On Ramsey Numbers Defined by Factorizations of Regular Complete Multi-Partite Graphs

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James M. Benedict
For Rolla Ann
who worked harder than I.
For Nathan and Laura,
who missed their Daddy,
and never did understand why.
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OF REGULAR COMPLETE MULTI-PARTITE GRAPHS.

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CHAPTER I

PRELIMINARIES

The central purpose of this dissertation is to investigate certain generalizations of existing Ramsey numbers. The focus of Chapter I is to provide a unifying framework within which all previous Ramsey numbers occur as special cases. Arguments are given to support the contention that only those numbers which occur within this framework should be called Ramsey numbers.

In the final three chapters of this dissertation a particular restriction of the general case is studied. New Ramsey numbers are discovered all of which are extensions of currently known Ramsey numbers. A frequent procedure of generalization in graph theory is the extension of theorems involving the complete graphs to theorems involving a larger family of complete multipartite graphs. Many examples of this type of extension from topological graph theory are found in White [36]. The target of such extensions is often the octahedral graphs [20] or arbitrary regular complete multipartite graphs [36]. Similar results from factorization theory are contained in [24, 25]. This trend is followed in the final three chapters of this dissertation since the notion of the complete...
graph in Ramsey theory is extended to all regular complete multi-partite graphs. Special attention is given to the octahedral graphs.

The next section provides an overview of Ramsey theory. The standard graph theoretical terms and notation (as found in Behzad and Chartrand [1] or White [36]) are used. It is simply noted here that graphs are the usual (single-edged, finite, loopless, undirected) combinatorial entities built upon vertices and edges.

Section 1.1

An Overview of Ramsey Numbers

A brief overview of Ramsey numbers is given in this section. A more thorough treatment may be found in the excellent surveys by Burr [5] and Harary [21]. Even today these surveys nearly represent the state of the art.

In 1930 [31] Ramsey proved (in a non-graphical context) that if \( G \) is an infinite graph having a countably infinite vertex set, then either \( G \) or \( \bar{G} \) contains the complete infinite graph of order \( \aleph_0 \) as a subgraph. Moreover, for positive integers \( m \) and \( n \) there is a smallest positive integer \( p \) such that given any graph \( G \) of order \( p \) either \( K_m \) is a subgraph of \( G \) or \( K_n \) is a subgraph of \( \bar{G} \). Determining these numbers \( p \) for various \( m \) and \( n \) is the classical Ramsey problem. The notation and concepts have advanced as follows.
A graph $G$ is said to have a factorization into the $k$ (al) subgraphs (factors) $F_1, F_2, \ldots, F_k$ (written $G = F_1 \oplus F_2 \oplus \ldots \oplus F_k$ or more compactly $G = \bigoplus_{j=1}^{k} F_j$) if

(i) $V(F_j) = V(G)$ for $1 \leq j \leq k$, (ii) $\bigcup_{j=1}^{k} E(F_j) = E(G)$,

(iii) $E(F_s) \cap E(F_t) = \emptyset$ whenever $s \neq t$. A semi-partition of a set $S$ is a collection of sets $S_1, S_2, \ldots, S_k$ having the properties (i) $\bigcup_{j=1}^{k} S_j = S$ and (ii) $S_j \cap S_t = \emptyset$ whenever $j \neq t$. It may be said then that a factorization of a graph $G$ is a finite collection of spanning subgraphs whose edge sets form a semi-partition of $E(G)$.

If the graph $F$ is a subgraph of the graph $G$ we write $F \subseteq G$. For positive integers $n_1, n_2, \ldots, n_k$ the Ramsey number $r(n_1, n_2, \ldots, n_k)$ is the least positive integer $p$ such that whenever $K_p = \bigoplus_{j=1}^{k} F_j$ then $K_{n_j} \subseteq F_j$ for at least one $j$ with $1 \leq j \leq k$. Thus the classical Ramsey problem (in two factors) has been to determine $r(m, n)$ for every two positive integers $m$ and $n$. The degree of difficulty of the classical Ramsey problem is reflected in the fact that only seven non-trivial such numbers are known (see Figure 1.1).

The difficulties of the classical Ramsey problem probably account for the growth of the scope of the
Classical Ramsey Numbers

\[
\begin{align*}
    r(3,3) &= 6 & r(3,3,3) &= 17 \\
r(3,4) &= 9 & r(3,6) &= r(4,4) = 18 \\
r(3,5) &= 14 & r(3,7) &= 23
\end{align*}
\]
Figure 1.1

problem. Starting in 1967 [19], the focus of the problem has changed to include the generalized Ramsey number.

For the \( k \) (\( \geq 1 \)) graphs \( G_1, G_2, \ldots, G_k \), the generalized Ramsey number \( r(G_1, G_2, \ldots, G_k) \) is the least positive integer \( p \) such that whenever \( K_p = \bigoplus_{j=1}^{k} \mathcal{F}_j \) then there is some \( j \) with \( 1 \leq j \leq k \) for which \( G_j \subseteq \mathcal{F}_j \). That \( r(G_1, G_2, \ldots, G_k) \) exists for any \( k \) (\( \geq 1 \)) graphs \( G_1, G_2, \ldots, G_k \) follows directly from Ramsey's original work. The generalized Ramsey problem is to find \( r(G_1, G_2, \ldots, G_k) \) for every \( k \) graphs \( G_1, G_2, \ldots, G_k \) (or at least the standard interesting graphs). As seen in Burr's survey [5] much has been accomplished along these lines.

Since 1974, evolutions from the generalized Ramsey numbers have occurred in three differing varieties:

(i) properties of factors of \( K_p \) other than graph theoretical inclusion; (ii) factors of graphs other than \( K_p \); and (iii) restricted factorizations of \( K_p \) have been studied. Examples of (i), (ii), and (iii) are presented.
Chartrand and Polimeni [8] define \( \chi(n_1, n_2, \ldots, n_k) \) for the \( k \) (\( \geq 1 \)) positive integers \( n_1, n_2, \ldots, n_k \) to be the least positive integer \( p \) such that whenever
\[
K_p = \bigoplus_{j=1}^k F_j,
\]
then there is some \( j \) with \( 1 \leq j \leq k \) for which \( \chi(F_j) \geq n_j \) (where \( \chi(G) \) denotes the chromatic number of the graph \( G \)). Further generalizations are given in [3,26,27,28]. Beineke and Schwenk [2] define \( R(K(m,n), K(m,n)) \) for the positive integers \( m \) and \( n \) to be the least positive integer \( p \) such that whenever
\[
K(p,p) = F_1 \oplus F_2 \quad \text{it follows that} \quad K(m,n) \subseteq F_1 \quad \text{or} \quad K(m,n) \subseteq F_2.
\]
Similar generalizations are given in [17,18].

As a concluding example, Sumner [33] defines \( r_c(G_1, G_2) \) for the graphs \( G_1 \) and \( G_2 \) to be the least positive integer \( p \) such that whenever \( K_p = F_1 \oplus F_2 \), where both \( F_1 \) and \( F_2 \) are connected graphs, it follows that \( G_1 \subseteq F_1 \) or \( G_2 \subseteq F_2 \).

It is the purpose of the next section to define a general Ramsey number and to develop some suggestive and unifying notation. It is hoped that both the definition and notation are broad enough to contain the mainstream Ramsey numbers as special cases, yet narrow enough to be meaningful. Before this can be accomplished, it must be known when a given number (which arises as a result of a factorization of a graph) is to be considered a Ramsey number. Graham has been quoted (see [22]) as
regarding "any criterion for partitioning the lines of a graph as a ramsey-type problem." The author disagrees with this assessment. Every ramsey number to date has been discovered with the aid of a certain proof technique, an example of which now follows.

To show that \( r(3,3,3) = 17 \), it is demonstrated that whenever \( K_{17} = \bigoplus F_j \) then \( K_3 \subseteq F_j \) for some \( j \) \((1 \leq j \leq 3)\). It follows that \( r(3,3,3) \leq 17 \). To see that \( r(3,3,3) \geq 17 \) it suffices to display a factorization \( K_{16} = \bigoplus F_j \) with \( K_3 \not\subseteq F_j \) for every \( j \) \((1 \leq j \leq 3)\). The main point is this: since \( K_3 \subseteq F' \subseteq F \) implies \( K_3 \subseteq F \) and since \( p \leq s \) implies \( K_p \subseteq K_s \) then for every \( 1 \leq p \leq 16 \) there is a factorization \( K_p = \bigoplus F_j \) with \( K_3 \not\subseteq F_j \) for every \( j \) \((1 \leq j \leq 3)\).

Moreover, for every \( p \geq 17 \) and every factorization \( K_p = \bigoplus F_j \) there is \( K_3 \subseteq F_j \) for some \( j \) \((1 \leq j \leq 3)\). This phenomenon can be thought of as "continuity in the integers". Every known ramsey number has implicitly used this phenomenon within its proof. It seems reasonable to require this continuity in the integers for a factorization problem to be classified as a ramsey problem. Care will be taken in the general definition to see that the desired "continuity" is preserved.
Section 1.2

General Ramsey Numbers

The purpose of this section is to define the general ramsey number for graphs. Some attention is paid to the three main components of ramsey number definitions which are: (i) a family of graphs is being factored; (ii) certain specialized properties (of a type described below) of each factor may be required; and (iii) arbitrary restrictions may be placed on the entire factorization.

Attention is primarily directed toward items (i) and (ii). It is suggested that ramsey numbers which arise from restricted factorizations be called restricted ramsey numbers. This will allow us to think of the general ramsey number as an unrestricted ramsey number. Sumner's work [33] is the only known example of restricted ramsey number theory to date. The types of families of graphs and properties of factors to be considered are now specified.

The indexed family of graphs \( \mathfrak{G} = \{G_p | p \in \mathbb{Z}^+ \} \) is called an ascending family of graphs if \( p \leq s \) implies \( G_p \subseteq G_s \). For example \( \{K_p | p \in \mathbb{Z}^+ \} \) is an ascending family whereas \( \{C_{p+2} | p \in \mathbb{Z}^+ \} \) (the family of cycles) is not. Only ascending families will be considered for the general ramsey number.
Now let $\pi$ be a property of graphs such that every given graph $G$ either has or fails to have property $\pi$. For example, $\pi$ could be the property of having chromatic number at least five. We see that $K_5$ has property $\pi$ while $K_4$ does not. The property $\pi$ is called an ascending property if whenever the graph $G$ has property $\pi$ and $G \subset H$ then it follows that the graph $H$ has property $\pi$. It may be noted that the above example is an ascending property while "having chromatic number equal to five" is not. The general ramsey number is now defined.

**Definition 1.1** Let $\mathcal{F} = \{G_p | p \in \mathbb{Z}^+\}$ be an ascending family of graphs, and let $\pi_1, \pi_2, \ldots, \pi_k$ be $k$ (all) ascending properties of graphs. Define the general ramsey number $r_{\mathcal{F}}(\pi_1, \pi_2, \ldots, \pi_k)$ to be the least integer $p$ (if it exists) such that whenever $G_p = \bigoplus F_j$ then $F_j$ has property $\pi_j$ for some $j$ ($1 \leq j \leq k$).

Some notation is developed after which it is shown that the desired continuity in the integers is present in the general ramsey number.

For graphs $G$ and $H$ with $V(G) \cap V(H) \neq \emptyset$ we define the graph $G \cap H$ by

(i) $V(G \cap H) = V(G) \cap V(H)$ and

(ii) $E(G \cap H) = E(G) \cap E(H)$.
Note that if $V(G) \cap V(H) \neq \emptyset$ and if $G = \bigoplus_{j=1}^{k} F_j$ then
\[G \cap H = \bigoplus_{j=1}^{k} (F_j \cap H).\]

Again for the graphs $G$ and $H$, we can always insure that $V(G) \cap V(H) = \emptyset$ by relabeling the vertices of one of the graphs if necessary. With this convention in mind, we define $G \cup H$ only for graphs with disjoint vertex sets by

1. $V(G \cup H) = V(G) \cup V(H)$ and
2. $E(G \cup H) = E(G) \cup E(H)$.

For the positive integer $n$, we write $nG$ to denote the graph $\bigcup_{j=1}^{n} G = G \cup G \cup \ldots \cup G$ (n copies of $G$). We often write $G^{(j)}$ to denote the $j$-th copy of the graph $G$. That is, $G = G^{(j)}$ up to a possible relabeling of the vertices. Using these ideas, we can describe that $C_6$ is constructed from two 1-factors each of order 6 by the notation $C_6 = \bigoplus_{j=1}^{2} (3K_2)^{(j)}$.

**Theorem 1.1 (continuity in $Z^+$)**

Let $\mathcal{G} = \{G_\lambda | \lambda \in Z^+\}$ be an ascending family of graphs and let $\pi_1, \pi_2, \ldots, \pi_k$ be $k$ (at least 1) ascending properties of graphs.

Let $p$ be an integer such that whenever $G_p = \bigoplus_{j=1}^{k} F_j$ then $F_j$ has property $\pi_j$ for some $j$ with $1 \leq j \leq k$. 

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Let $b$ be an integer such that there exists a factorization $G_b = \bigoplus_{j=1}^k H_j$ for which $H_j$ fails to have property $\pi_j$ for every $j$ with $1 \leq j \leq k$.

Then for every $n \geq p$ it holds that whenever $G_n = \bigoplus_{j=1}^k F_j$ there is an integer $j$ with $1 \leq j \leq k$ for which $F_j$ has property $\pi_j$. Also, for every $n \leq b$ there exists a factorization $G_n = \bigoplus_{j=1}^k F_j$ where $F_j$ fails to have property $\pi_j$ for every $j$ with $1 \leq j \leq k$.

**Proof:** Let $n \geq p$ and let $G_n = \bigoplus_{j=1}^k F_j$. Since $G_p \subseteq G_n$ we have $G_p = \bigoplus_{j=1}^k (F_j \cap G_p)$. By the hypothesis on $p$ we can assume $F_1 \cap G_p$ has property $\pi_1$. Since $F_1 \cap G_p \subseteq F_1$ and since $\pi_1$ is an ascending property then $F_1$ has property $\pi_1$.

Now let $n \leq b$. Since $G_n \subseteq G_b$ we have the factorization $G_n = \bigoplus_{j=1}^k (H_j \cap G_n)$. Since (i) $H_j$ does not have property $\pi_j$ ($1 \leq j \leq k$), (ii) $\pi_j$ is an ascending property ($1 \leq j \leq k$), and (iii) $H_j \cap G_n \subseteq H_j$ ($1 \leq j \leq k$) then $H_j \cap G_n$ cannot have property $\pi_j$ for each $j$ with $1 \leq j \leq k$.

The importance of Theorem 1.1 is that we are guaranteed that $b+1 \leq r_\pi(\pi_1, \pi_2, \ldots, \pi_k) \leq p$. 

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It is easy to concoct examples of numbers defined in a manner similar to the general ramsey number which fail to be continuous in the integers since either the family of graphs or the properties are not ascending. As an example of the former case, take 
\[ \mathfrak{G} = \{ G_p | G_p = K_p \text{ if } p \neq 10 \} \cup \{ G_{10} \} , \] where \( G_{10} = 10K_1 \). Further, let \( \pi_j \) be the property of having three or more edges for \( j = 1 \) and for \( j = 2 \). Then \( r_{\mathfrak{G}}(\pi_1, \pi_2) = 4 \) even though \( G_{10} = 10K_1 \ominus 10K_1 \) is a factorization in which both factors fail to have three or more edges.

As an example of the latter case, let \( \mathfrak{G} = \{ K_p | p \in \mathbb{Z}^+ \} \) and let \( \pi \) be the property of having order equal to three. Then \( r_{\mathfrak{G}}(\pi, \pi) = 3 \) even though \( K_4 = P_4 \ominus P_4 \) where both factors fail to have property \( \pi \).

More theorems basic to "general ramsey theory" are now presented.

**Theorem 1.2** (symmetry)

Let \( \mathfrak{G} \) be an ascending family of graphs, let \( \pi_1, \pi_2, \ldots , \pi_k \) be \( k \) \((\geq 1)\) ascending properties of graphs, and let \( r_{\mathfrak{G}}(\pi_1, \pi_2, \ldots , \pi_k) = p \).

Then for any permutation \( \tau \) of \( [1, 2, \ldots , k] \), it follows that \( r_{\mathfrak{G}}(\pi_\tau(1), \pi_\tau(2), \ldots , \pi_\tau(k)) = p \).

**Proof:** Let \( \mathfrak{G} = \{ G_t | t \in \mathbb{Z}^+ \} \) be an ascending family of graphs. To show that \( r_{\mathfrak{G}}(\pi_\tau(1), \pi_\tau(2), \ldots , \pi_\tau(k)) \leq p \),
let \( G_p = \bigoplus_{j=1}^{k} F_j \) be a factorization of \( G_p \) and let
\[
\sigma = \tau^{-1}.
\]
Consider \( G_p = \bigoplus_{j=1}^{k} F_{\sigma(j)} \). By the definition of \( \sigma \) there exists \( j_0 \) with \( 1 \leq j_0 \leq k \) for which the graph \( F_{\sigma(j_0)} \) has property \( \pi_{j_0} \). Let \( j = \sigma(j_0) \). Then
\[
\pi_{j_0} = \pi_{\sigma^{-1}(j)} = \pi_{\tau}(j).
\]
This implies that \( F_j \) has property \( \pi_{\tau}(j) \) which establishes an upper bound for
\[
r_{\pi_{\tau}(1)}, \pi_{\tau}(2), \ldots, \pi_{\tau}(k).
\]
The desired lower bound is similarly established.

Theorems 1.1 and 1.2 are generalizations of corresponding results from previously existing Ramsey theory as is Theorem 1.3, which needs the following definition.

For the ascending properties \( \pi \) and \( \psi \) of graphs we write \( \psi \prec \pi \) (\( \psi \) given \( \pi \)) if \( \psi \) is true for a given graph \( G \) whenever \( \pi \) is true for that graph \( G \). For example, "\( \chi(G) \geq 3 \)" \( \prec \) "\( \chi(G) \geq 4 \).

**Theorem 1.3 (monotonicity)**

Let \( \psi \) be an ascending family of graphs, let \( \pi_1', \pi_2', \ldots, \pi_k' \) and \( \psi_1', \psi_2', \ldots, \psi_k' \) be two pairs of \( k \geq 1 \) ascending properties of graphs, let
\[
r_{\psi'}(\pi_1', \pi_2', \ldots, \pi_k') = p,
\]
and assume \( \psi_j \prec \pi_j \) for \( 1 \leq j \leq k \).

Then \( r_{\psi'}(\psi_1', \psi_2', \ldots, \psi_k') \leq p \).
Proof: We specify $\xi = \{G_t | t \in Z^+\} \text{ and assume } G_p = \bigoplus_{j=1}^k F_j$. There is some $j$ for which the graph $F_j$ has property $\pi_j$. Hence, $F_j$ has property $\psi_j$.

The factorization is arbitrary so the conclusion follows $\blacksquare$

Theorem 1.3 makes a statement concerning the general ramsey number for a fixed family and changing properties. The situation is reversed in Theorem 1.4 which needs the following definition. For the ascending families $\xi = \{G_p | p \in Z^+\}$ and $\eta = \{H_p | p \in Z^+\}$ of graphs we write $\xi \rightarrow \eta$ if $G_p \subseteq H_p$ for each $p \in Z^+$. Theorem 1.4 has no analogous result in ordinary ramsey theory.

**Theorem 1.4 (anti-monotonicity)**

Let $\xi$ and $\eta$ be ascending families of graphs for which $\xi \rightarrow \eta$, let $\pi_1, \pi_2, \ldots, \pi_k$ be ascending properties of graphs, and let $r_\xi(\pi_1, \pi_2, \ldots, \pi_k) = p$.

Then $r_\eta(\pi_1, \pi_2, \ldots, \pi_k) \leq p$.

Proof: Let $\xi = \{G_t | t \in Z^+\}$, let $\eta = \{H_t | t \in Z^+\}$, and let $H_p = \bigoplus_{j=1}^k F_j$. Since $G_p \subseteq H_p$ it follows that

$$G_p = (G_p \cap H_p) = \bigoplus_{j=1}^k (F_j \cap G_p).$$

However, by the definition of $p$ there exists some $j$ for which $F_j \cap G_p$ has property $\pi_j (1 \leq j \leq k)$. 

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Since $\pi_j$ is an ascending property and since $F_j \cap G_p \subset F_j$, then $F_j$ has property $\pi_j$. This implies the validity of the theorem.

