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An Analytic Study of a System of Nonlinear Ordinary Differential Equations at an Irregular Type Singularity

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AN ANALYTIC STUDY
OF
A SYSTEM OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS
AT AN IRREGULAR TYPE SINGULARITY

by

James M. Lamb

A Project Report
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Specialist in Arts Degree

Western Michigan University
Kalamazoo, Michigan
December 1973
ACKNOWLEDGEMENTS

For his inspiration, encouragement, criticisms and, above all, unlimited patience during the preparation of this paper, I am ever thankful to Dr. Po-Fang Hsieh. I would also like to acknowledge the generous financial support from Western Michigan University, enabling me to pursue and complete my studies toward the degree of Specialist in Arts in Mathematics.
LAMB, James Morton

AN ANALYTIC STUDY OF A SYSTEM OF NONLINEAR
ORDINARY DIFFERENTIAL EQUATIONS AT AN
IRREGULAR TYPE SINGULARITY.

Western Michigan University, Sp.A., 1973
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan
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I. INTRODUCTION

The notion of asymptotic expansions was founded by H. Poincaré [1] 1886, who proved that formal power series solutions (which are divergent) are asymptotic solutions for nth order linear differential equations of a particular type with an irregular singular point. W. J. Trjitzinsky [2] 1938 proved that formal power series solutions are always asymptotic solutions for a general case, and in 1941, J. Malmquist [3] succeeded in enlarging a sector in which asymptotic expansions of solutions are valid. M. Hukuhara [4] 1941 completed the asymptotic theory in the sense that his sector has the widest opening among these authors'. Hukuhara's work was extended to the non-linear case by M. Iwano [5] 1957 who tried to apply a general theory about how to construct analytic expressions of bounded solutions for equations of the form

\[(0.1) \quad x^{\alpha+1} y' = f(x, y)\]

where \(x\) is a complex independent variable, \(f\) and \(y\) are \(n\)-column vectors.

Encouraged by the results P. Hartman [6] obtained in solving a third-order boundary layer equation, Iwano considered two systems of \(m + n\) differential equations of the form
\[(0.2) \quad x^{q+1} y' = f(x, y, z) \quad xz' = g(x, y, z)\]

where \(f(0, 0, 0) = 0, g(0, 0, 0) = 0\), and he succeeded in improving the method of constructing analytic expressions.

In order to explain Iwano's work and our problem, let us consider a simple system of \((0.2)\) with \(m = n = 1\), and \(g_z(0, 0, 0) = \mu\) where \(\mu\) is not zero, nor a positive integer. In this case, by Hukuhara's theory, we can assume without loss of generality that the equations have the form

\[(0.3) \quad x^{q+1} y' = f(x, y, z) \quad xz' = \mu z.\]

The key element of Iwano's method was the determination of a stable domain. "Stable" means that the values of a general solution for \(xz' = \mu z\) always remain in a small neighborhood of the origin of the \(z\)-plane as \(x\) approaches the origin along a suitable curve \(\Gamma\) in the \(x\)-plane. His method, however, was not useful for a system of the form

\[(0.4) \quad x^{q+1} y' = f(x, y, z) \quad xz' = \mu z + bx^{\mu} \]

where \(\mu\) is a positive integer and \(f(x, y, z)\) is a scalar function, holomorphic in \((x, y, z)\) at \((0, 0, 0)\), vanishing there, and \(f_y(0, 0, 0) \neq 0\).
It is the purpose of this paper to exhibit a unique formal solution of the latter system and to show that there exists a unique solution which has that formal solution as an asymptotic expansion in an appropriate sector. Proof of the existence of a unique solution is accomplished by the method of successive approximations utilized by P. F. Hsieh [7] and the stable domain investigated by Iwano [8].

In Chapter II, the unique formal solution is constructed and statement of the main results is made in the form of two theorems.

In Chapter III, proof of the first main theorem is reduced to that of solving a transformed system of equations.

The idea of a stable domain similar to that used by Iwano is introduced and fundamental inequalities are established in Chapter IV. Then the completion of the main theorem is accomplished by solving the reduced problem in terms of successive approximations.

The asymptotic convergence of the formal solution is established in Chapter V.

It is noteworthy that during the course of this work, a similar result for a more general system of equations has been obtained by M. Iwano [9] using the Tychonoff type fixed point theorem.
II. MAIN THEOREMS

§ 1. Assumptions and preparatory propositions. Let there be given a system of differential equations:

\[(E) \quad x^{\sigma+1}y' = f(x, y, z) \quad xz' = \mu z + bx^\mu.\]

Here we assume that:

i) \(x\) is a complex independent variable and \(\mu\) and \(\sigma\) are positive integers, \(b\) is a nonzero complex constant,

ii) \(f(x, y, z)\) is holomorphic in a domain \(|x| < \xi, |y| < \delta, |z| < \delta\)

where \(\xi, \delta\), and \(\delta\) are positive constants, and \(f(0, 0, 0) = 0\).

iii) \(f_y(0, 0, 0) = \lambda \neq 0\).

Before stating the main results to be proved, we introduce two propositions concerning a formal solution of the first equation of \((E)\).

PROPOSITION 1. The first equation of \((E)\) has a unique formal solution

\[(1.1) \quad y \sim \sum_{k+l=1}^{\infty} a_{kl} x^k z^l.\]

If \(\mu\) is not a positive integer, Hukuhara theory assures that \((0.2)\) can be reduced to \((0.3)\), namely \(b = 0\) in \((0.4)\). Then, \(y\) can be expressed as a formal solution of the form
(1.2) \[ y \sim \sum_{t=0}^{\infty} Q_t(x) z^t \]

where \( Q_t(x) \) can be determined successively and is analytic and possesses asymptotic expansions in a certain sector of the \( x \)-plane. As a matter of fact (1.2) can be proved to be uniformly convergent.

However, an expression of the form (1.2) cannot be obtained for equation (E), due to the fact that \( b \neq 0 \). Thus we shall attempt to find a formal solution in the form of

(1.3) \[ y \sim \sum_{k=0}^{\infty} P_k(z) x^k \]

Let \( V(x) = x^\mu (C + b \log x) \), where \( C \) is an arbitrary constant. Then \( V(x) \) is a holomorphic solution of \( x z' = \mu z + bx^\mu \). We shall prove:

**PROPOSITION 2.** *The first equation of (E) has a formal solution*

(1.4) \[ y \sim \sum_{k=0}^{\infty} P_k(V(x)) x^k \]

where \( P_k(v) \) is holomorphic in \( |v| < \delta_1 \) and

\[
P_k(z) = \sum_{t=0}^{\infty} a_{kt} z^t, \quad P_0(0) = a_{00} = 0.
\]

The proofs of propositions 1 and 2 will be given in sections § 3 and § 4.
§ 2. Main theorems. The main results of this paper are the following two theorems:

THEOREM A. (E) has a unique solution \[ \{\phi(x, V(x)), V(x)\} \] where \( \phi(x, v) \) is holomorphic in a domain

\[
0 < |x| < \xi^0, \quad 0 < \arg x < \theta, \quad |v| < \delta^0
\]

where \( \theta, \delta, \xi^0, \delta^0 \) are constants and \( \xi^0, \delta^0 \) are positive.

