Average Solid Angle for Detection of Photon Pair Positron Annihilations

Albert S. Ferrari

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AVERAGE SOLID ANGLE
FOR DETECTION OF PHOTON
PAIR POSITRON ANNIHILATIONS

by

Albert S. Ferrari

A Thesis
Submitted to the
Faculty of The Graduate College
in partial fulfillment
of the
Degree of Master of Arts

Western Michigan University
Kalamazoo, Michigan
December 1971
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Albert S. Ferrari
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Western Michigan University, M.A., 1971
Physics, solid state

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>SECTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>INTRODUCTION TO THE PROBLEM</td>
</tr>
<tr>
<td>II</td>
<td>COINCIDENT DETECTION</td>
</tr>
<tr>
<td>III</td>
<td>SYMMETRY CONSIDERATIONS</td>
</tr>
<tr>
<td>IV</td>
<td>FORMULATION OF THE AVERAGE SOLID ANGLE</td>
</tr>
<tr>
<td>V</td>
<td>CALCULATION OF SOLID ANGLE</td>
</tr>
<tr>
<td>VI</td>
<td>ANALYSIS OF ERROR</td>
</tr>
<tr>
<td>VII</td>
<td>SAMPLE CALCULATION</td>
</tr>
<tr>
<td>VIII</td>
<td>APPENDIX A</td>
</tr>
<tr>
<td></td>
<td>Computer Program</td>
</tr>
<tr>
<td>IX</td>
<td>APPENDIX B</td>
</tr>
<tr>
<td></td>
<td>Numerical Double Integration</td>
</tr>
<tr>
<td>X</td>
<td>BIBLIOGRAPHY</td>
</tr>
</tbody>
</table>
INTRODUCTION TO THE PROBLEM

Presently, at Western Michigan University, research is being carried out by Kusmiss and Oppliger involving gamma ray emissions from positron-electron annihilations. They are mainly concerned with two special cases: two-gamma ray emissions and the case of three-gamma ray emissions where the three gamma rays share equally the energy released in annihilation. They are interested in experimentally determining the ratio of the cross sections for these two processes. To do this, the average solid angles subtended by the detector at the source of annihilations for these two cases must be calculated (see figure 1).

In this thesis, the average solid angle subtended by the detector for the case of two-gamma ray emissions was calculated using numerical approximations and a high speed computer. The calculation was made for the particular experiment currently being undertaken at Western Michigan University. When approximations were required, the dimensions involved in this experiment were used to determine whether the accuracy of the approximations was reasonable. Although the calculation is
designed for this one particular experiment, the results are in a form which may be used for any similar experiment. All that is necessary is to read into the computer the appropriate dimensions used; however, it should be understood that any error introduced due to approximations will be a function of the dimensions used. This should be taken into consideration before using these results.
Figure 1 shows the geometry of the apparatus used in the detection of two-gamma ray emission annihilations. The two detectors are wired for coincident detection (each detector must sense a photon at the same time before an event is recorded). The electron and positron are assumed to be nearly at rest when annihilation occurs; thus, this geometry singles out two-gamma ray emission events from other possibilities. Figure 2 illustrates this selection of events. If a three-gamma ray emission takes place where each of the photons are given off 120 degrees apart (this is the case where they share equally the annihilation energy), there is no possibility that two of the photons could hit the detectors at the same time regardless of where in the source annihilation takes place.

Coincident detection is what makes the calculation of the average solid angle so difficult. Depending on where in the source the annihilation takes place, only the area of the detector that will project through the source point of annihilation and onto the other detector is used in the calculation. This also means either
detector can be used for the calculation since, if the solid angles subtended by the two detectors were different, one detector could sense a photon that the other could not, but coincident detection prohibits this. Figure 3 shows coincident detection areas for two different source points of annihilation.
3-gamma ray emission: Both detectors do not sense a gamma ray coincidently; therefore, the event is not recorded.

2-gamma ray emission: Both detectors sense a gamma ray; therefore, the event is recorded.
SYMMETRY CONSIDERATIONS

Before starting to calculate the average solid angle the symmetry of the system should be considered. There are two axes of symmetry, Al and A2 (see figure 4). Any point P can be rotated into a point P', P'', or P''' by rotations of 180 degrees about these axes without changing the geometry of the system. Thus, every point on the source can be represented by some point in one quadrant and some combination of symmetry rotations. This enables the integration to be performed over only one quadrant of the source when averaging the solid angle.

