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Sum-Defined Colorings in Graphs

James Hallas
Western Michigan University, hallasjr@gmail.com

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James Hallas

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

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ABSTRACT

There have been numerous studies using a variety of methods for the purpose of uniquely distinguishing every two adjacent vertices of a graph. Many of these methods have involved graph colorings. The most studied colorings are proper colorings. A proper coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned distinct colors. The minimum number of colors required in a proper coloring of G is the chromatic number of G. In our work, we introduce a new coloring that induces a (nearly) proper coloring. Two vertices u and v in a nontrivial connected graph G are twins if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, they are referred to as true twins; while if u and v are nonadjacent, they are false twins. For a positive integer k, let c be a coloring of a graph G using colors in the set $\mathbb{N}_k = \{1, 2, \ldots, k\}$. Define another coloring s of G such that the color $s(v)$ of a vertex v is the sum of the colors of all vertices in the closed neighborhood of v. Then c is called a closed sigma k-coloring if $s(u) \neq s(v)$ for all pairs u, v of adjacent vertices that are not true twins. The minimum k for which G has a closed sigma k-coloring is the closed sigma chromatic number of G, denoted by $\chi_s(G)$. We study closed sigma colorings of graphs and the relationship among closed sigma colorings and other graphical parameters. Closed sigma chromatic numbers have been determined for several well-known classes of connected graphs. Other results and open questions are presented.

1 Introduction

A *neighbor-distinguishing coloring* is a coloring in which every pair of adjacent vertices are colored differently. Such a coloring is more commonly called a *proper coloring*. The minimum number of colors in a proper vertex coloring of a graph G is its chromatic number and is denoted by $\chi(G)$. A number of neighbor-distinguishing vertex colorings different from standard proper colorings have been introduced in the literature (see [4, pp.379-385], for example). In many situations, vertex or edge colorings (proper or not) of a graph give rise to neighbor-distinguishing colorings of the graph. In fact, two books [5, 6], published in 2015 and 2016, respectively, are devoted to the study of color-induced colorings in graphs. We refer to [1] for graph theory notation and terminology not described in this paper. All graphs under consideration here are nontrivial connected graphs.

In 2010 the concept of sigma vertex colorings was introduced in [3]. This is an example of a nonproper vertex coloring of a graph that induces a proper vertex coloring of the graph. More precisely, let G be a nontrivial connected graph and let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* $\sigma(v)$ of a vertex v in G is the sum of the colors of the vertices in the neighborhood $N(v)$ of v in G. That is,

$$\sigma(v) = \sum_{u \in N(v)} c(u) \ \text{ in } \mathbb{N}. \tag{1}$$

If $\sigma(u) \neq \sigma(v)$ for every two adjacent vertices u and v of G, then c is neighbor-distinguishing and c is a *sigma coloring* of G. The minimum number of colors required of a sigma coloring of G is the *sigma chromatic number* of G and is denoted by $\sigma(G)$. It was shown in [3] that if G is a connected graph of order $n \geq 2$, then $\sigma(G)$ exists and

$$1 \leq \sigma(G) \leq \chi(G) \leq n. \tag{2}$$

Furthermore, $\sigma(K_n) = n$ for every positive integer n and $\sigma(G) = 1$ if and only if every two adjacent vertices of G have different degrees. It was also shown in [3] that for each pair a, b of positive integers with $a \leq b$, there is a connected graph G with $\sigma(G) = a$ and $\chi(G) = b$.

In 2012 another neighbor-distinguishing vertex coloring was introduced in [2] that is closely related to the colorings discussed above. For a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring where adjacent vertices may be assigned the same color. The coloring c induces another vertex coloring $c' : V(G) \to \mathbb{Z}_k$, where

$c'(v) = \sum_{u \in N[v]} c(u)$, where $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v and the sum is performed in $\mathbb{Z}_k$. A coloring c of G is called a *closed modular k-coloring* if for every pair x, y of adjacent vertices in G either $c'(x) \neq c'(y)$ or $N[x] = N[y]$, where, in the latter case, we must have $c'(x) = c'(y)$. The minimum k for which G has a closed modular k-coloring is called the *closed modular chromatic number* of G. Closed modular colorings of graphs were introduced in [2] and arose from a domination problem.

Inspired by sigma colorings and numerous sum-defined neighbor-distinguishing colorings in the literature, we introduce another "neighbor-distinguishing" vertex coloring, one that also depends on the color of a vertex as well as those of its neighbors, but where the colors are positive integers. For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. The *closed color sum* $s(v)$ of a vertex v in G is defined as

$$s(v) = \sum_{u \in N[v]} c(u) \quad \text{in } \mathbb{N}. \tag{3}$$

That is, $s(v)$ is the sum of the colors of the vertices in the closed neighborhood $N[v] = N(v) \cup \{v\}$ of v. The vertex coloring c then induces another vertex coloring $s : V(G) \to \mathbb{N}$. If u and v are two vertices of G such that $N[u] = N[v]$, then $s(u) = s(v)$. On the other hand, if $s(u) \neq s(v)$ for every two adjacent vertices u and v of G for which $N[u] \neq N[v]$, then c is said to be (almost) *neighbor-distinguishing* and is called a *closed sigma coloring* of G. In a sigma coloring of a graph, the color of a vertex plays no role in the color sum of that vertex in (1), while in a closed sigma coloring, the initial color assigned to a vertex contributes to the color sum of the vertex in (3).

Two vertices u and v in a connected graph G are *twins* if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, they are referred to as *adjacent twins* (or *true twins*); while if u and v are nonadjacent, they are *nonadjacent twins* (or *false twins*).

Thus, in a closed sigma coloring of a graph, $s(u) = s(v)$ if u and v are true twins, $s(u) \neq s(v)$ if u and v are adjacent vertices that are not twins and no condition is placed on $s(u)$ and $s(v)$ otherwise. The minimum number of colors required in a closed sigma coloring of a graph G is called the *closed sigma chromatic number* of G and is denoted by $\chi_s(G)$. We first show that every graph has a closed sigma coloring and so $\chi_s(G)$ exists for every graph G.

Proposition 1.1 *If G is a connected graph of order n, then $\chi_s(G)$ exists and*

$$1 \leq \chi_s(G) \leq n.$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $n \geq 2$. Define a coloring $c : V(G) \to \mathbb{N}$ of G by $c(v_i) = 2^{i-1}$ for $1 \leq i \leq n$. If v_i and v_j are adjacent vertices that are not twins, then $N[v_i] \neq N[v_j]$ and so $s(v_i) \neq s(v_j)$. Thus c is a closed sigma n-coloring of G (since the base 2 representation of an integer is unique). Therefore, $\chi_s(G)$ exists and $\chi_s(G) \leq n$. ■

A closed sigma coloring using k colors is called a *closed sigma k-coloring*. A graph G is *closed sigma k-colorable* if $\chi_s(G) \leq k$. Thus, every connected graph of order n is closed sigma n-colorable. It is known that if $\chi(G) = k$, then G has a proper k-coloring using the colors $1, 2, \ldots, k$; that is, $\chi(G)$ is also the largest color used in a proper $\chi(G)$-coloring of G. This is not necessarily the case for closed sigma colorings of a graph, as we will see later.

2 Graphs Having Small Closed Sigma Chromatic Numbers

Since every two vertices in a nontrivial complete graph K_n of order n are adjacent twins, the coloring that assigns the color 1 to each vertex of K_n is a closed sigma coloring and so $\chi_s(K_n) = 1$ for all $n \geq 2$. We now characterize those connected graphs whose closed sigma chromatic number is 1.

Proposition 2.1 *Let G be a nontrivial connected graph. Then $\chi_s(G) = 1$ if and only if $\deg(u) \neq \deg(v)$ for every pair u, v of adjacent vertices that are not twins of G.*

Proof. First, assume that $\chi_s(G) = 1$. Then there is a closed sigma coloring $c : V(G) \to \mathbb{N}$ using exactly one color, say a. Note that $s(x) = a(\deg(x) + 1)$ for each $x \in V(G)$. If u and v are adjacent vertices that are not twins of G, then $s(u) \neq s(v)$ and so $\deg(u) \neq \deg(v)$.

For the converse, assume that $\deg(u) \neq \deg(v)$ for every pair u, v of adjacent vertices that are not twins of G. Then the coloring that assigns the same color to every vertex in G is a closed sigma coloring of G and so $\chi_s(G) = 1$. ■

Theorem 2.2 *If T is a nontrivial tree, then $\chi_s(T) \leq 2$ and T has a closed sigma coloring using only the colors 1 and 2.*

Proof. It suffices to show that every nontrivial tree has a closed sigma coloring using only the colors 1 and 2. First, we consider stars. By the proof of Proposition 2.1, the coloring that assigns the color 1 to each vertex in a star is a closed sigma coloring and so the statement holds for all stars. Assume, for some integer $n \geq 3$, that every tree of order at most n has a closed sigma coloring using only the colors 1 and 2. Let T be a tree of order $n+1$ and we may assume that T is not a star. Let x be a peripheral vertex and let u be the vertex that is adjacent to x. Thus, every vertex adjacent to u is an end-vertex with exactly one exception. Let $U = \{x = u_1, u_2, \ldots, u_k\}$ $(k \geq 1)$ be the set of end-vertices adjacent to u. Since T is not a star, $T^* = T - U$ is a tree of order at least 3 and at most n. By the inductive hypothesis, T^* has a closed sigma coloring $c_{T^*} : V(T^*) \to \{1, 2\}$. Let w be the vertex in T^* that is adjacent to u. Since c_{T^*} is a closed sigma coloring, $s_{c_{T^*}}(w) > s_{c_{T^*}}(u)$. Suppose that $s_{c_{T^*}}(w) - s_{c_{T^*}}(u) = \ell$, where then $\ell \in \mathbb{N}$. We now extend the closed sigma coloring $c_{T^*} : V(T^*) \to \{1, 2\}$ to a coloring $c_T : V(T) \to \{1, 2\}$ as follows. For each $v \in V(T) - U$, define $c_T(v) = c_{T^*}(v)$.

- If $k \neq \ell$, then define $c_T(u_i) = 1$ for $1 \leq i \leq k$. Then $s_{c_T}(u) = s_{c_{T^*}}(u) + k \neq s_{c_T}(w)$. Since $c_T(w) \neq 0$, it follows that $s_{c_T}(u) > s_{c_T}(u_i)$ for $1 \leq i \leq k$.
- If $k = \ell$, then define $c_T(u_1) = 2$ and $c_T(u_i) = 1$ for $2 \leq i \leq k$. Then $s_{c_T}(u) = s_{c_{T^*}}(u) + k + 1 \neq s_{c_T}(w)$ and $s_{c_T}(u) > s_{c_T}(u_i)$ for $1 \leq i \leq k$.

In each case, if $v \in V(T) - (U \cup \{u\})$, then $s_{c_T}(v) = s_{c_{T^*}}(v)$. Therefore, $c_T : V(T) \to \{1, 2\}$ is a closed sigma coloring using colors 1 and 2. ■

As an example, we consider a path P_n of order $n \geq 3$. If $n = 3$, then assign the color 1 to each vertex of P_n; while if $n \geq 4$, define a coloring $c : V(P_n) \to \{1, 2\}$ by $c(v_i) = 1$ if i is odd and $c(v_i) = 2$ if i is even. Then $s(v_1) = s(v_n) = 3$ and for $2 \leq i \leq n-1$ we have $s(v_i) = 5$ if i is odd and $s(v_i) = 4$ if i is even. Thus c is a closed sigma coloring and so

$$\chi_s(P_n) = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } n \geq 4 \end{cases}$$

By Theorem 2.2, every tree is closed sigma 2-colorable. This fact can be extended to all connected bipartite graphs.

Proposition 2.3 *Every nontrivial connected bipartite graph is closed sigma 2-colorable.*

Proof. Let G be a connected bipartite graph of order n. Since the result is trivial when $n = 2$, we may assume that $n \geq 3$. Let U and V be the partite sets of G and let $d = \Delta(G)$. Define a coloring $c : V(G) \to \mathbb{N}$ by $c(u) = 1$ for all $u \in U$ and $c(v) = d$ for all $v \in V$. Let u and v be adjacent vertices such that $u \in U$ and $v \in V$, where $\deg(u) = a$ and $\deg(v) = b$. It follows that $s(u) = 1 + ad$ and $s(v) = d + b$. Assume, to the contrary, that $s(u) = s(v)$ in $\mathbb{N}$ and so $1 + ad = d + b$. Thus, $d(a-1) = b - 1$ or $a - 1 = (b-1)/d$. Since $b - 1 < d$, it follows that $0 \leq (b-1)/d < 1$. Now $a - 1 \geq 0$ is an integer and so $a = b = 1$. Since G is connected and the order of G is at least 3, this is a contradiction, and so $s(u) \neq s(v)$ in $\mathbb{N}$ for every two adjacent vertices u and v. Therefore, c is a closed sigma 2-coloring of G. ■

Remark. Although a connected bipartite graph G is closed sigma 2-colorable by Proposition 2.3, it is not known whether G has a closed sigma 2-coloring using colors 1 and 2.

For the cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ of order $n \geq 4$ and a coloring $c : V(C_n) \to \mathbb{N}$, define the *color sequence* $\mathcal{S}_c$ *of* c as $\mathcal{S}_c = (c(v_1), c(v_2), \ldots, c(v_n))$.

Proposition 2.4 *For each integer* $n \geq 4$*,* $\chi_s(C_n) = \chi(C_n)$.

Proof. By Proposition 2.1, $\chi_s(C_n) \geq 2$ for each $n \geq 4$. If n is even, then $\chi_s(C_n) \leq 2$ by Proposition 2.3 and so $\chi_s(C_n) = 2 = \chi(C_n)$. Thus, we may assume that n is odd. First, we show that there is a closed sigma 3-coloring of C_n.