We end Section 1.2 with a final observation concerning general Ramsey numbers. Given $\xi$ and $\pi_1, \pi_2, \ldots, \pi_k$ as usual, let $\pi_{k+1}$ be the ascending property "the edge set is non-empty". Then

$$r_\xi(\pi_1, \pi_2, \ldots, \pi_k) = r_\xi(\pi_1, \pi_2, \ldots, \pi_k, \pi_{k+1}).$$

Section 1.3

The $i$-th Ramsey Numbers

We narrow the discussion to examine general Ramsey numbers with respect to a certain type of ascending family.

A graph $G$ having vertex set $V(G)$ is called $p$-partite if the vertex set can be partitioned into $p$ ($\geq 1$) partite sets $V_1, V_2, \ldots, V_p$ so that each partite set is an independent set of vertices of $G$. Note that every graph $G$ can be thought of as $p$-partite for each integer $p$ such that $\chi(G) \leq p \leq |V(G)|$. A graph $K$ is called complete $p$-partite if $K$ is a $p$-partite graph such that for each two vertices $u$ and $v$ of $K$ we have $uv \in E(K)$ if and only if $u$ and $v$ belong to distinct partite sets. Given that $|V_j| = n_j$ for
1 ≤ j ≤ p we write $K = K(n_1, n_2, \ldots, n_p)$. Each such $K$ is unique up to a reordering of partite sets. If the number of partite sets of $K$ is to be left unspecified, we refer to $K$ as a complete **multi-partite** graph. As examples, note that $\bar{K}^n_n = nK_1 = K(n)$ for $n \in \mathbb{Z}^+$, and the **star** graphs are $K(1, n)$ for $n \in \mathbb{Z}^+$.

If each partite set of the complete $p$-partite graph $K(n_1, n_2, \ldots, n_p)$ has exactly $i$ ($≥ 1$) vertices we write $K_p(i)$ rather than $K(i, i, \ldots, i)$ to denote this graph. Note that $K_p(1) = K_p$ for each $p \in \mathbb{Z}^+$. If $p - t$ ($≥ 1$) of the partite sets have $i$ ($≥ 1$) vertices and the remaining $t$ ($≥ 1$) partite sets have $j$ ($≥ 1$) vertices we write $K_{[p-t]}(i), t(j)$ for $K(i, i, \ldots, i, j, j, \ldots, j)$. We specifically define

$$K_{t(i), s(o)} = K_t(i) = K_{t(i), o(s)}$$

for the positive integers $t, i$ and $s$, and

$$K_{o(i), t(j)} = K_t(j)$$

for the positive integers $t, i$ and $j$.

With this notation, define $\xi_i$ for each $i \in \mathbb{Z}^+$ by $\xi_i = \{K_p(i) | p \in \mathbb{Z}^+\}$. It is seen that each $\xi_i$ is an ascending family of graphs. In particular, the family of octahedral graphs is thus denoted $\xi_2$. The main definition of this section is now given.
Definition 1.2 Let $\pi_1, \pi_2, \ldots, \pi_k$ be ascending properties of graphs, let $i$ be a positive integer, and assume $r_{\pi_i}(\pi_1, \pi_2, \ldots, \pi_k)$ exists. The $i$-th general ramsey number $r_i(\pi_1, \pi_2, \ldots, \pi_k)$ is defined to be $r_{\pi_i}(\pi_1, \pi_2, \ldots, \pi_k)$. This number is sometimes called the ramsey number with respect to $K_p(i)$ for the properties $\pi_1, \pi_2, \ldots, \pi_k$. We shall often write $r(\pi_1, \pi_2, \ldots, \pi_k)$ for $r_1(\pi_1, \pi_2, \ldots, \pi_k)$.

For the integers $i$ and $t$ with $t \geq i$ we have $\pi_i \rightarrow \pi_t$ since $K_p(i) \subseteq K_p(t)$ for every $p \in \mathbb{Z}^+$. It then follows by anti-monotonicity that if $t \geq i$ then $r_t(\pi_1, \pi_2, \ldots, \pi_k) \leq r_i(\pi_1, \pi_2, \ldots, \pi_k)$. The two main consequences of this observation are: (1) If $r_i(\pi_1, \pi_2, \ldots, \pi_k)$ exists then so does $r_t(\pi_1, \pi_2, \ldots, \pi_k)$ for every $t \geq i$. (2) The sequence $\left\{r_t(\pi_1, \pi_2, \ldots, \pi_k)\right\}_{t=i}^{\infty}$ is a non-increasing sequence of integers (given that it is non-empty) bounded below by 1. Hence there exists $i_0$, the smallest integer $i$ for which $r_i(\pi_1, \pi_2, \ldots, \pi_k) = \lim_{t \to \infty} r_t(\pi_1, \pi_2, \ldots, \pi_k)$. We are led to the following definition.

Definition 1.3 Let $\pi_1, \pi_2, \ldots, \pi_k$ be ascending properties for which $r(\pi_1, \pi_2, \ldots, \pi_k)$ exists. The ramsey index $i(\pi_1, \pi_2, \ldots, \pi_k)$ is the least integer
i for which \( r_i(\pi_1, \pi_2, \ldots, \pi_k) = \lim_{t \to \infty} r(t; \pi_1, \pi_2, \ldots, \pi_k) \).

The definition of the Ramsey index is slightly restrictive in that it is conceivable that \( r(\pi_1, \pi_2, \ldots, \pi_k) \) might not exist while \( r_i(\pi_1, \pi_2, \ldots, \pi_k) \) does exist for some \( i \geq 2 \). In this case a natural alternative definition of a Ramsey index could follow. That no such definition is given reflects the fact that the previously described situation never arises in this dissertation.

The main work of Chapter I has been completed. We conclude this chapter with a section of observations and problems.

Section 1.4

Closing Comments

The original ascending property studied in the context of Ramsey numbers is that of graph theoretic inclusion. For the \( k (\geq 1) \) graphs \( G_1, G_2, \ldots, G_k \) and the positive integer \( i \), define \( r_i(G_1, G_2, \ldots, G_k) \) to be the least integer \( p \) such that if \( K_p(i) = \bigoplus_{j=1}^{k} F_j \) then it follows that for some \( j \) with \( 1 \leq j \leq k \) we have \( G_j \subset F_j \). That \( r_i(G_1, G_2, \ldots, G_k) \) exists for every positive integer \( i \) follows directly from antimonotonicity and the fact that \( r_1(G_1, G_2, \ldots, G_k) \) is known to exist. The existence of the corresponding
ramsey index denoted \( i(G_1, G_2, ..., G_k) \) is thereby guaranteed. We compare and contrast these notions with a "ramsey number" defined by Beineke and Schwenk.

In [2], for the positive integers \( m \) and \( n \), Beineke and Schwenk define (in the current notation) the bipartite ramsey number \( R(m,n) \) to be the least positive integer \( i \) such that if \( K_2(i) = F_1 \circ F_2 \) then either \( K(m,n) \subseteq F_1 \) or \( K(m,n) \subseteq F_2 \). It is submitted here that \( R(m,n) \) should be thought of as a (bipartite) ramsey index (in the sense just described) rather than a ramsey number for the following two reasons. First, since all other known ramsey numbers (with a single possible exception to be discussed presently) have been defined by minimizing the number of partite sets in a factored graph, it is most reasonable that ramsey numbers should continue to enjoy this property. Second, it is an easy consequence of definitions to prove that \( R(m,n) = i(K(m,n), K(m,n)) \). Hence \( R(m,n) \) fits neatly and naturally into general ramsey theory as a ramsey index.

The "single exception" mentioned above is now considered. It seems likely that the most one could expect (within a ramsey context) is to know for the graphs \( G_1, G_2, ..., G_k \) exactly which complete p-partite graphs \( K = K(n_1, n_2, ..., n_p) \) have the property that if
K = \bigoplus_{j=1}^{k} F_j \text{ then } G_j \subseteq F_j \text{ for at least one } j \text{ (} 1 \leq j \leq k\).}

(This claim is made since one must have knowledge of which edges are present in the factored graph if one is to be able to prove something about the factors. Therefore, in general, the factored graph must be a complete multipartite graph.) Indeed, it may be that most generalized ramsey numbers to date consider only pairs of graphs due to a lack of the above type of knowledge.

Faudree and Schelp observed in [17] that in order to find \( r(P_{n_1}, P_{n_2}, P_{n_3}) \) in [18] it is helpful to know for which integers \( m \) and \( n \) it is the case that if \( K(m,n) = F_1 \oplus F_2 \) then either \( P_{n_2} \subseteq F_1 \) or \( P_{n_3} \subseteq F_2 \).

(The graph \( P_{n} \) is the path of length \( n-1 \) on \( n \) vertices.) For the ordered graphs \((G_1, G_2)\) they defined (in [17]) the Ramsey bipartite number pair \( B(G_1, G_2) \) to be (if it exists) the lattice point \((n,m)\) with \( n \geq m \) such that the statement "whenever \( K(r,s) = F_1 \oplus F_2 \) it follows that \( G_1 \subseteq F_1 \) or \( G_2 \subseteq F_2 \)" is true if and only if \( r \geq n \) and \( s \geq m \). It is submitted that \( B(G_1, G_2) \) should be thought of as a generalized ramsey index (as defined below) rather than a ramsey number.

**Definition 1.4** For the \( k \geq 1 \) graphs \( G_1, G_2, \ldots, G_k \) let \( i(G_1, G_2, \ldots, G_k) = I \) and let \( r_I(G_1, G_2, \ldots, G_k) = p_0 \). For the integer \( p \geq p_0 \),
define the generalized ramsey index \( p(I)(G_1, G_2, \ldots, G_k) \) to be the lattice point \((n_1, n_2, \ldots, n_p)\) with \(n_1 \geq n_2 \geq \ldots \geq n_p \geq 1\) such that the statement "whenever \( K(m_1, m_2, \ldots, m_p) = \bigoplus_{j=1}^{k} F_j \) it follows that \( G_j \subseteq F_j \) for at least one \( j \) with \( 1 \leq j \leq k \)" is true if and only if \( m_j \geq n_j \) for \( 1 \leq j \leq p \).

We note that \( p(I)(G_1, G_2, \ldots, G_k) \) exists for every \( k \) graphs \( G_1, G_2, \ldots, G_k \) and for every integer \( p \geq p_0 \). In fact, applying "\( \geq \)" to real vectors in the usual way we have

\[
\underbrace{(I, I, \ldots, I)}_{p \text{ copies}} \geq p(I)(G_1, G_2, \ldots, G_k).
\]

Moreover, \( p'(I)(G_1, G_2, \ldots, G_k) \), when defined in a similar fashion, can never exist if \( p' < p_0 \). It is thought that the generalized ramsey index will closely approximate its associated ramsey index in the case \( p = p_0 \). It could very well be the case that a detailed study of ramsey indices is needed to enable significant further advances in generalized ramsey theory (with \( k \geq 3 \)) to occur. This dissertation presents no such study, but ramsey indices will be mentioned from time to time. We end Chapter I by presenting a few problems.
Problem 1.1

Characterize those ascending properties \( \pi_1, \pi_2, \ldots, \pi_k \) for which \( r_i(\pi_1, \pi_2, \ldots, \pi_k) \) exists \((i \geq 2)\) if and only if \( r(\pi_1, \pi_2, \ldots, \pi_k) \) exists.

Problem 1.2

Conduct an in-depth study of the ramsey index. Try to apply the results to obtain progress in generalized ramsey theory.

Problem 1.3

For the positive integers \( i, m \) and \( n \) define \( r_i(m,q(n)) \) to be the least integer \( p \) such that whenever \( K_p(i) = F_1 \oplus F_2 \) it necessarily follows that \( K_m \subseteq F_1 \) or else \( |E(F_2)| \geq n \).

It is a direct result of Turán's theorem (see [1, p. 237]) and elementary calculus that \( r(m,q(n)) = c(m-1) + s \) where

\[
c = \left\lfloor \left( 1 + \sqrt{1 + 8n/(m - 1)} \right)/2 \right\rfloor - 1 \quad \text{and} \quad s = \left\lfloor n/c - (c - 1)(n - 1)/2 \right\rfloor.
\]

Extend Turán's theorem in such a way that a corollary of the new result yields the value of \( r_i(m,q(n)) \) for all \( i \in \mathbb{Z}^+ \).
CHAPTER II

ON SUBGRAPH RAMSEY NUMBERS
FOR THE REGULAR COMPLETE MULTI-PARTITE GRAPHS

For the \( k \geq 1 \) graphs \( G_1, G_2, \ldots, G_k \), it follows from anti-monotonicity that \( r_i(G_1, G_2, \ldots, G_k) \leq r(G_1, G_2, \ldots, G_k) \) for every \( i \in \mathbb{Z}^+ \). It is natural to seek a criterion which produces a lower bound for \( r_i(G_1, G_2, \ldots, G_k) \) for every positive integer \( i \).

Section 2.1 makes progress toward finding such a criterion and offers a conjecture whose validity would establish the criterion. Section 2.2 presents special cases which support the conjecture. One of the formulas of Section 2.2 finds all \( i \)-th ramsey numbers for every finite collection of stars.

Section 2.1

Lower Bounds

Recall that for the \( k \geq 1 \) graphs \( G_1, G_2, \ldots, G_k \) and the positive integer \( i \), we define \( r_i(G_1, G_2, \ldots, G_k) \) to be the least integer \( p \) such that whenever

\[
P = \bigoplus_{j=1}^{k} F_j,
\]

then for some \( j \) with \( 1 \leq j \leq k \) it follows that \( G_j \subseteq F_j \). This number is known to exist.

Hence the corresponding ramsey index, the least integer...
I for which \( r_i(G_1, G_2, \ldots, G_k) = \lim_{i \to \infty} r_i(G_1, G_2, \ldots, G_k) \),
denoted \( i(G_1, G_2, \ldots, G_k) \), exists. If each graph \( G_j \)
is the complete graph \( K_{n_j} \) we define

\[
\begin{align*}
    r_i(n_1, n_2, \ldots, n_k) &= r_i(K_{n_1}, K_{n_2}, \ldots, K_{n_k}) \\
    i(n_1, n_2, \ldots, n_k) &= i(K_{n_1}, K_{n_2}, \ldots, K_{n_k}).
\end{align*}
\]

In Section 2.1 we seek a lower bound for \( r_i(G_1, G_2, \ldots, G_k) \)
in terms of Ramsey numbers of simpler yet closely related graphs.

The following theorem is a lower bound theorem of the type desired which allows us in general to consider only graphs \( G_j \) having no isolated vertices. It is divided into two parts to simplify its statement.

**Theorem 2.1 (part a)**

Let \( i \) be a positive integer, and let \( G_1, G_2, \ldots, G_k \)
be \( k (\geq 1) \) graphs, some of which are empty. In fact (by symmetry) assume that \( G_1 = mK_1 \) is an empty graph of smallest order from among the graphs \( G_j, 1 \leq j \leq k \).

Then \( r_i(G_1, G_2, \ldots, G_k) = \{m/i\} \).

**Proof:** If \( K_{\{m/i\}}(i) = \bigoplus_{j=1}^{k} F_j \), then \( |V(F_1)| = \{m/i\} \cdot i \geq m \).

Therefore \( mK_1 \subset F_1 \), establishing \( \{m/i\} \) as an upper bound.
Now consider \( n = (m/i) - 1 \) and the factorization 
\[
K_n(i) = \bigoplus_{j=1}^{k} F_j \quad \text{where only } F_1 \text{ has edges. Since } n \cdot i < m
\]
then if \( |V(G_j)| \geq m \) it follows that \( G_j \not\subset F_j \) for every \( j \) with \( 1 \leq j \leq k \). (In particular \( G_1 \not\subset F_1 \).) If \( 2 \leq j \leq k \) and \( |V(G_j)| < m \), then \( |E(G_j)| \geq 1 \) implying \( G_j \not\subset F_j \). Hence \( r_i(G_1, G_2, \ldots, G_k) > (m/i) - 1 \).

**Theorem 2.1 (part b)**

Let \( i \) be a positive integer, and let \( G_1, G_2, \ldots, G_k \) be \( k \) (\( \geq 1 \)) graphs, none of which has isolated vertices. For each \( j \) with \( 1 \leq j \leq k \) let \( m_j \) be a non-negative integer, and let \( H_j = m_j K_1 \cup G_j \). (If \( m_j = 0 \) then \( H_j = G_j \).) Let \( p = \max\{|V(H_j)| : 1 \leq j \leq k\} \). Then
\[
r_i(H_1, H_2, \ldots, H_k) = \max\{r_i(G_1, G_2, \ldots, G_k), \{p/i\}\}.
\]

**Proof:** Let \( m = \max\{r_i(G_1, G_2, \ldots, G_k), \{p/i\}\} \). There are two cases.

**Case 1:** \( m = r_i(G_1, G_2, \ldots, G_k) \).

By definition there is a factorization
\[
K_{(m-1)}(i) = \bigoplus_{j=1}^{k} F_j \quad \text{in which } G_j \not\subset F_j \quad \text{for every } j, 1 \leq j \leq k.
\]
Since \( G_j \subset H_j \) then \( H_j \not\subset F_j \) for every \( j, 1 \leq j \leq k \).

**Case 2:** \( m = \{p/i\} \).

It is without loss of generality that we assume \( m' = \left(\{V(G_1)\} + m_1\right)/i \). As in part a, let
\[ K_{(m-1)}(i) = \bigoplus_{j=1}^{k} F_j \] where \( F_j \) is empty if \( j \geq 2 \). For each \( j \) with \( 2 \leq j \leq k \) it follows that \( H_j \not\subset [i]_{K_1} \) since \( |E(H_j)| \geq 1 \). Moreover, \( H_1 \not\subset F_1 \) since
\[ |V(F_1)| = [m-1] \cdot i < p = |V(H_1)|. \]

It therefore follows that \( r_i(H_1, H_2, \ldots, H_k) \geq m \).

Conversely, for any factorization \( K_m(i) = \bigoplus_{j=1}^{k} F_j \) there is some \( j_0 \) with \( 1 \leq j_0 \leq k \) such that \( G_{j_0} \subset F_{j_0} \)
(by continuity in the integers) since
\[ m \geq r_i(G_1, G_2, \ldots, G_k). \] Since we also have \( m \geq \frac{p}{i} \geq \left( |V(G_{j_0})| + m_{j_0} \right)/i \), then \( |V(F_{j_0})| = m \cdot i \geq |V(G_{j_0})| + m_{j_0} \).
Thus \( m_{j_0} K_1 \subset F_{j_0} - V(G_{j_0}) \) from which we have \( H_{j_0} \subset F_{j_0} \), concluding the proof of Theorem 2.1 (parts a and b).

We may now concentrate exclusively on finding \( i \)-th Ramsey numbers for graphs without isolated vertices.

Theorem 2.1 essentially states that the \( i \)-th Ramsey number for graphs having isolated vertices is bounded below (due to anti-monotonicity) by the \( i \)-th Ramsey number of the "cores" of the graphs. A result is now presented which makes a similar type of statement. Note that for the graph \( G \) we denote the clique number (the order of the largest complete subgraph of \( G \)) by \( \omega(G) \).
Theorem 2.2 (lower bound)

Let \( i \) be a positive integer, let \( G_1, G_2, \ldots, G_k \)
be \( k \) (\( \geq 1 \)) graphs, and let \( w(G_j) = n_j \) for every \( j \)
(\( 1 \leq j \leq k \)).

Then \( r_i(G_1, G_2, \ldots, G_k) \geq r(n_1, n_2, \ldots, n_k) \).

Proof: Let \( r_i(n_1, n_2, \ldots, n_k) = p \). It follows that
there is a factorization \( K_{(p-1)}(i) = \bigoplus_{j=1}^{k} F_j \) for which
\( K_{n_j} \not\subset F_j \) for every \( j \) with \( 1 \leq j \leq k \). Since
\( K_{n_j} \subset G_j \) it follows that \( G_j \not\subset F_j \) where \( 1 \leq j \leq k \).
This implies \( r_i(G_1, G_2, \ldots, G_k) \geq p \). It suffices to
show that \( p \geq r(n_1, n_2, \ldots, n_k) \).

To establish that \( p \geq r(n_1, n_2, \ldots, n_k) \), let
\( K_p = \bigoplus_{j=1}^{k} F_j \) be any arbitrary factorization of \( K_p \) into
\( k \) factors. Label the vertices of \( K_p \) (and hence of
each factor) with the elements of \( \{v_1, v_2, \ldots, v_p\} \).
We construct \( K_p(i) \) from \( K_p \) by the following process.

To obtain the vertex set of \( K_p(i) \), replace each
vertex \( v_j \) of \( K_p \) with a copy of \( K_i \), denoted here by
\( (K_i)v_j \). The vertices \( u_s \in (K_i)v_t \) and \( u_a \in (K_i)v_b \)
form an edge of \( K_p(i) \) if and only if \( v_t \neq v_b \).

