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Field Radiation of Spin I Massless Particle

Ralph Alvin Mudgett

*Western Michigan University*

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FIELD RADIATION OF SPIN 1
MASSLESS PARTICLE

by
Ralph Alvin Mudgett

A Thesis
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Master of Arts

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Kalamazoo, Michigan
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Ralph Alvin Mudgett
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I. INTRODUCTION

Einstein began a short time after finishing the general theory of relativity, to work on an unified field theory.

He combined the laws of electromagnetic theory and gravitation in one system of formulae. He hoped that he could obtain in this way not only a formal unification but an explanation of the existence of the elementary particles. These particles are commonly described with the help of quantum theory.

We follow on this way of unification to establish relations equivalent to the Maxwell equations for the electromagnetic field, which are invariant under general coordinate transformations. Quantum Mechanics and Special Relativity were explicitly taken into consideration in development of our theory. In particular, an equivalency between energy-momentum relation for a massless particle in Special Relativity and time-space derivative operators in Quantum Mechanics is used.

In Chapter II, the fundamental relation between the total angular momentum of a system and its infinitesimal rotation operators was summarized. A matrix which described an infinitesimal rotation, was found to be related to the spin angular momentum. This matrix was then transformed into a conventional form for the Spin I=1 matrix.

In Chapter III, the curl operator was found to be expressed by a spin angular momentum operator. The Maxwell equations were...
then represented in terms of this spin matrix. This expression was similar to Schrödinger's wave equation used in Quantum Mechanics. The relation of the Maxwell equation to the Klein-Gordon equation, which plays an important role in Relativistic Quantum Theory, was also discussed. Finally, gradient and divergence operators were also expressed in matrix forms.

In Chapter IV, a new Sigma Spin matrix was defined. We combined the four Maxwell equations into two field equations with the help of a Sigma matrix. We found the Sigma Spin matrix satisfied commutation relations of exactly the same form as that of the Pauli spin matrices, and guaranteed the energy-momentum relation for the electromagnetic field in relativistic form. The z-component of the Sigma Spin matrix was diagonalized; then the x and y-components were transformed into the same coordinate system. We found a form which is a reasonable extension from the Pauli spin matrices.

To generalize the Maxwell equations to the Spin I field, it was necessary to find a general form of the Sigma Spin I matrix. These generalized Sigma Spin I matrices were then required to satisfy the commutation relations (as above) as that of the Pauli spin matrices. This generalized formalism was derived by Professor M. Soga.

In Chapter V, we calculated the energy flow (Poynting's Vector) associated with the radiation of Spin I field. This is an application of the field equation which was developed in previous chapters.

In Chapters VI through XII, explicit forms were shown of
the Sigma Spin matrix for Spin $I=3/2$ and 2 and Poynting's Vector
for Spin $I=1/2$, 1, 3/2 and 2.

In Chapter XIII, some discussions about a developed field
equation and its application were presented.
II. A VECTOR UNDER INFINITESIMAL ROTATION

In Euclidean geometry, length plays an essential role. Hence, we seek those linear transformations of Euclidean vector spaces which preserve the lengths $|\mathbf{r}|$ of all vectors $\mathbf{r}$. The first consists in rotating the reference axis, keeping fixed each point $\mathbf{r}$ of space and the physical quantities attached to it. The second consists in keeping the axes fixed and rotating the physical system itself. The two points of view are equivalent. We rotate the coordinate axes or we rotate the physical system itself in the opposite direction, amounts to exactly the same thing. Unless otherwise specified in what follows, we shall adopt the second of these viewpoints (rotation of the physical system).

The total angular momentum $\mathbf{J}$ can also be defined in a way that permits of generalization to more complicated systems. We suppose that the system is specified by a Hamiltonian $H$ that is unaffected by rotations $\mathbf{R}$ of the coordinate system. For an arbitrary function, we then have $\mathbf{R}Hf = H\mathbf{R}f$, so that $\mathbf{R}$ commutes with $H$. Thus any rotation $\mathbf{R}$ is a constant of the motions, and its constancy is a direct consequence of the invariance of $H$ with respect to rotations. We, therefore, expect that there is a relation between $\mathbf{R}$ and $\mathbf{J}$.

We are now in a position to establish the fundamental relation between the total angular momentum of a system and its infinitesimal rotation operators.

In Figure 1., we first consider the case of the Rotational Matrix $\mathbf{R}_\phi$ of rotation $\phi$ around the $n$ axis.
A rotation about the $n$-axis consists of a transformation from $(i, j, k)$ coordinates to $(i', j', k')$ coordinates where $k' = \bar{n}$.
A rotation about the \( n \)-axis consists of a transformation from \((i, j, k)\) coordinates to \((i', j', k')\) coordinates where \(k'=n\).

\[
\begin{align*}
\vec{r} &= x\vec{i} + y\vec{j} + z\vec{k} \\
\vec{r}' &= x'\vec{i}' + y'\vec{j}' + z'\vec{k}'
\end{align*}
\]  

hence

\[
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} = 
\begin{pmatrix}
\vec{x}' \cdot \vec{i} & \vec{x}' \cdot \vec{j} & \vec{x}' \cdot \vec{k} \\
\vec{y}' \cdot \vec{i} & \vec{y}' \cdot \vec{j} & \vec{y}' \cdot \vec{k} \\
\vec{z}' \cdot \vec{i} & \vec{z}' \cdot \vec{j} & \vec{z}' \cdot \vec{k}
\end{pmatrix} 
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]  

(2.1)

similarly;

\[
\begin{align*}
\vec{r}_R &= X_R \vec{i} + Y_R \vec{j} + Z_R \vec{k} \\
\vec{r}'_R &= X'_R \vec{i}' + Y'_R \vec{j}' + Z'_R \vec{k}'
\end{align*}
\]

hence

\[
\begin{pmatrix}
X'_R \\
Y'_R \\
Z'_R
\end{pmatrix} = 
\begin{pmatrix}
\vec{x} \cdot \vec{i} & \vec{x} \cdot \vec{j} & \vec{x} \cdot \vec{k} \\
\vec{y} \cdot \vec{i} & \vec{y} \cdot \vec{j} & \vec{y} \cdot \vec{k} \\
\vec{z} \cdot \vec{i} & \vec{z} \cdot \vec{j} & \vec{z} \cdot \vec{k}
\end{pmatrix} 
\begin{pmatrix}
X_R \\
Y_R \\
Z_R
\end{pmatrix}
\]  

(2.2)

(2.3)

The transformation from \((\vec{i}, \vec{j}, \vec{k})\) to \((\vec{i}', \vec{j}', \vec{k}')\) coordinate system is expressed by the matrix

\[
\begin{pmatrix}
\vec{x}' \cdot \vec{i} & \vec{x}' \cdot \vec{j} & \vec{x}' \cdot \vec{k} \\
\vec{y}' \cdot \vec{i} & \vec{y}' \cdot \vec{j} & \vec{y}' \cdot \vec{k} \\
\vec{z}' \cdot \vec{i} & \vec{z}' \cdot \vec{j} & \vec{z}' \cdot \vec{k}
\end{pmatrix}
\]  

(2.5)
The concept of a transformation may become more explicit if we consider a specific rotation $\phi$ about the $z'$-axis. For a rotation around $\vec{k}'$ axis connecting $\vec{r}'(\vec{i}', \vec{j}', \vec{k}')$ to $\vec{r}_R(\vec{i}', \vec{j}', \vec{k}')$ coordinate frame, the matrix has the form:

\[
\tilde{R}_{\vec{k}' \vec{r}'} = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

(2.6)

The general form can be obtained by

\[
\tilde{R}(\vec{x}, \vec{j}, \vec{k}) \rightarrow \tilde{R}(\vec{x}', \vec{j}', \vec{k}') \rightarrow \tilde{R}_R(\vec{x}', \vec{j}', \vec{k}') \rightarrow \tilde{R}_R(\vec{x}, \vec{j}, \vec{k}) \]  

(2.7)

and referring directly to equations 2.5 and 2.6.

\[
\tilde{R}_{\vec{r} \vec{r}'} = \prod_{\vec{k} \vec{r}'} \prod_{\vec{r} \vec{k}} \prod_{\vec{k}' \vec{r}'}
\]

\[
\begin{pmatrix}
\tilde{x}' \tilde{x} & \tilde{j}' \tilde{j} & \tilde{k}' \tilde{k}
\end{pmatrix}
\begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \tilde{x} & \tilde{j} \tilde{j} & \tilde{k} \tilde{k}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\]

(2.8)

The elements of the matrix $\tilde{R}_{\vec{r} \vec{r}'}$ can be obtained with the aid of the following vector identities:
\[(\vec{\alpha} \cdot \vec{\beta})(\vec{\gamma} \cdot \vec{\delta}) + (\vec{\gamma} \cdot \vec{\alpha})(\vec{\delta} \cdot \vec{\beta}) + (\vec{\beta} \cdot \vec{\alpha})(\vec{\gamma} \cdot \vec{\delta}) = (\vec{\alpha} \cdot \vec{\beta})\] (2.9)

\[(\vec{\alpha} \times \vec{\beta}) \cdot (\vec{\gamma} \times \vec{\delta}) = (\vec{\alpha} \cdot \vec{\gamma})(\vec{\beta} \cdot \vec{\delta}) - (\vec{\alpha} \cdot \vec{\delta})(\vec{\beta} \cdot \vec{\gamma})\] (2.10)

The particular elements are:

\[R_{11} = \left[ (\vec{x} \cdot \vec{x})^2 + (\vec{y} \cdot \vec{y})^2 \right] \cos \phi + (\vec{k} \cdot \vec{x})^2\]

\[= (\vec{k} \cdot \vec{x})^2 (1 - \cos \phi) + \cos \phi\]

\[R_{12} = \left[ (\vec{x} \cdot \vec{y}) + (\vec{y} \cdot \vec{x}) \right] \cos \phi + (\vec{k} \cdot \vec{x}) (1 - \cos \phi)\]

\[+ \left[ - (\vec{x} \cdot \vec{y}) + (\vec{x} \cdot \vec{y}) + (\vec{y} \cdot \vec{x}) \right] \sin \phi + (\vec{k} \cdot \vec{x}) (\vec{k} \cdot \vec{x})\]

\[= (\vec{k} \cdot \vec{x}) (\vec{k} \cdot \vec{x}) (1 - \cos \phi) - (\vec{k} \cdot \vec{x}) \sin \phi\]

\[R_{13} = \left[ (\vec{x} \cdot \vec{z}) + (\vec{y} \cdot \vec{z}) \right] \cos \phi + (\vec{k} \cdot \vec{x}) (1 - \cos \phi)\]

\[+ \left[ - (\vec{x} \cdot \vec{z}) + (\vec{x} \cdot \vec{z}) + (\vec{y} \cdot \vec{z}) \right] \sin \phi + (\vec{k} \cdot \vec{x}) (\vec{k} \cdot \vec{x})\]

\[= (\vec{k} \cdot \vec{x}) (\vec{k} \cdot \vec{x}) (1 - \cos \phi) + (\vec{k} \cdot \vec{x}) \sin \phi\]

\[R_{21} = \left[ (\vec{x} \cdot \vec{y}) + (\vec{y} \cdot \vec{x}) \right] \cos \phi + (\vec{k} \cdot \vec{x}) (1 - \cos \phi)\]

\[+ \left[ - (\vec{x} \cdot \vec{y}) + (\vec{x} \cdot \vec{y}) + (\vec{y} \cdot \vec{x}) \right] \sin \phi + (\vec{k} \cdot \vec{x}) (\vec{k} \cdot \vec{x})\]

\[= (\vec{k} \cdot \vec{x}) (\vec{k} \cdot \vec{x}) (1 - \cos \phi) + (\vec{k} \cdot \vec{x}) \sin \phi\]

\[R_{22} = \left[ (\vec{x} \cdot \vec{y})^2 + (\vec{y} \cdot \vec{x})^2 \right] \cos \phi + (\vec{k} \cdot \vec{x})^2\]

\[= (\vec{k} \cdot \vec{x})^2 (1 - \cos \phi) + \cos \phi\]
\[ R_{23} = \left[ (x' \cdot k)(i' \cdot j) + (\bar{j}' \cdot \bar{k})(\bar{\bar{i}}' \cdot \bar{j}) \right] \cos \phi \\
+ \left[ - (x' \cdot j)(\bar{j}' \cdot \bar{k}) + (\bar{j}' \cdot \bar{k})(x' \cdot \bar{i}) \right] \sin \phi + (k' \cdot \bar{k})(\bar{j}' \cdot \bar{j}) \\
= (k' \cdot j)(\bar{k}' \cdot \bar{k})(1 - \cos \phi) - (k' \cdot \bar{i}) \sin \phi \]

\[ R_{31} = \left[ (x' \cdot \bar{k})(x' \cdot \bar{x}) + (\bar{j}' \cdot \bar{k})(\bar{j}' \cdot \bar{x}) \right] \cos \phi \\
+ \left[ - (x' \cdot \bar{k})(\bar{j}' \cdot \bar{x}) + (\bar{j}' \cdot \bar{x})(x' \cdot \bar{x}) \right] \sin \phi + (k' \cdot \bar{k})(\bar{k}' \cdot \bar{j}) \\
= (k' \cdot \bar{k})(\bar{k}' \cdot \bar{j})(1 - \cos \phi) - (k' \cdot \bar{x}) \sin \phi \]

\[ R_{32} = \left[ (x' \cdot \bar{k})(\bar{x}' \cdot j) + (\bar{j}' \cdot \bar{k})(\bar{x}' \cdot \bar{j}) \right] \cos \phi \\
+ \left[ - (x' \cdot \bar{k})(\bar{x}' \cdot \bar{j}) + (\bar{j}' \cdot \bar{k})(x' \cdot \bar{j}) \right] \sin \phi + (k' \cdot \bar{k})(\bar{k}' \cdot \bar{j}) \\
= (k' \cdot \bar{k})(\bar{k}' \cdot \bar{j})(1 - \cos \phi) + (k' \cdot \bar{x}) \sin \phi \]

\[ R_{33} = \left[ (x' \cdot \bar{k})^2 + (\bar{j}' \cdot \bar{k})^2 \right] \cos \phi + (k' \cdot \bar{k})^2 \\
= (k' \cdot \bar{k})^2 (1 - \cos \phi) + \cos \phi \]  
(2.11)

The matrix associated with a rotation is uniquely defined if we choose \( k' = \bar{n} \) then

\[(\bar{k}' \cdot \bar{x}) = \eta_x, \quad (\bar{k}' \cdot \bar{j}) = \eta_y, \quad (\bar{k}' \cdot \bar{\bar{i}}) = \eta_z \]  
(2.12)

The matrix \( \tilde{R}_{n\phi} \) can now be expressed as:

\[
\tilde{R}_{n\phi} = 
\begin{bmatrix}
\eta_x(1 - \cos \phi) + \cos \phi & \eta_x \eta_y (1 - \cos \phi) - \eta_z \sin \phi & \eta_x \eta_z (1 - \cos \phi) + \eta_y \sin \phi \\
\eta_y \eta_x (1 - \cos \phi) + \eta_z \sin \phi & \eta_y^2 (1 - \cos \phi) + \cos \phi & \eta_y \eta_z (1 - \cos \phi) - \eta_x \sin \phi \\
\eta_z \eta_x (1 - \cos \phi) - \eta_y \sin \phi & \eta_z \eta_y (1 - \cos \phi) + \eta_x \sin \phi & \eta_z^2 (1 - \cos \phi) + \cos \phi
\end{bmatrix}
\]
A theorem concerning the orthogonal properties of the matrix states that the reciprocal matrix is to be identified as the transposed matrix, hence:\n\[ \hat{R}_{\hat{R}_\phi}^{-1} \approx \begin{pmatrix} \eta_x & \eta_y & \eta_z \\ \eta_y & \eta_z & -\eta_x \\ -\eta_z & \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} \cos \phi & -\eta_z \sin \phi & \eta_y \sin \phi \\ \eta_z \sin \phi & \cos \phi & -\eta_x \sin \phi \\ -\eta_y \sin \phi & \eta_x \sin \phi & \cos \phi \end{pmatrix} \] (2.13)

In particular, the infinitesimal rotation \( \hat{R}_{\hat{R}_\phi} \) gives only terms of the first order in \( \phi \). The Taylor's expansion of the sine and cosine functions lead to the following matrix form:
\[ \hat{R}_{\hat{R}_\phi} \approx \begin{pmatrix} 1 & -\eta_z \phi & \eta_y \phi \\ \eta_z \phi & 1 & -\eta_x \phi \\ -\eta_y \phi & \eta_x \phi & 1 \end{pmatrix} \] (2.15a)

Similarly:
\[ \hat{R}_{\hat{R}_\phi}^{-1} \approx \begin{pmatrix} 1 & \eta_z \phi & -\eta_y \phi \\ -\eta_z \phi & 1 & \eta_x \phi \\ \eta_y \phi & -\eta_x \phi & 1 \end{pmatrix} \] (2.15b)
\( \tilde{R}_\phi \) and \( \tilde{R}^{-1}_\phi \) are expressions for infinitesimal rotation and can be related to the angular momentum of a system.

A vector field may be defined by the following transformation property

\[
\vec{A}_R(\tilde{R}_\phi \vec{r}) = \tilde{R}_\phi \vec{A}(\vec{r})
\]

where \( \vec{A}_R \) is a vector created by an infinitesimal rotation from original vector \( \vec{A} \). Applying this definition to the vector \( \tilde{R}_\phi \vec{r} \), we obtain:

\[
\vec{A}_R(\vec{r}) = \tilde{R}_\phi \vec{A}(\tilde{R}^{-1}_\phi \vec{r}) \quad (2.16)
\]

When the arguments in the right hand vector are expanded in incremental values, we obtain the following results:

\[
\tilde{R}^{-1}_\phi \vec{r} = \begin{bmatrix}
    x + \phi(\eta_z y - \eta_y z) \\
    y + \phi(\eta_x z - \eta_z x) \\
    z + \phi(\eta_y x - \eta_x y)
\end{bmatrix}
\]

\[
\vec{A}_R(x, y, z) = \begin{bmatrix}
    A_x(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(\eta_z A_y - \eta_y A_z) \\
    A_y(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(\eta_x A_z - \eta_z A_x) \\
    A_z(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(\eta_y A_x - \eta_x A_y)
\end{bmatrix} \quad (2.17)
\]

We represent the incremental values as:

\[
\begin{bmatrix}
    \Delta x \\
    \Delta y \\
    \Delta z
\end{bmatrix} = \begin{bmatrix}
    \phi(\eta_z y - \eta_y z) \\
    \phi(\eta_x z - \eta_z x) \\
    \phi(\eta_y x - \eta_x y)
\end{bmatrix} \quad (2.19)
\]
The Taylor expansion of a function $f(x)$ about the point $x(x, y, z)$ takes the following form, to terms of the first order:

$$f(x + \Delta x, y + \Delta y, z + \Delta z) \approx f(x, y, z) + \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \frac{\partial f}{\partial z} \Delta z$$

$\approx f(x, y, z) + \phi \left\{ \left( \eta_z \eta_y - \eta_y \eta_z + (\eta_x \eta_y - \eta_y \eta_x) \right) \frac{\partial}{\partial x} + \left( \eta_y \eta_z - \eta_z \eta_y + (\eta_z \eta_x - \eta_x \eta_z) \right) \frac{\partial}{\partial y} + \left( \eta_x \eta_y - \eta_y \eta_x + (\eta_z \eta_y - \eta_y \eta_z) \right) \frac{\partial}{\partial z} \right\} f(x, y, z)$

$\approx f(x, y, z) + \phi \left\{ \eta_z \eta_y - \eta_y \eta_z \right\} \frac{\partial}{\partial x} f(x, y, z) + \phi \left\{ \eta_y \eta_z - \eta_z \eta_y \right\} \frac{\partial}{\partial y} f(x, y, z) + \phi \left\{ \eta_x \eta_y - \eta_y \eta_x \right\} \frac{\partial}{\partial z} f(x, y, z)$

(2.20)

The symbol $\nabla$, called del, is the vector operator:

$$\nabla = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

(2.21)

Hence, the expanded function about the point $x(x, y, z)$ has the following form:

$$f(x + \Delta x, y + \Delta y, z + \Delta z)$$

$$= f(x, y, z) - \phi \left\{ \eta_z \eta_y - \eta_y \eta_z \right\} \frac{\partial}{\partial x} \left[ f(x, y, z) \right]$$

(2.22)

The vector field $A_r(r)$ then has the expanded form:

$$\Delta_r(x, y, z) = \left[ \begin{array}{ccc} 1 - \phi \eta_z \eta_y - \phi \eta_z & -\phi \eta_z & \phi \eta_y \\ \phi \eta_z & 1 - \phi \eta_z \eta_y & -\phi \eta_x \\ -\phi \eta_y & \phi \eta_x & 1 - \phi \eta_z \eta_y \end{array} \right] A(x, y, z)$$

$$= \left[ 1 - \phi \eta_z \eta_y \right] A(x, y, z)$$

(2.23)

Where the matrix $M = i M_x + j M_y + k M_z$ is defined.
If $J$ is the total angular momentum of a system, its component along any axis $n$ is related to the operator of infinitesimal rotations about that axis. The vector field becomes:

$$
\vec{A}_\mu(x,y,z) = [1 - i \phi \vec{n} \cdot \vec{J}] \vec{A}(x,y,z)
$$

where the total angular momentum $J = L + S$:

$$
\vec{A}_\mu(x,y,z) = [1 - i \phi \vec{n} \cdot \vec{L} - i \vec{n} \cdot \vec{S}] \vec{A}(x,y,z)
$$

and where the angular momentum operator is $\vec{L} = \hbar \vec{r} \times \vec{V}$ \( \hbar = 1 \)

consequently:

$$
\vec{A}_\mu(x,y,z) = [1 - i \phi \vec{n} \cdot (\vec{r} \times \vec{V}) - i \phi \vec{n} \cdot \vec{S}] \vec{A}(x,y,z)
$$

By comparing this expression with equation 2.24, the components of the matrix $\vec{M}$ are found to be proportional to the corresponding components of the spin angular momentum.

$$
\vec{M} = i\vec{S} \quad \text{or} \quad \vec{S} = i\vec{M}
$$

Therefore:

$$
\begin{align*}
S_x &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, &
S_y &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, &
S_z &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{align*}
$$

We denote this angular momentum by $\vec{S}$. Its square is:
\[ \mathbf{S}^2 = \mathbf{S}_x^2 + \mathbf{S}_y^2 + \mathbf{S}_z^2 = 2 \mathbf{\hat{I}} = I(I+1) \mathbf{\hat{I}} \]  
(2.29)

which corresponds to angular momentum \( I=1 \). By definition, we shall say that \( \mathbf{S} \) is the intrinsic angular momentum, or spin, of the vector field\(^2\). Therefore, a vector field \( \mathbf{A}(\mathbf{r}) \) corresponds to a spin \( I=1 \) particle. One often uses the representation where the base vectors are the simultaneous eigen vectors of \( \mathbf{S}^2 \) and \( S_z \) operators, belonging respectively to the eigenvalues \( I=1 \) and \( m=0, \pm 1 \).

Whether all eigenvalues are distinct or not, whenever \( \mathbf{S} \) is Hermitian; a unitary matrix \( \mathbf{U} \) can be found such that

\[ \mathbf{U} \mathbf{\hat{S}} \mathbf{U} = \mathbf{\Lambda} \]

a diagonal matrix. The spin functions, where \( \mathbf{S}^2 \) and \( S_z \) are diagonal, are expressed by an irreducible spherical tensor with rank 1; namely:

\[
\chi_{+1} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad \chi_{0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}
\]

(2.30)

Then the spin \( I=1 \) operators can be defined by:

\[
\mathbf{S}_x \chi_{+1} = 2 \chi_{+1}, \quad \mathbf{S}_z \chi_{+1} = \chi_{+1}
\]

\[
\mathbf{S}_x \chi_{0} = 2 \chi_{0}, \quad \mathbf{S}_z \chi_{0} = 0
\]

\[
\mathbf{S}_x \chi_{-1} = 2 \chi_{-1}, \quad \mathbf{S}_z \chi_{-1} = -\chi_{-1}
\]

(2.31)

where \((+1, 0, -1)\) are eigenvalues of \( S_z \).

Now we can find another basis; namely, the \( S_z \) matrix, a diagonal matrix\(^3\):
The transformation matrix between equations 2.32 and 2.34 (in the spin I plane) which transforms $X_m$ coordinates to $X'_m$ are:

$$
\begin{align*}
\chi'_{+1} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\
\chi'_0 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\
\chi'_{-1} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{align*}
$$

(2.32)

Then we can easily find the unitary matrix $U$:

$$
U = \begin{pmatrix}
-\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\
0 & 0 & 1 \\
\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}, \\
U^{-1} = \begin{pmatrix}
-\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
0 & 1 & 0 \\
-\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}}
\end{pmatrix}
$$

(2.33)

(2.34)

Where we have used the definition of a unitary matrix:

$$
U^{-1} = U^\dagger
$$

(2.35)

With this transformation matrix, we can obtain a conventional form for spin I=1 matrix:

$$
S_x' = \tilde{U} S_x U = \begin{pmatrix}
0 & \frac{i}{\sqrt{2}} & 0 \\
\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\
0 & \frac{i}{\sqrt{2}} & 0
\end{pmatrix}
$$

$$
S_y' = \tilde{U} S_y U = \begin{pmatrix}
0 & -\frac{i}{\sqrt{2}} & 0 \\
-\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\
0 & -\frac{i}{\sqrt{2}} & 0
\end{pmatrix}
$$
\[
S_z' = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

These spin 1=1 matrices satisfy the commutation relations:

\([S_x', S_y'] = i S_z', \quad [S_y', S_z'] = i S_x', \quad [S_z', S_x'] = i S_y'\]
\([S_z', S_\pm'] = \pm S_\pm', \quad [S_+, S_-'] = 2 S_z'\]

(2.37)

where \(S_\pm' = S_x' \pm i S_y'\)

The basis vectors of a standard representation \(\{S^2, S_z\} \mid \text{Im} \rangle\) are:

\[S_z' \mid \text{Im} \rangle = (I (I+1)) \mid \text{Im} \rangle\]
\[S_x' \mid \text{Im} \rangle = m \mid \text{Im} \rangle\]
\[S_\pm' \mid \text{Im} \rangle = (I \mp m) (I \mp m + 1) \mid \text{Im} \mp 1 \rangle\]

\[\chi \langle \text{Im} \mid \text{Im}' \rangle = \delta_{m', m} \]

(2.38)

(I integral or half-integral \(\geq 0\), \(m = -I, -I+1, \ldots, +I\), \(m' = m \pm 1\))

From equation 2.38 we get Hermitian matrices in \(m, m'\) for \(S_x', S_y'\). In table 1-3 we have illustrated the \(m, m'\) matrices of \(S_x', S_y', S_z'\). Using these matrices one sees that the commutation relations 2.37 hold.