For $n = 5$, define $c : V(G) \to \mathbb{N}$ such that the color sequence of c is $\mathcal{S}_c = (3, 2, 2, 1, 1)$. Thus the color sequence of the induced coloring s is $\mathcal{S}_s = (6, 7, 5, 4, 5)$. Thus c is a closed sigma 3-coloring of G. For $n \geq 7$, define $c : V(G) \to \{1, 2, 3\}$ by

$$c(v_i) = \begin{cases} 3 & \text{if } i \in \{1, 2, 3\} \\ 1 & \text{if } i \equiv 0, 4, 5 \pmod{6} \\ 2 & \text{if } i \equiv 1, 2, 3 \pmod{6} \text{ and } i \geq 4. \end{cases}$$

For $n = 7$, $\mathcal{S}_c = (3, 3, 3, 1, 1, 1, 2)$ and so $\mathcal{S}_s = (8, 9, 7, 4, 3, 4, 6)$. Then c is a closed sigma 3-coloring of G for $n = 7$. Thus, we may assume that $n \geq 9$. Let

$$\begin{aligned} \mathcal{S}_0 &= (\underline{3,3,3},\ \underline{1,1,1},\ \underline{2,2,2},\ \underline{1,1,1},\ \underline{2,2,2}, \ldots, \underline{1,1,1},\ \underline{2,2,2}). \\ \mathcal{S}_1 &= (\underline{3,3,3},\ \underline{1,1,1},\ \underline{2,2,2},\ \underline{1,1,1},\ \underline{2,2,2}, \ldots, \underline{1,1,1},\ \underline{2,2,2}, \underline{1,1,1}). \end{aligned}$$

Then the color sequence of c is

$$\mathcal{S}_c = \begin{cases} \mathcal{S}_0 & \text{if } n = 3k \text{ and } k \text{ is odd} \\ (\mathcal{S}_0, 1, 1) & \text{if } n = 3k+2 \text{ and } k \text{ is odd} \\ (\mathcal{S}_1, 2) & \text{if } n = 3k+1 \text{ and } k \text{ is even.} \end{cases}$$

For example then,

$$\mathcal{S}_c = \begin{cases} (3,3,3,1,1,1,2,2,2) & \text{if } n = 9 \\ (3,3,3,1,1,1,2,2,2,1,1) & \text{if } n = 11 \\ (3,3,3,1,1,1,2,2,2,1,1,1,2) & \text{if } n = 13. \end{cases}$$

If $n = 9$, then $\mathcal{S}_s = (8, 9, 7, 4, 3, 4, 5, 6, 7)$. For $n \geq 11$, in $\mathbb{Z}_3$ let

$$\begin{array}{rcl} \mathcal{S}_{0,s} & = & (0, 1, \underline{2, 0, 1},\ \underline{2, 0, 2},\ \underline{1, 0, 1},\ \underline{2, 0, 2},\ \underline{1, 0, 1}, \ldots, 2, 0) \\ \mathcal{S}_{1,s} & = & (0, 1, \underline{2, 0, 1},\ \underline{2, 0, 2},\ \underline{1, 0, 1},\ \underline{2, 0, 2},\ \underline{1, 0, 1}, \ldots, \underline{2, 0, 2}, 1, 0). \end{array}$$

Then the color sequence of s is

$$\mathcal{S}_s = \begin{cases} (2, \mathcal{S}_{0,s}, 1) & \text{if } n = 3k \text{ and } k \text{ is odd} \\ (1, \mathcal{S}_{0,s}, 2, 1, 2) & \text{if } n = 3k+2 \text{ and } k \text{ is odd} \\ (2, \mathcal{S}_{1,s}, 1, 0) & \text{if } n = 3k+1 \text{ and } k \text{ is even} \end{cases}$$

Thus c is a closed sigma coloring and so $\chi_s(C_n) \leq 3$.

We show next that C_n has no closed sigma 2-coloring. Assume, to the contrary, that there is a closed sigma 2-coloring c using colors a and b. By a *block of a color* is meant a largest sequence of consecutive vertices of C_n that are assigned the same color. Note that no block can have length 4 or more. Also, if a block has length 2, then those two vertices will have the same closed color sum. Thus, every block must have length 1 or 3. Since the number of blocks colored a and the number of blocks colored b are the same, it follows that the number of vertices in C_n is even, which is a contradiction. Thus there is no closed sigma 2-coloring, and so $\chi_s(C_n) \geq 3$. Therefore, $\chi_s(C_n) = 3 = \chi(C_n)$ when n is odd. ■

3 Complete Multipartite Graphs

By Proposition 2.3, every nontrivial connected bipartite graph G is closed sigma 2-colorable. However, it is not known whether G has a closed sigma 2-coloring using colors

1 and 2 in general. We have seen in Theorem 2.2 that every nontrivial tree has a closed sigma 2-coloring using colors 1 and 2. This is also true for complete bipartite graphs, as we show next.

Proposition 3.1 *For positive integers r and s with $r \leq s$, the complete bipartite graph $K_{r,s}$ has a closed sigma coloring using only the colors 1 and 2. Furthermore,*

$$\chi_s(K_{r,s}) = \begin{cases} 1 & \text{if } r < s \text{ or } r = s = 1 \\ 2 & \text{if } r = s \geq 2. \end{cases}$$

Proof. Let $G = K_{r,s}$ where $1 \leq r \leq s$. If $r < s$, then $\deg x \neq \deg y$ for every pair x, y of adjacent vertices of G. It then follows by Proposition 2.1 that $\chi_s(G) = 1$. If $r = s = 1$, then $G = K_2$ and so $\chi_s(G) = 1$. If $r = s \geq 2$, then $\chi_s(G) \geq 2$, again by Proposition 2.1. Next, we show that there is a closed sigma coloring using the colors 1 and 2 when $r = s \geq 2$. Let U and V are partite sets of G with $|U| = |V| = r$. Define a coloring $c : V(G) \to \{1, 2\}$ by $c(u) = 1$ if $u \in U$ and $c(v) = 2$ if $v \in V$. Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$ defined by

$$\begin{aligned} s(u) &= 2r + 1 \quad \text{for } u \in U \\ s(v) &= r + 2 \quad \text{for } v \in V \end{aligned}$$

Since $r \geq 2$, it follows that $s(u) > s(v)$ for all $u \in U$ and $v \in V$ and so s is proper. Thus, c is a closed sigma 2-coloring of G using colors 1 and 2. Therefore, $\chi_s(G) \leq 2$ and so $\chi_s(G) = 2$ when $r = s \geq 2$. ■

In the case of complete tripartite graphs, the situation is more complicated.

Theorem 3.2 *For positive integers r, s and t with $r \leq s \leq t$, the complete tripartite graph $K_{r,s,t}$ has a closed sigma coloring using colors in the set $\{1, 2, 3\}$. Furthermore,*

$$\chi_s(K_{r,s,t}) = \begin{cases} 1 & \text{if } r < s < t \text{ or } r = s = 1 \leq t \\ 2 & \text{if (i) } |\{r, s, t\}| \leq 2 \text{ and } s \geq 4, \\ & \text{(ii) } r = s \in \{2, 3\} \text{ and } s < t \text{ or} \\ & \text{(iii) } r < s = t \in \{2, 3\} \\ 3 & \text{if } r = s = t \in \{2, 3\}. \end{cases}$$

Proof. Let $G = K_{r,s,t}$ whose partite sets are

$$U = \{u_1, u_2, \ldots, u_r\},\ V = \{v_1, v_2, \ldots, v_s\} \text{ and } W = \{w_1, w_2, \ldots, w_t\}.$$

First, suppose that $r < s < t$ or $r = s = 1 \le t$. We show that $\chi_s(G) = 1$.

- ⋆ If $r < s < t$, then $\deg u = s + t$ for each $u \in U$, $\deg v = r + t$ for each $v \in V$ and $\deg w = r + s$ for each $w \in W$. Since $s + t \ne r + t$, $r + s \ne r + t$ and $t + s \ne r + s$, it follows that then $\deg x \ne \deg y$ for every pair x, y of adjacent vertices that are not twins of G. It then follows by Proposition 2.1 that $\chi_s(G) = 1$.
- ⋆ For $r = s = 1 \le t$, $G = K_{1,1,t}$. If $t = 1$, then $G = K_3$ and so $\chi_s(G) = 1$ by Proposition 2.1. If $t > 1$, then $\deg u_1 = \deg v_1 = 1 + t \ge 3$ and $\deg w = 2$ for each $w \in W$. Since u_1 and v_1 are true twins, it follows that $\deg x \ne \deg y$ for every pair x, y of adjacent vertices that are not twins of G. It then again follows by Proposition 2.1 that $\chi_s(G) = 1$.

We now consider the three cases when $\chi_s(G) = 2$, namely,

(i) $|\{r, s, t\}| \le 2$ and $s \ge 4$, (ii) $r = s \in \{2, 3\}$ and $s < t$ and (iii) $r < s = t \in \{2, 3\}$.

Case 1. $|\{r, s, t\}| \le 2$ *and* $s \ge 4$. First, suppose that $|\{r, s, t\}| = 1$. Then $r = s = t$ and $G = K_{r,r,r}$ is a $(2r)$-regular graph. Since there are two adjacent vertices that are not twins of G, it follows by Proposition 2.1 that $\chi_s(G) \ge 2$. Next, we show that there is a closed sigma coloring using two colors. Define a coloring $c : V(G) \to \{1, 2\}$ by

$$c(x) = \begin{cases} 1 & \text{if } x \in U \cup \{v_3, v_4, \ldots, v_s\} \\ 2 & \text{otherwise.} \end{cases} \tag{4}$$

Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$ defined by

$$\begin{aligned} s(u_i) &= 3 + s + 2t = 3 + 3r \quad \text{for } 1 \le i \le r \\ s(v_j) &= \begin{cases} 1 + r + 2t = 1 + 3r & \text{if } 3 \le j \le r \\ 2 + r + 2t = 2 + 3r & \text{if } j = 1, 2 \end{cases} \\ s(w) &= 4 + s + r = 4 + 2r \le 3r \ \text{ (since } r \ge 4). \end{aligned}$$

Hence $s(w) < s(v) < s(u)$ where $u \in U$, $v \in V$ and $w \in W$. Since s is a proper vertex coloring of G, it follows that c is a closed sigma 2-coloring of G using colors 1 and 2. Thus, $\chi_s(G) \le 2$ and so $\chi_s(G) = 2$ when $r = s = t \ge 4$.

Next, suppose that $|\{r,s,t\}| = 2$. Thus, either $r = s < t$ or $r < s = t$. We consider these two subcases.

Subcase 1.1. $r = s < t$ *and* $s \geq 4$. If $u \in U$ and $v \in V$, then u and v are not true twins and $\deg u = \deg v$; it follows by Proposition 2.1 that $\chi_s(G) \geq 2$. Next, we show that there is a closed sigma coloring using two colors. Consider the coloring $c : V(G) \to \{1,2\}$ as described in (4). Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$ defined by

$$\begin{aligned} s(u_i) &= 3 + s + 2t \quad \text{ for } 1 \leq i \leq r \\ s(v_j) &= \begin{cases} 1 + r + 2t & \text{if } 3 \leq j \leq r \\ 2 + r + 2t & \text{if } j = 1, 2 \end{cases} \\ s(w) &= 4 + s + r. \end{aligned}$$

Since $r = s < t$ and $s \geq 4$, it follows that $s(w) < s(v) < s(u)$ where $u \in U$, $v \in V$ and $w \in W$. Hence s is a proper vertex coloring of G and so c is a closed sigma 2-coloring of G using colors 1 and 2. Therefore, $\chi_s(G) \leq 2$ and so $\chi_s(G) = 2$ when $r = s < t$ and $s \geq 4$.

Subcase 1.2. $r < s = t$ *and* $s \geq 4$. If $v \in V$ and $w \in V$, then v and w are not true twins and $\deg v = \deg w$; it follows by Proposition 2.1 that $\chi_s(G) \geq 2$. Next, we show that there is a closed sigma coloring using two colors. Consider the coloring $c : V(G) \to \{1,2\}$ as described in (4). Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$ defined by

$$\begin{aligned} s(u_i) &= 3 + s + 2t \text{ for } 1 \leq i \leq r \\ s(v_j) &= \begin{cases} 1 + r + 2t & \text{if } 3 \leq j \leq r \\ 2 + r + 2t & \text{if } j = 1, 2 \end{cases} \\ s(w) &= 4 + s + r. \end{aligned}$$

Since $r < s = t$ and $s \geq 4$, it follows that (1) $s(u) > s(v)$ when $u \in U$ and $v \in V$ and (2) $s(v) \geq 1 + r + 2t \geq 5 + r + t > s(w)$. Hence s is a proper vertex coloring of G and so c is a closed sigma 2-coloring of G using colors 1 and 2. Therefore, $\chi_s(G) \leq 2$ and so $\chi_s(G) = 2$ when $r < s = t$ and $s \geq 4$.

Case 2. $r = s \in \{2,3\}$ *and* $s < t$. Since $r = s$, it follows that $\deg u = \deg v$ for all $u \in U$ and $v \in V$. By Proposition 2.1, $\chi_s(G) \geq 2$. Next, we show that there is a closed sigma coloring using two colors. Define a coloring $c : V(G) \to \{1, 2\}$ by

$$c(x) = \begin{cases} 1 & \text{if } x \in U \\ 2 & \text{if } x \in V \cup W. \end{cases} \tag{5}$$

Consider the vertex coloring $s : V(G) \to \mathbb{N}$ induced by c.

- ⋆ If $r = s = 2$, then

$$\begin{array}{rcll} s(u) & = & 5 + 2t & \text{for } u \in U \\ s(v) & = & 4 + 2t & \text{for } v \in V \\ s(w) & = & 8 & \text{for } w \in W. \end{array}$$

- ⋆ If $r = s = 3$, then

$$\begin{array}{rcll} s(u) & = & 7 + 2t & \text{for } u \in U \\ s(v) & = & 5 + 2t & \text{for } v \in V \\ s(w) & = & 11 & \text{for } w \in W. \end{array}$$

Since $t \geq 3$, it follows that $s(w) < s(v) < s(u)$ where $u \in U$, $v \in V$ and $w \in W$ in each case. Hence, s is a proper vertex coloring of G and so c is a closed sigma 2-coloring of G using colors 1 and 2. Therefore, $\chi_s(G) \leq 2$ and so $\chi_s(G) = 2$ when $r = s \in \{2, 3\}$ and $s < t$.

Case 3. $r < s = t \in \{2, 3\}$. Since $s = t$, it follows that $\deg u = \deg w$ for all $u \in U$ and $w \in W$. By Proposition 2.1, $\chi_s(G) \geq 2$. Next, we show that there is a closed sigma coloring using two colors. Define a coloring $c : V(G) \to \{1, 2\}$ by

$$c(x) = \begin{cases} 1 & \text{if } x \in U \cup V \\ 2 & \text{if } x \in W. \end{cases} \tag{6}$$

Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$ defined by

$$\begin{array}{rcll} s(u) & = & 1 + s + 2t & \text{for } u \in U \\ s(v) & = & 1 + r + 2t & \text{for } v \in V \\ s(w) & = & 2 + r + s & \text{for } w \in W. \end{array}$$

- ⋆ If $s = t = 2$, then $r = 1$ and so $1 + s + 2t = 7$, $1 + r + 2t = 6$ and $2 + r + s = 5$. Thus, $s(w) < s(v) < s(u)$ where $u \in U$, $v \in V$ and $w \in W$.