Moreover,

\[
\phi(x, v) \sim \sum_{k=0}^{\infty} P_k(v) x^k
\]

uniformly in \( v \) for \( |v| < \delta^0 \), as \( x \to 0 \) in \( 0 < \arg x < \theta \),

where \( P_k(v) \) is holomorphic in \( |v| < \delta^0 \).

The proof of this theorem will be given in Chapter III and Chapter IV.

By the use of Theorem A and Cauchy's theory, the solution \( \phi(x, V(x)) \) satisfies the following theorem which gives a uniformly convergent series expansion of the form (1.2).

THEOREM B. The solution \( \phi(x, V(x)) \) of the first equation of (E) satisfies

\[
\phi(x, V(x)) = \sum_{l=0}^{\infty} \phi_l(x) V(x)^l
\]

where the right hand side is uniformly convergent for \( |V(x)| < \delta^0 \), provided \( (x, V(x)) \) is in (2.1). The
coefficients \( \phi_t(x) \) are holomorphic in \( 0 < |x| < \xi^0 \), 
\( \theta < \arg x < \theta \) and

\[(2.4) \quad \phi_t(x) \sim \sum_{k=0}^{\infty} a_{k\ell} x^k \]
as \( x \to 0 \) in \( \theta < \arg x < \theta \).

The complete proof of this theorem will be given in Chapter V.

§ 3. Proof of Proposition 1. Let

\[(3.1) \quad y \sim \sum_{k+\ell=0}^{\infty} a_{k\ell} x^k z^\ell \]

Differentiating (3.1) formally, we get

\[y' \sim \sum_{k+\ell=0}^{\infty} a_{k\ell} \left\{ kx^{k-1}z + lx^{k-1}xz^\ell \right\} \]

\[= \sum_{k+\ell=0}^{\infty} a_{k\ell} \left\{ kx^{k-1}z + \mu lx^{k-1}z + \mu lx^{k+\mu-1}z^{\ell-1} \right\} \]

Therefore,

\[x^{\sigma+1}y' \sim x^{\sigma} \sum_{k+\ell=0}^{\infty} a_{k\ell} \left\{ kx^kz + \mu lx^kz + \mu lx^{k+\mu}z^{\ell-1} \right\} \]

\[(3.2) \quad = \sum_{k+\ell=0}^{\infty} \left\{ a_{k-\sigma,\ell} x^{k-\sigma} + \mu la_{k-\sigma,\ell} + b(t+1)a_{k-\sigma-\mu,\ell} \right\} x^k z^\ell \]

where \( a_{k\ell} = 0 \) if \( k < 0 \).
On the other hand, by assumption (ii), \( f(x, y, z) \) can be expressed as a convergent power series

\[
(3.3) \quad f(x, y, z) = f_0(x, z) + f_1(x, z)y + \sum_{p=2}^{\infty} f_p(x, z)y^p
\]

where

\[
\begin{align*}
  f_0(x, z) &= \sum_{k+t=1}^{\infty} f_{0kt}x^k z^t \\
  f_1(x, z) &= (\lambda + \sum_{k+t=1}^{\infty} A_{k+t}x^k z^t) \\
  f_p(x, z) &= \sum_{k+t=0}^{\infty} f_{pkt}x^k z^t.
\end{align*}
\]

Now, substitute (3.1) into (3.3) with (3.4) and compare the coefficients of \( x^k z^t \) with those in (3.2). First we see, by the implicit function theorem, since \( f(0, 0, 0) = 0 \) and \( f_y(0, 0, 0) = \lambda \neq 0 \), that \( y(0, 0) = a_{00} = 0 \). And, in general,

\[
\lambda a_{kt} + H_{kt}(a_{k't'}) = 0
\]

for \( k, t = 0, 1, 2, \ldots, k+t \geq 1 \), where \( H_{kt} \) is a polynomial in \( a_{k't'} \) (\( k'+t' < k+t \)). Hence, \( a_{kt} \) can be determined uniquely in the successive order of \( k+t \).
§ 4. Proof of Proposition 2. Put

\[(4.1)\]
\[
y \sim \sum_{k=0}^{\infty} P_k(V(x))x^k
\]

where \(V(x)\) is a holomorphic solution of the second equation of (E). Differentiating formally,

\[(4.2)\]
\[
x^{\sigma+1}y' \sim x^{\sigma+1} \frac{dP_0}{dz} z' + x^{\sigma+1} \sum_{k=1}^{\infty} \left( \frac{dP_k}{dz} z'x^k + kP_kx^{k-1} \right)
\]

\[
= x^{\sigma} \frac{dP_0}{dz} (\mu z + bx^l) +
\]

\[
x^{\sigma} \sum_{k=1}^{\infty} \left\{ \frac{dP_k}{dz} (\mu z + bx^l)x^k + kP_kx^k \right\}
\]

\[
= \sum_{k=0}^{\infty} x^{\sigma+k} (\mu z \frac{dP_k}{dz} + bx^l \frac{dP_k}{dz} + kP_k)
\]

\[
= \sum_{k=0}^{\infty} (\mu z \frac{dP_{k-\sigma}}{dz} + b \frac{dP_{k-\mu-\sigma}}{dz} + (k-\sigma)P_{k-\sigma})x^k
\]

where \(P_j(z) = 0\) if \(j < 0\). Next, substitute (4.1) into (E) with

\[(4.3)\]
\[
f(x, y, z) = f_0(y, z) + \sum_{k=1}^{\infty} f_k(y, z)x^k.
\]

Then, first of all, by the implicit function theorem, since \(f_0(0, 0) = 0, f_{0y}(0, 0) = \lambda \neq 0\), there exists a unique function \(\varphi(z) = P_0(z)\), holomorphic in \(|z| < \delta\), with \(f(0, \varphi(z), z) = f_0(\varphi(z), z) = 0\) and \(\varphi(0) = 0\).
But, (1.1) is the unique formal solution of the first equation of (E); hence, \( P_0(z) \) has a convergent power series expression

\[
P_0(z) = \sum_{t=1}^{\infty} a_0 t^t
\]

for \(|z| < \delta\).

Now, equating coefficients of \( x^k \) from (4.2) and (4.3) we find in general

\[
\sum_{k=0}^{\infty} f_{0y}(P_0, z) P_k(z) + H_k(z; P_0, P_1, \ldots, P_{k-1}) = 0
\]

where \( H_k(z; P_0, P_1, \ldots, P_{k-1}) \) is a polynomial of the functions \( P_i(z) \), \( i = 0, 1, \ldots, k-1 \), and their derivatives. Thus, \( H_k \) is known if \( P_0, P_1, \ldots, P_{k-1} \) are known. Moreover, since \( f_{0y}(0, 0) \neq 0 \), there exists a neighborhood \(|z| < \delta_1 < \delta\) such that \( f_{0y}(P_0(z), z) \neq 0 \). Hence,

\[
P_k(z) = \frac{H_k(z; P_0, P_1, \ldots, P_{k-1})}{f_{0y}(P_0(z), z)}
\]

can be determined successively and each \( P_k(z) \) is holomorphic in \(|z| < \delta_1, k = 1, 2, \ldots \).

