Properties one and two follow, simply, by our convention of labeling edges (current with the arrow, inverse against the arrow). The index of \( \Omega \) in \( \Gamma \) is surely one. Figure 4.4 actually represents \( G_n \) imbedded in a surface of genus \( 1+s \) with \( r_{3n/2} = r_{6s+3} = 1 \). To verify this, let \( G' = G_n = G_{\Delta_1}(\Gamma_1) \), where \( \Gamma_1 = Z_n \) and \( \Delta_1 = \{1,n/2\} \), as given before. Let \( T \) be the imbedding of \( G' \) given by the quotient structure in Figure 4.5 (this imbedding is guaranteed to be a 2-cell imbedding). We have seen that this imbedding has exactly 3 regions (each region boundary is an \( n \)-cycle) on a surface of genus \( s \). Let \( v \) and \( w \) be any two adjacent vertices of \( G' \) and form \( G = G' - [v,w] \) (note that one of \( v \) and \( w \) becomes the hollow vertex in Figure 4.4). This reduces the number of regions in \( T \) to two. Since each of \( v \) and \( w \) lies in all three region boundaries of \( G' \) (recall each region boundary is an \( n \)-cycle), each lies in the remaining two region boundaries of \( G = G' - [v,w] \). Call the two distinct regions \( F_v \) and \( F_w \). Now apply Ringeisen's lemma to get a 2-cell imbedding of \( G' = G_n \) with one region; hence \( r_{3n/2} = r_{6s+3} = 1 \). Thus the index

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of our imbedding is one so that property three follows. Property four readily follows from the assignment of labels (currents) given by the functions C and D, as described previously. Because we have an index-one imbedding, property five is immediate. Therefore, \( M(\Gamma/\Omega) \) satisfies properties one through five, and so, \( M(\Gamma/\Omega) \) is a quotient structure. Next, we verify that the KCL holds at the \( k \)th vertex of \( G_n \) \((1 \leq k \leq 4s+2)\) by exhibiting an equation (for each \( k \)) in which both sides represent the sum of the currents (directed away) at the \( k \)th vertex. Since one side of the equation will always be zero, that the KCL holds will be immediate. We single out two unusual vertices, the \((2s+2)\)nd and the \((4s+2)\)nd, and divide the remaining \(4s\) vertices into four major cases:

1) \(1 \leq k \leq s + 1\)
2) \(s + 2 \leq k \leq 2s + 1\)
3) \(2s + 3 \leq k \leq 3s + 2\)
4) \(3s + 3 \leq k \leq 4s + 1\)

That the KCL holds for vertex \( k \), \( k = 2s + 2 \) or \( 4s + 2 \), follows from the equations:

\[
\begin{align*}
    k &= 2s + 2:\quad -[8s+2] + 3 + [8s-1] = 0 \\
    k &= 4s + 2:\quad [8s-2] - [8s-3] - 1 = 0
\end{align*}
\]

where the three terms of each equation represent the
directed current on the kth diameter edge, kth circle-edge, and (k+1)st circle-edge, respectively.

**Case 1.** $1 \leq k \leq s + 1$

First, we consider $k$ even. Using the functions $C$ and $D$ defined before, it is easy to see that the three edges incident with the kth vertex $v$ are labeled and directed as follows:

a) the kth diameter-edge--labeled $D(k) = 8s + 2 - 8(k-1)$
   and directed toward $v$.

b) the kth circle-edge--labeled $C(k) = 8s + 3 - 4(k-2)$
   and directed away from $v$.

c) the (k+1)st circle-edge--labeled $C(k+1) = 1 + 4(k+1-1)$
   and directed toward $v$.

The corresponding equation is then

$$-[8s + 2 - 8(k-1)] + [8s + 3 - 4(k-2)] - [1 + 4k] = 0.$$ 

Therefore the KCL holds at the kth vertex for $k$ even and $1 \leq k \leq s + 1$. Second, suppose that $k$ is odd. Again using the functions $C$ and $D$, we write the label and direction of each of the three edges incident with the kth vertex $v$. Namely,

a) the kth diameter-edge--labeled $D(k) = 8s + 2 - 8(k-1)$
   and directed away from $v$. 

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b) the $k$th circle-edge—labeled $C(k) = 1 + 4(k-1)$
and directed away from $v$.

c) the $(k+1)$st circle-edge—labeled $C(k+1) = 8s+3-4(k+1-2)$
and directed toward $v$.

The sum of the currents at $v$ is then

\[ [8s + 2 - 8(k-1)] + [1+4(k-1)] - [8s + 3 - 4(k-1)] = 0. \]

Therefore the KCL holds for case 1.

In cases 2 through 4 we only list the equations representing the sum of the currents (directed away) at the $k$th vertex $v$. In each equation the first term represents the current directed away from $v$ of the $k$th diameter-edge; the second term, the $k$th circle-edge; and the third term, the $(k+1)$st circle-edge. The cases are:

**Case 2.** $s + 2 \leq k \leq 2s + 1$

$k$ even:

\[ [6 + 8(k - (s+2))] + [8s + 3 - 4(k - 2)] - [1 + 4k] = 0 \]

$k$ odd:

\[ -(6 + 8(k - (s+2))] + [1 + 4(k - 1)] - [8s + 3 - 4(k - 1)] = 0 \]

**Case 3.** $2s + 3 \leq k \leq 3s + 2$
\[ k \text{ even:} \]
\[-[8s+2-8(k-(2s+2))] - [5+4(k-(2s+4))] + [(8s-1)-4(k+1-(2s+3))] = 0\]

\[ k \text{ odd:} \]
\[[8s+2-8(k-(2s+2))] - [8s-1-4(k-(2s+3))] + [5+4(k+1-(2s+4))] = 0\]

**Case 4.** \(3s + 3 \leq k \leq 4s + 1\)

\[ k \text{ even:} \]
\[[6+8(k-(3s+3))] - [5+4(k-(2s+4))] + [8s-1-4(k+1-(2s+3))] = 0\]

\[ k \text{ odd:} \]
\[-[6+8(k-(3s+3))] - [8s-1-4(k-(2s+3))] + [5+4(k+1-(2s+4))] = 0\]

Hence the KCL holds at each vertex of \(G_n\). Using this along with Theorem 3.3, it follows that in the imbedding of \(G = G_{\Delta}(\Gamma)\) there are exactly \(n \cdot (|\Omega|/\nu) = n \cdot (4n/1) = 4n^2\) triangular regions. Thus in the imbedding of \(G = G_{\Delta}(\Gamma) = K_{n,n,n,n}\), we have \(p = 4n\), \(q = 6n^2\), and \(r = 4n^2\); applying Euler's formula, we get

\[ 4n - 6n^2 + 4n^2 = 2 - 2\gamma \]

or

\[ \gamma = n^2 - 2n + 1 \]

\[ \gamma = (n-1)^2. \]

Since every region in the imbedding of \(K_{n,n,n,n}\) is triangular, the imbedding is minimal and it follows that

\[ \gamma(K_{n,n,n,n}) = (n-1)^2 \text{ for } n \equiv 2 \pmod{4}. \]


6. R. D. Ringeisen, Determining all compact orientable 2-manifolds which K_{m,n} has 2-cell imbedding, J. Combinatorial Theory, B 12 (1972), 101-104.