Now for each \( j \) with \( 1 \leq j \leq k \) we construct the
graphs \( F_j^i \) from the graphs \( F_j \) in a similar manner.
That is, the vertex set of \( F_j^i \) is formed by replacing

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each vertex $v_t$ of $F_j$ with a copy of $K_i$ denoted $(K_i)_{v_t}$ where $1 \leq t \leq p$. The vertices $u_s \in (K_i)_{v_t}$ and $u_a \in (K_i)_{v_b}$ are adjacent in $F_j$ if and only if $v_t$ and $v_b$ are adjacent in $F_j$.

It now follows that $K_P(i) = \bigoplus_{j=1}^{k} F'_j$. By the definition of $p$, we may assume, without loss of generality, that $K_{n_1} \subset F'_1$. No two vertices of this copy of $K_{n_1}$ can come from the same partite set of $K_P(i)$. Hence we can form the set $U = \{u_1, u_2, \ldots, u_p\}$ of vertices so that (i) each partite set of $K_P(i)$ contributes exactly one vertex to the set $U$ and (ii) $V(K_{n_1}) \subset U$. By relabeling if necessary, we specify $u_t \in (K_i)_{v_t}$ for each $t$ with $1 \leq t \leq p$.

Recall that $V(K_P) = \{v_1, v_2, \ldots, v_p\}$. Define the graph $G$ by

(i) $V(G) = U$

(ii) $u_s u_t \in E(G)$ if and only if $u_s u_t \in E(F'_1)$.

It is a routine matter to verify that the map $u_t \to v_t$ (for every $t$ with $1 \leq t \leq p$) yields an isomorphism for the graphs $G$ and $F'_1$. Hence $K_{n_1} \subset G = F'_1$.

Since $K_P = \bigoplus_{j=1}^{k} F_j$ is an arbitrary factorization, and since there is some $j$ for which $K_{n_j} \subset F_j$ with $1 \leq j \leq k$, then the conclusion is that
\[ r(n_1, n_2, \ldots, n_k) \leq p \]

which concludes the proof ■

Two immediate corollaries of Theorem 2.2 concern the classical Ramsey problem and the Ramsey index.

**Corollary 2.2 A**

For the positive integers \( i \) and \( n_1, n_2, \ldots, n_k \) it follows that \( r_i(n_1, n_2, \ldots, n_k) = r(n_1, n_2, \ldots, n_k) \).

**Proof:** By anti-montonicity, \( r_i(n_1, n_2, \ldots, n_k) \leq r(n_1, n_2, \ldots, n_k) \). By Theorem 2.2 \( r_i(n_1, n_2, \ldots, n_k) \geq r(w(K_{n_1}), w(K_{n_2}), \ldots, w(K_{n_k})) = r(n_1, n_2, \ldots, n_k) \).

**Corollary 2.2 B**

For the \( k \) (\( k \geq 1 \)) integers \( n_1, n_2, \ldots, n_k \) it follows that \( i(n_1, n_2, \ldots, n_k) = 1 \).

It is natural to ask if the lower bound is always attained. That is, given the graphs \( G_1, G_2, \ldots, G_k \) and given \( I = i(G_1, G_2, \ldots, G_k) \), is it the case that

\[ r_i(G_1, G_2, \ldots, G_k) = r(w(G_1), w(G_2), \ldots, w(G_k)) \]

It is easily seen that \( r(K_2, C_5) = 5 \) and \( r_i(K_2, C_5) = 3 \) whenever \( i \geq 2 \). In general then, the answer is no. We may note however that
Erdős has conjectured [4,9] that for every graph $G$,

$$r(G,G) \geq r(\chi(G),\chi(G)),$$

which (by anti-monotonicity) is consistent with the following conjecture.

**Conjecture**

For the $k \geq 1$ graphs $G_1, G_2, \ldots, G_k$ it holds that $r_i(G_1, G_2, \ldots, G_k) \geq r(\chi(G_1), \chi(G_2), \ldots, \chi(G_k))$ for every $i \in \mathbb{Z}^+$. Moreover, if $I = i(G_1, G_2, \ldots, G_k)$ then $r_I(G_1, G_2, \ldots, G_k) = r(\chi(G_1), \chi(G_2), \ldots, \chi(G_k))$.

The next section presents some formulas in support of the conjecture.

**Section 2.2**

**Exact Results**

The next result presents the exact value of $i$-th Ramsey numbers for many pairs of graphs. We note that for the graph $G$, the independence number $\beta(G)$ is the maximum cardinality among the independent sets of vertices of $G$. For the disjoint graphs $G$ and $H$ we denote the join of the graphs by $G + H$ where
(i) \( V(G+H) = V(G) \cup V(H) \) and
(ii) \( E(G+H) = E(G) \cup E(H) \cup \{ uv \mid u \in V(G) \text{ and } v \in V(H) \} \).

Finally, if \( U \subseteq V(G) \) then the subgraph induced by \( U \) is denoted \( <U>_G \) where

(i) \( V(<U>_G) = U \) and
(ii) vertices \( u \) and \( v \) from \( U \) are adjacent in \( <U>_G \) if and only if \( uv \in E(G) \).

That \( H \) is an induced subgraph of \( G \) is denoted by \( H < G \). Note that if \( H < G \) and \( G = \bigoplus_{j=1}^{k} F_j \), then
\[
H = \bigoplus_{j=1}^{k} <V(H)>_{F_j}.
\]

**Theorem 2.3**

Let \( n \in \mathbb{Z}^+ \), let \( G \) be a graph, and let \( I = i(K(1,n), G) \).

Then \( r_i(K(1,n), G) = \chi(G) \).

**Proof:** For each graph \( H \) define
\[
M(H) = \beta(H)(n-1)(\chi(H)-1) \quad \text{and} \quad I(H) = \beta(H)(M(H)+1).
\]

It suffices to show \( r_i(K(1,n), G) = \chi(G) \) for every positive integer \( i \geq I(G) \). This fact is established by induction on \( \chi(G) = p \).
If \( \chi(G) = p = 1 \), then \( G = K_s \) for some \( s \in Z^+ \).

By Theorem 2.1 a, \( r_i(K(l,n),G) = \lceil s/i \rceil \) whenever \( i \in Z^+ \).

In particular \( r_i(K(l,n),G) = 1 = \chi(G) \) for every \( i \geq s \).

Since \( I(G) = I(K_s) = s \), then the induction is anchored.

We assume \( p \geq 2 \) and that \( \chi(G) = p \). Also assume that whenever \( H \) is a graph with \( 1 \leq \chi(H) \leq p - 1 \) it follows that \( r_i(K(l,n),H) = \chi(H) \) for every positive integer \( i \geq I(H) \). It is shown that \( r_i(K(l,n),G) = \chi(G) \) whenever \( i \geq I(G) \).

Let \( i \) be an integer for which \( i \geq I(G) \). Since \( p \geq 2 \), consider \( K_{[p-1]}(i) = F_1 \otimes F_2 \) where \( F_1 \) is empty. Since \( |E(F_1)| < |E(K(l,n))| \) then \( K(l,n) \not\subset F_1 \).

Since \( \chi(F_2) < \chi(G) \) then \( G \not\subset F_2 \). Hence \( r_i(K(l,n),G) > p - 1 = \chi(G) - 1 \).

To show \( r_i(K(l,n),G) \leq \chi(G) \), let \( K_{\chi(G)}(i) = F_1 \otimes F_2 \) with \( K(l,n) \not\subset F_1 \). It suffices to show \( G \subset F_2 \). To this end let a \( \chi(G) \)-coloring of \( G \) be specified, and let the resulting color classes be \( V_1, V_2, \ldots, V_{\chi(G)} \). Let \( H = G - V_1 \). Then \( G \subset H + \beta(G)K_l \) so it suffices to show \( H + \beta(G)K_l \subset F_2 \) to conclude \( G \subset F_2 \). Let the partite sets of \( K_{\chi(G)}(i) \) be \( U_1, U_2, \ldots, U_{\chi(G)} \) and consider the factorization

\[
K_{[\chi(G)-1]}(i) = K_{\chi(G)}(i) - U_1 = (F_1 - U_1) \oplus (F_2 - U_1).
\]

We have \( \chi(H) = \chi(G) - 1 \geq 1 \) and \( H \subset G \) so \( \beta(H) \leq \beta(G) \). It follows that \( I(H) \leq I(G) \leq i \).
Moreover by the inductive hypothesis \( r_1(K(l,n), H) = \chi(H) = \chi(G) - 1 \). Since \( r_1(K(l,n), H) = \chi(G) - 1 \) and since \( K(l,n) \not\subseteq F_1 - U_1 \), then it follows that \( H \subseteq F_2 - U_1 \).

We now show \( H + \beta(G)K_1 \subseteq F_2 \) to conclude the proof.

Recall that \( |U_1| = i \geq I(G) = \beta(G)(M(G) + 1) \). Hence we may partition \( U_1 \) into \( M(G) + 1 \) sets \( W_1, W_2, \ldots, W_{M(G)+1} \) where \( |W_j| \geq \beta(G) \) for every \( j \) with \( 1 \leq j \leq M(G) + 1 \).

If for each \( j \) (with \( 1 \leq j \leq M(G) + 1 \)) there exist vertices \( w_j \in W_j \) and \( u_j \in V(H) \) for which \( w_ju_j \in E(F_1) \), then the average degree in \( F_1 \) of a vertex of \( H \) is at least

\[
\frac{M(G) + 1}{|V(H)|} \geq \frac{M(G) + 1}{\beta(H) \cdot \chi(H)} \geq \frac{M(G) + 1}{\beta(G) (\chi(G) - 1)} = \frac{\beta(G)(n-1)(\chi(G) - 1) + 1}{\beta(G)(\chi(G) - 1)} > n - 1.
\]

The above assumption is thus seen to imply \( K(l,n) \subseteq F_1 \).

This contradiction implies that there is some \( j \) with \( 1 \leq j \leq M(G) + 1 \) for which every vertex of \( W_j \) is adjacent in \( F_2 \) with every vertex of \( H \), which in turn implies that \( H + \beta(G)K_1 \subseteq F_2 \), concluding the proof.

The following corollary follows implicitly from the proof of Theorem 2.5.
Corollary 2.3 A

Let $n$ be a positive integer, and let $G$ be a graph.

Then $i(K(l,n),G) \leq \left\lceil \chi(G) \right\rceil^2 (n - 1) (\chi(G) - 1) + \beta(G)$.

The bound from Corollary 2.3 A on $i(K(l,n),G)$ is sharp in the cases (i) $G = \overline{K_s}$ for some $s \in \mathbb{Z}^+$ and (ii) $n = 1$ and $G = K_p$ for some $p \in \mathbb{Z}^+$. In general the bound is not particularly good. For example, it will be shown in Chapter III that $i(K(1,3),K_4 - e) = 4$. However,

$$\left\lceil \beta(K_4 - e) \right\rceil^2 (3-1) (\chi(K_4 - e) - 1) + \beta(K_4 - e) = 4 \cdot 2 \cdot 2 + 2 = 18.$$

We note that $\chi(K(1,n)) = 2$ for $n \in \mathbb{Z}^+$ and that $r(2,m) = m$ for $m \in \mathbb{Z}^+$. Hence Theorem 2.3 states that $r_1(K(1,n),G) = r(\chi(K(1,n)),\chi(G))$ which supports the conjecture of the last section. We now begin the task of establishing the final main result of Chapter II.

For the graph $G$, the edge-chromatic number is denoted by $\chi_1(G)$ and the maximum degree is denoted by $\Delta(G)$. The following five results will be used in the proof of the main theorem.

Theorem A (Vizing [35])

For every graph $G$ it follows that

$$\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1.$$
**Theorem B** (Laskar and Hare [24]; Himelwright and Williamson, unpublished)

Let $p$ and $i$ be positive integers. Then

$$\chi_1(K_p(i)) = \Delta(K_p(i))$$

if and only if $p \cdot i$ is even.

**Theorem C** (Petersen [30])

Let $p$ and $i$ be positive integers with $p \geq 2$. Then $K_p(i)$ is 2-factorable if and only if $(p-1)i$ is even.

**Lemma 2.4 (a)**

Let $n_1, n_2, \ldots, n_k$ be $k$ ($\geq 1$) positive integers, let

$$S = \sum_{j=1}^{k} (n_j - 1),$$

and let $G$ be a graph such that

$$\chi_1(G) \leq S.$$

Then there is a factorization $G = \bigoplus_{j=1}^{k} H_j$ with

$$\Delta(H_j) \leq n_j - 1$$

for each $j$ with $1 \leq j \leq k$.

**Proof:** Since $\chi_1(G) \leq S$, there exist graphs $F_1, F_2, \ldots, F_S$ (obtained from the $\chi_1(G)$ edge color classes of some $\chi_1(G)$-edge coloring of $G$) such that

(i) $V(F_j) = V(G)$ for each $j$ with $1 \leq j \leq S$,

(ii) $\Delta(F_j) \leq 1$ for each $j$ with $1 \leq j \leq S$, and

(iii) $G = \bigoplus_{j=1}^{S} F_j$.
Partition \{F_1, F_2, \ldots, F_g\} into sets \(U_1, U_2, \ldots, U_k\) such that \(|U_j| = n_j - 1\) for every \(j\) with \(1 \leq j \leq k\).

For each \(j\) (with \(1 \leq j \leq k\)) define \(H_j = \bigoplus_{F \subseteq U_j} F\). It follows that \(\Delta(H_j) \leq |U_j| = n_j - 1\) for every \(j\) with \(1 \leq j \leq k\). Moreover, \(G = \bigoplus_{j=1}^k F_j = \bigoplus_{j=1}^k H_j\).

**Lemma 2.4 (b)**

Let \(i, k, n_1, n_2, \ldots, n_k\) be positive integers. Then

\[
\sum_{j=1}^{k} (n_j - 1) \geq \left(\frac{\left(\sum_{j=1}^{k} (n_j - 1)\right)}{i}\right) - 1 \cdot i
\]

with equality if and only if \(i|\sum_{j=1}^{k} (n_j - 1)\).

**Proof:** Let \(S = \sum_{j=1}^{k} (n_j - 1)\), and let \(l + S = t \cdot i + m\) with \(0 \leq m \leq i - 1\). We proceed by cases on the value of \(m\) to show that \(S \geq \left(\frac{(l + S)}{i}\right) - 1\) with equality if and only if \(i|S\).

**Case 0:** \(m = 0\).

We have \(1 + S = t \cdot i\). It follows that

\[
S \geq 1 + S - i \quad \text{(with equality if and only if } i = 1\)
\]

\[
= t \cdot i - i
\]

\[
= (t - 1) \cdot i
\]

\[
= \left(\frac{(l + S)}{i}\right) - 1 \cdot i.
\]
Equality holds if and only if \( i = 1 \). That is to say, equality holds if and only if \( i \mid S \) and \( i \mid S + 1 \). Since \( i \mid S + 1 \) then equality holds if and only if \( i \mid S \).

**Case 1: \( m = 1 \).**

We have \( 1 + S = t \cdot i + 1 \) so that \( S = t \cdot i \). Since \( i \mid S \), it must be shown that equality holds. Note that

\[
S = t \cdot i \\
= ((1/i + t) - 1) \cdot i \\
= [[[1 + t/i] - 1] - 1] \cdot i.
\]

**Case 2: \( 2 \leq m \leq i - 1 \).**

We have \( 1 + S = t \cdot i + m \) with \( 2 \leq m \leq i - 1 \) which implies \( S = t \cdot i + m' \) with \( 1 \leq m' \leq i - 2 \). It follows that \( i \mid S \) so that strict inequality must be shown. Note that

\[
S = t \cdot i + (m - 1) \\
> t \cdot i \\
= (t + m/i - 1) \cdot i \\
= (t + (1 + S)/i - 1) \cdot i.
\]

For the positive integers \( i, k, \) and \( n_1, n_2, \ldots, n_k \) let \( t \) be the number of \( n_j \) that are even, let \( k \)

\[
S = \sum_{j=1}^{k} (n_j - 1), \text{ and define}
\]
Define $r_i(n_1, n_2, \ldots, n_k/*/)\) to be the least positive integer $p$ such that whenever $K_p(i) = \bigoplus_{j=1}^{k} F_j$ then there is some $j$ with $1 \leq j \leq k$ for which the star graph $K(1,n_j) \subset F_j$. A complete evaluation of this Ramsey number for stars is now presented.

**Theorem 2.4**

Let $i, k$, and $n_1, n_2, \ldots, n_k$ be positive integers. Let $t$ be the number of $n_j$ that are even, let $S = \sum_{j=1}^{k} (n_j - 1)$, and define $\theta$ as in (2.1).

Then $r_i(n_1, n_2, \ldots, n_k/*/) = [(1 + S)/i] + \theta$.

**Proof:** Let $p = [(1 + S)/i]$. It is first shown that $r_i(n_1, n_2, \ldots, n_k/*/) \leq p + 1$.

Let $K_{[p+1]}(i) = \bigoplus_{j=1}^{k} F_j$ and assume that $K(1,n_j) \not\subset F_j$ for every $j$ with $1 \leq j \leq k$. Then $\Delta(F_j) \leq n_j - 1$ for every $j$ with $1 \leq j \leq k$. For every $j$ with $1 \leq j \leq k$, the average value of the degree of a vertex of $F_j$ does not exceed $S/k$. However, this value is known to be
given by

\[ p \cdot \frac{i}{k} = \left( \frac{(1 + S)/i}{i/k} \right) \cdot \frac{i}{k} \geq \frac{1}{k} + \frac{S}{k}. \]

This contradiction implies \( K(1, n_j) \subseteq F_j \) for some \( j \) with \( 1 \leq j \leq k \). Hence, in general,

\[ r_1(n_1, n_2, \ldots, n_{k/\ast}) \leq p + 1. \]

Now in particular, let (1), (2), and (3) of (2.1) be satisfied. It is shown that \( r_1(n_1, n_2, \ldots, n_{k/\ast}) \leq p \).

Let \( K_p(i) = \bigoplus F_j \) and assume \( K(1, n_j) \subseteq F_j \) for each \( j \) with \( 1 \leq j \leq k \). It follows that

\[ |E(K_p(i))| = \sum_{j=1}^{k} |E(F_j)| \]

\[ \leq \frac{1}{2} \sum_{j=1}^{k} |V(F_j)| \cdot \Delta(F_j) \tag{2.2} \]

\[ \leq \frac{D \cdot i}{2} \sum_{j=1}^{k} (n_j - 1) \]

\[ = \frac{D \cdot i}{2} (p - 1)i \quad \text{(by Lemma 2.4 (b) and (2))} \]

\[ = \left( \frac{p}{2} \right)^2 \]

\[ = |E(K_p(i))|. \]

Hence the inequalities of (2.2) are equalities. In particular, each \( F_j \) must be regular of degree \( n_j - 1 \).
By (1) $p \cdot i$ is odd and by (3) we may assume $n_1$ is even. Hence the assumption forces the conclusion that $F_1$ is a graph of odd order which is regular of odd degree. This contradiction implies $K(l,n_j) \subseteq F_j$ for some $j$ with $1 \leq j \leq k$.

We have established that $r_i(n_1, n_2, \ldots, n_k/\times/) \leq p+\theta$. To show that $r_i(n_1, n_2, \ldots, n_k) > p - 1 + \theta$ we proceed by cases on the value of $\theta$.

**Case 0: $\theta = 0$**

It follows that (1) $p \cdot i$ is odd, (2) $i | S$, and (3) $t$ is positive. From (1) both $p$ and $i$ are odd. If $p = 1$ the claim surely is true; hence assume $p > 3$.

It suffices to show the existence of a factorization

$$K(p-1)(i) = \bigoplus_{j=1}^{k} F_j$$

where $\Delta(F_j) \leq n_j - 1$ for each $j$ with $1 \leq j \leq k$. By an application of Lemma 2.4 (a) it suffices to show $\chi_1(K(p-1)(i)) \leq S$. However, we have

$$S = (p - 1) \cdot i \quad (\text{by Lemma 2.4 (b) and (2)})$$

$$\geq (p - 2) \cdot i + 1$$

$$= \Delta(K(p-1)(i)) + 1$$

$$\geq \chi_1(K(p-1)(i)) \quad (\text{by Theorem A}).$$

**Case 1: $\theta = 1$**

It follows that either (1)' $p \cdot i$ is even, or (2)' $i / S$, or (3)' $t = 0$. We show the existence of a
factorization $K_p(i) = \bigoplus_{j=1}^{k} F_j$ with $\Delta(F_j) \leq n_j - 1$ for every $j$ such $1 \leq j \leq k$.

