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Uniform Factorizations of Graphs

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UNIFORM FACTORIZATIONS OF GRAPHS

by

David Burns

A Dissertation
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Doctor of Philosophy

Western Michigan University
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David Burns
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CHAPTER I
PRELIMINARIES

In this initial chapter, we present some background on the problems we are about to investigate and outline the topics to be treated in each of the following chapters. We also define some of the terminology and notation that we will require in this dissertation.

Section 1.1
Introduction

Many problems in graph theory involve factorizations of graphs. Generalized Ramsey Theory, for example, treats the question: Given graphs $G_1, G_2, \ldots, G_k$ ($k \geq 2$), what is the least integer $p$ such that for any factorization $K_p = F_1 \oplus F_2 \oplus \cdots \oplus F_k$, the graph $G_i$ is a subgraph of $F_i$ for at least one $i = 1, 2, \ldots, k$.

Many problems dealt with in this thesis involve what might be termed 'uniform' factorizations of graphs, these being factorizations where each factor possesses some given graph theoretical property.

The first such common property to be considered is line-symmetry, and a study of line-symmetric graphs themselves is presented in Chapter II, extending the work begun by Foster [14] in 1932. Most of the early work...
with line-symmetric graphs also involved point-symmetric graphs. These are graphs where any vertex may be mapped to any other vertex by some automorphism. An unpublished manuscript by Dauber and Harary [10] investigated the relationship between line-symmetry and point-symmetry. They gave examples of graphs that are line-symmetric but not point-symmetric and vice versa. Each of their line-symmetric graphs which was not point-symmetric was not regular. This observation motivated Folkman [12] to investigate the extent to which line-symmetry and regularity imply point-symmetry. Folkman produced several infinite classes of line-symmetric graphs one of which was inspired by a single line-symmetric graph produced by Gray (see [15]) in 1932. Responding to Folkman's interests, Bouwer [4] produced a cubic line-symmetric graph which was not point-symmetric and, four years later, several additional results [5] on regular line-symmetric graphs which are not point-symmetric. This latter work by Bouwer involved the theory of configurations (see [11]) and their associated Levi graphs (see [9]).

Our work in Chapter II starts with an attempt to organize line-symmetric graphs by categorizing them into three essentially different types. Following this, we produce several non-structural characterizations of line-symmetric graphs. For the first such characteri-
zation, we define a graph $G$ to be uniquely edge extendible if the graphs $G + e_1$ and $G + e_2$ are isomorphic for any pair of edges $e_1$ and $e_2$ belonging to the complement of $G$. We then prove that a graph is line-symmetric if and only if its complement is uniquely edge extendible. It follows that if a graph $G$ is called uniquely edge retractible when the graphs $G - e_1$ and $G - e_2$ are isomorphic for any pair of edges $e_1$ and $e_2$ belonging to $G$, then a graph is line-symmetric if and only if it is uniquely edge retractible. The vertex analog of this result is then established; we introduce the notion of a uniquely vertex retractible graph and prove that a graph is point-symmetric if and only if it is uniquely vertex retractible.

The line graph function is then introduced and it is shown that, with some restrictions, a graph is line-symmetric if and only if its line graph is point-symmetric.

Several structural characterizations providing complete lists of line-symmetric graphs that fall into particular categories are then presented, where the categories are specified by various connectivity properties of either the graphs themselves or their complements.
Next, we term a graph biregular if the set of degrees of its vertices has cardinality 2 and recall an early result stating that every line-symmetric graph which contains no isolated vertices is either regular or biregular. Working towards characterizations of connected biregular line-symmetric graphs, we introduce the notion of a neighborhood symmetric graph; a graph $G$ is neighborhood symmetric if for any two pairs $(u_1, v_1)$ and $(u_2, v_2)$ of adjacent vertices of $G$ where $u_1$ and $u_2$ are of the same degree and $v_1$ and $v_2$ are of the same degree, there exists an automorphism $\alpha$ of $G$ such that $\alpha(N(u_1)) = N(u_2)$ and $\alpha(N(v_1)) = N(v_2)$ where $N(w)$ denotes the set of vertices of $G$ which are adjacent with $w$. We then prove that if $G$ is a connected graph and if $a$ and $b$ are unequal integers which are realized as the degrees of vertices of $G$, then $G$ is biregular and line-symmetric if and only if $G$ is neighborhood symmetric and has a bipartition with one partite set consisting of all the vertices of degree $a$ and the other partite set consisting of all the vertices of degree $b$. Examples are provided to show that this theorem is best possible.

For a graph containing vertices of degree $a$ and $b$ we denote by $V_a$ (respectively $V_b$) the set of all
vertices in the graph of degree \( a \) (respectively \( b \)). We define an equivalence relation on \( V_a \cup V_b \) calling two vertices \( u \) and \( v \) equivalent if \( N(u) = N(v) \). It is immediate that if two vertices are equivalent then they are of the same degree. Thus the equivalence classes so formed are subsets of either \( V_a \) or \( V_b \). We denote these classes by \( V_a(1), V_a(2), \ldots, V_a(n_a), V_b(1), V_b(2), \ldots, V_b(n_b) \) and prove that if \( G \) is a biregular line-symmetric graph with vertex set \( V_a \cup V_b \) then there exist positive integers \( c \) and \( d \) with 
\[
|V_a(i)| = c \quad \text{and} \quad |V_b(j)| = d \quad \text{for} \quad i = 1, 2, \ldots, n_a \quad \text{and} \quad j = 1, 2, \ldots, n_b.
\]

A second characterization of connected biregular line-symmetric graphs is given using the following definition. A transitive bipartition of a bipartite graph \( G \) is a bipartition of \( G \) where \( \mathcal{G}(G) \) acts transitively on both of the partite sets. It is shown that if \( G \) is a connected graph which is not regular then \( G \) is line-symmetric and biregular if and only if \( G \) has a transitive bipartition and is neighborhood symmetric.

In Chapter III we discuss problems related to line-symmetric graphs. In the first section of this chapter we present three generalizations of line-symmetric graphs. In each case we provide a corresponding general-
ization of the notion of a uniquely edge extendible graph, establish a theorem analogous to the result that a graph is line-symmetric if and only if its complement is uniquely edge extendible, and structurally characterize the graphs so defined. In the second section of Chapter III we investigate line-symmetric graphs which have line-symmetric complements. In view of an earlier theorem, such graphs are equivalently described as being both line-symmetric and uniquely edge extendible (LSUEE). We provide a structural characterization of all LSUEE graphs containing isolated vertices and then, using the three part categorization theorem presented early in Chapter II, we establish the following results: a characterization of LSUEE graphs which are biregular and bipartite, a characterization of LSUEE graphs that are regular and bipartite, and several partial results concerning LSUEE graphs that belong to the third and final category, namely LSUEE graphs that are regular and point-symmetric but not bipartite. We also characterize those LSUEE graphs that are disconnected.

In Chapter IV we introduce two new problems involving what might be called 'almost uniform' factorizations of graphs. For an integer \( n \) and a graph \( G \) satisfying \( 2 \leq n \leq \chi(G) \), this latter symbol denoting the chromatic number of \( G \), we define the
n-minimal chromatic multiplicity (respectively n-maximal chromatic multiplicity) of $G$, denoted $m(x, \chi, G)$ (respectively $M(n, \chi, G)$) as the least (respectively greatest) positive integer $k$ such that $G$ can be expressed as the edge sum of edge-disjoint spanning subgraphs $G_1, G_2, \ldots, G_k, R$ where $\chi(G_i) = n$ for $i = 1, 2, \ldots, k$ and $\chi(R) < n$.

We begin our work with $m(n, \chi, G)$ by borrowing a definition from Harary, Hsu, and Miller [21]. For an integer $n \geq 2$, the $n$-particity of a graph $G$, denoted $\beta_n(G)$, is the minimum number of spanning subgraphs in a partition of $E(G)$ into $n$-partite graphs. We will say that a graph $G$ has lower $n$-particity if there exists a collection of $\beta_n(G)$ edge-disjoint $n$-partite spanning subgraphs of $G$ whose edge sum is $G$, not all of which have chromatic number $n$.

Using these ideas we prove that for an integer $n$ and a graph $G$ satisfying $2 \leq n \leq \chi(G)$ the parameter $m(n, \chi, G) = \{\log_n \chi(G)\} - 1$ if $G$ has lower $n$-particity and $m(n, \chi, G) = \{\log_n \chi(G)\}$ otherwise.

As a corollary of this result we establish an equality stated but not proved by Harary, Hsu, and Miller [21]: for an integer $n$ and a graph $G$ satisfying $2 \leq n \leq \chi(G)$, $\beta_n(G) = \{\log_n \chi(G)\}$.
The second section of Chapter IV is devoted to the parameter $M(n, \chi, G)$. An explicit result determining $M(n, \chi, G)$ for every graph $G$ and every integer $n$ satisfying $2 \leq n \leq \chi(G)$ has not been obtained but we have been able to derive several results computing this parameter for particular classes of graphs and results providing inequalities and other information about it. We show that if $G$ is a disconnected graph with components $G_1, G_2, \ldots, G_k$ and $n$ is an integer satisfying $2 \leq n \leq \min\{\chi(C_i) | 1 \leq i \leq k\}$, then

$$M(n, \chi, G) = \sum_{i=1}^{k} M(n, \chi, G_i).$$

We also prove that if $G$ is a graph with blocks $B_1, B_2, \ldots, B_k$, then for every possible value of $n$, the equality $M(n, \chi, G) = \sum_{i=1}^{k} M(n, \chi, B_i)$ holds. With the aid of a theorem borrowed from the theory of balanced incomplete block designs [17] we determine $M(n, \chi, K_p)$ for those values of $n$ and $p$ where $n$ divides $p$ and $n - 1$ divides $p - 1$. For $n = 3$ we find $M(n, \chi, K_p)$ for all $p \geq 3$. The case of $n = 2$ is covered by the more general observation that if $G$ is any nonempty graph then $M(2, \chi, G) = |E(G)|$. Other observations about the numbers $M(n, \chi, K_p)$ are reported.
Our results in Chapter V were motivated by recent work of Harary, Robinson, and Wormald [23]. They considered the question of the existence of uniform factorizations of $K_p$, where all of the factors are isomorphic. In particular, they proved that there exists a $(p, q)$ graph $G$ such that $K_p = G_1 \oplus G_2 \oplus \cdots \oplus G_t$ with $G \cong G_i$ for $i = 1, 2, \ldots, t$ if and only if $q$ divides $\binom{p}{2}$. We view this result as providing a necessary and sufficient condition for the existence of a graph $G$ where $\overline{G}$ is the edge sum of $t - 1$ subgraphs each isomorphic with $G$. In Chapter V we work with the related question: which graphs appear as subgraphs of their complements at least once? We show that every $(p, p - 2)$ graph, with $p \geq 2$, is isomorphic with a subgraph of its complement, and we characterize those $(p, p - 1)$ graphs which have this property.

Section 1.2

Definitions and Notation

In this section we give several definitions in graph theory that we require throughout this discussion. Additional more specialized definitions will be given later as needed. The basic terminology not defined here follows that of Behzad, Chartrand, and
Lesniak-Foster [2] or of Harary [19].

Two (nonempty) graphs $G_1$ and $G_2$ are isomorphic (respectively edge-isomorphic) if there exists a one-to-one mapping $\phi$ from $V(G_1)$ onto $V(G_2)$ ($E(G_1)$ onto $E(G_2)$) such that $\phi$ preserves adjacency and nonadjacency of vertices (edges). If $H_1$ and $H_2$ are nonempty graphs and $\phi: V(H_1) \rightarrow V(H_2)$ is an isomorphism then $\hat{\phi}: E(H_1) \rightarrow E(H_2)$ defined by $\hat{\phi}(uv) = \phi(u)\phi(v)$ is an edge-isomorphism. An automorphism (respectively edge-automorphism) of a (nonempty) graph $G$ is an isomorphism (edge-isomorphism) of $G$ with itself.

The automorphism group of a graph $G$, denoted $\text{Aut}(G)$ (respectively edge-automorphism group of a nonempty graph $G$, denoted $\text{Aut}_e(G)$) is the set of all automorphisms (edge-automorphisms) of $G$ under the operation of composition. A particular (and possibly proper) subgroup of $\text{Aut}(G)$ called the induced edge-automorphism group and denoted $\text{Aut}_e^*(G)$ will be of interest. $\text{Aut}_e^*(G) = \{\phi \in \text{Aut}_e(G) | \phi \in \text{Aut}(G)\}$.

Let $G$ and $H$ be graphs such that $E(G) \cap E(H) = \emptyset$. If $V(G) \cap V(H) = \emptyset$ then the union of $G$ and $H$, denoted $G \cup H$, is that graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $V(G) = V(H)$ then the edge sum of $G$ and $H$, denoted $G \oplus H$, has vertex set $V(G)$ and edge set $E(G) \cup E(H)$. If $n$
is a positive integer then \( nG \) denotes the union of \( n \) graphs each isomorphic with \( G \). The join of disjoint graphs \( G \) and \( H \), denoted \( G + H \), is defined as follows: \( V(G + H) = V(G) \cup V(H) \) and \( E(G + H) = E(G) \cup E(H) \cup X \) where \( X = \{uv \mid u \in V(G) \text{ and } v \in V(H)\} \). The wheel of order \( n + 1 \), denoted \( W_n \), is the graph \( C_n + K_1 \). The cartesian product \( G \times H \) of disjoint graphs \( G \) and \( H \) has vertex set \( V(G) \times V(H) \) and two vertices \( u_1u_2 \) and \( v_1v_2 \) are adjacent in \( G \times H \) if and only if either \( u_1 = v_1 \) and \( u_2v_2 \in E(H) \) or \( u_2 = v_2 \) and \( u_1v_1 \in E(G) \).
CHAPTER II

LINE-SYMMETRIC GRAPHS

In this chapter we investigate line-symmetric graphs both as objects worthy of study in their own right and to produce theorems for later use when we consider uniform factorizations of complete graphs where every factor is line-symmetric. A graph \( G \) is line-symmetric if for any two edges \( e \) and \( f \) of \( G \) there exists \( \alpha \in G^*(G) \) such that \( \alpha(e) = f \). For continuity of terminology we have chosen to maintain the name 'line-symmetric' as used by other workers in this area over the other possibility 'edge-symmetric' which would perhaps be more consistent with our other name choices.

Line-symmetric graphs have been studied by Bouwer [4][5], Duber and Harary [10], Folkman [12], and Foster [14][15]. We observe that for all positive integers \( m \) and \( n \), the graphs \( nK_m \), \( K(m,n) \), and \( K_m(n) \) are line-symmetric, as are all cycles, together with a number of other well-known graphs such as the Heawood and Petersen graphs and the graphs of the five regular polyhedra.
Section 2.1

Non-structural Characterizations of Line-Symmetric Graphs

As the above examples indicate, there is great diversity among line-symmetric graphs. We attempt to organize these graphs by presenting some results which categorize them into three essentially different types. As a further aid in beginning to understand line-symmetric graphs we present in this section several non-structural characterizations of them both in terms of classical concepts from graph theory and in terms of notions introduced here for the first time.

To facilitate the upcoming three part categorization theorem it is convenient to adopt some new terminology. A graph will be called biregular if its degree set has cardinality two. A transitive bipartition of a bipartite graph $G$ is bipartition of $G$ such that for any two vertices $u$ and $v$ of $G$ belonging to the same partite set there exists $\theta \in \mathcal{G}(G)$ such that $\theta(u) = v$. The vertex analog of line-symmetry is called point-symmetry; a graph $G$ is point-symmetric if for any two vertices $u$ and $v$ of $G$ there exists $\pi \in \mathcal{G}(G)$ such that $\pi(u) = v$.

The first two results presented are attributed by
Folkman [12] to an unpublished manuscript of Dauber and Harary [10], and they are attributed by Harary [19] to Dauber alone. We provide their proofs for the sake of completeness.

**Theorem 2-1.** Let $G$ be a line-symmetric graph which has no isolated vertices.

(i) If $G$ is bipartite, then $G$ has a transitive bipartition.

(ii) If $G$ is not bipartite, then $G$ is point-symmetric.

**Proof.** Let $u_1$ and $u_2$ be adjacent vertices of $G$ and let $V_i = \{ \pi(u_i) | \pi \in \phi(G) \}$ for $i = 1, 2$. Then of course, $\phi(G)$ acts transitively on each of the sets $V_1$ and $V_2$ (meaning that for $i = 1$ or $i = 2$ any pair of vertices $u$ and $v$ belonging to $V_i$ satisfies $\beta(u) = v$ for some $\beta \in \phi(G)$). We consider the following two cases.

**Case 1.** Assume that $V_1$ and $V_2$ are disjoint. Let $e_0 = u_1u_2$, and let $e = uv$ be an arbitrary edge of $G$. Then there exists an automorphism $\alpha$ of $G$ such that $\hat{\alpha}(e_0) = e$. Hence $\{\alpha(u_1), \alpha(u_2)\} = \{u, v\}$, so $e$ is incident with one vertex in each of the sets $V_1$ and $V_2$. Moreover for an arbitrary vertex $z$ of $G$, $z$ is not isolated so there is an edge $f$ of $G$ which

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is incident with \( z \). Then there exists an automorphism \( \pi \) of \( G \) such that \( \pi(e_0) = f \), so that

\[ z \in \{\pi(u_1), \pi(u_2)\} \subset V_1 \cup V_2 \]. Thus \( V(G) = V_1 \cup V_2 \) is a transitive bipartition of \( G \).

**Case 2.** Assume that \( V_1 \) and \( V_2 \) are not disjoint. It is readily verified that \( V_1 = V_2 = V(G) \) so that \( G(G) \) acts transitively on \( V(G) \). Thus \( G \) is point-symmetric; and in the event that \( G \) is also bipartite, every bipartition of \( G \) is transitive because \( G \) is point-symmetric.

We note from Case 2 of the preceding proof that if \( G \) is a line-symmetric graph it is possible for \( G \) to be both bipartite and point-symmetric. Indeed all cycles of even order are of this type. Regularity or lack thereof provides an alternative scheme for categorizing line-symmetric graphs.

**Theorem 2-2.** If \( G \) is a line-symmetric graph which has no isolated vertices then either \( G \) is regular or \( G \) is biregular.

**Proof.** Since \( G \) has no isolated vertices, \( G \) must be nonempty. Let \( e \) be an edge of \( G \) and let \( \{d_1, d_2\} \) be the degrees of the two vertices of \( G \) incident with \( e \). Now let \( u \) be any vertex of \( G \). There must exist...
some edge \( f \) of \( G \) incident with \( u \). We know that there exists \( \pi \in \text{G}^*(\text{G}) \) with \( \pi(f) = e \). Hence the degrees of the two vertices of \( G \) incident with \( f \) must be \( \{d_1, d_2\} \). Therefore \( \deg u \in \{d_1, d_2\} \). Hence \( G \) is regular or biregular according as \( d_1 = d_2 \) or \( d_1 \neq d_2 \).

The common hypothesis in Theorems 2-1 and 2-2 that \( G \) have no isolated vertices is not a significant hindrance since if \( G \) and \( H \) are graphs with \( G = H \cup K_1 \) then \( G \) is line-symmetric if and only if \( H \) is line-symmetric. In short, isolated vertices do not effect the line-symmetry of a graph; they only complicate the description of its structure.

The two classification criteria of the preceding theorems may be combined to produce a three-way categorization of line-symmetric graphs.

Theorem 2-3. If \( G \) is a line-symmetric graph which has no isolated vertices then \( G \) satisfies exactly one of the following conditions.

(i) \( G \) is biregular and bipartite, in which case \( G \) has a unique transitive bipartition, namely, the unique partition of \( V(G) \) into two subsets so that the vertices of equal degree are in the same subset.
(ii) \( G \) is regular and bipartite, in which case \( G \) might or might not be point-symmetric.

(iii) \( G \) is regular but not bipartite, in which case \( G \) must be point-symmetric.

**Proof.** By Theorem 2-2 we know that \( G \) is either regular or biregular. If \( G \) is biregular then, by Theorem 2-1, \( G \) must be bipartite since \( G \) cannot be point-symmetric. For any transitive bipartition of \( G \), vertices in the same partite set must have the same degree. Since two different degrees occur in \( G \) there can only be one such partition of \( V(G) \). Thus if \( G \) is biregular it satisfies condition (i) but neither (ii) nor (iii).

If \( G \) is regular then either \( G \) is bipartite or it is not. If \( G \) is bipartite it may be point-symmetric, as in the case of even cycles, or it may fail to be point-symmetric as in the case of the graphs constructed by Folkman [12]. If \( G \) is regular and not bipartite then, by Theorem 2-1, \( G \) must be point-symmetric. Thus regular line-symmetric graphs without isolated vertices satisfy exactly one of the conditions (ii) and (iii).

We now work towards presenting several non-structural characterizations of line-symmetric graphs. The
graph $G$ in Figure 2-1 is easily seen to be edge-re-
constructible since the graph $H$ of Figure 2-1 can be
obtained from $G$ by the removal of the edge $e$ and
$H + e_1 \cong H + e_2 \cong G$ for any two edges $e_1$ and $e_2$ in $E(H)$. We focus our attention on the graph $H$ and in-
troduce terminology to describe those graphs $J$ with
the property that $J + e_1$ is isomorphic with $J + e_2$
for any two edges $e_1$ and $e_2$ in $E(J)$. A graph $G$
will be called an edge extension of a graph $H$ if
there exists an edge $e \in E(G)$ such that $G - e \cong H$.