Unless otherwise specified, the diagonalized spin I matrices \(\overline{S}'\) will simply be represented by unprimed notations \(\overline{S}\).
Table 1. Matrix Representation of $S_x(I)$. We have illustrated the $m, m'$ - matrices of $S(I)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
The Matrix Representation of $S_x(I)$

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>I-1</th>
<th>I-2</th>
<th>...</th>
<th>-I+1</th>
<th>-I</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0</td>
<td>$\frac{1}{2} \sqrt{2I} \cdot 1$</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>I-1</td>
<td>$\frac{1}{2} \sqrt{2I} \cdot 1$</td>
<td>0</td>
<td>$\frac{1}{2} \sqrt{(2I-1) \cdot 2}$</td>
<td>...</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>I-2</td>
<td>0</td>
<td>$\frac{1}{2} \sqrt{(2I-1) \cdot 2}$</td>
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<td>$\frac{1}{2} \sqrt{1 \cdot 2I}$</td>
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</tbody>
</table>

Table 1
Table 2. Matrix Representation of $S_y(I)$. We have illustrated the $m, m'$ - matrices of $S(I)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
The Matrix Representation of $S_y(I)$

<table>
<thead>
<tr>
<th>$m'$</th>
<th>$I$</th>
<th>$I-1$</th>
<th>$I-2$</th>
<th>...</th>
<th>$-I+1$</th>
<th>$-I$</th>
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</thead>
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<tr>
<td>$I$</td>
<td>0</td>
<td>$-\frac{i\sqrt{2I+1}}{2}$</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$I-1$</td>
<td>$\frac{i\sqrt{2I+1}}{2}$</td>
<td>0</td>
<td>$-\frac{i\sqrt{(2I-1)+2}}{2}$</td>
<td>...</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$I-2$</td>
<td>0</td>
<td>$\frac{i\sqrt{(2I-1)+2}}{2}$</td>
<td>0</td>
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<td>$-\frac{i\sqrt{1+2I}}{2}$</td>
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<td>0</td>
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Table 2
Table 3. Matrix Representation of $S_z(I)$. We have illustrated the $m, m'$ - matrices of $\tilde{S}(I)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
The Matrix Representation of $S_z(I)$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$m'$</th>
<th>$I$</th>
<th>$I-1$</th>
<th>$I-2$</th>
<th>$\ldots$</th>
<th>$-I+1$</th>
<th>$-I$</th>
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</thead>
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<td>$I-1$</td>
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<td>$I-2$</td>
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<td>0</td>
<td>$\ldots$</td>
<td>0</td>
<td>$-I$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3

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III. THE MAXWELL EQUATIONS

Electromagnetic theory can be based on fundamental laws of Maxwell. The study of mechanics is founded on three basic concepts - space, time, and mass. In the theory of electromagnetism, the additional concept of change is introduced. The mutual interactions of charges and currents are described in terms of fields of electric intensity $E$ and magnetic induction $B$.

The electromagnetic field in a given region of space depends on the kind of matter occupying the region as well as on the distribution of charge and currents giving rise to the field. This thesis will assume the electromagnetic fields to be in a vacuum region.

The fundamental laws of electricity and magnetism can be summarized in differential form by the four Maxwell equations, in gaussian units;

Coulomb's Law:

$$\nabla \cdot \vec{E} = 4\pi \rho$$

(3.1)

Gauss's Law (absence of magnetic mono-poles):

$$\nabla \cdot \vec{B} = 0$$

(3.3)

Faraday's Law:

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

(3.4)

The continuity equation between charge and current is ex-
pressed by; conservation of charge:

\[ \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \]  

(3.5)

The quantity of electric current density is denoted by \( j \), and the total charge density by \( \rho \).

The curl or rotation operator is expressed as:

\[
\begin{pmatrix}
0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\
-\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0
\end{pmatrix}
\]  

(3.6)

We shall now transform this curl matrix operator into a form of a spin angular momentum operator:

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix}
\]  

(3.7a)

equivalently:

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  

(3.7b)

hence the new curl matrix:

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
S_x \frac{\partial}{\partial x} \\
S_y \frac{\partial}{\partial y} \\
S_z \frac{\partial}{\partial z}
\end{pmatrix}
\]  

(3.7c)

Ampere's and Faraday's Law can now be represented in terms of
the new spin matrix; we obtain

\[
\frac{i}{c} \frac{\partial \vec{E}}{\partial t} + i \vec{S} \cdot \nabla \vec{B} = 0
\]  
\[(3.8)\]

\[
\frac{i}{c} \frac{\partial \vec{B}}{\partial t} - i \vec{S} \cdot \nabla \vec{E} = 0
\]  
\[(3.9)\]

Since in a vacuum region the electric current density \( \vec{j} \) and the charge density \( \rho \) are by definition equal to zero.

Let us now define a new field vector \( \psi \) the following way:

\[
\psi = \begin{pmatrix} \psi_t \\ \psi_B \end{pmatrix}
\]

where

\[
\psi_E = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}
\]  
and

\[
\psi_B = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}
\]  
\[(3.10)\]

With this field vector we can combine Ampere's and Faraday's Laws into one equation; namely:

\[
\frac{i}{c} \frac{\partial \psi}{\partial t} + i \begin{pmatrix} 0 & \vec{S} \cdot \nabla \\ -\vec{S} \cdot \nabla & 0 \end{pmatrix} \psi = 0
\]  
\[(3.11)\]

Let us now define a new spin matrix:

\[
\vec{\alpha} = \begin{pmatrix} 0 & \vec{S} \\ -\vec{S} & 0 \end{pmatrix}
\]  
\[(3.12)\]

then the field vector equation becomes:

\[
\frac{i}{c} \frac{\partial \psi}{\partial t} + i (\vec{\alpha} \cdot \nabla) \psi = 0
\]

or
\[-\hbar \frac{\partial \Psi}{\partial t} = i \hbar \mathcal{C} (\mathbf{\hat{x}} \cdot \mathbf{\hat{u}}) \Psi \quad (3.13)\]

This expression is similar to Schrödinger's wave equation used in Quantum Mechanics:

\[-\hbar \frac{\partial \Psi}{\partial t} = \mathcal{H} \Psi \quad (3.14)\]

The result 3.13, in fact, represents a system for which the Hamiltonian \( \mathcal{H} \) is:

\[\mathcal{H} = \mathcal{C} (\mathbf{\hat{x}} \cdot \mathbf{\hat{p}}) \quad (3.15)\]

where

\[\mathbf{\hat{p}} = \frac{\hbar}{\epsilon} \mathbf{\hat{u}} \quad (3.16)\]

as used in quantum mechanics.

We shall confirm that the Hamiltonian is rotationally invariant; i.e., the total angular momentum operator \( \mathbf{\hat{J}} \) commutes with \( \mathcal{H} \):

\[\mathbf{\hat{J}} = \hbar (\mathbf{\hat{L}} + \mathbf{\hat{S}'}), \quad \text{where} \quad \mathbf{\hat{S}'} = \begin{pmatrix} \mathbf{\hat{s}} & 0 \\ 0 & \mathbf{\hat{s}} \end{pmatrix} \quad (3.17a)\]

We shall consider a particular representation for the angular momentum (refer to appendix B):

\[\mathbf{\hat{J}_x} = \hbar \mathbf{\hat{L}_x} + \hbar \begin{pmatrix} S_x & 0 \\ 0 & S_x \end{pmatrix} \quad (3.17b)\]

then:

\[\llbracket \mathbf{\hat{J}_x}, \mathcal{H} \rrbracket = \mathcal{C} \hbar \mathbf{\hat{a}} \cdot \mathbf{[L_x, \mathbf{\hat{p}}]} + \mathcal{C} \hbar \mathbf{[S_x', \mathbf{\hat{a}}]} \cdot \mathbf{\hat{p}}\]
Thus, for canonical coordinates and momenta $q_i, p_i$, we get in addition to the quantum conditions

$$[q_i, p_j] = i\hbar \delta_{ij}, \quad [q_i, q_j] = 0, \quad [p_i, p_j] = 0 \quad (3.19)$$

the further condition that

$$[J_x, H] = iC \left( \alpha_y \hat{h} \frac{P_z}{2} - \alpha_z \hat{h} P_y \right)$$

$$+ i C \hbar \begin{pmatrix} 0 & [S_x, S_y] & 0 \\ -[S_x, S_y] & 0 & -[S_x, S_z] \\ 0 & [S_z, S_y] & 0 \end{pmatrix}$$

Since we have shown that the spin matrices satisfy the commutation relations

$$[S_i, S_j] = i S_k \quad (3.20)$$

therefore: $(i, j, k)$ are in orders of $(x, y, z)$

$$[J_x, H] = -C \hbar (\bar{\alpha} x \bar{P})_x$$

$$-C \hbar \begin{pmatrix} 0 & S_z & 0 \\ -S_z & 0 & S_x \end{pmatrix} + S_y \begin{pmatrix} 0 & -S_y \\ S_y & 0 \end{pmatrix} = 0 \quad (3.18c)$$

Since no special property of $J_x$ was involved, this result implies that each component of $\bar{J}$ commutes with the Hamiltonian $H$: more compactly,
We look for a solution $\psi$ representing a dynamical state of well-defined energy $E$. Such a wave $\psi$ can be represented by the so-called time-independent Schrödinger's equation:

$$E \psi = H \psi \quad (3.21)$$

where

$$E = \frac{i\hbar \partial}{\partial t} \quad (3.22)$$

and equivalence between energy and time-derivative operator in quantum mechanics.

The Hamiltonian $H$ is defined by equation 3.15. Consequently, the operation of equation 3.21 by $E$ results in the form:

$$E^2 \psi = i \hbar C \begin{vmatrix} 0 & \bar{S} \\ -\bar{S} & 0 \end{vmatrix} \cdot \mathbf{PE} \psi$$

$$= C^2 \begin{vmatrix} 0 & \bar{S} \cdot \bar{P} \\ -\bar{S} \cdot \bar{P} & 0 \end{vmatrix} \begin{vmatrix} 0 & \bar{S} \cdot \bar{P} \\ -\bar{S} \cdot \bar{P} & 0 \end{vmatrix} \psi$$

$$= C^2 (\bar{S} \cdot \bar{P})^2 \psi \quad (3.23)$$

The curl matrix is expressed by:

$$\bar{S} \cdot \bar{P} = S_x P_x + S_y P_y + S_z P_z$$
\[
\begin{pmatrix}
0 & \frac{1}{2} P_x & 0 \\
\frac{1}{2} P_x & 0 & \frac{1}{2} P_y \\
0 & \frac{1}{2} P_y & 0
\end{pmatrix} + 
\begin{pmatrix}
0 & -\frac{1}{2} P_y & 0 \\
\frac{1}{2} P_y & 0 & -\frac{1}{2} P_x \\
0 & \frac{1}{2} P_x & 0
\end{pmatrix} + 
\begin{pmatrix}
P_z & 0 & 0 \\
0 & 0 & -P_z \\
0 & 0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
P_z & \frac{1}{2} P_+ & 0 \\
\frac{1}{2} P_+ & 0 & \frac{1}{2} P_- \\
0 & \frac{1}{2} P_+ & -P_z
\end{pmatrix}
\]

(3.24)

where \( P_\pm = P_x \pm i P_y \) therefore:

\[
(\vec{S} \cdot \vec{P})^2 = \vec{\Pi} \cdot \vec{P}^2 + \vec{A}
\]

(3.25)

\[
\vec{A} = - \begin{pmatrix}
\frac{1}{2} P_- \\
0 \\
\frac{1}{2} P_+
\end{pmatrix} \begin{pmatrix}
-P_z & \frac{1}{2} P_+ & \frac{1}{2} P_- \\
0 & \frac{1}{2} P_+ & \frac{1}{2} P_- \\
0 & \frac{1}{2} P_+ & \frac{1}{2} P_- \\
\end{pmatrix}
\]

(3.26)

We then define the matrices:

\[
(\vec{G} \cdot \vec{P}) = \begin{pmatrix}
\frac{1}{2} P_- \\
P_z \\
\frac{1}{2} P_+
\end{pmatrix}
\]

(3.27)

\[
(\vec{D} \cdot \vec{P}) = \begin{pmatrix}
-\frac{1}{2} P_+ & P_z & \frac{1}{2} P_-
\end{pmatrix}
\]

where:
\[(\vec{G} \cdot \vec{P}) = (\vec{D} \cdot \vec{P})^\dagger\]
\[(\vec{G} \cdot \vec{P})^\dagger = (\vec{D} \cdot \vec{P})\] (3.28)

since \(P^\pm = P_x \mp ip_y = P_x \mp ip_y\)

We then obtain the squared Hermitian matrix in the form:
\[ (\vec{S} \cdot \vec{P})^2 = P^2 - (\vec{G} \cdot \vec{P})(\vec{D} \cdot \vec{P}) \] (3.29a)

which is equivalent to
\[ -\text{curl} \cdot \text{curl} = \nabla^2 - (\text{grad})(\text{div}) \] (3.29b)

Consequently; we designate new gradient and divergence operators; namely:

\[
\begin{align*}
\text{Gradient} &= \vec{G} \cdot \vec{V} \\
\text{Divergence} &= \vec{D} \cdot \vec{V}
\end{align*}
\] (3.30)

Since the divergence of \(\psi\) is zero, then;
\[(\vec{D} \cdot \vec{P})\psi = 0 \] (3.31)

The results of equation 3.23 have shown that the following relation is satisfied when acting on \(\psi\);
\[E^2 = C^2 P^2 \] (3.32)

We can interpret this wave equation to correspond to the Klein-Gordon equation, which plays an important role in Relativistic Quantum Theory\(^7\). The Klein-Gordon equation states that
\[ E^2 \Psi = \left[ C^2 P^2 + (mc)^2 \right] \Psi \] (3.33a)

or equivalently:
\[ \frac{1}{C^2} \frac{\partial^2 \Psi}{\partial t^2} - \nabla^2 \Psi = \frac{m^2 c^2 \Psi}{\hbar^2} \] (3.33b)
This is the Klein-Gordon equation but it does not describe any particle with spin.

We note that the equation 3.33a can be written as the product of two linear expressions in $E$ and $\bar{P}$:

$$E = \pm c \sqrt{\left|\bar{P}\right|^2 + m^2 c^2}$$

(3.34)

Characteristic is the ambiguity in sign of equation 3.34. If we choose the positive sign, we are in agreement with the classical theory where the energy $E$, is always positive, corresponding to a positive rest mass. The negative sign is, however, also possible and corresponds to a negative rest mass.

How far and how exactly can one consistently compare the radiation field with an ensemble of independent particles?

When in 1924 de Broglie suggested that material particles should show wave phenomena such a comparison was of great historic importance. Now that wave mechanics has become a consistent formalism one could ask whether it is possible to consider the Maxwell equations to be a kind of Schrödinger equation for a massless particle, instead of considering them, as we have done up to now, to be classical equations of motion which formally look like a wave equation.

According to the latest investigation of the properties of the vacuum, the quantum theory of a charge-free radiation field must in its present status be considered to be an approximation. This must be kept in mind as we show the analogy of the Klein-Gordon equation with the theory of electromagnetic field. It enables us to obtain a relativistically invariant theory contain-
ing positive eigenvalues only, and it makes possible the existence of a field which as far as its space-time properties are concerned is characterized by a differential equation.

We shall show that only the transverse waves contribute to the radiation field in a vacuum. Let us transcribe the Maxwell and continuity equations (3.1 to 3.5) into a quantum mechanical form with the aid of equations 3.16 and 3.22. The new forms are:

Coulomb's Law:
\[
\overrightarrow{P} \cdot \overrightarrow{E} = \frac{4\pi}{\lambda c} (c\rho) 
\]

Gauss's Law:
\[
\overrightarrow{P} \cdot \overrightarrow{B} = 0 
\]

Ampere's Law:
\[
\overrightarrow{P} \times \overrightarrow{B} = \frac{4\pi}{\lambda c} \overrightarrow{J} - \overrightarrow{E} \overrightarrow{E} 
\]

Faraday's Law:
\[
\overrightarrow{P} \times \overrightarrow{E} = \frac{\overrightarrow{E}}{\lambda c} \overrightarrow{B} 
\]

Conservation of charge (continuity equation)
\[
\overrightarrow{P} \cdot \overrightarrow{J} = \frac{\overrightarrow{E}}{\lambda c} (c\rho) 
\]

We resolve the fields into longitudinal and transverse parts:
\[
\overrightarrow{E}(r,t) = E_L(r,t) + E_T(r,t) \\
\overrightarrow{B}(r,t) = B_L(r,t) + B_T(r,t) 
\]

Whereby convention, we define a component of the field which is parallel to the momentum vector \( \overrightarrow{P} \) as longitudinal; perpendicular to the momentum vector \( \overrightarrow{P} \) as transverse. Then, because of the
properties of the cross products, we find that only the transverse parts of \( E \) and \( B \) propagate in space time. The transverse relations are:

\[
P \times B_T = \frac{4\pi}{c} J_T - \frac{E}{c} E_T
\]
\[
P \times E_T = \frac{E}{c} B_T
\]

(3.41)

The longitudinal relations are;

\[
P \cdot E_L = \frac{4\pi}{\lambda} \rho \hspace{1cm} \frac{E}{c} E_L = \frac{4\pi}{\lambda c} J_L
\]
\[
P \cdot B_L = 0 \hspace{1cm} \frac{E}{c} B_L = 0
\]

(3.42)

In a vacuum region the electric current density \( \mathbf{j} \) and the charge density \( \rho \), are by definition equal to zero. We then conclude longitudinal fields cannot propagate, although static longitudinal fields can exist.
IV. SIGMA SPIN MATRIX

The Maxwell equations for vacuum (spin I=1) can be written in the following form:

\[
\begin{align*}
\frac{\epsilon}{\kappa c} \vec{E} &= + \left( \vec{S} \cdot \vec{\nabla} \right) \vec{B} \quad , \quad \vec{\nabla} \cdot \vec{E} = 0 \\
\frac{\mu}{\kappa c} \vec{B} &= - \left( \vec{S} \cdot \vec{\nabla} \right) \vec{E} \quad , \quad \vec{\nabla} \cdot \vec{B} = 0
\end{align*}
\]  

(4.1)

The vectors $\vec{E}$ and $\vec{B}$ can be expressed in irreducible tensor form which resulted from Cartesian coordinate by canonical transformations.

\[
\begin{bmatrix}
W_{+1} \\
W_{0} \\
W_{-1}
\end{bmatrix} =
\begin{bmatrix}
-\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\
0 & 0 & 1 \\
\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0
\end{bmatrix}
\begin{bmatrix}
W_{x} \\
W_{y} \\
W_{z}
\end{bmatrix}
\]

(4.2)

Let us define: $\vec{W} = \vec{E}$ or $\vec{B}$.

We shall equate new notations; namely:

\[
\vec{U} \equiv \vec{E} \quad , \quad \vec{\nabla} \equiv \vec{B}
\]  

(4.3)

The Maxwell equations can now combine and form new field vector equations:

\[
\begin{align*}
\frac{\epsilon}{\kappa c} \begin{bmatrix}
U_{+1} \\
U_{0} \\
U_{-1} \\
0
\end{bmatrix} &= + \begin{bmatrix}
\nabla_{x} & \frac{i}{\sqrt{2}} \nabla_{z} & 0 & -\frac{i}{\sqrt{2}} \nabla_{z} \\
\frac{i}{\sqrt{2}} \nabla_{z} & 0 & \frac{i}{\sqrt{2}} \nabla_{x} & \nabla_{x} \\
0 & \frac{i}{\sqrt{2}} \nabla_{z} & -\nabla_{x} & \frac{i}{\sqrt{2}} \nabla_{z} \\
-\frac{i}{\sqrt{2}} \nabla_{z} & \frac{i}{\sqrt{2}} \nabla_{x} & \nabla_{x} & 0
\end{bmatrix} \begin{bmatrix}
V_{+1} \\
V_{0} \\
V_{-1} \\
V_{s}
\end{bmatrix}
\end{align*}
\]
and similarly:

$$
\begin{pmatrix}
V_{+1} \\
V_0 \\
V_{-1} \\
0
\end{pmatrix} = \begin{pmatrix}
\nabla_+ & \frac{i}{\sqrt{2}} \nabla_+ & 0 & -\frac{i}{\sqrt{2}} \nabla_-\\
\frac{1}{\sqrt{2}} \nabla_+ & 0 & \frac{i}{\sqrt{2}} \nabla_- & \frac{i}{\sqrt{2}} \nabla_z \\
0 & \frac{1}{\sqrt{2}} \nabla_+ & -\nabla_z & \frac{i}{\sqrt{2}} \nabla_+ \\
-\frac{i}{\sqrt{2}} \nabla_+ & \frac{i}{\sqrt{2}} \nabla_z & \frac{i}{\sqrt{2}} \nabla_- & 0
\end{pmatrix}
\begin{pmatrix}
U_{+1} \\
U_0 \\
U_{-1} \\
U_s
\end{pmatrix}
$$

(4.4)

Where we define the del operator $\nabla_\pm$ and $\nabla_z$:

$$
\nabla_\pm = \frac{2}{\partial x} \pm i \frac{2}{\partial y}, \quad \nabla_z = \frac{2}{\partial z}
$$

(4.5)

The $V_s$ and $U_s$ are static vacuum fields. Let us show designate $\Sigma^\xi \cdot \nabla$ as the Sigma Spin operator:

$$
\begin{align*}
\frac{E}{\hbar c} \tilde{U}' &= + (\Sigma^\xi \cdot \nabla) \tilde{V}' \\
\frac{E}{\hbar c} \tilde{V}' &= - (\Sigma^\xi \cdot \nabla) \tilde{U}'
\end{align*}
$$

where

$$
\tilde{U}' = \begin{pmatrix}
\tilde{U} \\
U_s
\end{pmatrix} \text{ and } \tilde{V}' = \begin{pmatrix}
\tilde{V} \\
V_s
\end{pmatrix}
$$

(4.6)

The Sigma Spin matrix $\Sigma^\xi$ is defined by:

$$
\Sigma^\xi = \begin{pmatrix}
S & \xi G \\
\xi D & 0
\end{pmatrix}
$$

where $\xi = -1$

(4.7)

The property then satisfies:
\[ \left| \sum_{n} \right|^{2} = \begin{bmatrix} \tilde{S}_{n}^{2} + \tilde{G}_{n} \tilde{D}_{n} & \frac{1}{2} \tilde{S}_{n} \tilde{G}_{n} \\ \frac{1}{2} \tilde{D}_{n} \tilde{S}_{n} & \tilde{D}_{n} \tilde{G}_{n} \end{bmatrix} = 1 \]  

(4.8)

Because of \( \tilde{S}_{n} \tilde{G}_{n} = 0, \tilde{D}_{n} \tilde{S}_{n} = 0, \tilde{S}_{n}^{2} + \tilde{G}_{n} \tilde{D}_{n} = 1, \tilde{D}_{n} \tilde{G}_{n} = 1 \)

and

\[ \sum_{n} = \begin{bmatrix} \tilde{S}_{m} \tilde{S}_{m} + \tilde{G}_{m} \tilde{D}_{m} & \frac{1}{2} \tilde{S}_{m} \tilde{G}_{m} \\ \frac{1}{2} \tilde{D}_{m} \tilde{S}_{m} & \tilde{D}_{m} \tilde{G}_{m} \end{bmatrix} \]  

(4.9)

(\( \ell, m, n \)) are in orders of (x, y, z)

Consequently:

\[ \sum_{\ell} \sum_{m} + \sum_{m} \sum_{\ell} = 0 \]  

(4.10)

We define:

\[ \tilde{S}_{n} = (\tilde{S} \cdot \tilde{n}), \tilde{G}_{n} = (\tilde{G} \cdot \tilde{n}), \tilde{D}_{n} = (\tilde{D} \cdot \tilde{n}) \]  

(4.11)

and \( \tilde{n} \) in terms of unit vectors (\( n_{x}, n_{y}, n_{z} \))

We shall now diagonalize the z-component of the Sigma Spin Matrix, then transform the x and y-components. The diagonalization is

the

\[ \lim_{n} \sum_{n} = \lambda_{i} \delta_{ij} \]  

(4.12)

The transformation matrix is then:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & A_{\ell} & 0 & A_{\ell} \\
0 & 0 & 1 & 0 \\
0 & A_{\ell} & 0 & -A_{\ell}
\end{pmatrix}
\]  

(4.13)
where $A=\pm 1$, $\xi=\pm 1$

The $x$ component of the transformed Sigma Spin matrix is:

$$
\vec{\Sigma}'_x = U \hat{\Sigma}_x U^{-1}
$$

$$
\begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & A \\
0 & 0 & A \xi & 0 \\
0 & A \xi & 0 & 0 \\
A & 0 & 0 & 0
\end{pmatrix}
$$

(4.14)

The transformed $x$-component of the Sigma Spin matrix is required to have a form similar to the Pauli spin matrix; namely:

$$
\vec{\Sigma}'_x = \begin{pmatrix}
0 & 0 & 1 \\
\tilde{\vec{\sigma}}_x(1/2) & 0 \\
1 & 0 & 0
\end{pmatrix}
$$

(4.15)

The $\tilde{\vec{\sigma}}_x(1/2)$ is the Pauli spin matrix for $I=1/2$. We then conclude that for spin $I=1$, we choose this solution $A=+1$ and $\xi=+1$.

The $y$-component of the transformed Sigma Spin matrix is:

$$
\vec{\Sigma}'_y = U \hat{\Sigma}_y U^{-1}
$$

$$
\begin{pmatrix}
0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & -iA \\
0 & 0 & -iA \xi & 0 \\
iA \xi & 0 & 0 & 0 \\
iA & 0 & 0 & 0
\end{pmatrix}
$$

(4.16)
The transformed $y$ component of the Sigma Spin matrix is required to have a form similar to the Pauli spin matrix; namely:

\[
\Sigma'_y = \begin{pmatrix}
0 & 0 & -i \\
\sim & \sim_{y(1/2)} & \sim \\
i & 0 & 0
\end{pmatrix}
\]  

(4.17)

The $\sim_{y}(1/2)$ is the Pauli spin matrix for $I=1/2$. We conclude that for spin $I=1$, we choose this solution $\xi=+1$ and $\xi=+1$. This is consistent with the previous results.

The diagonalized Sigma Spin matrix for the $z$-component is:

\[
\Sigma'_z = \sum_z \Sigma_z \sum_z^{-1}
\]

\[
\Sigma'_z = \sum_z^{-1} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & -1 & 0 \\
0 & i & 0 & 0
\end{pmatrix} \sum_z = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]  

(4.18)

Consequently; the form becomes:

\[
\Sigma'_z = \begin{pmatrix}
1 & 0 & 0 \\
\sim & \sim_{z(1/2)} & \sim \\
0 & 0 & -1
\end{pmatrix}
\]  

(4.19)

The $\sim_{z}(1/2)$ is the Pauli spin matrix for $I=1/2$.

Therefore, to generalize the Maxwell equations to the Spin I field, it is necessary to find a general form of the Sigma Spin...
matrix. This general form has dimensions \((4I) \times (4I)\):

\[
\sum_{n}^{I} = \begin{pmatrix}
\bar{C}(I) & \bar{G}(I) \\
\bar{D}(I) & \bar{F}(I)
\end{pmatrix}
\]  

(4.20)

If the \(E_{n}(I)\) matrix is transformed into the convenient form where \(E_{z}(I)\) is diagonalized, the transformed components \(E'_{n}(I) = U E_{n}(I) U\) are required to have a form similar to the Pauli spin matrix:

\[
\begin{pmatrix}
\bar{0} & \bar{0} & \bar{1} \\
\bar{0} & \bar{0} & -i\bar{1} \\
\bar{1} & \bar{0} & \bar{0}
\end{pmatrix}
\]

(4.21)

where \(\bar{U}\) is a \((2I-1) \times (2I-1)\) matrix and \(\bar{\sigma}(1/2)\) is the Pauli spin matrix.

The generalized vector operators are then represented in the following form:

The Curl matrix \(\bar{C}(I)\), which has dimensions \((2I+1) \times (2I+1)\):

\[
\bar{C}(I) = \frac{i}{I} \bar{\mathbf{S}}(I)
\]  

(4.22)

The Divergence matrix \(\bar{D}(I)\), which has dimensions \((2I-1) \times (2I+1)\):

The Gradient matrix \(\bar{G}(I)\), which has a dimensions \((2I+1) \times (2I-1)\):

\[
\bar{G}_{\alpha}(I) = \bar{D}^{\dagger}_{\alpha}(I)
\]  

(4.23)
where $\tilde{G}_n(I)$, $\tilde{D}_n(I)$ and $\tilde{C}_n(I)$ are required to satisfy the relation:

$$\tilde{G}_n(I) \cdot \tilde{D}_n(I) = \mathbb{1} - \tilde{C}_n(I)$$

(4.24)

The free-spaced matrix $\tilde{F}(I)$ which represents a region free from spin I waves has a dimension $(2I-1) \times (2I-1)$.

$$\tilde{F}(I) = \frac{-1}{I^2} \tilde{S}(I-I) = -(I^{-1}) \tilde{C}(I-I)$$

(4.25)

The sign of $\xi$ must be determined to satisfy relations similar to that of the Pauli spin matrices.