The symmetry of the detector (see figure 3) allows integration over only half its area if the results are multiplied by two. It is also much easier to integrate over the detector that has its complete area available for detection than the one that has a small circular off-centered portion of its surface area available (again see figure 3).
Rotation about $A_2$

Rotation about $A_1$

Rotation about $A_2$

Rotation about $A_1$ followed by a rotation about $A_2$

*figure 4*
FORMULATION OF THE AVERAGE SOLID ANGLE

This section is devoted to setting up the integrations needed in the calculation of the average solid angle. Although the symmetry of the system indicates the use of cylindrical coordinates, more success was obtained using a Cartesian coordinate system. Figure 5, along with equations 1 through 4, shows the setting up of the solid angle, \( \Omega \), subtended by some arbitrary point in the source.

\[ \begin{align*}
(1) \quad d\Omega &= \mathbf{n} \cdot dS/r^3 = \cos \theta \, dS/r^2 \\
(2) \quad \cos \theta &= (d-y)/r \\
(3) \quad r^2 &= (d-y)^2 + \ell^2 \text{ where } \ell^2 = (x'-x)^2 + z'^2 \\
(4) \quad d\Omega &= (d-y) \, dS/r^3 \\
&= (d-y) \, dz' \, dx' / \left( (d-y)^2 + (x'-x)^2 + z'^2 \right)^{3/2}
\end{align*} \]

\( \text{ figure 5 } \)
Previously it was mentioned that the area of the detector available for detection depended upon where in the source the annihilations took place. If the annihilations occurred in the region $x \leq -ay/d$, denoted by region I on figure 6, the entire surface area of the detector was used in the integration. When the annihilations occurred in the region $x > -ay/d$, denoted by region II on figure 7, the area integrated over was the intersection of two circles, also shown in figure 7.

Figure 8, along with equations 8 through 12, establishes the limits of integration for region II. When the average solid angle for region I is to be calculated, the results of region II can be used; all that is necessary is to let $S$ equal $-a$ (see figure 8 and equation 12) and use the results of II-B.
Point of annihilation

Detector used in calculation

Entire surface of detector is used

figure 6
Same dimensions as in figure 6

(5) \[ \tan \theta = \frac{x}{(d+y)} = \frac{f}{2d} \]
   therefore \( f = \frac{2xd}{(d+y)} \)

(6) \[ \tan \phi = \frac{(a-x)}{(d+y)} = \frac{g}{(d-y)} \]
   therefore \( g = \frac{(a-x)(d-y)}{(d+y)} \)

(7) \[ b = g - x + f \]
   therefore \( b = a(d-y)/(d+y) \)

figure 7
Negative S

\( b^2 = a^2 + f^2 - 2af \cos \theta' \)
where \( \cos \theta' = S/a \)

\( S = (a^2 + f^2 - b^2) / 2f \)
现在使用（5）和（7）

\( S = (x^2 d + a^2 y) / x(d+y) \)
\( S \) 等于 \(-a\) 当
\( x \) 等于 \(-ay/d\)

Positive S

Limits of Integration
（拆成两部分A和B）

(11) A: \( z' \) 从 0 到 \((b^2 - x''^2)^{1/2}\)
\( x'' \) 从 \(-b\) 到 \(S-f\)
where \( x'' = x'-f \)

(12) B: \( z' \) 从 0 到 \((a^2 - x'^2)^{1/2}\)
\( x' \) 从 \(S\) 到 \(a\)

figure 8
All that remains before the actual calculations can be performed is to establish integration limits for the source. Figure 9, along with equations 13 through 20, indicates how averaging the solid angle over the source is to be done.

\[
\begin{align*}
\int_{(13)} \int_{(14)} \int_{(15)}
\end{align*}
\]

Limits of Integration

Region I
(16) \( y: \ - (R^2-x^2) \) to \(-\alpha d/a\)
\( x: \ 0 \) to \( x_0 \)

Region II
(17) \( x: \ -\alpha y/d \) to \((R^2-y^2)\)
\( y: \ y_0 \) to \( 0 \)

(18) \( \overline{\Omega} = (\overline{\Omega}_{I} \bar{S}_I + \overline{\Omega}_{II} \bar{S}_{II}) / (\bar{S}_I + \bar{S}_{II}) \)

(19) \( \overline{\Omega}_{I} \bar{S}_I = \int \Omega_I dS_I \) limits given in equation 16

(20) \( \overline{\Omega}_{II} \bar{S}_{II} = \int (\Omega_{II-A} + \Omega_{II-B}) dS \) limits given in equation 17

figure 9
CALCULATION OF SOLID ANGLE

Previously it was stated that the calculations for region II would be performed first, since the results could be used for region I. Using equations 4 and 11 the following is obtained:

\[
\Omega_{II-A} = 2 \int_{-b}^{b} \frac{z'(x')}{((d-y)^2+(x'-x)^2+z'^2)^{3/2}} \, dx'
\]

where \( z'(x') = \sqrt{(b-(x'-f)} \).