Next, a graph $J$ will be termed uniquely edge extend-
ible (UEE) if all edge extensions of $J$ are
isomorphic. The graphs $W_4$, $W_5$, $C_4$, $C_5$, $K_n \cup K_m$,
$nK_m$, $K(l,n)$ as well as the graph $G$ of Figure 2-1
are UEE. A graph of order $p$ will be called irre-
ducible if $p = 1$ or the graph contains no vertices of
degree $p - 1$. The following lemma will help establish
the connection between line-symmetric graphs and UEE
graphs.
Lemma 2-4. If G is an irreducible UEE graph then exactly one of the following is true.

(i) G is trivial.

(ii) G is regular and not complete.

(iii) G is biregular and \( \deg_G^x \neq \deg_G^y \) for every edge \( xy \in E(G) \).

Proof. Let G be an irreducible UEE graph. Then, if G is complete, G must be trivial. Assume that G is not complete. Then there exist vertices u and v in G such that \( uv \in E(G) \). We consider two cases.

Case 1. Assume that \( \deg_u^G = \deg_v^G = n \). In this case the integer \( n + 1 \) occurs as a degree in \( G + uv \) twice more than it does in G. Now let x be an arbitrary vertex of G. Since G is irreducible and nontrivial, there exists a vertex w of G such that \( wx \in E(G) \). Then \( G + uv \cong G + wx \), the number \( n + 1 \) occurs as a degree twice more in \( G + wx \) than it does in G. It follows that \( \deg_x^G = n \). Thus G is regular and not complete.

Case 2. Assume that \( m = \deg_u^G < \deg_v^G = n \).

In this case each of the integers \( m + 1 \) and \( n + 1 \) occur as a degree once more in \( G + uv \) than they do in G. Let x be an arbitrary vertex of G.
As in Case 1, there exists a vertex $w$ in $G$ such that $wx \in E(G)$ and, by the same reasoning as in Case I, $\deg_G x$ must be either $m$ or $n$ and $\deg_G w \neq \deg_G x$.

**Theorem 2-5.** Let $G$ be a graph having no isolated vertices. Then $G$ is line-symmetric if and only if $\overline{G}$ is UEE.

**Proof.** First let $G$ be a line-symmetric graph having no isolated vertices. Let $e_1$ and $e_2$ be edges of $G$. We know that there exists an automorphism $\pi$ of $G$ whose induced edge automorphism $\hat{\pi}$ maps $e_1$ to $e_2$. Since $G(G) = G(\overline{G})$, the map $\pi$ is also an automorphism of $\overline{G}$. It follows that $\pi$ is an isomorphism of $\overline{G} + e_1$ with $\overline{G} + e_2$ and thus $\overline{G}$ is UEE.

Conversely assume that $G$ is a graph having no isolated vertices whose complement is UEE. It is immediate that $\overline{G}$ is nontrivial and that $\overline{G}$ contains no vertices of degree $|V(\overline{G})| - 1$. Thus $\overline{G}$ is irreducible and, by Lemma 2-4, is either regular or biregular.

**Case 1.** Assume that $\overline{G}$ is regular of degree $n$. Let $e_1$ and $e_2$ be edges of $G$ where $e_i = u_iv_i$ for $i = 1, 2$. By assumption there exists an isomorphism
\[ \pi: V(\overline{G} + e_1^+) \rightarrow V(\overline{G} + e_2^+) \]. For \( i = 1, 2 \), note that \( u_i \) and \( v_i \) are the only vertices of \( \overline{G} + e_i^+ \) of degree \( n + 1 \). Therefore \( \pi \) maps \( \{u_1^+, v_1^+\} \) onto \( \{u_2^+, v_2^+\} \). Thus \( \pi \) is an automorphism of \( \overline{G} \) and hence, also of \( G \), and \( \pi \) induces the edge automorphism \( \hat{\pi} \in \text{ Aut}(G) \) where \( \hat{\pi}(e_1^+) = e_2^+ \). Thus \( G \) is line-symmetric.

**Case 2.** Assume that \( \overline{G} \) is biregular with \( \Delta \overline{G} = \{a, b\} \) and \( b > a \).

Let \( e_1 \) and \( e_2 \) be edges of \( G \) where \( e_i = u_i^+v_i^+ \) for \( i = 1, 2 \). Since \( \overline{G} \) is UEE there exists an isomorphism \( \pi: V(\overline{G} + e_1) \rightarrow V(\overline{G} + e_2) \). Let \( V_a = \{v \in V(\overline{G}) | \deg_{\overline{G}}v = a\} \) and \( V_b = \{v \in V(\overline{G}) | \deg_{\overline{G}}v = b\} \). Since \( \overline{G} \) is irreducible and nontrivial, none of its vertices are of degree \( |V(\overline{G})| - 1 \). By Lemma 2-4 (iii), \( <V_a> \) and \( <V_b> \) are complete subgraphs of \( \overline{G} \) and we may assume that \( u_i \in V_a \) and \( v_i \in V_b \) for \( i = 1, 2 \). We now consider two subcases to complete the proof.

**Subcase 2A.** Assume that \( b > a + 1 \). Here, for \( i = 1, 2 \), \( u_i \) is the only vertex of \( \overline{G} + e_i \) of degree \( a + 1 \) and \( v_i \) is the only vertex of \( \overline{G} + e_i \) of degree \( b + 1 \). Hence \( \pi \) maps the set \( \{u_1^+, v_1^+\} \) onto the set \( \{u_2^+, v_2^+\} \) and the same reasoning as in
Case 1 may be employed to show that $G$ is line-symmetric.

**Subcase 2B.** Assume that $b = a + 1$.

In this situation, for $i = 1, 2$, let $H_i = \overline{G} + e_i$ and note that $v_i$ is the only vertex of $H_i$ with degree $a + 2$. Hence $\pi(v_i) = v_2$. Let $\pi(u_1) = w \in V(\overline{G} + e_2)$. Then $w \in (V_b \cup \{u_2\}) - \{v_2\}$. Now $u_1$ is adjacent to every vertex of degree $a$ in $H_i$ because $\langle V_a \rangle$ is a complete subgraph of $\overline{G}$; hence $w$ must be adjacent to every vertex in the set $V_a - \{u_2\}$ in the graph $H_2$.

![Diagram](image)

$H_1 = \overline{G} + e_1$

$H_2 = \overline{G} + e_2$

Figure 2-2
If \( w \neq u_2 \), then \( w \in V_b - \{v_2\} \). Since \( \bar{G} \) is irreducible and \( \langle V_b \rangle \) is a complete subgraph of \( \bar{G} \), \( \deg_{\bar{G}} w = b = p - 2 \), where \( p \) is the order of \( \bar{G} \). Thus \( a = p - 3 \) and it follows that \( G = mK(1, 2) \) where \( m = p/3 \), a line-symmetric graph.

The only remaining possibility is that \( w = u_2 \). In this case \( \pi \) maps \( \{u_1, v_1\} \) onto \( \{u_2, v_2\} \), and the same reasoning as in Case 1 may be employed to conclude that \( G \) is line-symmetric.

Let \( G \) and \( H \) be graphs with \( G = H \cup K_1 \). It is straightforward to verify that \( G \) is line-symmetric if and only if \( H \) is line-symmetric and that \( \bar{G} \) is UEE if and only if \( \bar{H} \) is UEE. These observations, together with Theorem 2-5, yield the following more general result.

**Corollary 2-6.** A graph is line-symmetric if and only if its complement is UEE.

We have defined the concept of an edge extension of a graph, and have called a graph uniquely edge extendible if all its edge extensions are isomorphic. It seems natural to introduce the notion of an edge retraction of a graph, and to consider the collection of graphs which might be termed uniquely edge retractilbe. A graph \( G \) is an edge retraction of a
graph $H$ if there exists an edge $e$ of $H$ such that $G \cong H - e$. We will call a graph $J$ uniquely edge retractable (UER) if all edge retractions of $J$ are isomorphic. Line-symmetric graphs and UER graphs are indistinguishable as we now show.

**Theorem 2-7.** A graph is line-symmetric if and only if it is UER.

**Proof.** Let $G$ be a graph. Using Corollary 2-6, $G$ is line-symmetric if and only if $\overline{G}$ is UEE which is, in turn, equivalent to the assertion that $\overline{G} + e_1 \cong \overline{G} + e_2$ for every pair of edges $e_1$ and $e_2$ of $G$. Now $\overline{G} + e_i = \overline{G - e_i}$ for $i = 1, 2$ so that the assertion of the previous sentence is equivalent to the statement $\overline{G - e_1} \cong \overline{G - e_2}$ for every pair of edges $e_1$ and $e_2$ of $G$. This statement holds if and only if $G - e_1 \cong G - e_2$ for every pair of edges $e_1$ and $e_2$ of $G$ which is the definition of a UER graph.

In Harary [19, p. 171] a graph is called symmetric if it is both line-symmetric and point-symmetric. Because of the preceding theorem it seems likely that the vertex analog of the concept of a UER graph would be related to the notion of point-symmetry. This con-
nection is made in our next result with the help of the following definitions. A graph $G$ is a vertex re-
traction of a graph $H$ if there exists a vertex $v$ of $H$ such that $G \cong H - v$. A graph $J$ is called uniquely vertex retractable (UVR) if all vertex re-
tractions of $J$ are isomorphic. Sergio Ruiz, in a private communication, made suggestions incorporated in the following proof.

**Theorem 2-8.** A graph is point-symmetric if and only if it is UVR.

**Proof.** Let $G$ be a point-symmetric graph containing vertices $u$ and $v$. If $\alpha$ is an automorphism of $G$ and if we denote the restriction of $\alpha$ to $V(G) - \{u\}$ by $\alpha'$, then $\alpha' : V(G - u) \rightarrow V(G - \alpha(u))$ is an isomorphism. Since $G$ is point-symmetric there exists an automorphism $\beta$ of $G$ with $\beta(u) = v$. Thus $\beta' : V(G - u) \rightarrow V(G - \beta(u)) = V(G - v)$ is an isomorphism and $G$ is UVR.

Conversely, let $G$ be a UVR graph. We first show that $G$ is regular. Let $v_1$ and $v_2$ be vertices of $G$. Let $q_i = |E(G - v_i)|$ for $i = 1, 2$. Because $G$ is UVR, the graphs $G - v_1$ and $G - v_2$ are isomorphic and therefore $q_1 = q_2$. Now $\deg_{G} v_1 = |E(G)| - q_1 = |E(G)| - q_2 = \deg_{G} v_2$. Thus $G$ is
regular, say of degree $r$.

Next let $v$ and $w$ be vertices of $G$. We seek an automorphism of $G$ that maps $v$ to $w$. Because $G$ is UVR, there exists an isomorphism $\phi$ from the graph $G - v$ to the graph $G - w$. Since $G$ is $r$-regular, the set of vertices of $G - v$ with degree different from $r$ is $N(v)$, the neighborhood of $v$. Similarly, the set of vertices of $G - w$ of degree different from $r$ is $N(w)$. Because isomorphism of graphs preserves the degrees of vertices, it follows that $\phi(N(v)) = N(w)$. Define a map

$$\pi: V(G) \rightarrow V(G)$$

by

$$\pi(x) = \begin{cases} 
\phi(x) & \text{if } x \neq v \\
w & \text{if } x = v.
\end{cases}$$

It is straightforward to verify that $\pi$ is the desired automorphism of $G$, completing the proof.

Defining a graph to be uniquely retractable (UR) if it is both UVR and UER allows us to characterize symmetric graphs using the previous two theorems.

**Corollary 2-9.** A graph is symmetric if and only if it is UR.
We next examine the relationship between line-symmetry in graphs and the line graph function, probably the best known of the graph-valued functions. The line graph \( L(G) \) of a nonempty graph \( G \) is that graph whose vertex set is in one-to-one correspondence with the edges of \( G \), and where two vertices of \( L(G) \) are adjacent if and only if the corresponding edges of \( G \) are adjacent. Whitney [34] was the first to determine to what extent the isomorphism of the line graphs of two graphs implies the isomorphism of the graphs.

**Whitney's Theorem.** Let \( G_1 \) and \( G_2 \) be nontrivial connected graphs. Then \( L(G_1) \cong L(G_2) \) if and only if \( G_1 \cong G_2 \), or one of \( G_1 \) and \( G_2 \) is the graph \( K_3 \) and the other is \( K(1,3) \).

Information about the edges of a graph can often be equivalently interpreted as information about the vertices of its line graph. Because the result we are working toward addresses disconnected line-symmetric graphs the following result concerning disconnected point-symmetric graphs will be useful.

**Lemma 2-10.** Let \( G \) be a disconnected graph. Then \( G \) is point-symmetric if and only if there is a connected point-symmetric graph \( H \) and an integer \( t \geq 2 \) such
that \( G = tH \).

**Proof.** It is readily verified that if \( G \cong tH \) with \( t \geq 2 \) and \( H \) is point-symmetric, then \( G \) is a disconnected point-symmetric graph. Conversely suppose that \( G \) is a disconnected point-symmetric graph. Let \( G_1, G_2, \ldots, G_t, t \geq 2, \) be the components of \( G \). Let \( H = G_i \) and let \( u \in V(H) \) and \( v \in V(G_i) \) for some \( i \) with \( 2 \leq i \leq t \). Then there exists an automorphism \( \pi \) of \( G \) such that \( \pi(v) = u \). It follows that \( \pi(V(G_i)) = V(H) \) and \( \pi \) restricted to \( V(G_i) \) is an isomorphism of \( G_i \) with \( H \). Thus \( G_j \cong H \) for every \( 2 \leq j \leq t \) and \( G \cong tH \). \( \blacksquare \)

Using these tools we may present our main result.

**Theorem 2-11.** If \( G \) is a nonempty graph which does not contain both \( K_3 \) and \( K(1,3) \) as components then \( G \) is line-symmetric if and only if \( L(G) \) is UVR.

**Proof.** Let \( G \) be a nonempty line-symmetric graph which does not contain both \( K_3 \) and \( K(1,3) \) as components. By Theorem 2-7 we know that \( G \) is UER. Let \( v_1 \) and \( v_2 \) be vertices of \( L(G) \) which correspond to the edges \( e_1 \) and \( e_2 \) respectively in \( G \). The isomorphism \( G - e_1 \cong G - e_2 \) immediately yields
L(G - e₁) ≅ L(G - e₂). It is straightforward [2, p.198] to establish that \( L(G - e_i) = L(G) - v_i \) for \( i = 1, 2 \). Hence \( L(G) - v_1 \cong L(G) - v_2 \). Because \( v_1 \) and \( v_2 \) were arbitrarily chosen in \( V(L(G)) \), the line graph \( L(G) \) is UVR.

Conversely assume that \( G \) is a nonempty graph which does not contain both \( K_3 \) and \( K(1, 3) \) as components and whose line graph is UVR. By Theorem 2-8 we conclude that \( L(G) \) is point-symmetric. We consider two cases according as \( G \) is connected or not.

**Case 1.** Assume that \( G \) is connected.

We note that \( L(C_4 + e) \cong K_1 + C_4 \) is not point-symmetric and that \( L(K(1, 3) + e) \cong C_4 + f \) is not point-symmetric. Thus \( G \) is neither of the graphs \( C_4 + e \), \( K(1, 3) + e \). If \( G \) is \( K_4 \) then \( G \) is line-symmetric and we are done, so we assume in addition that \( G \) is not isomorphic to \( K_4 \). These observations and assumptions allow us to conclude that \( G_1(G) = G^*(G) \) by a well known theorem [2, p. 179]. We note in addition that \( G_1(G) \cong G(L(G)) \), a relationship that holds for any nonempty graph. Now let \( e \) and \( f \) be arbitrary edges of \( G \) and let \( v \) and \( w \) be the corresponding vertices of \( L(G) \). Because \( L(G) \) is
point-symmetric, there exists $\alpha \in \mathcal{G}(L(G))$ such that $\alpha(v) = w$. Let $\beta \in \mathcal{G}_1(G)$ be the edge automorphism of $G$ corresponding to $\alpha$ under the isomorphism $\mathcal{G}_1(G) \cong \mathcal{G}(L(G))$. Then $\beta(e) = f$ and, because $\mathcal{G}_1(G) = \mathcal{G}^*(G)$, the function $\beta$ is an element of $\mathcal{G}^*(G)$. Thus $G$ is line-symmetric.

Case 2. Assume that $G$ is disconnected.

Let $G_1, G_2, \ldots, G_k$ be the components of $G$ ($k \geq 2$). If $L(G)$ is connected, then $G$ has exactly one non-trivial component to which Case 1 may be applied to show that $G$ is line-symmetric. If $L(G)$ is disconnected then its components, without loss of generality, are $L(G_1), L(G_2), \ldots, L(G_m)$ for some integer $m$ in the range $2 \leq m \leq k$. By Lemma 2-10, $L(G_1) \cong L(G_2) \cong \cdots \cong L(G_m)$. Since not both of $K_3$ and $K(1,3)$ are components of $G$ it follows by Whitney's Theorem that $G_1 \cong G_2 \cong \cdots \cong G_m$. Also $G_{m+1} \cong G_{m+2} \cong \cdots \cong G_k \cong K_1$. Hence, if $H \cong G_1$, then $G \cong mH \cup (k - m)K_1$ and $L(G) \cong mL(H)$. By again applying Lemma 2-10 and using the fact that $L(G)$ is point-symmetric, we know that $L(H)$ is point-symmetric. Then, by applying the argument of Case 1 to $H$, we conclude that $H$ is line-symmetric. Because every non-trivial component of $G$ is isomorphic to $H$, the graph $G$ must also be line-symmetric. 

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We note that the restriction in Theorem 2-11 that $G$ should not contain both $K_3$ and $K(1,3)$ as components is essential since if $G = K_3 \cup K(1,3)$ then $G$ is not line-symmetric but $L(G) \cong 2K_3$ is UVR. The following corollary is an immediate consequence of Theorems 2-8 and 2-11.

**Corollary 2-12.** If $G$ is a nonempty graph which does not contain both $K_3$ and $K(1,3)$ as components then $G$ is line-symmetric if and only if $L(G)$ is point-symmetric.

**Section 2.2**

**Structural Characterizations of Certain Classes of Line-Symmetric Graphs**

In this section we continue our investigation of line-symmetric graphs by providing complete lists of such graphs that fall into particular categories, where the categories are specified by various connectivity properties of either the graph or its complement.

Our first result is the edge analog of Lemma 2-10.

**Theorem 2-13.** Let $G$ be a disconnected graph. Then $G$ is line-symmetric if and only if there exists a connected line-symmetric graph $H$ such that every non-trivial component of $G$ is isomorphic to $H$. 

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Proof. Let $G$ be a disconnected line-symmetric graph. It is easily seen that each component of $G$ must be line-symmetric. If $G$ has exactly one nontrivial component, the result is true. Otherwise, let $G_1$ and $G_2$ be distinct nontrivial components of $G$. Let $e_i \in E(G_i)$ for $i = 1, 2$. Since $G$ is line-symmetric, there exists $\alpha \in \text{Aut}(G)$ such that $\alpha(e_1) = e_2$. It follows that $\alpha(E(G_1)) = E(G_2)$ and that $\alpha$ restricted to $V(G_1)$ is an isomorphism of $G_1$ with $G_2$.

It is readily verified that if there exists a connected line-symmetric graph $H$ such that every nontrivial component of $G$ is isomorphic to $H$, then $G$ is line-symmetric.

Theorem 2-13 allows us to restrict our attention to nontrivial connected line-symmetric graphs when convenient. The next result permits us to further restrict our study to line-symmetric blocks.

Theorem 2-14. Let $G$ be a nontrivial connected graph which is not a block. Then $G$ is line-symmetric if and only if $G \cong K(1,n)$ for some $n \geq 2$.

Proof. If $G \cong K(1,n)$ for some $n \geq 2$, then $G$ is a nontrivial connected line-symmetric graph which is not a block.
Conversely let \( G \) be a nontrivial connected line-symmetric graph which is not a block. Then \( G \) contains at least one cut-vertex. Assume that \( G \) contains two or more cut-vertices. Then \( G \) contains end-blocks and \( G \) contains blocks that are not end-blocks. Let \( e \) be an edge in an end-block of \( G \), and let \( f \) be an edge belonging to a block of \( G \) which is not an end-block. Then for all \( \alpha \in G^*(G) \), the edge \( \alpha(e) \) must belong to some end-block of \( G \) and therefore \( \alpha(e) \neq f \). This contradicts the fact that \( G \) is line-symmetric. Therefore \( G \) has exactly one cut-vertex \( v \). Let \( g \) be an edge of \( G \) incident with \( v \), and assume that \( G \) contains an edge \( h \) not incident with \( v \). Then for all \( \alpha \in G^*(G) \), the edge \( \alpha(g) \) must be incident with some cut-vertex of \( G \); and hence \( \alpha(g) \) must be incident with \( v \). It follows that \( \alpha(g) \neq h \) for all \( \alpha \in G^*(G) \) again contradicting the line-symmetry of \( G \). Thus every edge of \( G \) is incident with \( v \) and \( G \) is isomorphic with \( K(1, n) \) for some \( n \geq 1 \). But since \( G \) is not a block, we must have \( n \geq 2 \).

We now characterize certain classes of line-symmetric graphs defined in terms of the connectivities.
of their complements. Our first result lists all line-
symmetric graphs whose complements are not connected.
Recall that the complete $t$-partite graph $K(n, n, \ldots, n)$
is denoted $K_t(n)$.