If $p \cdot i$ is even then

$$x_1(K_p(i)) = \Delta(K_p(i)) \quad \text{(by Theorem B)}$$
$$= (p - 1) \cdot i$$
$$\leq S \quad \text{(by Lemma 2.4 (b))}.$$ 

An application of Lemma 2.4 (a) demonstrates the existence of the desired factorization. We may assume $p \cdot i$ is odd.

If $i \nmid S$ then by Lemma 2.4 (b) $S > (p - 1) \cdot i$. It follows that

$$S \geq (p - 1) \cdot i + 1$$
$$= \Delta(K_p(i)) + 1$$
$$\geq x_1(K_p(i)).$$

Again the desired factorization exists as a consequence of Lemma 2.4 (a), and we assume $i \mid S$.

In case $t = 0$ we proceed in a slightly different manner. Since $p \cdot i$ is odd then $K_p(i)$ is regular of the even degree $(p - 1) \cdot i$. There is nothing to prove if $p = 1$ so we assume $p \geq 3$. From Theorem C we conclude that $K_p(i)$ is 2-factorable into $(p - 1) \cdot i/2$ 2-factors. Since $i \mid S$ then $(p - 1) \cdot i/2 = S/2$, by Lemma 2.4 (b).
Now $t = 0$ so every $n_j$ is odd. Hence we can partition this set of 2-factors into the classes
$U_1, U_2, \ldots, U_k$ so that $|U_j| = (n_j - 1)/2$ for every $j$ with $1 \leq j \leq k$. Define the graph $F_j$ to be the "edge sum" of the 2-factors in $U_j$ for each $j$ with $1 \leq j \leq k$. It follows that each $F_j$ is regular of degree $2|U_j| = n_j - 1$ and that $K_p(i) = \bigoplus_{j=1}^{k} F_j$.

We have established that

$$r_i(n_1, n_2, \ldots, n_k) > p + \theta - 1$$

which concludes the proof of the theorem.

The cases $i = 1$ and $i = 2$ are stated explicitly.

**Corollary 2.4 A** (Burr and Roberts [6])

Let $k$ and $n_1, n_2, \ldots, n_k$ be positive integers, and let $t$ be the number of $n_j$ that are even. Then

$$r(n_1, n_2, \ldots, n_k) = \sum_{j=1}^{k} n_j - k + 1 + \begin{cases} 0 & \text{if } t \text{ is even and positive} \\ 1 & \text{otherwise} \end{cases}$$

**Proof:** Conditions (1), (2), and (3) of (2.1) are true if and only if $t$ is even and positive in the case $i = 1$.

**Corollary 2.4 B** (the octahedral case)

Let $k$ and $n_1, n_2, \ldots, n_k$ be positive integers.
Then
\[ r_2(n_1, n_2, \ldots, n_k/*/) = \left\{ \left(1 + \sum_{j=1}^{k} (n_j - 1)\right)/2 \right\} + 1. \]

**Proof:** By the definition of \( \Theta \), it follows that \( \Theta = 1 \) whenever \( i \) is even \( \blacksquare \).

For the positive integers \( k \) and \( n_1, n_2, \ldots, n_k \) define
\[ i(n_1, n_2, \ldots, n_k/*/) = i(K(1,n_1), K(1,n_2), \ldots, K(1,n_k)). \]

**Corollary 2.4 C**

Let \( k \) and \( n_1, n_2, \ldots, n_k \) be positive integers, and let \( I = i(n_1, n_2, \ldots, n_k/*/) \). Then
\[ r_i(n_1, n_2, \ldots, n_k/*/) = 2. \]

**Proof:** By the definition of \( \Theta \), it follows that \( \Theta = 1 \) if \( i \geq S + 1 \) whence \( r_i(n_1, n_2, \ldots, n_k/*/) \leq 2 \). Of course \( r_i(n_1, n_2, \ldots, n_k/*/) > 1 \) for every \( i \in \mathbb{Z}^+ \) \( \blacksquare \).

**Corollary 2.4 D**

Let \( k \) and \( n_1, n_2, \ldots, n_k \) be positive integers. Then
\[ i(n_1, n_2, \ldots, n_k/*/) = \sum_{j=1}^{k} (n_j - 1) + 1. \]

**Proof:** Let \( S = \sum_{j=1}^{k} (n_j - 1) \). If \( S = 0 \) the result is evident, so take \( S \) to be positive. By Theorem 2.4.
\( r_S(n_1, n_2, \ldots, n_k/\ast/) = \left\lceil \frac{S + 1}{S} \right\rceil + 1 = 3 \) since condition (1) of (2.1) is false. By anti-monotonicity and Corollary 2.4 C it follows that
\[
i(n_1, n_2, \ldots, n_k/\ast/) > S.
\]

However, \( r_{S+1}(n_1, n_2, \ldots, n_k/\ast/) = \left\lceil \frac{S + 1}{S + 1} \right\rceil + 1 = 2 \) since condition (2) of (2.1) fails to be true. Then by definition and Corollary 2.4 C we have \( i(n_1, n_2, \ldots, n_k/\ast/) = S + 1 \).

It seems reasonable to assert that examples of particular ramsey numbers with respect to the graphs \( K_p(i) \) are needed before much more progress can be made. In order to provide evidence for conjectures and possible base cases for inductive arguments, over 90% of the "small" \( i \)-th ramsey numbers and over 90% of the "small" ramsey indices are computed in Chapter III.

Chapter II is concluded with some problems.

**Problem 2.1**

Support or disprove the conjecture of Section 2.1 by finding \( r_i(C_5, C_5) \) for every \( i \in \mathbb{Z}^+ \).

**Problem 2.2**

Extend Theorem 2.3 so that the star is replaced by an arbitrary tree. Extend this new result so the tree
is replaced by a forest. Make a final extension which solves Erdös' conjecture given in Section 2.1.

**Problem 2.3**

The next most accessible formulas would seem to come from paths [19], matchings [15], and stars and matchings [16]. Find the Ramsey numbers of these families of graphs with respect to the symmetric complete $p$-partite graphs. (The graph $nK_2$ is a matching.)
CHAPTER III

MOST SMALL i-TH RAMSEY NUMBERS

In order to establish information for future study, most of the "small" i-th ramsey numbers are determined in Chapter III.

Section 3.1

Preliminary Results

The motivation for the work of this chapter is given by Harary [21, p. 11].

"It is useful to obtain the ramsey numbers for small graphs in order to have data for making conjectures. Also, as Burr [5] observed, these small cases often provide the starting point for inductive proofs, but must be proved independently."

As an extension of a definition by Chvátal and Harary [12], the small i-th ramsey numbers are those ramsey numbers \( r_i(F,G) \) for which \( i \in \mathbb{Z}^+ \) and \( F \) and \( G \) are graphs of order less than five having no isolated vertices. The small ramsey indices are those ramsey indices \( i(F,G) \) for which \( r_i(F,G) \) is a small i-th ramsey number. If \( F = G \), then \( r_i(G,G) \) is called a diagonal i-th ramsey number.
The need for the data supplied by small (1-st) ramsey numbers has generated many papers. For example Harary and Chvátal found all small (1-st) diagonal ramsey numbers in [12] and all small (1-st) non-diagonal ramsey numbers in [13]. Also, Clancy [14] has recently determined most of the (1-st) ramsey numbers \( r(F,G) \) where \( F \) and \( G \) are graphs of orders at most four and five (respectively) having no isolated vertices.

Chapter III continues the trend outlined in the preceding paragraph by finding nearly all small i-th ramsey numbers (as well as their corresponding ramsey indices). The only unsolved cases are \( r_i(C_4,K_4) \), \( r_i(K_4-e,K_4-e) \), and \( r_i(K_4-e,K_4) ; i \geq 2 \).

The graphs to be considered and their symbolic names are listed in Figure 3.1. A summary of the ramsey numbers found in this chapter is given in Figure 3.2. With three exceptions (which are each indicated by "?") the ramsey index \( i(F,G) \) is the largest value of \( i \) for which \( r_i(F,G) \) is given in Figure 3.2. The small ramsey indices are explicitly given in Figure 3.3. It may be noted that for the graphs \( G \) to be considered in this chapter \( \omega(G) = \chi(G) \). Hence it may not be surprising that for \( I = i(F,G) \), it follows that \( r_I(F,G) = r(\chi(F),\chi(G)) \) in every known case for \( I \) in Chapter III.
All Graphs of Order Less Than Five Having No Isolated Vertices

Figure 3.1
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<th>2K₂</th>
<th>P₃</th>
<th>P₄</th>
<th>C₄</th>
<th>K₁,3</th>
<th>K₃</th>
<th>K₁,3+e</th>
<th>K₄-e</th>
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Figure 3.2

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The Small Ramsey Indices $i(F, G)$

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Figure 3.3
Complete proofs of all results in this chapter are not given. All lower bounds will be established. If the proof of an upper bound is omitted, then it may be assumed that its proof is quite similar to a given proof. A brief discussion of some of the tools used in the computations of the small i-th ramsey numbers is now presented.

The general technique used to show \( r_i(F, G) \leq p \) is to assume \( K_p(i) = F \uplus F_2 \) with \( F \not\subseteq F_1 \). It is then shown that \( G \subseteq F_2 \). The argument often proceeds by cases on the value of \( \Delta(F_1) \).

Given that \( \Delta(F_1) = m \) the argument usually proceeds by subcases which depend upon the form of certain restricted partitions of integers. That is, given that \( F_1 \subseteq K_p(i) \) and \( \deg_{F_1}(u) = \Delta(F_1) = m \), then \( u \) is adjacent in \( F_1 \) with vertices \( a_1, a_2, \ldots, a_m \) where (i) no more than \( i \) of the adjacencies can occur in the same given partite set of \( K_p(i) \) and (ii) no more than \( p - 1 \) of the partite sets can contain vertices adjacent to \( u \) in \( F_1 \). This is entirely analogous to the restricted partition \( m = \sum_{j=1}^{k} s_j \) where each \( s_j \) is an integer with (i) \( 0 \leq s_j \leq i \) and (ii) \( k \leq p - 1 \).

For example when it is written that \( \Delta(F_1) = 6 = 4 + 2 \), what is meant is that for \( \deg_{F_1}(u) = 6 \), then there is a
partite set containing 4 vertices adjacent with \( u \) and there is a different partite set containing 2 vertices adjacent with \( u \).

In general, for \( K_p(i) = F_1 \oplus F_2 \) and \( \Delta(F_1) = m \), there are as many cases to consider (to show that \( G \subseteq F_2 \)) as there are restricted partitions of \( m \) into \( p - 1 \) or fewer summands of non-negative integers with each summand not exceeding \( i \) in value. It becomes helpful to have results at hand which restrict the allowable values of \( \Delta(F_1) \) (or equivalently, \( \delta(F_2) \)). Such a restriction is the purpose of the next result.

For the graph \( G \), let \( u \in V(G) \) and let \( B \subseteq V(G) \). Define the number \( \|uB\|_G \) to be the cardinality of \( \{uv \in E(G) | v \in B\} \). When there is no chance for ambiguity, \( \|uB\| \) is written for \( \|uB\|_G \). Note that \( \|uB\| = \deg_{\langle[u]UB\rangle_G}(u) \). Given that two copies of \( P_3 \) are distinct when they have unequal edge sets, then \( \left(\|uB\|/2 \right) \) counts the number of distinct copies of \( P_3 \) in \( G \) having both end vertices in \( B \) and having \( u \) for the vertex of degree two. It follows that \( \sum_{u \in A} \left(\|uB\|/2 \right) \) counts the number of distinct copies of \( P_3 \) in \( G \) with both end vertices in \( B \) and with the vertex of degree two in \( A \).
Theorem 3.1

Let \( G \) be a graph. Then \( C_4 \subseteq G \) if and only if there are sets \( A \) and \( B \) with \( A \subseteq V(G) \), \( B \subseteq V(G) \), and \( \sum (\|uB\|) > \binom{|B|}{2} \).

Proof: If \( C_4 \subseteq G \), then let \( v_1, v_2, v_3, v_4, v_1 \) denote a copy of \( C_4 \) in \( G \). Let \( A = \{u_1, u_3\} \) and let \( B = \{u_2, u_4\} \). Note that for \( u \in A \) then \( \|uB\| = 2 \). Therefore, \( \sum (\|uB\|) = 2(2) = 2 > 1 = \binom{2}{2} = \binom{|B|}{2} \).

Conversely, let \( A \subseteq V(G) \), let \( B \subseteq V(G) \), and let \( \sum (\|uB\|) > \binom{|B|}{2} \). Note by cases that \( |A| \geq 2 \) and \( |B| \geq 2 \). The goal is to show that \( C_4 \subseteq G \) by showing the existence of two distinct copies of \( P_3 \) in \( G \) which have for end vertices the elements of the same subset of \( B \) of cardinality 2. This existence follows if \( n \), the number of distinct copies of \( P_3 \) in \( G \) having both end vertices in \( B \), exceeds \( \binom{|B|}{2} \).

However, \( n \geq \sum (\|uB\|) \) which by hypothesis exceeds \( \binom{|B|}{2} \). It follows that \( C_4 \subseteq G \).

The usual application of Theorem 3.1 is in the form of the following corollary.
Corollary 3.1 A (Chvátal and Harary [13])

Let $G$ be a graph of order $p$ such that
\[ \delta(G)(\delta(G) - 1) \geq p. \]

Then $C_4 \subseteq G$.

Proof: Letting $A$ and $B$ both be $V(G)$, we note that
\[ ||uB|| \geq \delta(G) \quad \text{for every vertex } u \text{ of } G. \]
Hence,
\[
\sum_{u \in A} \left( \frac{||uB||}{2} \right) \geq \sum_{u \in V(G)} \left( \frac{\delta(G)}{2} \right)
= p\delta(G)(\delta(G) - 1)/2
\geq p^2/2 \quad \text{(by hypothesis)}
> \left( \frac{p}{2} \right) \quad \text{(necessarily } p \geq 2)
= \left( \frac{|V(G)|}{2} \right) = \left( \frac{|B|}{2} \right). \]

Occasionally the proof that $r_i(F, G) = p$ follows directly from the fact that $r_j(F', G') = p'$ for $j \leq i$, $F' \subseteq F$, $G' \subseteq G$, and $p' \leq p$. More often it is of use to know the nature of certain factorizations in the case certain subgraph relationships are avoided. A specific example of this type of knowledge is given in the next result.

Corollary 3.1 B

Let $K_2(3) = F_1 \oplus F_2$. Then $F_1 = 3K_2$ and $F_2 = C_6$ if and only if $P_3 \not\subseteq F_1$ and $C_4 \not\subseteq F_2$. 

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Proof: The necessity is trivial. Assume then that
\( K_3(2) = F_1 \oplus F_2 \) with \( P_3 \not\subset F_1 \) and \( C_4 \not\subset F_2 \). Since
\( C_4 \not\subset F_2 \), then \( \delta(F_2) \neq 3 \) by Corollary 3.1 A. Hence
\( \delta(F_2) \leq 2 \) and so \( \Delta(F_1) \geq 1 \). Since \( K(1,2) = P_3 \not\subset F_1 \)
then \( \Delta(F_1) \leq 1 \). It follows that \( \Delta(F_1) = 1 \) and
\( \delta(F_2) = 2 \). Hence \( F_2 \) is regular of degree 2 which implies every vertex of \( F_2 \) lies on a cycle. Since \( F_2 \)
is bipartite, the only possible cycles in \( F_2 \) are \( C_4 \) and \( C_6 \). Hence \( C_6 \subset F_2 \). Moreover, if the cycle \( C_6 \)
has any diagonals then \( K_3 \) or \( C_4 \) is a subgraph of \( F_2 \) which is impossible. Since \( C_6 \) spans \( F_2 \) then \( F_2 = C_6 \).
Hence \( F_1 = 3K_2 \).

The value of \( r_i(F,G) \) where \( F \) and \( G \) are stars
is computed by Theorem 2.4. All small ramsey numbers
\( r_1(F,G) \) are either computed or referenced in [13]. The
small ramsey numbers \( r_i(F,G) \) for \( i \geq 2 \) where \( F \) and
\( G \) are complete follows as a corollary to [13] and Corol-
lary 2.2 A. As noted earlier, \( r_i(K_2,G) \) is the least
integer \( p \) for which \( G \subset K_p(i) \). No specific refer-
ences to this case will be made. Figure 3.4 shows ex-
plicitly the ramsey numbers \( r_i(F,G) \) to be proved or
commented upon in Section 3.2. No comments will be made
on ramsey indices since these value follow immediately
from the computations of the corresponding ramsey numbers.
The Ramsey Numbers $r_1(F,G)$
to be Discussed in Section 3.2

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<td>2</td>
<td></td>
<td></td>
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<td>2</td>
<td>3</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$2K_2$</td>
<td>3</td>
<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
<td></td>
<td>2</td>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>

Figure 3.4

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Section 3.2

Computations

The ramsey numbers indicated in Figure 3.4 are listed in this section. While all the results have been proved, we omit tedious details here.

**Theorem 3.2 A**

For the integer $i \geq 2$, $r_i(2K_2, P_3) = 2$.

**Partial Proof:** It follows from the lower bound theorem that $r_i(2K_2, P_3) \geq r(2,2) = 2$.

**Theorem 3.2 B**

For the integer $i \geq 2$,

$$
r_i(2K_2, K_3) = r_i(2K_2, K(1,3)+e) = r_i(2K_2, K_4-e) =
$$

$$
r_i(P_3, K_3) = r_i(P_3, K(1,3)+e) = r_i(P_3, K_4-e) = 3.$$

**Partial Proof:** It follows from the lower bound theorem that each of the six ramsey numbers of this result are bounded below by $r(2,3) = 3$.

**Theorem 3.2 C**

$$
r_2(2K_2, 2K_2) = r_2(2K_2, P_4) = r_2(2K_2, C_4) =
$$

$$
r_2(2K_2, K(1,3)) = r_2(P_3, P_4) = r_2(P_4, P_4) = 3.$$
Partial Proof: The factorizations

\[ K_2(2) = \bigoplus_{j=1}^{2} (P_3 \cup K_1)(j) \]

and

\[ K_2(2) = \bigoplus_{j=1}^{2} (2K_2)(j) \]

establish 3 as a lower bound for each of the six Ramsey numbers of this theorem.

Theorem 3.2 D

\[ r_i(P_3, P_3) = 3 \text{ for } i = 1 \text{ or } 2. \]

Proof: By anti-montonicity and [12], \( r_2(P_3, P_3) \leq r(P_3, P_3) = 3. \) The factorization \( K_2(2) = \bigoplus_{j=1}^{2} (2K_2)(j) \)
establishes the inequality \( 2 < r_2(P_3, P_3) \).

Theorem 3.2 E

For the integer \( i \geq 2, \)

\[ r_i(2K_2, K_4) = r_i(P_3, K_4) = 4. \]

Partial Proof: Both of the Ramsey numbers of this theorem are bounded below by \( r(2, 4) = 4 \).

Theorem 3.2 F

\[ r_2(P_4, C_4) = r_2(P_4, K(1, 3)) = r_2(P_4, K_3) = \]

\[ r_2(P_4, K(1, 3)+e) = r_2(C_4, C_4) = r_2(C_4, K(1, 3)) = \]

\[ r_2(C_4, K_3) = r_2(C_4, K(1, 3)+e) = 4. \]
Proof: That each of the eight octahedral ramsey numbers of this theorem exceeds 3 follows from the factorization

$$K_3(2) = 2K_3 \oplus C_6.$$ 

To show that $r_2(P_4, C_4) = r_2(P_4, K(1,3)) = r_2(P_4, K_3) = r_2(P_4, K(1,3)+e) = 4$, assume $K_4(2) = F_1 \oplus F_2$ with $P_4 \not\subset F_1$. It suffices to show $K_4-e \subset F_2$. (The stronger consequence of this fact will be recorded later.) We show $K_4-e \subset F_2$ by taking cases on the value of $\Delta(F_1)$.

If $\Delta(F_1) \leq 1$ it is easily checked that $K_4-e \subset F_2$. Hence assume $\Delta(F_1) \geq 2$. Let $V(K_4(2)) = \{u_1, u_2, a_1, a_2, b_1, b_2, c_1, c_2\}$ with vertices of the same partite set receiving the same alphabetic label. Let $\deg_{F_1}(u_1) \geq 2$. The applicable restricted partitions of 2 are $2 = 2 + 0$ and $2 = 1 + 1$.