In order to satisfy these relations, the factor $\frac{1}{I}$ is also necessary for defining the Curl matrix as well as the free-space matrix. Then we require the following relations:

$$\left| \sum_{n}^{t} \right|^2 = \left| \begin{array}{cc}
\tilde{C}_n^2(I) + G_n(I) \cdot \tilde{D}_n(I) & \frac{1}{2} \tilde{C}_n(I) \tilde{G}_n(I) + \frac{1}{2} \tilde{G}_n(I) \tilde{F}_n(I) \\
\frac{1}{2} \tilde{D}_n(I) \tilde{C}_n(I) + \frac{1}{2} \tilde{F}_n(I) \tilde{D}_n(I) & \tilde{D}_n(I) \tilde{G}_n(I) + \tilde{F}_n^2(I) 
\end{array} \right| = \mathbb{1}$$

(4.26)

and

$$\sum_{n}^{t} = \left| \begin{array}{cc}
\tilde{C}_n^t(I) \cdot \tilde{C}_n(I) + G_n(I) \cdot \tilde{D}_n(I) & \frac{1}{2} \tilde{C}_n(I) \tilde{G}_n(I) + \frac{1}{2} \tilde{G}_n(I) \tilde{F}_n(I) \\
\frac{1}{2} \tilde{D}_n(I) \tilde{C}_n(I) + \frac{1}{2} \tilde{F}_n(I) \tilde{D}_n(I) & \tilde{D}_n(I) \tilde{G}_n(I) + \tilde{F}_n^t(I) 
\end{array} \right|$$

(4.27)

$(\ell, m, n)$ are in orders of $(x, y, z)$

Consequently:

$$\sum_{g}^{t} \sum_{m}^{t} (I) + \sum_{m}^{t} (I) \sum_{g}^{t} (I) = 0$$

(4.28)

We note the similarity to the Pauli spin matrices anticommutative quantities; namely:
The new generalized field vector equations for spin I are then represented in the following form:

\[
\begin{align*}
\vec{\mathcal{E}}_x(Y_2) &= \vec{\mathcal{E}}_y(Y_2) = \vec{\mathcal{E}}_z(Y_2) = \vec{\mathcal{I}} \\
\vec{\mathcal{G}}_x(Y_2) \vec{\mathcal{G}}_y(Y_2) &= -\vec{\mathcal{G}}_y(Y_2) \vec{\mathcal{G}}_x(Y_2) = \mathcal{I} \vec{\mathcal{G}}_z(Y_2)
\end{align*}
\]

(4.29)

The dimensions of the field vectors \(\psi'(I)\) are \((41)\); namely: they are the field vectors \(\psi(I)\) of dimensions \((2I+1)\), and the static vacuum fields \(\psi_s(I-1)\) of dimensions \((2I-1)\). Therefore: \(\psi'=U'\) or \(V'\)

\[
\psi'(I) = \begin{bmatrix} \psi(I) \\ \psi_s'(I) \end{bmatrix}
\]

(4.31)

We shall now demonstrate the complexity of the static vacuum field.

From Maxwell equations of Gauss's Law and Faraday's Law, it follows that:

\[
\vec{B} = \vec{\nabla} \times \vec{A} \quad (\vec{A}=\text{vector potential})
\]

(4.32)

and

\[
\vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \quad (\phi=\text{scalar potential})
\]

(4.33)

Since \(\vec{B}\) is defined in terms of a vector potential \(\vec{A}\), an arbitrary fourth component of scalar function \(\Lambda\) can be added. Thus, \(\vec{B}\) is left unchanged by the transformation;
In order that the electric field be changed as well, the scalar potential must be simultaneously transformed:

$$A' = A + \nabla \Lambda$$  \hspace{1cm} (4.34)

The freedom implied means that we must choose a set of potentials $(\vec{A}, \phi)$ such that "Lorentz gauge condition".

$$\left( \nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right) = 0$$  \hspace{1cm} (4.35)

is not gauge-invariant. To see that potentials can always be found to satisfy the Lorentz condition, suppose that the potentials $\vec{A}, \phi$ which satisfy the equations of vector and scalar potentials do not satisfy the "Lorentz gauge condition". Then we shall make a gauge transformation of the potentials $\vec{A}', \phi$ and demand that $\vec{A}', \phi'$ satisfy the Lorentz condition:

$$\left( \nabla \cdot \vec{A}' + \frac{1}{c} \frac{\partial \phi'}{\partial t} \right) = 0 = \nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2}$$  \hspace{1cm} (4.36)

But since $\Lambda$ is arbitrary and does not affect the Maxwell equations, we are permitted to make this choice. Obviously the new gauge transformation

$$\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = 0$$  \hspace{1cm} (4.37)

preserves the Lorentz condition, provided $\vec{A}, \phi$ satisfies it initially. The advantage of having the "Lorentz gauge condition" satisfied is that we can solve the equation for its highest time derivatives of the potentials and obtain a system of differential equations. In such a system the initial conditions on a function

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guarantee its unique continuation into the future (and its past).

Using the definition $\nabla^2 = \nabla \nabla \cdot - \nabla \nabla \times$, the differential equations relating the potential functions to their sources follow from Maxwell's equations. These equations are:

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{1}{c} \frac{\partial}{\partial t} \left( \nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right) - 4 \pi \rho$$  \hspace{1cm} (4.39a)

$$\nabla^2 \vec{A} = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} + \nabla \left( \nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right) - \frac{4 \pi}{c} \vec{J}$$  \hspace{1cm} (4.40a)

Equivalently:

$$\nabla \cdot \left( \frac{1}{c^2} \frac{\partial \vec{A}}{\partial t} - 4 \pi \vec{P} + \nabla \phi \right) = 0$$  \hspace{1cm} (4.39b)

$$\frac{\partial}{\partial t} \left( \nabla \times \nabla \times \vec{A} + \frac{1}{c^2} \frac{\partial \vec{A}}{\partial t} - \frac{4 \pi}{c} \vec{P} + \frac{1}{c} \nabla \phi \right) = 0$$  \hspace{1cm} (4.40b)

where the polarization $\vec{P} = \int \vec{J} dt$, $\nabla \cdot \vec{P} = \rho$. When $\nabla \cdot \vec{P} = 0$, the equations assume a simple form if $A$ is chosen so that $\phi$ vanishes.

Equation 4.39b has a simpler form when expressed in terms of the field vector $\vec{\Pi}$:

$$\vec{\Pi} = c \int \vec{A} dt$$  \hspace{1cm} (4.41)

In terms of the field vector, the equation becomes:

$$\nabla \times \nabla \times \vec{\Pi} + \frac{1}{c^2} \frac{\partial^2 \vec{\Pi}}{\partial t^2} - 4 \pi \vec{P} + \nabla \phi = 0$$  \hspace{1cm} (4.42)

If the gauge $\phi = -\nabla \cdot \vec{\Pi}$ is chosen, the equation reduces to a form particularly convenient when $\vec{\Pi}$ is expressed in rectangular coordinates

$$\nabla^2 \vec{\Pi} - \frac{1}{c^2} \frac{\partial^2 \vec{\Pi}}{\partial t^2} = -4 \pi \vec{P}$$
\[ \mathbf{B} = \frac{i}{c} \mathbf{\nabla} \times \frac{\partial \mathbf{\Pi}}{\partial t} \]
\[ \mathbf{E} = \mathbf{\nabla} \cdot \mathbf{\Pi} - \frac{1}{c^2} \frac{\partial^2 \mathbf{\Pi}}{\partial t^2} \]

(4.43)

In a vacuum region \( \mathbf{P} \) vanishes; therefore, Maxwell's equations are symmetrical in \( \mathbf{E} \) and \( \mathbf{B} \). If one works with the field strengths \( \mathbf{E} \) and \( \mathbf{B} \) themselves rather than with the potentials, then the transformed generalized fields equations are unchanged, regardless of any gauge condition. We introduce potentials primarily for mathematical convenience. Consequently, the field vector equation

4.30

\[ \frac{\mathbf{E}}{hc} \begin{bmatrix} \mathbf{U}(I) \\ \mathbf{\nabla} \end{bmatrix} = \left[ \mathbf{\Sigma}(I) \cdot \mathbf{\nabla} \right] \begin{bmatrix} \mathbf{V}(I) \\ \mathbf{V}_s(I^{-1}) \end{bmatrix} \]

has an expanded form

\[ \frac{\mathbf{E}}{hc} \begin{bmatrix} \mathbf{U}(I) \\ \mathbf{\nabla} \end{bmatrix} = \mathbf{C}(I) \cdot \mathbf{\nabla} \mathbf{V}(I) + \xi \mathbf{G}(I) \cdot \mathbf{\nabla} \mathbf{V}_s(I^{-1}) \]

\[ \mathbf{\nabla} = \xi \mathbf{D}(I) \cdot \mathbf{\nabla} \mathbf{V}(I) - \frac{(I^{-1}) \mathbf{C}(I^{-1}) \cdot \mathbf{\nabla} \mathbf{V}_s(I^{-1})}{I} \]

The reciprocal relationship is:

\[ \frac{\mathbf{E}}{hc} \begin{bmatrix} \mathbf{V}(I) \\ \mathbf{\nabla} \end{bmatrix} = -\left[ \mathbf{\Sigma}(I) \cdot \mathbf{\nabla} \right] \begin{bmatrix} \mathbf{U}(I) \\ \mathbf{U}_s(I^{-1}) \end{bmatrix} \]

has an expanded form

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\[ E \delta \vec{\nabla}(I) = \overline{C}(I) \cdot \vec{\nabla} \vec{U}(I) - \xi \overline{G}(I) \cdot \vec{\nabla} \vec{U}_s(I-l) \]

\[ \overline{0} = -\xi \overline{D}(I) \cdot \vec{\nabla} \vec{U}(I) + (I-l) \overline{C}(I-l) \cdot \vec{\nabla} \vec{U}_s(I-l) \] (4.44)

For a fluidic field, let \( \vec{\psi}_s(l-l) \) equal the velocity vector \( \vec{\omega}_s \) times the density \( \rho_s \); \( \vec{\psi}_s = \vec{u}_s \) or \( \vec{v}_s \):

\[ \overline{\psi}_s \cdot \vec{\nabla} f_s = \text{div grad} f_s \]

\[ \overline{\psi}_s \cdot (\vec{\nabla} \times \vec{\psi}_s) = \text{div curl} \vec{\psi}_s = 0 \]

\[ \vec{\nabla} \times (\vec{\nabla} f_s) = \text{curl grad} f_s = 0 \] (4.45)

And \( \vec{\nabla} \times \vec{\psi}_s = \text{curl} \vec{\psi}_s = 0 \) is a necessary and sufficient condition for \( \vec{\psi}_s \) to be the gradient of some scalar function \( f_s \) defined by:

\[ \vec{\psi}_s = \vec{\nabla} f_s \] (4.46)

The general equation of continuity for any fluidic field is:

\[ \vec{\nabla} \cdot \vec{\psi}_s = -\frac{\partial \rho_s}{\partial t} \] (4.47)

The flux of an incompressible fluidic field has the divergence zero,

\[ \vec{\nabla} \cdot \vec{\psi}_s = 0 \] (4.48)

These static vacuum fields (fluidic in nature) are analyzed as a dilation and a rigid displacement. For the latter, the angular velocity (spin) is zero. Consequently; the generalized divergence of the field vector \( \vec{\psi}(I) \) is zero; \( \vec{\psi} = \vec{u} \) or \( \vec{v} \):

\[ \overline{D}(I) \cdot \vec{\nabla} \vec{\psi}(I) = 0 \] (4.49)
V. POYNTING'S VECTOR

The forms of the laws of conservation of energy and momentum are important results to establish for the electromagnetic field. We begin by considering conservation of energy, often called "Poynting's theorem (1884)". He found that a plane electromagnetic wave in free space propagates in the direction of the vector $\mathbf{E} \times \mathbf{B}$. Let us calculate the divergence of this vector for any electromagnetic field in vacuum:

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$$  \hspace{1cm} (5.1)$$

From Maxwell's equations it can be shown that:

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = -\frac{1}{c} \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{J})$$  \hspace{1cm} (5.2)$$

If there exists a continuous distribution of charge and current, the total rate of doing work by the fields in a finite volume $V$ is

$$\int_V \mathbf{J} \cdot \mathbf{E} \, dV$$  \hspace{1cm} (5.3)$$

This power represents a conversion of electromagnetic energy into mechanical or thermal energy.

Let us now represent the total energy density $\varepsilon$ by the relation:

$$\varepsilon = \frac{1}{8\pi} \left( \mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B} \right)$$  \hspace{1cm} (5.4)$$

then its time derivatives are:

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{8\pi} \left( \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{E}}{\partial t} \right)$$  \hspace{1cm} (5.5)$$
The use of Maxwell's equations of Ampere's and Faraday's Law represented by the new spin matrix resolve the energy density loss per time into:

\[
\frac{\partial \epsilon}{\partial t} = \frac{i}{\hbar} \left[ \mathbf{\nabla} \cdot (\mathbf{S} \cdot \mathbf{V}) \mathbf{B}^* \mathbf{E} - i \mathbf{C} (\mathbf{S} \cdot \mathbf{V}) \mathbf{B}^* \mathbf{E}^+ \\
- i \mathbf{C} (\mathbf{S} \cdot \mathbf{V}) \mathbf{E}^+ \mathbf{B} + i \mathbf{C} (\mathbf{S} \cdot \mathbf{V}) \mathbf{E} \mathbf{B}^+ \right]
\]

\[
= \frac{i}{\hbar \pi} \mathbf{\nabla} \cdot [(\mathbf{B}^* \mathbf{S} \mathbf{E}) - (\mathbf{E}^* \mathbf{S} \mathbf{B})]
\]

(5.6)

We use the law of conservation of energy (equation 3.5) to determine the vector \( \mathbf{\Pi} \), which represents energy flow;

\[
\mathbf{\Pi} = \frac{i}{\hbar \pi} \left[ (\mathbf{E}^+ \mathbf{S} \mathbf{B}) - (\mathbf{B}^+ \mathbf{S} \mathbf{E}) \right]
\]

(5.7)

and has the dimensions (energy/area x time).

We shall now prove this energy flow to be equivalent to the classical Poynting's vector for spin \( I=1 \). We shall now examine each Cartesian component of the energy flow. The x-component is:

\[
\mathbf{\Pi}_x = \frac{i}{\hbar \pi} \left[ (\mathbf{E}^+ \mathbf{S}_x \mathbf{B}) - (\mathbf{B}^+ \mathbf{S}_x \mathbf{E}) \right]
\]

\[
= \frac{i}{\hbar \pi} \left[ \begin{array}{ccc}
0 & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0
\end{array} \right]
\]

\[
= \frac{i}{\hbar \pi} \left[ \begin{array}{ccc}
0 & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0
\end{array} \right] \left[ \begin{array}{c}
\mathbf{B}_{+1} \\
\mathbf{B}_0 \\
\mathbf{B}_{-1}
\end{array} \right] + \left[ \begin{array}{c}
\mathbf{E}_{+1} \\
\mathbf{E}_0 \\
\mathbf{E}_{-1}
\end{array} \right]
\]

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\[
\Pi_x = \frac{i\hbar}{8}\left[ +E_0^* B_{+1} + E_{+1}^* B_0 + E_{-1}^* B_{-1} + E_0^* B_{-1} \\ -E_{+1}^* B_0 - E_0^* B_{+1} - E_{-1}^* B_{-1} - E_0^* B_{+1}\right] (5.8a)
\]

The irreducible spherical tensor forms can be expressed in Cartesian coordinates using equation 4.2.

\[
\Pi_x = \frac{i\hbar}{16}\left[ -E_z^* (B_x - i B_y) - (E_x^* + i E_y^*) B_z \\ + (E_x^* - i E_y^*) B_z + E_z^* (B_x + i B_y) \\ + (E_x - i E_y) B_z^* + E_z (B_x + i B_y) \\ -E_z (B_x - i B_y) - (E_x + i E_y) B_z^* \right]
\]

\[
= \frac{\hbar}{8\pi} \left[ (E_y B_z^* + E_y B_z) - (E_z B_y^* + E_z B_y) \right] (5.8b)
\]

The y-component is:

\[
\Pi_y = \frac{i\hbar}{8\pi} \left[ \left( \bar{E}^* S_y \bar{B} \right) - \left( \bar{B}^* S_y \bar{E} \right) \right]
\]

\[
= \frac{i\hbar}{8\pi}\left[ \left( \begin{array}{cc}
0 & -\frac{i}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & 0
\end{array} \right) \left( \begin{array}{c}
B_{+1} \\
B_0 \\
B_{-1}
\end{array} \right) \\
+ \left( \begin{array}{cc}
E_0 & E_{+1} \\
E_{-1} & -E_0
\end{array} \right) \left( \begin{array}{cc}
\frac{i}{\sqrt{2}} & 0 \\
0 & \frac{i}{\sqrt{2}}
\end{array} \right) \left( \begin{array}{c}
B_{+1} \\
B_0 \\
B_{-1}
\end{array} \right) \\
- \left( \begin{array}{cc}
B_{+1} & B_0 \\
B_{-1} & -E_0
\end{array} \right) \left( \begin{array}{cc}
\frac{i}{\sqrt{2}} & 0 \\
0 & \frac{i}{\sqrt{2}}
\end{array} \right) \left( \begin{array}{c}
E_{+1} \\
E_0 \\
E_{-1}
\end{array} \right) \right]
\]

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The irreducible spherical tensor forms can be expressed in Cartesian coordinates using equation 4.2.

\[
\Pi_y = \frac{c}{8\sqrt{2}\pi} \begin{bmatrix}
-E_0^* B_{t1} + E_{t1}^* B_0 - E_{-1}^* B_0 + E_0^* B_{-1} \\
+E_{t1}^* B_0^* - E_0 B_{t1}^* + E_0^* B_{-1}^* - E_{-1} B_0^*
\end{bmatrix} 
\] (5.9a)

\[
\Pi_y = \frac{c}{16\pi} \begin{bmatrix}
+E_z^* (B_x - iB_y) - (E_x^* + iE_y^*) B_z \\
-(E_x^* - iE_y^*) B_z^* + E_z^* (B_x + iB_y) \\
-(E_x - iE_y) B_z^* + E_z (B_x^* + iB_y^*) \\
+E_z (B_x^* - iB_y^*) - (E_x + iE_y) B_z^*
\end{bmatrix} 
\] (5.9b)

The z-component is:

\[
\Pi_z = \frac{ic}{8\pi} \left[ \begin{bmatrix}
\bar{E}^+ S_z B \\
+ (E_{t1}^* E_0^* E_{-1}^*) \\
-(B_{t1}^* B_0^* B_{-1}^*)
\end{bmatrix} \right] 
\] (5.10a)
The irreducible spherical tensor forms can be expressed in Cartesian coordinates using equation 4.2.

\[
\Pi_z = \frac{ic}{4\pi} \begin{pmatrix}
+(E_x^*+iE_y^*)(B_x-iB_y) - (E_x^*-iE_y^*)(B_x+iB_y) \\
-(E_x-iE_y)(B_x^*+iB_y^*) + (E_x+iE_y)(B_x^*-iB_y^*)
\end{pmatrix}
\]

Consequently, the energy flow \( \Pi \) for spin \( I=1 \) is equivalent to Poynting's vector:

\[
\text{Re} \left\{ \frac{ic}{4\pi} \left[ E \cdot B^* \right] \right\} = \frac{ic}{8\pi} \left[ (E \cdot B) - (B \cdot E) \right]
\] (5.11)

The generalized energy flow \( \Pi \) for spin \( I \) field has the form:

\[
\Pi = \frac{ic}{8\pi} \left\{ \left[ \mathcal{U}(I) \mathcal{C}(I) \mathcal{V}(I) \right] - \left[ \mathcal{V}(I) \mathcal{C}(I) \mathcal{U}(I) \right] \right\}
\] (5.12a)

which is equivalently represented by:

\[
\Pi = \frac{ic}{8\pi} \begin{pmatrix}
\mathcal{U}(I) & \mathcal{V}(I) \\
-\mathcal{C}(I) & \mathcal{U}(I)
\end{pmatrix}
\] (5.12b)

Let us find the representation of energy flow in spherical coordinates. The transformation from Cartesian to spherical coordinates is represented by:

\[
\mathbf{S} = S_x \mathbf{i} + S_y \mathbf{j} + S_z \mathbf{k} \quad \text{(Cartesian)}
\]

\[
\mathbf{S} = S_\phi \mathbf{\hat{\rho}} + S_\theta \mathbf{\hat{\theta}} + S_\phi \mathbf{\hat{\phi}} \quad \text{(Spherical)}
\] (5.13)
We then obtain the transformation:

\[
\begin{pmatrix}
S_x \\
S_y \\
S_z
\end{pmatrix} =
\begin{pmatrix}
\overrightarrow{p} \cdot \overrightarrow{i} & \overrightarrow{p} \cdot \overrightarrow{j} & \overrightarrow{p} \cdot \overrightarrow{k} \\
\overrightarrow{m} \cdot \overrightarrow{i} & \overrightarrow{m} \cdot \overrightarrow{j} & \overrightarrow{m} \cdot \overrightarrow{k} \\
\overrightarrow{n} \cdot \overrightarrow{i} & \overrightarrow{n} \cdot \overrightarrow{j} & \overrightarrow{n} \cdot \overrightarrow{k}
\end{pmatrix}
\begin{pmatrix}
S_x \\
S_y \\
S_z
\end{pmatrix}
\]  

(5.14)

The spherical coordinate transformations are:

\[
\begin{align*}
X &= r \cos \phi \sin \theta \\
Y &= r \sin \phi \sin \theta \\
Z &= r \cos \theta \\
\overrightarrow{r} &= X \overrightarrow{i} + Y \overrightarrow{j} + Z \overrightarrow{k}
\end{align*}
\]  

(5.15)

The unit vectors are then represented:

\[
\begin{align*}
\overrightarrow{l} &= \frac{\partial \overrightarrow{r}}{\partial r} = \cos \phi \sin \theta \overrightarrow{i} + \sin \phi \sin \theta \overrightarrow{j} + \cos \theta \overrightarrow{k} \\
\overrightarrow{m} &= \frac{1}{r} \frac{\partial \overrightarrow{r}}{\partial \theta} = \cos \phi \cos \theta \overrightarrow{i} + \sin \phi \cos \theta \overrightarrow{j} - \sin \theta \overrightarrow{k} \\
\overrightarrow{n} &= \frac{1}{rsin \theta} \frac{\partial \overrightarrow{r}}{\partial \phi} = -\sin \phi \overrightarrow{i} + \cos \phi \overrightarrow{j}
\end{align*}
\]

(5.16)

Consequently:

\[
\begin{pmatrix}
S_x \\
S_y \\
S_z
\end{pmatrix} =
\begin{pmatrix}
\cos \phi \sin \theta & \sin \phi \sin \theta & \cos \theta \\
\cos \phi \cos \theta & \sin \phi \cos \theta & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{pmatrix}
\begin{pmatrix}
S_x \\
S_y \\
S_z
\end{pmatrix}
\]  

(5.17)

This represents the spin matrix for \( I=1 \).
The $\lambda$ component (radial) of the matrix is represented by:

$$S_\lambda = (\vec{L} \cdot \vec{r}) S_x + (\vec{L} \cdot \vec{j}) S_y + (\vec{L} \cdot \vec{k}) S_z$$

$$= \sin \theta \cos \phi \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$+ \sin \theta \sin \phi \begin{pmatrix} 0 & -i \frac{1}{\sqrt{2}} & 0 \\ i \frac{1}{\sqrt{2}} & 0 & -i \frac{1}{\sqrt{2}} \\ 0 & i \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$+ \cos \theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

therefore:

$$S_\lambda = \begin{pmatrix} \cos \theta & \frac{1}{\sqrt{2}} \sin \theta e^{-i\phi} & -i \phi \\ \frac{1}{\sqrt{2}} \sin \theta e^{i\phi} & 0 & \frac{1}{\sqrt{2}} \sin \theta e^{-i\phi} \\ 0 & \frac{1}{\sqrt{2}} \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}$$

(5.18)

The $m$ component of the matrix is represented by:

$$S_m = (\vec{m} \cdot \vec{r}) S_x + (\vec{m} \cdot \vec{j}) S_y + (\vec{m} \cdot \vec{k}) S_z$$
The $n$ component (azimuthal) of the matrix is represented by:

$$S_n = \begin{pmatrix} -\sin \theta & \frac{1}{\sqrt{2}} \cos \theta e^{-i\phi} & 0 \\ \frac{1}{\sqrt{2}} \cos \theta e^{i\phi} & 0 & \frac{1}{\sqrt{2}} \cos \theta e^{i\phi} \\ 0 & \frac{1}{\sqrt{2}} \cos \theta e^{i\phi} & \sin \theta \end{pmatrix}$$

(5.19)

We can now represent the generalized energy flow in spherical coordinates for spin $I$:

$$\Pi_\lambda = \frac{i \lambda c}{8 \pi} \left[ (\hat{U}(I)^+ C_\lambda(I) \hat{V}(I)) - (\hat{V}(I)^+ C_\lambda(I) \hat{U}(I)) \right]$$

(5.21)

where $\lambda = l, m, n$.
VI. SIGMA SPIN $I = 3/2$ OPERATOR MATRIX

Let us now derive the Sigma Spin operator matrix for spin-$I=3/2$, and then examine its properties.

\[
\sum_i^{i} \frac{\mathbf{S} \cdot \nabla}{3} = \begin{pmatrix}
\mathbf{C}(3/2) & \frac{i}{\sqrt{3}} \mathbf{G}(3/2) \\
\frac{i}{\sqrt{3}} \mathbf{D}(3/2) & 3(3/2) \\
\end{pmatrix} \cdot \nabla
\]  

(6.1)

The Curl matrix $C(3/2)$ is then:

\[
\mathbf{C}(3/2) = \frac{2}{3} \mathbf{S}(3/2)
\]

\[
C_x(3/2) = \begin{pmatrix}
0 & \frac{1}{\sqrt{3}} & 0 & 0 \\
\frac{1}{\sqrt{3}} & 0 & \frac{2}{3} & 0 \\
0 & \frac{2}{3} & 0 & \frac{1}{\sqrt{3}} \\
0 & 0 & \frac{1}{\sqrt{3}} & 0
\end{pmatrix}
\]

\[
C_y(3/2) = \begin{pmatrix}
0 & -\frac{i}{\sqrt{3}} & 0 & 0 \\
\frac{i}{\sqrt{3}} & 0 & -\frac{2}{3} i & 0 \\
0 & \frac{2}{3} i & 0 & -\frac{i}{\sqrt{3}} \\
0 & 0 & \frac{1}{\sqrt{3}} & 0
\end{pmatrix}
\]
The Curl operator is defined by:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & -\frac{1}{3} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

(6.2)

The Curl operator squared has the form:

\[
\begin{pmatrix}
\nabla_z & \frac{1}{3} \nabla_+ & 0 & 0 \\
\frac{1}{3} \nabla_+ & \frac{1}{3} \nabla_+ & 0 & 0 \\
0 & 0 & \frac{2}{3} \nabla_+ & -\frac{1}{3} \nabla_+ \\
0 & 0 & \frac{1}{3} \nabla_+ & -\nabla_z
\end{pmatrix}
\]

(6.3)

Let us now examine the operator \( \widetilde{A} \) define as:

\[
\widetilde{A} \equiv [\widetilde{C}(3/2) \cdot \nabla]^2 - \nabla^2
\]
Equivalently:

\[
\vec{A} = -\left[ \bar{G}(\frac{3}{2}) \cdot \vec{\nabla} \right] \left[ \bar{D}(\frac{3}{2}) \cdot \vec{\nabla} \right]
\]  

(6.5b)

This is the analogous relation to that which is proved for the case \( I = 1 \). The elements of the gradient and divergence matrix are required to have the following form. The matrix dimensions are expressed in equations 4.23.

\[
\bar{A} = -\left( \begin{array}{cccc}
\alpha^* & e^* & \beta^* & f^* \\
\gamma^* & g^* & \delta^* & h^*
\end{array} \right)
\]

(6.6)

Then the solutions are:

\[
\begin{align*}
\alpha^* a + e^* e &= \frac{2}{3} \nabla_+ \nabla_-
\\
\alpha^* b + e^* f &= -\frac{4}{3\sqrt{3}} \nabla_\varepsilon \nabla_-
\\
\alpha^* c + e^* g &= -\frac{2}{3\sqrt{3}} \nabla^2
\\
\alpha^* d + e^* h &= 0
\end{align*}
\]
\[ b^* a + f^* e = - \frac{4}{3 \sqrt{3}} \nabla^2 \]
\[ b^* b + f^* f = \frac{8}{9} \nabla^2 + \frac{2}{9} \nabla_+ \nabla_- \]
\[ b^* c + f^* g = 0 \]
\[ b^* d + f^* h = - \frac{2}{3 \sqrt{3}} \nabla^2 \]
\[ c^* a + g^* e = - \frac{2}{3 \sqrt{3}} \nabla^2_+ \]
\[ c^* b + g^* f = 0 \]
\[ c^* c + g^* g = \frac{8}{9} \nabla^2 + \frac{2}{9} \nabla_+ \nabla_- \]
\[ c^* d + g^* h = \frac{4}{3 \sqrt{3}} \nabla^2 \]
\[ d^* a + h^* e = 0 \]
\[ d^* b + h^* f = - \frac{2}{3 \sqrt{3}} \nabla^2_+ \]
\[ d^* c + h^* g = \frac{4}{3 \sqrt{3}} \nabla^2_+ \]
\[ d^* d + h^* h = \frac{2}{3} \nabla^2_+ \]

(6.7)

We find that we can resolve eight solutions to the matrix equation. We then restrict our solutions to satisfy the commutation relation of the Sigma Spin matrix. We then find we have two solutions, differing only by a sign \( \xi = \pm 1 \). The solutions are:
\[ a = -\sqrt{\frac{\alpha}{3}} \nabla_4, \quad e = 0 \]

\[ b = \sqrt{\frac{\alpha}{3}} \nabla_2, \quad f = -\sqrt{\frac{\alpha}{3}} \nabla_1 \]

\[ c = \sqrt{\frac{\alpha}{3}} \nabla_1, \quad g = \sqrt{\frac{\alpha}{3}} \nabla_2 \]

\[ d = 0, \quad h = \sqrt{\frac{\alpha}{3}} \nabla_1 \quad (6.8) \]

The divergence matrix \((2\times4)\) is expressed by:

\[
\begin{bmatrix}
\alpha & b & c & d \\
e & f & g & h
\end{bmatrix}
\]

\[ (6.9) \]

The gradient matrix \((4\times2)\) is expressed by:

\[
\begin{bmatrix}
\alpha^* & e^* \\
b^* & f^* \\
c^* & g^* \\
d^* & h^*
\end{bmatrix}
\]

\[ (6.10) \]

The free-space matrix \((2\times2)\) is expressed by:

\[
\begin{bmatrix}
-\frac{1}{3} \nabla_2 & -\frac{1}{3} \nabla_4 \\
-\frac{1}{3} \nabla_1 & \frac{1}{3} \nabla_2
\end{bmatrix}
\]

\[ (6.11) \]

The final Sigma Spin operator matrix is then:
We shall now diagonalize the z-component of the Sigma Spin matrix, then transform the x and y-components. The components are required to have a form similar to the Pauli spin matrices.