Theorem 2.15. If $G$ is a graph whose complement is dis­
connected then $G$ is line-symmetric if and only if
either there exist distinct positive integers $m$ and
$n$ such that $G \cong K(m, n)$ or there exist integers
$n \geq 1$ and $t \geq 2$ such that $G \cong K_t(n)$.

Proof. It is easily verified that if $G$ has either of
the specified forms then $G$ is line-symmetric and $\overline{G}$
is disconnected. Suppose conversely that $G$ is a line-
symmetric graph whose complement is disconnected. Let
$G_1, G_2, \ldots, G_t$ be the components of $\overline{G}$ with $t \geq 2$.
We claim that each $G_i$ is a complete subgraph of $\overline{G}$.
To establish this claim, assume to the contrary that
there are two nonadjacent vertices $x$ and $y$ in $G_i$
for some $1 \leq i \leq t$. Then $e = xy$ is an edge of $G$.
Also, if we choose arbitrary vertices $u$ in $G_1$ and
$v$ in $G_2$ then $f = uv$ is an edge of $G$. Because
$G$ is line-symmetric, there exists $\pi \in \text{Aut}(G)$ with
$\overline{\pi}(e) = f$. But $\pi$ is also an automorphism of $\overline{G}$ so
that there exists an index $j$ satisfying $1 \leq j \leq t$
such that $\pi(V(G_1)) = V(G_j)$, contradicting
\[ f(e) = f. \] Thus every component of \( \overline{G} \) is complete.

If \( t = 2 \) and \( m = |V(G_1)| \) and \( n = |V(G_2)| \) we have \( \overline{G} = K_m \cup K_n \) so that \( G = K(m, n) \) and either \( m \neq n \) or \( m = n \) in which case \( G = K_2(n) \). If \( t \geq 3 \) then-

because \( \overline{G} \) is UEE, \( |V(G_1)| = |V(G_2)| = \cdots = |V(G_t)| \). If \( n \) denotes the common values of \( |V(G_i)| \) then \( \overline{G} = tK_n \) and \( G = K_t(n) \).

Theorem 2-15 lists all nontrivial line-symmetric graphs whose complements have connectivity zero. We build on this result by characterizing those nontrivial line-symmetric graphs whose complements have connectivity 1.

**Theorem 2-16.** If \( G \) is a nontrivial graph whose complement is connected and not a block then \( G \) is line-symmetric if and only if there exist distinct positive integers \( m \) and \( n \) such that

\[ G = K_1 \cup K(m, n) \] or there exist integers \( n \geq 1 \) and \( t \geq 2 \) such that

\[ G = K_1 \cup K_t(n). \]

**Proof.** It is easily verified that if \( G \) has either of the specified froms then \( G \) is a nontrivial line-symmetric graph whose connected complement contains a cut-vertex. Suppose conversely that \( G \) is a nontrivial line-symmetric graph whose complement is connected and not a block. It follows that \( \overline{G} \) must contain a cut-
vertex \( v \). We claim that \( v \) is the only cut-vertex of \( \overline{G} \). To establish this claim, assume, to the contrary, that \( \overline{G} \) contains at least two cut-vertices. Then there exist two end-blocks \( B_1 \) and \( B_2 \) of \( \overline{G} \) containing cut-vertices \( u_1 \) and \( u_2 \) respectively, where \( u_1 \neq u_2 \). For \( i = 1, 2 \) let \( w_i \) be a vertex of \( B_i \) different from \( u_i \). Then \( e = w_1 u_2 \) and \( f = w_1 w_2 \) are edges of \( G \). Because \( G \) is line-symmetric, there exists \( \pi \in \text{Aut}(G) \) such that \( \pi(e) = f \).

It follows that \( \pi(u_2) \in \{w_1, w_2\} \). The function \( \pi \) is also an automorphism of \( \overline{G} \) and \( u_2 \) is a cut-vertex of \( \overline{G} \) so that \( \pi(u_2) \) must also be a cut-vertex of \( \overline{G} \). However, neither \( w_1 \) nor \( w_2 \) are cut-vertices of \( \overline{G} \) and a contradiction is reached. Thus \( v \) is, in fact, the only cut-vertex of \( \overline{G} \).

Suppose next that there exists a vertex \( w \in V(\overline{G}) - \{v\} \) such that \( vw \in E(\overline{G}) \).

Let \( x \) be any vertex in \( V(G) - \{v\} \) belonging to a block of \( \overline{G} \) not containing \( w \). Because \( v \) is the only cut-vertex of \( \overline{G} \), vertices \( v \) and \( w \) belong to the same block of \( \overline{G} \) and therefore the graph \( \overline{G} + vw \) has the same number of blocks as the graph \( \overline{G} \). Also, because \( w \) and \( x \) belong to different blocks of \( \overline{G} \), \( wx \notin E(\overline{G}) \) and the graph \( \overline{G} + wx \) has fewer blocks than \( \overline{G} \). It follows that \( \overline{G} + vw \notin \overline{G} + wx \).
and that $\bar{G}$ is not UEE. By Corollary 2-6, $G$ is not line-symmetric, a contradiction. Thus for every vertex $w \in V(\bar{G}) - \{v\}$ we conclude that $vw \in E(\bar{G})$.

Now $\bar{G}$ is a UEE graph and $v$ is a cut-vertex of $\bar{G}$ which is adjacent, in $\bar{G}$, to every vertex other than itself. It follows that $\bar{G} - v$ is a disconnected UEE graph. Thus $\bar{G} - v$ is a line-symmetric graph whose complement is disconnected. Employing Theorem 2-15, we see that there exist distinct positive integers $m$ and $n$ such that $\bar{G} - v \cong K(m, n)$ or there exist integers $n \geq 1$ and $t > 2$ such that $\bar{G} - v \cong K_t(n)$. Hence $G \cong K_1 \cup K(m, n)$ or $G \cong K_1 \cup K_t(n)$.

Our listing of line-symmetric graphs would be complete if we could continue the work begun by Theorems 2-15 and 2-16 by characterizing those line-symmetric graphs whose complements have connectivity $s$ for $s \geq 2$. Unfortunately, we can produce such a theorem only with some additional restrictions imposed.

**Theorem 2-17.** Let $s$ be an integer with $s \geq 2$ and let $G$ be a graph with at least $s - 1$ isolated vertices such that $\kappa(\bar{G}) = s$. Then $G$ is line-symmetric if and only if either $G \cong K_{s+1}$ or there exist distinct positive integers $m$ and $n$ such that...
$$G = \overline{K_s} \cup K(m, n)$$ or there exist integers $n \geq 1$ and $t \geq 2$ such that $G \cong \overline{K_s} \cup K_t(n)$.

**Proof.** It is easily seen that if $G$ has any of the three forms specified then $G$ is a line-symmetric graph containing at least $s - 1$ isolated vertices whose complement has connectivity $s$.

Conversely suppose that $G$ is a line-symmetric graph containing at least $s - 1$ isolated vertices such that $\kappa(G) = s$. Let $I$ be a set of $s - 1$ isolated vertices in $G$. If $G$ is complete, then, $\overline{G} \cong K_{s+1}$, the only complete graph with connectivity $s$. Thus $G \cong \overline{K_{s+1}}$ and our result holds. Assume then, that $G$ is not complete. If $G - I$ is the trivial graph, then $G \cong \overline{K_s}$ which implies $\kappa(G) = s - 1$, a contradiction. Therefore $G - I$ is nontrivial. Note also that $G - I$ is line-symmetric.

With a view towards applying Theorem 2-16 to the graph $G - I$, we establish some facts about the graph $\overline{G - I} = \overline{G} - I$. If $\overline{G} - I$ is disconnected then, because $\overline{G}$ is connected, $\kappa(\overline{G}) < |I| = s - 1$, a contradiction. Thus $\overline{G - I}$ is connected. Because $\overline{G}$ is not complete, it must contain a cut-set $I'$ of cardinality $\kappa(\overline{G}) = s$. Being a cut-set of $\overline{G}$, the set $I'$ must contain all the isolated vertices of $G$ so that $I \subseteq I'$ and $I' = I \cup \{w\}$ for some vertex $w$. 

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of $G - I$. Now $G - I' = G - I - \{w\}$ is disconnected so that $w$ is a cut-vertex of the connected graph $G - I$. Thus $G - I$ is not a block. Using Theorem 2-16 either $G - I \cong K_1 \cup K(m, n)$ or $G - I \cong K_1 \cup K_t(n)$; and $G \cong K_s \cup K(m, n)$ or $G \cong K_s \cup K_t(n)$ for appropriate values of $m$ and $n$ or $t$ and $n$.

A similar result holds for graphs whose complements contain isolated vertices.

**Theorem 2-18.** Let $G$ be a graph whose complement contains an isolated vertex. Then $G$ is line-symmetric if and only if either $G \cong K_t$ for some positive integer $t$ or $G \cong K(1, n)$ for some integer $n \geq 2$.

**Proof.** It is easily seen that complete graphs and stars are line-symmetric and that their complements contain at least one isolated vertex. Conversely, let $G$ be a line-symmetric graph whose complement contains an isolated vertex. If $G$ is of order 1 then $G \cong K_1$ and our result holds. Assume, then, that $G$ has order at least two. Then $\overline{G}$ is disconnected, so by Theorem 2-15 we need consider only two cases.
Case 1. Assume that $G \cong K(m, n)$ for some positive integers $m < n$. Since $G$ has a vertex which is isolated in $G$, we must have $m = 1$ and $G \cong K(1, n)$ for some $n \geq 2$.

Case 2. Assume that $G \cong K_t(n)$ for some integers $n \geq 1$ and $t \geq 2$. In this case $G \cong tK_n$ and because $G$ has an isolated vertex we must have $n = 1$ and $G \cong K_t(1) \cong K_t$.

Section 2.3

Biregular Line-Symmetric Graphs

Recall from Theorem 2-2 that if $G$ is a line-symmetric graph which contains no isolated vertices then either $G$ is regular or $G$ is biregular. In this section we present two characterizations of biregular line-symmetric graphs.

We begin work towards that goal with some new terminology and notation. For unequal positive integers $a$ and $b$ and for a graph $G$ with $\{a, b\} \subseteq \mathcal{D}_G$, we denote by $V_a$ (respectively $V_b$) the set of vertices of $G$ with degree $a$ (respectively $b$). For such a graph we define an equivalence relation on $V_a \cup V_b$ as follows: two vertices $v_1$ and $v_2$ are equivalent if and
only if \( N(v_1) = N(v_2) \). Note that if two vertices are equivalent then they have the same degree. Hence each equivalence class so formed is a subset of either \( V_a \) or \( V_b \). We denote these classes by \( V_a(1), V_a(2), \ldots, V_a(n_a), V_b(1), V_b(2), \ldots, V_b(n_b) \). 

A graph \( G \) will be called neighborhood symmetric if for any two pairs \( \{u_1, v_1\}, \{u_2, v_2\} \) of adjacent vertices of \( G \) with \( \deg u_1 = \deg u_2 \) and \( \deg v_1 = \deg v_2 \), there exists an automorphism \( \alpha \) of \( G \) such that \( \alpha(N(u_1)) = N(u_2) \) and \( \alpha(N(v_1)) = N(v_2) \).

Note that every line-symmetric graph is neighborhood symmetric and that the tree \( G \) in Figure 2-3 is neighborhood symmetric but not line-symmetric.

![Figure 2-3](image-url)
Theorem 2-19. Let $G$ be a connected graph and let $a$ and $b$ be unequal integers with $\{a, b\} \subset \mathcal{S}_G$. Then $G$ is line-symmetric and biregular if and only if $G$ has a bipartition $V_a \cup V_b$ and $G$ is neighborhood symmetric.

Proof. Let $G$ be a connected line-symmetric graph and let $a$ and $b$ be unequal positive integers with $\mathcal{S}_G = \{a, b\}$. It is immediate from Theorem 2-3 that $G$ has a transitive bipartition $V_a \cup V_b$. To show that $G$ is neighborhood symmetric, let $\{u_1, v_1\}$ and $\{u_2, v_2\}$ be two pairs of adjacent vertices in $G$ such that $\deg u_1 = \deg u_2$ and $\deg v_1 = \deg v_2$. Without loss of generality we may assume that $\deg u_1 = \deg u_2 = a$ and that $\deg v_1 = \deg v_2 = b$. Because $G$ is line-symmetric, there exists an automorphism $\alpha$ of $G$ whose induced edge automorphism maps the edge $u_1v_1$ to the edge $u_2v_2$. Thus $\alpha(\{u_1, v_1\}) = \{u_2, v_2\}$ and because $a \neq b$, $\alpha(u_1) = u_2$ and $\alpha(v_1) = v_2$. These equalities immediately yield $N(\alpha(u_1)) = N(u_2)$ and $N(\alpha(v_1)) = N(v_2)$. An automorphism preserves adjacency and non-adjacency of vertices so that $N(\alpha(u_1)) = \alpha(N(u_1))$ and $N(\alpha(v_1)) = \alpha(N(v_1))$. Thus $\alpha(N(u_1)) = N(u_2)$ and $\alpha(N(v_1)) = N(v_2)$.

Suppose conversely that $G$ is a connected neighborhood symmetric graph with a bipartition $V_a \cup V_b$ where $a \neq b$. It is obvious that $G$ is biregular so
we work towards showing that $G$ is line-symmetric. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be edges of $G$. Without loss of generality $u_i \in V_a$ and $v_i \in V_b$ for $i = 1, 2$. We seek an induced edge automorphism of $G$ that maps $e_1$ to $e_2$. We consider two cases depending on whether or not $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are equivalent pairs of vertices.

Case 1. Assume that $N(u_1) = N(u_2)$ and that $N(v_1) = N(v_2)$.

In this case there exist integers $i$ and $j$ with $1 \leq i \leq n_a$ and $1 \leq j \leq n_b$ where $\{u_1, u_2\} \subseteq \mathcal{V}_a(i)$ and $\{v_1, v_2\} \subseteq \mathcal{V}_b(j)$. Let $\pi : \mathcal{V}_a(i) \to \mathcal{V}_a(i)$.

Figure 2-4

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be a bijection such that \( \pi(u_1) = u_2 \). Define \( \phi: V(G) \to V(G) \) as follows:

\[
\phi(v) = \begin{cases} 
\pi(v) & \text{if } v \in V_a(i) \\
v & \text{if } v \in V(G) - V_a(i).
\end{cases}
\]

The bijection \( \phi \) is an automorphism of \( G \) because vertices in \( V_a(i) \) have the same neighbors. Similarly, let \( \lambda: V_b(j) \to V_b(j) \) be a bijection with \( \lambda(v_1) = v_2 \) and define \( \theta: V(G) \to V(G) \) as follows:

\[
\theta(v) = \begin{cases} 
\lambda(v) & \text{if } v \in V_b(j) \\
v & \text{if } v \in V(G) - V_b(j).
\end{cases}
\]

Then \( \theta \) is an automorphism of \( G \) and \( \theta \phi \in \text{Aut}(G) \) with \( \theta \phi(u_1) = u_2 \) and \( \theta \phi(v_1) = v_2 \). Therefore \( \theta \phi \) induces an edge automorphism of \( G \) which maps \( e_1 \) to \( e_2 \).

**Case 2.** Assume that either \( N(u_1) \neq N(u_2) \) or \( N(v_1) \neq N(v_2) \).

Because \( G \) is neighborhood symmetric and since \( \deg u_1 = \deg u_2 = a \) and \( \deg v_1 = \deg v_2 = b \), we know there exists an automorphism \( \alpha \) of \( G \) such that \( \alpha(N(u_1)) = N(u_2) \) and \( \alpha(N(v_1)) = N(v_2) \). Since \( \alpha \)
is an automorphism, \( \alpha(N(u_1)) = N(\alpha(u_1)) \) and
\( \alpha(N(v_1)) = N(\alpha(v_1)) \). Thus \( N(\alpha(u_1)) = N(u_2) \) and
\( N(\alpha(v_1)) = N(v_2) \). Applying the argument of Case 1 to
the equivalent pairs of vertices \( \{\alpha(u_1), u_2\} \) and
\( \{\alpha(v_1), v_2\} \) we conclude that there exists an auto­
morphism \( \beta \) of \( G \) such that \( \beta(\alpha(u_1)) = u_2 \) and
\( \beta(\alpha(v_1)) = v_2 \). Thus \( \beta \alpha \) is an automorphism of \( G \)
whose induced edge automorphism maps \( e_1 \) to \( e_2 \).

We note that Theorem 2-19 is best possible in the
sense that there exist connected bipartite biregular
neighborhood symmetric graphs which are not line-sym­
metric. The graph \( G \) in Figure 2-3 is such a graph.
In view of Theorem 2-19, \( G \) fails to be line-sym­
metric because \( V_G = \{1, 3\} \) and \( G \) has no bi­
partition \( V_1 \cup V_3 \).

In fact the graph formed by joining the two ver­
tices of degree \( n \) in \( 2K(1,n) \) for any \( n \geq 2 \) is
bipartite, biregular, and neighborhood symmetric yet
not line-symmetric because it has no bipartition
\( V_1 \cup V_{n+1} \).

The requirement that \( G \) be neighborhood symmetric
is also necessary to Theorem 2-19 because there exist
connected bipartite biregular graphs with degree set
\( \{a, b\} \) and bipartition \( V_a \cup V_b \) which fail to be line-
symmetric. One such graph appears in Figure 2-5. We first verify that this graph is not neighborhood symmetric by considering the two pairs of adjacent vertices \{a, 1\} and \{a, 2\}. Note that \(N(a) = \{1, 2, 3\}\), \(N(1) = \{a, d\}\), and \(N(2) = \{a, b\}\). If there exists an automorphism \(\alpha\) of this graph with \(\alpha(N(a)) = N(a)\) and \(\alpha(N(1)) = N(2)\) then \(\alpha(\{1, 2, 3\}) = \{1, 2, 3\}\) and \(\alpha(\{a, d\}) = \{a, b\}\). If \(\alpha(d) = b\) then \(\alpha(N(d)) = N(b)\) which may be rewritten \(\alpha(\{1, 5, 6\}) = \{2, 3, 4\}\). Thus either 2 or 3 is the image of 5 or 6 under \(\alpha\) contradicting \(\alpha(\{1, 2, 3\}) = \{1, 2, 3\}\). The only other possibility is that \(\alpha(d) = a\) which forces \(\alpha(a) = b\) implying \(\alpha(N(a)) = N(b)\) contradicting \(\alpha(N(a)) = N(a)\). Thus this graph is not neighborhood symmetric. It also fails to be line-symmetric because, for example, the edge \(e_1\) belongs to an induced subgraph isomorphic to \(K(2,2)\) while the edge \(e_2\) belong to no such subgraph. Another way to
easily see that this graph is not line-symmetric is to note that the equivalence classes belonging to $V_2$ ($\{1\}, \{2, 3\}, \{4\}, \{5, 6\}$) are not all of the same cardinality, as the following corollary indicates.

**Corollary 2-20.** If $G$ is a biregular line-symmetric graph with $\mathcal{G} = \{a, b\}$ then there exist positive integers $c$ and $d$ with $|V_a(i)| = c$ and $|V_b(j)| = d$ for $i = 1, 2, \ldots, n_a$ and $j = 1, 2, \ldots, n_b$.

**Proof.** Let $i$ and $j$ be integers with $1 \leq i, j \leq n_a$. We show that $|V_a(i)| = |V_a(j)|$. Let $v_1 \in V_a(i)$ and $v_2 \in V_a(j)$. Then $v_1$ and $v_2$ belong to $V_a$.

![Figure 2-6](image-url)
By Theorem 2-3, the bipartition \( V_a \cup V_b \) is transitive so that there exist automorphisms \( \alpha \) and \( \beta \) of \( G \) with \( \alpha(v_1) = v_2 \) and \( \beta(v_2) = v_1 \). Let \( v_3 \in V_a(i) \). Then \( N(v_3) = N(v_1) \) and we have \( N(v_2) = N(\alpha(v_1)) = \alpha(N(v_1)) = \alpha(N(v_3)) = N(\alpha(v_3)) \). Thus \( v_2 \) and \( \alpha(v_3) \) are equivalent which yields \( \alpha(v_3) \in V_a(j) \), the equivalence class containing \( v_2 \). Because \( v_3 \) was arbitrarily chosen in \( V_a(i) \), it follows that \( \{\alpha(w) | w \in V_a(i)\} \subseteq V_a(j) \). The automorphism \( \alpha \) is a one-to-one function so that \( |V_a(i)| \leq |V_a(j)| \).

We next make a similar argument using a vertex \( v_4 \in V_a(j) \). As before, \( N(v_4) = N(v_2) \) so that \( N(v_1) = N(\beta(v_2)) = \beta(N(v_2)) = N(\beta(v_4)) \) and thus \( v_1 \) and \( \beta(v_4) \) are equivalent, yielding \( \beta(v_4) \in V_a(i) \). Hence \( \{\beta(w) | w \in V_a(j)\} \subseteq V_a(i) \)
and \( |V_a(j)| \leq |V_a(i)| \). We conclude \( |V_a(i)| = |V_a(j)| \) and that there exists a positive integer \( c \) such that \( |V_a(i)| = c \) for \( i = 1, 2, \ldots, n_a \).

In a similar fashion it can be shown that there exists a positive integer \( d \) such that \( |V_b(j)| = d \) for \( j = 1, 2, \ldots, n_b \).