- ⋆ If $s = t = 3$, then $1 + s + 2t = 10$, $1 + r + 2t = 7 + r$ and $2 + r + s = 5 + r$. Since $r \leq 2$, it follows that $10 > 7 + r > 5 + r$. Therefore, $s(w) < s(v) < s(u)$ where $u \in U$, $v \in V$ and $w \in W$.

In each case, s is a proper vertex coloring of G and so c is a closed sigma 2-coloring of G using colors 1 and 2. Therefore, $\chi_s(G) \leq 2$ and so $\chi_s(G) = 2$ when $r < s = t \in \{2, 3\}$.

Finally, we consider the case when $r = s = t \in \{2, 3\}$. We show that $\chi_s(G) = 3$ in this case. Assume, to the contrary, that there is a closed sigma coloring $c : V(G) \to \{a, b\}$ where $a, b \in \mathbb{N}$ and $a \neq b$. Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$. We consider two cases, according to whether $r = s = t = 2$ or $r = s = t = 3$.

Case 1. $r = s = t = 2$. Then $G = K_{2,2,2}$. Observe that

$$\begin{aligned}
s(u_1) &= c(u_1) + c(v_1) + c(v_2) + c(w_1) + c(w_2) \\
s(v_1) &= c(u_1) + c(u_2) + c(v_1) + c(w_1) + c(w_2) \\
s(w_1) &= c(u_1) + c(u_2) + c(v_1) + c(v_2) + c(w_1).
\end{aligned}$$

Since $s(u_1) \neq s(v_1)$, it follows that $c(v_2) \neq c(u_2)$; since $s(u_1) \neq s(w_1)$, it follows that $c(u_2) \neq c(w_2)$; and since $s(v_1) \neq s(w_1)$, it follows that $c(v_2) \neq c(w_2)$. Thus, $c(u_2), c(v_2), c(w_3)$ are three distinct colors used by c, which is a contradiction.

Case 2. $r = s = t = 3$. Then $G = K_{3,3,3}$. An argument similar to the one used in Case 1 shows the following: For each pair i, j of integers $i, j \in \{1, 2, 3\}$ and $i < j$,

$$c(w_i) + c(w_j) \neq c(u_i) + c(u_j) \neq c(v_i) + c(v_j) \neq c(w_i) + c(w_j).$$

Since $c(x) \in \{a, b\}$ for each vertex x of G, we may assume, without loss of generality, that $c(u_1) = c(w_1) = c(w_2) = a$ and $c(u_2) = c(v_1) = c(v_2) = b$. We may further assume that $c(w_3) = a$ and $c(u_3) = b$. However then, since $c(u_3) = c(v_3) = b$, it follows that $c(u_2) + c(u_3) = 2b = c(w_2) + c(w_3)$, which is impossible.

Therefore, G has no closed sigma coloring using two colors and so $\chi_s(G) \geq 3$.

Next, we show that G has a closed sigma coloring using the three colors 1, 2 and 3. Define a coloring $c : V(G) \to \{1, 2, 3\}$ by

$$c(x) = \begin{cases} 1 & \text{if } x \in U \\ 2 & \text{if } x \in V \\ 3 & \text{if } x \in W. \end{cases} \tag{7}$$

Let $s : V(G) \to \mathbb{N}$ be the vertex coloring of G induced by c. Since $r = s = t \in \{2, 3\}$, it follows that

$$\begin{aligned} s(u) &= 1 + 5r \quad \text{for } u \in U \\ s(v) &= 2 + 4r \quad \text{for } v \in V \\ s(w) &= 3 + 3r \quad \text{for } w \in W. \end{aligned}$$

Since $r \in \{2, 3\}$, it follows that $s(w) < s(v) < s(u)$ where $u \in U$, $v \in V$ and $w \in W$ and so s is a proper vertex coloring of G. Thus, c is a closed sigma 2-coloring of G using the three colors 1, 2 and 3. Therefore, $\chi_s(G) \leq 3$ and so $\chi_s(G) = 3$ when $r = s = t \in \{2, 3\}$. ■

By Theorem 3.2, $\chi_s(K_{1,2,3}) = 1$, $\chi_s(K_{1,2,2}) = 2$ and $\chi_s(K_{2,2,2}) = 3$. In general, for an integer $k \geq 2$, let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph with partite sets $V_1, V_2, \ldots, V_k$ where $|V_i| = n_i$ for $1 \leq i \leq k$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. If $n_1 = n_2 = \cdots = n_k = r$, then G is denoted by $K_{k(r)}$.

Conjecture 3.3 *If G is a complete k-partite graph for some integer $k \geq 2$, then $\chi_s(G) \leq k$ and G has a closed sigma coloring using colors in the set $\{1, 2, \ldots, k\}$.*

4 Cartesian Products of Graphs

The *Cartesian product* G of two graphs F and H, commonly denoted by $F \square H$ or $F \times H$, has vertex set $V(G) = V(F) \times V(H)$, where two distinct vertices (u, v) and (x, y) of $F \square H$ are adjacent if either (1) $u = x$ and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(F)$. In particular, if $H = K_2$, then $F \square K_2$ is obtained from two copies F_1 and F_2 of F, where $V(F_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(F_2) = \{v_1, v_2, \ldots, v_n\}$, by adding the n edges u_iv_i for $1 \leq i \leq n$. In this section, we study closed sigma colorings of $C_n \square K_2$, $P_n \square K_2$ and $K_n \square K_2$ for each integer $n \geq 3$, beginning with $C_n \square K_2$.

For the cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ of order $n \geq 4$ and a coloring $c : V(C_n) \to \mathbb{N}$, the *color sequence of* c is defined as $(c(v_1), c(v_2), \ldots, c(v_n))$.

Proposition 4.1 *For each integer $n \geq 4$, the graph $C_n \square K_2$ has a closed sigma coloring using only the colors 1 and 2 and so $\chi_s(C_n \square K_2) = 2$.*

Proof. Let $G = C_n \square K_2$, where $C = (u_1, u_2, \ldots, u_n, u_1)$ and $C' = (v_1, v_2, \ldots, v_n, v_1)$ are the two copies of C_n and $u_iv_i \in E(G)$ for $1 \leq i \leq n$. Since G is 3-regular and has no twins, it follows by Proposition 2.1 that $\chi_s(C_n \square K_2) \geq 2$. It remains to show that

G has a closed sigma coloring using only the colors 1 and 2. We consider two cases, according to whether n is even or n is odd.

Case 1. *n is even.* Define the proper coloring $c : V(G) \to \{1, 2\}$ by

$$c(u_i) = \begin{cases} 1 & \text{if } i \text{ is even and } 2 \le i \le n \\ 2 & \text{if } i \text{ is odd and } 1 \le i \le n-1, \end{cases}$$

$$c(v_j) = \begin{cases} 2 & \text{if } j \text{ is even and } 2 \le j \le n \\ 1 & \text{if } j \text{ is odd and } 1 \le j \le n-1. \end{cases}$$

Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$ defined by

$$s(u_i) = \begin{cases} 7 & \text{if } i \text{ is even and } 2 \le i \le n \\ 5 & \text{if } i \text{ is odd and } 1 \le i \le n-1, \end{cases}$$

$$s(v_j) = \begin{cases} 5 & \text{if } j \text{ is even and } 2 \le j \le n \\ 7 & \text{if } j \text{ is odd and } 1 \le j \le n-1. \end{cases}$$

Since s is a proper vertex coloring of G, it follows that c is a closed sigma 2-coloring of G using colors 1 and 2.

Case 2. *n is odd.* Define the proper coloring $c : V(G) \to \{1, 2\}$ by

$$c(u_i) = \begin{cases} 1 & \text{if } i \text{ is even and } 2 \le i \le n-3 \text{ or } i = n \\ 2 & \text{if } i \text{ is odd and } 1 \le i \le n-2 \text{ or } i = n-1 \end{cases}$$

$$c(v_j) = \begin{cases} 1 & \text{if } j \text{ is odd and } 1 \le j \le n-2 \text{ or } j = n \\ 2 & \text{if } j \text{ is even and } 2 \le j \le n-3 \text{ or } j = n-1. \end{cases}$$

Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$ defined by

$$s(u_i) = \begin{cases} 5 & \text{if } i \text{ is odd and } 1 \leq i \leq n-4 \\ 6 & \text{if } i = n-2 \text{ or } i = n \\ 7 & \text{if } i \text{ is even and } 2 \leq i \leq n-1, \end{cases}$$

$$s(v_j) = \begin{cases} 5 & \text{if } j \text{ is even and } 2 \leq j \leq n-3 \text{ or } j = n \\ 6 & \text{if } j = 1 \text{ or } j = n-1 \\ 7 & \text{if } j \text{ is odd and } 3 \leq j \leq n-2. \end{cases}$$

For the two cycles $C = (u_1, u_2, \ldots, u_n, u_1)$ and $C' = (v_1, v_2, \ldots, v_n, v_1)$ in G, the color sequences $s(C)$ and $s(C')$ of s are, respectively,

$$\begin{aligned} s(C) &= (s(u_1), s(u_2), \ldots, s(u_n)) = (\underline{5,7},\ \underline{5,7},\ \ldots,\ \underline{5,7},\ 6,7,6) \\ s(C') &= (s(v_1), s(v_2),\ \ldots, s(v_n)) = (\underline{6,5},\ \underline{7,5},\ \ldots,\ \underline{7,5},\ 7,6,5). \end{aligned}$$

Hence s is a proper vertex coloring of G and so c is a closed sigma 2-coloring of G using colors 1 and 2. ■

Proposition 4.2 *For each integer $n \geq 3$, the n-cube Q_n has a closed sigma coloring using only the colors 1 and 2.*

Proof. The graph Q_n is an n-regular bipartite graph of order 2^n. By Propositions 2.1 and 2.3, $\chi_s(Q_n) = 2$ for each $n \geq 3$. We show, in fact, that Q_n has a closed sigma coloring using only the colors 1 and 2. Let U and W be two partite sets of Q_n. Define the proper coloring $c : V(Q_n) \to \{1, 2\}$ by $c(u) = 1$ for each $u \in U$ and $c(w) = 2$ for each $w \in W$. Then the induced vertex coloring $s : V(Q_n) \to \mathbb{N}$ is defined by

$$\begin{aligned} s(u) &= 1 + 2(n-1) = 2n+1 \text{ for each } u \in U \\ s(w) &= 2 + (n-1) = n+1 \text{ for each } w \in W. \end{aligned}$$

Since $s(u) \neq s(w)$ for all $u \in U$ and $w \in W$, it follows that s is a proper vertex coloring of Q_n. Therefore, c is a closed sigma 2-coloring of Q_n using colors 1 and 2. ■

Proposition 4.3 *For each integer $n \geq 3$, the grid $P_n \,\square\, K_2$ has a closed sigma 2-coloring using colors 1 and 2.*

Proof. Since $G = P_n \square K_2$ is bipartite, it follows by Propositions 2.1 and 2.3 that $\chi_s(G) = 2$. We show that G has a closed sigma 2-coloring c using colors 1 and 2. Let $P = (u_1, u_2, \ldots, u_n)$ and $P' = (v_1, v_2, \ldots, v_n)$ be the two copies of P_n in G where $u_iv_i \in E(G)$ for $1 \leq i \leq n$.

- ⋆ For $n = 3$, let $(c(u_1), c(u_2), c(u_3)) = (1, 1, 1)$ and $(c(v_1), c(v_2), c(v_3)) = (1, 2, 2)$;
- ⋆ For $n = 4$, let $(c(u_1), c(u_2), c(u_3), c(u_4)) = (1, 1, 2, 1)$ and $(c(v_1), c(v_2), c(v_3), c(v_4)) = (1, 2, 1, 1)$;
- ⋆ For $n = 5$, let $(c(u_1), c(u_2), \ldots, c(u_5)) = (1, 1, 2, 1, 1)$ and $(c(v_1), c(v_2), \ldots, c(v_5)) = (1, 2, 1, 2, 1)$;

Since this is a closed sigma 2-coloring using colors 1 and 2 for each $n = 3, 4, 5$, we may assume that $n \geq 6$. We consider two cases, according to whether n is even or n is odd.

Case 1. *$n \geq 6$ is even.* Define the proper coloring $c : V(G) \to \{1, 2\}$ by

$$c(u_i) = \begin{cases} 1 & \text{if } i \text{ is even and } 2 \leq i \leq n \text{ or } i = 1 \\ 2 & \text{if } i \text{ is odd and } 3 \leq i \leq n-1, \end{cases}$$

$$c(v_j) = \begin{cases} 1 & \text{if } j \text{ is odd and } 1 \leq j \leq n-1 \text{ or } j = n \\ 2 & \text{if } j \text{ is even and } 2 \leq j \leq n-2. \end{cases}$$

Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$ defined by

$$s(u_i) = \begin{cases} 3 & \text{if } i = 1 \\ 4 & \text{if } i = n \\ 5 & \text{if } i \text{ is odd and } 3 \leq i \leq n-1 \\ 6 & \text{if } i = 2 \\ 7 & \text{if } i \text{ is even and } 4 \leq i \leq n-2 \end{cases}$$

and

$$s(v_j) = \begin{cases} 3 & \text{if } j = n \\ 4 & \text{if } j = 1 \\ 5 & \text{if } j \text{ is even and } 2 \leq j \leq n-2 \\ 6 & \text{if } j = n-1 \\ 7 & \text{if } j \text{ is odd and } 3 \leq j \leq n-3 \end{cases}$$

The color sequences $s(P)$ and $s(P')$ of s are, respectively,

$$\begin{aligned} s(P) &= (\underline{3,6},\ \underline{5,7},\ \ldots,\ \underline{5,7},\ 5,4) \\ s(P') &= (\underline{4,5},\ \underline{7,5},\ \ldots,\ \underline{7,5},\ 6,3). \end{aligned}$$

Case 2. *$n \geq 7$ is odd.* Define the proper coloring $c : V(G) \to \{1,2\}$ by

$$c(u_i) = \begin{cases} 1 & \text{if } i \text{ is even and } 2 \leq i \leq n-1 \text{ or } i = 1, n \\ 2 & \text{if } i \text{ is odd and } 3 \leq i \leq n-2, \end{cases}$$

$$c(v_j) = \begin{cases} 1 & \text{if } j \text{ is odd and } 1 \leq j \leq n \\ 2 & \text{if } j \text{ is even and } 2 \leq j \leq n-1. \end{cases}$$

Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$ defined by

$$s(u_i) = \begin{cases} 3 & \text{if } i = 1 \text{ or } i = n \\ 5 & \text{if } i \text{ is odd and } 3 \leq i \leq n-2 \\ 6 & \text{if } i = 2 \text{ or } i = n-1 \\ 7 & \text{if } i \text{ is even and } 4 \leq i \leq n-3 \end{cases}$$

$$s(v_j) = \begin{cases} 4 & \text{if } j = 1 \text{ or } j = n \\ 5 & \text{if } j \text{ is even and } 2 \leq j \leq n-1 \\ 7 & \text{if } j \text{ is odd and } 3 \leq j \leq n-2 \end{cases}$$

The color sequences $s(P)$ and $s(P')$ of s are, respectively,

$$\begin{aligned} s(P) &= (\underline{3,6},\ \underline{5,7},\ \ldots,\ \underline{5,7},\ 5,6,3) \\ s(P') &= (\underline{4,5},\ \underline{7,5},\ \ldots,\ \underline{7,5},\ 7,5,4). \end{aligned}$$

In each case, s is a proper vertex coloring of G and so c is a closed sigma 2-coloring of G using colors 1 and 2. ■

Proposition 4.4 *For each integer $n \geq 2$, the graph $K_n \square K_2$ has a closed sigma n-coloring of G using colors $1, 2, \ldots, n$ and $\chi_s(K_n \square K_2) = n$.*

Proof. Let $G = K_n \square K_2$, where $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ are the vertex sets of the two copies of K_n in G and $u_iv_i \in E(G)$ for $1 \leq i \leq n$. We show that G

has a closed sigma n-coloring of G using colors $1, 2, \ldots, n$. Define the proper coloring $c : V(G) \to \{1, 2, \ldots, n\}$ by $c(u_i) = 1$ for $1 \le i \le n$, $c(v_j) = j + 1$ for $1 \le j \le n - 1$ and $c(v_n) = 1$. Then $\sum_{i=1}^{n} c(u_i) = \sum_{j=1}^{n} c(v_j) = \frac{n(n+1)}{2} = a$. The induced vertex coloring $s : V(G) \to \mathbb{N}$ is then defined by $s(u_i) = a + i + 1$ if $1 \le i \le n - 1$, $s(u_n) = a + 1$ and $s(v_j) = a + j$ for $1 \le j \le n$. Observe that

$$\begin{aligned}(s(u_1), s(u_2), \ldots, s(u_n)) &= (a+2,\ a+3, \ldots, a+n-1,\ a+n,\ a+1)\\ (s(v_1), s(v_2),\ \ldots, s(v_n)) &= (a+1,\ a+2, \ldots, a+n-2,\ a+n-1,\ a+n).\end{aligned}$$

Since s is a proper vertex coloring of G, it follows that c is a closed sigma n-coloring of G using colors $1, 2, \ldots, n$ and so $\chi_s(G) \le n$.

Next, we show that $\chi_s(G) = n$. Assume, to the contrary, that $\chi_s(G) = k \le n - 1$ and let $c^* : V(G) \to S$ be a closed sigma k-coloring of G where S consists of k positive integers. Then there are two vertices u_i and u_j such that $c^*(u_i) = c^*(u_j) = b$ where $1 \le i < j \le n$. However then, the induced vertex coloring $s : V(G) \to \mathbb{N}$ has

$$s(v_i) = s(v_j) = b + \sum_{i=1}^{n} c^*(v_i),$$

which contradicts that s is a proper coloring of G. Hence $\chi_s(G) \ge n$ and so $\chi_s(G) = n$. ■

We conclude this section with the following question.

Problem 4.5 *For two vertex-disjoint connected graphs F and H of order at least 2, what is the relationship among $\chi_s(F \,\square\, H)$, $\chi_s(F)$ and $\chi_s(H)$?*

5 Joins of Graphs

The *join* $G \vee H$ of two vertex-disjoint graphs G and H is the graph whose vertex set is $V(G) \cup V(H)$ and edge set is

$$E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

If G has order p and H has order q, then the subgraph induced by the set $[V(G), V(H)]$ of edges between G and H is the complete bipartite graph $K_{p,q}$. If c is a vertex coloring of a graph G and F is a subgraph of G, the *restriction* c_F of c to F is the vertex coloring of F defined by $c_F(v) = c(v)$ for each vertex v of F.

Proposition 5.1 *If G and H are vertex-disjoint connected graphs, then*

$$\chi_s(G \vee H) \geq \max\{\chi_s(G), \chi_s(H)\}.$$

Proof. Suppose that $\chi_s(G) = k$ and $\chi_s(H) = \ell$, where $k \geq \ell$. Assume, to the contrary, that $\chi_s(G \vee H) = t \leq k - 1$. Then $G \vee H$ has a closed sigma t-coloring c using colors $\alpha_1, \alpha_2, \ldots, \alpha_t$. The coloring c then induces a sum vertex coloring $s_c : V(G) \to \mathbb{N}$ such that $s_c(x) \neq s_c(y)$ for every two adjacent vertices x and y of $G \vee H$ for which $N[x] \neq N[y]$. Let $a = \sum_{v \in V(H)} c(v)$ and let c_G be the restriction of c to G. Observe that $s_{c_G}(u) = s_c(u) - a$ for each vertex u of G. Since c is a closed sigma t-coloring of $G \vee H$, it follows that $s_{c_G}(x) \neq s_{c_G}(y)$ for every two adjacent vertices x and y of G for which $N_G[x] \neq N_G[y]$ and so c_G is a closed sigma t-coloring of H. However, because $t \leq \chi_s(G) - 1$, this is a contradiction. ■

It is not known whether there exist infinitely many pairs G, H of noncomplete connected graphs such that $\chi_s(G \vee H) = \max\{\chi_s(G), \chi_s(H)\}$. On the other hand, there is reason to believe that the following is true.

Conjecture 5.2 *If G and H are vertex-disjoint connected graphs, then*

$$\chi_s(G \vee H) \leq \chi_s(G) + \chi_s(H).$$

Next, we verify Conjecture 5.2 for two well-known classes of graphs, namely, the joins of two paths. If Conjecture 5.2 is true, then it is the best possible. For example, $\chi_s(P_4) = 2$ by Propositions 2.1 and 2.3, while $\chi_s(P_4 \vee P_4) = 4$, as we show next.

Proposition 5.3 *The graph $P_4 \vee P_4$ has a closed sigma coloring using the colors $1, 2, 3, 4$ and $\chi_s(P_4 \vee P_4) = 4$.*

Proof. Let $G = P_4 \vee P_4$, where (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4) are the two copies of P_4 in G. Define a coloring $c : V(G) \to \{1, 2, 3, 4\}$ by $c(u_1) = 1$, $c(u_4) = 3$, $c(v_4) = 4$ and $c(x) = 2$ for each $x \in V(G) - \{u_1, u_4, v_4\}$. Since the induced sum coloring $s : V(G) \to \mathbb{N}$ satisfies

$$(s(u_1), s(u_2), s(u_3), s(u_4)) = (13, 15, 17, 15)$$
$$(s(v_1), s(v_2), s(v_3), s(v_4)) = (12, 14, 16, 14),$$

it follows that c is a closed sigma coloring using the colors $1, 2, 3, 4$ and so $\chi_s(G) \leq 4$.

Next, we show that $\chi_s(G) \geq 4$. Let c be a closed sigma t-coloring of G. We claim that all of $c(u_1), c(u_4), c(v_1), c(v_4)$ are distinct. Observe that

- if $c(u_1) = c(v_4)$ (or $c(u_4) = c(v_1)$), then $s(v_2) = s(u_3)$ (or $s(u_2) = s(v_3)$);
- if $c(u_1) = c(u_4)$ (or $c(v_1) = c(v_4)$), then $s(u_2) = s(u_3)$ (or $s(v_2) = s(v_3)$);
- if $c(u_1) = c(v_1)$ (or $c(u_4) = c(v_4)$), then $s(u_3) = s(v_3)$ (or $s(u_2) = s(v_2)$).

Thus, all of $c(u_1), c(u_4), c(v_1), c(v_4)$ are distinct, as claimed. Therefore, $\chi_s(G) \geq 4$ and so $\chi_s(G) = 4$. ■

Proposition 5.4 *For integers n and m with $n \geq m+2 \geq 6$, the join $P_n \vee P_m$ of paths P_n and P_m has a closed sigma coloring using only the colors 1 and 2 and so $\chi_s(P_n \vee P_m) = 2$.*

Proof. Let $G = P_n \vee P_m$. By Proposition 2.1, $\chi_s(G) \geq 2$. We show that G has a closed sigma coloring using only the colors 1 and 2. Let $P_n = (u_1, u_2, \ldots, u_n)$ and $P_m = (v_1, v_2, \ldots, v_m)$ be the two paths of order n and m, respectively, in G. Define a coloring $c : V(G) \to \{1, 2\}$ as follows: For $1 \leq i \leq n$ and $1 \leq j \leq m$, let

$$c(u_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad c(v_j) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even} \end{cases}$$

Let

$$a = \sum_{i=1}^{n} c(u_i) \quad \text{and} \quad b = \sum_{j=1}^{n} c(v_j). \tag{8}$$

Since $n \geq m + 2$, it follows that $a \geq b + 3$. For the induced sum coloring $s : V(G) \to \mathbb{N}$, observe that for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$s(u_i) = \begin{cases} 3 + b & \text{if } i = 1, n \\ 4 + b & \text{if } i \text{ is even} \\ 5 + b & \text{if } i \text{ is odd} \end{cases} \quad \text{and} \quad s(v_j) = \begin{cases} 3 + a & \text{if } j = 1, m \\ 4 + a & \text{if } j \text{ is even} \\ 5 + a & \text{if } j \text{ is odd.} \end{cases}$$

Since $a \geq b + 3$, it follows that $s(v_j) \geq 6 + b > s(u_i)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus, s is a proper vertex coloring of G and so c is a closed sigma coloring using only the colors 1 and 2 of G and so $\chi_s(P_n \vee P_m) = 2$. ■

Proposition 5.5 *If $n = 2, 3$, then $\chi_s(P_n \vee P_2) = 1$ and if $n \geq 4$, then $\chi_s(P_n \vee P_2) = 2$.*

Proof. Let $G = P_n \vee P_2$. For $n = 2, 3$, the coloring that assigns color 1 to each vertex of G is a closed sigma coloring and so $\chi_s(P_n \vee P_2) = 1$ if $n = 2, 3$. For $n \geq 3$, it follows by Proposition 2.1 that $\chi_s(G) \geq 2$. Next, define a vertex coloring $c : V(G) \to \{1, 2\}$ such that (1) c assigns the color 1 to the two vertices of P_2 and (2) c is a proper coloring of P_n with $c(v_1) = 1$. Then c is a closed sigma coloring using only the colors 1 and 2. Hence, $\chi_s(G) = 2$ for $n \geq 3$. ■

Proposition 5.6 *If $n \geq 3$ and $n \neq 4$, then $\chi_s(P_n \vee P_3) = 2$.*

Proof. Let $G = P_n \vee P_3$. For $n = 3$, define a vertex coloring $c : V(G) \to \{1, 2\}$ such that c assigns the color 1 to the three vertices in one copy of P_3 and the color 2 to the three vertices in the other copy of P_3. For $n \geq 5$, define a vertex coloring $c : V(G) \to \{1, 2\}$ such that (1) c assigns the color 1 to the three vertices of P_3 and (2) c is a proper coloring of P_n with $c(v_1) = 1$. Then c is a closed sigma coloring using only the colors 1 and 2. By Proposition 2.1, $\chi_s(G) \geq 2$ and so $\chi_s(G) = 2$. ■

Proposition 5.7 *If m and n are integers with $m \in \{n, n+1\}$ and $n \geq 5$, then $P_n \vee P_m$ has a closed sigma coloring using the colors $1, 2, 3$ and $2 \leq \chi_s(P_n \vee P_m) \leq 3$.*

Proof. Let $G = P_n \vee P_m$, where $m \in \{n, n + 1\}$ and $n \geq 6$. Since $\chi_s(G) \geq 2$ by Proposition 2.1, it remains to show that G has a closed sigma coloring using the colors $1, 2, 3$. Let $P_n = (u_1, u_2, \ldots, u_n)$ and $P_m = (v_1, v_2, \ldots, v_m)$ be the two paths of order n and m, respectively, in G.

First, suppose that $n = m$. Define a coloring $c : V(G) \to \{1, 2, 3\}$ as follows: For $1 \leq i, j \leq n$, let

$$c(u_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad c(v_j) = \begin{cases} 2 & \text{if } j \text{ is even} \\ 3 & \text{if } j \text{ is odd.} \end{cases}$$

Let a and b be defined as in (8). For the induced sum coloring $s : V(G) \to \mathbb{N}$, observe that for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$s(u_i) = \begin{cases} 3 + b & \text{if } i = 1, n \\ 4 + b & \text{if } i \text{ is even} \\ 5 + b & \text{if } i \text{ is odd} \end{cases} \quad \text{and} \quad s(v_j) = \begin{cases} 5 + a & \text{if } j = 1, m \\ 7 + a & \text{if } j \text{ is odd} \\ 8 + a & \text{if } j \text{ is even.} \end{cases}$$

Since $n \geq 5$, it follows that $b \geq a+6$ and so $s(u_i) \geq 3+b \geq 9+a > 8+a \geq s(v_j)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus, s is a proper vertex coloring of G and so c is a closed sigma coloring using only the colors $1, 2, 3$. Hence, $\chi_s(P_n \vee P_n) \leq 3$.

Next, suppose that $n = m+1$. Define a coloring $c : V(G) \to \{1, 2, 3\}$ as follows: For $1 \leq i, j \leq n$ and $1 \leq j \leq m$, let

$$c(u_i) = \begin{cases} 2 & \text{if } i \text{ is even} \\ 3 & \text{if } i \text{ is odd} \end{cases} \quad \text{and} \quad c(v_j) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even} \end{cases}$$

Let a and b be defined as in (8). For the induced sum coloring $s : V(G) \to \mathbb{N}$, observe that for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$s(u_i) = \begin{cases} 5+b & \text{if } i = 1, n \\ 8+b & \text{if } i \text{ is even} \\ 7+b & \text{if } i \text{ is odd} \end{cases} \quad \text{and} \quad s(v_j) = \begin{cases} 3+a & \text{if } j = 1, m \\ 4+a & \text{if } j \text{ is even} \\ 5+a & \text{if } j \text{ is odd.} \end{cases}$$

Since $n \geq 5$, it follows that $a \geq b+6$ and so $s(v_j) \geq 3+a \geq 3+(b+6) = 9+b > 8+a \geq s(u_i)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus, s is a proper vertex coloring of G and so c is a closed sigma coloring using only the colors $1, 2, 3$. Hence $\chi_s(P_n \vee P_n) \leq 3$. ■

Next, we show that $\chi_s(P_n \vee P_n) = 3$ for each integer $n \geq 5$. First, we present some preliminary results.