The diagonalization is then:

\[
\sum_{z} \lambda_{V} \sigma_{z} = \sum_{z} \lambda_{V} \sigma_{z}
\]  

(6.13)

The eigenvalues of a Hermitian matrix, \( \lambda_{V} \), are all real.

The unitary matrix is expressed by:
\[ U(3/2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{3} A^4 & 0 & 0 & \frac{1}{3} A & 0 \\
0 & 0 & \frac{1}{3} B^4 & 0 & 0 & \sqrt{3} B \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & \frac{1}{3} A & 0 & 0 & -\frac{1}{3} A^4 & 0 \\
0 & 0 & \sqrt{3} B & 0 & 0 & -\frac{1}{3} B^4 \\
\end{pmatrix} \]

(6.14)

where \( A = \pm 1, \ B = \pm 1, \ \xi = \pm 1 \)

The \( x \)-component of the transformed Sigma Spin matrix is:

\[ \Sigma_x' = U \Sigma_x U \]

\[ \Sigma_x' = U \begin{pmatrix}
0 & \frac{1}{3} & 0 & 0 & -\sqrt{3} \xi & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & -\sqrt{3} \xi \\
0 & \frac{2}{3} & 0 & \frac{1}{3} & \sqrt{3} \xi & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & \sqrt{3} \xi \\
-\sqrt{3} \xi & 0 & \sqrt{3} \xi & 0 & 0 & -\frac{1}{3} \\
0 & -\sqrt{3} \xi & 0 & \sqrt{3} \xi & -\frac{1}{3} & 0 \\
\end{pmatrix} U \]
The x-component of the transformed Sigma Spin matrix is required to have a form similar to the Pauli spin matrix; namely:

\[
\sum_x' = \begin{pmatrix}
0 & 0 & 0 & 0 & A & 0 \\
0 & 0 & 0 & 0 & 0 & AB_x^x \\
0 & 0 & B_x^x & 0 & 0 & 0 \\
0 & 0 & B_x^x & 0 & 0 & 0 \\
A & 0 & 0 & 0 & 0 & 0 \\
0 & AB_x^x & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]  \quad (6.15)

We then conclude that for spin $I=3/2$, the two solutions are $A=+1$ and $B_x^x=+1$. The $\sigma_x(1/2)$ is the Pauli spin matrix for $I=1/2$.

The y-component of the transformed Sigma Spin matrix is:
The \( y \)-component of the transformed Sigma Spin matrix is required to have a form similar to the Pauli spin matrix; namely:

\[
\sum_y' = \sum_y U
\]

\[
\begin{pmatrix}
0 & -\sqrt{3} & 0 & 0 & \sqrt{3}i & 0 \\
\sqrt{3} & 0 & -\frac{2}{3}i & 0 & 0 & \sqrt{3}i \\
0 & \frac{2}{3}i & 0 & -\frac{1}{\sqrt{3}} & \sqrt{3}i \\
0 & 0 & \frac{1}{\sqrt{3}} & 0 & \sqrt{3}i \\
-\frac{2}{3}i & 0 & -\sqrt{3}i & 0 & 0 & \frac{1}{3} \\
0 & -\frac{2}{3}i & 0 & -\sqrt{3}i & \frac{1}{3} & 0
\end{pmatrix}U
\]

(6.17)

\[
\sum_y = \begin{pmatrix}
0 & 0 & 0 & \cdots & -A & 0 \\
0 & 0 & 0 & \cdots & 0 & -AB \gamma \\
0 & 0 & 0 & \cdots & -B \gamma & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
A & 0 & 0 & \cdots & 0 & 0 \\
0 & AB \gamma & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(6.18)
We then conclude that for spin $I=3/2$, the two solutions are $A=+1$ and $B=+1$. The $\gamma_I^{(1/2)}$ is the Pauli spin matrix for $I=1/2$.

The $z$-component of the diagonalized Sigma Spin matrix is:

$$\Sigma^z = \mathbf{U}^{-1} \Sigma^z \mathbf{U}$$

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & \frac{\sqrt{6}}{3} & 0 \\
0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{\sqrt{6}}{3} \\
0 & \frac{\sqrt{3}}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 0 & \frac{\sqrt{3}}{3} & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(6.19)

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Consequently the form becomes:

$$\sum'_{\mathcal{z}} = \begin{pmatrix}
\tilde{1} & \tilde{0} & \tilde{0} \\
\tilde{0} & \tilde{u}_z(y_z) & \tilde{0} \\
\tilde{0} & \tilde{0} & -\tilde{1}
\end{pmatrix}$$

(6.20)

The $\tilde{\sigma}_z(1/2)$ is the Pauli spin matrix for $I=1/2$. 

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VII. SIGMA SPIN I=2 OPERATOR MATRIX

Let us now derive the Sigma Spin operator matrix for spin I=2, and then examine its properties.

$$\sum_{\xi} (2) \cdot \nabla = \begin{pmatrix} \bar{C}(2) & \xi \bar{G}(2) \\ \xi \bar{D}(2) & F(2) \end{pmatrix} \cdot \nabla$$

(7.1)

The curl matrix $$\bar{C}(2)$$ is then:

$$\bar{C}(2) = \frac{1}{2} \bar{S}(2)$$

$$C_x(2) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$$C_y(2) = \begin{pmatrix} 0 & -\frac{i}{2} & 0 & 0 & 0 \\ \frac{i}{2} & 0 & -\frac{i \sqrt{2}}{4} & 0 & 0 \\ 0 & \frac{i \sqrt{2}}{4} & 0 & -\frac{i \sqrt{2}}{4} & 0 \\ 0 & 0 & \frac{i \sqrt{2}}{4} & 0 & -\frac{i}{2} \\ 0 & 0 & 0 & \frac{i}{2} & 0 \end{pmatrix}$$
\[ C_z(2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix} \]  

(7.2)

The Curl operator is defined by:

\[ \overline{C}(2) \cdot \overline{\nabla} = \begin{pmatrix}
\nabla_{\frac{z}{4}} & \frac{1}{2} \nabla_{\frac{z}{4}} & 0 & 0 & 0 \\
\frac{1}{2} \nabla_{+} & \frac{1}{2} \nabla_{\frac{z}{4}} & \frac{\sqrt{2}}{4} \nabla_{0} & 0 & 0 \\
0 & \frac{\sqrt{2}}{4} \nabla_{+} & 0 & \frac{\sqrt{2}}{4} \nabla_{0} & 0 \\
0 & 0 & \frac{\sqrt{2}}{4} \nabla_{+} & -\frac{1}{2} \nabla_{z} & \frac{1}{2} \nabla_{-} \\
0 & 0 & 0 & \frac{1}{2} \nabla_{+} & -\nabla_{z}
\end{pmatrix} \]  

(7.3)

The Curl operator squared has the form:

\[ [\overline{C}(2) \cdot \overline{\nabla}]^2 = \begin{pmatrix}
\nabla_{\frac{z}{4}}^2 + \frac{1}{4} \nabla_{+} \nabla_{-} & \frac{3}{4} \nabla_{\frac{z}{4}} \nabla_{-} & \frac{\sqrt{2}}{8} \nabla_{0}^2 & 0 & 0 \\
\frac{3}{4} \nabla_{\frac{z}{4}} \nabla_{+} & \frac{1}{4} \nabla_{\frac{z}{4}}^2 + 2 \nabla_{+} \nabla_{-} & \frac{\sqrt{2}}{8} \nabla_{0}^2 & \frac{3}{8} \nabla_{-}^2 & 0 \\
\frac{\sqrt{2}}{8} \nabla_{+}^2 & \frac{\sqrt{2}}{8} \nabla_{\frac{z}{4}} \nabla_{+} & \frac{3}{4} \nabla_{+} \nabla_{-} & \frac{1}{4} \nabla_{z}^2 & \frac{\sqrt{2}}{8} \nabla_{0}^2 \\
0 & \frac{3}{8} \nabla_{+}^2 & -\frac{\sqrt{2}}{8} \nabla_{\frac{z}{4}} \nabla_{+} & 1 \nabla_{2}^2 + 5 \nabla_{0} \nabla_{2} & -\frac{3}{4} \nabla_{z} \nabla_{2} \\
0 & 0 & \frac{\sqrt{2}}{8} \nabla_{+}^2 & -\frac{3}{4} \nabla_{\frac{z}{4}} \nabla_{+} & \frac{\sqrt{2}}{4} \nabla_{2}^2 + \frac{1}{4} \nabla_{0} \nabla_{2}
\end{pmatrix} \]  

(7.4)
Let us now examine the operator $A$ is defined as:

$$
\overline{A} = \left[ \overline{C}(z) \cdot \overline{\nabla} \right]^2 - \nabla^2
$$

$$
A = -\begin{pmatrix}
\frac{3}{4} \nabla_+ \nabla_+ & -\frac{2}{4} \nabla_+ \nabla_- & -\frac{\sqrt{6}}{8} \nabla_-^2 & 0 & 0 \\
-\frac{3}{4} \nabla_- \nabla_+ & \frac{3}{4} \nabla_-^2 + \frac{3}{8} \nabla_+ \nabla_- & -\frac{\sqrt{6}}{8} \nabla_- \nabla_- & -\frac{2}{8} \nabla_-^2 & 0 \\
-\frac{\sqrt{6}}{8} \nabla_+^2 & -\frac{\sqrt{6}}{8} \nabla_- \nabla_+ & \frac{\sqrt{6}}{8} \nabla_+ \nabla_- & 0 & 0 \\
0 & -\frac{3}{8} \nabla_-^2 & \frac{\sqrt{6}}{8} \nabla_+ \nabla_+ & \frac{3}{4} \nabla_- \nabla_- & \frac{3}{4} \nabla_+ \nabla_- \\
0 & 0 & -\frac{\sqrt{6}}{8} \nabla_+^2 & \frac{3}{4} \nabla_- \nabla_+ & \frac{3}{4} \nabla_+ \nabla_- \\
\end{pmatrix}
$$

(7.5a)

Equivalently:

$$
\overline{A} = -\left[ \overline{G}(z) \cdot \overline{\nabla} \right] \left[ \overline{D}(z) \cdot \overline{\nabla} \right]
$$

(7.5b)

The elements of the gradient and divergence matrix are required to have the following form. The matrix dimensions are expressed in equations 4.23.

$$
\overline{A} = \begin{pmatrix}
\alpha^* f^* k^* \\
b^* g^* l^* \\
c^* h^* m^* \\
d^* i^* n^* \\
e^* j^* p^*
\end{pmatrix}
\begin{pmatrix}
a & b & c & d & e \\
f & g & h & i & j \\
k & l & m & n & p
\end{pmatrix}
$$

(7.6)

Then the solutions are:
\[ a^*a + f^*f + k^*k = \frac{3}{4} \nabla_4 \nabla_4 \]
\[ b^*a + g^*f + l^*k = -\frac{3}{4} \nabla_z \nabla_4 \]
\[ c^*a + h^*f + m^*k = -\frac{\sqrt{6}}{8} \nabla_2 \nabla_4 \]
\[ d^*a + i^*f + n^*k = 0 \]
\[ e^*a + j^*f + p^*k = 0 \]
\[ a^*b + f^*g + k^*l = -\frac{3}{4} \nabla_z \nabla_4 \]
\[ b^*b + g^*g + l^*l = \frac{3}{4} \nabla_2^2 + \frac{3}{8} \nabla_4 \nabla_4 \]
\[ c^*b + h^*g + m^*l = -\frac{\sqrt{6}}{8} \nabla_z \nabla_4 \]
\[ d^*b + i^*g + n^*l = -\frac{3}{8} \nabla_2^2 \]
\[ e^*b + j^*g + p^*l = 0 \]
\[ a^*c + f^*h + k^*m = -\frac{\sqrt{6}}{8} \nabla_2^2 \]
\[ b^*c + g^*h + l^*m = -\frac{\sqrt{6}}{8} \nabla_z \nabla_4 \]
\[ c^*c + h^*h + m^*m = \nabla_2^2 + \frac{i}{4} \nabla_4 \nabla_4 \]
\[ d^*c + i^*h + n^*m = \frac{\sqrt{6}}{8} \nabla_z \nabla_4 \]
\[ e^*c + j^*h + p^*m = -\frac{\sqrt{6}}{8} \nabla_2^2 \]
\[ a^*d + f^*i + k^*n = 0 \]
\[ b^*d + g^*i + l^*n = -\frac{3}{8} \nabla_2^2 \]
\[ c^*d + h^*i + m^*n = \frac{\sqrt{6}}{8} \nabla_z \nabla_4 \]
\[ \begin{align*}
d^*d + \lambda^* \lambda + \eta^* n &= \frac{3}{4} \nabla_z^2 + \frac{3}{8} \nabla_+ \nabla_- \\
e^*d + j^* \lambda + p^* n &= \frac{3}{4} \nabla_z \nabla_+ \\
a^*e + f^* j + k^* p &= 0 \\
b^*e + g^* j + l^* p &= 0 \\
c^*e + h^* j + m^* p &= -\frac{\sqrt{2}}{8} \nabla_-^2 \\
d^*e + \lambda^* j + n^* p &= \frac{3}{4} \nabla_z \nabla_- \\
e^*e + f^* j + p^* p &= \frac{3}{4} \nabla_+ \nabla_- \\
\end{align*} \]

(7.7)

We find that we can resolve forty-eight solutions to the matrix equation. We then restrict our solutions to satisfy the commutation relation of the Sigma Spin matrix. We then find we have two solutions, differing only by a sign \( \xi = \pm 1 \). The solutions are:

\[ \begin{align*}
a &= -\frac{\sqrt{2}}{4} \nabla_+ , & f &= 0 , & k &= 0 \\
b &= \frac{\sqrt{2}}{4} \nabla_z , & g &= -\frac{\sqrt{2}}{8} \nabla_+ , & \lambda &= 0 \\
c &= \frac{\sqrt{2}}{8} \nabla_+ , & h &= \nabla_z , & m &= -\frac{\sqrt{2}}{8} \nabla_+ \\
d &= 0 , & \lambda &= \frac{\sqrt{2}}{8} \nabla_- , & n &= \frac{3}{4} \nabla_z \\
e &= 0 , & j &= 0 , & p &= \frac{\sqrt{2}}{4} \nabla_- \\
\end{align*} \]

(7.8)

The divergence matrix (3x5) is expressed by:
\[ \left[ \xi \, \overline{D}(z) \cdot \nabla \right] = \frac{1}{4} \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & p \end{pmatrix} \]  

(7.9)

The gradient matrix (5x3) is expressed by:

\[ \left[ \xi \, \overline{G}(z) \cdot \nabla \right] = \frac{1}{4} \begin{pmatrix} a^* & f^* & k^* \\ b^* & g^* & l^* \\ c^* & h^* & m^* \\ d^* & j^* & n^* \\ e^* & j^* & p^* \end{pmatrix} \]  

(7.10)

The free-space matrix (3x3) is expressed by:

\[ \left[ \overline{F}(z) \cdot \nabla \right] = \begin{pmatrix} -\frac{1}{2} \nabla_\perp & -\frac{i}{\sqrt{8}} \nabla_\perp & 0 \\
-\frac{i}{\sqrt{8}} \nabla_\perp & 0 & -\frac{i}{\sqrt{8}} \nabla_\perp \\
0 & -\frac{i}{\sqrt{8}} \nabla_\perp & \frac{1}{2} \nabla_\parallel \end{pmatrix} \]  

(7.11)
The final Sigma Spin operator matrix is then:

\[ \sum_{z}^{(2)} \cdot \nabla = \]

\[
\begin{pmatrix}
\n \nabla_{z} & \frac{1}{2} \nabla_{-} & 0 & 0 & 0 & -\frac{i}{\sqrt{8}} \nabla_{z} & 0 & 0 \\
\n\frac{1}{2} \nabla_{+} & \frac{1}{2} \nabla_{z} & \frac{\sqrt{3}}{2} \nabla_{-} & 0 & 0 & \frac{i}{\sqrt{8}} \nabla_{z} & -\frac{i}{\sqrt{8}} \nabla_{z} & 0 \\
\n0 & \frac{\sqrt{3}}{2} \nabla_{+} & 0 & \frac{\sqrt{3}}{2} \nabla_{-} & 0 & \frac{i}{\sqrt{8}} \nabla_{z} & \frac{i}{\sqrt{8}} \nabla_{z} & -\frac{i}{\sqrt{8}} \nabla_{z} \\
\n0 & 0 & \frac{3}{2} \nabla_{+} & -\frac{1}{2} \nabla_{z} & \frac{3}{2} \nabla_{-} & 0 & \frac{i}{\sqrt{4}} \nabla_{z} & \frac{i}{\sqrt{4}} \nabla_{z} \\
\n-\frac{i}{\sqrt{4}} \nabla_{+} & \frac{i}{\sqrt{4}} \nabla_{z} & \frac{i}{\sqrt{4}} \nabla_{-} & 0 & 0 & -\frac{i}{\sqrt{8}} \nabla_{z} & -\frac{i}{\sqrt{8}} \nabla_{z} & 0 \\
\n0 & -\frac{i}{\sqrt{8}} \nabla_{+} & \frac{i}{\sqrt{8}} \nabla_{z} & \frac{i}{\sqrt{8}} \nabla_{-} & 0 & -\frac{i}{\sqrt{8}} \nabla_{z} & 0 & -\frac{i}{\sqrt{8}} \nabla_{z} \\
\n0 & 0 & -\frac{i}{\sqrt{8}} \nabla_{+} & \frac{i}{\sqrt{8}} \nabla_{z} & \frac{i}{\sqrt{8}} \nabla_{-} & 0 & -\frac{i}{\sqrt{8}} \nabla_{z} & \frac{i}{\sqrt{8}} \nabla_{z} \\
\n\end{pmatrix}
\]

(7.12)

We shall now diagonalize the z-component of the Sigma Spin matrix, then transform the x and y components. These transformations must satisfy relations similar to the Pauli spin matrices. The diagonalization is then:

\[ \prod_{z} \sum_{z}^{(2)} \nabla = \| \lambda_i \delta_{ij} \| \]

(7.13)
The unitary matrix is then:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{1/4}A & 0 & 0 & 0 & \sqrt{1/2}B & 0 \\
0 & 0 & \sqrt{1/2}B & 0 & 0 & 0 & \sqrt{3}C \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3}C \\
0 & A & 0 & 0 & 0 & \sqrt{1/4}A & 0 \\
0 & 0 & \sqrt{1/2}B & 0 & 0 & 0 & -\sqrt{1/2}B \\
0 & 0 & 0 & 0 & -\sqrt{3}C & 0 & 0 \\
\end{pmatrix}
\]  

(7.14)

where \( A=\pm1 \), \( B=\pm1 \), \( C=\pm1 \) and \( \xi=\pm1 \).

The \( x \)-component of the transformed Sigma Spin matrix is:

\[
\sum'_{x} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{x} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]
\[ \sum_{x} \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\ -\frac{1}{2} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

(7.15)

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The x-component of the transformed Sigma Spin matrix is required to have a form similar to the Pauli spin matrix; namely:

\[
\Sigma_x' = \begin{pmatrix}
\sigma & 0 & \tilde{1} \\
0 & \widetilde{\sigma}_{x}(\xi) & 0 \\
\tilde{1} & 0 & \sigma
\end{pmatrix}
\]  

(7.16)

We then conclude for spin \( I=2 \), the solution is \( A=+1 \), \( B=+1 \), \( C=+1 \) and \( \xi=+1 \). The \( \widetilde{\sigma}_{x}(1/2) \) is the Pauli spin matrix for \( I=1/2 \).

The y-component of the transformed Sigma Spin matrix is:

\[
\Sigma_y' = U \Sigma_y U^{-1}
\]

\[
\Sigma_y' = U
\left[
\begin{array}{cccccccc}
0 & -\frac{i}{2} & 0 & 0 & 0 & i\sqrt{\frac{2}{3}} & 0 & 0 \\
\frac{i}{2} & 0 & -i\sqrt{\frac{1}{3}} & 0 & 0 & 0 & i\sqrt{\frac{2}{3}} & 0 \\
0 & i\sqrt{\frac{2}{3}} & 0 & -i\sqrt{\frac{1}{3}} & 0 & i\sqrt{\frac{2}{3}} & 0 & i\sqrt{\frac{2}{3}} \\
0 & 0 & i\sqrt{\frac{2}{3}} & 0 & -\frac{i}{2} & 0 & i\sqrt{\frac{2}{3}} & 0 \\
0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & i\sqrt{\frac{2}{3}} \\
-i\sqrt{\frac{2}{3}} & 0 & -i\sqrt{\frac{1}{3}} & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\
0 & -i\sqrt{\frac{2}{3}} & 0 & -i\sqrt{\frac{1}{3}} & 0 & -i\sqrt{\frac{2}{3}} & 0 & i\sqrt{\frac{2}{3}} \\
0 & 0 & -i\sqrt{\frac{2}{3}} & 0 & -i\sqrt{\frac{1}{3}} & 0 & -i\sqrt{\frac{2}{3}} & 0
\end{array}
\right] U
\]
The $y$-component of the transformed Sigma Spin matrix is required to have a form similar to the Pauli spin matrix; namely:

$$\sum_{y}' = \begin{pmatrix} \tilde{\sigma} & \tilde{\sigma} & -i\tilde{A} \\ \sigma & \tilde{\sigma} & \tilde{\sigma} \\ i\tilde{\sigma} & \tilde{\sigma} & \tilde{\sigma} \end{pmatrix}$$

(7.18)

We then conclude for spin $I=2$, the solution is $A=+1$, $B=+1$, $C=+1$ and $\xi=+1$. The $\tilde{\sigma}_y(1/2)$ is the Pauli spin matrix for $I=1/2$.

The $z$-component of the diagonalized Sigma Spin matrix is:

$$\sum_{z}' = U \sum_{z} U^\dagger$$
\[ \sum' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & \sqrt{\frac{\xi}{\eta}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \sqrt{\frac{\xi}{\eta}} & 0 \\
0 & \sqrt{\frac{\xi}{\eta}} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & \xi & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{\xi}{\eta}} & 0 & 0 & 0 & \frac{1}{2} \\
\end{bmatrix} \]

\[ \sum'' = \begin{bmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ddots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & 0 & 0 \\
\end{bmatrix} \]

(7.19)
The diagonalized Sigma Spin matrix has a form similar to the Pauli spin matrix; namely:

\[
\sum^' = \begin{pmatrix}
\Tilde{1} & \Tilde{0} & \Tilde{0} \\
\Tilde{0} & \Tilde{\sigma_x(1/2)} & \Tilde{0} \\
\Tilde{0} & \Tilde{0} & -\Tilde{1}
\end{pmatrix}
\] (7.20)

The \( \Tilde{o_z(1/2)} \) is the Pauli spin matrix for \( I=1/2 \).
VIII. RADIATION FIELD \(( j \ m)\) FOR SPIN I=1/2

We shall demonstrate the use of the Sigma Spin operator for spin I=1/2. The field vector equation for spin I=1/2 is obtained from equation 4.30.

\[
\frac{E}{\hbar c} \overrightarrow{U}(\gamma_2) = + \left[ \sum (\gamma_2) \cdot \overrightarrow{\nabla} \right] \overrightarrow{V}(\gamma_2)
\]

Equivalently:

\[
\begin{vmatrix}
E \frac{\gamma_2}{\hbar c} U_{j_m}^{\gamma_2} \\
E \frac{\gamma_2}{\hbar c} U_{-j_m}^{\gamma_2} \\
\end{vmatrix} = + \begin{vmatrix}
\nabla_+ & \nabla_- \\
\nabla_+ & -\nabla_- \\
\end{vmatrix} \begin{vmatrix}
V_{j_m}^{\gamma_2} \\
V_{-j_m}^{\gamma_2} \\
\end{vmatrix}
\]

The reciprocal relationship is:

\[
\frac{E}{\hbar c} \overrightarrow{V}(\gamma_2) = - \left[ \sum (\gamma_2) \cdot \overrightarrow{\nabla} \right] \overrightarrow{U}(\gamma_2)
\] (8.1)

Under the condition of rotational invariance, the vector field can be represented by \(\psi=U\) or \(V\):

\[
\psi_{\Omega}^{j_m}(\vec{r}) = \sum_{j=1}^{\gamma_2} h_j(\vec{r}) \left< j^{j} \cdot \mathcal{M} \cdot \gamma_2 \cdot \mathcal{M} \cdot j_m \right>_{\gamma_2}(\mathcal{M})
\] (8.2)

where \(h, f\) and \(g\) are the radial part of \(\psi, U\) and \(V\) respectively.

The solution of the field equations require the use of Spherical Harmonics found in Appendix B, and the use of tabulated Clebsch-Gordon coefficients found in Appendix G. The solution of the field equation requires sixteen angular momentum operators, twenty-four expansions of angular functions into Tesseral harmonics, and the reduction of four Tesseral harmonic terms to two.
The solutions of the reciprocal field equation can be expressed simply by interchanging signs and arguments ($f_{\pm \ell}^{\pm \ell}$).