We close this section with a slight modification of Theorem 2-19.
Corollary 2-21. Let $G$ be a connected graph and let $a$ and $b$ be unequal integers with $\{a, b\} \subset \Delta_G$. Then $G$ is line-symmetric and biregular if and only if $G$ has a transitive bipartition and is neighborhood symmetric.

Proof. If $G$ is connected, biregular with $\Delta_G = \{a, b\}$, and line-symmetric then, by Theorem 2-19, $G$ is neighborhood symmetric and has a bipartition $V_a \cup V_b$. By Theorem 2-3, this bipartition is transitive. Conversely suppose that, under the given conditions, $G$ has a transitive bipartition and is neighborhood symmetric. Let $x$ and $y$ be vertices of $G$ belonging to the same partite set. Then there exists an automorphism $\alpha$ of $G$ with $\alpha(x) = y$ implying that $\deg x = \deg y$. Thus the transitive bipartition of $G$ must be $V_a \cup V_b$ and the result follows by Theorem 2-19.
CHAPTER III

PROBLEMS RELATED TO LINE-SYMMETRIC GRAPHS

Having investigated line-symmetric graphs and characterizing some but not all of them in Chapter II, we denote the first section of this chapter to three generalizations of line-symmetric graphs and their complete characterizations.

We then return, in the second section of this chapter, to uniform factorizations of graphs. In particular we consider decompositions of complete graphs into two edge-disjoint spanning subgraphs each of which is line-symmetric.

Section 3.1

Generalizations of Line-Symmetric Graphs

The concept of a line-symmetric graph can be extended in many ways. We present three such generalizations in this section. In each case we provide a corresponding generalization of the notion of a uniquely edge extendible graph, establish a result analogous to Theorem 2-5, and structurally characterize the graphs so defined. This work will be accomplished in a sequence of lemmas which, taken together, will
constitute a proof Theorem 3.8 which is the main result of this section.

Let $k$ be a positive integer. A graph $G$ is $k$-line-symmetric if $|E(G)| \geq k + 1$ and for any pair $A, B$ of sets of edges of $G$ with $|A| = |B| = k$ there exists an $\alpha$ in $G^*(G)$ such that $\alpha(A) = B$. Note that if $G$ is a graph with at least two edges then $G$ is 1-line-symmetric if and only if it is line-symmetric. A graph $G$ is uniquely $k$ edge extendible (denoted U$k$EE) if $|E(G)| \geq k + 1$ and for any pair $A, B$ of sets of edges of $\overline{G}$ with $|A| = |B| = k$, the graphs $G + A$ and $G + B$ are isomorphic. Note that a graph whose complement has size at least two if U$k$EE if and only if it is UEE.

Lemma 3-1. Let $G$ be a graph of size $q \geq 2$ and let $k$ be an integer satisfying $1 \leq k \leq q - 1$. Then $G$ is $k$-line-symmetric if and only if $G$ is $(q - k)$-line-symmetric.

Proof. Assume that $G$ is a $k$-line-symmetric graph of size $q$ and let $k$ be an integer satisfying $1 \leq k \leq q - 1$. Since $k \geq 1$, we have $q \geq (q - k) + 1$ so that $G$ has sufficient size to be $(q - k)$-line-symmetric. Let $A$ and $B$ be sets of edges of $G$
with \(|A| = |B| = q - k\). Let \(C = E(G) - A\) and \(D = E(G) - B\). Then \(|C| = |D| = k\), so there exists \(\alpha\) in \(G^*(G)\) such that \(\alpha(C) = D\). It follows that \(\alpha(E(G) - C) = E(G) - D\) which yields \(\alpha(A) = B\). Thus \(G\) is \((q - k)\)-line-symmetric. Because this implication is self-converse, the lemma is established.

Since a graph of size \(q \geq 2\) is line-symmetric if and only if it is \((q - 1)\)-line-symmetric, and because we have already studied line-symmetric graphs in Chapter II, we will limit our attention here to \(k\)-line-symmetric graphs of size \(q\) where \(2 \leq k \leq q - 2\). Moreover, we are working toward the result that with suitable restrictions on \(k\), a graph \(G\) is \(U_{kEE}\) if and only if \(\overline{G}\) is \(k\)-line-symmetric. Thus we also limit our study of \(U_{kEE}\) graphs to those whose complements have size \(\overline{q}\) with \(2 \leq k \leq \overline{q} - 2\). This does not seriously restrict the scope of our study since the only other case permitted by the definition of \(U_{kEE}\) graphs is that of \(k = \overline{q} - 1\), and every graph is \(U(\overline{q} - 1)EE\). Our next result shows that subject to these restrictions an irreducible graph \(G\) which is \(U_{kEE}\) must take one of two forms, which surprisingly are independent of the value of \(k\).
Lemma 3-2. Let $G$ be an irreducible graph and let $k$ be an integer satisfying $2 \leq k \leq q - 2$. If $G$ is UkEE then for $n = \overline{q}$ either $G \cong K_n(2)$ or $G \cong K_1 \cup K_n$.

Proof. Let $G$ be a graph and $k$ an integer satisfying the hypotheses of the lemma, and assume that $G$ is UkEE. We consider two cases according as $E(G)$ is or is not an independent set of edges in $\overline{G}$.

Case 1. Assume that no two edges of $\overline{G}$ are adjacent. It follows that $\Delta(\overline{G}) \leq 1$. Since $G$ is irreducible, $\overline{G}$ has no isolated vertices and thus $\delta(\overline{G}) \geq 1$. Hence $\overline{G}$ is 1-regular and, for $n = \overline{q}$, we conclude $\overline{G} \cong nK_2$ which yields $\overline{G} \cong K_n(2)$.

Case 2. Assume that $\overline{G}$ contains a pair of adjacent edges. Let $e_1, f_1$ be edges of $\overline{G}$ that are incident with a common vertex $u$. Let $e$ and $f$ be two arbitrary distinct edges of $\overline{G}$ and suppose that $e$ and $f$ are independent. Since $\overline{q} \geq k + 2$ there exists a subset $C$ of $E(\overline{G}) - \{e_1, f_1, e, f\}$ with $|C| = k - 2$. Let $A = C \cup \{e_1, f_1\}$ and let $B = C \cup \{e, f\}$. Then the graphs $G + A$ and $G + B$ have different degree sequences implying that $G + A$ and $G + B$ are not isomorphic. Because $|A| = |B| = k$, this contradicts the assumption that $G$ is UkEE. Thus not two edges of...
Our next result ties together our first pair of generalizations.

**Lemma 3-3.** Let $G$ be a graph whose complement is $k$-line-symmetric for some $k \geq 2$. Then $G$ is $UkEE$.

**Proof.** Assume that $\overline{G}$ is $k$-line-symmetric. Then $|E(\overline{G})| \geq k + 1$ and $\overline{G}$ has sufficient size for $G$ to be $UkEE$. Let $A$ and $B$ be subset of $E(\overline{G})$ with $|A| = |B| = k$. Because $\overline{G}$ is $k$-line-symmetric, there exists $\pi \in G(\overline{G})$ such that $\hat{\pi}(A) = B$ where $\hat{\pi} \in G^*(\overline{G})$ is induced by $\pi$. Since $G(\overline{G}) = G(G)$, the map $\pi$ is also an automorphism of $G$. It follows that $\pi$ is an isomorphism of $G + A$ with $G + B$ and that $G$ is $UkEE$.

The elementary observation that for positive integers $n$ and $k$ with $n \geq k + 1$, the graphs $nK_2$ and $K(l, n)$ are $k$-line-symmetric together with the two preceding lemmas show that, under the hypotheses of Lemma 3-2, the following statements are equivalent:
1. G is U\text{KEE}.
2. \overline{G} is k-line-symmetric.
3. For \( n = q \), either \( G \cong K_n(2) \) or \( G \cong K_1 \cup K_n \).

We now introduce our second generalization of line-symmetric graphs along with its associated generalization of UEE graphs. For a positive integer \( k \), a graph \( G \) is \textit{totally k-line-symmetric} if \(|E(G)| \geq k + 1\) and for any pair \( A, B \) of sets of edges of \( G \) with \(|A| = |B| = n\) and \( 1 \leq n \leq k \), there exists an \( \alpha \) in \( G \ast (G) \) such that \( \alpha(A) = B \). Thus a graph is totally \( k \)-line-symmetric if and only if it is \( n \)-line-symmetric for all \( n \) such that \( 1 \leq n \leq k \). A graph \( G \) is \textit{totally} \( k \) \textit{edge extendible} (denoted TkEE) if \(|E(\overline{G})| \geq k + 1\) and for any pair \( A, B \) of sets of edges of \( \overline{G} \) with \(|A| = |B| = n\) and \( 1 \leq n \leq k \), the graphs \( G + A \) and \( G + B \) are isomorphic. Therefore a graph is TkEE if and only if it is UnEE for all \( n \) such that \( 1 \leq n \leq k \).

\textbf{Lemma 3-4.} Let \( G \) be an irreducible graph and let \( k \) be an integer satisfying \( 2 \leq k \leq q - 2 \). If \( G \) is TkEE, then \( \overline{G} \) is totally \( k \)-line-symmetric.

\textbf{Proof.} Let \( G \) be an irreducible TkEE graph with \( 2 \leq k \leq q - 2 \). Then \( q = |E(\overline{G})| \geq k + 2 \) and \( \overline{G} \) has sufficient size to be totally \( k \)-line-symmetric.
Let \( n \) be an integer with \( 1 \leq n \leq k \). Note that \( G \) is UnEE. Using Corollary 2-6 if \( n = 1 \), or employing earlier work in this chapter if \( n \geq 2 \), we conclude that \( \bar{G} \) is \( n \)-line-symmetric. Since \( n \) was arbitrarily chosen in the range \( 1 \leq n \leq k \), the graph \( \bar{G} \) is totally \( k \)-line-symmetric.

Our third pair of generalizations can now be stated.

For a positive integer \( k \), a graph \( G \) is \( k \)-fold line-symmetric if \( |E(G)| \geq k + 1 \) and for any two ordered \( k \)-tuples of edges of \( G \), \( A = (e_1, e_2, \ldots, e_k) \) and \( B = (f_1, f_2, \ldots, f_k) \), there exists an induced edge automorphism \( \alpha \) of \( G \) such that \( \alpha(e_i) = f_i \) for all \( i = 1, 2, \ldots, k \). Note that if \( G \) has at least two edges, then \( G \) is \( 1 \)-fold line-symmetric if and only if \( G \) is line-symmetric. A graph \( G \) is uniquely \( k \)-fold edge-extendible (denoted \( U_{k\text{EE}} \)) if \( |E(G)| \geq k + 1 \) and for every two ordered \( k \)-tuples \( A = (e_1, e_2, \ldots, e_k) \) and \( B = (f_1, f_2, \ldots, f_k) \) of edges of \( G \), there exists an isomorphism \( \alpha \) from \( G + A \) to \( G + B \) such that \( \alpha(e_i) = f_i \) for all \( i = 1, 2, \ldots, k \).

**Lemma 3-5.** If \( \bar{G} \) is \( k \)-fold line-symmetric for \( k \geq 1 \) then \( G \) is \( U_{k\text{EE}} \).
Proof. Because \( \overline{G} \) is \( k \)-fold line-symmetric with \( k \geq 1 \), we know \(|E(\overline{G})| \geq k + 1\) and thus \( \overline{G} \) has sufficient size for \( G \) to be UkFEE. Let \( A = (e_1, e_2, \ldots, e_k) \) and \( B = (f_1, f_2, \ldots, f_k) \) be ordered \( k \)-tuples of edges of \( \overline{G} \). Because \( \overline{G} \) is \( k \)-fold line-symmetric, there exists \( \pi \in \text{Aut}(\overline{G}) \) such that \( \pi(e_i) = f_i \) for \( 1 \leq i \leq k \). Since \( \pi \) is also an automorphism of \( G \), the map \( \pi \) is an isomorphism of \( G + A \) with \( G + B \).

The following two direct results, together with Lemmas 3-2 through 3-5, constitute a proof of Theorem 3-8.

Lemma 3-6. For positive integers \( n \) and \( k \) with \( n \geq k + 1 \), the graphs \( nK_2 \) and \( K(1, n) \) are \( k \)-fold line-symmetric and have \( TkEE \) complements.

Lemma 3-7. For \( k \geq 1 \), each UkFEE graph is UkEE and each totally \( k \)-line-symmetric graph is \( k \)-line-symmetric.

Theorem 3-8. Let \( G \) be an irreducible graph and let \( k \) be an integer satisfying \( 2 \leq k \leq \overline{q} - 2 \). Then the following statements are equivalent.
1. \( G \) is UkEE.
2. \( \overline{G} \) is \( k \)-line-symmetric.
3. $G$ is $TkEE$.
4. $\overline{G}$ is totally $k$-line-symmetric.
5. $G$ is $UkFE$.
6. $\overline{G}$ is $k$-fold line-symmetric.
7. For $n = q$, either $G \cong K_n(2)$ or $G \cong K_1 \cup K_n$.

Section 3.2

Line-Symmetric Graphs with Line-Symmetric Complements

Recall that many problems dealt with in this dissertation concern uniform factorizations of graphs; for a given graph $G$, we study equations of the type $G = G_1 \oplus G_2 \oplus \cdots \oplus G_t$ where each factor $G_i$ possesses a given graphical property $P$. In this section we apply some results from Chapter II to this graphical equation with $t = 2$, $G$ a complete graph, and $P$ the property of being line-symmetric. Stated another way, we investigate line-symmetric graphs whose complements are also line-symmetric. In view of Corollary 2-6, such graphs are equivalently described as being both line-symmetric and uniquely edge extendible, and will therefore be called LSUEE graphs. This study may be considered an extension of recent work by Akiyama and Harary [1] involving graphs that share a given property with their complements.
For positive integers \( m \) and \( n \), the graphs \( K(1, n) \), \( mK_n \), \( K(n, n) \), \( K_1 \cup K_n \), and \( K_m(n) \) are examples of LSUEE graphs.

Using results from Chapter II, we obtain various characterizations of LSUEE graphs. For example, the following statements are equivalent for a graph \( G \).

1. \( G \) is LSUEE.
2. \( \overline{G} \) is LSUEE.
3. Both \( G \) and \( \overline{G} \) are UEE.
4. \( G \) is UEE and UER.

This last equivalent deserves highlighting; a graph \( G \) is LSUEE if and only if \( G + e \cong G + f \) and \( G - x \cong G - y \) for all \( \{e, f\} \subseteq E(G) \) and for all \( \{x, y\} \subseteq E(G) \).

We organize our study of LSUEE graphs using the framework for line-symmetric graphs given in Theorem 2-3. We first consider, in a category denoted (a), those LSUEE graphs containing isolated vertices. Then, according to Theorem 2-3, every LSUEE graph not containing isolated vertices must belong to exactly one of the categories (b), (c), or (d) given below. Each LSUEE graph \( G \) will, then, satisfy one of the following statements.
(a) G contains isolated vertices.
(b) G is biregular and bipartite and G has a unique transitive bipartiton where a set of vertices of equal degree constitutes each of the partite sets.
(c) G is regular and bipartite, in which case G may or may not be point-symmetric.
(d) G is regular but not bipartite, in which case G must be point-symmetric.

We will provide structural characterizations of all LSUEE graphs that belong to categories (a), (b), and (c). Partial information and examples will be provided for LSUEE graphs in category (d).

The first step is our study is a characterization of LSUEE graphs containing more than one isolated vertex.

**Theorem 3-9.** A graph G of order p with at least two isolated vertices is LSUEE if and only if $G \cong \overline{K}_p$.

**Proof.** Let G be an LSUEE graph of order p which contains at least two isolated vertices. Our intent is to show that $\overline{G}$ is complete, so suppose to the contrary that $\overline{G}$ is not complete. Then G has at least one vertex with positive degree, say degree n, as
well as two vertices of degree 0. It follows that \( \overline{G} \) contains at least two vertices of degree \( p - 1 \) and at least one vertex of degree \( p - 1 - n \). Thus \( \overline{G} \) is a line-symmetric graph containing no isolated vertices with \( |\overline{G}| > 2 \). By Theorem 2-3, \( \overline{G} \) must be biregular and bipartite with a transitive bipartition \( V_{p-1} \cup V_{p-1-n} \). Because any two vertices of degree \( p - 1 \) are adjacent, we must have \( |V_{p-1}| = 1 \), contradicting the hypothesis that \( G \) has at least two isolated vertices. Thus \( \overline{G} \) must be complete, i.e. \( \overline{G} = K_p \), and \( G = \overline{K_p} \). The converse implication that \( \overline{K_p} \) is LSUEE is easily verified.

There exist LSUEE graphs containing exactly one isolated vertex, as our next result indicates. This theorem also serves the purpose of characterizing all biregular LSUEE graphs.

**Theorem 3-10.** A biregular graph \( G \) of order \( p \geq 3 \) is LSUEE if and only if, for \( n = p - 1 \), either \( G \cong K(1, n) \) or \( G \cong K_1 \cup K_n = \overline{K(1, n)} \).

**Proof.** Let \( G \) be a biregular LSUEE graph. We claim that either \( G \) or \( \overline{G} \) must have an isolated vertex.

Suppose, to the contrary, that neither \( G \) nor \( \overline{G} \) has an isolated vertex. By Theorem 2-3, \( G \) is bipartite with a transitive bipartition \( V(G) = W_1 \cup W_2 \) where
all vertices in $W_1$ have the same degree, say $a$, and all vertices in $W_2$ have the same degree, say $b$. Now $\overline{G}$ is also a biregular line-symmetric graph with no isolated vertices and in $\overline{G}$ every vertex of $W_1$ has degree $p-1-a$ and every vertex of $W_2$ has degree $p-1-b$. By another application of Theorem 2-3, the graph $\overline{G}$ is bipartite with a unique transitive bipartition $W_1 \cup W_2$. Since $W_1$ and $W_2$ are partite sets in both $G$ and $\overline{G}$, it follows that $|W_1| = |W_2| = 1$ implying that $a = b$. This contradicts the hypothesis that $G$ is biregular. Thus we need only consider the following two cases.

**Case 1.** Assume that $\overline{G}$ contains an isolated vertex $v_o$.

Let $V(G) = W_1 \cup W_2$ be the unique transitive bipartition of $G$. Without loss of generality we may assume that $v_o \in W_1$. Then $v_o$, being isolated in $\overline{G}$, is of degree $p-1$ in $G$, implying that $|W_1| = 1$ and $W_1 = \{v_o\}$. Therefore every vertex in $W_2$ is adjacent in $G$ only to $v_o$ and $G \cong K(1,n)$ where $n = |W_2| \geq 2$ in order that $G$ be biregular.

**Case 2.** Assume that $G$ contains an isolated vertex.

An argument similar to that of Case 1 may be employed to conclude that $\overline{G} \cong K(1,n)$ and $G \cong K_1 \cup K_n$ with $n \geq 2$. 

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Conversely, it is readily verified that graphs of these forms are LSUEE and biregular.

Using Theorems 3-9 and 3-10 we may now characterize those LSUEE graphs containing at least one isolated vertex.

Theorem 3-11. If a graph $G$ of order $p$ has an isolated vertex then $G$ is LSUEE if and only if either $G \cong \overline{K}_p$ or $G \cong K_1 \cup K_{p-1}$.

Proof. It is readily verified that all graphs of the forms specified are LSUEE. Suppose conversely that $G$ is an LSUEE graph with an isolated vertex. If $G$ has two or more isolated vertices then, by Theorem 3-9, $G \cong \overline{K}_p$. We suppose, then, that $G$ has exactly one isolated vertex. If $G$ is of order 1, then $G \cong \overline{K}_1$ and our result holds. If, on the other hand, $G$ is nontrivial, then $G$ has at least one vertex of positive degree and $|\bar{\theta}_G| \geq 2$. If $|\bar{\theta}_G| \geq 3$ then $|\bar{\theta}_G| \geq 3$ where $\overline{G}$ is a line-symmetric graph containing no isolated vertices. This contradicts Theorem 2-3. Thus $|\bar{\theta}_G| = 2$ and $G$ is both biregular and LSUEE. It follows that $p \geq 3$ and, by Theorem 3-10, $G \cong K(1, n)$ or $G \cong K_1 \cup K_n$ with $n = p - 1$. Since $G$ contains an isolated vertex, only
the second possibility can occur.

This completes our description of LSUEE graphs that belong to categories (a) and (b) of our re-statement of Theorem 2-3. We also have a characterization of regular biparitite LSUEE graphs, namely those in category (c). To prepare for that result we have a theorem about regular LSUEE graphs involving the following concept. A regular graph G which is neither complete nor empty is called strongly regular if there exist nonnegative integers s and t such that every pair of adjacent vertices of G have exactly s common neighbors and every pair of nonadjacent vertices of G have exactly t common neighbors. For vertices u and v in a graph G we let A(u, v) = {z ∈ V(G) | zu and zv ∈ E(G)} denote the set of vertices which are adjacent to both u and v.

**Theorem 3-12.** If G is a regular LSUEE graph that is neither complete nor empty, then G is strongly regular.

**Proof.** Let G be a regular LSUEE graph that is neither complete nor empty. Let e_o = u_o v_o ∈ E(G) and let f_o = x_o y_o ∈ E(\overline{G}) . Let u and v be arbitrary adjacent vertices of G , with e = uv . Then there exists \pi ∈ G(G) with \hat{\pi}(e) = e_o . Thus 
\{\pi(u), \pi(v)\} = \{u_o, v_o\} and \pi(A(u, v)) = A(u_o, v_o) .