Lemma 5.8 *For an integer $n \geq 5$, let $G = P_n \vee P_n$, where $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ be the two paths of order n in G. If c is a closed sigma coloring of G, where $n \geq 5$, then $c(u_i) \neq c(u_{i+3})$ and $c(v_i) \neq c(v_{i+3})$ for $1 \leq i \leq n-3$.*

Proof. Assume, to the contrary, that $c(u_i) = c(u_{i+3})$ for some integer i with $1 \leq i \leq n-3$. Since

$$\begin{aligned} s(u_{i+1}) &= c(u_i) + c(u_{i+1}) + c(u_{i+2}) + \sum_{i=1}^{n} c(v_i) \\ s(u_{i+2}) &= c(u_{i+3}) + c(u_{i+1}) + c(u_{i+2}) + \sum_{i=1}^{n} c(v_i), \end{aligned}$$

it follows that $s(u_{i+1}) = s(u_{i+2})$, which is a contradiction. Similarly, $c(v_i) \neq c(v_{i+3})$ for $1 \leq i \leq n-3$. ■

Proposition 5.9 *If $n \geq 8$ and $n \equiv 2, 3 \pmod 6$, then $\chi_s(P_n \vee P_n) = 3$.*

Proof. Let $G = P_n \vee P_n$, where $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ be the two paths of order n in G. By Proposition 5.7, it suffices to show that $\chi_s(G) \neq 2$. Assume, to the contrary, that there is a closed sigma 2-coloring c of G. Let $n = 6k + r$, where $k \geq 1$ and $r = 2, 3$. By Lemma 5.8, we may assume that $A = \sum_{i=1}^{6k} c(u_i) = \sum_{i=1}^{6k} c(v_i)$. If $n \equiv 2 \pmod 6$, then $s(u_1) = c(u_1) + c(u_2) + c(v_1) + c(v_2) + A = s(v_1)$, while if $n \equiv 3 \pmod 6$, then $s(u_2) = c(u_1) + c(u_2) + c(u_3) + c(v_1) + c(v_2) + c(v_3) + A = s(v_2)$. In either case, a contradiction is produced. ■

Proposition 5.10 *The graph $P_5 \vee P_5$ has a closed sigma coloring using the colors $1, 2, 3$ and $\chi_s(P_5 \vee P_5) = 3$.*

Proof. Let $G = P_5 \vee P_5$ where $(u_1, u_2, \ldots, u_5)$ and $(v_1, v_2, \ldots, v_5)$ are two copies of P_5 in G. We first show that $\chi_s(G) \geq 3$. Assume, to the contrary, that there is a closed sigma coloring $c : V(G) \to \{a, b\}$ where $a, b \in \mathbb{N}$ and $a \neq b$. Then c induces the vertex coloring $s : V(G) \to \mathbb{N}$. Let $\alpha = \sum_{i=1}^{5} c(u_i)$ and $\beta = \sum_{i=1}^{5} c(v_i)$. Observe that

$$\begin{aligned} s(u_2) &= c(u_1) + c(u_2) + c(u_3) + \beta \\ s(u_3) &= c(u_2) + c(u_3) + c(u_4) + \beta. \end{aligned}$$

Since u_2 and u_3 are adjacent and are not true twins, it follows that $s(u_2) \neq s(u_3)$. It is possible only when $c(u_1) \neq c(u_4)$. By symmetry, we must have $c(u_2) \neq c(u_5)$, $c(v_1) \neq c(v_4)$, and $c(v_2) \neq c(v_5)$. Also, observe the following:

(1) If $c(u_1) + c(u_2) = c(v_1) + c(v_2)$, then $s(u_4) = s(v_4)$

(2) If $c(u_1) + c(u_5) = c(v_1) + c(v_2)$, then $s(u_3) = s(v_4)$

(3) If $c(u_1) + c(u_2) = c(v_4) + c(v_5)$, then $s(u_4) = s(v_2)$

(4) If $c(u_1) + c(u_5) = c(v_4) + c(v_5)$, then $s(u_3) = s(v_2)$.

We may assume, without loss of generality, that $c(u_1) = c(u_2) = a$. Thus, either $c(v_1) = c(v_2) = b$ or $c(v_1) = a$ and $c(v_2) = b$.

- ⋆ If $c(v_1) = c(v_2) = b$, then $c(v_4) = a$. By (3), this implies that $c(v_5) = b$. So, by (4), $c(u_5) = a$. However, this gives us that $c(u_2) = c(u_5)$, which is a contradiction.

- ⋆ If $c(v_1) = a$ and $c(v_2) = b$, then by (2), we have that $c(u_5) = a$. Once again, $c(u_2) = c(u_5)$, which is a contradiction.

Hence, there is no closed sigma 2-coloring of G. Thus, $\chi_s(G) \geq 3$.

It remains to show that G has a closed sigma coloring using the colors $1, 2, 3$. Let $c : V(G) \to \{1, 2, 3\}$ be defined by

$$c(u_i) = \begin{cases} 1 & \text{if i is odd} \\ 2 & \text{if i is even} \end{cases} \quad \text{and} \quad c(v_j) = \begin{cases} 3 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

Then the induced vertex coloring $s : V(G) \to \mathbb{N}$ satisfies that

$$s(u_i) \quad = \quad \begin{cases} 16 & \text{if } i = 1 \text{ or } i = n \\ 17 & \text{if } 2 \leq \text{i} \leq n-1 \text{ and } i \text{ is even} \\ 18 & \text{if } 3 \leq \text{i} \leq n-2 \text{ and } i \text{ is odd} \end{cases}$$

$$s(v_j) \quad = \quad \begin{cases} 12 & \text{if } j = 1 \text{ or } j = n \\ 14 & \text{if } 3 \leq \text{j} \leq n-2 \text{ and } j \text{ is odd} \\ 15 & \text{if } 2 \leq \text{j} \leq n-1 \text{ and } j \text{ is even.} \end{cases}$$

Since s is a proper coloring of G, it follows that c is a closed sigma 3-coloring of G and so $\chi_s(G) \leq 3$. Therefore, $\chi_s(G) = 3$. ■

By Propositions 5.9 and 5.10, we may assume that $n \geq 6$ and $n \not\equiv 2, 3 \pmod 6$.

Lemma 5.11 *For an integer $n \geq 6$ with $n \not\equiv 2, 3 \pmod 6$, let $G = P_n \vee P_n$, where $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ be the two paths of order n in G. If c is a closed sigma coloring of G using two distinct colors a and b, then $c(u_1) \neq c(v_1)$ and $c(u_n) \neq c(v_n)$.*

Proof. Assume, to the contrary, that $c(u_1) = c(v_1)$ or $c(u_n) = c(v_n)$, say $c(u_1) = c(v_1) = a$. Since $n \geq 6$, we write $n = 6k + r$ for some integers k and r with $k \geq 1$ and $r \in \{0, 1, 4, 5\}$. Let

$$c(P_u) = (c(u_1), c(u_2), \ldots, c(u_n)) \;\; \text{and} \;\; c(P_v) = (c(v_1), c(v_2), \ldots, c(v_n)).$$

By Lemma 5.8, there are only four possibilities for $c(P_u)$ and for $c(P_v)$, denoted by $c(P)$, whose first $6k$ terms are shown and T_r denotes the tail-subsequence of this sequence consisting of the last r terms in the sequence:

$$c(P) = \begin{cases} (a,a,a,b,b,b,a,a,a,b,b,b,\ldots,a,a,a,b,b,b,\ T_r) & \text{if 1. occurs} \\ (a,a,b,b,b,a,a,a,b,b,b,a,\ldots,a,a,b,b,b,a,\ T_r) & \text{if 2. occurs} \\ (a,b,a,b,a,b,a,b,a,b,a,b,\ldots,a,b,a,b,a,b,\ T_r) & \text{if 3. occurs} \\ (a,b,b,b,a,a,a,b,b,b,a,a,\ldots,a,b,b,b,a,a,\ T_r) & \text{if 4. occurs.} \end{cases} \tag{9}$$

Let $\alpha = \sum_{i=1}^{n} c(u_i)$ and $\beta = \sum_{i=1}^{n} c(v_i)$. A triple (x,y,z) is said to be *available* for $c(P_u)$ if there is $u_i \in V(P_u)$, where $2 \le i \le n-1$, such that $(c(u_{i-1}), c(u_i), c(u_{i+1})) = (x,y,z)$. In this case, $s(u_i) = \beta + x + y + z$. Similarly, a triple (x,y,z) is said to be *available* for $c(P_v)$ if there is $v_j \in V(P_v)$, where $2 \le j \le n-1$, such that $(c(v_{j-1}), c(v_j), c(v_{j+1})) = (x,y,z)$ and so $s(v_j) = \alpha + x + y + z$.

First, suppose that $\alpha = \beta$. Since (a,b,b) or (b,a,b) is available for each choice of $c(P)$ as described in (9), there are $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = s(v) = a + 2b + \alpha$, which is impossible. Hence, $\alpha \ne \beta$ and so $n \not\equiv 0, 1 \pmod 6$ and $c(P_u) \ne c(P_v)$. Thus, may assume that $c(P_u)$ is the ℓth sequence in (9) for $\ell = 1, 2, 3$. We consider these three cases when $n \equiv 4, 5 \pmod 6$.

Case 1. $c(P_u) = (a,a,a,b,b,b,a,a,a,b,b,b,\ldots,a,a,a,b,b,b,\ T_r)$ *is the first sequence.* Let $c(P_v)$ be a sequence in (9) such that $\alpha \ne \beta$.

- ⋆ If $n = 6k+4$, then $c(P_v)$ can be the ℓth sequence for $\ell = 2, 3, 4$. Thus, (i) $\alpha = A_k + 3a + b$ and (ii) $\beta = A_k + 2a + 2b$ or $\beta = A_k + a + 3b$. If $\beta = 2a + 2b$, then $\ell = 2, 3$ and (a,a,b) is available for $c(P_u)$ and (a,b,b) or (b,a,b) is available for $c(P_v)$. If $\beta = a + 3b$, then $\ell = 4$ and (a,a,a) is available for $c(P_u)$ and (a,b,b) is available for $c(P_v)$. Thus, there are $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = s(v)$, which is impossible.
- ⋆ If $n = 6k+5$, then $c(P_v)$ is the ℓth sequence for $\ell = 2, 4$. Thus, (i) $\alpha = A_k + 3a + 2b$ and $\beta = A_k + 2a + 3b$. Then (a,a,b) is available for $c(P_u)$ and (a,b,b) is available for $c(P_v)$. Thus, there are $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = s(v)$, which is impossible.

Case 2. $c(P_u) = (a,a,b,b,b,a,a,a,b,b,b,a,\ldots,a,a,b,b,b,a,\ T_r)$ *is the second sequence.* By Case 1, we may assume that $c(P_v)$ is the third sequence or the four sequence such that $\alpha \ne \beta$.

- ⋆ If $n = 6k+4$, then $c(P_v)$ is the 4th sequence. Thus, $\alpha = A_k + 2a + 2b$ and $\beta = A_k + a + 3b$. Since (a,a,a) is available for $c(P_u)$ and (a,a,b) is available for $c(P_v)$, there are $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = s(v)$, which is impossible.

- ⋆ If $n = 6k + 5$, then $c(P_v)$ is the 4th sequence. Thus, $\alpha = A_k + 3a + 2b$ and $\beta = A_k + 2a + 3b$. Since (a, a, b) is available for $c(P_u)$ and (a, b, b) is available for $c(P_v)$, there are $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = s(v)$, which is impossible.

Case 3. *$c(P_u) = (a, b, a, b, a, b, a, b, a, b, a, b, \ldots, a, b, a, b, a, b,\ T_r)$ is the third sequence.* By Cases 1 and 2, we may assume that $c(P_v)$ is the fourth sequence.

- ⋆ If $n = 6k+4$, then $\alpha = A_k+2a+2b$ and $\beta = A_k+a+3b$. Since (a, b, a) is available for $c(P_u)$ and (a, b, b) is available for $c(P_v)$, there are $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = s(v)$, which is impossible.

- ⋆ If $n = 6k+5$, then $\alpha = A_k+3a+2b$ and $\beta = A_k+2a+3b$. Since (a, b, a) is available for $c(P_u)$ and (a, b, b) is available for $c(P_v)$, there are $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = s(v)$, which is impossible. ■

We are now prepare to present the following result.