The solutions are:

\[ \nabla_{+} V_{j/2}^{im} + \nabla_{-} V_{-j/2}^{im} \]

\[ = L(1) A(j + j/2, j/2) + L(2) A(-j - j/2, j/2) \]

\[ = \frac{E}{\hbar c} U_{j/2}^{im} \]

\[ = \frac{E}{\hbar c} \left\{ \int_{j + j/2}^{r} A(j + j/2, j/2) + \int_{-j - j/2}^{r} A(-j - j/2, j/2) \right\} \]

\[ \nabla_{+} V_{j/2}^{im} - \nabla_{-} V_{-j/2}^{im} \]

\[ = L(1) A(j + j/2, -j/2) + L(2) A(-j - j/2, -j/2) \]

\[ = \frac{E}{\hbar c} U_{-j/2}^{im} \]

\[ = \frac{E}{\hbar c} \left\{ \int_{j + j/2}^{r} A(j + j/2, -j/2) + \int_{-j - j/2}^{r} A(-j - j/2, -j/2) \right\} \]

The linear functions $L(k)$ and $A(j', \mu)$ are defined:

\[ L(1) = -\left[ \frac{3}{3r} - \frac{(j - j/2)}{r} \right] g_{j/2}(r) \]

\[ L(2) = -\left[ \frac{3}{3r} + \frac{(j + 3/2)}{r} \right] g_{j/2}(r) \]

\[ A(j', \mu) = \langle j', m' \mu \mu' j/2 | j m \rangle Y_{j'}^{m' \mu}(\Omega) \]

(8.4)
The solutions satisfy the linear relations:

\[ \frac{E}{k_c} \int f_j(r) = L(1) \]
\[ \frac{E}{k_c} \int f_{j+\frac{1}{2}}(r) = L(2) \]

Reciprocally:

\[ \frac{E}{k_c} q_j(r) = -L(f \rightarrow g) \quad (8.5) \]

The solutions of the radial functions are obtained by solving the differential equations; namely:

\[ \left[ \frac{1}{r} \frac{d^2}{dr^2} - \left( \frac{j}{r} \right) \left( \frac{j+\frac{1}{2}}{r} \right) + \left( \frac{E}{k_c} \right)^2 \right] \frac{f_j(r)}{q_j(r)} = 0 \]

and,

\[ \left[ \frac{1}{r} \frac{d^2}{dr^2} - \left( \frac{j+\frac{1}{2}}{r} \right) \left( \frac{j+\frac{3}{2}}{r} \right) + \left( \frac{E}{k_c} \right)^2 \right] \frac{f_{j+\frac{1}{2}}(r)}{q_{j+\frac{1}{2}}(r)} = 0 \quad (8.6) \]

The solutions of these equations satisfies the Spherical Bessel functions found in Appendix C. The solutions for the asymptotic region are:

\[ f_j(r) \sim \frac{A}{r} e^{\pm \frac{i}{2} \left[ k r - \left( j + \frac{3}{2} \right) \frac{\pi}{2} \right]} \]

\[ q_j(r) \sim \frac{A'}{r} e^{\pm \frac{i}{2} \left[ k r - \left( j + \frac{1}{2} \right) \frac{\pi}{2} \right]} \quad \text{where} \quad k = \frac{E}{k_c} \quad (8.7) \]

We obtain the following relations when we substitute the solution for the asymptotic region (8.7) into equations 8.5.

\[ \frac{E}{k_c} q_j(r) = + \left[ k' - \left( j + \frac{1}{2} \right) \right] f_j(r) \]
The solutions have the following asymptotic form \( (r \to \infty) \):

\[
\frac{1}{r} f_{J+\frac{1}{2} Y_2}(r) \to 0, \quad \frac{1}{r} g_{J-\frac{1}{2} Y_2}(r) \to 0
\]  

(8.9)

We obtain the solutions for the asymptotic region

\[
g_{J+\frac{1}{2} Y_2} \sim \frac{i}{2} f_{J-\frac{1}{2} Y_2}
\]

\[
f_{J+\frac{1}{2} Y_2} \sim -i g_{J-\frac{1}{2} Y_2}
\]  

(8.10)

We shall demonstrate the energy flow in spherical coordinates for spin \( I=\frac{1}{2} \). The Energy flow equation is expressed by:

\[
\Pi_\lambda(J, m') = \frac{i}{2} \frac{\mathbf{C}_\lambda \mathbf{C}_{\lambda}^\dagger}{\mathbf{U}} \left[ \tilde{U}^\dagger \mathbf{C}_\lambda (y_2) \tilde{V} - \tilde{V}^\dagger \mathbf{C}_{\lambda} (y_2) \tilde{U} \right]
\]

where \( \lambda = \ell, m, n \)  

(8.11)

and has the dimensions (energy/area x time).

The \( \lambda \) component (radial direction) of the matrix is expressed by equation 5.17:

\[
\mathbf{C}_\ell (y_2) = \begin{pmatrix}
\cos \theta & -i \phi \\
\sin \theta e^{i \phi} & \cos \theta
\end{pmatrix}
\]  

(8.12)

The Energy flow of the \( \lambda \) component is then equal to:
The expansion of the angular functions into Tesseral harmonics (Appendix B) and substitution of the vector field equation 8.2 into equation 8.13 results in an expression:

\[
\mathcal{P}_{j}(j,m) = \frac{\lambda c}{8\pi} \left\{ \begin{array}{c}
+ \cos \theta U_{j}\, V_{j}^{m} + \sin \theta e^{-i\phi} U_{j}\, V_{j}^{m} \\
- \cos \theta U_{-j}\, V_{-j}^{m} - \sin \theta e^{i\phi} U_{-j}\, V_{-j}^{m} \\
+ \sin \theta e^{-i\phi} U_{j}\, V_{-j}^{m} - \cos \theta U_{j}\, V_{-j}^{m} \\
- \sin \theta e^{i\phi} U_{-j}\, V_{j}^{m} + \cos \theta U_{-j}\, V_{j}^{m}
\end{array} \right\} (8.13)
\]

\[
\mathcal{P}_{j}(j,m) = \frac{\lambda c}{8\pi} \left\{ \begin{array}{c}
\left( g_{j+\frac{1}{2}}^{\ell} \right) \langle j+\frac{1}{2}, m-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} | j \rangle Y_{j}^{m-\frac{1}{2}}(\Omega) \\
+ g_{j-\frac{1}{2}}^{\ell} \langle j-\frac{1}{2}, m-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} | j \rangle Y_{j-\frac{1}{2}}^{m-\frac{1}{2}}(\Omega) \\
x f_{j+\frac{1}{2}}^{\ell} \langle j+\frac{1}{2}, m-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} | j \rangle Y_{j+\frac{1}{2}}^{m-\frac{1}{2}}(\Omega) \\
+ f_{j-\frac{1}{2}}^{\ell} \langle j-\frac{1}{2}, m-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} | j \rangle Y_{j-\frac{1}{2}}^{m-\frac{1}{2}}(\Omega) \\
\end{array} \right\} - \left\{ g \rightarrow f \right\}
\]
The use of the tabulated Clebsch-Gordon coefficients (Appendix G) result in:

\[
\prod_f (J, m) = \frac{i}{\sqrt{\pi}} \left\{ \left[ g^* (r) f^* (r) - f^* (r) g^* (r) \right] \left( \frac{J_{m+1}}{2J+2} \right) Y^*_{m+1/2} \hat{Y}_{m-1/2} \right. \\
- \left[ g^* (r) f^* (r) - f^* (r) g^* (r) \right] \left( \frac{(J+M)(J+m)}{2J(2J+2)} \right) Y_{m+1/2} \hat{Y}_{m-1/2} \\
- \left[ g^* (r) f^* (r) - f^* (r) g^* (r) \right] \left( \frac{J_{m+1}}{2J+2} \right) Y_{m+1/2} \hat{Y}_{m-1/2} \\
+ \left[ g^* (r) f^* (r) - f^* (r) g^* (r) \right] \left( \frac{J_{m+1}}{2J+2} \right) Y_{m+1/2} \hat{Y}_{m-1/2} \\
+ \left[ g^* (r) f^* (r) - f^* (r) g^* (r) \right] \left( \frac{J_{m+1}}{2J+2} \right) Y_{m+1/2} \hat{Y}_{m-1/2} \\
\left. \right\} 
\] (8.14)
The substitution of the radial solutions for the asymptotic region (equations 8.10) into the equation 8.15 result in an expression for Energy flow in the radial direction for spin \( I=1/2 \).

\[
\Pi_{r}(J, m) = \frac{C}{8\pi r^2} \left\{ \left| f_{0}(r) \right|^2 + \left| g_{0}(r) \right|^2 \right\} \\
	imes \left\{ \left( \frac{1+m}{2J+2} \right) \left| Y_{J+m+1}^{(m+1)}(\hat{r}) \right|^2 + \left( \frac{1-m}{2J+2} \right) \left| Y_{J-m}^{(m)}(\hat{r}) \right|^2 \\
+ \left( \frac{1+m}{2J} \right) \left| Y_{J-m}^{(m+1)}(\hat{r}) \right|^2 + \left( \frac{1-m}{2J} \right) \left| Y_{J+m}^{(m)}(\hat{r}) \right|^2 \right\} 
\]  
(8.16)
The Energy flow of the m component is then equal to:

\[
\Pi_m(j,m) = \frac{i c}{2 \pi} \left\{ \begin{array}{c}
- e^{i \phi} (\sin \theta e^{i \phi} V_{y_2}^{j m} - \cos \theta V_{-y_2}^{j m} ) U_{y_2}^{j m} \\
+ e^{i \phi} (\sin \theta e^{i \phi} U_{y_2}^{j m} - \cos \theta U_{-y_2}^{j m} ) V_{y_2}^{j m} \\
+ e^{i \phi} (\cos \theta V_{y_2}^{j m} + \sin \theta e^{i \phi} V_{-y_2}^{j m} ) U_{y_2}^{j m} \\
- e^{i \phi} (\cos \theta U_{y_2}^{j m} + \sin \theta e^{i \phi} U_{-y_2}^{j m} ) V_{y_2}^{j m} \end{array} \right\} (8.20)
\]

The expansion of the angular functions into Tesseral harmonics and substitution of the vector field equation 8.2 into equation 8.20 results in:

\[
\Pi_m(j,m) = \frac{i c}{2 \pi} \left\{ \begin{array}{c}
- \frac{e^{i \phi}}{2} \left\{ g_{j+\frac{1}{2},m+\frac{1}{2}}^{j m} \right\} Y_{j+m+\frac{1}{2}}^{m+\frac{1}{2}} \\
+ g_{j-\frac{1}{2},m+\frac{1}{2}}^{j m} \right\} Y_{j-m+\frac{1}{2}}^{m+\frac{1}{2}} \\
\end{array} \right\} (8.20)
\]

\[
\times \left\{ \begin{array}{c}
\frac{e^{i \phi}}{2} \left\{ f_{j-\frac{1}{2},m-\frac{1}{2}}^{j m} \right\} Y_{j+m-\frac{1}{2}}^{m-\frac{1}{2}} \\
+ f_{j+\frac{1}{2},m-\frac{1}{2}}^{j m} \right\} Y_{j+m+\frac{1}{2}}^{m+\frac{1}{2}} \\
\end{array} \right\}
\]

\[
- \left\{ f_{\rightarrow 0} \right\}
\]
The substitution of the radial solutions for the asymptotic region (equation 8.10) into equation 8.21 result in the following result:

\[ \Pi_n (j, m) = 0 \quad (8.22) \]

The n component (azimuthal) of the matrix is expressed by equation 5.17:

\[ C_n (\gamma) = \begin{pmatrix} 0 & -i e^{-i \phi} \\ i e^{i \phi} & 0 \end{pmatrix} \quad (8.23) \]

The Energy flow of the n component is then equal to:

\[ \Pi_n (j, m) = \begin{Bmatrix} i e^{i \phi} U_{j-1/2}^{*m+1/2} V_{j+1/2}^m - i e^{i \phi} U_{j+1/2}^{*m-1/2} V_{j-1/2}^m \\ -i e^{i \phi} U_{j+1/2}^{*m+1/2} V_{j-1/2}^m + i e^{i \phi} U_{j-1/2}^{*m-1/2} V_{j+1/2}^m \end{Bmatrix} \quad (8.24) \]
Substitution of the vector field equation 8.2 into equation 8.24 results in an expression:

\[
\Pi_n(j, m) = \frac{i c}{\eta^4} \left\{ i e^{i \Phi} \left( g_{j+1/2}^{(r)} \langle j+1/2 m-1/2 y_2 1/2 | j m \rangle \right)^{m-1/2} \\
+ g_{j+1/2}^{(r)} \langle j+1/2 m+1/2 y_2 1/2 | j m \rangle \right\}^{m+1/2} \\
\times \left\{ f_{j+1/2}^{(r)} \langle j+1/2 m+1/2 y_2 1/2 | j m \rangle \right\}^{m+1/2} \\
+ g_{j+1/2}^{(r)} \langle j+1/2 m+1/2 y_2 1/2 | j m \rangle \right\}^{m+1/2} \\
+ g_{j+1/2}^{(r)} \langle j+1/2 m+1/2 y_2 1/2 | j m \rangle \right\}^{m+1/2} \\
- \left\{ f_{j+1/2}^{(r)} \langle j+1/2 m+1/2 y_2 1/2 | j m \rangle \right\}^{m+1/2} \\
- \left\{ i e^{i \Phi} \left( g_{j-1/2}^{(r)} \langle j-1/2 m+1/2 y_2 1/2 | j m \rangle \right)^{m+1/2} \\
+ g_{j-1/2}^{(r)} \langle j-1/2 m+1/2 y_2 1/2 | j m \rangle \right\}^{m+1/2} \\
\times \left\{ f_{j-1/2}^{(r)} \langle j-1/2 m+1/2 y_2 1/2 | j m \rangle \right\}^{m+1/2} \\
+ g_{j-1/2}^{(r)} \langle j-1/2 m+1/2 y_2 1/2 | j m \rangle \right\}^{m+1/2} \\
- \left\{ f_{j-1/2}^{(r)} \langle j-1/2 m+1/2 y_2 1/2 | j m \rangle \right\}^{m+1/2}
\right\}
\]

(8.25)
The substitution of the radial solutions for the asymptotic region (equation 8.10) into equations 8.25 result in the following result:

\[ T_{n}(j, m) = 0 \]  
\[ (8.26) \]

We can conclude that the Energy flow is only along the radial direction and is equivalent to the following equation

\[ \Pi(j, m) = \frac{c}{8\pi} \sum_{\kappa} \left\{ |U_{\kappa}^{m}|^2 + |V_{\kappa}^{m}|^2 \right\} \]  
\[ (8.27) \]

The Energy flow \( \Pi \) has the dimensions (energy/ area x time).

The Energy density (energy/volume) is denoted by:

\[ \epsilon(j, m) = \frac{1}{8\pi} \sum_{\kappa} \left\{ |U_{\kappa}^{m}|^2 + |V_{\kappa}^{m}|^2 \right\} \]  
\[ (8.28) \]
IX. RADIATION FIELD (j m) FOR SPIN I=1

We shall demonstrate the use of the Sigma Spin operator for spin I=1. The field vector equations are defined as:

\[ \frac{E}{\hbar c} \mathbf{U}(l) = \left[ \sum (l) \cdot \nabla \right] \mathbf{V}(l) \]  \hspace{1cm} (9.1a)

The Sigma Spin operator \( \Sigma(l) \cdot \nabla \) is defined by equation 4.4.

The reciprocal relation is:

\[ \frac{E}{\hbar c} \mathbf{V}(l) = - \left[ \sum (l) \cdot \nabla \right] \mathbf{U}(l) \]  \hspace{1cm} (9.1b)

Under the condition of rotational invariance, the vector field can be represented by \( \psi=U \) or \( V \):

\[ \psi_{j^m} = \sum_{j_{m-1}} h_{j^m}(r) \langle j^m | j_{m-1} \rangle Y_{j^m}(\Omega) \]

\[ \psi_{s^m} = h_{s}(r) Y_{j^m}(\Omega) \]  \hspace{1cm} (9.2)

where \( h, f \) and \( g \) are the radial part of \( \psi, U \) and \( V \) respectively.

The solutions of the field vector equations require the use of Spherical Harmonics found in Appendix B, and the use of tabulated Clebsch-Gordon coefficients found in Appendix G. The solutions of the field vector equation require sixty angular momentum operators, ninety expansions of angular functions into Tesseral harmonics, and the reduction of five Tesseral harmonics terms to three.

The solutions of the reciprocal field vector equation can be expressed simply by interchanging signs and arguments (\( f^*g \)).
The solutions are:

\[
\nabla_z V_i^j + \sqrt{\frac{1}{2}} \nabla_- V_0^j - \sqrt{\frac{1}{2}} \nabla_- V_s^j \\

= L(1) A(J+1,1) + \left[ L(2) + L(3) \right] A(J,1) + L(4) A(J-1,1)
\]

\[
E_{\frac{1}{R_C}} U_i^j
\]

\[
= \frac{E}{R_C} \left\{ f_{J+1}(r) A(J+1,1) + f_J(r) A(J,1) + f_{J-1}(r) A(J-1,1) \right\} 
\]

\[
\frac{\sqrt{1}}{\sqrt{2}} \nabla_+ V_i^j + \frac{\sqrt{1}}{\sqrt{2}} \nabla_- V_0^j + \nabla_z V_s^j
\]

\[
= L(1) A(J+1,0) + \left[ L(2) + L(3) \right] A(J,0) + L(4) A(J-1,0)
\]

\[
E_{\frac{1}{R_C}} U_0^j
\]

\[
= \frac{E}{R_C} \left\{ f_{J+1}(r) A(J+1,0) + f_J(r) A(J,0) + f_{J-1}(r) A(J-1,0) \right\} 
\]

\[
\frac{1}{\sqrt{2}} \nabla_+ V_0^j - \nabla_\bar{z} V_1^j + \frac{1}{\sqrt{2}} \nabla_+ V_s^j
\]

\[
= L(1) A(J+1,-1) + \left[ L(2) + L(3) \right] A(J,-1) + L(4) A(J-1,-1)
\]

\[
E_{\frac{1}{R_C}} U_{-1}^j
\]

\[
= \frac{E}{R_C} \left\{ f_{J+1}(r) A(J+1,-1) + f_J(r) A(J,-1) + f_{J-1}(r) A(J-1,-1) \right\} 
\]

\[
\frac{1}{\sqrt{2}} \nabla_+ V_1^j + \nabla_\bar{z} V_0^j + \frac{1}{\sqrt{2}} \nabla_- V_1^j
\]

\[
= \left[ (J+1) L(2) - J L(3) \right] A(J,0)
\]

\[
= 0
\]
The linear functions \( L(n) \) and \( A(j', \mu) \) are:

\[
L(1) = -\left[ \frac{2}{r} - \frac{j}{r^2} \right] \left[ \sqrt{\frac{j}{2j+1}} g_j(r) + \sqrt{\frac{j+1}{2j+1}} g_{j+1}(r) \right]
\]

\[
L(2) = -\sqrt{\frac{j}{2j+1}} \left[ \frac{2}{r} + \frac{j+2}{r} \right] g_{j+1}(r)
\]

\[
L(3) = -\sqrt{\frac{j+1}{2j+1}} \left[ \frac{2}{r} - \frac{j-1}{r^2} \right] g_j(r)
\]

\[
L(4) = -\left[ \frac{2}{r} + \frac{j+1}{r} \right] \left[ \sqrt{\frac{j+1}{2j+1}} g_j(r) - \sqrt{\frac{j}{2j+1}} g_{j+1}(r) \right]
\]

\[
A(j', \mu) = \langle j', m - \mu | 1 \mu | j \mu \rangle Y_{j'}(\Omega 2)
\]  

(9.4)

The solutions satisfy the linear relations and utilizing the solution of equation 9.3d result in:

\[
\frac{E}{\hbar c} f_j(r) = L(1)
\]

\[
\frac{E}{\hbar c} f_j(r) = L(2) + L(3) = \left( \frac{j+1}{j} \right) L(2) = \left( \frac{j+1}{j+1} \right) L(3)
\]

\[
\frac{E}{\hbar c} f_{j+1}(r) = L(4)
\]

Reciprocally:

\[
\frac{E}{\hbar c} g_{j'}(r) = -L(\chi \rightarrow \nu)
\]  

(9.5)

The solutions of the radial functions are obtained by solving the differential equations; namely:

\[
\left( \frac{E}{\hbar c} \right)^2 f_j(r) = -\left[ \frac{1}{r} \frac{2}{r^2} - \frac{j(j+1)}{r^2} \right] f_j(r) + \sqrt{\frac{j+1}{j}} f_{j+1}(r)
\]

\[
\left( \frac{E}{\hbar c} \right)^2 f_{j+1}(r) = -\left[ \frac{1}{r} \frac{2}{r^2} - \frac{j(j+1)}{r^2} \right] f_{j+1}(r) + \sqrt{\frac{j}{j+1}} f_j(r)
\]
When the equations are separated, we obtain:

\[
\left( \frac{E}{hc} \right)^2 f_j(r) = -\left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{J(j+1)}{r^2} \right] f_j(r) - \sqrt{\frac{J}{J+j+1}} f_j^S(r) \tag{9.6}
\]

The solution of this equation satisfies the Spherical Bessel function found in Appendix C. The solution for the asymptotic region is:

\[
f_j(r) \sim \frac{C}{r} e^{i \left[ kr - \frac{J\pi}{2} \right]}
\tag{9.8a}
\]

secondly;

\[
0 = -\left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{J(j+1)}{r^2} \right] f_j^S(r)
\tag{9.9}
\]

The solution of this equation satisfies the Laplace equation found in Appendix D. The static vacuum field equation is:

\[
\frac{\delta^2}{\partial r^2} f_j^S(r) - \left( \frac{J(j+1)}{r^2} \right) f_j^S(r) = A_j r^j + B_j \frac{1}{r^{j+1}} \tag{9.10a}
\]

The solutions of the reciprocal equations are similarly obtained for the asymptotic region:

\[
g_j(r) \sim \frac{C'}{r} e^{i \left[ kr - \frac{J\pi}{2} \right]}
\tag{9.8b}
\]

and

\[
g_j^S(r) = A'_j r^j + B'_j \frac{1}{r^{j+1}} \tag{9.10b}
\]
We obtain the following relations when we substitute the static vacuum solutions 9.10 into equations 9.5:

\[
\frac{E}{\hbar c} f_J^J(r) = -\sqrt{\frac{j}{2j+1}} \left[ \frac{2}{2j+1} \frac{\alpha}{r} - \frac{j}{r} \right] g_J(r) \\
+ \sqrt{(j+1)(2j+1)} B_J \frac{1}{r^{j+2}}
\]

\[
\frac{E}{\hbar c} f_J^{J-1}(r) = -\sqrt{\frac{j+1}{2j+1}} \left[ \frac{\alpha}{2j+1} + \frac{j+1}{r} \right] g_J(r) \\
+ \sqrt{j(2j+1)} A_J' r^{-j-1}
\]

Reciprocally:

\[
\frac{E}{\hbar c} g_J(r) = \frac{1}{(f \to g \text{ and deleting the primes})} (9.11)
\]

The static vacuum fields approaches the following solutions in the asymptotic region:

\[
f_J^s(r) = B_J \frac{1}{r^{j+1}} \longrightarrow 0 \quad A_J = 0
\]

\[
g_J^s(r) = B_J' \frac{1}{r^{j+1}} \longrightarrow 0 \quad A_J' = 0 \quad (9.12)
\]

The solutions have the following asymptotic form \((r \to \infty)\):

\[
\frac{1}{r} f_J^s(r) \longrightarrow 0 \quad \frac{1}{r} g_J^s(r) \longrightarrow 0 \quad (9.13)
\]

We obtain the solutions for the asymptotic region by utilizing the solutions 9.8 and equations 9.11; consequently:
We shall demonstrate the Energy flow for spin $s=1$ in the asymptotic region. The Energy flow is expressed as:

$$\mathcal{G}(J,m) = \frac{C_s}{8\pi} \sum_{\mu} \left\{ |U_{\mu}^{jm}|^2 + |V_{\mu}^{jm}|^2 \right\}$$  \hspace{1cm} (9.15)

The vector field 9.2 can now be expressed utilizing the solutions 9.14. The vector fields are:

$$U_{\mu}^{jm}(r) \sim \left\{ \begin{array}{c}
-\sqrt{\frac{j+1}{2j+1}} g_j(r) \langle j+1 \ m-\mu \ 1 \mu | j m \rangle Y_{j+1}^{m-\mu} \\
+ f_j(r) \langle j \ m-\mu \ 1 \mu | j m \rangle Y_{j}^{m-\mu} \\
-\sqrt{\frac{j+1}{2j+1}} g_j(r) \langle j-1 \ m-\mu \ 1 \mu | j m \rangle Y_{j-1}^{m-\mu}
\end{array} \right\}$$

$$f_j(r) \sim \frac{C}{r} e^{i[\sqrt{r}-J\frac{r}{2}]}$$

$$g_{j+1}(r) \sim i\sqrt{\frac{j}{2j+1}} f_j(r)$$

$$g_{j-1}(r) \sim i\sqrt{\frac{j+1}{2j+1}} f_j(r)$$

$$g_j(r) \sim \frac{C}{r} e^{i[\sqrt{r}-J\frac{r}{2}]}$$

$$f_{j+1}(r) \sim -i\sqrt{\frac{j}{2j+1}} g_j(r)$$

$$f_{j-1}(r) \sim -i\sqrt{\frac{j+1}{2j+1}} g_j(r)$$
The substitution of the vector field 9.16 into the equation 9.15 result in an expression for Energy flow for spin $I=1$.

\[
\Pi^\mu (j,m) = \frac{c}{8\pi^2} \left[ |f_j(r)|^2 + |g_j(r)|^2 \right]
\]
\[
\times \sum_{\mu} \left\{ \left( \frac{j+1}{2j+1} \right) < j+1 \, m-\mu \, 1 \, M |JM \rangle \left| \chi^m_{j+1} \right|^2 
\right.
\]
\[
+ \left( j \, m-\mu \, 1 \, M |JM \rangle \left| \chi^m_j \right|^2
\right.
\]
\[
+ \frac{\left( j+1 \right)}{2j+1} \left[ < j+1 \, m-\mu \, 1 \, M |JM \rangle < j-1 \, m-\mu \, 1 \, M |JM \rangle
\right.
\]
\[
\left[ \chi^m_{j+1} \chi^m_j + \chi^m_{j-1} \chi^m_{j+1} \right]
\right\}
\]

The normalization requirement is:

\[
\sum_{\mu} \Pi^\mu (j,m) = \frac{c}{4\pi^2} r^2
\]
The total Energy flow is then:

\[ \int \sum_{\mu} \Pi(j, \mu) r^2 d\Omega = c \]  

(9.19)
We shall demonstrate the use of the Sigma Spin operator for spin $I=3/2$. The field vector equations are defined as:

$$\vec{E}_{nC}(3/2) = -\left[\sum (\frac{3}{2}) \cdot \vec{V}\right] \vec{U}(3/2)$$  \hspace{1cm} (10.1a)

The Sigma Spin operator $\Sigma(3/2) \cdot \vec{V}$ is expressed by equation 6.12. We have selected the case for $\xi=+1$.

The reciprocal relation is:

$$\vec{E}_{nC}(3/2) = -\left[\sum (\frac{3}{2}) \cdot \vec{V}\right] \vec{U}(3/2)$$  \hspace{1cm} (10.1b)

Under the condition of rotational invariance, the vector field $\psi_{J\mu}^m(r)$, $\psi_{J\mu}^m(r)$, and $\psi_{J\mu}^{m\mu}(r)$ are expressed by equations 6.12. We have selected the case for $\xi=+1$.

$$\psi_{J\mu}^m(r) = \sum_{J=m+\frac{3}{2}}^{J+\frac{3}{2}} h_{J\mu}(r) \langle J^- m^- 3/2 \mu^- | J^m \rangle Y_{J\mu}^m(\Omega)$$

$$\psi_{J\mu}^m(r) = \sum_{J=m+\frac{3}{2}}^{J+\frac{3}{2}} h_{J\mu}(r) \langle J^- m^- 3/2 \mu^- | J^m \rangle Y_{J\mu}^m(\Omega)$$

$$\psi_{J\mu}^{m\mu}(r) = \sum_{J=m+\frac{3}{2}}^{J+\frac{3}{2}} h_{J\mu}(r) \langle J^- m^- \frac{3}{2} \mu^- | J^m \rangle Y_{J\mu}^{m\mu}(\Omega)$$  \hspace{1cm} (10.2)

where $h$, $f$, and $g$ are the radial part of $\psi$, $U$, and $V$ respectively.

The solution of the field vector equations requires the use of Spherical Harmonics found in Appendix B, and the use of tabulated Clebsch-Gordon coefficients found in Appendix G. The solution of the field vector equation requires one-hundred-sixty-eight angular momentum operations, two-hundred-fifty-two expansions of angular functions into Tesseral harmonics, and the reduction of six Tesseral harmonic terms to four.
The solutions of the reciprocal field vector equation can be expressed simply by interchanging signs and arguments ($f_{+g}$).