Integrating the above over \( z' \) gives

\[
\Omega_{II-A} = 2 \int_{-b}^{b} \frac{z'(x')}{((d-y)^2+(x'-x)^2+z'^2)^{3/2}} \, dx'
\]

Expanding \( (d-y)^2+(x'-x)^2 \) and \( (d-y)^2+(x'-x)^2+z'^2 \) in binomial expansions and keeping the first two terms yields

\[
\Omega_{II-A} = 2 \int_{-b}^{b} \left\{ \frac{z'(x')}{(d-y)^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left( \frac{(x'-x)^2}{(d-y)^2} \right)^m \right\} \, dx'
\]

Combining terms gives

\[
\Omega_{II-A} = \int_{-b}^{b} \left\{ A_1 z' + A_2 x' z' + A_3 x'^2 z' + A_4 x'^3 z' + A_5 x'^4 z' + A_6 x'^3 z' + A_7 x'^2 z' + A_8 x'^2 z' \right\} \, dx'
\]
where

\begin{align*}
A_1 &= \frac{1}{(d-y)^2} - \frac{5x^2}{(d-y)^2} = \frac{3-x^2}{(d-y)^2} \quad \text{and} \quad A_5 = \frac{1}{(d-y)^4} \\
A_2 &= \frac{x}{(d-y)^2} \left\{ \frac{3-2x^2}{(d-y)^2} \right\} \quad \text{and} \quad A_6 = \frac{5}{(d-y)^2} \left\{ \frac{x^2}{(d-y)^2} - 1 \right\} \\
A_3 &= \frac{3}{(d-y)^2} \left\{ \frac{x^2}{(d-y)^2} - \frac{3}{2} \right\} \quad \text{and} \quad A_7 = \frac{x}{(d-y)^2} \\
A_4 &= \frac{-2x}{(d-y)^4} \quad \text{and} \quad A_8 = \frac{1}{(d-y)^4}.
\end{align*}

Making a change of variables, \( x' = x' + f \), \( dx' = dx' \) and \( z'(x') = z'(x') = (b^2-x'^2)^{1/2} \), yields

\begin{align*}
\sum_{II-A}^{} &= \int_{II-A}^{} \left\{ B_1 z' + B_2 x'^2 + B_3 x'^2 z' + B_4 x'^3 z' + B_5 x'^4 z' + B_6 x'^3 + B_7 x'^3 z' + B_8 x'^2 z' \right\} dx',
\end{align*}

where

\begin{align*}
B_1 &= A_1 + fA_2 + f^2 A_3 + f^3 A_4 + f^4 A_5 \quad B_5 = A_5 \\
B_2 &= A_2 + 2fA_3 + 3f^2 A_4 + 4f^3 A_5 \quad B_6 = A_6 + fA_7 + f^2 A_8 \\
B_3 &= A_3 + 3fA_4 + 6f^2 A_5 \quad B_7 = A_7 + 2fA_8 \\
B_4 &= A_4 + 4fA_5 \quad B_8 = A_8.
\end{align*}

Upon term by term integration equation 26 becomes

\begin{align*}
\sum_{II-A}^{} &= 2 \left\{ C_1 \arcsin(x'/b) + C_2 x' \left( b^2-x'^2 \right)^{1/2} + C_3 (b^2-x'^2)^{3/2} + C_4 x' \left( b^2-x'^2 \right)^{3/2} + C_5 (b^2-x'^2)^{5/2} \right\},
\end{align*}

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where

\[ C_1 = \frac{b^2}{8(d-y)} \left\{ 4B_1 + b^2B_3 + b^4B_5 + 3b^2B_6 \right\} \]

\[ C_2 = \frac{1}{b^2} \]

\[ C_4 = \frac{(B_6 - b^2B_5 - B_3)}{4(d-y)^2} \]

\[ C_3 = \frac{-(B_2 + b^2B_4)}{3(d-y)^2} \]

\[ C_5 = \frac{(B_4 - B_7)}{5(d-y)^2} \]