Theorem 5.12 *For each integer $n \geq 5$ and, the graph $P_n \vee P_n$ has a closed sigma coloring using the colors $1, 2, 3$ and $\chi_s(P_n \vee P_n) = 3$.*

Proof. Since $2 \leq \chi_s(P_n \vee P_n) \leq 3$ and $P_n \vee P_n$ has a closed sigma coloring using the colors $1, 2, 3$ by Proposition 5.7, it suffices to show that $\chi_s(P_n \vee P_n) \neq 2$ for $n \geq 5$. Assume, to the contrary, that the statement is false. Then there is a smallest integer $n \geq 5$ such that $\chi_s(P_n \vee P_n) = 2$. By Propositions 5.9 and 5.10, we may assume that $n \geq 6$ and $n \not\equiv 2, 3 \pmod 6$. Let $G = P_n \vee P_n$ and let $c : V(G) \to \{a, b\}$ be a closed sigma coloring of G, where $a, b \in \mathbb{N}$ and $a \neq b$. Furthermore, let

$$P_u = (u_1, u_2, \ldots, u_n) \text{ and } P_v = (v_1, v_2, \ldots, v_n)$$

be the two paths of order n in G and let

$$c(P_u) = (c(u_1), c(u_2), \ldots, c(u_n)) \text{ and } c(P_v) = (c(v_1), c(v_2), \ldots, c(v_n)).$$

By Lemma 5.11, we may assume, without loss of generality, that

$$\{c(u_1), c(v_1)\} = \{c(u_n), c(v_n)\} = \{a, b\}.$$

We may further assume that $c(u_1) = a$ and $c(v_1) = b$. Since $c(u_1) = a$, there are only four possibilities of $(c(u_1), c(u_2), c(u_3))$, namely

$$1.\quad (a, a, a) \qquad 2.\quad (a, a, b) \qquad 3.\quad (a, b, a) \qquad 4.\quad (a, b, b).$$

Similarly, since $c(v_1) = b$, there are only four possibilities of $(c(v_1), c(v_2), c(v_3))$, namely

$$1^*.\quad (b, b, b) \qquad 2^*.\quad (b, b, a) \qquad 3^*.\quad (b, a, b) \qquad 4^*.\quad (b, a, a).$$

Since $n \geq 6$, we write $n = 6k + r$ for some integers k and r with $k \geq 1$ and $r \in \{0, 1, 4, 5\}$. By Lemma 5.8, there are only four possibilities for $c(P_u)$, whose first $6k$ terms are shown and T_r denotes the tail-subsequence of $c(P_u)$ consisting of the last r terms in the sequence:

$$c(P_u) = \begin{cases} (a, a, a, b, b, b, a, a, a, b, b, b, \ldots, a, a, a, b, b, b,\ T_r) & \text{if 1. occurs} \\ (a, a, b, b, b, a, a, a, b, b, b, a, \ldots, a, a, b, b, b, a,\ T_r) & \text{if 2. occurs} \\ (a, b, a, b, a, b, a, b, a, b, a, b, \ldots, a, b, a, b, a, b,\ T_r) & \text{if 3. occurs} \\ (a, b, b, b, a, a, a, b, b, b, a, a, \ldots, a, b, b, b, a, a,\ T_r) & \text{if 4. occurs.} \end{cases} \tag{10}$$

By interchanging a and b in the four sequences in (10), we obtain the following four possibilities for $c(P_v)$, where T_r^* denotes the tail-subsequence of $c(P_v)$ consisting of the last r terms in the sequence:

$$c(P_v) = \begin{cases} (b, b, b, a, a, a, b, b, b, a, a, a, \ldots, b, b, b, a, a, a,\ T_r^*) & \text{if 1*. occurs} \\ (b, b, a, a, a, b, b, b, a, a, a, b, \ldots, b, b, a, a, a, b,\ T_r^*) & \text{if 2*. occurs} \\ (b, a, b, a, b, a, b, a, b, a, b, a, \ldots, b, a, b, a, b, a,\ T_r^*) & \text{if 3*. occurs} \\ (b, a, a, a, b, b, b, a, a, a, b, b, \ldots, b, a, a, a, b, b,\ T_r^*) & \text{if 4*. occurs.} \end{cases} \tag{11}$$

Let $\alpha = \sum_{i=1}^{n} c(u_i)$ and $\beta = \sum_{i=1}^{n} c(v_i)$. We verify the following claim.

Claim 2. *There exists a constant C such that $\{\alpha, \beta\} = \{C + ta, C + tb\}$ for some integer $t \in \{0, 1, 2, 3\}$.*

To verify Claim 2 holds, observe for each of the 8 sequences in (10) and (11) that

$$\sum_{i=1}^{6k} c(u_i) = \sum_{i=1}^{6k} c(v_i) = 3ka + 3kb = 3k(a + b). \tag{12}$$

Now, let $A_k = 3k(a+b)$. Since Claim 2 is clear if $r = 0, 1$, we may assume that $r = 4, 5$. Let $T_r = (t_1, t_2, \ldots, t_r)$ and $T_r^* = (t_1^*, t_2^*, \ldots, t_r^*)$, where $t_s, t_s^* \in \{a, b\}$ for $1 \le s \le r$. Next, let $\sigma_r = \sum_{i=1}^r t_i$ and $\sigma_r^* = \sum_{i=1}^r t_i^*$. Hence, $\alpha = A_k + \sigma_r$ and $\beta = A_k + \sigma_r^*$. Suppose that $r = 4, 5$. Since neither T_r nor T_r^* contains (a, a, a, a) or (b, b, b, b) as a subsequence, it follows that σ_r and σ_r^* have the form $xa + by$ where $1 \le x, y \le 3$ and $x + y = r$. Hence, $\{\alpha, \beta\} = \{C + ta, C + tb\}$ where $C \in \{A_k, A_k + a + b, A_k + 2a + 2b\}$ and $t \in \{0, 1, 2, 3\}$. Therefore, Claim 2 holds.

By Claim 2, it remains to verify the following.

Claim 3. *If $\{\alpha, \beta\} = \{C + ta, C + tb\}$, where C is a constant and $t \in \{0, 1, 2, 3\}$, then there exist $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = s(v)$.*

To verify Claim 3, we first introduce some additional notation and definitions. For each pair i, j^* of integers with $1 \le i, j^* \le 4$, we use $(c(P_u), c(P_v)) = (i, j^*)$ to indicate that $c(P_u)$ is the *ith* sequence in (10) when i occurs and $c(P_v)$ is the *jth* sequence in (11) when j^* occurs. For example, $(c(P_u), c(P_v)) = (1, 2^*)$ means that $c(P_u)$ is the first sequence in (10) when 1. occurs and $c(P_v)$ is the second sequence in (11) when 2^*. occurs; namely

$$c(P_u) = (a, a, a, b, b, b, a, a, a, b, b, b, \ldots, a, a, a, b, b, b,\ T_r)$$
$$c(P_v) = (b, b, a, a, a, b, b, b, a, a, a, b, \ldots, b, b, a, a, a, b,\ T_r^*).$$

A sequence (x, y, z) is said to be *available* for $c(P_u)$ if there is $u_i \in V(P_u)$, where $2 \le i \le n-1$, such that $(c(u_{i-1}), c(u_i), c(u_{i+1})) = (x, y, z)$. In this case, $s(u_i) = \beta + x + y + z$. Similarly, a sequence (x, y, z) is said to be *available* for $c(P_v)$ if there is $v_j \in V(P_v)$, where $2 \le j \le n-1$, such that $(c(v_{j-1}), c(v_j), c(v_{j+1})) = (x, y, z)$ and so $s(v_j) = \alpha + x + y + z$.

We are now prepared to verify Claim 3. There are four cases, according to the values of α and β. In each case, we use C to indicate a constant.

Case 1. $\alpha = \beta = C$. Since (a, b, b), (b, b, a) or (b, a, b) is available for $c(P_u)$ and $c(P_v)$, there are $u_i \in V(P_u)$ and $v_j \in V(P_v)$ such that $s(u_i) = s(v_j) = C + a + 2b$.

Case 2. $\alpha = C + a$ *and* $\beta = C + b$. Since (a, b, b), (b, b, a) or (b, a, b) is available for $c(P_u)$ and (b, a, a) or (a, b, a) is available for $c(P_v)$, there are $u_i \in V(P_u)$ and $v_j \in V(P_v)$ such that $s(u_i) = s(v_j) = C + 2a + 2b$.

Case 3. $\alpha = C + 2a$ *and* $\beta = C + 2b$. Then $(c(P_u), c(P_v)) \ne (3, 3^*)$; for otherwise, $\{\alpha, \beta\} = \{C + ta, C + tb\}$ for some integer $t = 0, 1$. Thus, we may assume that $c(P_u)$ is not the third sequence in (10). Thus, (b, b, b) is available for $c(P_u)$ and either (b, a, a)

or (a,b,a) is available for $c(P_v)$. Hence, there are $u_i \in V(P_u)$ and $v_j \in V(P_v)$ such that $s(u_i) = s(v_j) = C + 2a + 3b$.

Case 4. $\alpha = C + 3a$ *and* $\beta = C + 3b$. Hence, $(c(P_u), c(P_v)) = (i, j^*)$ where $i \neq 3$ and $j \neq 3$. Thus, (b,b,b) is available for $c(P_u)$ and (a,a,a) is available for $c(P_v)$. Hence, there are $u_i \in V(P_u)$ and $v_j \in V(P_v)$ such that $s(u_i) = s(v_j) = C + 3a + 3b$.

Notice that, in each case, every available triple belongs to the first six terms in every sequence. Thus, Claim 3 holds. Therefore, c is not a closed sigma coloring of G. Therefore, $\chi_s(P_n \vee P_n) = 3$ for each integer $n \geq 5$. ∎

We have seen in Proposition 5.7, for each integer $n \geq 3$, that $2 \leq \chi_s(P_{n+1} \vee P_n) \leq 3$ and $P_{n+1} \vee P_n$ has a closed sigma coloring using the colors $1, 2, 3$. Next, we determine the exact value of $\chi_s(P_{n+1} \vee P_n)$ for each integer $n \geq 3$. First, we establish some notation and definitions used in this section. For an integer $n \geq 3$, let $G = P_{n+1} \vee P_n$, where $P_u = (u_1, u_2, \ldots, u_n)$ and $P_v = (v_1, v_2, \ldots, v_{n+1})$ are the two paths of order n and $n+1$, respectively, in G.

For a closed sigma coloring c of G, let

$$c(P_u) = (c(u_1), c(u_2), \ldots, c(u_n)) \quad \text{and} \quad c(P_v) = (c(v_1), c(v_2), \ldots, c(v_{n+1})).$$

Furthermore, let

$$\alpha = \sum_{i=1}^{n} c(u_i) \quad \text{and} \quad \beta = \sum_{i=1}^{n+1} c(v_i). \tag{13}$$

Lemma 5.13 *For an integer $n \geq 4$, let $G = P_n \vee P_{n+1}$. If c is a closed sigma coloring of G, where $n \geq 5$, then $c(u_i) \neq c(u_{i+3})$ and $c(v_j) \neq c(v_{j+3})$ where $1 \leq i \leq n-3$ and $1 \leq j \leq n-2$.*

Proposition 5.14 *If $n = 3, 4, 5$, then $\chi_s(P_{n+1} \vee P_n) = 3$.*

Proof. We consider three cases.

Case 1: $n = 3$. By lemma 5.13, we have that $c(u_1) \neq c(u_4)$. Also,

$$s(v_3) = c(v_2) + c(v_3) + c(v_4) + c(u_2) + c(u_3) + c(u_1)$$

$$s(u_3) = c(v_2) + c(v_3) + c(v_4) + c(u_2) + c(u_3) + c(v_1)$$

Hence, we must have that $c(u_1) \neq c(v_1)$. A similar argument shows that $c(u_1) \neq c(v_4)$. Thus, we have that $\chi_s(P_4 \vee P_3) = 3$.

Case 2: $n = 4$. Observe that,

1. If $c(u_1) + c(u_2) = c(v_1) + c(v_2)$, then $s(u_4) = s(v_4)$
2. If $c(u_1) + c(u_2) = c(v_1) + c(v_5)$, then $s(u_4) = s(v_3)$
3. If $c(u_1) + c(u_2) = c(v_4) + c(v_5)$, then $s(u_4) = s(v_2)$
4. If $c(u_3) + c(u_4) = c(v_1) + c(v_2)$, then $s(u_1) = s(v_4)$
5. If $c(u_3) + c(u_4) = c(v_1) + c(v_5)$, then $s(u_1) = s(v_3)$
6. If $c(u_3) + c(u_4) = c(v_4) + c(v_5)$, then $s(u_1) = s(v_2)$

Also, by Lemma 5.13, we have that

1. $c(u_1) \neq c(u_4)$ 2. $c(v_1) \neq c(v_4)$ 3. $c(v_2) \neq c(v_5)$

This gives us that $c(v_1) + c(v_2) \neq c(v_1) + c(v_5)$ and $c(v_1) + c(v_5) \neq c(v_4) + c(v_5)$.

If we have that $c(u_1) + c(u_2) \neq c(u_3) + c(u_4)$, then we have a contradiction since we have four distinct pairwise sums of a and b when the only possibilities are $a + a$, $a + b$, and $b + b$.

Similarly, we have that $c(v_1) + c(v_2) = c(v_4) + c(v_5)$ is forced.

So, $c(u_1) + c(u_2) = c(u_3) + c(u_4)$. Without loss of generality, let $c(u_1) = a$. Then it is forced that $c(u_2) = c(u_4) = b$ and $c(u_3) = a$ We consider two subcases:

Subcase 0.1: $c(v_1) = a$. Then, it is forced that $c(v_4) = b$.
Since $c(v_1) + c(v_2) = c(v_4) + c(v_5)$, we have that $c(v_2) = b$ and $c(v_5) = a$. However, then we have that $c(v_1) + c(v_2) = a + b = c(u_1) + c(u_2)$ which is a contradiction.

Subcase 0.2: $c(v_1) = b$. Then, we have that $c(v_4) = a$.
Since $c(v_1) + c(v_2) = c(v_4) + c(v_5)$, we have that $c(v_2) = a$ and $c(v_5) = b$. Again, we have

that $c(v_1) + c(v_2) = a + b = c(u_1) + c(u_2)$.
Hence we have can see that $\chi_s(P_5 \vee P_4) \geq 3$

Case 3: $n = 5$. First, we need an intermediate result. For a two coloring of P_5 using a and b, either $c(u_1), c(u_2), c(u_3) \in \{a, b\}$ with $|\{(c(u_1), c(u_2), c(u_3)\}| = 2$ or $c(u_3), c(u_4), c(u_5) \in \{a, b\}$ and $|\{(c(u_3), c(u_4), c(u_5)\}| = 2$.

Proof. If $c(u_1) \neq c(u_2)$, then this is true. If $c(u_1) = c(u_2)$, then without loss of generality, let $c(u_1) = c(u_2) = a$, then by lemma 5.13, $c(u_4) = c(u_5) = b$. So regardless of our choice for $c(u_3)$, we will have either $c(u_1), c(u_2), c(u_3) \in \{a, b\}$ with $|\{(c(u_1), c(u_2), c(u_3)\}| = 2$ or $c(u_3), c(u_4), c(u_5) \in \{a, b\}$ with $|\{(c(u_3), c(u_4), c(u_5)\}| = 2$.

Assume to the contrary that $P_5 \vee P_6$ has a closed sigma two coloring. In our coloring of $P_5 \vee P_6$, without loss of generality, let $c(u_1) + c(u_2) + c(u_3) = 2a + b$. Then, we the following:

1. $c(v_1) + c(v_2) + c(v_3) \neq 2a + b$
2. $c(v_1) + c(v_2) + c(v_6) \neq 2a + b$
3. $c(v_1) + c(v_5) + c(v_6) \neq 2a + b$
4. $c(v_4) + c(v_5) + c(v_6) \neq 2a + b$

Suppose that $c(v_1) = a$. Since $c(v_1) + c(v_2) + c(v_3) \neq 2a + b$, we consider two cases.

Subcase 1.1: $c(v_2) = c(v_3) = a$. Then, $s(u_5) = s(v_4)$.

Subcase 1.2: $c(v_2) = c(v_3) = b$. Then, $s(u_5) = s(v_2)$.

Suppose that $c(v_1) = b$. Since $c(v_1) + c(v_2) + c(v_3) \neq 2a + b$, we consider three cases.