The solutions are:

\[
\nabla_{z} V_{3/2}^{jm} + \frac{1}{3} \nabla_{x} V_{3/2}^{jm} - \frac{1}{3} \nabla_{y} V_{3/2}^{jm} \\
= L(1) A(J+3/2,3/2) + \left[ L(2) + L(3) \right] A(J+1/2,3/2) \\
+ \left[ L(4) + L(5) \right] A(J-1/2,3/2) + L(6) A(J-3/2,3/2) \\
= \frac{E}{\hbar c} \mathcal{U}_{3/2}^{jm} \\
= \frac{E}{\hbar c} \left\{ \int \frac{\partial \varphi}{\partial z} A(J+3/2,3/2) + \int \frac{\partial \varphi}{\partial y} A(J+1/2,3/2) + \int \frac{\partial \varphi}{\partial y} A(J-1/2,3/2) \\
+ \int \frac{\partial \varphi}{\partial z} A(J-3/2,3/2) \right\} \\
(10.3a)
\]

\[
\frac{1}{3} \nabla_{x} V_{3/2}^{jm} + \frac{1}{3} \nabla_{y} V_{3/2}^{jm} - \frac{1}{3} \nabla_{z} V_{3/2}^{jm} \\
= L(1) A(J+3/2,1/2) + \left[ L(2) + L(3) \right] A(J+1/2,1/2) \\
+ \left[ L(4) + L(5) \right] A(J-1/2,1/2) + L(6) A(J-3/2,1/2) \\
= \frac{E}{\hbar c} \mathcal{U}_{1/2}^{jm} \\
= \frac{E}{\hbar c} \left\{ \int \frac{\partial \varphi}{\partial z} A(J+3/2,1/2) + \int \frac{\partial \varphi}{\partial y} A(J+1/2,1/2) + \int \frac{\partial \varphi}{\partial y} A(J-1/2,1/2) \\
+ \int \frac{\partial \varphi}{\partial z} A(J-3/2,1/2) \right\} \\
(10.3b)
\]
\[
\begin{align*}
  &= \frac{E}{k_c} \mathcal{U} - y_2 \\
  &\equiv \frac{E}{k_c} \left\{ \int_{j+3/2} A(j+3/2, j, -j) + \int_{j+1/2} A(j+1/2, j, -j) \\
  &\quad + \int_{j-1/2} A(j-1/2, j, -j) \right\} \\
  \ &\quad + \sqrt{\frac{5}{3}} \nabla_+ V_{j/2}^{\text{m}} - \sqrt{\frac{8}{9}} \nabla_2 V_{j/2}^{\text{m}} \\
  \ &\quad + \sqrt{\frac{8}{9}} \nabla_2 V_{j/2}^{\text{m}} - \frac{1}{3} \nabla_2 V_{j}^{\text{m}} - \frac{1}{3} \nabla_{-2} V_{j}^{\text{m}} \\
  \ &= \sqrt{\frac{j-m+1}{2j+2}} \left[ \sqrt{\frac{j}{2j+3}} L(2) - \sqrt{\frac{2j+3}{j}} L(3) \right] A(j+1/2, j, -j, y_2) \\
  \ &\quad + \sqrt{\frac{j+m}{2j}} \left[ \sqrt{\frac{j+1}{2j-1}} L(4) - \sqrt{\frac{2j-1}{j+1}} L(5) \right] A(j-1/2, j, -j, y_2) \\
  \ &= 0 \\
\end{align*}
\]
- \sqrt{\frac{2}{3}} \nabla_+ V_{\gamma_2}^\mu + \sqrt{\frac{2}{3}} \nabla_\gamma V_{\gamma_2}^\mu + \sqrt{\frac{2}{3}} \nabla_\gamma V_{\gamma_3}^\mu - \frac{1}{3} \nabla_+ V_{\gamma_2}^{\mu_3} + \frac{1}{3} \nabla_\gamma V_{-\gamma_2}^{\mu_3} = - \sqrt{\frac{j+m+1}{2j+2}} \left[ \sqrt{j+3} \, L(2) - \sqrt{2j+3} \, L(3) \right] A(J+\gamma_2, -\gamma_2) + \sqrt{\frac{j-m}{2j}} \left[ \sqrt{\frac{j+1}{2j-1}} \, L(4) - \sqrt{\frac{2j-1}{J+1}} \, L(5) \right] A(J-\gamma_2, \gamma_2) = 0 \quad (10.3f)

The linear functions \( L(n) \) and \( A(j, \mu) \) are defined:

\[
L(1) = -\sqrt{\frac{1}{3(j+1)}} \left[ \frac{2}{3} - \frac{1}{j+\gamma_2} \right] \left[ \sqrt{j} \, g_j^{(r)}(r) + \sqrt{2j+3} \, g_{j+\gamma_2}^{(r)}(r) \right]
\]

\[
L(2) = -\frac{1}{3} \sqrt{\frac{2j+3}{J}} \left[ \frac{2}{3} - \frac{1}{j-\gamma_2} \right] \left[ \sqrt{2j-1} \, g_j^{(r)}(r) + g_{j+\gamma_2}^{(r)}(r) \right]
\]

\[
L(3) = -\sqrt{\frac{j}{3(j+1)}} \left[ \frac{2}{3} + \frac{j+5/2}{r} \right] g_{j+\gamma_2}^{(r)}
\]

\[
L(4) = -\frac{1}{3} \sqrt{\frac{2j-1}{j+1}} \left[ \frac{2}{3} + \frac{j+3/2}{r} \right] \left[ \sqrt{j+3} \, g_j^{(r)}(r) - g_{j+\gamma_2}^{(r)}(r) \right]
\]

\[
L(5) = -\sqrt{\frac{j+1}{3j}} \left[ \frac{2}{3} + \frac{j-3/2}{r} \right] g_j^{(r)}
\]

\[
L(6) = -\sqrt{\frac{1}{3j}} \left[ \frac{2}{3} + \frac{j+3/2}{r} \right] \left[ \sqrt{j+1} \, g_j^{(r)}(r) - \sqrt{2j-1} \, g_{j-\gamma_2}^{(r)}(r) \right]
\]

\[
A(j', \mu) = \left< j' m-\mu | \frac{3}{2} \mu | j m > Y_j^{\mu-\mu}(\Omega)
\]
The solutions satisfy the linear relations and utilizing
the solutions of equations 10.3e and 10.3f result in:

\[ \frac{E}{\hbar \mathcal{C}} f_{1/2} (r) = L(1) \]

\[ \frac{E}{\hbar \mathcal{C}} f_{3/2} (r) = L(2) + L(3) = 3 \left( \frac{j+1}{2j+3} \right) L(2) = 3 \left( \frac{j+1}{j} \right) L(3) \]

\[ \frac{E}{\hbar \mathcal{C}} f_{j-1/2} (r) = L(4) + L(5) = 3 \left( \frac{j}{2j-1} \right) L(4) = 3 \left( \frac{j}{j+1} \right) L(5) \]

\[ \frac{E}{\hbar \mathcal{C}} f_{j-3/2} (r) = L(6) \]

Reciprocally:

\[ \frac{E}{\hbar \mathcal{C}} g_j (r) = -L(f \rightarrow g) \] \hspace{1cm} (10.5)

The solutions of the radial functions are obtained by
solving the differential equations; namely:

\[ \left( \frac{E}{\hbar \mathcal{C}} \right)^2 f_j (r) = -\sqrt{\frac{2(1+j)}{j}} \left[ \frac{\partial}{\partial r} + \frac{j+5/2}{r} \right] \frac{E}{\hbar \mathcal{C}} g_j (r) \]

\[ = -\left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{j+1/2}{r^2} \right] f_j (r) + \sqrt{\frac{2j+3}{j}} \frac{E}{\hbar \mathcal{C}} g_j (r) \] \hspace{1cm} (10.6)

When the equations are separated, we obtain:

\[ \left( \frac{E}{\hbar \mathcal{C}} \right)^2 f_j (r) = -\left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{j+1/2}{r^2} \right] f_j (r) \] \hspace{1cm} (10.7)

The solution of this equation satisfies the Spherical
Bessel function in Appendix C. The solution for the asymptotic
region is:

\[ f_{j+1/2} (r) \sim \frac{C}{r} e^{i \left[ K r - (j+1/2) \pi \right]} \] \hspace{1cm} (10.8a)

where \( K = \frac{E}{\hbar \mathcal{C}} \)
secondly;

\[ 0 = - \left[ \frac{1}{r} \frac{\partial^2 r}{\partial r^2} - \frac{(J+\nu_2)(J+3/2)}{r^2} \right] f_{J+\nu_2}^S(r) \]  
(10.9)

The solution of this equation satisfies the Laplace equation found in Appendix D. The static vacuum field equation is:

\[ f_{J+\nu_2}^S(r) = A_{J+\nu_2} r^J \nu_2 + B_{J+\nu_2} \frac{1}{r^J+\nu_2} \]  
(10.10a)

The solutions of the reciprocal equations are similarly obtained for the asymptotic region:

\[ g_{J+\nu_2}^S(r) \sim \frac{C^J}{r^J} e^{i\left[ k r - (J+\nu_2) \frac{\pi}{2} \right]} \]  
(10.8b)

and

\[ g_{J+\nu_2}^S(r) = A_{J+\nu_2} r^J \nu_2 + B_{J+\nu_2} \frac{1}{r^J+\nu_2} \]  
(10.10b)

Corresponding:

\[
\left( \frac{E}{\kappa c} \right)^2 f_{J-\nu/2}^S(r) = -\sqrt{\frac{3J}{J+1}} \left[ \frac{2}{3} \frac{r^3}{r^3} - \frac{(J-\nu/2)}{r^2} \right] \frac{E}{\kappa c} g_{J-\nu/2}^S(r) \\
= -\left[ \frac{1}{r} \frac{\partial^2 r}{\partial r^2} - \frac{(J-\nu/2)(J+\nu/2)}{r^2} \right] \left[ f_{J-\nu/2}^S(r) - \sqrt{\frac{2J-1}{3J}} f_{J-\nu/2}^S(r) \right] 
\]  
(10.11)

When the equations are separated, we obtain:

\[
\left( \frac{E}{\kappa c} \right)^2 f_{J-\nu/2}^S(r) = -\left[ \frac{1}{r} \frac{\partial^2 r}{\partial r^2} - \frac{(J-\nu/2)(J+\nu/2)}{r^2} \right] f_{J-\nu/2}^S(r) 
\]  
(10.12)
The solution of this equation satisfies the Spherical Bessel function. The solution for the asymptotic region is:

\[ f_{j-\nu_2}(r) \sim \frac{E}{r} e^{i \left[ k r - (j-\nu_2) \frac{r^2}{2} \right]} \quad \text{where} \quad k = \frac{E}{hc} \quad (10.13a) \]

secondly,

\[ 0 = + \left[ \frac{i}{r} \frac{2}{r^2} - \frac{(j-\nu_2)(j+\nu_2)}{r^2} \right] f_{j-\nu_2}^5(r) \quad (10.14) \]

The solution of this equation satisfies the Laplace equation. The static vacuum fields is:

\[ f_{j-\nu_2}^5(r) = D_{j-\nu_2} r^{j-\nu_2} + E_{j-\nu_2} \frac{1}{r^{j+\nu_2}} \quad (10.15a) \]

The solutions of the reciprocal equations are similarly obtained for the asymptotic region:

\[ g_{j-\nu_2}(r) \sim \frac{E'}{r} e^{i \left[ k r - (j-\nu_2) \frac{r^2}{2} \right]} \quad (10.13b) \]

\[ g_{j-\nu_2}^5(r) = D_{j-\nu_2} r^{j-\nu_2} + E_{j-\nu_2} \frac{1}{r^{j+\nu_2}} \quad (10.15b) \]

We obtain the following relations when we substitute the static vacuum solutions 10.15 into equations 10.5.

\[ \frac{E}{hc} f_{j-\nu_2}(r) = -\sqrt{\frac{j}{3(j+1)}} \left[ \frac{2}{3} \frac{2}{r^2} - \frac{1+j+\nu_2}{r} \right] g_{j+\nu_2}(r) \]

\[ + 2 \sqrt{\frac{(j+1)(2j+3)}{3}} \left[ B_{j+\nu_2} \frac{1}{r^{j+5/2}} \right] \]

\[ \frac{E}{hc} f_{j+\nu_2}(r) = -\sqrt{\frac{(j+1)(2j-1)}{j(2j+3)}} \left[ \frac{2}{3} \frac{2}{r^2} - \frac{j-\nu_2}{r} \right] g_{j-\nu_2}(r) \]

\[ + 2 (j+1) \sqrt{\frac{j}{2j+3}} E_{j-\nu_2} \frac{1}{r^{j+3/2}} \]
\[
\frac{E}{\kappa c} f_j(r) = -\sqrt{\frac{j(j+1)}{3j}} \left[ \frac{2}{\partial r} + \frac{j+3/2}{r} \right] g_{j+1/2}(r) + 2J\sqrt{\frac{j+1}{2j-1}} A_{j+1/2} r^{j-1/2}
\]

\[
\frac{E}{\kappa c} g_{j'}(r) = -\sqrt{\frac{j+1}{3j}} \left[ \frac{2}{\partial r} + \frac{j+1/2}{r} \right] g_{j-1/2}(r) + 2\sqrt{\frac{j(2j-1)}{3}} D_{j-1/2} r^{j-3/2}
\]

Reciprocally:

\[
\frac{E}{\kappa c} g_{j'}(r) = -L(f \rightarrow g \text{ and deleting the primes}) \tag{10.16}
\]

The static vacuum field approaches the following solutions in the asymptotic region:

\[
f_{j+1/2}^{\infty}(r) = B_{j+1/2} \frac{1}{r^{j+1/2}} \rightarrow 0 \quad A_{j+1} = 0
\]

\[
g_{j+1/2}^{\infty}(r) = B_{j+1/2}' \frac{1}{r^{j+3/2}} \rightarrow 0 \quad A_{j+1}' = 0
\]

\[
f_{j-1/2}^{\infty}(r) = E_{j-1/2} \frac{1}{r^{j+1/2}} \rightarrow 0 \quad D_{j-1/2} = 0
\]

\[
g_{j-1/2}^{\infty}(r) = E_{j-1/2}' \frac{1}{r^{j+1/2}} \rightarrow 0 \quad D_{j-1/2}' = 0 \tag{10.17}
\]

The vector fields have the following asymptotic form \((r \rightarrow \infty)\):

\[
\frac{1}{r} f_j(r) \rightarrow 0, \quad \frac{1}{r} g_j(r) \rightarrow 0 \tag{10.18}
\]
We obtain the solutions for the asymptotic region by utilizing the solutions 10.8 and 10.13 and equations 10.16; consequently:

\[ f_{j+\frac{3}{2}} (r) \sim \frac{C}{r} e^{+i \left[ Kr - \left( j + \frac{3}{2} \right) \frac{H}{r} \right]} \]

\[ g_{j+\frac{3}{2}} (r) \sim +i \sqrt{\frac{J}{3(J+1)}} f_{j+\frac{3}{2}} (r) \]

\[ g_{j-\frac{3}{2}} (r) \sim +i \sqrt{\frac{J(2J+3)}{J(2J+3) - (2J-1)}} f_{j+\frac{3}{2}} (r) \]

\[ f_{j-\frac{3}{2}} (r) \sim -i \sqrt{\frac{J}{3(J+1)}} g_{j+\frac{3}{2}} (r) \]

\[ f_{j-\frac{3}{2}} (r) \sim -i \sqrt{\frac{J(2J+3)}{J(2J+3) - (2J-1)}} g_{j+\frac{3}{2}} (r) \]

\[ g_{j-\frac{3}{2}} (r) \sim -i \sqrt{\frac{J+1}{2J}} f_{j+\frac{3}{2}} (r) \]

\[ g_{j+\frac{3}{2}} (r) \sim -i \sqrt{\frac{J(2J+3)}{J(2J+3) - (2J-1)}} g_{j+\frac{3}{2}} (r) \]

\[ f_{j+\frac{3}{2}} (r) \sim -i \sqrt{\frac{J+1}{2J}} g_{j+\frac{3}{2}} (r) \]
\[ f_{j-\frac{3}{2}}(r) \sim -i\sqrt{\frac{j+1}{3j}} \, g_{j-\frac{3}{2}}(r) \]

Equivalency:

\[ f_{j+\frac{3}{2}}(r) \sim \sqrt{\frac{2j-1}{3(2j+3)}} \, f_{j-\frac{3}{2}}(r) \]

\[ g_{j+\frac{3}{2}}(r) \sim \sqrt{\frac{2j-1}{3(2j+3)}} \, g_{j-\frac{3}{2}}(r) \]

\[ f_{j+\frac{3}{2}}(r) \sim \sqrt{\frac{2j+3}{3(2j-1)}} \, f_{j-\frac{3}{2}}(r) \]

\[ g_{j+\frac{3}{2}}(r) \sim \sqrt{\frac{2j+3}{3(2j-1)}} \, g_{j-\frac{3}{2}}(r) \]  \hspace{1cm} (10.19)

We shall demonstrate the Energy flow for spin \( I=\frac{3}{2} \) in the asymptotic region. The Energy flow is expressed as:

\[ \mathcal{F}_{\Omega} (j, m) = \frac{c}{\hbar r} \sum_{\mu} \left\{ |U^m_{\mu}(r)|^2 + |V^m_{\mu}(r)|^2 \right\} \]  \hspace{1cm} (10.20)

The vector field 10.2 can now be expressed utilizing the solutions 10.19.

The vector fields are:

\[ U^m_{\mu}(r) \sim \begin{cases} \sqrt{\frac{2j-1}{3(2j+3)}} \, f_{j-\frac{3}{2}}(r) \begin{pmatrix} j+\frac{3}{2} \cr m-\mu \cr \frac{3}{2} \end{pmatrix} Y^m_{j+\frac{3}{2}}(\Omega) \\ -i\sqrt{\frac{(j+1)(2j-1)}{j(2j+3)}} \, g_{j-\frac{3}{2}}(r) \begin{pmatrix} j+\frac{3}{2} \cr m-\mu \cr \frac{3}{2} \end{pmatrix} Y^m_{j+\frac{3}{2}}(\Omega) \\ + f_{j-\frac{3}{2}}(r) \begin{pmatrix} j-\frac{3}{2} \cr m-\mu \cr \frac{3}{2} \end{pmatrix} Y^m_{j-\frac{3}{2}}(\Omega) \\ -i\sqrt{\frac{j+1}{3j}} \, g_{j-\frac{3}{2}}(r) \begin{pmatrix} j-\frac{3}{2} \cr m-\mu \cr \frac{3}{2} \end{pmatrix} Y^m_{j-\frac{3}{2}}(\Omega) \end{cases} \]
Substitution of the vector field 10.21 into the equation
10.20 result in an expression for Energy flow for spin I=3/2.

\[ \Pi'(J, M) = \frac{C}{8\pi} \left[ |f_{j-I/2}(r)|^2 + |g_{j-I/2}(r)|^2 \right] \]

\[
= \sum_{\mu} \left( \frac{2J-1}{3(2J+3)} \right) \left[ \begin{array}{l}
\langle j+3/2 m-\mu \frac{3}{2} \mu | jm \rangle Y_{j+3/2}^{m-\mu} \\
+ \left( \frac{(J+1)(2J-1)}{J(2J+3)} \right) \langle j+1 \mu m-\mu \frac{3}{2} \mu | jm \rangle Y_{j+1/2}^{m-\mu} \\
+ \langle j-\frac{1}{2} m-\mu \frac{3}{2} \mu | jm \rangle Y_{j-1/2}^{m-\mu} \\
+ \left( \frac{J+1}{3J} \right) \langle j-3/2 m-\mu \frac{3}{2} \mu | jm \rangle Y_{j-3/2}^{m-\mu} \\
+ \left( \frac{2J-1}{3(2J+3)} \right) \langle j+3/2 m-\mu \frac{3}{2} \mu | jm \rangle < j+3/2 m-\mu \frac{3}{2} \mu | jm \rangle \\
x \left[ Y_{j+3/2}^{m-\mu} Y_{j-3/2}^{m-\mu} + Y_{j+1/2}^{m-\mu} Y_{j-1/2}^{m-\mu} \right]
\right)
\]
\( + \sqrt{\frac{2j+1}{3(2j+3)}} \left< j+\frac{3}{2}, m-\mu \left| j, \mu \right> \left< j-\frac{3}{2}, m-\mu \left| j, \mu \right> \right. \)
\[
\times \left\{ Y_{j+3/2}^{m-\mu}(\theta, \phi) Y_{j-1/2}^{m-\mu}(\theta, \phi) + Y_{j+3/2}^{m-\mu}(\theta, \phi) Y_{j-1/2}^{m-\mu}(\theta, \phi) \right\} \]

The normalization requirement is:

\[
\sum_{\mu} \Pi_j(j, \mu) = \frac{c}{4\pi r^2} \]  
(10.23)

Consequently:

\[
\frac{1}{(2j+3)(2j-1)} \left[ \left| f_{j+3/2}(r) \right|^2 + \left| g_{j+3/2}(r) \right|^2 \right] = \left[ \left| f_{j+1/2}(r) \right|^2 + \left| g_{j+1/2}(r) \right|^2 \right] \]
(10.24)

The total energy flow is then:

\[
\int \sum_{\mu} \Pi_j(j, \mu) r^2 d\Omega = C \]  
(10.25)
XI. RADIATION FIELD (j m) FOR SPIN I=2

We shall demonstrate the use of the Sigma Spin operator for spin I=2. The field vector equations are defined as:

\[ \frac{E}{\hbar c} \mathbf{U}(2) = \pm \left( \sum (2) \cdot \mathbf{V} \right) \mathbf{V}(2) \]  

(11.1a)

The Sigma Spin operator \( \mathbf{E}(2) \cdot \mathbf{V} \) is expressed by equation 7.12. The reciprocal relation is:

\[ \frac{E}{\hbar c} \mathbf{V}(2) = -\left( \sum (2) \cdot \mathbf{U} \right) \mathbf{U}(2) \]  

(11.1b)

Under the condition of rotational invariance, the vector field can be represented by \( \psi=U \) or \( V \):

\[
\psi_{J^M}^{J^m}(r) = \sum_{j=j-2}^{j+2} h_j(r) < j', m-\mu, 2 \mu | j m > Y_{j'}^{m-\mu} \\
\psi_{J^S}^{J^m}(r) = \sum_{j'=j-1}^{j+1} h_j(r) < j', m-\mu, 1 \mu | j m > Y_{j'}^{m-\mu} 
\]

(11.2)

where \( h, f \) and \( g \) are the radial part (11.2) of \( \psi, U \) and \( V \) respectively.

The solution of the field vector equations requires the use of Spherical Harmonics found in Appendix B, and the use of tabulated Clebsch-Gordon coefficients found in Appendix G. The solution of the field vector equation requires three-hundred angular momentum operations, four-hundred-fifty expansions of angular functions into Tesseral harmonics, and the reduction of seven Tesseral harmonic terms to five.
The solutions of the reciprocal state field vector equation can be expressed simply by interchanging signs and arguments ($E^{\perp}$).

The solutions are:

\[
\begin{align*}
\nabla \nabla \nu_2 \nu_2^{jm} + \frac{1}{2} \nabla \nabla \nu_1^{jm} - \sqrt{\frac{2}{3}} \nabla \nu_{15}^{jm} \\
= L(1) A(J+2,2) + \left[ L(2) + L(3) \right] A(J+1,2) + \left[ L(4) + L(5) \right] A(J+2,2) \\
+ \left[ L(6) + L(7) \right] A(J-1,2) + L(8) A(J-2,2) \\
= \frac{E}{\xi} U_2^{jm} \\
= \frac{E}{\xi} \left\{ \int_{J+1}^{J+2} A(J+2,2) + \int_{J+1}^{J+2} A(J+1,2) + \int_{J+1}^{J+2} A(J+2,2) \\
+ \int_{J-1}^{J+2} A(J-1,2) + \int_{J-1}^{J-2} A(J-2,2) \right\} \\
(11.3a)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} \nabla \nu_2 \nu_2^{jm} + \frac{1}{2} \nabla \nu_1^{jm} + \sqrt{\frac{3}{8}} \nabla \nu_0^{jm} + \sqrt{\frac{3}{8}} \nabla \nu_{15}^{jm} - \sqrt{\frac{3}{8}} \nabla \nu_{15}^{jm} \\
= L(1) A(J+2,1) + \left[ L(2) + L(3) \right] A(J+1,1) + \left[ L(4) + L(5) \right] A(J+1,1) \\
+ \left[ L(6) + L(7) \right] A(J-1,1) + L(8) A(J-2,1) \\
= \frac{E}{\xi} U_1^{jm} \\
= \frac{E}{\xi} \left\{ \int_{J+1}^{J+2} A(J+2,1) + \int_{J+1}^{J+1} A(J+1,1) + \int_{J+1}^{J+2} A(J+2,1) \\
+ \int_{J-1}^{J+1} A(J-1,1) + \int_{J-2}^{J+1} A(J-2,1) \right\} \\
(11.3b)
\end{align*}
\]

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\[
\begin{align*}
\mathcal{E}_o & = \sum_{j} \mathcal{U}_j^m \\
\mathcal{E}_o & = \sum_{j} \left\{ f_{j+2}(r) A(j+2,0) + f_{j+1}(r) A(j+1,0) + f_{j}(r) A(j,0) \right. \\
& \left. + f_{j-1}(r) A(j-1,0) + f_{j-2}(r) A(j-2,0) \right\} \quad (11.3c) \\
\Delta_+ \mathcal{V}_0^m & - \frac{1}{2} \Delta_2 \mathcal{V}_0^m + \frac{1}{2} \Delta_+ \mathcal{V}_2^m + \frac{1}{2} \Delta_+ \mathcal{V}_0^m + \frac{1}{2} \Delta_2 \mathcal{V}_0^m \\
& = L(1) A(j+2,-1) + \left[ L(2) + L(3) \right] A(j+1,-1) + \left[ L(4) + L(5) \right] A(j,-1) \\
& \left. + \left[ L(6) + L(7) \right] A(j-1,-1) + L(8) A(j-2,-1) \right\} \quad (11.3d) \\
\frac{1}{2} \Delta_+ \mathcal{V}_0^m & \Delta_2 \mathcal{V}_2^m + \frac{1}{2} \Delta_+ \mathcal{V}_0^m + \frac{1}{2} \Delta_2 \mathcal{V}_0^m \\
& = L(1) A(j+2,-2) + \left[ L(2) + L(3) \right] A(j+1,-2) + \left[ L(4) + L(5) \right] A(j,-2) \\
& \left. + \left[ L(6) + L(7) \right] A(j-1,-2) + L(8) A(j-2,-2) \right\} \quad (11.3e) \\
\mathcal{E}_o & = \sum_{j} \mathcal{U}_j^m \\
\mathcal{E}_o & = \sum_{j} \left\{ f_{j+2}(r) A(j+2,-2) + f_{j+1}(r) A(j+1,-2) + f_{j}(r) A(j,-2) \right. \\
& \left. + f_{j-1}(r) A(j-1,-2) + f_{j-2}(r) A(j-2,-2) \right\} \quad (11.3e)
\end{align*}
\]
\[
- \sqrt{\frac{3}{4}} \nabla_+ V_2 + \sqrt{\frac{3}{4}} \nabla_2 V_1 + \frac{1}{6} \nabla_1 V_0 - \frac{1}{2} \nabla_2 V_{15} - \frac{1}{6} \nabla_1 V_{05} \\
= \left[ \frac{1}{13} \left( \frac{j + 1}{j + 2} \right) L(2) - \sqrt{3} \left( \frac{j + 2}{j + 2} \right) L(3) \right] A(J+1, 1) \\
\quad + \left[ -\frac{1}{\sqrt{3}} \left( \frac{2j - 1}{1 - 2m^2} \right) L(4) + \frac{1}{\sqrt{3}} \left( \frac{2j + 3}{1 - 2m^2} \right) L(5) \right] A(J, 1) \\
\quad + \left[ \frac{1}{\sqrt{3}} \left( \frac{j + 1}{j - 2m + 1} \right) L(6) - \sqrt{3} \left( \frac{j - 1}{j - 2m + 1} \right) L(7) \right] A(J-1, 1) \\
= 0 \quad (11.3f)
\]

\[
- \sqrt{\frac{3}{8}} \nabla_+ V'_1 + \nabla_2 V'_0 + \sqrt{\frac{3}{8}} \nabla_1 V'_1 - \frac{1}{6} \nabla_1 V'_{15} - \frac{1}{6} \nabla_1 V'_{05} \\
= \left[ \frac{1}{3} \left( \frac{1}{m} \right) L(2) - \left( \frac{j + 1}{m} \right) L(3) \right] A(J+1, 0) \\
\quad + \left[ \frac{m(2j - 1)}{3m^2 - j(j+1)} L(4) - \frac{m(2j + 3)}{3m^2 - j(j+1)} L(5) \right] A(J, 0) \\
\quad + \left[ -\frac{1}{3} \left( \frac{j + 1}{m} \right) L(6) + \left( \frac{j - 1}{m} \right) L(7) \right] A(J-1, 0) \\
= 0 \quad (11.3g)
\]
The linear functions $L(n)$ and $A(j', \mu)$ are defined:

\begin{align*}
L(1) &= -\frac{1}{2}(1)\frac{1}{\sqrt{2j+3}} [\frac{2}{3} - \frac{j+1}{r}] \left[ \sqrt{2j} g_{j+1}(r) + \sqrt{3j+1} g_{j+1}^s(r) \right] \\
L(2) &= \frac{1}{2}(2)\frac{1}{\sqrt{2j+3}} [\frac{2}{3} - \frac{j+1}{r}] \left[ \sqrt{2j+1} g_{j+1}(r) + \sqrt{3j+1} g_{j+1}^s(r) \right] \\
L(3) &= -\frac{1}{2}(3)\frac{1}{\sqrt{2j+3}} [\frac{2}{3} + \frac{j+3}{r}] g_{j+2}(r) \\
L(4) &= -\frac{1}{2}(4)\frac{1}{\sqrt{2j+3}} [\frac{2}{3} + \frac{j+1}{r}] \left[ \sqrt{3j+1} g_{j+1}(r) + \sqrt{j+1} g_{j+1}^s(r) \right] \\
L(5) &= -\frac{1}{2}(5)\frac{1}{\sqrt{2j+3}} [\frac{2}{3} + \frac{j+2}{r}] \left[ \sqrt{3j+2} g_{j+1}(r) - \sqrt{j} g_{j+1}^s(r) \right] \\
L(6) &= -\frac{1}{2}(6)\frac{1}{\sqrt{2j+3}} [\frac{2}{3} + \frac{j+1}{r}] \left[ \sqrt{2j+3} g_{j+1}(r) - g_{j+1}^s(r) \right] \\
L(7) &= -\frac{1}{2}(7)\frac{1}{\sqrt{2j+3}} [\frac{2}{3} - \frac{j+2}{r}] g_{j+2}(r)
\end{align*}