Evaluating equation 28 at the limits gives

\[ \Omega_{II-A} = 2\left\{ C_1\left(\arcsin(S-f) + \pi/2 \right) + C_2(S-f)\sqrt{(b^2-(S-f)^2)} + C_3\sqrt{(b^2-(S-f)^2)^3} + C_4(S-f)\sqrt{(b^2-(S-f)^2)^3} + C_5\sqrt{(b^2-(S-f)^2)^5} \right\} . \]

The next step in the calculation of the solid angle is the evaluation of \( \Omega_{II-B} \). \( \Omega_{II-B} \) is found by using equations 4 and 12:

\[ \Omega_{II-B} = \int_a^\sqrt{a^2-x^2} \frac{2(d-y) \, dz'}{s \circ ((d-y)^2+(x'-x)^2+z'^2)^{3/2}} . \]

Using the results of \( \Omega_{II-A} \) through equation 25 yields

\[ \Omega_{II-B} = \frac{2}{(d-y)^2} \int_a^{\sqrt{a^2-x^2}} \left\{ A_1z'^3 + A_2x'z'^2 + A_3x'z + A_4x'^3z + A_5x'4z' + A_6z'^3 + A_7x'z'^3 + A_8x'^2z + A_9x'^2z'^3 \right\} \, dx' . \]

The integration of equation 32 is identical to that performed in equation 26 except \( x'' \) and \( b \) are replaced by \( x' \) and \( a \):

\[ \Omega_{II-B} = 2\left\{ D_1\arcsin(x'/a) + D_2x'(a^2-x'^2)^{1/2} + \frac{D_3(a^2-x'^2)^{3/2}}{3^2} + D_4x'(a^2-x'^2)^{3/2} + D_5(a^2-x'^2)^{5/2} \right\} . \]
where

\begin{align*}
D_1 &= \frac{a^2}{8(d-y)^2} \left\{ 4A_1 + a^2 A_3 + a^4 A_5 + 3a^2 A_6 \right\} \\
D_2 &= \frac{D_1}{a^2} \\
D_3 &= \frac{-(A_2 + a^2 A_4)}{3(d-y)^2} \\
D_4 &= \frac{(A_6 - a^2 A_5 - A_3)}{4(d-y)^2} \\
D_5 &= \frac{(A_4 - A_7)}{5(d-y)^2}.
\end{align*}

Evaluating equation 33 at the limits gives

\begin{align*}
\Omega_{II-B} &= 2D_1 \left( \frac{\pi}{2} - \arcsin(S/a) \right) - D_2 S(a^2 - S^2)^{1/2} - \\
&\quad D_3 (a^2 - S^2)^{3/2} - D_4 S(a^2 - S^2)^{3/2} - D_5 (a^2 - S^2)^{5/2}.
\end{align*}

Combining equations 30 and 35 results in

\begin{align*}
\Omega_{II} &= 2 \left\{ C_1 \left( \arcsin(S-f) + \frac{\pi}{2} \right) + C_2 (S-f) \left( b^2 - (S-f)^2 \right) + \\
&\quad C_3 (b^2 - (S-f)^2)^3 + C_4 (S-f) \left( b^2 - (S-f)^2 \right)^3 + \\
&\quad C_5 (b^2 - (S-f)^2)^5 \right\} + D_1 \left( \frac{\pi}{2} - \arcsin(S/a) \right) - \\
&\quad D_2 S(a^2 - S^2) - D_3 (a^2 - S^2)^3 - D_4 S(a^2 - S^2)^3 - \\
&\quad D_5 (a^2 - S^2)^5.
\end{align*}

where

\begin{align*}
C_1 &= \frac{b^2}{8(d-y)^2} \left\{ 4 + 4x^4 + 9a^2 x^2 + a^4 - (6x^2 + 3a^2) \right\} \\
C_2 &= \frac{C_1}{b^2} \\
C_3 &= \frac{2x^2 + 2a^2 x - 3x(d+y)^2}{3(d^2 - y^2)^3} \\
C_4 &= \frac{2(d+y)^2 - 5x^2 - a^2}{8(d-y)^4(d+y)^2} \\
C_5 &= \frac{x}{5(d+y)(d-y)^5}.
\end{align*}
The averaging of $\Omega_{\text{II}}$ will be done using Simpson's Rule twice. Equation 20 and the computer program give further details on averaging $\Omega_{\text{II}}$.