Subcase 2.1: $c(v_2) = c(v_3) = b$. Then, $s(u_5) = s(v_3)$.

Subcase 2.2: $c(v_2) = b, c(v_3) = a$. Then, $s(u_5) = s(v_2)$.

subcase 2.3: $c(v_2) = a, c(v_3) = b$. Then, $s(u_5) = s(v_2)$.

Thus, we can see that there is no 2 closed sigma coloring of $P_5 \vee P_6$. Hence we have can see that $\chi_s(P_5 \vee P_6) = 3$

In what follows, we will assume $n \geq 6$. Write $n = 6k + r$ where $k \geq 0$ and $0 \leq r \leq 5$. If $c(u_1) = a$, then there are only four possibilities of $(c(u_1), c(u_2), c(u_3))$, namely

$$1.\ \ (a, a, a) \quad 2.\ \ (a, a, b) \quad 3.\ \ (a, b, a) \quad 4.\ \ (a, b, b).$$

Similarly, if $c(u_1) = b$, then there are only four possibilities of $(c(u_1), c(u_2), c(u_3))$, namely

$$5.\ \ (b, b, b) \quad 6.\ \ (b, b, a) \quad 7.\ \ (b, a, b) \quad 8.\ \ (b, a, a).$$

By Lemma 5.13, there are only eight possibilities for the color sequences $c(P_u)$, namely

$$\begin{aligned}
Q_1 &= (a, a, a, b, b, b, a, a, a, b, b, b, ..., a, a, a, b, b, b, T_r)\\
Q_2 &= (a, a, b, b, b, a, a, a, b, b, b, a, ..., a, a, b, b, b, a, T_r)\\
Q_3 &= (a, b, a, b, a, b, a, b, a, b, a, b, ..., a, b, a, b, a, b, T_r)\\
Q_4 &= (a, b, b, b, a, a, a, b, b, b, a, a, ..., a, b, b, b, a, a, T_r)\\
Q_5 &= (b, b, b, a, a, a, b, b, b, a, a, a, ..., b, b, b, a, a, a, T_r)\\
Q_6 &= (b, b, a, a, a, b, b, b, a, a, a, b, ..., b, b, a, a, a, b, T_r)\\
Q_7 &= (b, a, b, a, b, a, b, a, b, a, b, a, ..., b, a, b, a, b, a, T_r)\\
Q_8 &= (b, a, a, a, b, b, b, a, a, a, b, b, ..., a, b, a, b, a, b, T_r),
\end{aligned}$$

whose first $6k$ terms are shown and T_r denotes the tail-subsequence of $c(P_u)$ consisting of the last r terms in the sequence. Similarly, there are only eight possibilities for the color sequences $c(P_v)$, namely

$$\begin{aligned}
Q_1^* &= (a, a, a, b, b, b, a, a, a, b, b, b, ..., a, a, a, b, b, b, T_{r+1})\\
Q_2^* &= (a, a, b, b, b, a, a, a, b, b, b, a, ..., a, a, b, b, b, a, T_{r+1})\\
Q_3^* &= (a, b, a, b, a, b, a, b, a, b, a, b, ..., a, b, a, b, a, b, T_{r+1})\\
Q_4^* &= (a, b, b, b, a, a, a, b, b, b, a, a, ..., a, b, b, b, a, a, T_{r+1})\\
Q_5^* &= (b, b, b, a, a, a, b, b, b, a, a, a, ..., b, b, b, a, a, a, T_{r+1})\\
Q_6^* &= (b, b, a, a, a, b, b, b, a, a, a, b, ..., b, b, a, a, a, b, T_{r+1})\\
Q_7^* &= (b, a, b, a, b, a, b, a, b, a, b, a, ..., b, a, b, a, b, a, T_{r+1})\\
Q_8^* &= (b, a, a, a, b, b, b, a, a, a, b, b, ..., a, b, a, b, a, b, T_{r+1}),
\end{aligned}$$

whose first $6k$ terms are shown and T_{r+1} denotes the tail-subsequence of $c(P_v)$ consisting of the last $r+1$ terms in the sequence. Therefore,

$$\sum_{i=1}^{6k} c(u_i) = \sum_{i=1}^{6k} c(v_i) \text{ for each integer } k \geq 1. \tag{14}$$

Thus, let $A_k = \sum_{i=1}^{6k} c(u_i) = \sum_{i=1}^{6k} c(v_i)$ for each integer $k \geq 1$.

Proposition 5.15 *If $n \geq 7$ and $n \equiv 2 \pmod 6$, then $\chi_s(P_{n+1} \vee P_n) = 3$.*

Proof. Let $G = P_{n+1} \vee P_n$, where $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_{n+1})$ be the two paths of order n and order $n+1$, respectively, in G. By Proposition 5.7, it suffices to show that $\chi_s(G) \neq 2$. Assume, to the contrary, that there is a closed sigma 2-coloring c of G. By Lemma 5.13 or by (14), we may assume that $A = \sum_{i=1}^{n-2} c(u_i) = \sum_{i=1}^{n-2} c(v_i)$. Since $s(u_1) = c(u_1) + c(u_2) + c(v_1) + c(v_2) + c(v_3) + A = s(v_2)$, a contradiction is produced. ■

A pair (x, y) is *available* for $c(P_u)$ if ether $(c(u_1), c(u_2)) = (x, y)$ or $(c(u_n), c(u_{n-1})) = (x, y)$. Let α and β be defined in (13). In this case, $s(u_2) = \beta + x + y$. Similarly, a pair (x, y) is *available* for $c(P_v)$ if $(c(v_1), c(v_2)) = (x, y)$ or $(c(v_{n+1}), c(v_n)) = (x, y)$ and so $s(v_n) = \alpha + x + y$. A triple (x, y, z) is *available* for $c(P_u)$ if there is $u_i \in V(P_u)$, where $2 \leq i \leq n-1$, such that $(c(u_{i-1}), c(u_i), c(u_{i+1})) = (x, y, z)$. In this case, $s(u_i) = \beta + x + y + z$. Similarly, a triple (x, y, z) is *available* for $c(P_v)$ if there is $v_j \in V(P_v)$, where $2 \leq j \leq n-1$, such that $(c(v_{j-1}), c(v_j), c(v_{j+1})) = (x, y, z)$ and so $s(v_j) = \alpha + x + y + z$.

Lemma 5.16 *For an integer $n \geq 6$, let $G = P_n \vee P_{n+1}$. If c is an edge coloring of G using two distinct colors a and b such that $\beta - \alpha \in \{a, b\}$, then c is not a closed sigma coloring of G.*

Proof. By Proposition 5.15, we may assume that $n \not\equiv 2 \pmod 6$. Furthermore, since $\beta - \alpha \in \{a, b\}$, it follows that $n \not\equiv 0 \pmod 6$. Assume, to the contrary, that c is a closed sigma coloring of G. We may assume, without loss of generality, that $\beta - \alpha = a$. Let $\alpha = B$ and $\beta = B + a$ Thus,

$$(c(u_1), c(u_2)) \in \{(a, a), (a, b), (b, a), (b, b)\}.$$

We consider two cases, according to whether $c(u_1) \neq c(u_2)$ or $c(u_1) = c(u_2)$.

Case 1. $(c(u_1), c(u_2)) \in \{(a,b),(b,a)\}$. We may assume, without loss of generality, that $(c(u_1), c(u_2)) = (a,b)$. Hence, $s(u_1) = a + b + \beta = 2a + b + B$. However, one of (a,a,b), (b,a,a), or (a,b,a) is available for $c(P_v)$. Hence, there exists a $v \in V(P_v)$ such that $s(v) = 2a + b + \alpha = 2a + b + B_k = s(u_1)$, which is a contradiction.

Case 2. $(c(u_1), c(u_2)) \in \{(a,a),(b,b)\}$. If $(c(u_1), c(u_2)) = (b,b)$, then $s(u_1) = a + 2b + B_k$. An argument similar to the one used in Case 1 shows that there is a $v \in V(P_v)$ such that $s(u_1) = s(v)$. Thus, we may assume that $(c(u_1), c(u_2)) = (a,a)$. Hence, either $c(P_u) = Q_1$ or $c(P_u) = Q_2$. We consider these two subcases.

Subcase 2.1. $c(P_u) = Q_1$. Since $n \not\equiv 0, 2 \pmod 6$, it follows that $n \geq 7$. If $c(P_v) = Q_i^*$, for $i = 1,2,4,5,6,8$, then (a,a,a) is available for $c(P_v)$ and so there is a $v \in V(P_v)$ such that $s(u_1) = s(v) = 3a + B$. So, it remains to consider either $c(P_v) = Q_3^*$ or $c(P_v) = Q_7^*$. Recall that $n = 6k + r$. Let $\Sigma_{i=1}^{6k} c(u_i) = \Sigma_{i=1}^{6k} c(v_i) = A_k$ as described in (14).

- If $r = 1$, then $\alpha = A_k + a$ and $\beta = A_k + a + b$. Hence, $\beta - \alpha \neq a$, a contradiction.
- If $r = 3$, then $\alpha = A_k + 3a$ and $\beta = A_k + 2a + 2b$. Hence, $\beta - \alpha \neq a$, a contradiction.
- If $r = 4$, then $\alpha = A_k + 3a + b$ and either $\beta = A_k + 3a + 2b$ or $\beta = A_k + 2a + 3b$. Hence, $\beta - \alpha \neq a$, a contradiction.
- If $r = 5$, then $\alpha = A_k + 3a + 2b$ and $\beta = A_k + 3a + 3b$ Hence, $\beta - \alpha \neq a$, a contradiction.

Subcase 2.2. $c(P_u) = Q_2$. Similarly, if $c(P_v) = Q_i^*$, for $i = 1,2,4,5,6,8$, then (a,a,a) is available for $c(P_v)$ and so there is a $v \in V(P_v)$ such that $s(u_1) = s(v) = 3a + B$. Again, consider either $c(P_v) = Q_3^*$ or $c(P_v) = Q_7^*$.

- If $r = 1$, then $\alpha = A_k + a$ and $\beta = A_k + a + b$. Hence, $\beta - \alpha \neq a$, a contradiction.
- If $r = 3$, then $\alpha = A_k + 2a + b$ and $\beta = A_k + 2a + 2b$. Hence, $\beta - \alpha \neq a$, a contradiction.
- If $r = 4$, then $\alpha = A_k + 2a + 2b$ and either $\beta = A_k + 3a + 2b$ or $\beta = A_k + 2a + 3b$. If $\beta = A_k + 3a + 2b$, then $\beta - \alpha = a$. However, $(c(u_{n-1}), c(u_n)) = (b,b)$. So by a previous argument, there is a $v \in V(P_v)$ such that $s(u_n) = s(v) = 2b + a + B_k$. If $\beta = A_k + 2a + 3b$, then $\beta - \alpha \neq a$, a contradiction.

- If $r = 5$, then $\alpha = A_k + ba + ab$ and $\beta = A_k + 3a + 3b$ So, $\beta - \alpha = a$. However, then either $(c(u_{n-1}), c(u_n)) = (a, b)$ or $(c(u_{n-1}), c(u_n)) = (b, a)$. So by a previous argument, there is a $v \in V(P_v)$ such that $s(u_n) = s(v) = 2b + a + B_k$, a contradiction.

Therefore, if $\beta - \alpha \in \{a, b\}$, then c is not a closed sigma coloring. ■

The following is an immediate consequence of Lemma 5.16.

Proposition 5.17 *If $n \geq 6$ and $n \equiv 0, 5 \pmod 6$, then $\chi_s(P_n \vee P_{n+1}) = 3$.*

Proof. Assume, to the contrary, that there is a closed sigma coloring c of $G = P_n \vee P_{n+1}$ using two colors. By Lemma 5.13, $\beta - \alpha \in \{a, b\}$. However then, c is not closed sigma coloring c of G by Lemma 5.16, a contradiction. ■

By Propositions 5.14, 5.15 and 5.17, we will assume that $n \geq 7$ with $n \equiv 1, 3, 4 \pmod 6$.

Lemma 5.18 *For an integer $n \geq 7$ with $n \equiv 1, 3, 4 \pmod 6$, let $G = P_n \vee P_{n+1}$. If c is a closed sigma coloring of G using two distinct colors a and b, then $c(u_1) \neq c(v_1)$ and $c(u_n) \neq c(v_{n+1})$.*

Proof. Assume, to the contrary, that $c(u_1) = c(v_1)$ or $c(u_n) = c(v_{n+1})$, say $c(u_1) = c(v_1) = a$. $n \geq 7$ with $n \equiv 1, 3, 4 \pmod 6$, we write $n = 6k + r$ for some integers k and r with $k \geq 1$ and $r \in \{1, 3, 4\}$. By Lemma 5.13, there are only four possibilities for $c(P_u)$, namely

$$Q_1 = (a, a, a, b, b, b, a, a, a, b, b, b, ..., a, a, a, b, b, b, T_r)$$
$$Q_2 = (a, a, b, b, b, a, a, a, b, b, b, a, ..., a, a, b, b, b, a, T_r)$$
$$Q_3 = (a, b, a, b, a, b, a, b, a, b, a, b, ..., a, b, a, b, a, b, T_r)$$
$$Q_4 = (a, b, b, b, a, a, a, b, b, b, a, a, ..., a, b, b, b, a, a, T_r)$$

where the first $6k$ terms are shown and the tail sequence of the last r terms is represented by T_r. Similarly, there are only four possibilities for $c(P_v)$, which are

$$Q_1^\star = (a, a, a, b, b, b, a, a, a, b, b, b, ..., a, a, a, b, b, b, T_{r+1})$$
$$Q_2^\star = (a, a, b, b, b, a, a, a, b, b, b, a, ..., a, a, b, b, b, a, T_{r+1})$$
$$Q_3^\star = (a, b, a, b, a, b, a, b, a, b, a, b, ..., a, b, a, b, a, b, T_{r+1})$$
$$Q_4^\star = (a, b, b, b, a, a, a, b, b, b, a, a, ..., a, b, b, b, a, a, T_{r+1})$$

We consider four cases, according to whether $c(P_u) = Q_i$ for $i = 1, 2, 3, 4$.

Case 1 . $c(P_u) = Q_1 = (a, a, a, b, b, b, a, a, a, b, b, b, ..., a, a, a, b, b, b, T_r)$.