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Since $\pi$ is injective, $|A(u^, v^)| = |A(u_o^, v_o^)|$.

For arbitrary nonadjacent vertices $x$ and $y$ of $G$ with $f = xy \in E(G)$ there exists $a \in G(G)$ such that $a(f) = f_o$. Again $|A(x^, y)| = |A(x_o^, y_o^)|$ since $a$ is injective. Thus $G$ is strongly regular with parameters $s = |A(u_o^, v_o^)|$ and $t = |A(x_o^, y_o^)|$.

We can neither prove nor disprove the conjecture suggested by Theorem 3-12, namely, that all strongly regular graphs are LSUEE. However, Theorem 3-12 is useful in the characterization of regular bipartite LSUEE graphs.

Theorem 3-13. Let $G$ be a bipartite graph which is regular of degree $d \neq 0$. Then $G$ is LSUEE if and only if there exists a positive integer $n$ such that either $G \cong nK_2$ or $G \cong K(n, n)$.

Proof. Suppose that $G$ is a bipartite LSUEE graph that is regular of degree $d \neq 0$. Then $G$ is non-empty. If $G$ is complete then $G \cong K_2$ because $K_m$ is not bipartite for $m > 2$, and $K_1$ is 0-regular.

Thus we need only consider the case where $G$ is neither empty nor complete. By Theorem 3-12, $G$ is strongly regular. Hence there exist nonnegative integers $s$ and $t$ such that any two adjacent vertices of $G$ have exactly $s$ common neighbors and any two non-
adjacent vertices of G have exactly t common
neighbors. Let V(G) = V₁ ∪ V₂ be a bipartition of G.
Then |V₁|d = |E(G)| = |V₂|d so that |V₁| = |V₂| = n,
for some positive integer n. It follows that d ≤ n,
creating the following two cases.

Case 1. Assume that d < n.

In this case there exist vertices x ∈ V₁ and
y ∈ V₂ which are not adjacent. Being in different
partite sets, x and y have no common neighbors.
Thus t = 0. Also, since d ≠ 0, there exist verti­
ces u ∈ V₁ and v ∈ V₂ which are adjacent. Since
u and v have no common neighbors, we conclude that
s = 0. Thus no two vertices of G have common neigh­
bors. It follows that the edges of G are independent
and G = nK₂.

Case 2. Assume that d = n.

In this case every vertex in V₁ is adjacent to
every vertex in V₂ so that G ≥ K(n, n).

Conversely, it is easy to verify that the graphs
nK₂ and K(n, n) are LSUEE.

The final category (d) of LSUEE graphs are those
which are regular and not bipartite. For these graphs

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no characterization has been found. We present, however, a class of LSUEE graphs which fall partially in this category.

**Theorem 3-14.** If $G$ is a disconnected graph of order $p$, then $G$ is LSUEE if and only if either there exist integers $n \geq 1$ and $t \geq 2$ such that $G \cong tK_n$ or $G \cong K_1 \cup K_{p-1}$.

**Proof.** If either $G \cong tK_n$ ($t \geq 2$) or $G \cong K_1 \cup K_{p-1}$ then $G$ is disconnected and LSUEE. Conversely, let $G$ be a disconnected LSUEE graph of order $p$. If any component of $G$ is trivial, then, using Theorem 3-11, either $G \cong K_p = pK_1$ or $G \cong K_1 \cup K_{p-1}$. Thus we may assume that every component of $G$ is nontrivial. Let $G_1, G_2, \ldots, G_t$, $t \geq 2$, be the components of $G$ and let $V_i = V(G_i)$ for $1 \leq i \leq t$. We intend to show that each $G_i$ is complete. Assume, to the contrary, that some component is not complete. Without loss of generality we will assume that $G_1$ is not complete. Then there exist vertices $u$ and $v$ in $V_1$ such that $e = uv$ is an edge of $G$. Also, if $x$ is any vertex of $G_2$, then $f = ux$ is an edge of $G$. Let $\alpha$ be any automorphism of $G$. Then $\alpha$ is also an automorphism of $G$, so that there exists an index $j$ in the range $1 \leq j \leq t$ such that $\alpha(V_1) = V_j$. Thus
{α(u), α(v)} ⊂ V_j so that α(e) ≠ f. It follows that G̅ is not line-symmetric, a contradiction. Thus every component of G is complete. It is then immediate that the line-symmetry of G forces all components of G to be of the same order. Thus G ≅ tK_n where n = |V_1| = |V_2| = ... = |V_t|.

We know of the existence of LSUEE graphs whose structure is not specified by any of the preceding theorems. These are all in category (d); that is to say they are LSUEE, regular, and not bipartite. The graph C_5 is one such graph. This graph is unusual in that it is a self-complementary line-symmetric graph and thus immediately one which is LSUEE. This suggests a whole new subcategory, namely all self-complementary line-symmetric graphs.

The graph C_5 is one of an infinite family of line-symmetric self-complementary graphs known as the Paley graphs [6, p. 14] which are constructed as follows. Let s be a prime power which is congruent to 1 modulo 4. Let V_s be the field with s elements and let I_s be the set of non-zero squares in V_s. The Paley graph P(s) has as its vertex set V_s and its edge set E_s = {xy | x, y ∈ V_s and y - x ∈ I_s}. For each i ∈ V_s and j ∈ I_s define α_i: V_s → V_s.
and \( \beta_j : V_s + V_s \) by \( \alpha_i(x) = x + i \) and \( \beta_j(x) = x \cdot j \).

**Lemma 3-15.** For each \( i \) in \( V_s \) and each \( j \) in \( I_s \), \( \alpha_i \) and \( \beta_j \) are automorphisms of \( P(s) \).

**Proof.** Let \( i \in V_s \) and \( j \in I_s \). For any \( x, y \in V_s \), 
\[(y + i) - (x + i) = y - x \] . Thus if \( xy \in E_s \) then 
\( \alpha_i(xy) \in E_s \). Also \( yj - xj = (y - x) \cdot j \) and \( I_s \)
is closed under multiplication. Thus if \( xy \in E_s \), then 
\( \beta_j(xy) \in E_s \).

**Theorem 3-16.** For every prime power \( s \equiv 1 \mod 4 \),
the Paley graph \( P(s) \) is LSUEE.

**Proof.** All of the Paley graphs are self-complementary \([6]\) . We will show that for every edge \( e \) of \( P(s) \)
there is an automorphism \( \pi \) of \( P(s) \) such that 
\( \pi(01) = e \). Let \( e = xy \in E_s \). Then \( y - x = j \in I_s \).
Let \( \pi = \alpha_x \beta_j \in G(P(s)) \). Then 
\( \pi(01) = \alpha_x(0j) = xx + j = xy = e \).

As mentioned above, \( C_5 \) is one of the Paley graphs, namely \( C_5 = P(5) \). The only other Paley graph we can specifically identify is \( P(9) = K_3 \times K_3 \). We note that the Paley graphs all fall into the regular non-bipartite category of LSUEE graphs.
CHAPTER IV

CHROMATIC MULTIPLICITY OF GRAPHS

In this chapter we address problems involving what might be called 'almost uniform factorizations' of graphs. We will seek decompositions of a given graph into edge-disjoint spanning subgraphs where all the subgraphs, with exactly one exception, possess a specified graph theoretic property \( P \).

This property \( P \) will be that of being \( n \)-chromatic for a given integer \( n \). The one exceptional subgraph in any such factorization will be required to have chromatic number less than \( n \).

In the first section we will determine, for any given graph \( G \) and appropriate integer \( n \), the least number of factors in such a decomposition. The companion problem of finding the largest number of such factors will be investigated in the second section.

Section 4.1

\( n \)-Minimal Chromatic Multiplicity

For a given positive integer \( n \) and a graph \( G \) it may not be possible to express \( G \) as the edge-sum of spanning \( n \)-chromatic subgraphs. For example, if \( n > \chi(G) \) then \( G \) has no \( n \)-chromatic subgraphs and

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clearly no such factorization is possible. If $G$ is nonempty and $n = 1$ then each subgraph must be empty and certainly no collection of empty graphs can have a nonempty edge sum. Even if limit $n$ to the range $2 \leq n \leq \chi(G)$ it is still possible that no such decomposition of $G$ will exist. For example, let $G = K_4$ and $n = 3$. Any factorization of $K_4$ into 3-chromatic factors would include at least two factors each containing an odd cycle. The graph $K_4$ does not possess two edge-disjoint odd cycles, and thus no such factorization exists.

We overcome this problem by considering 'almost uniform' factorizations where all but one of the factors have chromatic number $n$ and the one (possible empty) exceptional factor has chromatic number less than $n$. We seek to minimize the number of factors in such a decomposition.

More formally then, let $n$ be an integer and let $G$ be a graph with $2 \leq n \leq \chi(G)$. The $n$-minimal chromatic multiplicity of $G$, denoted by $m(n, \chi, G)$, is the least positive integer $k$ such that

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R$$

can be expressed as the edge sum of spanning subgraphs $G_1, G_2, \ldots, G_k, R$ where

$$\chi(G_i) = n \text{ for } i = 1, k \text{ and } \chi(R) < n.$$
Example 4-1. Consider the graph $G = K_1 + 3K_2$.

(a) 

(b) 

(c) 

(d) 

Figure 4-1
Figure 4-1 displays four decompositions of $G$ into edge-disjoint spanning subgraphs. In each case the last subgraph has chromatic number less than three while each of the preceding subgraphs has chromatic number equal to three.

The decomposition given in Figure 4.1(a) proves that $m(3, \chi, G) \leq 1$. Since $m(3, \chi, G)$ is a positive integer we conclude that $m(3, \chi, G) = 1$. This example is a particular case of the equality $m(n, \chi, G) = 1$ which holds if and only if $n = \chi(G)$.

Suppose now that $G$ and $H$ are isomorphic graphs and $n$ is an integer with $2 \leq n \leq \chi(G) = \chi(H)$. We show that the parameter $m(n, \chi, G)$ is well-defined by verifying that $m(n, \chi, G) = m(n, \chi, H)$. Assume that $m(n, \chi, G) = k$. Then $G$ has a decomposition $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R$ where $\chi(G_i) = n$ for $i = 1, k$ and $\chi(R) < n$. It follows that $H$ has a decomposition $H = H_1 \oplus H_2 \oplus \cdots \oplus H_k \oplus R'$ where $H_i \cong G_i$ for $i = 1, k$ and $R \cong R'$. Therefore $\chi(H_i) = n$ for $i = 1, k$ and $\chi(R') < n$. Thus $m(n, \chi, H) \leq k = m(n, \chi, G)$. A similar argument shows that $m(n, \chi, G) \leq m(n, \chi, H)$.

The following definition, appearing in Harary, Hsu and Miller [21], will prove useful. Let $n \geq 2$ be an integer. The $n$-particity of a graph $G$, denoted $\beta_n(G)$,
is the minimum number of factors in any factorization of $G$ into $n$-partite spanning subgraphs. The number $\beta_2(G)$ was termed the biparticity of the graph $G$ by Harary [20]. Nieminen [27] noted that every graph $G$ satisfies the inequality $\beta_2(G) \leq \beta_2(K_\chi(G))$, an observation that was improved by Matula [25] who was the first to establish the equality $\beta_2(G) = \lfloor \log_2 \chi(G) \rfloor$.

We will make use of the parameter $\beta_n(G)$ for integers $n \geq 2$ together with the following definition. A graph $G$ has lower $n$-particity if there exists a factorization of $G$ into $\beta_n(G)$ nonempty $n$-partite spanning subgraphs not all of which have chromatic number $n$.

Example 4-2. Consider the graph $K_4$. Note that $\beta_3(K_4) = 2$. Figure 4-2 exhibits a factorization of $K_4$ into two nonempty 3-partite spanning subgraphs of $K_4$ where one of the subgraphs does not have chromatic number 3. Thus $K_4$ has lower 3-particity. Next note that

![Figure 4-2](image-url)
$\beta_2(K_4) = 2$. Assume that there exist two edge-disjoint bipartite spanning subgraphs $G$ and $H$ of $K_4$ whose edge sum is $K_4$ where $\chi(H) \not= 2$. Because $H$ is bipartite, it follows that $\chi(H) = 1$ which means that $G \not= K_4$, contradicting the fact that $G$ is bipartite. Hence $K_4$ does not have lower 2-particity.

**Theorem 4-1.** Let $n$ be an integer and let $G$ be a graph with $2 \leq n \leq \chi(G)$. Then

$$m(n, \chi(G)) = \begin{cases} 
\lfloor \log_n \chi(G) \rfloor - 1 & \text{if } G \text{ has lower } n\text{-particity} \\
\lfloor \log_n \chi(G) \rfloor & \text{otherwise.}
\end{cases}$$

**Proof.** Let $m(n, \chi(G)) = k$. Then $G$ contains a collection of $k + 1$ edge-disjoint spanning subgraphs $G_1, G_2, \ldots, G_k, R$ whose edge sum is $G$ and where $\chi(G_i) = n$ for all $i = 1, 2, \ldots, k$ and $\chi(R) < n$.

Since a graph $H$ is $n$-partite if and only if $\chi(H) \leq n \leq |V(H)|$ it follows that $G_1, G_2, \ldots, G_k, R$ are all $n$-partite. Therefore

$$\beta_n(G) \leq k + 1.$$  \hspace{1cm} (1)

Assume next that $\beta_n(G) = t$ and that $E(G) = E_1 \cup E_2 \cup \ldots \cup E_t$ is a partition of $E(G)$ where the graph $F_i$ with vertex set $V(G)$ and edge set $E_i$ is
n-partite for \( i = 1, 2, \ldots, t \). It follows that the vertices of each subgraph \( F_i \) can be partitioned into \( n \) partite sets. Now we label the vertices of \( G \) where the label on each vertex \( v \) is a \( t \)-tuple \((s_1, s_2, \ldots, s_t)\) with \( s_i = j \) if vertex \( v \) belongs to the \( j^\text{th} \) partite set when considered as a vertex of the graph \( F_i \) (for each \( i = 1, 2, \ldots, t \)).

If two vertices \((s_1, s_2, \ldots, s_t), (s'_1, s'_2, \ldots, s'_t)\) are adjacent in \( G \) then the edge joining them must belong to \( E_j \) for some subscript \( j \) in the range \( 1 \leq j \leq t \). Hence these two vertices belong to different partite sets in the graph \( F_j \) implying that \( s_j \neq s'_j \) and hence \((s_1, s_2, \ldots, s_t) \neq (s'_1, s'_2, \ldots, s'_t)\). This labeling of \( V(G) \) is therefore a coloring of \( G \).

Each of these vertex labels are \( t \)-tuples where each of the \( t \) co-ordinates can assume any one of \( n \) possible values. The set of all possible such labels therefore has cardinality \( n^t \). It follows that the number of different labels used to label \( V(G) \) is at most \( n^t \). Therefore \( \chi(G) \leq n^t \) which yields \( \log_n \chi(G) \leq t \).

Since \( \beta_n(G) = t \) is an integer we conclude that

\[
\beta_n(G) \geq \{\log_n \chi(G)\}.
\]

Combining this inequality with (1) produces \( k + 1 \geq \{\log_n \chi(G)\} \) or

\[
m(n, \chi, G) \geq \{\log_n \chi(G)\} - 1 \quad (2)
\]
Now recall that \( G = F_1 \oplus F_2 \oplus \cdots \oplus F_t \), where each \( F_i \) is \( n \)-partite and \( t = \beta_n(G) \). Note that \( \chi(F_i) \leq n \) for \( i = 1, 2, \ldots, t \). We now show that \( m(n, \chi, G) \leq t \).

If \( \chi(F_i) = n \) for each \( i \) with \( 1 \leq i \leq t \), then we write \( G = F_1 \oplus F_2 \oplus \cdots \oplus F_t \oplus R \) where \( R \) is the empty graph on \( V(G) \) so that \( \chi(R) = 1 < n \). By definition of \( n \)-minimal chromatic multiplicity, it follows that \( m(n, \chi, G) \leq t \).

The only other possibility is that some graphs in the set \( \{ F_i | 1 \leq i \leq t \} \) have chromatic number less than \( n \) (i.e. \( G \) has lower \( n \)-particity). Without loss of generality we may assume that

\[
n \geq \chi(F_1) \geq \chi(F_2) \geq \cdots \geq \chi(F_t) .
\]

Let \( s \) be the least positive integer with \( \chi(F_i) < n \) when \( i \) lies in the range \( s \leq i \leq t \). Let \( H = F_s \oplus F_{s+1} \oplus \cdots \oplus F_t \).

If \( s = 1 \) so that none of the \( F_i \) have chromatic number \( n \), then edges may be removed from \( F_t \) (and from \( F_{t-1}, F_{t-2}, \ldots, F_2 \) if need be) and allocated to \( F_1 \) until this augmented \( F_1 \) does have chromatic number \( n \). This augmentation is certainly possible since \( \chi(G) \geq n \).

Hence we may assume, at the onset, that \( \chi(F_1) = n \), that is to say, that \( s \geq 2 \). Under this assumption we may write \( G = F_1 \oplus F_2 \oplus \cdots \oplus F_{s-1} \oplus H \). If \( \chi(H) < n \) then, since \( \chi(F_1) = \chi(F_2) = \cdots = \chi(F_{s-1}) = n \) we
conclude \( m(n, \chi, G) \leq s - 1 \leq t - 1 < t \). If \( \chi(H) > n \) then \( H \) contains an \( n \)-chromatic subgraph \( H_1 \), formed by augmenting \( F_s \) with edges from \( F_{s+1}, F_{s+2}, \ldots, F_t \), so that \( H = H_1 \oplus H_2 \) where \( E(H_2) = E(H) - E(H_1) \) and thus \( G = F_1 \oplus \cdots \oplus F_{s-1} \oplus H_1 \oplus H_2 \). If \( \chi(H_2) < n \) then we conclude that \( m(n, \chi, G) \leq s - 1 \). If \( \chi(H_2) \geq n \) then \( H_2 \) contains an \( n \)-chromatic spanning subgraph.

This process may be continued and eventually we will write \( G = F_1 \oplus F_2 \oplus \cdots \oplus F_{s-1} \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_{\ell} \oplus H_{\ell+1} \) with \( \chi(H_{\ell+1}) < n \) and \( \chi(H_i) = n \) for \( i = 1, 2, \ldots, \ell \). Thus \( m(n, \chi, G) \leq s - 1 + \ell \).

Because \( H_1 \oplus H_2 \oplus \cdots \oplus H_{\ell} \oplus H_{\ell+1} = H \)

\( = F_s \oplus F_{s+1} \oplus \cdots \oplus F_t \) and each of the \( n \)-chromatic graphs \( H_1, H_2, \ldots, H_{\ell} \) were formed by augmenting a spanning subgraph of the corresponding graph in the list \( F_s, F_{s+1}, \ldots, F_t \) with edges from graphs following it in the list \( F_s, F_{s+1}, \ldots, F_t \), it follows that

\( \ell \leq t - s + 1 = |\{F_s, F_{s+1}, \ldots, F_t\}| \). This inequality implies \( s - 1 + \ell \leq t \) which produces, in all cases,

\[
m(n, \chi, G) \leq t = \beta_n(G) \quad (3)
\]
Our next step in this chain of inequalities is to prove that \( \beta_n(G) \leq \{\log_n \chi(G)\} \). To that end we first establish a result for graphs with chromatic number \( n^m \) for any positive integer \( m \). We claim that if \( \chi(H) = n^m \) then \( \beta_n(H) \leq m \).

This claim will be proved by induction on \( m \). If \( m = 1 \) then \( \chi(H) = n \) and thus \( H \) is \( n \)-partite and it follows that \( \beta_n(H) = 1 \) as desired. Assume the result holds for \( m = r \) and let \( J \) be a graph with \( \chi(J) = n^{r+1} \). Let these \( n^{r+1} \) colors used in coloring \( V(J) \) be partitioned into \( n \) sets with \( n^r \) colors in each set. Denote these sets of colors by \( C_1, C_2, \ldots, C_n \). For \( j = 1, 2, \ldots, n \) let \( J_j \) be the subgraph of \( J \) induced by those vertices of \( J \) colored with any of the \( n^r \) colors of \( C_j \). Clearly \( \chi(J_j) = n^r \) for \( j = 1, 2, \ldots, n \). By the inductive assumption, \( \beta_n(J_j) \leq r \) for \( j = 1, 2, \ldots, n \).

For \( q = 1, 2, \ldots, n \) the set \( E(J_q) \) may be partitioned into \( \beta_n(J_q) \) subsets \( E_1^q, E_2^q, \ldots, E_{\beta_n(J_q)}^q \) with each subset inducing an \( n \)-partite graph in \( J_q \). Note that \( E_1^d \cup E_2^d \cup \ldots \cup E_n^d \) induces an \( n \)-partite graph in \( J \) for \( 1 \leq d \leq \max\{\beta_n(J_q)\}_{q=1}^n \) since, for such an integer \( d \), \( E_1^d \), \( E_2^d \), \( E_3^d \), \ldots, \( E_n^d \) are each \( n \)-partite in \( J \) and have pairwise disjoint vertex sets.
Thus the graph $I$ with vertex set $V(J)$ and edge set $E(J_1) \cup E(J_2) \cup \cdots \cup E(J_n)$ satisfies $\beta_n(I) \leq \max\{\beta_n(J_q) | q = 1, n\} \leq r$. The remaining edges of $J$, which are $E(J) - E(I)$, induce an $n$-partite subgraph of $J$ with partite sets $V(J_1), V(J_2), \ldots, V(J_n)$. Thus $\beta_n(J) \leq 1 + r$, completing the induction and the proof of our claim.