If we consider $2 = 2 + 0$, then without loss of generality let $a_1$ and $a_2$ be adjacent with $u_1$ in $F_1$. Since $P_4 \not\subset F_1$, then $u_2, b_1, b_2, c_1$, and $c_2$ are all adjacent with both $a_1$ and $a_2$ in $F_1$. If $u_2$ is adjacent with any of $b_1, b_2, c_1$, or $c_2$ in $F_2$ then $K_4-e \subset F_2$. Hence, we assume all of these adjacencies occur in $F_1$. The path $P_3: b_1, u_2, b_2$ is contained in $F_1$. Since $P_4 \not\subset F_1$ then $b_2c_1 \in E(F_2)$ whence $\langle\{a_1, a_2, b_2, c_1\}\rangle F_2 = K_4-e$. Hence $K_4-e \subset F_2$ in the case $2 = 2 + 0$. 

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To consider the case $2 = 1 + 1$, let $u_1$ be adjacent in $F_1$ with $a_1$ and $c_2$. Necessarily $a_1$ and $c_2$ are both adjacent with $u_2$, $b_1$, and $b_2$ in $F_2$ since $F_4 \not\subseteq F_1$. If either $b_1$ or $b_2$ is adjacent with $u_2$ in $F_2$, then $K_4-e \subseteq F_2$. We assume both of these adjacencies occur in $F_1$. Hence, there is a $2+0$ partition of $2$ in $F_1$ which returns us to the prior case, implying $K_4-e \subseteq F_2$. This establishes the value of the first four Ramsey numbers of this theorem.

We now show $r_2(C_4, K(1,3)) = r_2(C_4, K_3) = r_2(C_4, K(1,3)+e) = 4$. It suffices to show $r_2(C_4, K(1,3)+e) \leq 4$. To this end let $K_4(2) = F_1 \oplus F_2$, let $V(K_4(2)) = \{u_1, u_2, a_1, a_2, b_1, b_2, c_1, c_2\}$ as before, and let $K(1,3)+e \not\subseteq F_2$. We show $C_4 \subseteq F_1$.

If $K_3 \subseteq F_2$ then without loss of generality assume $K_3 = \langle\{u_1, a_1, c_1\}\rangle_{F_2}$. Since $K(1,3)+e \not\subseteq F_2$ then the edges $u_1b_1, u_1b_2, a_1b_1$, and $a_1b_2$ are all edges of $F_1$ whence $C_4 \subseteq F_1$. We thus assume $K_3 \not\subseteq F_2$. Now for $u \in V(K_4(2))$ it follows that

$$\deg(u) = \deg_{F_1}(u) + \deg_{F_2}(u) = 6.$$ 

Hence $\delta(F_1) + \Delta(F_2) = 6$, so that if $\Delta(F_2) \leq 2$ then $\delta(F_1) \geq 4$. This implies $C_4 \subseteq F_1$ by Corollary 3.1 A. Hence we also assume $\Delta(F_2) \geq 3$. There are four cases: we may have $\Delta(F_2) \geq 4$ with restricted partitions.
4 = 2 + 2 and 4 = 2 + 1 + 1 or we may have \( \Delta(F_2) = 3 \) with restricted partitions 3 = 2 + 1 and 3 = 1 + 1 + 1.

If \( \Delta(F_2) \geq 4 \) and the restricted partition is 4 = 2 + 2, relabel \( V(K_4(2)) \) so that \( b_1 \) and \( c_1 \) form a partition set as do \( b_2 \) and \( c_2 \). No other labels are changed. If \( \Delta(F_2) \geq 4 \) and the restricted partition is 4 = 2 + 1 + 1 do not relabel. Either way, let \( u_1 \) be adjacent in \( F_2 \) with \( a_1, a_2, b_1, \) and \( c_1 \). Since \( K_3 \not\subseteq F_2 \) then under both of the two possible labelings it follows that \( c_1a_1, c_1a_2, b_1a_1, \) and \( b_1a_2 \) are edges of \( F_1 \). Hence \( C_4 \subseteq F_1 \).

If \( \Delta(F_2) = 3 \) and the restricted partition is 3 = 1 + 1 + 1, let \( u_1 \) be adjacent in \( F_2 \) with \( a_2, b_2, \) and \( c_2 \). Since \( \Delta(F_1) = 3 \) then \( u_1 \) is adjacent in \( F_1 \) with \( a_1, b_1, \) and \( c_1 \). Since \( K_3 \not\subseteq F_2 \) then \( (a_2, b_2, c_2)_{F_1} = K_3 \). Now if any two of \( u_2a_2, u_2c_2, \) or \( u_2b_2 \) are edges of \( F_1 \) then \( C_4 \subseteq F_1 \). Hence assume at least two of these edges are in \( F_2 \). It follows that \( \deg_{F_2}(u_2) \geq 2 \). Since \( \Delta(F_2) = 3 \) then at least two of the edges \( u_2a_1, u_2b_1, \) and \( u_2c_1 \) are in \( F_1 \) whence \( C_4 \subseteq F_1 \).

Finally if \( \Delta(F_2) = 3 \) and the pertinent restricted partition is 3 = 2 + 1, let \( u_1 \) be adjacent in \( F_2 \) with \( a_1, a_2, \) and \( c_2 \). Since \( \Delta(F_1) = 3 \) then \( u_1 \) is adjacent in \( F_1 \) with \( c_1, b_1, \) and \( b_2 \). Since \( K_3 \not\subseteq F_2 \)...
then \( \langle \{a_1, a_2, c_2\} \rangle_{F_1} = P_3 \). Now if both \( u_2a_1 \) and \( u_2a_2 \) are edges of \( F_1 \) then \( C_4 \subset F_1 \). We therefore assume that \( u_2a_2 \), say, is an edge of \( F_2 \). If any two of \( a_2c_1, a_2b_2 \), or \( a_2b_1 \) are edges of \( F_1 \) then \( C_4 \subset F_1 \). However, \( \Delta(F_2) = 3 \), so at most one of \( a_2c_1, a_2b_2 \), and \( a_2b_1 \) is an edge of \( F_2 \). Hence \( C_4 \subset F_1 \).

The last three Ramsey numbers of this theorem have thus been established. It only remains to show

\[ r_2(C_4, C_4) \leq 4. \]

We use the same labeling of \( V(K_4(2)) \) as before, and let \( K_4(2) = F_1 \oplus F_2 \) with \( C_4 \not\subset F_1 \). It follows from Corollary 3.1 A that if \( \delta(F_2) \geq 4 \) then \( C_4 \subset F_2 \). We therefore assume \( \delta(F_2) \leq 3 \) whence \( \Delta(F_1) \geq 3 \).

If \( \Delta(F_1) \geq 4 \) and the restricted partition is

\[ 4 = 2 + 2, \]

then we may assume \( a_1, a_2, c_1 \), and \( c_2 \) are adjacent in \( F_1 \) with \( u_1 \). Since \( C_4 \not\subset F_1 \), at most one of \( a_1, a_2, c_1 \), and \( c_2 \) is adjacent in \( F_1 \) with \( u_2 \). We assume that \( a_1, a_2 \), and \( c_2 \) are adjacent in \( F_2 \) with \( u_2 \). Since \( C_4 \not\subset F_1 \) then at most one of \( a_1, a_2 \), and \( c_2 \) is adjacent with \( b_1 \) in \( F_1 \). Hence at least two of \( a_1, a_2 \), and \( c_2 \) are adjacent with \( b_1 \) in \( F_2 \). This implies \( C_4 \subset F_2 \).

If \( \Delta(F_1) \geq 4 \) and the restricted partition is

\[ 4 = 2 + 1 + 1 \]

then we are able to assume that \( a_1, a_2, c_2 \), and \( b_2 \) are adjacent with \( u_1 \) in \( F_1 \). Hence,
as above, at least three of \( a_1, a_2, c_2, \) and \( b_2 \) are adjacent in \( F_2 \) with \( u_2 \). If both \( a_1 \) and \( a_2 \) are adjacent with \( u_2 \) in \( F_2 \), then we can conclude that \( C_4 \subset F_1 \) in a manner entirely analougous to that of the preceding case. Hence we may assume that \( a_1 \), say, is adjacent with \( u_2 \) in \( F_1 \) and that \[ \{a_2u_2, b_2u_2, c_2u_2\} \subset E(F_2). \]

Not both \( a_2b_2 \) and \( a_2c_2 \) are edges of \( F_1 \) since \( C_4 \not\subset F_1 \). If both \( a_2b_2 \) and \( a_2c_2 \) are edges of \( F_2 \) then \( C_4 \subset F_2 \). Hence we assume \( a_2c_2 \in E(F_1) \) and \( a_2b_2 \in E(F_2) \). Now \( P_4 \subset \{\{b_2, u_1, a_2, c_2\}\}F_1 \) and \( C_4 \not\subset F_1 \) so it follows that \( b_2c_2 \in E(F_2) \) whence \( C_4 \subset F_2 \).

If \( \Delta(F_1) = 3 \) and the restricted partition is \( 3 = 1 + 1 + 1 \), let \( u_1 \) be adjacent with \( a_1, b_1, \) and \( c_1 \) in \( F_1 \). Since \( C_4 \not\subset F_1 \) then we may assume \( u_2a_1 \) and \( u_2b_1 \) are edges of \( F_2 \). Moreover, not both \( c_1a_1 \) and \( c_1b_1 \) are edges of \( F_1 \). If both \( c_1a_1 \) and \( c_1b_1 \) are edges of \( F_2 \), then \( C_4 \subset F_2 \). We may assume that \( c_1a_1 \in E(F_1) \) and \( c_1b_1 \in E(F_2) \). Now \( P_4 \subset \{\{b_1, u_1, c_1, a_1\}\}F_1 \) and \( C_4 \not\subset F_1 \) implies \( b_1a_1 \in E(F_2) \). If \( u_2c_1 \in E(F_2) \) then \( C_4 \subset \{\{u_2, c_1, b_1, a_1\}\}F_2 \). Hence assume \( u_2c_1 \in E(F_1) \).

Since \( \Delta(F_1) = 3 \) then \( u_1a_2, u_1b_2, c_1a_2, \) and \( c_1b_2 \) are all edges of \( F_2 \) whence \( C_4 \subset F_2 \).
Finally, if \( \Delta(F_1) = 3 \) and the corresponding restricted partition is \( 3 = 2 + 1 \), let \( a_1, a_2, \) and \( c_1 \) be adjacent with \( u_1 \) in \( F_1 \). We may assume \( u_2c_1 \) is an edge of \( F_2 \). (If \( u_2c_1 \notin E(F_1) \), then \( u_2a_1 \) and \( u_2a_2 \) are edges of \( F_2 \) since \( C_4 \notin F_1 \). Hence, exactly one of \( b_1a_1 \) and \( b_1a_2 \) is in \( F_1 \). We assume that \( b_1a_1 \notin E(F_1) \). Similarly we assume \( c_1a_1 \notin E(F_1) \). Since \( \Delta(F_1) = 3 \) we have \( C_4 \subseteq \langle [u_1, b_1, b_2, c_1] \rangle_{F_2} \). If both \( u_2a_1 \) and \( u_2a_2 \) are in \( F_2 \) then since \( C_4 \notin F_2 \) at least two of \( b_1a_1, b_1a_2, \) and \( b_1c_1 \) are in \( F_2 \) whence \( C_4 \subseteq F_2 \). Hence we assume not both \( u_2a_1 \) and \( u_2a_2 \) are in \( F_2 \). Since \( C_4 \notin F_1 \) then not both \( u_2a_1 \) and \( u_2a_2 \) are edges in \( F_1 \). We may thus assume that \( u_2a_1 \notin E(F_1) \) and \( u_2a_2 \notin E(F_2) \). Now \( \Delta(F_1) = 3 \) so at least two of \( a_1b_1, a_1b_2, \) and \( a_1c_2 \) are edges of \( F_2 \). Moreover \( u_1b_1, u_1b_2, \) and \( u_1c_2 \) are all edges of \( F_2 \). It follows that \( C_4 \subseteq F_2 \).

It is noted that the partially proved theorems of Section 3.2 have full proofs which are similar in approach to the proof of Theorem 3.2 F.

Theorem 3.2 G

\[ r_2(C_4, K_4-e) = r_2(K(1,3), K_3) = r_2(K(1,3), K(1,3)+e) = r_2(K(1,3), K_4-e) = 5 \]
Partial Proof: To show that $r_2(C_4, K_4-e) > 4$, consider the factorization $K_4(2) = F_1 \oplus F_2$ where $F_1$ and $F_2$ are shown in Figure 3.5.

We see that $C_4 \not\subset F_1$. To see that $K_4-e \not\subset F_2$, note that only two copies of $K_3$ are present in $F_2$ and they do not share an edge.

The factorization

$$K_4(2) = 2K_2(2) \oplus K_2(4)$$

establishes 5 as a lower bound for the remaining three Ramsey numbers.

**Theorem 3.2**

For the integer $i \geq 2$, $r_i(K_3, K(1,3)+e) = r_i(K_3, K_4-e) = r_i(K(1,3)+e, K(1,3)+e) = r_i(K(1,3)+e, K_4-e) = 6$. 

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Partial Proof: The four Ramsey numbers of this theorem are all bounded below by \( r(3,3) = 6 \).

**Theorem 3.2 I**

\[ r_2(K(1,3),K_4) = 7. \]

Partial Proof: Since \( K_6(2) = 3K_2(2) \oplus K_3(4) \) then \( r_2(K(1,3),K_4) \geq 7 \).

**Theorem 3.2 J**

For the integer \( i \geq 2 \), \( r_i(K(1,3)+e,K_4) = 9. \)

Partial Proof: By the lower bound theorem, \( r_i(K(1,3)+e,K_4) \geq r(3,4) = 9 \).

**Theorem 3.2 K**

For the integer \( i \geq 3 \),

\[ r_i(2K_2,K(1,3)) = r_i(2K_2,2K_2) = r_i(2K_2,P_4) = \]

\[ r_i(2K_2,C_4) = r_i(P_3,P_4) = r_i(P_4,P_4) = 2. \]

Partial Proof: The six Ramsey numbers of this theorem are all bounded below by \( r(2,2) = 2 \).

**Theorem 3.2 L**

For the integer \( i \geq 3 \),

\[ r_i(P_3,K_4-e) = r_i(P_4,K_3) = r_i(P_4,K(1,3)+e) = \]

\[ r_i(C_4,K_3) = r_i(C_4,K(1,3)+e) = r_i(K(1,3),K_3) = \]

\[ r_i(K(1,3),K(1,3)+e) = 3. \]
**Partial Proof:** A lower bound for each of the seven Ramsey numbers of this theorem is \( r(2,3) = 3 \).  

**Theorem 3.2 M**

\[ r_3(P_4, C_4) = r_3(P_4, K(1,3)) = 3. \]

**Proof:** The factorization \( K_2(3) = 3K_2 \oplus C_6 \) implies \( 3 \) is a lower bound for each of the Ramsey numbers of this theorem.

Theorem 3.2 L establishes \( r_3(C_4, K(1,3)+e) = 3 \). Hence by monotonicity it follows that

\[ 3 \leq r_3(P_4, K(1,3)) \leq r_3(C_4, K(1,3)+e) = 3 \]

and by monotonicity and symmetry it follows that

\[ 3 \leq r_3(P_4, C_4) \leq r_3(K(1,3)+e, C_4) = 3 \]

**Theorem 3.2 N**

\[ r_i(P_3, C_4) = 3 \text{ for } i = 2 \text{ or } 3. \]

**Proof:** The factorization \( K_2(3) = 3K_2 \oplus C_6 \) implies \( 3 \leq r_3(P_3, C_4) \). If \( K_3(2) = F_1 \oplus F_2 \) with \( P_3 \not\subseteq F_1 \), then it follows that \( \Delta(F_1) \leq 1 \). Hence \( \delta(F_2) \geq 3 \) so that by Corollary 3.1 A it follows that \( C_4 \subseteq F_2 \). Hence by anti-monotonicity we have

\[ 3 \leq r_3(P_3, C_4) \leq r_2(P_3, C_4) \leq 3. \]
Theorem 3.2 0

\[ r_3(K(1,3), K_4-e) = 4. \]

Partial Proof: The factorization \( K_3(3) = F_1 \oplus F_2 \)
shown in Figure 3.6 establishes \( r_3(K(1,3), K_4-e) \geq 4. \)
To see that \( K_4-e \not\subset F_2 \), note that there are only three
copies of \( K_3 \) in \( F_2 \), no two of which share an edge.

\[ r_1(P_4, K_4-e) = 4 \text{ if } i = 2 \text{ or } 3. \]

Proof: The proof of Theorem 3.2 F establishes
\( r_2(P_4, K_4-e) \leq 4. \) The factorization \( K_3(3) = F_1 \oplus F_2 \)
where \( F_1 = 3K_3 \) establishes \( r_3(P_4, K_4-e) \geq 4. \) To see
that \( K_4-e \not\subset F_2 \) it can be shown that no two of the six
copies of \( K_3 \) in \( F_2 \) have a common edge. Hence, by
anti-monotonicity

\[ 4 \leq r_3(P_4, K_4-e) \leq r_2(P_4, K_4-e) \leq 4 \]
Theorem 3.2 Q

\[ r_3(K(1,3), K_4) = 5. \]

**Partial Proof:** Factor \( K_4(3) = F_1 \oplus F_2 \) with \( F_1 = 3C_4 \) as in Figure 3.7. We note \( K_4 \not\subset F_2 \) since every four vertices of \( F_1 \), with no two from the same partite set, induce at least one edge in \( F_1 \).

\[ \begin{array}{c}
\text{F}_1:
\end{array} \]

![Figure 3.7]

\[ \text{Theorem 3.2 R} \]

\[ r_i(P_4, K_4) = 5 \text{ for } i = 2 \text{ or } 3. \]

**Partial Proof:** Factor \( K_4(3) = F_1 \oplus F_2 \) with \( F_1 = 3K(1,3) \) as shown in Figure 3.8. We see \( K_4 \not\subset F_2 \) since every four vertices of \( K_4(3) \), with no two vertices in the same partite set, induces an edge in \( F_1 \).
Figure 3.8

**Theorem 3.2 S**

For the positive integer \( i \geq 4 \),

\[
\text{r}_i(P_3, C_4) = \text{r}_i(P_4, C_4) = \text{r}_i(P_4, K(1,3)) = 2.
\]

**Partial Proof:** The three Ramsey numbers of this theorem are bounded below by \( r(2,2) = 2 \).

**Theorem 3.2 T**

For the positive integer \( i \geq 4 \),

\[
\text{r}_i(P_4, K_4-e) = \text{r}_i(K(1,3), K_4-e) = 3.
\]

**Partial Proof:** Both Ramsey numbers of this theorem are bounded below by \( r(2,3) = 3 \).

**Theorem 3.2 U**

\[
\text{r}_i(C_4, C_4) = \text{r}_i(C_4, K(1,3)) = 3 \text{ for } i = 3 \text{ or } 4.
\]
Partial Proof: The factorization $K_2(4) = C_9 \ast C_8$ along with anti-monotonicity establishes 3 as the desired lower bound $\square$.

Theorem 3.2 V

For the integer $i \geq 4$,

$$r_i(P_4, K_4) = r_i(K(1,3), K_4) = 4.$$  

Partial Proof: Both of the Ramsey numbers of this theorem are bounded below by $r(2,4) = 4$ $\square$.

Theorem 3.2 W

$$r_i(C_4, K_4 - e) = 4$$ for $i = 3$ or $4$.

Partial Proof: The factorization $K_3(4) = F_1 \ast F_2$ (with the graphs $F_1$ and $F_2$ shown in Figure 3.9) provides the desired lower bound. That $C_4 \not\subset F_1$ is easily checked while $K_4 - e \not\subset F_2$ since no two of the seven copies of $K_3$ in $F_2$ have a common edge $\square$.

Theorem 3.2 X

For the integer $i \geq 5$,

$$r_i(C_4, C_4) = r_i(C_4, K(1,3)) = 2.$$  

Proof: Both of the numbers of this theorem are bounded below by $r(2,2) = 2$. By anti-monotonicity it suffices to show $r_5(C_4, C_4)$ and $r_5(C_4, K(1,3))$ are bounded above by 2.
Let $K_2(5) = F_1 \oplus F_2$, let $V(K_2(5)) = A \cup U$ where the partite sets are $A$ and $U$.

If $K(1,3) \not\subset F_2$ then $\Delta(F_2) \leq 2$ so that $\delta(F_1) \geq 3$.

Hence
\[
\sum_{a \in A} \binom{\deg F_1(a)}{2} \geq 5 \cdot \binom{3}{2} = 15 > 10 = \binom{5}{2} = \binom{|U|}{2}.
\]

By Theorem 3.1 it follows that $C_4 \subset F_1$.