- If $n = 6k + 1$, then $\alpha = A_k + a$ and either $\beta = A_k + 2a$ or $\beta = A_k + a + b$, so $\beta - \alpha \in \{a, b\}$ which is impossible.
- If $n = 6k + 3$, then $\alpha = A_k + 3a$ and, for $Q_1^\star, Q_2^\star$, and $Q_3^\star$ either $\beta = A_k + 3a + b$ or $\beta = A_k + 2a + 2b$. If $\beta = A_k + 3a + b$, $\beta - \alpha \in \{a, b\}$ which is impossible. Since (a, a) is the available in $c(P_u)$ and we have that (a, b, b),(b, b, a) or (b, a, b) is available in $c(P_v)$, then we have that there exists a $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = 4a + 2b + A_k = s(v)$ which is impossible. For $Q_4^\star$, $\beta = A_k + 3b + a$, however (b, b, b) is available in $Q_4^\star$, so there exists a $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = A_k + 3a + 3b = s(v)$, which is impossible.
- If $n = 6k+4$, then $\alpha = A_k+3a+b$ and either $\beta = A_k+3a+2b$ or $\beta = A_k+2a+3b$. Since (a, a) and (a, b) are both available in $c(P_u)$ and we have that (a, b, b) or (a, b, a) are available in $c(P_v)$, then we have that there exists a $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = 4a + 3b + A_k = s(v)$ which is impossible.

Case 2. $c(P_u) = Q_2 = (a, a, b, b, b, a, a, a, b, b, b, a, ..., a, a, b, b, b, a, T_r)$.

- If $n = 6k+1$, then $\alpha = A_k+a$ and $\beta = A_k+2a$ or $\beta = A_k+a+b$. So, $\beta-\alpha \in \{a, b\}$ which is impossible.
- If $n = 6k + 3$, then $\alpha = A_k + 2a + b$. For $Q_3^\star$, $Q_2^\star$,and $Q_1^\star$, $\beta = A_k + 2a + 2b$ or $\beta = A_k + 3a + b$, so $\beta - \alpha \in \{a, b\}$ which is impossible. For $Q_4^\star$, $\beta = A_k + 3b + a$. Since (a, a) is available in Q_2 and (a, b, b) is available in $Q_4^\star$, there exists $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = 3a + 3b + A_k = s(v)$.
- If $n = 6k + 4$, then $\alpha = A_k + 2a + 2b$ and $\beta = A_k + 3a + 2b$ or $\beta = A_k + 2a + 3b$, so $\beta - \alpha \in \{a, b\}$ which is impossible.

Case 3. $c(P_u) = Q_3 = (a, b, a, b, a, b, a, b, a, b, a, b, ..., a, b, a, b, a, b, T_r)$

- If $n = 6k + 1$, then $\alpha = A_k + a$ and $\beta = A_k + 2a$ or $\beta = A_k + a + b$, hence $\beta - \alpha \in \{a, b\}$ which is impossible.
- If $n = 6k + 3$, then $\alpha = A_k + 2a + b$ and, for $Q_3^\star$, $Q_2^\star$, and $Q_1^\star$ $\beta = A_k + 2a + 2b$ or $\beta = A_k + 3a + b$, so $\beta - \alpha \in \{a, b\}$ which is impossible. For $Q_4^\star$, $\beta = A_k + 3b + a$.

Since (a, b) is available in Q_3 and (b, b, b) is available in $Q_4^\star$, there exists $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = 2a + 4b + A_k = s(v)$.

- If $n = 6k + 4$, then $\alpha = A_k + 2a + 2b$ and $\beta = A_k + 3a + 2b$ or $\beta = A_k + 2a + 3b$, thus $\beta - \alpha \in \{a, b\}$ which is a contradiction.

Case 4. $c(P_u) = Q_4 = (a, b, b, b, a, a, a, b, b, b, a, a, ..., a, b, b, b, a, a, T_r)$

- If $n = 6k+1$, then $\alpha = A_k+a$ and $\beta = A_k+2a$ or $\beta = A_k+a+b$, thus $\beta-\alpha \in \{a, b\}$ which is a contradiction.

- If $n = 6k + 3$, then $\alpha = A_k + a + 2b$ and, for $Q_4^\star$, $Q_3^\star$, and $Q_2^\star$, $\beta = A_k + 2a + 2b$ or $\beta = A_k + 3b + a$, hence $\beta - \alpha \in \{a, b\}$ which is a contradiction. For $Q_1^\star$, $\beta = A_k + 3a + b$. Since (a, b) is available in $c(P_u)$ and (a, a, a) is available in $Q_1^\star$, there exists a $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = 4a + 2b + A_k = s(v)$.

- If $n = 6k+4$, then $\alpha = A_k+3b+a$ and $\beta = A_k+3a+2b$ or $\beta = A_k+2a+3b$. Since (b, b) and (a, b) is available in $c(P_u)$ and (a, a, b), (a, b, a), or (b, a, a) is available in $c(P_v)$, then there exists a $u \in V(P_u)$ and $v \in V(P_v)$ such that $s(u) = 3a+4b+A_k = s(v)$. ■

We are now prepared to verify the following result.

Theorem 5.19 *For each integer $n \geq 3$, the graph $P_n \vee P_{n+1}$ has a closed sigma coloring using the colors $1, 2, 3$ and $\chi_s(P_n \vee P_{n+1}) = 3$.*

Proof. Let $G = P_n \vee P_{n+1}$. By Propositions 5.14, 5.15, and 5.17, we may assume that $n \geq 7$ and $n \equiv 1, 3, 4 \pmod 6$. Since $2 \leq \chi_s(G) \leq 3$ and G has a closed sigma coloring using the colors $1, 2, 3$ by Proposition 5.7, it suffices to show that $\chi_s(G) \neq 2$ for each integer $n \geq 7$ and $n \equiv 1, 3, 4 \pmod 6$.

Assume, to the contrary, that the statement is false. Then there is a smallest integer $n \geq 7$ and $n \equiv 1, 3, 4 \pmod 6$ such that $\chi_s(G) = 2$. Let $c : V(G) \to \{a, b\}$ be a closed sigma coloring of G where $a, b \in \mathbb{N}$ and $a \neq b$. Furthermore, let

$$P_u = (u_1, u_2, \ldots, u_n) \text{ and } P_v = (v_1, v_2, \ldots, v_n, v_{n+1})$$

be the two paths of order n and $n + 1$, respectively, in G and let

$$c(P_u) = (c(u_1), c(u_2), \ldots, c(u_n)) \text{ and } c(P_v) = (c(v_1), c(v_2), \ldots, c(v_{n+1})).$$

By Lemma 5.18, we may assume that $c(u_1) \neq c(v_1)$ and $c(u_n) \neq c(v_{n+1})$. Thus, $\{c(u_1), c(v_1)\} = \{c(u_n), c(v_{n+1})\} = \{a, b\}$. We may further assume that $c(u_1) = a$ and $c(v_1) = b$. Write $n = 6k + r$ for some integers k and r with $k \geq 1$ and $r \in \{1, 3, 4\}$. So, our possible color sequences for $c(P_u)$ are

$$Q_1 = (a, a, a, b, b, b, a, a, a, b, b, b, ..., a, a, a, b, b, b, T_r)$$
$$Q_2 = (a, a, b, b, b, a, a, a, b, b, b, a, ..., a, a, b, b, b, a, T_r)$$
$$Q_3 = (a, b, a, b, a, b, a, b, a, b, a, b, ..., a, b, a, b, a, b, T_r)$$
$$Q_4 = (a, b, b, b, a, a, a, b, b, b, a, a, ..., a, b, b, b, a, a, T_r).$$

Similarly, we have that our possible color sequences for $c(P_v)$ are

$$Q_5^* = (b, b, b, a, a, a, b, b, b, a, a, a, ..., b, b, b, a, a, a, T_{r+1})$$
$$Q_6^* = (b, b, a, a, a, b, b, b, a, a, a, b, ..., b, b, a, a, a, b, T_{r+1})$$
$$Q_7^* = (b, a, b, a, b, a, b, a, b, a, b, a, ..., b, a, b, a, b, a, T_{r+1})$$
$$Q_8^* = (b, a, a, a, b, b, b, a, a, a, b, b, ..., a, b, a, b, a, b, T_{r+1}).$$

We consider four cases:

Case 1: $c(P_u) = Q_1$.

- If $n = 6k+1$, then $\alpha = A_k + a$ and $\beta = A_k + a + b$ or $\beta = A_k + 2b$. If $\beta = A_k + a + b$, then $\beta - \alpha \in \{a, b\}$, which is impossible by lemma 5.16. Hence, $\beta = A_k + 2b$. Since (b, a) is available in $c(P_u)$ and (b, b, b) is available in $c(P_v)$, we have that $s(u_n) = 3b + a + A_k = s(v)$, for some $v \in V(P_v)$, which is a contradiction.

- If $n = 6k + 3$, then $\alpha = A_k + 3a$ and either $\beta = A_k + 3b + a$ or $\beta = A_k + 2a + 2b$. If $\beta = A_k + 3b + a$, then $c(P_v) = Q_5^*$. We know (a, a) is available as a beginning sequence in $c(P_u)$ and (b, b, b) is available in $c(P_v)$. Hence, $s(u_1) = A_k + 3a + 3b = s(v)$, for some $v \in V(P_v)$, which is impossible. If $\beta = A_k + 2a + 2b$, then (a, a) is still available in $c(P_u)$ as a starting sequence and either (b, b, a), (b, a, b), or (a, b, b) is available in $c(P_v)$. Hence, $s(u_1) = 4a + 2b + A_k = s(v)$, for some $v \in V(P_v)$, which is also impossible.

- If $n = 6k+4$, then $\alpha = A_k + 3a + b$ and either $\beta = 3a + 2b + A_k$ or $\beta = 3b + 2a + A_k$. By lemma 5.16, $\beta = 3b + 2a + A_k$. So we have that either $c(P_v) = Q_5^*$ or $c(P_v) = Q_7^*$.

Since (a, a) is available in $c(P_u)$ and either (b, b, a) or (b, a, b) is available in $c(P_v)$, then we have that $s(u_1) = 3b + 4a + A_k = s(v)$, for some $v \in V(P_v)$. A contradiction.

Case 2: $c(P_u) = Q_2$.

- If $n = 6k + 1$, then $\alpha = A_k + a$ and $\beta = A_k + a + b$ or $\beta = A_k + 2b$. If $\beta = A_k + a + b$, then $\beta - \alpha \in \{a, b\}$, which is impossible by lemma 5.16. Hence, $\beta = A_k + 2b$. We have that (a, a) is available in $c(P_u)$ and (a, b, b) is available in $c(P_v)$. Thus, $s(u_1) = A_k + 2a + 2b = s(v)$, for some $v \in V(P_v)$, which is a contradiction.

- If $n = 6k + 3$, then $\alpha = A_k + 2a + b$ and either $\beta = A_k + 3b + a$ or $\beta = A_k + 2a + 2b$. If $\beta = A_k + 2a + 2b$, then we have a contradiction by lemma 5.16, so $\beta = A_k + 3b + a$. Hence, $c(P_v) = Q_5^*$. Since (a, a) is available in $c(P_u)$ and (b, b, a) is available in $c(P_v)$, then $s(u_1) = A_k + 3a + 3b = s(v)$, for some $v \in V(P_v)$, which is impossible.

- If $n = 6k + 4$, then $\alpha = A_k + 2a + 2b$ and either $\beta = 3a + 2b + A_k$ or $\beta = 3b + 2a + A_k$. By lemma 5.16, this is impossible.

Case 3: $c(P_u) = Q_3$.

- If $n = 6k + 1$, then similarly $\alpha = A_k + a$ and $\beta = A_k + 2b$. Note that $c(P_v) = Q_5^*$ or $c(P_v) = Q_6^*$. Since (a, b) is available in $c(P_u)$ and (b, b, b) is available in $c(P_v)$, we have that $s(u_1) = 3b + a + A_k = s(v)$, for some $v \in V(P_v)$, which is a contradiction.

- If $n = 6k + 3$, then $\alpha = A_k + 2a + b$ and either $\beta = A_k + 3b + a$ or $\beta = A_k + 2a + 2b$. If $\beta = A_k + 2a + 2b$, then we have a contradiction by lemma 0.6, so $\beta = A_k + 3b + a$ and $c(P_v) = Q_5^*$. We have that (a, b) is available as a starting sequence for $c(P_u)$ and (b, b, b) is available in $c(P_v)$. Thus, $s(u_1) = A_k + 2a + 4b = s(v)$, for some $v \in V(P_v)$, which is impossible.

- If $n = 6k + 4$, then $\alpha = A_k + 2a + 2b$ and either $\beta = 3a + 2b + A_k$ or $\beta = 3b + 2a + A_k$. By lemma 5.16, this is impossible.

Case 4: $c(P_u) = Q_4$.

- If $n = 6k + 1$, then by our previous arguments, $\alpha = A_k + a$ and $\beta = A_k + 2b$. Since (a, b) is available in $c(P_u)$ and (b, b, b) is available in $c(P_v)$, we have that $s(u_1) = 3b + a + A_k = s(v)$, for some $v \in V(P_v)$, which is a contradiction.

- If $n = 6k + 3$, then $\alpha = A_k + 2b + a$ and either $\beta = A_k + 3b + a$ or $\beta = A_k + 2a + 2b$, which is impossible by lemma 5.16.

- If $n = 6k+4$, then $\alpha = A_k + a + 3b$ and either $\beta = 3a+2b+A_k$ or $\beta = 3b+2a+A_k$. By lemma 5.16, $\beta = 3a + 2b + A_k$. So, $c(P_v) = Q_6^*$ or $c(P_v) = Q_8^*$. Since (a, b) is available in $c(P_u)$ and (a, a, a) is available in $c(P_v)$, then we have that $s(u_1) = A_k + 4a + 3b = s(v)$, for some $v \in V(P_v)$, which is impossible. ■

In summary, we have the following theorem.

Theorem 5.20 *For integers m and n with $n \geq m \geq 2$,*

$$\chi_s(P_n \vee P_m) = \begin{cases} 1 & \text{if } m = 2 \text{ and } n = 2, 3 \\ 2 & \text{if (i) } m = 2 \text{ and } n \geq 4 \\ & \quad \text{(ii) } n = m = 3 \text{ or} \\ & \quad \text{(iii) } n \geq m + 2 \geq 5 \\ 3 & \text{if } n = m \geq 5 \text{ or } n = m + 1 \geq 4 \\ 4 & \text{if } n = m = 4. \end{cases}$$

6 Problems

We conclude with some open questions in this area of study.

1. What is relationship between $\chi_s(G)$ and $\chi(G)$?
2. What is relationship between $\chi_s(G)$ and $\sigma(G)$?
3. For a connected graph G of order $n \geq 2$, how large can $\chi_s(G)$ be in terms of n?

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