Furthermore, since the formal solution (1.1) is unique, \( P_k(z) \) is expressible as the convergent power series

\[
P_k(z) = \sum_{t=0}^{\infty} a_k t^t.
\]
III. PROOF OF THEOREM A

§ 5. Reduction of Theorem A. In order to prove Theorem A, we first consider, for a positive integer N, the following transformation to the first equation of (E):

\[ y = P_N(x, v) + \eta_N \]

where

\[ P_N(x, v) = \sum_{k=0}^{N-1} p_k(v) x^k . \]

Then, the transformed equation can be written as

\[ x^{\sigma+1} \eta_N = \lambda \eta_N + \hat{f}(x, \eta_N, v) \]

where \( \hat{f}(x, \eta, v) \) is holomorphic and bounded for

\[ 0 < |x| < \epsilon_N, \quad |v| < \delta_N, \quad |\eta| < d_N. \]

Here \( \epsilon_N, \delta_N, d_N \) are constants depending on \( N, \epsilon_N < \epsilon, \delta_N < \delta, d_N \) depends on \( d, \epsilon_N \) and \( \delta_N \). Further, \( \hat{f}(0, 0, 0) = 0 \). Therefore, we have the inequality

\[ |\hat{f}(x, \eta, v)| \leq H|\eta| + B_N|x|^N \]

for \( (x, \eta, v) \) in domain (5.3), where \( H, B_N \) are positive constants and \( H \) is independent of \( N \). In fact, \( H \) can be taken arbitrarily small so that we can assume without loss of generality that \( H \) satisfies
(5.5) $4H < |\lambda| \sin 2\sigma \epsilon$

for a preassigned number $\epsilon$. Also, $f$ satisfies a Lipschitz condition with respect to $\eta$ in (5.3) with Lipschitz constant $H$.

Now, put

(5.6) $\eta_N = e^{\Omega(x)} T_N(x, v)$

where

$\Omega(x) = \frac{-\lambda}{\sigma x^\sigma}$.

Then, the system (E) is reduced to

(5.7) $T_N' = x^{-\sigma-1} e^{-\Omega(x)} f(x, e^{\Omega(x)} T_N', v)$

Thus, the proof of Theorem A is reduced to solving the following:

PROBLEM A. There exists a solution $[\varphi_N(x, V(x))]$ of (5.7) such that for suitable $\tilde{\alpha}, \tilde{\sigma}, \tilde{\xi}^\prime_N, \tilde{\delta}^\prime_N$ and $K_N$,

i) $\varphi_N(x, v)$ is holomorphic and bounded for

(5.8) $0 < |x| < \xi^\prime_N, \quad 0 < \arg x < \tilde{\sigma}, \quad |v| < \delta^\prime_N$;

ii) $\varphi_N(x, v)$ satisfies the inequality

$|\varphi_N(x, v)| \leq K_N |x|^N e^{-\text{Re} \Omega(x)}$

for $(x, v)$ in (5.8) $N$. 

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Moreover, a solution of (5.7) satisfying
\[
|T_N| = O(|x|^N e^{-Re \Omega(x)})
\]
is unique.

In fact, we can prove Theorem A by making use of
the solution to Problem A in the following manner.

Owing to the transformations (5.1) and (5.6), the
functions
\[
\sum_{k<N} \sum_{k<N} P_k(V(x)) x^k + e^{\Omega(x)} \varphi_N(x, V(x)),
\]
is a solution of (E) provided \((x, V(x))\) is in the domain
\((5.8)_N\). Let \(N'\) be an integer greater than \(N\). Then,
\[
e^{-\Omega(x)} \sum_{N \leq k < N'} P_k(V(x)) x^k + \varphi_N'(x, V(x)),
\]
is a solution of (5.7), satisfying (5.9) if \((x, V(x))\)
belongs to the common part of the domains \((5.8)_N\) and
\((5.8)_{N'}\). Hence, by the uniqueness of solution, the
solution (5.11) must coincide with \([\varphi_N(x, V(x)), V(x)]\).

From this, the solution of (E) expressed by (5.10) is
independent of \(N\). We denote this solution by \([\Phi(x, V(x)), V(x)]\). Then by analytic continuation, the functions
\(\Phi(x, v)\) and \(V(x)\) are defined in the domain of the form
(2.1) with $\xi^0 = \sup \xi^i_N$, $\delta^0 = \sup \delta^i_N$. Clearly, from
Problem A, we know that $f(x, V(x))$ admits the asymptotic
expansion (2.2) and that the expansion is uniform in $v$
for $|v| < \delta^0$ as $x \to 0$ in $\varrho < \arg x < \overline{\varrho}$.

IV. SOLUTION OF PROBLEM A

§ 6. Fundamental inequalities. In order to find $\psi_N(x, v)$,
satisfying the conditions prescribed in Problem A, it is
necessary to replace (5.8)$_N$ by an equivalent domain of
the form

$$0 < |x| < \xi^"_N \omega(\arg x), |v| < \delta^"_N f(\arg x) \omega(\arg x)^\mu$$

where $\varrho < \arg x < \overline{\varrho}$. Domain (6.1) and (5.8)$_N$ are equi-
valent in the sense that (6.1) is contained in (5.8)$_N$ if
$\xi^"_N$ and $\delta^"_N$ are suitably chosen and vice versa if $\xi^i_N$ and $\delta^i_N$
are sufficiently small. Moreover, we will want $\xi^"_N$ to
satisfy

$$\xi^"_N^\mu < \delta^"_N$$

and also, as we see later, two other restrictions given
by (6.5) and (10.5).

Here $\omega(\tau)$ is the scalar function defined by

$$\omega(\tau) = \exp \int_{\varrho_0}^{\tau} \cot a(t) \, dt, \quad \varrho_0 < \tau < \overline{\varrho}.$$
where $\theta_0$ is a fixed angle in $\theta < \tau < \overline{\theta}$ and $a(t)$ is defined in the next section. The function $f(\tau)$ is to be

$$f(\tau) = \begin{cases} f_{-}(\tau) & \Delta \leq \tau \leq \overline{\theta}, \\ f_{+}(\tau) & 0 \leq \tau \leq \Delta \end{cases}$$

where

$$f_{-}(\tau) = 1 + 2|b| \left\{ \int_{\theta}^{\Delta} \left| \cot a(t) \right| \, dt + \overline{\theta} - \tau \right\},$$

$$f_{+}(\tau) = 1 + 2|b| \left\{ \int_{0}^{\Delta} \left| \cot a(t) \right| \, dt + \tau - \overline{\theta} \right\}.$$

The existence of a unique solution to (5.7) in the domain (6.1) instead of (5.8) will be shown after defining the function $a(t)$ and proving the following:

**Proposition 3.** Let $(x^1, v^1)$ be an arbitrary point in the domain (6.1). Choose the constant $C$ so that

$$v(x^1) = v^1 \text{ (namely, } C = v^1/x_1^u - b \log x_1 \text{). Then,}$

i) There exists a curve $\Gamma_{x_1}$ joining the origin and the point $x_1$, which is entirely contained in the domain (6.1) except for the origin;

ii) As $x$ moves on $\Gamma_{x_1}$, inequalities of the form

$$\frac{d}{ds} \exp\{-\text{Re } \Omega(x)\} \geq$$

$$|x|^{-\sigma-1} \exp\{-\text{Re } \Omega(x)\} |\lambda| \sin 2\sigma$$

and

$$|x|^{-1} \frac{d}{ds} |x| \geq -|x|^{-1}$$
are satisfied. Here \( s \) denotes the arc length of the curve \( \Gamma_{x_1} \) measured from the origin to the variable point \( x \):

iii) The values of the function \( V(x) \) always remain in domain \( (6.1) \) as \( x \) is on \( \Gamma_{x_1} \);

iv) If \( \xi'' \) satisfies

\[
2N(\xi'' \max_{\theta \leq \tau \leq \theta} \omega(\tau))^{\sigma} \ll |\lambda| \sin 2\sigma \epsilon,
\]

then

\[
\int_{0}^{x_1} |x|^{N-\sigma-1} \exp\{ -\text{Re} \Omega(x) \} |dx| \ll \frac{2}{|\lambda| \sin 2\sigma \epsilon} |x_1|^N \exp\{ -\text{Re} \Omega(x_1) \}.
\]

Here the integration is carried along \( \Gamma_{x_1} \).