(11.3h)
The solution satisfy the linear system relations, and utilizing the solutions of equations 11.3f, 11.3g and 11.3h result in:

\[ L(8) = -\frac{1}{2} \sqrt{\frac{1}{2J_{-1}}} \left[ \frac{2}{2J_{+2}} \frac{1}{r} \right] \left[ \sqrt{2J_{+2}} g_{j+1} \left( \ell \right) - \sqrt{2J_{-2}} g_{j-1} \left( \ell \right) \right] \]

\[ A(j', m') = \langle j', m' \mid 2 \mid j \mid m \rangle \sum_{J} Y_{j'}(r) \]

(11.4)

Reciprocally:

\[ \mathcal{E} \frac{f}{\mathcal{C}} f_{j+2}(r) = L(1) \]

\[ \mathcal{E} \frac{f}{\mathcal{C}} f_{j+1}(r) = L(2) + L(3) = \frac{2}{3} \left( \frac{2j+3}{2j+1} \right) L(2) = 4 \left( \frac{2j+3}{2j} \right) L(3) \]

\[ \mathcal{E} \frac{f}{\mathcal{C}} f_{j}(r) = L(4) + L(5) = 2 \left( \frac{2j+1}{2j+3} \right) L(4) = 2 \left( \frac{2j+1}{2j+1} \right) L(5) \]

\[ \mathcal{E} \frac{f}{\mathcal{C}} f_{j-1}(r) = L(6) + L(7) = \frac{2}{3} \left( \frac{2j-1}{2j-1} \right) L(6) = 2 \left( \frac{2j-1}{2j+1} \right) L(7) \]

\[ \mathcal{E} \frac{f}{\mathcal{C}} f_{j-2}(r) = L(8) \]

The solutions of the radial functions are obtained by solving the differential equations; namely:

\[ \left( \mathcal{E} \frac{f}{\mathcal{C}} \right)^{2} f_{j+2}(r) = 4 \left( \frac{2j+3}{2j} \right) \mathcal{E} \frac{f}{\mathcal{C}} L(3) \]

\[ = -2 \sqrt{\frac{2j+3}{2j}} \left[ \frac{2}{2j} + \frac{j+3}{r} \right] \mathcal{E} \frac{f}{\mathcal{C}} g_{j+2}(r) \]
When the equations are separated, we obtain:

$$\left( \frac{\mathbf{E}}{\hbar c} \right) f_{J+1}^{\pm}(r) = - \left[ \frac{1}{r} \frac{\partial^2 r}{\partial r^2} - \frac{(J+1)(J+2)}{r^2} \right] f_{J+1}^{\pm}(r)$$  \hspace{1cm} (11.7)

The solution of this equation satisfies the Spherical Bessel function found in Appendix C. The solution for the asymptotic region is:

$$f_{J+1}^{\pm}(r) \sim \frac{C}{r} e^{\pm i \left[ k r - (J+1) \frac{\pi}{2} \right]}$$  \hspace{1cm} (11.8a)

secondly:

$$\mathcal{O} = - \left[ \frac{1}{r} \frac{\partial^2 r}{\partial r^2} - \frac{(J+1)(J+2)}{r^2} \right] f_{J+1}^{S}(r)$$  \hspace{1cm} (11.9)

The solution of this equation satisfies the Laplace equation found in Appendix D. The static vacuum field equation is:

$$f_{J+1}^{S}(r) = A_{J+1}^{\prime} r^{J+1} + B_{J+1}^{\prime} \frac{1}{r^{J+2}}$$  \hspace{1cm} (11.10a)

The solutions of the reciprocal relationships are similarly obtained for the asymptotic region:

$$g_{J+1}^{\prime}(r) \sim \frac{C^{\prime}}{r} e^{\pm i \left[ k r - (J+1) \frac{\pi}{2} \right]}$$  \hspace{1cm} (11.8b)

and:

$$g_{J+1}^{S}(r) = A_{J+1}^{\prime} r^{J+1} + B_{J+1}^{\prime} \frac{1}{r^{J+2}}$$  \hspace{1cm} (11.10b)
Corresponding:

\[
\left( \frac{E}{\hbar c} \right)^2 f_{j-1}(r) = 4 \left( \frac{2j-1}{2j+2} \right) \frac{E}{\hbar c} L(\gamma) \\
= -2\sqrt{\frac{2j-1}{2j+2}} \left[ \frac{3}{4} - \frac{j-2}{r^2} \right] \frac{E}{\hbar c} g_j(r) \\
= -\left[ \frac{1}{r^2} \frac{2r}{\partial r^2} - \frac{(j-1)j}{r^2} \right] \left[ f_{j-1}(r) + \sqrt{\frac{3(j-1)}{j+1}} f_{j-1}(r) \right]
\] (11.11)

When the equations are separated, we obtain:

\[
\left( \frac{E}{\hbar c} \right)^2 f_{j-1}(r) = -\left[ \frac{1}{r^2} \frac{2r}{\partial r^2} - \frac{(j-1)j}{r^2} \right] f_{j-1}(r)
\] (11.12)

The solution of this equation satisfies the Spherical Bessel function. The solution for the asymptotic region is:

\[
\tilde{f}_{j-1}(r) \sim \frac{E}{r} e^{i[Kr - (j-1)\frac{\gamma}{2}]}
\] where \( K = \frac{E}{\hbar c} \) (11.13a)

secondly;

\[
\tilde{g}_{j-1}(r) \sim \frac{E}{r} e^{i[Kr - (j-1)\frac{\gamma}{2}]}
\] (11.13b)

The solution of this equation satisfies the Laplace equation. The static vacuum field equation is:

\[
\tilde{f}_{j-1}(r) = D_{j-1} r^{-j-1} + E_{j-1} \frac{1}{r^j}
\] (11.15a)

The solutions of the reciprocal relationships are similarly obtained for the asymptotic region:
and

\[ g_j(r) = D_j^{-1} r^{j-1} + E_j^{-1} \frac{1}{r^j} \]  

(11.15b)

We obtain the following relationships when we substitute the static vacuum solutions 11.15 into equations 11.5.

\[
\frac{E}{\hbar c} f_j(r) = L(l)
\]

\[
= -\frac{1}{2} \sqrt{\frac{2j}{2j+3}} \left[ \frac{2}{3} r - \frac{j+1}{r} \right] g_{j+l}^{(r)}
\]

\[
+ \frac{1}{2} \sqrt{3(2j+3)(2j+4)} \ B_{j+l} \frac{1}{r^{j+3}}
\]

secondly;

\[
\frac{E}{\hbar c} f_j(r) = 2 \left( \frac{2j+1}{2j-1} \right) L(5)
\]

\[
= -\sqrt{\frac{3(2j+1)(2j+2)}{(2j-1)(2j+3)}} \left[ \frac{2}{3} r + \frac{j+2}{r} \right] g_{j+l}^{(r)}
\]

\[
+ \sqrt{\frac{j(2j+1)(2j+3)}{2j-1}} A_{j+l} \ r^j
\]

thirdly;

\[
\frac{E}{\hbar c} f_j(r) = 2 \left( \frac{2j+1}{2j+3} \right) L(4)
\]

\[
= -\sqrt{\frac{3(2j+1)(2j+1)}{(2j-1)(2j+3)}} \left[ \frac{2}{3} r - \frac{j-1}{r} \right] g_{j-1}^{(r)}
\]

\[
+ \sqrt{\frac{(2j-1)(2j+1)(j+1)}{2j+3}} E_{j-1} \frac{1}{r^{j+1}}
\]
fourthly;

$$
\left( \frac{E}{hc} \right)^2 f_j(r) = -\sqrt{\frac{2j+1}{(2j-1)(2j+3)}} \left[ \frac{2}{3} - \frac{j-1}{r} \right] \left[ \sqrt{3(j-1)} g_{JJ}(r) + \sqrt{j+1} \tilde{g}_{J+1}(r) \right] \frac{E}{hc}
$$

When we substitute:

$$
\frac{E}{hc} g_{JJ}(r) = + (2j-1) \sqrt{\frac{1}{3(j-1)(2j+3)}} \left[ \frac{2}{3} + \frac{j+1}{r} \right] \left[ \sqrt{\frac{2j+3}{2j-1}} f_j(r) - f_J^S(r) \right]
$$

we obtain this expression:

$$
\left( \frac{E}{hc} \right)^2 f_j(r) = -\sqrt{\frac{2j-1}{2j+3}} \left[ \frac{1}{r} \frac{2}{r^2} - \frac{j(j+1)}{r^2} \right] \left[ \sqrt{\frac{2j+3}{2j-1}} f_j(r) - f_J^S(r) \right]
$$

$$
+ \sqrt{\frac{(2j-1)(2j+3)(j+1)}{2j+3}} \left( \frac{E}{hc} \right) E_j \frac{1}{r^{j+1}}
$$

fifthly;

$$
\frac{E}{hc} f_j^S(r) = L(8)
$$

$$
= -\frac{1}{2} \sqrt{\frac{2j+2}{2j-1}} \left[ \frac{2}{2j-1} + \frac{j}{r} \right] g_{JJ}(r)
$$

$$
+ \frac{1}{2} \sqrt{3(2j-2)(2j-1)} D_j \frac{1}{r^{j-2}}
$$

Reciprocally:

$$
\frac{E}{hc} g_{JJ}(r) = -L(f \rightarrow g \text{ and deleting the primes}) \quad (11.16)
$$

The static vacuum field approaches the following solutions in the asymptotic region:

$$
f_{J+1}^S(r) = B_{J+1} \frac{1}{r^{J+2}} \longrightarrow 0 \quad \text{and} \quad A_{J+1} = 0
$$

$$
g_{J+1}^S(r) = B'_{J+1} \frac{1}{r^{J+2}} \longrightarrow 0 \quad \text{and} \quad A'_{J+1} = 0
$$
\[
f_{j-1}^{S}(r) = E_{j-1} \frac{1}{r} \to 0 \quad \text{and} \quad D_{j-1} = 0
\]
\[
g_{j-1}^{S}(r) = E_{j-1}' \frac{1}{r^3} \to 0 \quad \text{and} \quad D_{j-1}' = 0
\]  \hspace{1cm} (11.17)

Consequently
\[
-\left[ \frac{1}{r^2} \frac{\partial}{\partial r} - \frac{J(J+1)}{r^2} \right] f_{j}^{S}(r) = \sqrt{(2j+1)/(j+1)} \left( \frac{E}{\kappa c} \right) E_{j-1} \frac{1}{r^{j+1}}
\]
\[
f_{j}^{S}(r) \to 0
\]

and
\[
-\left[ \frac{1}{r^2} \frac{\partial}{\partial r} - \frac{J(J+1)}{r^2} \right] g_{j}^{S}(r) = \sqrt{(2j+1)/(j+1)} \left( \frac{E}{\kappa c} \right) E_{j-1} \frac{1}{r^{j+1}}
\]
\[
g_{j}^{S}(r) \to 0
\]  \hspace{1cm} (11.18)

The vector field having the following asymptotic for \((r \to \infty)\):
\[
\frac{1}{r} f_{j}^{S}(r) \to 0, \quad \frac{1}{r} g_{j}^{S}(r) \to 0
\]  \hspace{1cm} (11.19)

We obtain the solutions for the asymptotic region by utilizing the solutions 11.8 and 11.13 and equations 11.16; consequently:
\[
f_{j}^{S}(r) \sim \frac{\zeta}{r} e^{+i \left[ \kappa r - (j+1) \frac{\pi}{4} \right]}
\]
\[
g_{j+1}^{S}(r) \sim \frac{i}{2} \sqrt{\frac{2j}{2j+3}} f_{j}^{S}(r)
\]
\[
g_{j+2}^{S}(r) \sim \frac{i}{2} \sqrt{\frac{3(2j+1)(j+2)}{(2j-1)(2j+3)}} f_{j+1}^{S}(r)
\]
\[ f_{j_1}(r) \sim \frac{F}{r} e^{i \left[ Kr - (j_1 - \frac{1}{2}) \frac{\pi}{2} \right]} \]

\[ g_{j_1}(r) \sim +i\sqrt{\frac{3(j_1 - 1)(2j_1 + 1)}{(2j_1 - 1)(2j_1 + 3)}} f_{j_1}(r) \]

\[ g_{j_2}(r) \sim +i\frac{\sqrt{2j_2 + 1}}{2j_1 - 1} f_{j_1}(r) \]

\[ g_{j_3}(r) \sim \frac{F}{r} e^{i \left[ Kr - (j_1 + \frac{1}{2}) \frac{\pi}{2} \right]} \]

\[ f_{j_2}(r) \sim -\frac{i}{2} \sqrt{\frac{2j_1}{2j_1 + 3}} g_{j_1}(r) \]

\[ f_{j_3}(r) \sim -i\sqrt{\frac{3(2j_1 + 1)(j_1 + 2)}{(2j_1 - 1)(2j_1 + 3)}} g_{j_1}(r) \]

\[ g_{j_1}(r) \sim \frac{F}{r} e^{i \left[ Kr - (j_1 + \frac{1}{2}) \frac{\pi}{2} \right]} \]

\[ f_{j_1}(r) \sim -i\sqrt{\frac{3(j_1 - 1)(2j_1 + 1)}{(2j_1 - 1)(2j_1 + 3)}} g_{j_1}(r) \]

\[ f_{j_2}(r) \sim -\frac{i}{2} \sqrt{\frac{2j_1 + 2}{2j_1 - 1}} g_{j_1}(r) \]

**Equivalency:**

\[ f_{j_2}(r) = \frac{1}{2} \sqrt{\frac{(2j_1 - 1)(2j_1 + 2)}{3(2j_1 + 1)(j_1 + 2)}} f_{j_1}(r) \]

\[ g_{j_2}(r) = \frac{1}{2} \sqrt{\frac{(2j_1 - 1)(2j_1 + 2)}{3(2j_1 + 1)(j_1 + 2)}} g_{j_1}(r) \]
We shall demonstrate the Energy flow for spin $I=2$ in the asymptotic region. The Energy flow is expressed as:

\[ \Pi(J_m) = \frac{c}{8\pi} \sum_{J} \left\{ |U^m_J(r)|^2 + |V^m_J(r)|^2 \right\} \]  

(11.21)

The vector field 11.2 can now be expressed utilizing the solutions 11.20.

The vector fields are:

\[
U^m_J(r) = \begin{cases} 
\frac{1}{2} \sqrt{\frac{(2j-1)(2j)}{3(2j+1)(2j+2)}} f_J(r) \langle J+2 m-\mu 2 \mu | Jm \rangle Y^m_{J+2}(\Omega) \\
-i \sqrt{\frac{(2j-1)(2j+3)}{3(2j+1)(2j+2)}} g_J(r) \langle J+1 m-\mu 2 \mu | Jm \rangle Y^m_{J+1}(\Omega) \\
+f_J(r) \langle J m-\mu 2 \mu | Jm \rangle Y^m_J(\Omega) \\
-i \sqrt{\frac{(2j-1)(2j+3)}{3(2j-1)(2j+1)}} g_J(r) \langle J-1 m-\mu 2 \mu | Jm \rangle Y^m_{J-1}(\Omega) \\
+ \frac{1}{2} \sqrt{\frac{(2j+2)(2j+3)}{3(2j-1)(2j+1)}} f_J(r) \langle J-2 m-\mu 2 \mu | Jm \rangle Y^m_{J-2}(\Omega)
\end{cases}
\]
Substitution of the vector field $\mathbf{J}_{12}^{\text{m}}$ into the equation 11.21 result in an expression for Energy flow for spin $I=2$.

$$
\Pi^{(j,m)} = \frac{C}{g^{\text{ff}}_{\text{eff}}} \left[ |f_{j}(r)|^2 + |g_{j}(r)|^2 \right]
$$

$$
\times \sum_{m} \left\{ \left[ \frac{(2j+1)(2j+3)}{3(2j+1)(2j+3)} \right] \left| \langle j \pm 2 \ m \pm 2 \ | \ j m \rangle \right|^2 \left| Y_{j+2}^{m,\pm} \right|^2 \\
+ \left[ \frac{(2j)(2j+2)}{3(2j+1)(2j+3)} \right] \left| \langle j \pm 1 \ m \pm 1 \ | \ j m \rangle \right|^2 \left| Y_{j+1}^{m,\pm} \right|^2 \\
+ \left[ \frac{(2j-1)(2j+3)}{3(2j-1)(2j+3)} \right] \left| \langle j \pm 1 \ m \pm 1 \ | \ j m \rangle \right|^2 \left| Y_{j-1}^{m,\pm} \right|^2 \\
+ \left[ \frac{(2j+1)(2j+3)}{3(2j-2)(2j+3)} \right] \left| \langle j \pm 2 \ m \pm 2 \ | \ j m \rangle \right|^2 \left| Y_{j-2}^{m,\pm} \right|^2 \right\}
$$
The normalization requirement is:

\[ \sum_{\mu} T^\mu (j, \mu) = \frac{c}{4\pi r^2} \]  

(11.24)

The total Energy flow is then:

\[ \int \sum_{\mu} T^\mu (j, \mu) r^2 d\Omega = C \]  

(11.25)
XII. RADIATION PATTERNS FOR SPIN I FIELDS

The radiation from a vacuum source never has the same energy flow in all directions. The energy flow may even be zero in some directions from a reference z-axis; in others it may be greater than one would expect from a vacuum source that did radiate equally well in all directions. But even though no actual vacuum source radiates with equal energy flow in all directions, it is nevertheless convenient to assume such an vacuum source exists. Such a hypothetical vacuum source is called an isotropic radiator.

A graph showing the actual or relative energy flow, at a fixed distance, as a function of the direction from the vacuum source is called a radiation pattern. At the outset it must be realized that such a pattern is a three-dimensional figure rotated about the reference z-axis and therefore cannot be represented in a plane drawing. The solid radiation pattern of an vacuum source would be found by measuring the field strength at every point on the surface of an imaginary sphere having the vacuum source at its center. The information so obtained is then used to construct a solid figure such that the distance from a fixed point (source) to the surface, in any direction, is proportional to the field strength from the vacuum source in that direction. The solid pattern of an isotropic radiator, therefore, would be a sphere, since the field strength is the same in all directions.

The radial energy flow for spin I is represented by:

125
$$T^{\prime}(j,m) = \frac{i c}{\theta \pi} \left\{ \bar{U}(I) C_{j}(I) \bar{V}(I) - \bar{V}(I) C_{j}(I) \bar{U}(I) \right\}$$  \hspace{1cm} (5.21)$$

and is equivalent to the following equation:

$$T^{\prime}(j,m) = \frac{c}{\theta \pi} \sum_{\mu} \left\{ |\bar{U}(I)|^2 + |\bar{V}(I)|^2 \right\}$$  \hspace{1cm} (8.27)$$

Under the condition of rotational invariance, the vector field for spin I represented by \( \psi(I)=U(I) \) or \( V(I) \):$

$$\psi_{\mu}(r) = \sum_{j,\mu} h_{j\mu}(r) <j\eta,m,\mu | j\mu> Y_{j\mu}^m(\Omega)$$  \hspace{1cm} (12.1)$$

where h, f and g are the radial part of \( \psi \), U and V respectively.

The solution for the asymptotic region has the form:

$$h_{j\mu}(r) \sim A(h) \frac{e^{i j \phi - j \phi}}{r}$$

where \( \kappa = \frac{\omega}{\theta c} \)  \hspace{1cm} (12.2)$$

The normalization requirement is:

$$\sum_{\mu} T^{\prime}(j,\mu) = \frac{c}{4 \pi r}$$  \hspace{1cm} (12.3)$$

The total energy flow is then:

$$\int d\phi \int \sin \theta \ d\theta \sum_{\mu} T^{\prime}(j,\mu) = C$$  \hspace{1cm} (12.4)$$

Let us now represent a general source distribution field \( X_{j}^{\phi}(\eta) \) in the radiation zone by:

$$\sum_{\mu} T^{\prime}(j,\mu) = \frac{c}{(2j+1) \pi^2} \sum_{m=-j}^{j} \left| X_{j}^{\phi}(\theta,\phi) \right|^2$$  \hspace{1cm} (12.5)$$

It can be shown that the absolute squares of the vector Tesseval harmonics obey the sum rule; namely:
For a spin $I=0$ field, the Klein-Gordon equation for mass equal to zero was used.

Table 4 lists some of the simpler angular distributions for spin $I$ radiation patterns.

Figures 2-6 represent some of the simpler angular distributions plotted as polar intensity diagrams for spin $I$ radiation patterns.

\[ \sum_{n=-j}^{j} \left| \chi_{j}^{n}(\theta, \phi) \right|^{2} = \frac{2j+1}{4\pi} \]  

(12.6)
Table 4. Simpler Angular Distributions $|X^m_j, (\theta, \phi)|^2$. This table lists some of the simpler angular distributions for spin I radiation patterns.
| $|X_j^m(\theta, \phi)|^2$ |
|---|---|---|---|---|---|
| $j$ | $m$ | $I=0$ | $I=1/2$ | $I=1$ | $I=3/2$ | $I=2$ |
| 0 | 0 | $\frac{1}{4\pi}$ | $\frac{1}{4\pi}$ | $\frac{3}{16\pi}$ | $\frac{1}{8\pi}$ | $\frac{3}{8\pi}$ | $\frac{1}{8\pi}$ |
| 1/2 | $\pm 1/2$ | $\frac{3 \sin^2 \theta}{8\pi}$ | $\frac{3 (1+\cos^2 \theta)}{16\pi}$ | $\frac{3 \sin^2 \theta}{8\pi}$ | $\frac{1 (1+3\cos^2 \theta)}{8\pi}$ | $\frac{3 \sin^2 \theta}{8\pi}$ | $\frac{5 (1+\cos^2 \theta)^4}{64\pi}$ |
| 1 | $\pm 1$ | $\frac{3 \sin^2 \theta}{8\pi}$ | $\frac{3 (1+\cos^2 \theta)}{16\pi}$ | $\frac{3 \sin^2 \theta}{8\pi}$ | $\frac{1 (1+3\cos^2 \theta)}{8\pi}$ | $\frac{3 \sin^2 \theta}{8\pi}$ | $\frac{5 \sin^2 \theta (1+\cos^2 \theta)^2}{16\pi}$ |
| 2 | $\pm 1$ | $\frac{15 \sin^4 \theta}{32\pi}$ | $\frac{5 (1-\cos^4 \theta)}{16\pi}$ | $\frac{5 (1-3\cos^2 \theta + 4\cos^4 \theta)}{16\pi}$ | $\frac{5 \sin^2 \theta (1+\cos^2 \theta)^2}{16\pi}$ | $\frac{5 \sin^2 \theta}{32\pi}$ | $\frac{15 \sin^4 \theta}{32\pi}$ |
| 2 | 0 | $\frac{5 (3\cos^2 \theta - 1)^2}{16\pi}$ | $\frac{15 \sin^2 \theta \cos^2 \theta}{8\pi}$ | $\frac{15 \sin^2 \theta \cos^2 \theta}{8\pi}$ | $\frac{15 \sin^2 \theta}{32\pi}$ | $\frac{15 \sin^4 \theta}{32\pi}$ |
Figure 2. Radiation Pattern I=0

Simpler angular distributions plotted as polar intensity diagrams for spin I=0 radiation patterns.
\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{\textbf{Figure 2}}
\end{figure}

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Figure 3. Radiation Pattern I=1/2

Simpler angular distributions plotted as polar intensity diagrams for spin I=1/2 radiation patterns.
Figure 3

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Figure 4. Radiation Pattern I=1

Simpler angular distributions plotted as polar intensity diagrams for spin I=1 radiation patterns.
Figure 4

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Figure 5. Radiation Pattern I=3/2

Simpler angular distributions plotted as polar intensity

diagrams for spin I=3/2 radiation patterns.
Figure 5
Figure 6. Radiation Pattern I=2

Simpler angular distributions plotted as polar intensity diagrams for spin I=2 radiation patterns.
Figure 6
XIII. DISCUSSION

We have succeeded in finding a Sigma Spin I matrix which describes a rotational property of massless field. This Sigma Spin matrix is invariant under Lorentz transformations and is therefore in a relativistic form. The Sigma Spin I matrix satisfies the commutation relations in exactly the same as the Pauli spin matrices do. When the Sigma Spin matrix was transformed into a conventional form where the z-component was diagonal, the transformed forms were found to be a reasonable extension of the Pauli spin matrix.

The spin I field is accompanied by spin I-1 static vacuum fields except I=0 and 1/2. The motion of static vacuum field is interpreted as a dilatation and a rigid displacement, for the latter, the angular velocity is zero. The static vacuum field does not change in time.

The quanta known so far which have no mass and no charge which are described by the spin I fields are: neutrinos (spinor) and photons (vector).

Recently, an observation of two kinds of graviton (scalar and tensor) was reported.\textsuperscript{12}

When the Sigma Spin I matrix was transformed into a conventional form, a sign $\xi=\pm$ in the transformation matrix was determined. We shall speculate that the single-valuedness of sign represents a symmetric Sigma Spin I matrix and a two-valuedness of sign represents an antisymmetric Sigma Spin I matrix.
Quanta that obey Einstein-Bose statistics \((I=0, 1, 2)\), are described by symmetrical field equations. Quanta that obey Fermi-Dirac statistics \((I=1/2, 3/2)\) or (equivalently) the Pauli exclusion principle, are described by antisymmetrical field equations.

Results of calculations for Poynting's vectors show that for a given \(j, m\), their patterns are strongly dependent on a spin \(I\) of the field. A significant result was encountered for the case \(I=2\) and \(j=2\). These patterns were dependent on \(m\) values, not \(|m|\) values. All other cases depend only on \(|m|\).

Secondly, the patterns for \(j=3/2\), \(I=1/2\) and \(I=3/2\) were identical when the magnetic quantum numbers are interchanged.