If $C_4 \not\subset F_2$ then $\sum_{a \in A} \binom{\deg F_2(a)}{2} \leq \binom{|U|}{2} = 10$ which implies by cases that $\sum_{a \in A} \binom{\deg F_1(a)}{2} \geq 11$ so that $C_4 \subset F_1$. It follows that the desired upper bound holds.

Theorem 3.2 Y

For the integer $i \geq 5$, $r_i(C_4, K_4-e) = 3$.

Partial Proof: By the lower bound theorem,
\[
r_i(C_4, K_4-e) \geq r(2,3) = 3.
\]
Chapter III is concluded with a listing of some problems.
Problem 3.1

Bounds on the values of \( r_i(C_4, K_4), r_i(K_4-e, K_4-e) \), and \( r_i(K_4-e, K_4) \) for \( i \geq 2 \) are evident from Figure 3.2. Find the exact values.

Problem 3.2

A definition of Harary and Prins [23] may be extended. Define the \( i \)-th diagonal Ramsey multiplicity in two factors \( R_i(G,G) \) for the graph \( G \) to be the minimum number of occurrences of \( G \) in the factors \( F_j \) taken over all factorizations \( K_p = F_1 \otimes F_2 \) where \( p = r_i(G,G) \). It is strongly conjectured that \( R_i(G,G) = 1 \) if and only if \( G = K_2 \) or \( G = K(l,n) \) with \( n \) even [22]. This conjecture is false if \( i > 1 \), in general, as shown in Figure 3.10. Make a study of the \( i \)-th Ramsey multiplicity which extends the definition to the non-diagonal case in any number of factors.
A counterexample to the analogue of a conjecture of Harary in the case $i > 1$.

We have $r_3(C_4, C_4) = 3$, $K_3(3) = F_1 \oplus F_2$, $C_4 \not\cong F_2$, and $F_1$ contains exactly one copy of $C_4$.

![Figure 3.10](image-url)
CHAPTER IV

ON TWO VERTEX PARTITION
i-TH RAMSEY NUMBERS

In this chapter the i-th ramsey numbers for chromatic number and vertex arboricity are studied. The techniques of Chapter IV are at least partially extendable to other vertex partition parameters of graphs. The lexicographic product introduced by Sabidussi [32] proves to be of value in this context. We begin with a review of this graphical operation.

Section 4.1

A Review of the Lexicographic Product

A recent trend in graph theory has been the study of factorizations of $K_p$ with regard to the values of certain parameters of the factors. For example, Chartrand and Polimeni [8] define $\chi(n_1, n_2, \ldots, n_k)$ to be the least integer $p$ such that whenever $K_p = \bigoplus_{j=1}^{k} F_j$ it follows that $\chi(F_j) \geq n_j$ for some $j$ with $1 \leq j \leq k$. This chapter considers problems of this type when $K_p$ is replaced with $K_p(i)$.

There is a distinct similarity in the proof techniques belonging to this newer area of ramsey theory.
(See [3,8,26,27,28] for examples.) Part of the contribution of Chapter IV is that the delineation of this proof technique is shortened with the aid of the lexicographic product of graphs.

The lexicographic product (or composition) of the graphs $G_1$ and $G_2$ is a graph denoted $G_1[G_2]$ whose vertex set is $V(G_1) \times V(G_2)$. For vertices $u = (u_1,u_2)$ and $v = (v_1,v_2)$ we have $uv$ is an edge of $G_1[G_2]$ if and only if either

(a) $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or
(b) $u_1v_1 \in E(G_1)$.

A heuristic manner of defining $G_1[G_2]$ is:

(a) each vertex $u$ of $G_1$ is replaced by an entire copy of the graph $G_2$, denoted $G_2(u)$, and then
(b) each edge $uv$ of $G_1$ is replaced by the edges of the join $G_2(u) + G_2(v)$.

As an example, $K_3[K_2] = K_3(2)$ is illustrated in Figure 4.1.

The potential importance of the lexicographic product to the topic of Ramsey theory is indicated by the conclusions of the following lemma. The conclusions of the lemma are routine consequences of the definition of lexicographic product; hence the proofs are omitted.
Lemma 4.1

Let $f$, $i$, $p$, and $s$ be positive integers, and let $G_1$ and $G_2$ be graphs. It follows that

(i) $K_p(i) = K_p(\overline{K_i})$,

(ii) $K_{f\cdot s}(i) = K_f(K_{s(i)})$,

(iii) $K_f(s\cdot i) = K_f(\overline{K_s(K_i)})$

= $K_f(s)(\overline{K_i})$,

(iv) $K_{f\cdot p}(i), f(s) = K_{f\cdot p}(i) + K_f(s)$

= $K_f(K_p(i) + \overline{K_s})$

= $K_f(K_p(i), (s))$,

(v) $(fG_1)[G_2] = f(G_1[G_2])$,

(vi) $K_f(s)(i) = K_f(s\cdot i) \oplus fK_s(i)$, and
(vii) If \( G_1 \bigoplus_{j=1}^{i} F_j \) and \( G_2 \bigoplus_{j=1}^{i} H_j \), then

\[
G_1[G_2] = \bigoplus_{j=1}^{k} F_j[H_j].
\]

This concludes the review of the lexicographic products of graphs.

Section 4.2

The \( i \)-th Chromatic Ramsey Numbers

For positive integers \( i, k, \) and \( n_1, n_2, \ldots, n_k \), define \( r_i(n_1, n_2, \ldots, n_k/\chi/) \) to be the least integer \( p \) such that whenever \( K_p(i) = \bigoplus_{j=1}^{k} F_j \), then \( \chi(F_j) \geq n_j \) for some \( j \) with \( 1 \leq j \leq k \).

The main result of this section is the evaluation of \( r_i(n_1, n_2, \ldots, n_k/\chi/) \). Also, a construction which is of interest throughout Chapter IV will be given. Finally, the following result is also of use.

**Theorem D** (Chartrand and Polimeni [8])

Let \( G \) be a graph and let \( G = \bigoplus_{j=1}^{k} F_j \). Then

\[
\chi(G) \leq \bigoplus_{j=1}^{k} (\chi(F_j)).
\]
The following construction is also implicitly given in [8].

**Lemma 4.2 (a)**

Let i and k be positive integers, let $n_1, n_2, \ldots, n_k$ be integers exceeding 1 in value, and let $N = \prod_{j=1}^{k} (n_j - 1)$.

Then

$$K_N(i) = \bigoplus_{j=1}^{k} \prod_{s=1}^{j-1} (n_s - 1) K[n_{j-1}](i \cdot \prod_{s=j}^{k} (n_s - 1))$$

where $\prod_{s=a}^{b} m_s = 1$ by convention.

**Proof:** We induct on the value of $k$. To anchor the induction, note that when $k = 1$ the theorem claims that $K[n_1 - 1](i) = 1 K[n_1 - 1](i \cdot 1)$. Hence we assume that $k \geq 2$ and that for $N' = \prod_{j=1}^{k-1} (n_j - 1)$ we have

$$K_{N'}(i) = \bigoplus_{j=1}^{k-1} \prod_{s=1}^{j-1} (n_s - 1) K[n_{j-1}](i \cdot \prod_{s=j}^{k-1} (n_s - 1)).$$

Therefore,

$$K_N(i) = K_{N'}(i \cdot [n_k - 1]) \oplus N' K[n_k - 1](i)$$

(Lemma 4.1 (vi))
= \mathcal{K}_n(i)\left[\mathcal{K}[n_k - 1]\right] \oplus \mathcal{N}'K[n_k - 1](i)

\text{(Lemma 4.1 (iii))}

= \left(\oplus \prod_{j=1}^{k-1} (n_s - 1)K[n_j - 1](i \cdot \prod_{s=j+1}^{k-1} (n_s - 1))\right)\left[\mathcal{K}[n_k - 1]\right] \\
\quad \oplus \mathcal{N}'K[n_k - 1](i)

\text{(inductive hypothesis)}

= \left(\oplus \prod_{j=1}^{k-1} (n_s - 1)K[n_j - 1](i \cdot \prod_{s=j+1}^{k-1} (n_s - 1))\right)\left[\mathcal{K}[n_k - 1]\right] \\
\quad \oplus \mathcal{N}'K[n_k - 1](i)

\text{(Lemma 4.1 (vii))}

= \left(\oplus \prod_{j=1}^{k-1} (n_s - 1)K[n_j - 1](i \cdot \prod_{s=j+1}^{k-1} (n_s - 1))\right)\left[\mathcal{K}[n_k - 1]\right] \\
\quad \oplus \mathcal{N}'K[n_k - 1](i)

\text{(Lemma 4.1 (v))}

= \left(\oplus \prod_{j=1}^{k-1} (n_s - 1)K[n_j - 1](i \cdot \prod_{s=j+1}^{k} (n_s - 1))\right) \\
\quad \oplus \mathcal{N}'K[n_k - 1](i)

\text{(Lemma 4.1 (iii))}

= \left(\oplus \prod_{j=1}^{k} (n_s - 1)K[n_j - 1](i \cdot \prod_{s=j+1}^{k} (n_s - 1))\right)$
Before proceeding, we give an algorithm which determines a labeled version of the above factorization. Let

\[ V = \{ (p_1, p_2, \ldots, p_k) \mid 1 \leq p_j \leq n_j - 1 \text{ and } p_j \in \mathbb{Z}^+ \} \]

for \( 1 \leq j \leq k \). Then \( |V| = N = \prod_{j=1}^{k} (n_j - 1) \). Label the vertices of \( K_N \) with the vectors from \( V \). For \( 1 \leq j \leq k \), define the graph \( F'_j \) by \( V(F'_j) = V \). Also, for the vertices \( u = (p_1, p_2, \ldots, p_k) \) and \( v = (b_1, b_2, \ldots, b_k) \) of \( F'_j \), we have \( uv \) is an edge of \( F'_j \) if and only if

\begin{align*}
(a) & \quad p_j \neq b_j \quad \text{and} \\
(b) & \quad p_s = b_s \quad \text{for every } s \text{ where } 1 \leq s \leq j - 1.
\end{align*}

Now define \( F_j \) by \( F_j = F'_j[K_1] \). It is a routine matter to show that

\[ F_j = \prod_{s=1}^{j-1} (n_s - 1)K_{[n_j-1]}(i- \prod_{s=j+1}^{k} (n_s - 1)) \]

for every \( j \) with \( 1 \leq j \leq k \). The main theorem of this section will now be established.

Theorem 4.2

Let \( i, k, n_1, n_2, \ldots, n_k \) be positive integers. Then \( r_i(n_1, n_2, \ldots, n_k/\chi/) = 1 + \prod_{j=1}^{k} (n_j - 1) \).
**Proof:** It follows immediately that the theorem is true if \( n_j = 1 \) for some \( j \) with \( 1 \leq j \leq k \). Hence assume \( n_j \geq 2 \) for every \( j \) with \( 1 \leq j \leq k \). Let

\[
N = \prod_{j=1}^{k} (n_j - 1).
\]

It is first shown that

\[
\rho_i(n_1, n_2, \ldots, n_k) \leq 1 + N.
\]

Let \( K[1+N](i) = \bigoplus_{j=1}^{k} F_j \). If \( \chi(F_j) \leq n_j - 1 \) for every \( j \) with \( 1 \leq j \leq k \) then by Theorem D

\[
N + 1 = \chi(K[1+N](i)) \leq \prod_{j=1}^{k} \chi(F_j) \leq \prod_{j=1}^{k} (n_j - 1) = N.
\]

Hence \( \chi(F_j) \geq n_j \) for some \( j \) with \( 1 \leq j \leq k \), and it follows that \( \rho_i(n_1, n_2, \ldots, n_k) \leq 1 + N \).

Conversely, the factorization which serves as the conclusion of Lemma 4.2 (a) shows the existence of a factorization \( K_N(i) = \bigoplus_{j=1}^{k} F_j \) with \( \chi(F_j) \leq n_j - 1 \) for every \( j \) with \( 1 \leq j \leq k \). It follows that

\[
\rho_i(n_1, n_2, \ldots, n_k) > N,
\]

which establishes the theorem.

**Corollary 4.2 A** (Chartrand and Polimeni [8])

Let \( k \) and \( n_1, n_2, \ldots, n_k \) be positive integers.

Then

\[
\chi(n_1, n_2, \ldots, n_k) = 1 + \prod_{j=1}^{k} (n_j - 1).
\]
Proof: By definition,

$$\chi(n_1, n_2, \ldots, n_k) = r_l(n_1, n_2, \ldots, n_k/x/).$$

Defining $i(n_1, n_2, \ldots, n_k/x/)$ in a natural way, we have the following.

Corollary 4.2 B

The chromatic ramsey index $i(n_1, n_2, \ldots, n_k/x/) = 1$.

This section is concluded with a few observations which help to introduce the ideas of the next section.

For positive integers $i$ and $k$, define $\chi(i,k)$ to be the largest integer $p$ such that there exists a graph $G$ of order $p$ where

1. $G < K_{m(i)}$ for some $m \in \mathbb{Z}^+$, and
2. there is a factorization $G = \oplus F_j$ for $j = 1, 3, \ldots, k$ which $\chi(F_j) = 1$ for every $j$ (with $1 \leq j \leq k$).

By condition (ii) such a graph $G$ must be empty (since each $F_j$ is empty) regardless of the value of $k$. Hence by condition (i) $G < K_s$ for some integer $s$ with $1 \leq s \leq i$. It follows that $\chi(i,k) = i$. Moreover for positive integers $n_1, n_2, \ldots, n_k$

$$r_i(n_1, n_2, \ldots, n_k/x/) = 1 + \left[\chi(i,k) \cdot \prod_{j=1}^{k} (n_j - 1)/i\right].$$
The formula is written in this manner to suggest an extension. The next section evaluates a quantity similar to \( \overline{X}(i,k) \) but related to the concept of vertex arboricity.

Section 4.3

The Evaluation of \( \overline{a}(i,k) \)

For a given graph \( G \), the vertex arboricity of \( G \), denoted \( a(G) \), is the least positive integer \( n \) for which there exists a partition of \( V(G) \) into sets \( V_1, V_2, \ldots, V_n \) where \( \langle V_j \rangle_G \) is acyclic for every \( j \) with \( 1 \leq j \leq n \).

The edge arboricity \( a_1(G) \) for the nonempty graph \( G \) is the least positive integer \( m \) such that there is a partition of \( E(G) \) into sets \( E_1, E_2, \ldots, E_m \) where \( \langle E_j \rangle_G \) is acyclic for every \( j \) with \( 1 \leq j \leq m \). (See [1, p. 9] for the definition of the edge-induced subgraph \( \langle E_j \rangle_G \) of \( G \).) By convention, \( a_1(K_s) = 1 \) for every \( s \in \mathbb{Z}^+ \).

The statement "\( G < K_m(i) \) for some \( m \)" means only that \( G \) is some complete multi-partite graph in which each partite set of \( G \) contains at most \( i \) vertices.

Now for the positive integers \( i \) and \( k \), define \( \overline{a}(i,k) \) to be the largest integer \( p \) such that there exists a graph \( G \) of order \( p \) where
(i) $G < K_m(i)$ for some $m$, and

(ii) there is a factorization $G = \bigoplus_{j=1}^{k} F_j$ for which $a(F_j) = 1$ for every $j$ with $1 \leq j \leq k$.

Note that condition (ii) is equivalent to the condition

(ii)' $a_1(G) \leq k$.

The relevance of $\bar{a}(i,k)$ to Ramsey theory is that a vertex arboricity Ramsey formula will be developed similar to that for chromatic number, but in terms of $\bar{a}(i,k)$. In order to compute the value of $\bar{a}(i,k)$, we need to know the value of $a_1(G)$ whenever $G < K_m(i)$ for some $m$. To determine this number we make use of the following result.

**Theorem E** (Nash-Williams [29])

For any nonempty graph $G$,

$$a_1(G) = \max_{\substack{H < G \ \text{and} \ \bar{a}(H) \neq K_1}} \left\{ \frac{|E(H)|}{|V(H)| - 1} \right\}.$$

It will be presently shown that if $G$ is nonempty and $G < K_m(i)$ for some $m$, then $a_1(G)$ is given by the Nash-Williams formula evaluated at the graph $G$ itself. We proceed to this result by way of two lemmas.
Lemma 4.3 (a)

Let $n_1, n_2, \ldots, n_p$ be positive integers whose sum exceeds 2, and let $K = K(n_1, n_2, \ldots, n_p)$. Then for every $j$ with $1 \leq j \leq p$ it follows that

$$\frac{|E(K)| - |V(K)| + n_j}{|V(K)| - 2} \leq \frac{|E(K)|}{|V(K)| - 1}.$$ 

Proof: For notational convenience, let

$m = \max\{n_j | 1 \leq j \leq p\}$ and let $S = |V(K)| = \sum_{j=1}^{p} n_j \geq 3$.

We first show

$$m \leq S/2 + \binom{m}{2}/(S - 1).$$

By elementary calculus it follows that

$m^2 + (1 - 2S)m + S(S - 1) \geq 0$ since $m \leq S - 1$. Hence,

$$2m(S - 1) \leq S(S - 1) + m(m - 1)$$

and so

$$m \leq S/2 + \binom{m}{2}/(S - 1).$$

It now follows that

$$m \leq S/2 + \sum_{j=1}^{p} \frac{n_j}{(S - 1)}$$

so that

$$m(S - 1) \leq \binom{S}{2} + \sum_{j=1}^{p} \frac{n_j}{2}$$

which implies

$$\binom{S}{2} - \sum_{j=1}^{p} \frac{n_j}{2} \leq 2\binom{S}{2} - m(S - 1)$$

and

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\[ |E(K)| \leq (S - 1)(S - m) \text{ whence} \]
\[ (S - 1)(|E(K)| - S + m) \leq |E(K)| (S - 2). \]

Now given any \( j \) with \( 1 \leq j \leq p \) we have
\[ \frac{|E(K)| - S + n_j}{S - 2} \leq \frac{|E(K)| - S + m}{S - 2} \leq \frac{|E(K)|}{S - 1} \]

since \( S \geq 3 \).

Our primary interest is in a result which is slightly weaker than Lemma 4.3 (a).

**Lemma 4.3 (b)**

Let \( n_1, n_2, \ldots, n_p \) be \( p \) positive integers whose sum exceeds 2, and let \( K = K(n_1, n_2, \ldots, n_p) \).

Then for every \( j \) with \( 1 \leq j \leq p \) it follows that
\[
\left\{ \frac{|E(K)| - \sum_{s=1}^{p} n_s}{|V(K)| - 2} \right\} \leq \left\{ \frac{|E(K)|}{|V(K)| - 1} \right\}.
\]

**Proof:** If \( a \leq b \) then \( \{a\} \leq \{b\} \).

We are now prepared to present the desired theorem.

**Theorem 4.3**

Let \( K \) be a nonempty complete multi-partite graph. Then
\[ a_1(K) = \left\{ \frac{|E(K)|}{|V(K)| - 1} \right\}. \]
Proof: We induct on \( S = |V(K)| \geq 2 \). The induction is anchored in the case \( S = 2 \) by noting that
\[
a_1(K_2) = \lfloor \frac{1}{2} \rfloor = 1.
\]
Hence, we assume \( S \geq 3 \) and that
\[
a_1(K') = \left\lfloor \frac{|E(K')|}{|V(K')| - 1} \right\rfloor
\]
whenever \( K' \) is a nonempty complete multi-partite graph for which \( 2 \leq |V(K')| \leq S - 1 \). We let
\[
K = K(n_1, n_2, \ldots, n_p)
\]
be a nonempty complete multi-partite graph with \( |V(K)| = S \). We show
\[
a_1(K) = \left\lfloor \frac{|E(K)|}{|V(K)| - 1} \right\rfloor.
\]
Since \( K \) is not empty we have
\[
a_1(K) \geq \left\lfloor \frac{|E(K)|}{|V(K)| - 1} \right\rfloor \geq 1
\]
by Theorem E. Assume that
\[
a_1(K) > \left\lfloor \frac{|E(K)|}{|V(K)| - 1} \right\rfloor \geq 1.
\]
Then there is a graph \( H \) with \( H \neq K_1, H \subset K, H \neq K \) and
\[
a_1(K) = \left\lfloor \frac{|E(H)|}{|V(H)| - 1} \right\rfloor > \left\lfloor \frac{|E(K)|}{|V(K)| - 1} \right\rfloor \geq 1.
\]
Note that \( H \) is a nonempty complete multi-partite graph with \( 2 \leq |V(H)| < |V(K)| = S \). We may apply the inductive hypothesis to \( H \) to conclude \( a_1(H) = a_1(K) \).
Moreover, there exists some \( j \) with \( 1 \leq j \leq p \) such that
\[ H \subset K(n_1, n_2, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_p) \]. Denote this supergraph of \( H \) by \( K' \).