We now reconsider our original graph $G$. Recall that $\chi(G) \geq n$. Thus there exists a positive integer $r$ with $n^{r-1} < \chi(G) \leq n^r$. Let $J$ be a supergraph of $G$ with $\chi(J) = n^r$. By the claim just established, $\beta_n(J) \leq r = \log_n \chi(J)$. Since $G$ is a subgraph of $J$, $\beta_n(G) \leq \beta_n(J)$ yielding $\beta_n(G) \leq \log_n \chi(J) = r$. Now $n^{r-1} < \chi(G) \leq n^r$ implies $r - 1 < \log_n \chi(G) \leq r$ which forces $\{\log_n \chi(G)\} = r$. Hence

$$\beta_n(G) \leq \{\log_n \chi(G)\}. \quad (4)$$

Combining inequalities $(3)$ and $(4)$ produces

$$m(n, \chi, G) \leq \{\log_n \chi(G)\}. \quad (5)$$

Now joining $(2)$ and $(5)$ we may write

$$\{\log_n \chi(G)\} - 1 \leq m(n, \chi, G) \leq \{\log_n \chi(G)\}.$$

Finally we determine under what conditions $m(n, \chi, G)$ attains these bounds.
Assume first that $G$ has lower $n$-partiticy. Then there exists a collection of $\beta_n(G) = t$ $n$-partite spanning subgraphs of $G$, say $G_1, G_2, \ldots, G_t$ with $G = G_1 \oplus G_2 \oplus \cdots \oplus G_t$ and where not all of the $G_i$ have chromatic number $n$. Without loss of generality we assume that $\chi(G_t) < n$. By employing an edge re-allocation scheme similar to that used earlier in the proof, it follows that $m(n, \chi, G) \leq t - 1 = \beta_n(G) - 1$. Using (4) we have $m(n, \chi, G) \leq \{\log_n \chi(G)\} - 1$. Combining this last inequality with (2) yields $m(n, \chi, G) = \{\log_n \chi(G)\} - 1$.

Next assume that $G$ does not have lower $n$-partiticy. Hence for every collection of $\beta_n(G)$ $n$-partite edge-disjoint spanning subgraphs of $G$ whose edge sum is $G$, each subgraph has chromatic number $n$. Suppose, as before, that $m(n, \chi, G) = k$ and let $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R$ be a decomposition of $G$ into spanning subgraphs with $\chi(G_i) = n$ for $i = 1, 2, \ldots, k$ and $\chi(R) < n$. Assume, for the moment, that $k < \beta_n(G)$. In our argument so far, we have proved that $\beta_n(G) = \{\log_n \chi(G)\}$. Hence (2) may be rewritten $m(n, \chi, G) \leq \beta_n(G) - 1$. Under our assumption, we conclude that $m(n, \chi, G) = \beta_n(G) - 1$, or, equivalently, that $\beta_n(G) = k + 1$. Now $(G_1, G_2, \ldots, G_k, R)$ is a collection of $\beta_n(G)$ edge-disjoint $n$-partite spanning
subgraphs of $G$ whose edge sum is $G$. It follows that $\chi(R) = n$, a contradiction. Thus our assumption that $k < \beta_n(G)$ is false and we may write $m(n, \chi, G) \geq \beta_n(G) = \{\log_n \chi(G)\}$. Together with inequality (5) we conclude $m(n, \chi, G) = \{\log_n \chi(G)\}$ in the case where $G$ does not have lower $n$-particity.

The following consequence of the preceding proof was proved for $n = 2$ and stated for $n \geq 2$ in Harary, Hsu, and Miller [21].

**Corollary 4-2.** If $G$ is a graph and $n$ is an integer with $2 \leq n \leq \chi(G)$, then $\beta_n(G) = \{\log_n \chi(G)\}$.

We illustrate the use of Theorem 4-1 and its corollary by computing $m(n, \chi, W_t)$ for all $t \geq 3$ and $2 \leq n \leq \chi(W_t)$ where $W_t = C_t + K_1$ is the wheel of order $t + 1$.

**Proposition 4-3.**

$$m(n, \chi, W_t) = \begin{cases} 2 & \text{if } n = 2 \\ 1 & \text{otherwise.} \end{cases}$$
Proof. Note that $\chi(W_t) = 3$ or $4$ according as $t$ is even or odd. If $n = 2$ then for every $t \geq 4$ there is no factorization of $W_t$ into $\beta_2(W_t) = \{\log_2 \chi(W_t)\} = 2$ bipartite spanning subgraphs one of which is empty. Thus $W_t$ does not have lower bipartiticy so that $m(2, \chi, W_t) = \{\log_2 \chi(W_t)\} = 2$. If $n = 3$ and $t$ is odd then $(C_t \cup K_1) \oplus K(1, t)$ is a factorization of $W_t$ into $\beta_3(W_t) = \{\log_3 \chi(W_t)\} = 2$ 3-partite factors one of which has chromatic number less than 3 so that $W_t$ has lower 3-particity and $m(3, \chi, W_t) = \{\log_3 \chi(W_t)\} - 1 = 1$. The only remaining cases are $n = 3$ with $t$ even and $n = 4$ in which case the inequality $n \leq \chi(W_t)$ forces $t$ to be odd.

In both of these cases $n = \chi(W_t)$ so that $m(n, \chi, W_t) = 1$.

Section 4.2

$n$-Maximal Chromatic Multiplicity

In the previous section we determined the $n$-minimal chromatic multiplicity of a graph $G$ where $n$ and $G$ satisfied the inequality $2 \leq n \leq \chi(G)$. That work can be viewed as an effort to find spanning edge-disjoint $n$-chromatic subgraphs of $G$ with maximum size. In this section we modify this factorization problem by searching for spanning edge-disjoint $n$-chromatic sub-
graphs of \( G \) with minimum size

For a given positive integer \( n \) and graph \( G \) we will seek in this section the largest integer \( k \) for which there exists an 'almost uniform' factorization of \( G \) into \( k + 1 \) factors where \( k \) of the factors have chromatic number \( n \) and the one (possibly empty) remaining factor has chromatic number less than \( n \).

The necessity for requiring one factor to be less than \( n \)-chromatic and for the integer \( n \) to satisfy
\[ 2 \leq n \leq \chi(G) \]

is the same in this modified problem as it was for the determination of \( m(n, \chi, G) \) in Section 4.1.

The definition of our new parameter is as follows.

Let \( n \) be an integer and let \( G \) be a graph with \( 2 \leq n \leq \chi(G) \). The \( n \)-maximal chromatic multiplicity of \( G \), denoted \( M(n, \chi, G) \), is the largest positive integer \( k \) such that
\[ G = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R \]
can be expressed as the edge sum of spanning subgraphs \( G_1, G_2, \ldots, G_k, R \) where \( \chi(G_i) = n \) for \( i = 1, 2, \ldots, k \) and \( \chi(R) < n \).

**Example 4-3.** Consider the Petersen graph, denoted \( P \).

We compute \( M(n, \chi, P) \) for \( n = 2 \) and \( n = 3 = \chi(P) \).

Let \( E(P) = \{ e_1, e_2, \ldots, e_{15} \} \) and consider the factorization \( P = G_1 \oplus G_2 \oplus \cdots \oplus G_{15} \oplus R \) with
\[ E(G_i) = \{e_i\}, \quad E(R) = \emptyset, \quad \chi(G_i) = 2, \quad \text{and} \quad \chi(R) = 1 \]
for \( i = 1, 2, \ldots, 15 \). Thus \( M(2, \chi, P) \geq 15 \).

If \( M(2, \chi, P) = k \) and \( P = P_1 \oplus P_2 \oplus \cdots \oplus P_k \oplus R \)
is a corresponding almost uniform factorization, then each \( P_j \) is 2-chromatic and thus nonempty so that \( k \leq |E(P)| = 15 \). Thus \( M(2, \chi, P) = 15 \). This example is a particular case of the equality \( M(n, \chi, G) = |E(G)| \) which holds for any nonempty graph \( G \) if and only if \( n = 2 \). Next we take \( n = 3 \). Consider the factorization of \( P \) given in Figure 4-3.

\[ \text{Figure 4-3} \]

Note that the first two factors are 3-chromatic and that the third factor is 2-chromatic. Thus \( M(3, \chi, P) \geq 2 \). If \( M(3, \chi, P) \geq 3 \) then \( P \) must contain 3 edge-disjoint odd cycles. If any of these cycles has order at least 7 then, since \( |E(P)| = 15 \), at least one of the other two cycles must be \( C_3 \) an impossibility.
since $P$ has girth 5. Thus all three cycles have order 5 which implies, since $|E(P)| = 15$, that $E(P)$ can be partitioned into cycles, another impossibility for an 3-regular graph. Therefore $M(3, \chi, P) = 2$.

Our first result in this section verifies that $n$-maximal chromatic multiplicity is a well-defined parameter.

**Theorem 4-4.** If $G$ and $H$ are isomorphic graphs and $n$ is an integer with $2 \leq n \leq \chi(G) = \chi(H)$ then $M(n, \chi, G) = M(n, \chi, H)$.

**Proof.** Assume that $M(n, \chi, G) = k$ and that $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R$ is a factorization of $G$ with $\chi(G_i) = n$ for $i = 1, 2, \ldots, k$ and $\chi(R) < n$.

Because $G$ and $H$ are isomorphic, there exists a factorization $H = H_1 \oplus H_2 \oplus \cdots \oplus H_k \oplus R'$ with $H_i \cong G_i$ for $i = 1, 2, \ldots, k$ and $R' \cong R$. Therefore $\chi(H_i) = n$ for $i = 1, 2, \ldots, k$ and $\chi(R') < n$. It follows that $M(n, \chi, H) \geq k = M(n, \chi, G)$.

An analogous argument shows that $M(n, \chi, G) \geq M(n, \chi, H)$.
Note that for a graph $G$ and integer $n$ satisfying $2 \leq n \leq \chi(G)$, the number $M(n, \chi, G)$ exists and $1 \leq M(n, \chi, G) \leq |E(G)| = q$. This upper bound follows from the observation that an $n$-chromatic spanning subgraph of $G$ is nonempty when $n \geq 2$.

We have no single theorem that will specify $M(n, \chi, G)$ for every integer $n$ and graph $G$ satisfying $2 \leq n \leq \chi(G)$. We present, therefore, several results computing this parameter for particular classes of graphs and results providing inequalities and other partial information about it.

Our first result yields an improved upper bound for $M(n, \chi, G)$ when $n$ is large.

**Theorem 4-5.** If $G$ is a (p, q) graph and if $2 \leq n \leq \chi(G)$, then

$$M(n, \chi, G) \leq \left[ \frac{2q(p-1)}{p(n-1)^2} \right].$$

**Proof.** Assume that $M(n, \chi, G) = k$ and that $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R$ is a factorization of $G$ with $\chi(G_i) = n$ for $i = 1, 2, \ldots, k$ and $\chi(R) < n$. Let $q_i = |E(G_i)|$ for $i = 1, 2, \ldots, k$. Using Theorems by Wilf [35] and Harary, King, Mowshowitz, and Read [22] it can be show that for each $i$ satisfying $1 \leq i \leq k$, $\chi(G_i) \leq 1 + \frac{\sqrt{2q_i(p-1)}}{p}$. Thus for every $i$ satisfying $1 \leq i \leq k$ we have $p(n-1)^2/2(p-1) \leq q_i$. 

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Since each \( G_i \) has at least \( p(n-1)^2/(p-1) \) edges, it follows that \( k \leq \left[ q/(p(n-1)^2/(p-1)) \right] = \left[ 2q(p-1)/(p(n-1)^2) \right]. \)

This bound is slightly improved by our next result, which does not depend on the order of the given graph.

**Theorem 4-6.** If \( G \) is a graph of size \( q \) and \( n \) an integer with \( 2 \leq n \leq \chi(G) \), then \( M(n, \chi, G) \leq \left[ 2q/n(n-1) \right] \).

**Proof.** Let \( M(n, \chi, G) = k \) and let \( G = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R \) be an almost uniform factorization of \( G \). For \( i \) satisfying \( 1 \leq i \leq k \), the graph \( G_i \) is \( n \)-chromatic so that \( |E(G_i)| \geq \binom{n}{2} \). Let

\[
m = \left| \bigcup_{j=1}^{k} E(G_j) \right|.
\]

Then \( k \binom{n}{2} \leq m \leq q \) yielding

\[
k \leq \left[ q/\binom{n}{2} \right] = \left[ 2q/n(n-1) \right].
\]

Note that the bound in Theorem 4-6 is achieved for any nonempty graph \( G \) when \( n = 2 \). Next we illustrate how these bounds can be used to give sufficient conditions for \( M(n, \chi, G) \) to equal \( 1 \).

**Proposition 4-7.** If \( G \) is a graph of size \( q \) and \( n \) is an integer satisfying \( 2 \leq n \leq \chi(G) \) and
q < n(n - 1) then \( M(n, \chi, G) = 1 \).

**Proof.** Observe that \( q < n(n - 1) \) yields \( \frac{2q}{n(n - 1)} < 2 \). The result follows from Theorem 4-6 and the fact that \( M(n, \chi, G) \) is a positive integer. 

The next result follows in a similar fashion from Theorem 4-5.

**Proposition 4-8.** If \( G \) is a \((p, q)\) graph and \( n \) is an integer with \( 2 \leq n \leq \chi(G) \) and \( q < p(n - 1)^2/(p - 1) \) then \( M(n, \chi, G) = 1 \).

Recall now that the Petersen graph is 3-chromatic and that \( M(3, \chi, P) = 2 \). Thus \( n = \chi(G) \) is not a sufficient condition for \( M(n, \chi, G) \) to equal 1. For a certain class of \( n \)-chromatic graphs, however, this implication is valid.

**Theorem 4-9.** If \( G \) is a minimally \( n \)-chromatic graph with \( n \geq 2 \), then \( M(n, \chi, G) = 1 \).

**Proof.** Let \( G \) be a minimally \( n \)-chromatic graph with \( 2 \leq n \leq \chi(G) \) and assume, to the contrary, that \( M(n, \chi, G) = k \geq 2 \). Let \( G = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R \) be an almost uniform factorization of \( G \). Since \( n \geq 2 \), and both \( G_1 \) and \( G_2 \) are \( n \)-chromatic, these graphs are nonempty. Let \( e \in E(G_2) \). Then \( G_1 \) is
a spanning subgraph of $G - e$ implying

$$\chi(G_1) \leq \chi(G - e) = n - 1$$
contradicting $\chi(G_1) = n$.

Note that for $p \geq 2$ the equation

$$M(p, \chi, K_p) = 1$$
follows from Theorem 4-9. The notion of a minimally $n$-chromatic graph can be used to give an alternative definition of $M(n, \chi, G)$ as we now show.

Theorem 4-10. Let $G$ be a graph and $n$ an integer with $2 \leq n \leq \chi(G)$. Then $M(n, \chi, G)$ is the largest positive integer $k$ such that $G$ can be expressed as the edge sum of spanning subgraphs $G_1, G_2, \ldots, G_k, R$ where $G_i$ is minimally $n$-chromatic for $i = 1, 2, \ldots, k$ and $\chi(R) < n$.

Proof. Let $m$ be the largest positive integer such that $G$ can be expressed as the edge sum of spanning subgraphs $G_1, G_2, \ldots, G_m, R$ where $G_i$ is minimally $n$-chromatic for $i = 1, 2, \ldots, m$ and $\chi(R) < n$. Because $\chi(G_i) = n$ for $i = 1, 2, \ldots, m$ it follows immediately that $M(n, \chi, G) \geq m$. We establish the reverse inequality as follows. Let $M(n, \chi, G) = k$ and let $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k \oplus R'$ be an almost uniform factorization of $G$. Each $H_i$, being $n$-chromatic, has a minimally $n$-chromatic spanning subgraph $H_i^1$. 

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Thus \( G = H'_1 \oplus H'_2 \oplus \cdots \oplus H'_k \oplus R'' \) where \( E(R'') = E(R') \cup ( \bigcup_{i=1}^{k} E(H'_i - H'_i) ) \). It follows that \( m \geq k = M(n, \chi, G) \).

We next present a result that will allow us to work only with connected graphs.

**Theorem 4-11.** If \( G \) is a disconnected graph with components \( C_1, C_2, \ldots, C_k (k \geq 2) \) and if \( n \) is an integer satisfying \( 2 \leq n \leq \min \{\chi(C_i) | 1 \leq i \leq k\} \) then

\[
M(n, \chi, G) = \sum_{i=1}^{k} M(n, \chi, C_i).
\]

**Proof.** For each \( i = 1, 2, \ldots, k \) let \( M(n, \chi, C_i) = m_i \) and let \( C_i = G_{i1} \oplus G_{i2} \oplus \cdots \oplus G_{im_i} \oplus R_i \) be an almost uniform factorization of the component \( C_i \). For each ordered pair \( (i, j) \) satisfying \( 1 \leq i \leq k \) and \( 1 \leq j \leq m_j \) define a graph \( H(i, j) \) as follows.

\[ V(H(i, j)) = V(G) \quad \text{and} \quad E(H(i, j)) = E(G_{ij}). \]

For each \( i \) satisfying \( 1 \leq i \leq k \) define \( R'_i \) to be the spanning subgraph of \( G \) with edge set \( E(R'_i) \). Then \( G \) is the edge sum of the graphs in the set \( \{H(i, j) | 1 \leq i \leq k, 1 \leq j \leq m_j \} \cup \{R'_i | 1 \leq i \leq k\} \). Since \( \chi(H(i, j)) = \chi(G_{ij}) = n \) for all \( i \) and \( j \), it follows that
To verify that equality holds above we assume, to the contrary, that \( M(n, \chi, G) = \lambda > \sum_{i=1}^{k} M(n, \chi, C_i) \) and that \( G = G_1 \oplus G_2 \oplus \cdots \oplus G_\lambda \oplus R \) is an almost uniform factorization of \( G \). For \( i = 1, 2, \ldots, \lambda \) the graph \( G_i \) is a spanning subgraph of \( G \) so that

\[
G_i = \bigcup_{r=1}^{k} C_{ir} \quad \text{where} \quad V(C_{ir}) = V(C_r) \quad \text{and} \quad E(C_{ir}) = E(C_r) \cap E(G) \].

Note that \( n = \chi(G_i) = \max_{1 \leq r \leq k} \chi(C_{ir}) \) for \( i = 1, 2, \ldots, \lambda \). Thus for each \( i \) with \( 1 \leq i \leq \lambda \) there is an integer \( r \) satisfying \( 1 \leq r \leq k \) such that \( \chi(C_{ir}) = n \). Hence there are at least \( \lambda \) edge-disjoint spanning subgraphs of the components \( C_1, C_2, \ldots, C_k \) with chromatic number \( n \).

This contradicts the fact that \( \sum_{i=1}^{k} M(n, \chi, C_i) \) is the maximum number of spanning edge-disjoint subgraphs of \( C_1, C_2, \ldots, C_k \) with chromatic number \( n \).
Within the class of connected graphs we need only work with those that contain no cut-vertices as the following result shows.

**Theorem 4-12.** Let $G$ be a connected graph with blocks $B_1, B_2, \ldots, B_k$ ($k \geq 2$) and let $n$ be an integer satisfying $2 \leq n \leq \min\{\chi(B_i) \mid 1 \leq i \leq k\}$. Then

$$M(n, \chi, G) = \sum_{i=1}^{k} M(n, \chi, B_i).$$

**Proof.** For each $i = 1, 2, \ldots, k$ let $M(n, \chi, B_i) = m_i$ and let $B_i = G_{i1} \oplus G_{i2} \oplus \cdots \oplus G_{im_i} \oplus R_i$ be an almost uniform factorization of the block $B_i$. For each ordered pair $(i, j)$ satisfying $1 \leq i \leq k$ and $1 \leq j \leq m_j$ define a graph $H(i, j)$ as follows.

$V(H(i, j)) = V(G)$ and $E(H(i, j)) = E(G_{ij})$. For each $i$ satisfying $1 \leq i \leq k$ define $R_i'$ to be the spanning subgraph of $G$ with edge set $E(R_i)$. Then $G$ is the edge sum of the graphs in the set

$$\{H(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq m_j\} \cup \{R_i' \mid 1 \leq i \leq k\}.$$

Since $\chi(H(i, j)) = \chi(G_{ij}) = n$ for all $i$ and $j$ it follows that $M(n, \chi, G) \geq \sum_{i=1}^{k} M(n, \chi, B_i)$. 

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To verify that equality holds above we assume, to the contrary, that \( M(n, \chi, G) = \ell > \sum_{i=1}^{k} M(n, \chi, B_i) \)
and that \( G = G_1 \oplus G_2 \oplus \cdots \oplus G_\ell \oplus R \) is an almost uniform factorization of \( G \). For \( i = 1, 2, \ldots, \ell \) the graph \( G_i \) is a spanning subgraph of \( G \). For \( i = 1, 2, \ldots, \ell \) we identify \( k \) edge-disjoint subgraphs of \( G_i \) denoted \( B_{ir} \) for \( 1 \leq r \leq k \) as follows. \( V(B_{ir}) = V(B_r) \) and \( E(B_{ir}) = E(B_r) \cap E(G_i) \). Note that for all \( i \) and \( r \), the graph \( B_{ir} \) is a union of one or more blocks of \( G_i \) so that \( n = \chi(G_i) = \max_{1 \leq r \leq k} \chi(B_{ir}) \). Hence there are at least \( \ell \) edge-disjoint spanning subgraphs of the blocks \( B_1, B_2, \ldots, B_k \) with chromatic number \( n \). This contradicts the fact that \( \sum_{i=1}^{k} M(n, \chi, B_i) \) is the maximum number of spanning edge-disjoint subgraphs of \( B_1, B_2, \ldots, B_k \) with chromatic number \( n \).