§ 7. Determination of \( a(t) \). Let the sector \( \theta < \arg x < \bar{\theta} \) be defined by

\( S: \quad \theta = (1/\sigma) (\arg \lambda - 5\pi/2) + 4\epsilon, \)

\[
\bar{\theta} = (1/\sigma) (\arg \lambda + \pi/2) - 4\epsilon,
\]

or

\( S': \quad \theta = (1/\sigma) (\arg \lambda - \pi/2) + 4\epsilon, \)

\[
\bar{\theta} = (1/\sigma) (\arg \lambda + 5\pi/2) - 4\epsilon,
\]

where \( \epsilon \) is a sufficiently small positive constant. The
directions \( \arg x = \theta \) in the sector such that \( \text{Re } \Omega(x) = 0 \)
are called singular directions of \( \Omega(x) \) and are given by

\[
\theta_+ = \frac{1}{\sigma} (\arg(-\lambda) - \pi/2), \quad \theta_- = \frac{1}{\sigma} (\arg(-\lambda) + \pi/2).
\]

Then the angles \( \theta \) and \( \bar{\theta} \) are expressed as

\[
\theta = \theta_+ - \pi/\sigma + 4\varepsilon, \quad \bar{\theta} = \theta_- + \pi/\sigma - 4\varepsilon.
\]

Define the function \( a(\tau) \) by

\[
a(\tau) = \begin{cases} 
\sigma(\tau - \theta_+ + 2\varepsilon), & \theta_- + \pi/2\sigma - 2\varepsilon \leq \tau \leq \theta_+ - \theta_- + \pi/2\sigma - 2\varepsilon, \\
\pi/2, & \theta_- + \pi/2\sigma - 2\varepsilon \leq \tau \leq \theta_- + \pi/2\sigma - 2\varepsilon, \\
\sigma(\tau - \theta_+ - 2\varepsilon) + \pi, & \theta_- \leq \tau \leq \theta_- + \pi/2\sigma + 2\varepsilon.
\end{cases}
\]

We see that \( a(\tau) \) is a continuous function of \( \tau \) in

\( \theta \leq \tau \leq \bar{\theta} \) and satisfies the inequality

\[
2\sigma \varepsilon \leq a(\tau) \leq \pi - 2\sigma \varepsilon.
\]

Also, by inspection we can verify the following inequalities for \( f \):

\[
\begin{align*}
f_-(\arg x_1) & \leq f_-(\arg x) & \text{if } \arg x \leq \arg x_1 \\
f_+(\arg x_1) & \leq f_+(\arg x) & \text{if } \arg x_1 \leq \arg x
\end{align*}
\]

Further, the functions \( w(\tau) \) and \( f(\tau) \) are seen to be
strictly positive valued, bounded and continuous in \( \tau \) for
\( \theta \leq \tau \leq \bar{\theta} \). Moreover, we can assume (without loss of
generality since \( |b| \) may be made arbitrarily small by
the linear transformation \( x = kt \) that

\[
1 \leq f(\tau) \leq 2
\]

We now proceed with the proof of Proposition 3.

§ 8. **Proof of Proposition 3.**

**Assertion (i):**

Denote by \((\rho, \tau)\) the polar coordinates of the variable point on the curve \( \Gamma_{X_{1}} \) to be defined as follows:

\( \Gamma_{X_{1}} \) consists of a curvilinear part \( \Gamma' \):

\[
\rho = |x_{1}| \exp \int_{\arg x_{1}}^{\tau} \cot a(t) \, dt,
\]

for \( \theta_{-} - 2\varepsilon \leq \tau \leq \arg x_{1} \) or \( \arg x_{1} \leq \tau \leq \theta_{+} + 2\varepsilon \), and a rectilinear part \( \Gamma'' \):

\[
0 \leq \rho \leq |x_{1}| \exp \int_{\arg x_{1}}^{\tau} \cot a(t) \, dt,
\]

\( \tau = \theta_{-} - 2\varepsilon \) or \( \tau = \theta_{+} + 2\varepsilon \).

For \( \theta_{+} + 2\varepsilon \leq \arg x_{1} < \theta_{-} - 2\varepsilon \), \( \Gamma_{X_{1}} \) consists of a rectilinear part \( \Gamma'' \) only:

\[
0 \leq \rho \leq |x_{1}|, \quad \tau = \arg x_{1}.
\]

Now, for \( x \) on \( \Gamma' \),

\[
|x| \leq |x_{1}| \exp \int_{\arg x_{1}}^{\tau} \cot a(t) \, dt
\]
\[
\ar\frac{\arg x_1}{\exp \int_0^{\theta_0} \cot a(t) \ dt}\ (\exp \int_0^\tau \cot a(t) \ dt) = \frac{\pi}{2} \omega(\arg x).
\]

Similarly for \(x\) on \(\Gamma''\). Hence the curve \(\Gamma_{x_1}\) is contained in (6.1) except for the origin and Assertion (i) is proved.

Assertion (ii):

To prove inequality (6.3) for \(x\) on \(\Gamma'\), put \(x = \rho \, e^{i\tau}\).

Since on \(\Gamma'\) \(\rho\) is given by (8.1), it follows that

\[
\frac{d\rho}{d\tau} = \rho \cot a(\tau).
\]

Hence

\[
\frac{ds}{d\tau} = \pm \left[ \left( \frac{d}{d\tau} (\rho \cos \tau) \right)^2 + \left( \frac{d}{d\tau} (\rho \sin \tau) \right)^2 \right]^{\frac{1}{2}} = \pm \frac{\rho}{\sin a(\tau)}
\]

with the sign \(-\) for \(\arg x_1 < \tau < \theta_+ + 2\varepsilon\) and \(+\) for \(\theta_- - 2\varepsilon \leq \tau \leq \arg x_1\). Thus

\[
\frac{dx}{ds} = \rho \frac{d}{ds} e^{i\tau} + e^{i\tau} \frac{d\rho}{ds}
\]

(8.2)

\[
= \mp e^{i\tau} (\cos a(\tau) + i \sin a(\tau))
\]

\[
= \mp e^{i(\tau + a(\tau))}
\]

with the sign \(-\) or \(+\) according as \(\arg x_1 < \tau < \theta_+ + 2\varepsilon\) or \(\theta_- - 2\varepsilon \leq \tau \leq \arg x_1\). Hence, we have the equality
\[
- \frac{d}{ds} \left( \text{Re} \Omega(x) \right) = \pm r^{-\sigma-1} |\lambda| \cos(a(\tau) - \sigma \tau + \text{arg} \lambda)
\]

with the sign + for \( x_1 \leq \tau \leq \theta_+ + 2\epsilon \) and - for \( \theta_- - 2\epsilon \leq \tau \leq \text{arg} \ x_1 \), and consequently

\[
\frac{d}{ds} e^{-\text{Re} \Omega(x)} = \pm r^{-\sigma-1} |\lambda| e^{-\text{Re} \Omega(x)} \cos(a(\tau) - \sigma \tau + \text{arg} \lambda)
\]

according as \( x_1 \leq \tau \leq \theta_+ + 2\epsilon \) or \( \theta_- - 2\epsilon \leq \tau \leq \text{arg} \ x_1 \).