Despite extensive developments in the supposition of the Sigma Spin matrix for spin \(I\) fields, quantitative success has been achieved thus far only in the restricted domain where there are no sources. Furthermore, the existence of divergences, whether concealed or explicit, serves to emphasize that the present formalism for spin \(I\) fields should be extended to the cases where sources exist and/or particles have masses.
APPENDIX A

The Time-Independent Schrödinger Equation

In three dimensional space the spherical coordinates of a point are related to the cartesian coordinates of that point are expressed by equation 5.15 Whenever the potential energy \( V(r) \) in the Schrödinger equation is spherical symmetric, the angular dependency of the wave function can be separated out.

\[
-\frac{\hbar^2}{2\mu} \nabla^2 \Psi + V(r) = E \Psi
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{2 \mu \hbar^2}{r^2} [E - V(r)] \Psi = 0
\]  

(A.1)

when \( V(r) \) is constant, we define:

\[
\kappa^2 = \frac{2 \mu \hbar^2}{\hbar^2} (E - V)
\]

(A.2)

for which \( \mu \) is the reduced mass

The wave function:

\[
\Psi(r, \theta, \phi) = R_l(r) Y_{l,m}^m(\theta, \phi)
\]

(A.3)

\( l = 0, 1, 2, ..., m = -l, -l+1, ..., l-1, l \)

Tesseral harmonic \( Y_{l,m}^m(\Omega) \) functions (\( m > 0 \))

\[
Y_{l,m}^m(\theta, \phi) = (-)^m \left[ \frac{(2l+1)(l-m)!}{4\pi (l+m)!} \right]^{1/2} P_{l}^{m}\cos \theta e^{im\phi}
\]

(A.4)

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where \( P^\lambda_m(\cos \theta) \) is an associated Legendre function.

Here \( \lambda \) is the orbital (azimuthal) angular momentum quantum number, and \( m \) is the magnetic quantum number.

The radial operator can be written in an alternate form; namely:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}
\]  

(A.5)
APPENDIX B

Spherical Harmonics

1. Angular momentum operator in terms of \((x, y, z)\)

\[
\mathbf{L} = \frac{\hbar}{i} \bar{\mathbf{r}} \times \nabla = \bar{\mathbf{p}} \times \nabla
\]

\[
L_x = \frac{\hbar}{i} \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) = \gamma P_z - z P_y
\]

\[
L_y = \frac{\hbar}{i} \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) = \gamma P_x - x P_z
\]

\[
L_z = \frac{\hbar}{i} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = \gamma P_y - y P_x
\]

From the transformations in the equation 5.15 we obtain the reverse transformations:

\[
\mathbf{r}^2 = x^2 + y^2 + z^2
\]

\[
\cos \Theta = \frac{z}{r}
\]

\[
\tan \Phi = \frac{y}{x}
\]

from which we calculate:

\[
\frac{\partial}{\partial x} = \frac{2r}{2x} \frac{\partial}{\partial r} + \frac{\Theta}{\partial x} \frac{\partial}{\partial \Theta} + \frac{\Phi}{\partial x} \frac{\partial}{\partial \Phi}
\]

\[
= \sin \Theta \cos \Phi \frac{\partial}{\partial r} + \frac{\cos \Theta \cos \Phi}{r} \frac{\partial}{\partial \Theta} - \frac{\sin \Phi}{r \sin \Theta} \frac{\partial}{\partial \Phi}
\]

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\[
\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi}
\]
\[
= \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi}
\]
\[
\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi}
\]
\[
= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}
\]  
(B.3)

2. Angular momentum operator in terms of \((\theta, \phi)\)

\[
L_x = i \hbar \left( \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right)
\]
\[
L_y = i \hbar \left( -\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right)
\]
\[
L_z = L_0 = -i \hbar \frac{\partial}{\partial \phi}
\]
\[
L^\pm = L_x \pm i L_y = \hbar e^{\pm i \phi} \left[ \frac{\hbar}{\sin \theta} \cdot \frac{\partial}{\partial \theta} \mp i \cot \theta \frac{\partial}{\partial \phi} \right]
\]
\[
L^2 = L_x^2 + L_y^2 + L_z^2
\]
\[
= -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]
\]  
(B.4)
3. The divergence operators are:

\[
\nabla^\pm = \frac{2}{\partial x} \pm i \frac{2}{\partial y} = e^{\pm i \phi} \left[ \sin \theta \frac{2}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \pm i \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right] = \sin \theta e^{\pm i \phi} \left[ \frac{2}{\partial r} \pm \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right] \pm \frac{\cos \theta}{r} \frac{\partial}{\partial \phi} \nabla^\pm
\]

\[
\nabla^\pm = \frac{2}{\partial x} \pm i \frac{2}{\partial y} = e^{\pm i \phi} \left[ \sin \theta \frac{2}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right] = \frac{\cos \theta}{r} \frac{\partial}{\partial \phi} \nabla^\pm
\]

4. Eigenfunction common to the operators:

\[
L^2 Y^m_l(\Omega) = \ell(\ell + 1) \hbar^2 Y^m_l(\Omega)
\]

\[
L_0 Y^m_l(\Omega) = m \hbar Y^m_l(\Omega)
\]

(B.6)

5. Recursion relation

\[
L^\pm Y^m_l(\Omega) = \hbar \left[ (\ell \pm m)(\ell \pm m + 1) \right]^{1/2} Y^m_{\ell \pm 1}(\Omega)
\]

(B.7)

6. Orthonormality and Closure Relations

\[
\int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\phi \ Y^m_l(\theta, \phi) Y^{m'}_{l'}(\theta, \phi) = \delta_{m m'} \delta_{l l'}
\]

\[
\sum_{l=0}^\infty \sum_{m=-l}^l Y^m_l(\theta, \phi) Y^m_{l'}(\theta, \phi) = \frac{\delta(\theta - \theta') \delta(\phi - \phi')}{\sin \theta} = \delta(\Omega - \Omega')
\]

(B.8)

7. Parity (-)^l

\[
Y^m_l(\pi - \theta, \phi + \pi) = (-)^l Y^m_l(\theta, \phi)
\]

(B.9)

8. Complex Conjugation

\[
Y^m_l(\theta, \phi) = (-)^m Y^{-m}_l(\theta, \phi)
\]

(B.10)
9. Composition relation for the Tesseral harmonics

\[
\sum_{L \leq l_1, l_2} \sum_{M \leq m_1, m_2} \left[ \frac{(2L+1)(2l_1+1)}{4\pi (2L+1)} \right]^{1/2} \langle \mathcal{O}_{L_1} \mathcal{O}_{L_2} | \Omega \rangle \langle \mathcal{O}_{m_1} \mathcal{O}_{m_2} | L M \rangle Y_\ell^m (\Omega) \]

(B.11)

where \(|l_1-1_2| \leq L \leq l_1 + l_2\), \(-L \leq M \leq +L\)

10. Expansions of angular functions into Tesseral harmonics

\[
\cos \Theta \begin{pmatrix} \begin{array}{c} Y_{l_1}^m(\theta, \phi) \\ -i \Phi \end{array} \end{pmatrix} = \sqrt{\frac{4\pi}{3}} \begin{pmatrix} \begin{array}{c} Y_{l_1}^m(\theta, \phi) \\ Y_{l+1}^{-m}(\theta, \phi) \end{array} \end{pmatrix}
\]

\[
\sin \Theta e^{i\phi} \begin{pmatrix} \begin{array}{c} Y_{l_1}^m(\theta, \phi) \\ i \Phi \end{array} \end{pmatrix} = \sqrt{\frac{8\pi}{3}} \begin{pmatrix} \begin{array}{c} Y_{l_1}^m(\theta, \phi) \\ Y_{l+1}^{-m}(\theta, \phi) \end{array} \end{pmatrix}
\]
APPENDIX C

Spherical Bessel Functions

In polar coordinates, the Schrödinger equation for the free particle leads, for each value \( \ell \) of the orbital angular momentum, to the radial equation:

\[
\kappa^2 R_{\ell}^\ell(r) = -\left[ \frac{1}{r} \frac{d}{dr} \frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} \right] R_{\ell}^\ell(r)
\]  \hspace{1cm} (C.1)

The radial function can be expressed in terms of the Spherical Bessel functions of order \( \pm(\ell+1/2) \).

1. Regular, sometimes called Spherical Bessel functions of the first kind:

\[
J_{\ell}^\ell(\rho) = \left( \frac{\pi}{2\rho} \right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}(\rho)
\]  \hspace{1cm} (C.2)

2. Neuman functions, sometimes called Spherical Bessel functions of the second kind:

\[
\eta_{\ell}^\ell(\rho) = (-)^{\ell+1} \left( \frac{\pi}{2\rho} \right)^{\frac{1}{2}} J_{-\ell-\frac{1}{2}}(\rho)
\]  \hspace{1cm} (C.3)

3. Hankel functions, sometimes called Spherical Bessel functions of the third kind:

\[
h_{\ell}^{(1)}(\rho) = J_{\ell}^\ell(\rho) + i \eta_{\ell}^\ell(\rho)
\]

\[
h_{\ell}^{(2)}(\rho) = J_{\ell}^\ell(\rho) - i \eta_{\ell}^\ell(\rho)
\]  \hspace{1cm} (C.4)
The radial function for outgoing waves is then defined by:

$$R_{\ell}(r) = A \left( \frac{r}{\ell} \right)^{(i)} \text{ where } \rho = k r \tag{C.5}$$

The constant $A$ is determined by the requirements that $R_{\ell}$ and $dR/dr$ be continuous, and by the normalization requirement $\int_R^2 r^2 dr = 1$.

4. The asymptotic forms: $\rho \to \infty \left[ \rho \gg \ell(\ell+1) \right]$

$$j_{\ell}(\rho) \sim \frac{1}{\rho} \sin \left( \rho - \frac{1}{2} \rho \right)$$

$$n_{\ell}(\rho) \sim \frac{1}{\rho} \cos \left( \rho - \frac{1}{2} \rho \right)$$

$$R_{\ell}(r) \sim \frac{A e^{i \left( \rho - \frac{1}{2} \rho \right)}}{\rho} \left[ 1 + \frac{\ell(\ell+1)}{\rho} + \cdots \right] \tag{C.6}$$

5. The behavior at the origin: $\rho \to 0$

$$j_{\ell} \sim \frac{\rho}{(2\ell+1)!} \left[ 1 - \frac{\rho^2}{2(2\ell+3)} + \cdots \right]$$

$$n_{\ell} \sim \frac{(2\ell+1)!}{(2\ell+1)!} \left( \frac{1}{\rho} \right)^{\ell+1} \left[ 1 + \frac{\rho^2}{2(2\ell+1)} + \cdots \right] \tag{C.7}$$

As $\rho$ increases from 0 to $\infty$, the function $\rho j_{\ell}(\rho)$ increases first as $\rho^{\ell+1}$, then more and more rapidly (exponential behavior) up to the point $\rho = \sqrt{\ell(\ell+1)}$, where it has a point of inflection. The function then oscillates indefinitely between two extreme values which tend asymptotically toward +1 and -1, respectively. The asymptotic form is a good approximation when $\rho \gg \ell(\ell+1)/2$, but the amplitude of oscillation practically attains its asymptotic value (to within 10%) as soon as $\rho \geq 2\ell$. 

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APPENDIX D
Laplace Equation

The radial functions satisfies the Laplace equation in spherical coordinates when \( k=0 \):

\[
\nabla^2 \psi = 0
\]

\[
\psi(r, \theta, \phi) = R_\ell(r) Y^m_\ell(\theta, \phi)
\]

The Laplace equation:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} = 0
\]  

The function \( Y^m_\ell(\theta, \phi) \) satisfy the equation: \( \text{(D.1)} \)

\[
- \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial R_\ell(r)}{\partial r} \right) - \frac{\ell(\ell+1)}{r^2} \right] R_\ell(r) = 0
\]  

(D.2)

The solutions:

\[
R_\ell(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}
\]  

(D.3)

Here \( \ell \) is the orbital angular momentum quantum number.

The constant \( A_\ell \) and \( B_\ell \) are determined by the requirements that \( R_\ell(r) \) and \( dR_\ell(r)/dr \) be continuous, and by the boundary conditions.
Table 5. Table of Tesseral Harmonics $Y^m_l(\theta, \phi)$.
APPENDIX E

Table of Tesseral Harmonics \( Y^m_\ell(\theta, \phi) \)

\[
\begin{align*}
0 &= \sqrt{\frac{1}{4\pi}} Y^0_0, \quad 1 = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad 2 = \sqrt{\frac{5}{16\pi}} (3\cos^2 \theta - 1), \\
3 &= \sqrt{\frac{7}{16\pi}} (5\cos^3 \theta - 3\cos \theta), \quad 4 = \sqrt{\frac{9}{256\pi}} (35\cos^4 \theta - 30\cos^2 \theta + 3), \\
\pm 1 &= \pm i\sqrt{\frac{3}{8\pi}} \sin \theta e, \quad \pm 1 = \pm i\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e, \\
\pm 2 &= \pm 2i\sqrt{\frac{15}{32\pi}} \sin^2 \theta e, \quad \pm 2 = \pm 2i\sqrt{\frac{105}{32\pi}} \sin^2 \theta \cos \theta e, \\
\pm 3 &= \pm 3i\sqrt{\frac{35}{64\pi}} \sin^3 \theta e, \quad \pm 3 = \pm 3i\sqrt{\frac{315}{64\pi}} \sin^3 \theta \cos \theta e, \\
\pm 4 &= \pm 4i\sqrt{\frac{315}{512\pi}} \sin^4 \theta e
\end{align*}
\]

Table 5
APPENDIX F

Clebsch-Gordon (C.-G.) Coefficients

1. Addition Theorem for Two Angular Momenta

The simplest problem is that of adding two angular momenta. Suppose that \( J = j_1 + j_2 \) where \( j_1 \) and \( j_2 \) are the angular momenta of the separate systems 1 and 2 respectively which together form the system under study, and suppose that we have contracted complete sets of common eigenvectors.

The tensor product of the \((2j_1+1)\) vectors of system 1 \(|j_1 m_1> \) \((j_1 \text{ fixed}, m_1 = -j_1, \ldots, +j_1)\) by the \((2j_2+1)\) vectors of system 2 \(|j_2 m_2> \) \((j_2 \text{ fixed}, m_2 = -j_2, \ldots, +j_2)\) gives the \((2j_1+1)(2j_2+1)\) eigenvectors of \( j_1^2, j_2^2, j_1 z, j_2 z \), the vectors \(|j_1 m_1 j_2 m_2> \equiv |j_1 m_1> |j_2 m_2> \) from which we obtain, by a unitary transformation, the \((2j_1+1)(2j_2+1)\) simultaneous eigenvectors \(j_1^2, j_2^2, J^2, J_z\), the vectors where \( J \) is the total angular momentum and \( J_z \) is its \( z \)-component.

\(|j_1 j_2 JM> \ (J = |j_1-j_2|, \ldots, j_1+j_2; \ M = -J, \ldots, +J)\)

The elements of the transformation are called vector addition or Clebsch-Gordon or Wigner \(3J\) coefficients;

\[ <j_1 m_1 j_2 m_2|JM> \]  \( \text{(F.1)} \)

and are the coefficients of that unitary transformation:

\[ |j_1 j_2 JM> = \sum_{m_1} |j_1 m_1 j_2 m_2< <j_1 m_1 j_2 m_2|JM> \]  \( \text{(F.2)} \)

2. Principal Properties

Reality: They are all real:

\[ <j_1 m_1 j_2 m_2|JM> \ast = <j_1 m_1 j_2 m_2|JM> \]
Selection rules:

a. \( m_1 + m_2 = M; \)

b. \(|j_1 + j_2| \leq j \leq j_1 + j_2\) (triangular inequalities)

If these two conditions are not met; \( <j_1 m_1, j_2 m_2 | J M > = 0 \).

Consequences:

\[ \langle j_1 m_1, j_2 m_2 | J M > = (-)^{J_1 - J} \langle j_2 m_2, j_1 m_1 | J M > \]
\[ = (-)^{J_1 - J} \sqrt{\frac{2J_1 + 1}{2J_2 + 1}} \langle j_1 - m_1, j_2 + m_1 | J_1 M_2 | j_1 m_1 > \]
\[ = (-)^{J_2 - J} \sqrt{\frac{2J_2 + 1}{2J_1 + 1}} \langle j_1 + m_1, j_2 - m_1 | J_2 M_1 | j_2 m_2 > \]
\[ = (-)^{J_1 + J_2} \langle j_1 - m_1, j_2 - m_2 | J - M > \quad (F.3) \]

3. Orthogonality relations:

\[ \sum_{m_1 = -j_1}^{j_1} \sum_{m_2 = -j_2}^{j_2} \langle j_1 m_1, j_2 m_2 | J M > < j_1 m_1', j_2 m_2' | J' M' > = \delta_{JJ'} \delta_{M M'} \]

where \(|j_1 - j_2| \leq j \leq j_1 + j_2; -J < M < +J\)

\[ \sum_{J} \sum_{M} \langle j_1 m_1, j_2 m_2 | J M > < j_1 m_1', j_2 m_2' | J M > = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \quad (F.4) \]

where \((-j_1 \leq m_1 \leq j_1; -j_2 \leq m_2 \leq j_2)\)

4. Composition relation for the Tesseral harmonic

\[ \int Y_{l_2}^{m_2} (\theta) Y_{l_1}^{m_1} (\theta) Y_{l_3}^{m_3} (\theta) d\Omega \]
\[ = \left[ \frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l_3 + 1)} \right]^{\frac{1}{2}} < l_1 0 l_2 0 | l_3 0 > < l_1 m_1, l_2 m_2 | l_3 m_3 > \quad (F.5) \]
The general solutions of the Clebsch-Gordon coefficients have been given by Wigner by the use of group-theoretical methods as the following expression:

\[
<j_1 m_1 j_2 m_2 | J M> = \delta_{M_1 m_1 + m_2} \\
\times \sqrt{\frac{(J+j_1-j_2)! (J-j_1+j_2)! (j_1^2+j_2^2-J)! (J+M)! (J-M)! (2J+1)!}{(J+j_1+j_2+1)! (j_1-m_1)! (j_1+m_1)! (j_2-m_2)! (j_2+m_2)!}} \\
\times \sum_K (-1)^{K+j_2+m_2} \frac{(J+j_2+m_2-K)! (j_1-m_1+K)!}{(J-j_1+j_2-K)! (J+M-K)! K! (K+j_1-j_2-M)!} 
\]

(F.6)

In this summation K takes on all integral values consistent with the factorial notation, the factorial of a negative number being meaningless.
Table 6. Tabulated Functions of the Clebsch-Gordan Coefficients

\[<j_1 m_1 \ j_2 m_2 | j m>\].

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APPENDIX G

Tabulated Functions of the Clebsch-Gordan Coefficients

$$<j \, m \, j \, m \mid jm>$$

$$<j' \, m' \, 1/2 \, \mu \mid jm>$$

<table>
<thead>
<tr>
<th>$j'$=</th>
<th>$\mu$=+1/2</th>
<th>$\mu$=-1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j-1/2$</td>
<td>$\sqrt{\frac{j+m}{2j}}$</td>
<td>$\sqrt{\frac{j-m}{2j}}$</td>
</tr>
<tr>
<td>$j+1/2$</td>
<td>$\sqrt{\frac{j-m+1}{2j+2}}$</td>
<td>$\sqrt{\frac{j+m+1}{2j+2}}$</td>
</tr>
</tbody>
</table>

$$<j' \, m' \, -1 \, \mu \mid jm>$$

<table>
<thead>
<tr>
<th>$j'$=</th>
<th>$\mu$=+1</th>
<th>$\mu$=0</th>
<th>$\mu$=-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j-1$</td>
<td>$\sqrt{\frac{(j+m-1)(j+m)}{(2j-1)2j}}$</td>
<td>$\sqrt{\frac{(j-m)(j+m)}{(2j-1)j}}$</td>
<td>$\sqrt{\frac{(j-m-1)(j-m)}{(2j-1)2j}}$</td>
</tr>
<tr>
<td>$j$</td>
<td>$\sqrt{\frac{(j-m+1)(j+m)}{2j(j+1)}}$</td>
<td>$\frac{j}{\sqrt{3(j+1)}}$</td>
<td>$\sqrt{\frac{(j-m)(j+m+1)}{2j(j+1)}}$</td>
</tr>
<tr>
<td>$j+1$</td>
<td>$\sqrt{\frac{(j+m+1)(j+m+2)}{(2j+2)(2j+3)}}$</td>
<td>$\sqrt{\frac{(j-m+1)(j+m+1)}{(j+1)(2j+3)}}$</td>
<td>$\sqrt{\frac{(j+m+1)(j+m+2)}{(2j+2)(2j+3)}}$</td>
</tr>
</tbody>
</table>

Table 6
\(<j' \ m=\mu \ 3/2 \ u | jm>\)

<table>
<thead>
<tr>
<th>(j')</th>
<th>(\mu=+3/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{3}{2})</td>
<td>(\sqrt{\frac{(j+m)(j+m-1)(j+m-2)}{2j(2j-1)(2j-2)}})</td>
</tr>
<tr>
<td>(-\frac{1}{2})</td>
<td>(\sqrt{\frac{3(j+m-1)(j+m)(j-m+1)}{2j(2j+2)(2j-1)}})</td>
</tr>
<tr>
<td>(+\frac{1}{2})</td>
<td>(\sqrt{\frac{3(j+m)(j-m+2)(j-m)}{2j(2j+2)(2j+3)}})</td>
</tr>
<tr>
<td>(+\frac{3}{2})</td>
<td>(\sqrt{\frac{(j-m+1)(j-m+2)(j-m+3)}{2j(2j+2)(2j+3)(2j+4)}})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(j')</th>
<th>(\mu=+1/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{3}{2})</td>
<td>(\sqrt{\frac{3(j+m-1)(j+m)(j-m)}{2j(2j-1)(2j-2)}})</td>
</tr>
<tr>
<td>(-\frac{1}{2})</td>
<td>(-\frac{(j-3m+1)\sqrt{(j+m)}}{2j(2j+2)(2j-1)})</td>
</tr>
<tr>
<td>(+\frac{1}{2})</td>
<td>(-\frac{(j+3m-1)\sqrt{(j-m+1)}}{2j(2j+2)(2j+3)})</td>
</tr>
<tr>
<td>(+\frac{3}{2})</td>
<td>(\sqrt{\frac{3(j+m+1)(j-m+1)(j-m+2)}{2j(2j+2)(2j+3)(2j+4)}})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(j')</th>
<th>(\mu=-1/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{3}{2})</td>
<td>(\sqrt{\frac{3(j+m)(j-m-1)(j-m)}{2j(2j-1)(2j-2)}})</td>
</tr>
<tr>
<td>(-\frac{1}{2})</td>
<td>(\sqrt{\frac{(j-m+1)\sqrt{(j-m)}}{2j(2j+2)(2j-1)}})</td>
</tr>
<tr>
<td>(+\frac{1}{2})</td>
<td>(-\frac{(j+3m-1)\sqrt{(j+m+1)}}{2j(2j+2)(2j+3)})</td>
</tr>
<tr>
<td>(+\frac{3}{2})</td>
<td>(\sqrt{\frac{3(j+m+1)(j-m+1)(j-m+2)}{2j(2j+2)(2j+3)(2j+4)}})</td>
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<table>
<thead>
<tr>
<th>(j')</th>
<th>(\mu=-3/2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{3}{2})</td>
<td>(\sqrt{\frac{(j-m-2)(j-m-1)(j-m)}{2j(2j-1)(2j-2)}})</td>
</tr>
<tr>
<td>(-\frac{1}{2})</td>
<td>(\sqrt{\frac{3(j+m)(j-m+1)(j-m)}{2j(2j+2)(2j-1)}})</td>
</tr>
<tr>
<td>(+\frac{1}{2})</td>
<td>(\sqrt{\frac{(j+3m+1)(j-m+1)(j-m+2)}{2j(2j+2)(2j+3)})</td>
</tr>
<tr>
<td>(+\frac{3}{2})</td>
<td>(\sqrt{\frac{(j+m+1)(j+3m+1)(j-m+1)}{2j(2j+2)(2j+3)(2j+4)}})</td>
</tr>
</tbody>
</table>

Table 6 (cont.)
\[ <j' m - \mu | 2 \mu | jm> \]

<table>
<thead>
<tr>
<th>( j' = \mu + 2 )</th>
<th>( \mu = 1 )</th>
<th>( \mu = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \sqrt{\frac{(j+m-3)(j+m-2)(j+m-1)(j+m)}{(2j-3)(2j-2)(2j-1)2j}} ]</td>
<td>[ \sqrt{\frac{(j-m)(j+m-2)(j+m-1)(j+m)}{(2j-3)(j-1)(2j-1)j}} ]</td>
<td>[ \sqrt{\frac{3(j-m-1)(j-m)(j+m-1)(j+m)}{(2j-3)(2j-2)(2j-1)j}} ]</td>
</tr>
<tr>
<td>[ -(j-2m+1) \sqrt{\frac{(j+m-1)(j+m)}{(2j-2)(2j-1)j}} ]</td>
<td>[ (1-2m) \sqrt{\frac{3(j-m+1)(j+m)}{(2j-1)j(2j+2)(2j+3)}} ]</td>
<td>[ \frac{m}{3(j+m+1)} \sqrt{\frac{3(j-m)(j+m)}{(j-1)(2j-1)j(j+1)}} ]</td>
</tr>
<tr>
<td>[ \frac{(j+m)(j-m+1)(j+m+2)}{(2j-1)2j(j+1)(2j+3)} ]</td>
<td>[ \frac{(j+2m)}{j(j+1)(2j+3)(2j+4)} ]</td>
<td>[ -m \sqrt{\frac{3(j-m+1)(j+m+1)}{j(j+1)(2j+3)(j+2)}} ]</td>
</tr>
<tr>
<td>[ \sqrt{\frac{(j+m+1)(j-m+2)(j-m+3)(j-m+4)}{(2j+2)(2j+3)(2j+4)(2j+5)}} ]</td>
<td>[ -\sqrt{\frac{(j-m+1)(j-m+2)(j-m+3)(j+m+1)}{(j+1)(2j+3)(j+2)(2j+5)}} ]</td>
<td>[ \sqrt{\frac{3(j-m+1)(j-m+2)(j+m+1)(j+m+2)}{(2j+2)(2j+3)(j+2)(2j+5)}} ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( j' = \mu - 1 )</th>
<th>( \mu = -2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \sqrt{\frac{(j-m-2)(j-m-1)(j-m)(j+m)}{(2j-3)(j-1)(2j-1)j}} ]</td>
<td>[ \sqrt{\frac{(j-m-3)(j-m-2)(j-m-1)(j-m)}{(2j-3)(2j-2)(2j-1)2j}} ]</td>
</tr>
<tr>
<td>[ (j+2m+1) \sqrt{\frac{(j-m-2)(j-m-1)(j-m)(j+m+1)}{(j-1)(2j-1)j(2j+2)}} ]</td>
<td>[ \sqrt{\frac{(j-m-2)(j-m-1)(j-m)(j+m+1)}{(2j-1)j(2j+2)(2j+3)}} ]</td>
</tr>
<tr>
<td>[ (1+2m) \sqrt{\frac{3(j-m)(j+m+1)}{(2j-1)2j(j+1)}} ]</td>
<td>[ \sqrt{\frac{3(j-m+1)(j-m)(j+m+1)(j+m+2)}{(2j-1)j(2j+2)(2j+3)}} ]</td>
</tr>
<tr>
<td>[ -(j-2m) \sqrt{\frac{(j+m+1)(j+m+2)}{j(j+1)(2j+3)(2j+4)}} ]</td>
<td>[ \sqrt{\frac{(j-m)(j+m+1)(j+m+2)(j+m+3)}{j(j+1)(2j+3)(2j+4)}} ]</td>
</tr>
<tr>
<td>[ -(j-2m) \sqrt{\frac{(j+m+1)(j+m+2)}{j(j+1)(2j+3)(2j+4)}} ]</td>
<td>[ \sqrt{\frac{(j-m)(j+m+1)(j+m+2)(j+m+3)}{j(j+1)(2j+3)(2j+4)}} ]</td>
</tr>
</tbody>
</table>

Table 6 (cont.)
REFERENCES