In the next two results we describe the effect on \( M(n, \chi, G) \) of modifying either the integer \( n \) or the graph \( G \).
Theorem 4-13. Let $G$ be a graph and let $n_1$ and $n_2$ be integers with $2 \leq n_1 \leq n_2 \leq \chi(G)$. Then

$$M(n_2, \chi, G) \leq M(n_1, \chi, G).$$

Proof. Let $M(n_2, \chi, G) = k$ and let $G = G_1 \oplus \cdots \oplus G_k \oplus R$ be an almost uniform factorization of $G$ with $\chi(G_i) = n_2$ for $i = 1, 2, \ldots, k$ and $\chi(R) < n_2$. Each of the graphs $G_i$ has an $n_1$-chromatic spanning subgraph $H_i$ so that $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k \oplus R'$ where $V(R') = V(G)$ and

$$E(R') = E(R) \cup \left[ \bigcup_{i=1}^{k} E(G_i) - E(H_i) \right].$$

Thus $M(n_1, \chi, G) \geq k$.

Theorem 4-14. If $G$ is a spanning subgraph of $H$ and $n$ is an integer satisfying $2 \leq n \leq \chi(G)$, then

$$M(n, \chi, G) \leq M(n, \chi, H).$$

Proof. Let $M(n, \chi, G) = k$ and let $G = G_1 \oplus \cdots \oplus G_k \oplus R$ be an almost uniform factorization of $G$. Since $G$ is a spanning subgraph of $H$, it follows that $H = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R'$ where $R$ is a spanning subgraph of $R'$. Hence $M(n, \chi, H) \geq k$. 

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We now show that for any pair of positive integers \( n \) and \( t \) with \( n \geq 2 \), there exist many graphs \( G \) with \( M(n, \chi, G) = t \).

**Theorem 4-15.** Let \( t \) and \( n \) be positive integers with \( n \geq 2 \). Let \( G_1, G_2, \ldots, G_t \) be any collection of vertex-disjoint connected minimally \( n \)-chromatic graphs.

Let \( G = \bigcup_{i=1}^{t} G_i \) and let \( H \) be the graph obtained by identifying one vertex of each \( G_i \). Then

\[ M(n, \chi, G) = M(n, \chi, H) = t. \]

**Proof.** Note that \( 2 \leq n \leq \min\{\chi(G), \chi(H)\} = n \) so that \( M(n, \chi, G) \) and \( M(n, \chi, H) \) are defined. By Theorem 4-9 we have \( M(n, \chi, G_i) = 1 \) for \( i = 1, 2, \ldots, t \). Using Theorem 4-11 we conclude

\[ M(n, \chi, G) = \sum_{i=1}^{t} M(n, \chi, G_i) = t. \]

Now for \( i = 1, 2, \ldots, t \) the graph \( G_i \) is minimally \( n \)-chromatic for some \( n \geq 2 \) and is connected so that it can contain no isolated vertices. It follows that each \( G_i \) is a block and that the blocks of \( H \) are \( G_1, G_2, \ldots, G_t \). We employ Theorem 4-12 to conclude

\[ M(n, \chi, H) = \sum_{i=1}^{t} M(n, \chi, G_i) = t. \]
If \( G \) is a nonempty bipartite graph, then 
\( 2 \leq n \leq \chi(G) \) implies \( n = 2 \) so that 
\[ M(n, \chi, G) = |E(G)|. \]
It follows, of course, that for \( n = 2 \) and \( P_t \) the path of order \( t \) we have
\[ M(n, \chi, P_t) = t - 1. \]
Also, if \( Q_m \) denotes the \( m \)-cube, then
\[ M(n, \chi, Q_m) = m2^{m-1}. \]
Graphs with higher chromatic number allow more than one value for \( n \). For example, if \( C_t \) denotes the cycle of order \( t \) then
\[ M(2, \chi, C_t) = t \] and \[ M(3, \chi, C_t) = 1. \] The first equality follows from \( t = |E(C_t)| \) and the second is a consequence of the fact that \( C_t \) is minimally 3-chromatic when \( t \) is odd.

We now investigate a class of graphs some of which have chromatic number 4 in the following result.

**Proposition 4-16.** Let \( W_t \) denote the wheel of order \( t + 1 \) for integers \( t \geq 3 \). Then

\[ M(n, \chi, W_t) = \begin{cases} 
2t & \text{if } n = 2 \\
[t/2] & \text{if } n = 3 \\
1 & \text{if } n = 4.
\end{cases} \]

**Proof.** Note first that \( \chi(W_t) = \chi(C_t + K_1) \)
\[ = \chi(C_t) + \chi(K_1) = 3 \] or 4 according as \( t \) is even or odd. Thus \( M(n, \chi, W_t) \) is defined for \( 2 \leq n \leq 4 \).
If \( n = 2 \) then \( M(n, \chi, W_t) = |E(W_t)| = 2t \). If
n = 3 then let \( M(n, \chi, W_t) = k \) and let \( W_t = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R \) be an almost uniform factorization of \( W_t \). For \( i = 1, 2, \ldots, k \) \( \chi(G_i) = 3 \) so that \( G_i \) must contain an odd cycle. It follows that \( M(n, \chi, W_t) \) equals the maximum number of edge-disjoint odd cycles in \( W_t \) and that we may assume that \( <E(G_i)> \) is an odd cycle of \( W_t \) for \( i = 1, 2, \ldots, k \). It is straightforward to conclude that this maximum will occur when each \( <E(G_i)> \) \( \cong C_3 \), the odd cycle of minimum size, and that \( W_t \) contains at most \( \lfloor t/2 \rfloor \) edge-disjoint subgraphs isomorphic to \( C_3 \). Finally, if \( n = 4 \) then \( \chi(W_t) \geq n \) implies \( \chi(W_t) = 4 \) which forces \( t \) to be odd. Then \( W_t \) is minimally 4-chromatic so that \( M(n, \chi, W_t) = 1 \) using Theorem 4-9.

Clearly the difficulty of determining \( M(n, \chi, G) \) for all \( n \) satisfying \( 2 \leq n \leq \chi(G) \) increases sharply as \( \chi(G) \) increases. If \( G \) is a graph of order \( p \) then no matter how large the integral \( 2 \leq n \leq \chi(G) \), we have the upper bound \( M(n, \chi, G) \leq M(n, \chi, K_p) \) as was shown in Theorem 4-14. In an effort to make this bound useful, we now investigate the numbers \( M(n, \chi, K_p) \) where \( 2 \leq n \leq p \).

If \( n = 2 \) then \( M(n, \chi, K_p) = \binom{p}{2} \) so that the first substantial problem occurs when \( n = 3 \).
We solve this problem making use of results by Chartrand, Geller, and Hedetniemi [7], Fort and Hedlund [13], and Guy [16]. The following preliminary result is helpful.

**Theorem 4-17.** If $G$ is an $n$-chromatic graph with $n \geq 3$ then $M(3, \chi, G)$ is the maximum number of edge-disjoint odd cycles in $G$.

**Proof.** Suppose that $C_1, C_2, \ldots, C_k$ is a maximal collection of edge-disjoint odd cycles in $G$. Define the graph $D_i$ by $V(D_i) = V(G)$ and $E(D_i) = E(C_i)$ for $i = 1, 2, \ldots, k$. Then $G = D_1 \oplus D_2 \oplus \cdots \oplus D_k \oplus R$ where $R$ is that spanning subgraph of $G$ with edge set

$$E(G) - \left( \bigcup_{i=1}^{k} E(C_i) \right).$$

Thus $\chi(D_i) = 3$ for $i = 1, 2, \ldots, k$ and $M(3, \chi, G) \geq k$. Next let $M(3, \chi, G) = l$ and let $G = G_1 \oplus G_2 \oplus \cdots \oplus G_l \oplus R$ be an almost uniform factorization of $G$. Each $G_j$, being 3-chromatic, must contain an odd cycle so that $k \geq l = M(3, \chi, G)$.

**Theorem 4-18.** For integers $p \geq 3$,
\[
\begin{cases}
[p/3\lfloor (p-1)/2 \rfloor] - 1 & \text{when } p \equiv 5 \\
(p/3\lfloor (p-1)/2 \rfloor) & \text{otherwise}.
\end{cases}
\]

**Proof.** In the Chartrand, Geller, and Hedetniemi article it was shown that the maximum number of edge-disjoint cycles contained in \( K_p \) is \([p/3\lfloor (p-1)/2 \rfloor]\). By Theorem 4-17 it is immediate that \( M(3, \chi, K_p) \) is \([p/3\lfloor (p-1)/2 \rfloor]\). Fort and Hedlind show that when \( p \equiv 1 \) or \( p \equiv 3 \) (modulo 6) then \( E(K_p) \) can be partitioned into subsets of three edges each where every subset induces a triangle. Thus, for such values of \( p \), the maximum number of edge-disjoint odd cycles in \( K_p \) is \( (p/2)/3 = [p/3\lfloor (p-1)/2 \rfloor] \). Such partition of \( E(K_p) \) are equivalent to choosing maximal sets of triples from the set \( \{1, 2, \ldots, p\} \) so that no pair occurs in more than one triple. For \( p \equiv 1 \) or \( p \equiv 3 \) (modulo 6) these sets of triples are well known ([18], [26], [29], [32]) as Steiner triple systems. Next we consider those values of \( p \) equivalent to \( 0, 2, \) or \( 4 \) modulo 6. For such values of \( p \), Guy shows that \( K_p \) contains \([p/3\lfloor (p-1)/2 \rfloor]\) edge-disjoint triangles although, in this case, \( p/2 \) edges of \( K_p \) do not belong to any triangle. Thus \( M(3, \chi, K_p) = [p/3\lfloor (p-1)/2 \rfloor] \).
We now consider the remaining case; \( p \equiv 5 \pmod{6} \). Following Guy, \( p = 6t + 5 \) for some integer \( t \) so that
\[
\binom{p}{2} = 18t^2 + 27t + 10
\]
which is not a multiple of 3. Hence not all the edges of \( K_p \) are used in a maximal collection of \( k \) edge-disjoint triangles. Guy establishes that \( k = \frac{\binom{p}{2} - 4}{3} = \left\lfloor \frac{p}{3} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor - 1 \). Our proof will be complete if \( M(3, \chi, K_p) = k \). Using Theorem 4-17, let \( M(3, \chi, K_p) = \lambda \) be the maximum number of edge-disjoint odd cycles in \( K_p \). Clearly \( \lambda \geq k \). If \( \lambda > k \) then there exists a set \( S \) of \( \lambda \) edge-disjoint odd cycles in \( K_p \) not all of which are triangles. At least one cycle in \( S \) has size 5 or more. Thus the number of edges belonging to cycles in \( S \) is at least
\[
3k + 5 = 3\left(\frac{\binom{p}{2} - 4}{3}\right) + 5 = \binom{p}{2} + 1, \quad \text{an impossibility.}
\]

Based on the last result it is tempting to conjecture that
\[
M(n, \chi, K_p) \leq \left\lfloor \frac{p}{n} \right\rfloor \left\lfloor \frac{p-1}{n-1} \right\rfloor \quad \text{for} \quad 2 \leq n \leq p.
\]
However, we have neither a proof nor a counterexample. Wilson [36], working with balanced incomplete block designs, has established a theorem which we will employ to show that equality holds in the above conjecture for infinitely many pairs \( n \) and \( p \).
Theorem 4-19. (Wilson) If $n$ and $p$ are positive integers where $n$ divides $p$ and $n - 1$ divides $p - 1$, then $E(K_p)$ can be partitioned into $p(p - 1)/n(n - 1)$ sets each inducing a subgraph of $K_p$ isomorphic with $K_n$.

Theorem 4-20. If $n$ and $p$ are positive integers with $n^2$ where $n$ divides $p$ and $n - 1$ divides $p - 1$ then $M(n, \chi, K_p) = p(p - 1)/n(n - 1)$.

Proof. Clearly $M(n, \chi, K_p) \geq$ the maximum number of edge-disjoint subgraphs of $K_p$ each isomorphic to $K_n$. Using Wilson's Theorem, we have $M(n, \chi, K_p) \geq p(p - 1)/n(n - 1)$. Let $M(n, \chi, K_p) = k$ and let $K_p = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R$ be an almost uniform factorization of $K_p$. For $i = 1, 2, \ldots, k$, $\chi(G_i) = n$ so that $|E(G_i)| \geq \binom{n}{2}$. Thus $k \binom{n}{2} \leq \binom{p}{2}$ yielding $M(n, \chi, K_p) = k \leq \frac{p(p - 1)}{2} \leq \frac{n(n - 1)}{2} = p(p - 1)/n(n - 1)$.

Another observation from Theorem 4-18 is that $M(3, \chi, K_p)$ equals the maximum number of edge-disjoint subgraphs of $K_p$ each isomorphic to $K_3$. We make the following
CONJECTURE. For integers $n$ and $p$ satisfying $2 \leq n \leq p$, $M(n, x, K_p) =$ the maximum number of edge disjoin subgraphs of $K_p$ each isomorphic to $K_n$.

Richard Guy reported in a private communication that the determination of the maximum number of edge-disjoint subgraphs of $K_p$ each isomorphic to $K_n$ is "very hard. For $n = 3$ the answer is known, and for $n = 4$ a good deal of progress has been made." Richard Nowakowski, currently at Dalhousie University, wrote a thesis on this problem [28]. Related work has been done by Bermond and Sotteau [3], and Lovász and Simonovits [24].

Our conjecture is true under a condition motivated by the following famous Theorem [33].

Turan's Theorem. For positive integers $p$ and $n$ with $3 \leq n \leq p$, let $T(p,n)$ be the least positive integer such that every graph of order $p$ and size $T(p,n)$ contains $K_n$ as a subgraph. Then $T(p,n) = 1 + \left( \frac{p}{2} \right)$ $- t(p - n + 1 + r)/2$ where $p = t(n - 1) + r$ with $0 \leq r < n - 1$.

Theorem 4-21. Let $p$ and $n$ be positive integers with $3 \leq n \leq p$. If $n(n - 1) \geq 2 + p(p - 1)$ $- t(p - n + 1 + r)$ where $p = t(n - 1) + r$ and $0 \leq r < n - 1$ then $M(n, x, K_p) =$ the maximum number of edge-disjoint subgraphs of $K_p$ each isomorphic to $K_n$.
Proof. Let $\ell = \text{the maximum number of edge-disjoint subgraphs of } K_p \text{ each isomorphic to } K_n$. Clearly $M(n, \chi, K_p) \geq \ell$. Assume, to the contrary, that $M(n, \chi, K_p) = k > \ell$ and let $K_p = G_1 \oplus \cdots \oplus G_k \oplus R$ be an almost uniform factorization of $K_p$. For $i = 1, 2, \ldots, k$ the graph $G_i$ is $n$-chromatic so that $|E(G_i)| \geq \binom{n}{2}$. Let $H$ be a graph obtained by removing from $K_p$ $\ell$ edge-disjoint subgraphs each isomorphic to $K_n$. Since $k > \ell$, $H$ has size at least $\binom{n}{2} \geq 1 + \binom{p}{2} - t(p - n + 1 + r)/2$. By Turan's Theorem, $H$ contains $K_n$ as a subgraph, contradicting the maximality of $\ell$.

We next present an inequality that ties together a number appearing in the above conjecture and the conjecture following Theorem 4-18.

Theorem 4-22. Let $n$ and $p$ be integers with $2 \leq n \leq p$. Then the maximum number of edge-disjoint subgraphs of $K_p$ each isomorphic to $K_n$ is most

$$\left\lfloor \frac{p}{n} \left\lfloor \frac{p-1}{p-n} \right\rfloor \right\rfloor.$$

Proof. The graph $K_n$ is $(n-1)$-regular so that at any vertex $v$ of $K_p$ the maximum number of edge-disjoint subgraphs of $K_p$ each isomorphic to $K_n$ and containing vertex $v$ is at most $\left\lfloor (p-1)/(n-1) \right\rfloor$. Therefore to the
p vertices of $K_p$ there are associated at most $p[(p-1)/(n-1)]$ sets of $n - 1$ edges belonging to a maximal collection of subgraphs of $K_p$ isomorphic to $K_n$. Each such subgraph requires one set of $n - 1$ edges at each of $n$ vertices so that there are at most $[p/n][(p-1)/(n-1)]$ such subgraphs.

We now present some bounds on $M(n, x, K_p)$ applicable to all $n$ and $p$ satisfying $2 < n < p$.

**Proposition 4-23.** If $p$, $n$, and $k$ are positive integers with $2 < n < p$ and $\frac{p-1}{\sqrt{k}} + 1 < n$ then

$$M(n, x, K_p) < k.$$  

**Proof.** Using the given inequality, $(p-1)/(n-1) < \sqrt{k}$ yielding $((p-1)/(n-1))^2 < k$. Applying Theorem 4-5 to the complete graph $K_p$ produces $M(n, x, K_p) \leq \lfloor(p-1)^2/(n-1)^2\rfloor < k$.

Our next result gives a conclusion that must follow from the conjecture $M(n, x, K_p) = k$. Two preliminary theorems are needed, the first of which is due to Chartrand and Polimeni [8].

**Theorem 4-24.** If a graph $G$ is factored as $G = G_1 \oplus G_2 \oplus \cdots \oplus G_m$, then $\chi(G) \leq \prod_{i=1}^{m} \chi(G_i)$.
Theorem 4-25. If $K_p$ is factored as $K_p = G_1 \oplus G_2 \oplus \cdots \oplus G_m$ then

$$m \sqrt[p]{\prod_{i=1}^{m} \chi(G_i)} \leq \sum_{i=1}^{m} \chi(G_i).$$

Proof. Since the arithmetic mean of $m$ positive integers is at least as large as their geometric mean, we may write

$$\frac{m}{m} \sum_{i=1}^{m} \chi(G_i) \geq \sqrt[m]{\prod_{i=1}^{m} \chi(G_i)}.$$

By Theorem 4-24, $p = \chi(K_p) \leq \prod_{i=1}^{m} \chi(G_i)$ so that

$$\frac{\sum_{i=1}^{m} \chi(G_i)}{m} \geq \sqrt[m]{p}.$$

Proposition 4-26. If $n, p,$ and $k$ are positive integers with $2 \leq n \leq p$ and $M(n, \chi, K_p) = k$, then

$$p \leq (n-1/(k+1))^{k+1}.$$

Proof. Let $K_p = G_1 \oplus G_2 \oplus \cdots \oplus G_k \oplus R$ be an almost uniform factorization of $K_p$. Then $\chi(G_i) = n$ for $i = 1, 2, \ldots, k$ and $\chi(R) \leq n - 1$. By Theorem 4-25,

$$(k+1) \sqrt[p]{p} \leq \chi(R) + \sum_{i=1}^{k} \chi(G_i) \leq (k+1)n - 1.$$
Thus $\frac{k+1}{\sqrt{p}} \leq n - \frac{1}{k+1}$ yielding our result.

We remark that the bound $M(n, \chi, K_p) \leq \frac{p(p-1)/n(n-1)}{[p(p-1)/(n(n-1))]}$ produced from Theorem 4-6 is a slight improvement of $M(n, \chi, K_p) \leq \frac{(p-1)(n-1)}{(n-1)^2}$, corollary of Theorem 4-5. Moreover, this first bound is attained by any pair of positive integers $p$ and $n$ where $n \geq 2$, $n$ divides $p$, and $n - 1$ divides $p - 1$, as was shown in Theorem 4-20.

We now illustrate some further proof techniques in the determination of $M(n, \chi, K_p)$ by producing these numbers for $3 \leq p \leq 7$.

**Proposition 4-27.**

\[
\begin{align*}
M(n, \chi, K_3) & = \begin{cases} 3 & n = 2 \\ 1 & n = 3 \end{cases} & M(n, \chi, K_4) & = \begin{cases} 6 & n = 2 \\ 1 & n = 3, 4 \end{cases} \\
M(n, \chi, K_5) & = \begin{cases} 10 & n = 2 \\ 2 & n = 3 \\ 1 & n = 4, 5 \end{cases} & M(n, \chi, K_6) & = \begin{cases} 15 & n = 2 \\ 4 & n = 3 \\ 1 & n = 4, 5, 6 \end{cases} \\
M(n, \chi, K_7) & = \begin{cases} 21 & n = 2 \\ 7 & n = 3 \\ 2 & n = 4 \\ 1 & n = 5, 6, 7 \end{cases}
\end{align*}
\]
Proof. For $n = 2$, $M(n, \chi, K_p) = \binom{p}{2}$. For $n = 3$, $M(n, \chi, K_p)$ is determined using Theorem 4-18. $M(p, \chi, K_p) = 1$ by Theorem 4-9. If $n$ and $p$ are integers with $M(n, \chi, K_p) \geq 2$ then there are at least two edge-disjoint subgraphs in $K_p$ isomorphic with $K_n$. At most one of the $n$ vertices of $K_p$ used in one subgraph isomorphic to $K_n$ can also be used in another such subgraph so that $n + (n - 1) \leq p$. The failure of this inequality for $n = 4$ and $p = 5$, $n = 4$ and $p = 6$, $n = 5$ and $p = 7$, and $n = 6$ and $p = 7$ proves $M(4, \chi, K_5) = M(4, \chi, K_6) = M(5, \chi, K_7) = M(6, \chi, K_7) = 1$. $M(4, \chi, K_6) = 1$ implies $M(5, \chi, K_6) = 1$ by Theorem 4-13. It is easy to find two edge-disjoint subgraphs of $K_7$ isomorphic with $K_4$ so that $M(4, \chi, K_7) \geq 2$. Any such pair of subgraphs include all seven vertices of $K_7$ leaving no hope of finding a third. Thus $M(4, \chi, K_7) = 2$. 