Now, by the definition of the function \( a(\tau) \) and the angles \( \theta_+, \ \theta_- \) we have

\[
-3\pi/2 + 2\sigma \epsilon \leq a(\tau) - \sigma \tau + \text{arg} \ \lambda \leq -\pi + 2\sigma \epsilon,
\]

for \( \theta_- - 2\epsilon \leq \tau \leq \theta_+ \), and

\[
-2\sigma \epsilon \leq a(\tau) - \sigma \tau + \text{arg} \ \lambda \leq \pi/2 - 2\sigma \epsilon,
\]

for \( \theta_- + 2\epsilon \leq \tau \leq \theta_+ + 2\epsilon \).

Hence, for \( x \) on \( \Gamma' \)

\[
\frac{d}{ds} e^{-\text{Re} \Omega(x)} \geq \pm r^{-\sigma-1} |\lambda| e^{-\text{Re} \Omega(x)} \sin 2\sigma \epsilon.
\]

If \( x \) is on \( \Gamma'' \), \( s = p = |x| \) and \( \text{arg} \ x \) is constant and satisfies

\[
\theta_+ + 2\epsilon \leq \text{arg} \ x \leq \theta_- - 2\epsilon.
\]

An easy consideration shows that

\[
|\text{arg} \ \lambda - \sigma \tau + \pi| \leq \pi/2 - 2\sigma \epsilon.
\]
Since
\[- \Re \Omega(x) = |\lambda| (\alpha \rho)^{-1} \cos(\arg x - \sigma),\]
we have therefore
\[
\frac{d}{ds}( - \Re \Omega(x) ) = - |\lambda| \rho^{-c-1} \cos(\arg x - \sigma) = |\lambda| \rho^{-c-1} \sin \tau.
\]

which proves (6.3) for \(x\) on \(\Gamma''\). In order to prove (6.4), since \(s\) is real we have
\[
|\lambda|^{1-1} \frac{d|x|}{ds} = \frac{d}{ds} \log |x| = \frac{d}{ds} (\Re \log x) = \Re(\frac{d}{ds} \log x).
\]

Hence,
\[
\Re( x^{-1} \frac{dx}{ds} ) \geq - |x|^{-1}.
\]

Here we used (8.2) when \(x\) is on \(\Gamma'\). When \(x\) is on \(\Gamma''\), this inequality follows immediately because \(|x| = s\).

Thus Assertion (ii) is proved.

Assertion (iii):
A direct calculation yields
\[
V(x) = \frac{x}{x_1} + b x^{\mu} \log \frac{x}{x_1}.
\]

Hence,
\[
|V(x)| \leq \left| \frac{x}{x_1} \right|^\mu |v^1| + |b| \left| |x|^{\mu} \left| \log \frac{x}{x_1} \right| \right|.
\]

If \(x\) is on \(\Gamma'\),
\[
x = x_1 \exp\left( \int_{\arg x_1}^\tau \cot a(t) \, dt + i(t - \arg x_1) \right).
\]
Thus,

\[ |V(x)| < \exp\{\mu \int_{\arg x_1}^\tau \cot a(t) \, dt\} \times \]

\[ |v^1| + |b| |x_1|^\mu \left( \int_{\arg x_1}^\tau \cot a(t) \, dt + i(\tau - \arg x_1) \right). \]

Since \((x_1, v^1)\) is in domain (6.1), it follows by (6.2) and the definition of \(\omega(\tau)\) and \(f(\tau)\) that

\[ |V(x)| < \delta_N^{\mu} \omega(\tau) \left[ f(\arg x_1) + \right. \]

\[ |b| \left( |\int_{\arg x_1}^\tau |\cot a(t)| \, dt| + |\tau - \arg x_1| \right). \]

By using (7.1) and (7.2), we see that the expression in the bracket does not exceed \(f(\tau)\). Thus we have

\[ |V(x)| < \delta_N^{\mu} f(\arg x) \omega(\arg x)^\mu \]

for \(x\) on \(\Gamma'\).

For \(x\) on \(\Gamma''\), \(\arg x\) is fixed. Hence it will be sufficient to show that

\[ (8.3) \quad |V(rx_1^*)| < \delta_N^{\mu} f(\arg x_1^*) \omega(\arg x_1^*)^{\mu} \]

for all \(0 \leq r \leq 1\), where \(x_1^* = \Gamma' \cap \Gamma''\). First, we have

\[ V(x) = V(x; x_1, v_1) = (x/x_1)^{\mu} \{v^1 + b \cdot x_1 \log (x/x_1)\}. \]
Let $v^1 = V(x_1^*; x_1, v^1)$. Then it is easy to show that

$$V(x; x_1, v^1) = V(x; x_1^*, v^1).$$

The above equation implies that

$$V(rx_1^*) = r^μ \{v^1 + b x_1^* \log r\}.$$

Therefore,

$$|V(rx_1^*)| \leq r^μ \{ |v^1| + b |x_1^*|^μ |\log r| \}$$

$$\leq \delta_n^u r^μ \{ f(v) w(v)^μ + b |x_1^*|^μ |\log r| \},$$

$$= \delta_n^u [r^μ f(v) + r^μ γ |\log r|] w(v)^μ,$$

where

$$γ = \frac{|b| |x_1|^μ}{w(v)^μ}, \quad v = \arg x_1^*.$$

Hence,

$$|V(rx_1^*)| \leq \delta_n^u \{ f(v) + F(r) \} w(v)^μ,$$

where

$$F(r) = (r^μ - 1) f(v) + γ r^μ |\log r|.$$ 

Now, we wish to show that $F(r) \leq 0$ for $0 \leq r \leq 1$ and hence (8.3) will follow. For $0 < r \leq 1$,

$$\frac{df}{dr} = μ r^{μ-1} f(v) + γ( μ r^{μ-1} |\log r| - r^{μ-1} ),$$

$$= r^{μ-1} \left( μ f(v) + γ( μ |\log r| - 1 ) \right),$$

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\[ \geq r^{\mu-1} (\mu f(v) - \gamma), \]
\[ \geq r^{\mu-1} (\mu - \gamma), \quad \text{by (7.2)}. \]

Hence, \( \frac{dF}{dr} \geq 0 \) if \( \gamma \leq \mu \). So we choose \( b \) such that
\[ |b| \leq \frac{\mu \delta_n}{\xi_n}. \]

Using the fact that \( |x|^\mu < \xi_n \mu w(v)^\mu \), we conclude that, since \( F(1) = 0 \), \( F(r) \leq 0 \) for \( 0 < r \leq 1 \). By L'Hopital's rule, we see that
\[ \lim_{r \to 0} F(r) = -f(v). \]

Thus, the proof of Assertion (iii) is finished.