It must be the case that

\[ a_1(K') \geq a_1(H) = a_1(K) > \left\{ \frac{|E(K)|}{|V(K)| - 1} \right\} . \]

Since \( K' \) is a nonempty complete multi-partite graph where \( 2 \leq |V(H)| \leq |V(K')| = S - 1 \), then by the induction hypothesis

\[ a_1(K') = \left\{ \frac{|E(K')|}{|V(K')| - 1} \right\} = \left\{ \frac{|E(K)| - \sum_{S=1}^{P} n_S}{|V(K)| - 2} \right\} \]

Which contradicts Lemma 4.3 (b) since \( p \geq 2 \) and \( \sum_{j=1}^{P} n_j \geq 3 \).

Hence the assumption

\[ a_1(K) > \left\{ \frac{|E(K)|}{|V(K)| - 1} \right\} \]

is false, which implies

\[ a_1(K) = \left\{ \frac{|E(K)|}{|V(K)| - 1} \right\} = \]

For the positive integers \( i \) and \( k \) and given a specific graph \( G < K_{m(i)} \) for some \( m \), an efficient
method for deciding whether \( a_1(G) \leq k \) now exists as a result of Theorem 4.3.

In order to put a bound on the (conceivably high) number of graphs \( G \) for which \( a_1(G) \) is computed in the determination of \( \bar{a}(i,k) \), we are led to the next result.

**Theorem 4.4**

Let \( p \) and \( i \) be positive integers and let the graph \( H \) be a complete multi-partite graph of order \( p \) having no more than \( i \) vertices in each partite set. Let \( p = t \cdot i + s \) with \( 0 \leq s \leq i - 1 \).

Then \( |E(H)| \geq \binom{t}{2}i^2 + t \cdot i \cdot s \) with equality if and only if \( H = K_{t(i)}(s) \).

**Proof:** We may assume \( H = K(n_1, n_2, \ldots, n_m) \) and that \( i \geq n_1 \geq n_2 \geq \ldots \geq n_m \geq 1 \). Let the partite sets of \( H \) be \( V_1, V_2, \ldots, V_m \) with \( |V_j| = n_j \) for each \( j \) with \( 1 \leq j \leq m \). We let \( V_j = \{v_{j1}^1, v_{j1}^2, \ldots, v_{jn_j}^j\} \) for each \( j \) with \( 1 \leq j \leq p \). The theorem is now established in the case \( s = 0 \) (whence \( t \geq 1 \)).

If \( s = 0 \) then \( |V(H)| = p = t \cdot i \) and the theorem claims \( |E(H)| \geq \binom{t}{2}i^2 \) with equality if and only if \( H = K_{t(i)} \). Note that since a given vertex \( v \) of \( H \) is not adjacent in \( H \) with at most \( i \) vertices (including itself) of \( H \), then \( \deg_H(v) \geq p - i = t \cdot i - i \). Hence
\[ \delta(H) \geq (t - 1)i. \] By counting degrees, we see that
\[ |E(H)| \geq \delta(H) \cdot p/2 \geq (t - 1)\cdot i \cdot t/2 = \binom{t}{2}i^2. \] Equality holds if and only if \( H \) is regular of degree \((t - 1)\cdot i\) which holds if and only if \( H = K_t(i) \).

The theorem is true in the case \( s = 0 \). We thus assume \( 1 \leq s \leq i - 1 \). We fix the value of \( s \) for the remainder of the proof and proceed by induction on \( t \).

If \( t = 0 \) then \( |E(H)| = 0 = \binom{t}{2}i^2 + t \cdot i \cdot s \). Moreover, \( |E(H)| = 0 \) if and only if \( H = K_s = K_0(i),(s) \).

The induction is anchored, so we assume \( t \geq 1 \). We also assume that whenever \( H' \) is a complete multipartite graph with \( i \) or fewer vertices in each partite set having \( |V(H')| = (t - 1)\cdot i + s \) then it follows that \( |E(H')| \geq \binom{t-1}{2}i^2 + (t - 1)\cdot i \cdot s \) with equality if and only if \( H' = K_{[t-1]}(i),(s) \).

We show that for this (now fixed) value of \( t \) that \( |E(H)| \geq \binom{t}{2}i^2 + t \cdot i \cdot s \) with equality if and only if \( H = K_t(i),(s) \). This fact is established by descending induction on \( n^j \), the cardinality of a largest partite set of \( H \).

If \( n_1 = i \) (the largest possible theoretical value of the \( n_j \); \( 1 \leq j \leq m \)) then consider \( H - V_1 \). We have \( |V(H-V_1)| = |V(H)| - i = t \cdot i + s - 1 = (t - 1)\cdot i + s \).

Also, \( H - V_1 < H < K_m(i) \) so by the induction hypothesis (on \( t \)) it follows that
\[ |E(H-V_1)| \geq \binom{t-1}{2}i^2 + (t - 1) \cdot i \cdot s \quad (4.1) \]

with equality if and only if 
\[ H - V_1 = K_{[t-1]}(i),(s) \cdot \]

By (4.1) we have
\[
|E(H)| = |E(H-V_1)| + i(|V(H-V_1)|)
\geq \left[ \binom{t-1}{2}i^2 + (t - 1) \cdot i \cdot s \right] + \left[ (t - 1)i^2 + i \cdot s \right]
= \binom{t}{2}i^2 + t \cdot i \cdot s. \quad (4.2)
\]

Moreover, equality holds in (4.2) if and only if equality holds in (4.1). Hence equality holds in (4.2) if and only if
\[ H = K_{[t-1]}(i),(s) + \overline{K_i} \]
\[ = K_t(i),(s) \cdot \]

A basis for descending induction on the value of \( n_1 \) (with \( t \) and \( s \) fixed) has been established. If \( i = 1 \) then \( s = 0 \), so we are assured by a previous assumption that \( i \geq 2 \). Hence we assume \( 1 \leq n_1 \leq i - 1 \). Also we assume that whenever \( H' < K_m(i) \) with
\[ |V(H')| = t \cdot i + s \] and such that \( H' \) has a partite set of cardinality at least \( n_1 + 1 \) \((\leq i)\) then
\[ |E(H')| \geq \binom{t}{2}i^2 + t \cdot i \cdot s. \] By the hypothesis that \( n_1 \leq i - 1 \) we cannot have \( H = K_t(i),(s) \). Hence it is necessary (and at this stage of the proof sufficient)
to demonstrate that \(|E(H)| > \left(\frac{t}{2}\right)i^2 + t \cdot i \cdot s\). This
demonstration is made after the construction of a per-
tinent graph \(G\).

Let \(V(G) = (V(H) - \{v_1^2\}) \cup \{w\}\). (Note that \(v_1^2\)
exists in \(H\) since \(t \geq 1\) and \(s \geq 1\) implies \(m \geq 2\)
whence \(v_2\) exists.) Let

\[ E(G) = E(H-v_1^2) \cup \{wv_j^2 \mid 2 \leq j \leq n_2\} \cup \{wv_j^k \mid 3 \leq k \leq m, 1 \leq j \leq n_k\}. \]

(Heuristically, \(G\) is obtained from \(H\) by deleting the
edges between \(v_1^2\) and \(V_1\), adding all possible edges
between \(v_1^2\) and \(V_2 - \{v_1^2\}\), and fixing all other edges.)

It follows that \(G < K_m(i)\), \(|V(G)| = t \cdot i + s\), and that
\(G\) has a partite set of cardinality \(n_1 + 1\) \((\leq i)\). Hence
by the inductive hypothesis (on \(n_1\)),

\[ |E(G)| \geq \left(\frac{t}{2}\right)i^2 + t \cdot i \cdot s. \]

It now follows that

\[
\left(\frac{t}{2}\right)i^2 + t \cdot i \cdot s \leq |E(G)|
= |E(H)| + (n_2 - 1) - n_1
< |E(H)|
\]

since \(n_1 \geq n_2\) whence \(n_2 - n_1 - 1 < 0\) \(\ast\)

We note a property of \(K_t(i),(s)\) which is outside
the realm of ramsey theory.

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Given fixed positive integers \( m \) and \( p \) with \( m \leq p \), we ask

(i) how many complete multi-partite graphs \( G \) of order \( p \) exist having a maximum number of edges and having \( \beta(G) \leq m \) and

(ii) how many complete multi-partite graphs \( G \) of order \( p \) exist having a minimum number of edges and having \( \beta(G) \leq m \)?

The answer to (i) is one where the only such graph \( G \) is the well-known graph of Turán [34] given by

\[
G = K_{[m-s-1]}(t), s(t+1)
\]

with \( p = t(m - 1) + s \) and \( 0 \leq s \leq m - 2 \). The answer to (ii) is also one where by Theorem 4.4 the only such graph \( G \) is given by

\[
G = K_t(m), (s)
\]

with \( p = t \cdot m + s \) and \( 0 \leq s \leq m - 1 \).

A consequence of Theorem 4.3 and Theorem 4.4 is that we need only compute \( a_l(G) \) for relatively few graphs \( G \) to find \( \overline{a}(i,k) \).

**Corollary 4.4**

Let \( H < K_m(i) \) for some positive integers \( m \) and \( i \), and let \( |V(H)| = t \cdot i + s \) where \( 0 \leq s \leq i - 1 \). Then

\[
a_l(H) \geq a_l(K_t(i), (s)).
\]
Proof: If $H$ has only one partite set then $H = K_p$ where $1 \leq p \leq i$ and $p = t \cdot i + s$ with $0 \leq s \leq i - 1$.

If $s = 0$ then $t = 1$ and $H = K_t(i),(s)$. If $s > 0$ then $t = 0$, $s = p$, and $H = K_t(i),(s)$. The desired result follows.

We may thus assume $H$ has at least two partite sets. If $K_t(i),(s)$ has only one partite set the result follows. Hence, assume $K_t(i),(s)$ has at least two partite sets.

It follows from Theorem 4.3 that

$$a_1(H) = \left\lfloor \frac{|E(H)|}{|V(H)| - 1} \right\rfloor$$ and

$$a_1(K_t(i),(s)) = \left\lfloor \frac{(t)^2 + t \cdot i \cdot s}{t \cdot i + s - 1} \right\rfloor.$$

Since $|V(H)| = t \cdot i + s$, the desired result now follows by an application of Theorem 4.4.

We may now compute $a(i,k)$ for every pair of positive integers $i$ and $k$.

Lemma 4.5 (a)

Let $i$ and $k$ be positive integers. Then

$a(i,k) = t \cdot i + s$ where

(i) $t$ is the greatest integer such that $a_1(K_t(i)) \leq k$ and

(ii) $s$ is the greatest integer $0 \leq s \leq i - 1$ such that $a_1(K_t(i),(s)) \leq k$.
Proof: By the definition of $t$ and $s$ we have 

$$a_1(K_t(i), (s)) \leq k,$$

hence again by definition

$$a(i,k) \geq t \cdot i + s.$$ 

We show $a(i,k) < t \cdot i + s + 1$.

Let $H$ be any graph of order $t \cdot i + s + 1$ where $H < K_m(i)$ for some $m$. If $s + 1 = i$ then

$$a_1(H) \geq a_1(K_{[t+1]}(i)) \quad \text{(by Corollary 4.4 A)}$$

$$> k \quad \text{(by (i))}$$

while if $1 \leq s + 1 \leq i - 1$ then

$$a_1(H) \geq a_1(K_t(i), (s+1)) \quad \text{(by Corollary 4.4 A)}$$

$$> k \quad \text{(by (ii))}.$$ 

It follows that $a(i,k) < t \cdot i + s + 1$ from the definition of $a(i,k)$.

Theorem 4.5

Let $i$ and $k$ be positive integers. Then

$$a(i,k) = t \cdot i + s$$

where

(i) \quad \begin{align*} 
  t &= \left[ \frac{i^2 + 2 \cdot i \cdot k + \sqrt{(i^2 + 2 \cdot i \cdot k)^2 - 8 \cdot k \cdot i^2}}{2 i^2} \right] 
\end{align*}

and

(ii) \quad s = \left[ \frac{k(t \cdot i - 1) - \left( \frac{t}{2} \right) i^2}{t \cdot i - k} \right].

Proof: Let $p$ be any integer such that $a_1(K_p(i)) > k$.

By Lemma 4.5 (a) it suffices to show that the least such integer $p$ is $t + 1$ to validate the claim of (i).
Since $k \geq 1$ then $a_1(K_p(i)) \geq 2$ whence $p \geq 2$. Thus by Theorem 4.3 it follows that

$$a_1(K_p(i)) = \left\{ \frac{(p_2)^2}{p \cdot 1 - 1} \right\} \geq k + 1; \text{ hence}$$

$$\frac{(p_2)^2}{p \cdot 1 - 1} > k \text{ so that}$$

$$(i^2)p^2 - (i^2 + 2i \cdot k)p + 2k > 0.$$  

It can be routinely verified that

$$(i^2 + 2i \cdot k)^2 - 8k \cdot i^2 \geq 0$$

and that

$$2 \geq \frac{(i^2 + 2i \cdot k) - \sqrt{(i^2 + 2i \cdot k)^2 - 8k \cdot i^2}}{2i^2};$$

hence

$$p > \frac{(i^2 + 2i \cdot k) + \sqrt{(i^2 + 2i \cdot k)^2 - 8k \cdot i^2}}{2i^2}.$$  

The least such integer $p$ is

$$p_0 = \left\lfloor \frac{(i^2 + 2i \cdot k) + \sqrt{(i^2 + 2i \cdot k)^2 - 8k \cdot i^2}}{2i^2} \right\rfloor + 1$$

from which statement (i) follows.

To establish (ii), let $c$ be the least integer $1 \leq c \leq i$ such that $a_1(K_t(i), (c)) > k$. By the just established result for (i) we know

$$a_1(K_t(i), (i)) = a_1(K_{[t+1]}(i)) > k \text{ and } a_1(K_t(i)) \leq k.$$
Therefore it is clear that such $c$ exists. We have $k \geq 1$ so $t \geq 1$. Since $c \geq 1$ then $K_{t(i)}(c)$ has at least two partite sets.

By the definition of $c$ and by Theorem 4.3 we may conclude that

$$\frac{(t) i^2 + t \cdot i \cdot c}{t \cdot i + c - 1} > k$$

whence

$$c(t \cdot i - k) > k(t \cdot i - 1) - \frac{(t) i^2}{2}.$$  \hspace{1cm} (4.3)

Now if $t \cdot i - k \leq 0$ then $t \cdot i \leq k$ and

$$0 \geq c(t \cdot i - k) > k(t \cdot i - 1) - \frac{(t) i^2}{2}$$

$$\geq t \cdot i(t \cdot i - 1) - \frac{(t) i^2}{2}$$

$$= \frac{t^2 \cdot i^2}{2} + \frac{t \cdot i(i - 2)}{2}$$

$$\geq 0$$

since $t \geq 0$ and $i \geq 1$. This contradiction implies $t \cdot i - k \geq 1$ so from (4.3) we have

$$c > \frac{k(t \cdot i - 1) - \frac{(t) i^2}{2}}{t \cdot i - k}.$$

Of course the least such integer $c$ is

$$c_0 = \left\lceil \frac{k(t \cdot i - 1) - \frac{(t) i^2}{2}}{t \cdot i - k} \right\rceil + 1$$

whence (ii) is established.
In order to gain data concerning the nature of \( \bar{a}(i,k) \), the algorithm for computing \( \bar{a}(i,k) \) implicitly contained in Lemma 4.5 (a) was implemented in the following computer program.

\begin{verbatim}
10      FILES ARBNUM
15      SCRATCH #1
20      PRINT #1, "THE EDGE-ARBORICITY OF K(T[I],1[S])";
25      PRINT #1, "IS THE SMALLEST INTEGER ABOVE A."
30      PRINT #1,
35      PRINT #1, " P"," T";" I";" S" ,,A"
40      FOR I = 1 TO 10
45          FOR T = 1 TO 10
50              FOR S = 1 TO I
55                  A1 = T * (T-1) * I * I + 2 * T * I * S
60                  D1 = 2 * T * I + 2 * S - 2
65                  A = A1/D1
70                  P = T * I + S
75                  PRINT #1, P, T; I; S,,A
80          NEXT S
85      NEXT T
90      NEXT I
95      END
\end{verbatim}

The edge arboricity of \( \sum_{I=1}^{10} 10 \cdot I = 550 \) graphs was computed using 2.17 seconds of CPU time. It seemed
surprising that the data contained the value of \( \aleph(i,k) \) for 275 pairs of integers \( i \) and \( k \). Given the sequence of graphs

\[ \ldots, K_t(i)(i-2), K_t(i)(i-1), K_{t+1}(i), (1), \ldots \]

the data suggests that a value of \( \overline{a}(i,k) \) is determined at approximately every second entry of the sequence.

The data further suggested the following result which was quite surprising given the roles of \( t, i, s, \) and \( k \) in the computation of \( \overline{a}(i,k) \).

**Theorem 4.6**

Let \( i \) and \( k \) be positive integers. Then

\[ \aleph(i,k) \leq 2k + i - 1. \]

**Proof:** Let \( H < K_m(i) \) for some \( m \). It suffices to assume that \( |V(H)| = 2k + i \) and then demonstrate that \( a_1(H) \geq k + 1 \). The proof falls naturally into three cases.

**Case 1:** Assume \( 2k = i \).

We have \( |V(H)| = 2 \cdot i \) so by Corollary 4.4 it follows that

\[ a_1(H) \geq a_1(K_{2}(i)). \]

By Theorem 4.3 it suffices to show that
\[
\frac{C_i^2 \cdot i^2}{2 \cdot i - 1} > k = \frac{i}{2}
\]

which follows since \(i > 0\).

**Case 2:** Assume \(2k < i\).

It follows by assumption that \(|V(H)| = 1 \cdot i + 2k\) with \(2 \leq 2k < i\). Hence by Corollary 4.4 A we have

\[a_1(H) \geq a_1(K_{1(i)}, (2k))\]

It follows by Theorem 4.3 that it suffices to show that

\[\frac{2ki}{2k + i - 1} > k\]

This inequality follows since \(0 > k(2k - (i + 1))\) for \(1 \leq k < (i + 1)/2\).

**Case 3:** Assume \(1 \leq i < 2k\).

Let \(2k = t \cdot i + s\) where \(1 \leq s \leq i\). Then

\(|V(H)| = (t + 1) \cdot i + s\) so by Corollary 4.4 A

\[a_1(H) \geq a_1(K_{[t+1]}(i), (s))\]

Note that \(0 > s(s - (i + 1)) \geq s^2 - s(i + 1) - t \cdot i\) for \(1 \leq s \leq i\) and \(t \geq 0\). It follows that

\[(t + 1)t \cdot i^2 + [2(t + 1)i]s > s^2 + s(2t \cdot i + i - 1) + t^2 \cdot i^2 + t \cdot i^2 - t \cdot i.\]
so that
\[
\frac{(t + 1)(t^2 i^2) + [2(t + 1)i]s}{2} > \frac{(t \cdot i + s)(t \cdot i + s + i - 1)}{2}
\]
whereby
\[
\left(\frac{t+1}{2}\right)i^2 + (t + 1) \cdot i \cdot s > k[(t + 1)i + s - 1]
\]
and finally
\[
\frac{\left(\frac{t+1}{2}\right)i^2 + (t + 1) \cdot i \cdot s}{(t + 1)i + s - 1} > k \quad (4.4)
\]
since the denominator exceeds 0.

Now \( t > 0 \) so \( K[t+1](i), (s) \) has at least two partite sets whereby \( a_1(K[t+1](i), (s)) \geq k + 1 \) by Theorem 4.3 and (4.4).

A second computer program was written to explicitly calculate \( \overline{a}(i,k) \) for \( 1 \leq i \leq 13 \) and \( 1 \leq k \leq 50 \) using the formulas of Theorem 4.5. The results indicate that \( \overline{a}(i,k) = 2k + i - 1 \) approximately 40% of the time, \( \overline{a}(i,k) = 2k + i - 2 \) approximately 58% of the time, and \( \overline{a}(i,k) = 2k + i - 3 \) approximately 2% of the time (8 occurrences in 450 possibilities) given \( 5 \leq i \leq 13 \) and \( 1 \leq k \leq 50 \). The data also suggests \( \overline{a}(i,k) = 2k + i - 1 \) for \( i = 3 \) or \( i = 4 \). We verify in the next section that \( \overline{a}(i,k) = 2k + i - 1 \) for \( i = 1 \) or \( i = 2 \).
Section 4.4

On i-th Vertex Arboricity Ramsey Numbers

For positive integers $i$, $k$, and $n_1, n_2, \ldots, n_k$ define $r_i(n_1, n_2, \ldots, n_k/a)$ to be the least integer $p$ such that if $K_p(i) = \bigoplus_{j=1}^{k} F_j$ then for some $j$ (with $1 \leq j \leq k$) $a(F_j) \geq n_j$.