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CHAPTER V

EMBEDDING GRAPHS IN THEIR COMPLEMENTS

Harary, Robinson, and Wormald [23] considered the question of the existence of uniform factorizations of \( K_p \) where all of the factors are isomorphic. They proved that there exists a \((p, q)\) graph \( G \) such that \( K_p = G_1 \oplus G_2 \oplus \cdots \oplus G_t \) with \( G \cong G_i \) for \( i = 1, 2, \ldots, t \) if and only if \( q \) divides \( \binom{p}{2} \).

This result can be viewed as providing a necessary and sufficient condition for the existence of a graph \( G \) where \( \overline{G} \) is the edge sum of \( t - 1 \) subgraphs each isomorphic with \( G \). In this chapter we work with the related question: which graphs appear as subgraphs of their complements at least once?

If a graph \( G \) is isomorphic with a subgraph of a graph \( F \) we will say that \( G \) is contained in \( F \) or that \( G \) is embeddable in \( F \), and we will write \( G \subseteq F \).

The first of our two main results determines the largest size \( q \) of graphs of order \( p \) for which we can assert that all \((p, q)\) graphs are contained in their complements. Note that the star \( K(1, p - 1) \) is a \((p, p - 1)\) graph not embeddable in its complement.
Theorem 5-1. If G is a \((p, p-2)\) graph with \(p \geq 2\), then \(G\) is contained in its complement \(\overline{G}\).

Proof. We employ induction on \(p\), the order of \(G\). The theorem is easily seen to be true for \(2 \leq p \leq 4\) by inspecting the four \((p, p-2)\) graphs in question.

We assume that the theorem is true for \(2 \leq p < k\), where \(k \geq 5\), and now consider the graph \(G\), an arbitrary \((k, k-2)\) graph.

Case 1. Assume that \(G\) contains an isolated vertex \(u\).

Then \(G\) must contain a vertex \(v\) of degree greater than one. Thus \(H = G - \{u, v\}\) is a \((k-2, r)\) graph, where \(r \leq k-4\). If \(r = k-4\), the inductive assumption is directly applicable to \(H\); hence, we know that there exists an embedding \(\alpha\) of \(H\) into its complement \(\overline{H}\). Otherwise, if \(r < k-4\), we may add \(k-4-r\) edges to \(H\), forming a \((k-2, k-4)\) graph \(H'\) so that the inductive assumption may be applied to \(H'\). By this assumption there exists an embedding \(\alpha\) of \(H'\) into \(\overline{H'}\). In this case the mapping \(\alpha\), restricted to \(H\), is an embedding of \(H\) into \(\overline{H}\).

Now, by defining \(\alpha(v) = u\) and \(\alpha(u) = v\), the mapping \(\alpha\) becomes an embedding of \(G\) into \(\overline{G}\).
Case 2. Assume that $G$ contains no isolated vertices.

Since a cyclic component of order $n$ must have size at least $n$, $G$ must contain at least two components which are trees. Moreover, in this case, these trees are nontrivial. Let $v_1$ and $v_2$ be end-vertices of different trees of $G$ and let $w_1$ and $w_2$ be the vertices adjacent with $v_1$ and $v_2$, respectively. Consider $G' = G - \{v_1, v_2\}$. By the inductive assumption, there exists an embedding $\alpha$ of $G'$ into $\overline{G'}$.

If $\alpha(w_1) = w_2$ or $\alpha(w_2) = w_1$, we extend $\alpha$ by defining $\alpha(v_1) = v_1$ and $\alpha(v_2) = v_2$. If, on the other hand, $\alpha(w_1) \neq w_2$ and $\alpha(w_2) \neq w_1$, we extend $\alpha$ by defining $\alpha(v_1) = v_2$ and $\alpha(v_2) = v_1$. In either situation we obtain an embedding of $G$ into $\overline{G}$.

The preceding theorem was proved independently by Sauer and Spencer [30]. We shall have occasion to use the following strengthened version of it which is proved by Schuster in [31].

**Theorem 5-2.** If $G$ is a labeled $(p, p - 2)$ graph with $p \geq 2$, then there exists an embedding $\phi$ of $G$ into $\overline{G}$ such that $\phi$ has no fixed vertices.
Clearly the embeddings of Theorem 5-1 and 5-2 exist for any \((p, p - n)\) graph if \(n \geq 2\).

Consider now the \((p, p - 1)\) graphs \(G = C_4 \cup K_1\) and \(H = C_5 \cup K_1\) and note that \(H \subseteq \overline{H}\) while \(G \not\subseteq \overline{G}\). We next present a characterization of the class of \((p, p - 1)\) graphs which can be embedding in their complements.

H. Joseph Straight was the first to observe, in an unpublished report, that nearly all trees belong in this class. For completeness we present his theorem with proof.

**Theorem 5-3.** Every tree which is not a star is a subgraph of its complement.

**Proof.** We employ induction on \(p\), the order of the tree. Note that the theorem holds for \(p = 1\). If \(T\) is a nonempty tree which is not a star, then the order of \(T\) is at least 4. Figure 5-1 shows the three trees of order 4 or 5 which are not stars where the edges of each tree are indicated with solid lines. In each case dashed lines are used to indicate the tree as a subgraph of its complement.
Thus the lemma follows for \( p \leq 5 \). Assume the result holds for all trees which are not stars and which have orders less than \( k \) (where \( k \) is at least 6) and let \( T \) be a tree of order \( k \) which is not a star.

**Case 1.** Assume that \( T \) contains a vertex \( v \) of degree \( k - 2 \). Let the vertices of \( T \) adjacent with \( v \) be denoted \( v_1, v_2, \ldots, v_{k-2} \). Let \( w \) be the remaining vertex of \( T \). Clearly \( w \) is not adjacent with \( v \).

Since \( T \) is both acyclic and connected, it follows that \( w \) is adjacent to exactly one vertex of the set \( \{v_1, v_2, \ldots, v_{k-2}\} \). Without loss of generality, assume \( wv_{k-2} \) is an edge in \( T \). Then, in \( \overline{T} \), the vertex \( w \) is adjacent with the \( k - 2 \) vertices \( v, v_1, \ldots, v_{k-3} \). Since \( v_1v_{k-2} \) is an edge in \( \overline{T} \), we conclude that \( T \subseteq \overline{T} \).
Case 2. Assume that \( \Delta(T) \) is at most \( k - 3 \). In this case, \( T \) contains two end-vertices \( x \) and \( y \) such that \( d_T(x, y) \) is at least 3 and such that \( T - \{x, y\} \) is not a star. Since \( x \) and \( y \) are end-vertices of \( T \), the subgraph \( T - \{x, y\} \) is a tree and we may conclude by the inductive assumption that \( T - \{x, y\} \) is contained in \( \overline{T - \{x, y\}} \). Let \( u \) and \( v \) be the vertices of \( T \) adjacent with \( x \) and \( y \) respectively. Since \( d_T(x, y) > 2 \), the vertices \( u \) and \( v \) are distinct. Because \( T - \{x, y\} \) is isomorphic with a subgraph of \( \overline{T - \{x, y\}} \), there exist distinct vertices \( u' \) and \( v' \) in \( \overline{T - \{x, y\}} \) corresponding (under the isomorphism) to \( u \) and \( v \) respectively in \( T - \{x, y\} \). If \( u' \neq u \) and \( v' \neq v \), then \( xu' \) and \( yv' \) are edges of \( \overline{T} \), and hence \( T \) is contained in \( \overline{T} \). The only remaining possibility is that \( u' = u \) or \( v' = v \). If \( u' = u \) we conclude that \( v' \neq u \) and \( u' \neq v \), since \( u' \neq v' \) and \( u \neq v \). If \( v' = v \), we may again conclude that \( v' \neq u \) and \( u' \neq v \) by the same reasoning. Hence if either \( u' = u \) or \( v' = v \) then \( v' \neq u \) and \( u' \neq v \) which means that \( xv' \) and \( yu' \) are edges in \( \overline{T} \) and thus \( T \) is contained in \( \overline{T} \).
Our upcoming characterization theorem for \((p, p-1)\) graphs is a consequence of the following theorem involving a set \(\mathcal{F}\) of forbidden graphs. The graphs in \(\mathcal{F}\) are \(K_1 \cup C_3\), \(K_1 \cup C_4\), \(K(l, 1) \cup C_3\), \(k(l, p-1)\), \(K_1 \cup 2C_3\), and \(K(1, n) \cup C_3\) where \(n \geq 4\) (see Figure 5-2).

**Theorem 5-4.** Let \(T\) be a tree. If \(G\) is the union of \(T\) and \(m \geq 1\) disjoint cycles and \(G \notin \mathcal{F}\), then \(G \subseteq \overline{G}\).

**Proof.** Suppose that among the \(m\) cycles of \(G\) there is a cycle \(C_r\) with \(r \geq 5\). Let \(V(C_r) = \{v_1, v_2, \ldots, v_r\}\) and consider the \((k, k-2)\) graph...
$G_1 = G - \{v_1, v_2, \ldots, v_{r-1}\}$. By Theorem 5-2 we know that there exists an embedding $\alpha$ of $G_1$ into $\overline{G_1}$ such that $\alpha$ has no fixed vertices. Further, by Theorem 5-3, we know that there is an embedding $\beta$ of $C_r - v_r$ into $\overline{C_r - v_r}$. The union of mappings $\alpha$ and $\beta$ provides an embedding $\phi$ of $G$ into $\overline{G}$ as follows:

$$
\phi(v) = \alpha(v) \text{ for } v \in V(G_1)
$$

$$
\phi(v_i) = \beta(v_i) \text{ for } 1 < i < r - 1
$$

Since $\alpha(v_r) \neq v_r$, the edges $\phi(v_{r-1})\phi(v_r)$ and $\phi(v_1)\phi(v_r)$ are in $\overline{G}$. Certainly all other edges of $\phi(G)$ are in $\overline{G}$, so $\phi$ is the desired embedding.

Having proved the theorem for graphs with a cycle $C_r$, with $r \geq 5$, as one of its components, we assume henceforth that the $m$ cycles of $G$ are 3-cycles or 4-cycles. The remainder of the proof proceeds by induction on $m$.

If $m = 1$ then $G = T \cup C_3$ or $G = T \cup C_4$. Consider $G = T \cup C_3$. Since $G \notin \mathcal{J}$, $T$ is of order at least 3. If $T$ is a star, then $G = K(1,2) \cup C_3$ or $G = K(1,3) \cup C_3$; in each case it is easily verified that $G \subset \overline{G}$. If $T$ is not a star, then $T$ is of order $t \geq 4$. For $t = 4, 5$ the embeddings of $G$ into $\overline{G}$ are shown in Figure 5-3, where solid lines...
If $t \geq 6$, then there are end-vertices $x, y \in V(T)$ such that $T \setminus \{x, y\}$ is not at star; hence, there is an embedding $\alpha : T \setminus \{x, y\} \rightarrow \overline{T \setminus \{x, y\}}$. Calling $V(C_3) = \{u_1, u_2, u_3\}$, we define the mapping $\phi$ as follows:

$$
\phi(u_1) = x, \; \phi(u_2) = y, \; \phi(u_3) = u_3, \; \phi(x) = u_1,
$$

$$
\phi(y) = u_2, \; \text{and} \; \phi(v) = \alpha(v)
$$

for $v \in V(T \setminus \{x, y\})$.

Then $\phi$ is an embedding of $T \cup C_3$ into $\overline{T \cup C_3}$.

Consider $G = T \cup C_4$. Since $G \notin \mathcal{F}$, $T$ is non-trivial. If $T$ is a star, it is easy to see that $G \subseteq \overline{G}$. If $T$ is not a star, then let $\alpha$ be an embedding of $T$ into $\overline{T}$. Also, let $v_1, v_2 \in V(T)$.
and \( V(C_4) = \{u_1, u_2, u_3, u_4\} \). We define \( \phi \) as follows:

\[
\begin{align*}
\phi(u_1) &= u_1, & \phi(u_2) &= \alpha(v_1), & \phi(u_3) &= u_3, & \phi(u_4) &= \alpha(v_2), \\
\phi(v_1) &= u_2, & \phi(v_2) &= u_4, & \text{and } & \phi(v) &= \alpha(v)
\end{align*}
\]

for \( v \in V(T) \) and \( v \neq v_1, v_2 \).

Then, \( \phi \) is an embedding of \( G \) into \( \overline{G} \) as indicated in Figure 5-4. This completes the argument for \( m = 1 \).

Assume, now, that \( G \subset \overline{G} \) for any graph \( G \) satisfying the hypothesis of the theorem, where \( m < k \) and \( k \geq 2 \). Let \( H \) be a \((p, p - 1)\) graph satisfying the hypothesis, where \( H \) is the union of a tree and \( k \) cycles \( C_r \) with \( r = 3 \) or \( 4 \). We consider two cases depending on whether \( C_4 \) is a component of \( H \).
Case 1. Assume one component of $H$ to be a $C_4$.

Then $H$ has another cycle as a component. Consider $H_1 = H - \{C_4, C_r\}$, with $r = 3$ or $4$. Before we may apply the induction hypothesis to $H_1$, we must tediously eliminate those cases in which $H_1 \notin \mathcal{S}$ or $H_1$ degenerates to become $K_1$.

If $H_1 = K_1$ then $H = K_1 \cup C_4 \cup C_3$ or $H = K_1 \cup 2C_4$; in both cases, the embeddings of $H$ into $\overline{H}$ are simple to construct. If $H_1 = K_1 \cup C_3$ then $H = K_1 \cup C_4 \cup 2C_3$ or $H = K_1 \cup 2C_4 \cup C_3$; again it is easily verified that $H \subseteq \overline{H}$. If $H_1$ is a star, then the embeddings of $H$ into $\overline{H}$ are exhibited in Figure 5-5, where a solid vertex indicates the presence of dashed edges between that vertex and all other vertices in the diagram which are not already incident with dashed edges.

![Figure 5-5](image-url)
If \( H_1 = K(1, 1) \cup C_3 \), then \( H = K(1, 1) \cup 2C_4 \cup C_3 \) or \( H = K(1, 1) \cup C_4 \cup 2C_3 \); if \( H_1 = K_1 \cup C_4 \), then \( H = K_1 \cup 3C_4 \) or \( H = K_1 \cup 2C_4 \cup C_3 \); and if \( H_1 = K_1 \cup 2C_3 \), then \( H = K_1 \cup 2C_4 \cup 2C_3 \) or \( H_1 = K_1 \cup C_4 \cup 3C_3 \). In all these cases, it is a simple matter to verify that \( H \subseteq \overline{H} \).

Finally, if \( H_1 = K(1, n) \cup C_3 \), with \( n \geq 4 \), then \( H = K(1, n) \cup 2C_4 \cup C_3 \) or \( H = K(1, n) \cup 2C_3 \cup C_4 \). In the latter case, the embedding of \( H \) in \( \overline{H} \) is obvious. In the former case, we consider the graph \( H_2 = H - \{C_4, C_3\} = K(1, n) \cup C_4 \), in which case we may apply the induction hypothesis to \( H_2 \); since \( H_2 \subseteq \overline{H_2} \) and \( C_4 \cup C_3 \subseteq \overline{C_4 \cup C_3} \), we have \( H \subseteq \overline{H} \).

In all the remaining cases, \( H_1 \) satisfies the induction hypothesis, so \( H_1 \subseteq \overline{H_1} \). Since \( C_4 \cup C_r \subseteq \overline{C_4 \cup C_r} \) for any \( r \geq 3 \), we conclude that \( H \subseteq \overline{H} \).

**Case 2.** Assume that every cyclic component of \( H \) is a 3-cycle; i.e. \( H = T \cup kC_3 \).

Suppose that \( T = K_1 \). Then \( k > 2 \), otherwise \( H \in \mathcal{J} \). Since \( kC_3 \subseteq \overline{kC_3} \) for \( k > 2 \) we have \( H \subseteq \overline{H} \).

Suppose \( T \) is the star \( K(1, n) \). If \( n = 1 \), the embedding of \( H = K(1, 1) \cup kC_3 \) in its complement is
obvious for $k = 2, 3,$ and $4$. For $n > 1$ and $k = 2, 3, 4$ we exhibit the embeddings in Figure 5-6. If $k > 4$, the graph $H_2 = H - 3C_3$ obeys the induction hypothesis; hence, $H_2 \subseteq \overline{H_2}$. And, since $3C_3 \subseteq \overline{3C_3}$, we have $H \subseteq \overline{H}$. This completes the analysis for $T$ a star.

Figure 5-6
We now turn to the general case in which $T$ is neither $K_1$ nor a star $K(1,n)$. Let $v_1, v_2 \in V(T)$ and $\alpha : T \to \overline{T}$ be any embedding in which $\alpha(v_1) = v_1$ and $\alpha(v_2) = v_2$. If $k = 2$, we call the 3-cycles $u_1, u_2, u_3$ and $w_1, w_2, w_3$, and then define $\phi$ as follows (see Figure 5-7):

$$
\begin{align*}
\phi(u_1) &= u_1, & \phi(u_2) &= w_2, & \phi(u_3) &= v_1, \\
\phi(w_1) &= v_1, & \phi(w_2) &= u_2, & \phi(w_3) &= v_2, \\
\phi(v_1) &= u_3, & \phi(v_2) &= w_3, & \text{and } \phi(v) &= \alpha(v)
\end{align*}
$$

for the remaining $v \in V(T)$. The mapping $\phi$ is an embedding of $T \cup 2C_3$ into its complement.

Figure 5-7
If \( k = 3 \), then \( 3C_3 \subseteq \overline{3C_3} \) coupled with \( T \subseteq \overline{T} \) implies that \( H = T \cup 3C_3 \) is contained in its complement. For \( k > 3 \), \( H_3 = H - 3C_3 \) satisfies the induction hypothesis; hence, \( H_3 \subseteq \overline{H_3} \) and \( 3C_3 \subseteq \overline{3C_3} \) completes the proof that \( H \subseteq \overline{H} \).

We are now able to present a complete characterization of those \((p, p-1)\) graphs which are contained in their complements.

**Theorem 5-5.** Let \( G \) be a \((p, p-1)\) graph with \( p \geq 3 \). Then \( G \) is contained in its complement if and only if \( G \notin \mathcal{F} \).

**Proof.** Clearly if \( G \notin \mathcal{F} \) then \( G \) cannot be embedded in \( \overline{G} \). We proceed with the converse.

Our attention will be restricted to the case in which \( G \) is disconnected, for if \( G \) is connected it is a tree in which case Theorem 5-3 applies.

If \( v \) is an isolated vertex of \( G \), then \( G - v \) is a \((p-1, p-1)\) graph. Hence \( G - v \) is either a union of cycles or else it contains a vertex \( u \) of degree \( \geq 3 \). The former case is covered by Theorem 5-4.

In the latter case, \( G - \{v, u\} \) is a \((p-2, p-k)\) graph with \( k \geq 4 \). Thus, by the remark following Theorem 5-2, we know that there is an embedding.
\[ \phi : G - \{v, u\} \rightarrow \overline{G - \{v, u\}}. \] Defining \( \phi(v) = u \) and \( \phi(u) = v \) provides an embedding of \( G \) in \( \overline{G} \).

If \( G \) possesses no isolated vertices, then it must have a tree \( T \) of order \( t \geq 2 \) as one of its components (for every cyclic component with \( n \) vertices has at least \( n \) edges). Then \( G - T \) is a \((p - t, p - t)\) graph. Either \( G - T \) is a disjoint union of cycles or \( G - T \) contains a vertex \( w \) whose degree is at least \( 3 \). The former alternative, in which \( G \) is the union of a tree and cycles, is covered by Theorem 5-4. In the second alternative, \( G - T - w \) is a \((p - t - 1, p - t - s)\) graph with \( s \geq 3 \); hence, there is an isomorphic embedding \( \alpha \) of \( G - T - w \) into its complement. Also, if \( z \) is a vertex of maximal degree in \( T \). There there is an embedding \( \beta : T - z \rightarrow \overline{T - z} \), because \( T - z \) is either a \((t - 1, t - n)\) graph with \( n \geq 3 \) or it is \( K_1 \).

By defining

\[
\begin{align*}
\phi(e) &= z, \\
\phi(z) &= w, \\
\phi(x) &= \alpha(x) \quad \text{for} \quad x \in V(G - T - w), \\
\phi(y) &= \beta(y) \quad \text{for} \quad y \in V(T - z)
\end{align*}
\]

we obtain an embedding \( \phi \) of \( G \) into \( \overline{G} \). This completes the proof of our characterization theorem.
We wish to thank Richard Schelp for observing that a rearrangement of the cases in an earlier version of this theorem greatly simplified its proof. Indeed, following Schelp's suggestion enabled us to eliminate an induction argument, the anchor of which required the verification that $G \subseteq \bar{G}$ for 343 $(9, 8)$ graphs.