\textbf{Assertion (iv):}

From (6.3) and (6.4) we have
\[ \frac{d}{ds} |x|^N e^{-Re \Omega(x)} \geq |x|^{N-\sigma-1} e^{-Re \Omega(x)} (|\lambda| \sin 2\sigma e^{-N|x|^\sigma}) \]
for \( x \) in (6.1). Therefore,
\[ \frac{d}{ds} (|x|^N e^{-Re \Omega(x)}) \geq \frac{|\lambda| \sin 2\sigma e^{-N|x|^\sigma}}{2} |x|^{N-\sigma-1} e^{-Re \Omega(x)} \]
for \( x \) on \( \Gamma_{x_1} \) if \( \xi_n \) is chosen so that (6.5) is satisfied. Then Assertion (iv) follows and Proposition 3 is completely proved.
§ 9. Successive Approximations. We shall solve Problem A with (5.8) replaced by (6.1) by means of successive approximations.

Let \((x_1, v^1)\) be an arbitrary point in the domain (6.1) where \(\xi_N^n\) and \(\xi_N^n\) will be specified later. Let \(V(x) = x^\mu(C + b \log x)\) where \(C\) is chosen such that \(V(x_1) = v^1\). The first equation of the system (5.7) is equivalent to the following integral equation:

\[
(9.1) \quad \phi(x_1, v^1) = \int_0^{x_1} x^{-\nu-1} e^{-\Omega(x)} f(x, e^{\Omega(x)}) T(x, V(x)), V(x) \, dx.
\]

The successive approximations for (9.1) are defined to be the sequence of functions

\[
(9.2) \quad \phi(0)(x_1, v^1) = 0
\]

(9.3.m) \[
\phi(m+1)(x_1, v^1) = \int_0^{x_1} x^{-\nu-1} e^{-\Omega(x)} f(x, e^{\Omega(x)}) \phi(m)(x, V(x)), V(x) \, dx
\]

\(m = 0, 1, 2, \ldots\). The path of integration is \(\Gamma_{x_1}\) as defined in section § 8.

We shall prove the sequence thus defined converges to the desired solution in the following steps:

(I) Each term of the sequence \(\{\phi(m)(x_1, v^1)\}\) is well defined and holomorphic in \((x_1, v^1)\) for (6.1).
(II) The sequence converges uniformly to a function \( \phi_N(x_1, v^1) \) in any compact subset of (6.1).

(III) The limit function, \( \phi(x, v^1) \), satisfies the integral equation (9.1).

(IV) The function \( \phi_N(x, V(x)) \) is a solution of the first differential equation of system (5.7) satisfying the properties described in Problem A.

§ 10. The function \( \phi^{(1)}(x, v^1) \) shall prove Step (I) by mathematical induction.

Let \((x_1, v^1)\) be an arbitrary point in a domain of the form (6.1), \( V(x) \) a holomorphic solution of the equation \( \mu x = \beta z + bx^\mu \) such that \( V(x_\alpha) = v^1 \). Then \( \phi^{(1)}(x_1, v^1) \) is given by

\[
\phi^{(1)}(x_1, v^1) = \int_0^{x_1} f^{(1)}(x, V(x)) \, dx
\]

where

\[
f^{(1)}(x, v) = x^{-\alpha-1} e^{-\Omega(x)} (x, 0, v).
\]

1. Existence of integral. By the definition of the path \( \Gamma_{x_1} \), the integrand of (10.1) is holomorphic on \( \Gamma_{x_1} \) except at \( x = 0 \). We shall first prove the convergence of the integral at \( x = 0 \).

Let \( x_1^* \) be the last point of \( \Gamma_{x_1} \) on \( \tau = \theta_+ + 2\varepsilon \) or on \( \tau = \theta_- - 2\varepsilon \) according as whether \( \theta < \arg x_1 \leq \theta_+ + 2\varepsilon \).
or $\theta - 2\epsilon \leq \arg x^1 \leq \bar{\theta}$, when $x$ moves from $0$ to $x^1$ along $\Gamma_{x^1}$. Let $r^* = |x^1|$. Then, from the definition of $\Gamma_{x^1}$, it is readily seen that

$$r^* = |x^1|\left\{\frac{1}{\sin a(\arg x^1)}\right\}^{1/\sigma} \geq |x^1| > 0.$$ 

This means that no matter where $x^1$ is located in (6.1), the path $\Gamma_{x^1}$ always has rectilinear portion $\Gamma''$ of positive length. Furthermore, 

$$\Re \Omega(x) > 0 \quad \text{on } \Gamma''.$$ 

Then, by (5.4) we have

$$|f^{(1)}(x, V(x))| = O(|x|^{N-\sigma-1} e^{-\Re \Omega(x)}).$$ 

So the right hand side of the above equation tends to zero exponentially as $x \to 0$ along $\Gamma''$. Thus the integral exists at $x = 0$. Furthermore, the integrand is bounded on $\Gamma_{x^1}$, whether there is a curvilinear portion $\Gamma'$ or not, and length of $\Gamma_{x^1}$ is finite. Thus, $\xi^{(1)}(x^1, v^1)$ exists in (6.1) for arbitrary $(x^1, v^1)$.

2. Upper Bounds. By (5.4) we have

$$|f^{(1)}(x, V(x))| \leq B_{N}|x|^{N-\sigma-1} e^{-\Re \Omega(x)}$$ 

for

$$0 < |x| < \xi_N \omega(\arg x), \quad \theta < \arg x < \bar{\theta}.$$
Choose $\xi''_N$ so small that (6.5) is satisfied. Then, by (6.6) we get

\begin{equation}
|\hat{\psi}^{(1)}(x_1, v^1)| \leq \frac{2B_N}{|\lambda| \sin 2\sigma \varepsilon} |x_1|^N e^{-Re \Omega(x)}.
\end{equation}

Now, put $K_N$ as

\begin{equation}
K_N = \frac{4B_N}{|\lambda| \sin 2\sigma \varepsilon}
\end{equation}

and choose $\xi''_N$ such that

\begin{equation}
K_N \left[ \xi''_N \max_{\omega(\tau)} \right] N < d_N.
\end{equation}

Analyticity. When $x_1$ is fixed, (10.3) implies that the integral (10.1) converges uniformly with respect to $v^1$. Thus $\hat{\psi}^{(1)}(x_1, v^1)$ is holomorphic in $v^1$ for $|v^1| \leq \xi''_N f(\arg x) \omega(\arg x)^\mu$ when $x_1$ is fixed.

Next, we shall prove that $\hat{\psi}^{(1)}(x_1, v^1)$ is holomorphic in $x_1$ for (10.2) when $v^1$ is fixed. Let $x_0$ be a point in (10.2) and sufficiently near $x_1$. Observe that

\begin{equation}
\int_0^{x_1} f^{(1)}(x, v(x)) \, dx = \int_0^{x_0} f^{(1)}(x, v(x)) \, dx + \int_{x_0}^{x_1} f^{(1)}(x, v(x)) \, dx.
\end{equation}

Here the first integral in the sum is carried along the path $\Gamma_{x_0}$ and that in the second is carried along $x_0^{x_1}$.
For the proof of the above relation let $t_0$ and $t_1$ be respectively the intersection points of the paths $\Gamma_{x_0}$ and $\Gamma_{x_1}$ with a circle $|x| = \ell$ of small radius. Since $v^1$ is fixed and $f^{(1)}(x, V(x))$ is holomorphic in (10.2), by use of Cauchy's theorem, (10.6) is an immediate consequence of

\begin{equation}
(10.7) \quad \left| \int_{t_0}^{t_1} f^{(1)}(x, V(x)) \, dx \right| \to 0 \text{ as } \ell \to 0.
\end{equation}

Here the path of integration is taken along the circular arc $|x| = \ell$ in (10.2). However, from the construction of $\Gamma_{x_0}$ and $\Gamma_{x_1}$ we know that $\Re \Omega(x) > 0$ for $x$ on $\overline{t_0t_1}$. Thus, (10.7) follows and (10.6) is proved.