Lemma 4.7 (a)

Let $i$, $k$, and $n_1, n_2, \ldots, n_k$ be positive integers. Then

$$r_i(n_1, n_2, \ldots, n_k/a) \leq 1 + \left\lceil \frac{\overline{a}(i,k) \cdot \sum_{j=1}^{k} (n_j - 1)}{i} \right\rceil.$$  

Proof: The theorem holds if $n_j = 1$ for some $j$ with $1 \leq j \leq k$. Hence assume $n_j \geq 2$ for every $j$ with $1 \leq j \leq k$. Let $N = \sum_{j=1}^{k} (n_j - 1)$. We show that

$$r_i(n_1, n_2, \ldots, n_k/a) \leq 1 + \left\lceil \frac{\overline{a}(i,k) \cdot N}{i} \right\rceil.$$  

Let $K[1 + \left\lceil \frac{\overline{a}(i,k) \cdot N}{i} \right\rceil](i) = \bigoplus_{j=1}^{k} F_j$ and assume that $a(F_j) \leq n_j - 1$ for each $j$ with $1 \leq j \leq k$.

Since $a(F_1) \leq n_1 - 1$, then let $V(F_1)$ be partitioned into sets $V_1^1, V_2^1, \ldots, V_{n_1-1}^1$ such that $\langle V_i^1 \rangle_{F_1}$
is acyclic where $1 \leq t \leq n_1 - 1$. Since the cardinality of one of these sets must meet or exceed the average cardinality we may assume that

$$|V_1^1| \geq \frac{(1 + \lceil a(i,k) \cdot N/i \rceil) \cdot i}{n_1 - 1}$$

$$= \frac{i}{n_1 - 1} + \frac{\lceil a(i,k) \cdot N/i \rceil \cdot i}{n_1 - 1}$$

$$= \frac{1}{n_1 - 1} + a(i,k) \cdot \frac{k}{j=2} (n_j - 1).$$

Hence there is a set $U_1 \subseteq V_1^1$ such that

$$|U_1| = 1 + a(i,k) \cdot \frac{k}{j=2} (n_j - 1)$$

and $\langle U_1 \rangle_{F_1}$ is acyclic. Let

$$G_1 = \langle U_1 \rangle_{K[1 + \lceil a(i,k) \cdot N/i \rceil]}.$$

Then we have $G_1 = \bigoplus_{j=1}^{k} \langle U_1 \rangle_{F_j}$ with $a(\langle U_1 \rangle_{F_1}) = 1$ and $a(\langle U_1 \rangle_{F_j}) \leq n_j - 1$ for $2 \leq j \leq k$.

Since $a(\langle U_1 \rangle_{F_2}) \leq n_2 - 1$, then let

$V_1^2, V_2^2, \ldots, V_{n_2-1}^2$ partition $U_1$ so that $\langle V_i^2 \rangle_{F_2}$ is acyclic for $1 \leq t \leq n_2 - 1$. We may assume that
\[ |v_2^2| = \frac{|u_1|}{n_2 - 1} \]

\[ > \bar{a}(i,k) \cdot \prod_{j=3}^{k} (n_j - 1) \cdot f_j(n_2 - 1) \cdot j_3^3 \]

Hence there is a set \( U \subseteq v_2 \subseteq u_1 \subseteq v_1 \) such that

\[ |U_2| = 1 + \bar{a}(i,k) \cdot \prod_{j=3}^{k} (n_j - 1) \]

and \( \langle U_2 \rangle_{F_2} \) is acyclic. Let \( G_2 = \langle U_2 \rangle_{G_1} \). Then we have

\[ G_2 = \bigoplus_{j=1}^{k} \langle U_2 \rangle_{F_j} \] with \( a(\langle U_2 \rangle_{F_j}) = 1 \) for \( 1 \leq j \leq 2 \) and \( a(\langle U_2 \rangle_{F_j}) \leq n_j - 1 \) for \( 3 \leq j \leq k \).

Continuing in an inductive manner, we find a set \( U_k \) and a graph \( G_k \) where

\[ |U_k| = 1 + \bar{a}(i,k) \]

and where \( G_k = \bigoplus_{j=1}^{k} \langle U_k \rangle_{F_j} \) with \( a(\langle U_k \rangle_{F_j}) = 1 \) for \( 1 \leq j \leq k \). Since \( G_k < K_m(i) \) for some \( m \) and since \( a_1(G_k) \leq k \) then \( \bar{a}(i,k) \geq 1 + \bar{a}(i,k) \). This contradiction implies that

\[ r_{i}(n_1, n_2, \ldots, n_k/a) \leq 1 + \lfloor \bar{a}(i,k) \cdot n/i \rfloor \]

The remainder of this section is devoted to showing cases in which the upper bound is sharp and to finding a general lower bound.
Lemma 4.7 (b)

Let \( k \) be a positive integer, let \( n_1, n_2, \ldots, n_k \) be integers (exceeding 1), let \( N = \prod_{j=1}^{k} (n_j - 1) \), and let \( G \) be any graph for which \( a_1(G) \leq k \).

Then there is a factorization

\[
K_{N}[G] = \bigoplus_{j=1}^{k} F'_j
\]

for which \( a(F'_j) \leq n_j - 1 \) for every \( j \) with \( 1 \leq j \leq k \).

Proof: By hypothesis there is a factorization \( G = \bigoplus_{j=1}^{k} H_j \) such that \( a(H_j) = 1 \) for every \( j \) with \( 1 \leq j \leq k \).

Moreover by Lemma 4.2 (a) we have the factorization

\[
K_{N} = \bigoplus_{j=1}^{k} F_j
\]

where

\[
F_j = \prod_{l=1}^{j-1} (n_l - 1) K_{[n_j-1]} \left( \prod_{l=j+1}^{k} (n_l - 1) \right)
\]

whenever \( 1 \leq j \leq k \). It follows that \( K_{N}[G] = \bigoplus_{j=1}^{k} F_j[H_j] \).

It suffices to show that \( a(F_j[H_j]) \leq n_j - 1 \).

Let the partite sets of \( F_j \) be \( V_1^j, V_2^j, \ldots, V_{n_j-1}^j \) for every \( j \) with \( 1 \leq j \leq k \). Let \( U_l^j = V_l^j \times V(H_j) \) for every \( l \) with \( 1 \leq l \leq n_j - 1 \). It follows that \( U_1^j, U_2^j, \ldots, U_{n_j-1}^j \) forms a partition of \( V(F_j[H_j]) \).
Moreover, \( \langle u_j^i \rangle_{j=1}^n \) for every \( j \) with \( 1 \leq j \leq k \).

**Lemma 4.7 (c)**

Let \( s, i, \) and \( N \) be positive integers with \( 1 \leq s \leq i - 1 \) and \( [s \cdot N/i] > 0 \).

If \( \lfloor i/s \rfloor \cdot [s \cdot N/i] \leq N \) then \( K_{[s \cdot N/i]} \subseteq K_N(s) \).

**Proof:** Let \( x \) be the least positive integer such that \( x \cdot s \geq i \), that is, let \( x = \lfloor i/s \rfloor \). Let \( y = [s \cdot N/i] \).

Let the partite sets of \( K_N(s) \) be \( V_1, V_2, \ldots, V_N \).

Of course \( |V_j| = s \) for \( 1 \leq j \leq N \).

Since \( x \cdot y \leq N \) these partite sets of \( K_N(s) \) may be relabeled to have at least the \( x \cdot y \) distinct sets \( V_1^1, V_2^1, \ldots, V_1^x, V_2^x, \ldots, V_1^1, V_2^1, \ldots, V_x^y \).

For each \( j \) with \( 1 \leq j \leq y \) let \( U_j \subseteq \bigcup_{j=1}^{x} V_j^j \) such that \( |U_j| = i \). This can be done since \( x \cdot s \geq i \).

It suffices to show \( K_y(i) \subseteq \bigcup_{j=1}^{y} U_j \).

Let \( u \in U_j \) and \( v \in U_{j'} \) with \( j \neq j' \). Then \( u \in V_a \) and \( v \in V_b \) with \( a \neq b \) and \( 1 \leq a,b \leq N \). It follows that \( uv \in E(K_N(s)) \) which concludes the proof.

We have reached the main result of this section.
Theorem 4.7

Let $i$, $k$, and $n_1, n_2, \ldots, n_k$ be positive integers, let $a(i,k) = t \cdot i + s$ where $0 \leq s \leq i - 1$, and let $N = \prod_{j=1}^{k} (n_j - 1)$.

If either

(i) $[s \cdot N/i] = 0$ or
(ii) $[i/s][s \cdot N/i] \leq N$,

then $r_i(n_1, n_2, \ldots, n_k/a) = 1 + [a(i,k) \cdot N/i]$.

**Proof:** If $n_j = 1$ for some $j$ where $1 \leq j \leq k$ the theorem holds independently of conditions (i) and (ii). Hence we assume $n_j \geq 2$ whenever $1 \leq j \leq k$. By Lemma 4.7 (a) it suffices to show that

$$r_i(n_1, n_2, \ldots, n_k/a) > [a(i,k) \cdot N/i].$$

By Lemma 4.5 (a), $a_1(K_t(i),(s)) \leq k$ so by Lemma 4.7 (b) there is a factorization

$$K_N[K_t(i),(s)] = \bigoplus_{j=1}^{k} F_j$$

where $a(F_j) \leq n_j - 1$ for every $j$ with $1 \leq j \leq k$. Therefore, it suffices to show that

$$K[a(i,k) \cdot N/i](i) \subset K_N[K_t(i),(s)].$$
We have

\[ K[a(i,k) \cdot N/i](i) = K[t \cdot N + [s \cdot N/i]](i) \]

\[ = \begin{cases} 
K[t \cdot N](i) + K[s \cdot N/i](i) & \text{if } [s \cdot N/i] \neq 0 \\
K[t \cdot N](i) & \text{if } [s \cdot N/i] = 0
\end{cases} \]

and

\[ K_N[K_t(i)(s)] = \begin{cases} 
K[t \cdot N](i) + K_N(s) & \text{if } s \neq 0 \\
K[t \cdot N](i) & \text{if } s = 0
\end{cases} \]

If condition (i) holds then the desired subgraph relationship is automatic. We thus assume condition (i) does not hold whence \( s \neq 0 \). It suffices to show that

\[ K[s \cdot N/i] \subseteq K_N(s) \cdot \]

This fact however is an immediate consequence of Lemma 4.7 (c) given that condition (ii) holds.

A number of corollaries follow from the above theorem. In particular, we have exact formulas for the complete graphs and the octahedral graphs.

**Corollary 4.7 A**

Let \( i, k, \) and \( n_1, n_2, \ldots, n_k \) be positive integers and let \( a(i,k) = t \cdot i + s \) with \( 0 \leq s \leq i - 1 \).
If \( s = 0 \) or if \( s = 1 \), then
\[
\rho_i(n_1, n_2, \ldots, n_k/a) = 1 + \left[ \sigma(i,k) \times \prod_{j=1}^{k} (n_j - 1)/i \right].
\]

Proof: If \( s = 0 \) (respectively \( s = 1 \)) it follows that condition (i) (respectively condition (ii)) of Theorem 4.7 is satisfied.

Corollary 4.7 B (Chartrand and Polimeni [8])

Let \( k \) and \( n_1, n_2, \ldots, n_k \) be positive integers. Then
\[
\rho_1(n_1, n_2, \ldots, n_k/a) = 1 + 2k \prod_{j=1}^{k} (n_j - 1).
\]

Proof: Since \( i = 1 \) here, then \( \sigma(i,k) = t \cdot i + 0 \), and it suffices to show \( \sigma(1,k) = 2k \).

By Theorem 4.5 it follows that
\[
\sigma(1,k) = t = 1 + \sqrt{(1 + 2k)^2 - 8k}
\]
\[
= 1 + 2k + \sqrt{(2k - 1)^2}
\]
\[
= 2k
\]

It follows from Corollary 4.7 A that
\( \sigma(i,k) = 2k + i - 1 \) when \( i = 1 \).

Corollary 4.7 C (the octahedral case)

Let \( k \) and \( n_1, n_2, \ldots, n_k \) be positive integers. Then
\[
\rho_2(n_1, n_2, \ldots, n_k/a) = 1 + \left[ (k + \frac{1}{2}) \prod_{j=1}^{k} (n_j - 1) \right].
\]
Proof: Since $i = 2$ here, then $a(i,k) = t \cdot i + s$ with $0 \leq s \leq 1$, and it suffices to show $a(2,k) = 2k + 1$.

By Theorem 4.5 it follows that

$$t = \left\lfloor \frac{4 + 4k + \sqrt{(4 + 4k)^2 - 32k}}{8} \right\rfloor$$

$$= \left\lfloor \frac{1 + k + \sqrt{k^2 + 1}}{2} \right\rfloor$$

$$= k.$$

It also follows that

$$s = \left\lfloor \frac{k(2k - 1) - \binom{k}{2} \cdot 4}{2k - k} \right\rfloor$$

$$= \left\lfloor \frac{k}{k} \right\rfloor = 1.$$

Hence we have that $a(2,k) = t \cdot 2 + s = 2k + 1$.

Note that $a(i,k) = 2k + i - 1$ if $i = 2$. The section is concluded by presenting two bounds.

**Corollary 4.7**

Let $i, k,$ and $n_1, n_2, \ldots, n_k$ be positive integers, and let $a(i,k) = t \cdot i + s$ with $2 \leq s \leq i - 1$.

Then

$$r_i(n_1, n_2, \ldots, n_k/a) \geq 1 + \left\lfloor (t \cdot i + 1) \prod_{j=1}^{k} (n_j - 1)/i \right\rfloor.$$

**Proof:** A factorization which admits the desired lower bound can be constructed in a manner similar to that of Theorem 4.7.
We observe that this lower bound need be applied only in the case both

\[(i) \left[ s \cdot \prod_{j=1}^{k} \left( n_j - 1 \right) / i \right] > 0 \text{ and} \]

\[(ii) \left[ i / s \right] \left[ s \cdot \prod_{j=1}^{k} \left( n_j - 1 \right) / i \right] > \prod_{j=1}^{k} \left( n_j - 1 \right) > 0 . \]

Defining the Ramsey index \( i(n_1, n_2, \ldots, n_k/a) \) in a natural fashion, we now show that

\[\lim_{i \to \infty} r_i(n_1, n_2, \ldots, n_k/a) = \lim_{i \to \infty} r_1(n_1, n_2, \ldots, n_k/a) . \]

That is letting \( I = i(n_1, n_2, \ldots, n_k/a) \), we show

\[r_I(n_1, n_2, \ldots, n_k/a) = r_1(n_1, n_2, \ldots, n_k/a) . \]

**Corollary 4.7**

Let \( k \) and \( n_1, n_2, \ldots, n_k \) be positive integers and let \( I = i(n_1, n_2, \ldots, n_k/a) \).

Then \( r_I(n_1, n_2, \ldots, n_k/a) = r(n_1, n_2, \ldots, n_k/a) \)

and \( I \leq (2k - 1) \prod_{j=1}^{k} (n_j - 1) + 1 . \)

**Proof:** If \( n_j = 1 \) for some \( j \) with \( 1 \leq j \leq k \), then the theorem is true. Hence let \( N = \prod_{j=1}^{k} (n_j - 1) \) be positive. By definition and using the factorization of Lemma 4.2 (a) we have
\[ r_i(n_1, n_2, \ldots, n_k/a) > N \]
\[ = r(n_1, n_2, \ldots, n_k/\chi) - 1 \]

for every \( i \in \mathbb{Z}^+ \) (and in particular \( i = 1 \)). Hence, letting \( i \geq (2k - 1)\cdot N + 1 \), it suffices to show that
\[ r_i(n_1, n_2, \ldots, n_k/a) \leq r(n_1, n_2, \ldots, n_k/\chi). \]

By Lemma 4.7 (a) and Theorem 4.6 we have
\[
\begin{align*}
 r_i(n_1, n_2, \ldots, n_k/a) &\leq 1 + \lceil a(i,k)\cdot N/i \rceil \\
 &\leq 1 + [(2k + i - 1)\cdot N/i] \\
 &= 1 + [(2k - 1)\cdot N/i] + N \\
 &= 1 + N \\
 &= r(n_1, n_2, \ldots, n_k/\chi). 
\end{align*}
\]

As a consequence of Corollary 4.7 E we have that
\[ r_i(n_1, n_2, \ldots, n_k/a) = 1 + \prod_{j=1}^{k} (n_j - 1) \]
for the positive integers \( n_1, n_2, \ldots, n_k \).

The discussion concludes with the following problems.

**Problem 4.1**

Each of the ramsey numbers \( r_i(n_1, n_2, \ldots, n_k) \) studied in this dissertation through Section 4.4 has been of a form whereby all the properties \( n_1, n_2, \ldots, n_k \) were essentially the same property. A **mixed** ramsey number is one where the properties are of at least two different types.
For example, for the positive integers $i$, $l_1$, $l_2$, $k = l_1 + l_2$, and $n_1, n_2, \ldots, n_k$ the mixed Ramsey number $r_i(n_1, n_2, \ldots, n_{k+1}, \ldots, n_k)$ is the least integer $p$ such that whenever $K_p(i) = \bigoplus_{j=1}^k F_j$ then it follows that either $\chi(F_j) \geq n_j$ for some $j$ with $1 \leq j \leq l_1$ or else $K_{n_j} \subset F_j$ for some $j$ with $l_1 + 1 \leq j \leq k$.

In a similar fashion we define the mixed Ramsey numbers $r_i(n_1, n_2, \ldots, n_{k+1}, \ldots, n_k)$ and $r_i(n_1, n_2, \ldots, n_{k+1}, \ldots, n_k)$. For positive integers $l_1, l_2, l_3$, and $k = l_1 + l_2 + l_3$ we similarly define the mixed Ramsey number $r_i(n_1, \ldots, n_{l_1+1}, \ldots, n_{l_1+l_2+1}, \ldots, n_k)$.

In a straightforward manner, the following can be proved.

Let $l, m, k = l + m$, and $n_1, n_2, \ldots, n_k$ be positive integers. Then

(i) $r_2(n_1, \ldots, n_{l+1}, \ldots, n_k) =$

$$1 + \sum_{j=1}^l (n_j - 1) \left[ r(n_{l+1}, n_{l+2}, \ldots, n_k) - 1 \right]$$

(ii) $r_2(n_1, \ldots, n_{l+1}, \ldots, n_k) =$

$$1 + \left[ (k + \frac{1}{2}) \sum_{j=1}^l (n_j - 1) \right] \left[ r(n_{l+1}, \ldots, n_k) - 1 \right]$$
If in addition \( k = l + m + n \) for \( n \geq 0 \) then

\[
(iii) \quad r_2(n_1, \ldots, n_l/a/n_{l+1}, \ldots, n_k) = 1 + \left[ \left( k + \frac{1}{2} \right) \prod_{j=1}^{k} (n_j - 1) \right] \prod_{j=l+1}^{k} (n_j - 1).
\]

The formulas all state that the Ramsey number for the "concatenation" is the product of the "individual Ramsey numbers minus 1" plus 1. The proof technique of Theorem 4.7 works here since the "second" Ramsey numbers are independent of \( i \). It seems that in general, a "new" proof technique must be developed to accommodate mixed \( i \)-th Ramsey numbers. Find this technique and develop some interesting formulas.

**Problem 4.2**

Let \( \rho \) be any property of graphs such that \( K_1 \) has property \( \rho \). For the graph \( G \) define the vertex partition parameter \( \rho(G) \) to be the least integer \( n \) such that there exists a partition \( V(G) = \bigcup_{j=1}^{n} V_j \) for which \( \langle V_j \rangle \) has property \( \rho \) whenever \( 1 \leq j \leq n \).

For interesting properties \( \rho \) define

\[
r_i(n_1, n_2, \ldots, n_k/\rho) \quad \text{and} \quad \overline{\rho}(i,k)
\]

in a natural manner.
Carry out a study of these quantities in a manner similar to that of Chapter IV. Make a similar study for edge partition parameters. (See [28] before starting.)
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