Now, let $V(x)$ be specifically denoted by $W(x)$; namely $W(x_1, x_1, v^1) = v^1$. Let $\hat{V}^1 = W(x_1, x_1, v^1)$. Then,

\begin{equation}
\hat{\phi}^{(1)}(x_1, v^1) - \hat{\phi}^{(1)}(\hat{x}_1, v^1) = \left[ \hat{\phi}^{(1)}(x_1, v^1) - \hat{\phi}^{(1)}(\hat{x}_1, \hat{v}^1) \right] + \left[ \hat{\phi}^{(1)}(\hat{x}_1, \hat{v}^1) - \hat{\phi}^{(1)}(\hat{x}_1, v^1) \right]
\end{equation}

\begin{equation}
= \int_{0}^{x_1} f^{(1)}(x, W(x, x_1, v^1)) \, dx - \int_{0}^{\hat{x}_1} f^{(1)}(x, W(x, \hat{x}_1, \hat{v}^1)) \, dx
\end{equation}

\begin{equation}
+ \left[ \hat{\phi}^{(1)}(\hat{x}_1, \hat{v}^1) - \hat{\phi}^{(1)}(\hat{x}_1, v^1) \right].
\end{equation}

Here we use (10.6) and the paths of integration are
taken accordingly. Thus, the limit

\[
\lim_{x_1 \to x_1} \frac{\hat{\xi}^{(1)}(x_1, v^1) - \hat{\xi}^{(1)}(\hat{x}_1, v^1)}{x_1 - x_1}
\]

exists, since we proved in the first place that

\[ \hat{\xi}_v^{(1)}(x_1, v^1) \]

is well defined. Therefore, \( \hat{\xi}^{(1)}(x_1, v^1) \)

is holomorphic with respect to \( x_1 \) for (10.2) when \( v^1 \)

is fixed.

By Hartog's theorem, \( \hat{\xi}^{(1)}(x_1, v^1) \)

is holomorphic in

\( (x_1, v^1) \)

for (6.1).

\section{11. The function \( \hat{\xi}^{(m)}(x, v) \).}

With \( K_N \)

defined in

(10.4), we have seen that \( \hat{\xi}^{(1)}(x_1, v^1) \)

is holomorphic in \( (x_1, v^1) \)

for (6.1) and satisfies

\[ |\hat{\xi}^{(1)}(x_1, v^1)| \leq \frac{K_N}{2} |x_1|^N e^{-\Re \Omega(x_1)}. \]

Let \( S(m) \)

denote the conjunction of the following

predicates (i) and (ii):

(i) \( \hat{\xi}^{(m)}(x_1, v^1) \)

is well defined and holomorphic

for \( (x_1, v^1) \)

in (6.1).

(ii) \( \hat{\xi}^{(m)}(x_1, v^1) \)

satisfies
We have seen that proposition $S(1)$ is true. Suppose that $S(m)$ is true for $m = 1, 2, \ldots, k$. We want to show that $S(k+1)$ is true.

First of all, the function $f(x, e^\Omega(x), x, V(x))$ is well defined and holomorphic in (10.2) by (5.3), (10.5) and (11.3). Thus, $\hat{\xi}^{(k+1)}(x_1, v_1)$ as given by (9.3) exists, by the same reason as that in 1o for $\hat{\xi}^{(1)}(x_1, v_1)$.

Since $f(x, \eta, v)$ satisfies a Lipschitz condition with respect to $\eta$ with Lipschitz constant $H$, by using (11.2) and (6.6) we have the inequality

$$|\hat{\xi}^{(k+1)}(x_1, v_1) - \hat{\xi}^{(k)}(x_1, v_1)| \leq$$

$$\frac{2}{|\lambda| \sin 2\sigma \epsilon} \cdot H \cdot \frac{K_N}{2^k} |x_1|^N e^{-\Re \Omega(x_1)} \leq$$

$$\frac{K_N}{2^{(k+1)}} |x_1|^N e^{-\Re \Omega(x_1)}.$$
Furthermore, by the use of (11.3.k) and (11.2.k+1) we have

\[ |\varphi^{(k+1)}(x_1, v^1)| \leq K_N \left( \frac{1}{2} + \ldots + \frac{1}{2}^{(k+1)} \right) |x_1|^N e^{-\text{Re } \Omega(x_1)}. \]

Now, by (11.3.k+1) and the same reason as that in 3.0, \( \varphi^{(k+1)}(x_1, v^1) \) is holomorphic in \((x_1, v^1)\) for (6.1). Thus, proposition \( S(k+1) \) is true.

Therefore, by means of mathematical induction, \( S(m) \) is true for all positive integers \( m \). Hence Step (I) is proved and we have inequalities (11.2.m) and (11.3.m).

§ 12. *Convergence of \{ \varphi^{(m)}(x, v) \}.* Since

\( \varphi^{(m)}(x_1, v^1) = \varphi^{(0)}(x_1, v^1) + \sum_{k=0}^{m-1} \left\{ \varphi^{(k+1)}(x_1, v^1) - \varphi^{(k)}(x_1, v^1) \right\}, \)

the sequence \( \{ \varphi^{(m)}(x_1, v^1) \} \) converges if and only if the series in the right hand side of (12.1) does. However, by (11.2.k), the series converges absolutely and uniformly in any compact subset of (6.1).

Since each term of the sequence \( \{ \varphi^{(m)}(x_1, v^1) \} \) is holomorphic in \((x_1, v^1)\) for (6.1), the limit, denoted by \( \varphi_N(x_1, v^1) \), is also holomorphic in \((x_1, v^1)\) for (6.1).

Moreover, due to (11.2.k) we have

\( |\varphi_N(x_1, v^1)| \leq K_N |x_1|^N e^{-\text{Re } \Omega(x_1)} \)
for $(x^1, v^1)$ in (6.1).

Thus, Step (II) is proved.

§ 13. Integral expression of $\phi_N(x, v)$. We shall prove that the limit function $\phi_N(x^1, v^1)$ satisfies integral equation (9.1). We want to show that, given $\varepsilon > 0$, there exists an integer $M(\varepsilon, x^1)$, depending on $\varepsilon$ and $x^1$, such that

\[
|\int_0^{x^1} x^{-\sigma-1} e^{-\Omega(x)} \left[ f(x, e^{\Omega(x)} \phi_N(x, V(x)), V(x)) \right. - \left. f(x, e^{\Omega(x)} \phi^{(m)}(x, V(x)), V(x)) \right] \, dx | < \varepsilon
\]

holds for $m \geq M(\varepsilon, x^1)$. From (11.3), we know satisfies

\[
|\phi^{(m)}(x^1, v^1)| \leq K_N |x^1|^N e^{-Re \Omega(x^1)}
\]

for $(x^1, v^1)$ in (6.1), independently of $m$.

Since $f(x, \eta, v)$ satisfies a Lipschitz condition in $\eta$ with Lipschitz constant $H$, the left-hand side of (13.1) is dominated by

\[
2 H K_N \int_{x^1} |x|^{N-\sigma-1} e^{-Re \Omega(x)} \, dx,
\]

independently of $m$. By the same reason as we have seen in (10.4), the integral (13.2) exists and, moreover, we can choose a point $x^0_1$ on $\Gamma^s$ independent of $m$, such that
(13.3) \[ \left| \int_{x_1^0}^{x_1^1} x^{-\sigma-1} e^{-\Omega(x)} \left[ f(x, e^{\Omega(x)} \phi_N(x, V(x)), V(x)) - f(x, e^{\Omega(x)} \phi_m(x, V(x)), V(x)) \right] \right| \; dx \leq \varepsilon/2 \]

holds. On the other hand, since the arc \( \Gamma_{x_1^1} \) from \( x_1^0 \) to \( x_1^1 \) has finite length, and \( \phi_m(x, V(x)) \) converges uniformly in any compact subset of \( \Omega \), we can choose a compact subset of \( \Omega \), containing the portion of \( \Gamma_{x_1^1} \) from \( x_1^0 \) to \( x_1^1 \), and an integer \( M(\varepsilon, x_1^1) \) such that

(13.4) \[ \left| \int_{x_1^0}^{x_1^1} x^{-\sigma-1} e^{-\Omega(x)} \left[ f(x, e^{\Omega(x)} \phi_N(x, V(x)), V(x)) - f(x, e^{\Omega(x)} \phi_m(x, V(x)), V(x)) \right] \right| \; dx \leq \varepsilon/2 \]

holds for \( m \geq M(\varepsilon, x_1^1) \).

By (13.3) and (13.4), (13.1) is proved. Thus \( \phi_N(x_1^1, V(x)) \) satisfies the integral equation (9.1) and Step (III) is proved.

§ 14. \( \phi_N(x, V(x)), V(x) \) as a solution of system (5.7).

We shall prove that \( \phi_N(x, V(x)), V(x) \) is a solution of (5.7) whenever \( (x, V(x)) \) belongs to \( \Omega \).

To prove this, rewrite the integral equation satisfied by \( \phi_N(x, V(x)) \) as

(14.1) \[ \phi_N(x_1^1, V(x)) = \int_{0}^{x_1^1} \psi(x, V(x)) \; dx \]
where
\[ \psi(x, v) = x^{-\sigma-1} e^{-\Omega(x)} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial v} \right) \varphi_N(x, v). \]

Also, rewrite \( V(x) = W(x, x_1, v^1) \). Then it is sufficient to prove that

(14.2) \[ \frac{d}{dx_0} \varphi_N(x_0, v^0) = \psi(x_0, v^0) \]

where \( v^0 = W(x_0, x_1, v^1) \).

Since \( W(x, x_0, v^0) = W(x, x_1, v^1) \), it follows from (14.1) that

\[ \varphi_N(x_0, v^0) = \int_0^{x_0} \psi(x, W(x, x_0, v^0)) \, dx. \]

Hence,

(14.3) \[ \frac{d}{dx_0} \varphi_N(x_0, v^0) = \psi(x_0, v^0) + \int_0^{x_0} \frac{\partial \psi(x, W)}{\partial W} \left\{ \frac{\partial W(x, x_0, v^0)}{\partial x_0} + \frac{\partial W(x, x_0, v^0)}{\partial v^0} \cdot \frac{\partial W(x_0, x_1, v^1)}{\partial x_0} \right\} \, dx. \]

However, for any constants \( \hat{\xi} \) and \( \hat{\eta} = W(\hat{\xi}, x, v) \) is an integral of the equation \( xz' = \mu z + bx^\mu \). Thus \( \hat{\eta} = W(\hat{\xi}, x_0, v^0) = W(\hat{\xi}, x_1, v^1) \), and

\[ \frac{\partial W(\hat{\xi}, x_0, v^0)}{\partial x_0} = 0. \]

That is, the expression in the braces of the integrand in (14.3) vanishes identically. Therefore, equation (14.2)
§ 15. **Uniqueness.** To complete the solution of Problem \( A \), it remains to prove that a solution of the first equation of system (5.7) satisfying (5.9) is unique.

Suppose that there are two solutions satisfying (5.9). Let \( P(x, V(x)) \) be the difference of these two solutions. Then, there exists a positive constant \( K \) such that

\[
|P(x_1, v^1)| \leq K |x_1|^N e^{-\Re \Omega(x_1)}
\]

for \( (x_1, v^1) \) in (6.1). Since \( f(x, \eta, v) \) satisfies a Lipschitz condition with respect to \( \eta \) with Lipschitz constant \( H \), we have

\[
|P(x_1, v^1)| \leq H K \int_{x_1}^{x} |x|^{N-\sigma-1} e^{-\Re \Omega(x)} |dx|
\]

\[
\leq \frac{2HK}{|\lambda| \sin 2\epsilon} |x_1|^N e^{-\Re \Omega(x_1)}
\]

\[
\leq \frac{K}{2} |x_1|^N e^{-\Re \Omega(x_1)}.
\]

Here we use (6.6) and (5.5). Repeating this process, we have, for any positive integer \( p \),

\[
|P(x_1, v^1)| \leq K/2^p |x_1|^N e^{-\Re \Omega(x_1)}
\]

for \( (x_1, v^1) \) in (6.1). Hence \( P(x_1, v^1) = 0 \) and this proves the uniqueness and Step IV.
V. PROOF OF THEOREM B

Since \( v = 0 \) is an interior point of the domain (2.1) in which \( \xi(x, v) \) is defined, by Cauchy's theorem \( \xi(x, V(x)) \) can be expanded into a uniformly convergent power series of \( V(x) \) whenever \( (x, V(x)) \) is in the domain (2.1). Hence, \( \xi(x, V(x)) \) can be written as

\[
\xi(x, V(x)) = \sum_{t=0}^{\infty} \xi_t(x) V(x)^t
\]

where

\[
\xi_t(x) = \frac{1}{2\pi i} \int_{|v|=\gamma} \frac{\xi(x, v)}{v^{t+1}} dv
\]

for any fixed \( \gamma < \delta^0, \gamma \neq 0 \).

From PROPOSITION 2, we have

\[
a_{k\ell} = \frac{1}{2\pi i} \int_{|v|=\gamma} \frac{P_k(v)}{v^{\ell+1}} dv.
\]

Hence,

\[
\xi_t(x) = \sum_{k=0}^{N-1} a_{k\ell} x^k = \frac{1}{2\pi i} \int_{|v|=\gamma} \frac{\xi(x, v)}{v^{\ell+1}} dv,
\]

and, by Theorem A,
\[ |\dot{\phi}_t(x) - \sum_{k=0}^{N-1} a_k t^k | \leq \frac{1}{2N} K_N \int_{|v| = \gamma} \frac{d\nu}{\nu^{t+1}} |x|^N \]

\[ \approx K_N |x|^N. \]

Therefore,

\[ \dot{\phi}_t(x) \sim \sum_{k=0}^{\infty} a_k t^k. \]
BIBLIOGRAPHY