The final step in calculating the solid angle is the evaluation of $\Omega_{\text{I}}$. This is simply done by letting $S = -a$ in equation 35:

$$
\Omega_{\text{I}} = 2\pi D_1
= 2\pi a^2 \left\{ \frac{4x^4 + 9a^2 x^2 + a^4}{2(d-y)^6} + \frac{4}{(d-y)^2} \right\}.
$$

The first integration needed in averaging $\Omega_{\text{I}}$ is easily performed without the aid of the computer (refer to equation 19):

$$
\Omega_{\text{I}} = \int_{-\pi/2}^{\pi/2} 2\pi a^2 \left\{ \frac{4x^4 + 9a^2 x^2 + a^4}{2(d-y)^6} + \frac{4}{(d-y)^2} \right\} \sin x dx.
$$

The final integration will be done using Simpson's Rule. This integration and the averaging of $\Omega_{\text{I}}$ with $\Omega_{\text{II}}$ are done with the aid of the computer (see computer program).
ANALYSIS OF ERROR

When analyzing the accuracy of the calculations previously made, one must consider three sources of error: (1) the error due to assuming the electrons and positrons to be at rest when the annihilations occurred, (2) the neglecting of second and higher order terms in expansions of $\Omega$, and (3) the error due to truncation and round off during numerical integration.

If the electrons and positrons were not at rest when the annihilations occurred, the photons would not necessarily be emitted 180 degrees apart, since the total linear momentum after the annihilations would not necessarily be zero. Figure 10 shows the angular relations of these emitted gamma rays and the consequent variations in detection area. There are some gamma rays that would miss the detector if given off 180 degrees apart that now are detected. This would make the detector appear larger than it would be if only emissions 180 degrees apart were considered. On the other hand, there will be some gamma rays that should be detected but are not. This would make the detection area appear smaller than it would be if only emissions...
180 degrees apart were considered. In either case the variation from a 180 degree separation is small, since the velocities of the electrons and positrons are small relative to the velocity of the emitted photons.

Figure 10 also illustrates the distribution\(^1\) of these gamma rays emitted at various angles \(\theta\) apart. This distribution does depend upon the source of the annihilations, but for most metals these angular distributions have full widths at half maximum, no greater than \(16 \times 10^{-3}\) radians. Due to the symmetry in the angular distribution, many of the apparent variations in the detector area will cancel. Those variations which do not cancel may still be neglected due to the small number of gamma rays responsible for them.

The error introduced when representing a function \(f(x)\) by another function \(p(x)\) is defined to be \(E(x) = f(x) - p(x)\). Equations 40 through 46 apply this definition of error to the two first order expansions made in equation 23 on page 16:

Distribution of photons

Angle between annihilation photons in $10^{-3}$ radians (for aluminum)

Angular relation between emitted photons

Variation in detection area

figure 10
\[ f(x, y, x') = ((d-y)^2+(x'-x)^2)^{-1} \]
\[ = \sum_{m=0}^{\infty} \left(-\frac{(x'-x)^2}{(d-y)^2}\right)^m, \]

\[ p(x, y, x') = \sum_{m=0}^{\infty} \left(-\frac{(x'-x)^2}{(d-y)^2}\right)^m, \]

\[ E(x, y, x') = \sum_{m=2}^{\infty} \left(-\frac{(x'-x)^2}{(d-y)^2}\right)^m, \]

\[ E_{\text{max}} = \sum_{m=2}^{\infty} \left(-\frac{(2a)^2}{d^2}\right)^m. \]

The maximum percent error is

\[ E_{\text{max}} = \frac{\sum_{m=2}^{\infty} \left(-\frac{(2a)^2}{d^2}\right)^m}{\sum_{m=0}^{\infty} \left(-\frac{(2a)^2}{d^2}\right)^m} \times 100\%. \]

Equation 44 is simply a quotient of two geometric series. Evaluating for 2a = 3.81 cm, 2R = 2.54 cm, and d = 15.0 cm (see figure 1) results in a maximum percent error of 0.302%. A similar analysis of

\[ f'(x, y, x') = ((d-y)^2+(x'-x)^2+z'^2)^{-1/2} \]
\[ = \sum_{n=0}^{\infty} \left(-1\right)^n \frac{|2n-1|!! \left\{(x'-x)^2+z'^2\right\}^n}{2n!! \ (d-y)^2} \]

yields

\[ E'_{\text{max}} = \sum_{n=2}^{\infty} \left(-1\right)^n \frac{|2n-1|!! \left\{(2a)^2\right\}^n}{2n!! \ d^2}, \]

which gives a maximum percent error of 0.113%. These two expansions are multiplied together; therefore, to a good approximation the percent errors may be added together to give a total maximum percent error of 0.415%. Thus, for dimensions with a ratio of detector diameter
to distance between detectors of 3.18 to 30 or less, the error due to the first order expansions can be considered very small.

The final error to be considered is that due to truncation and round off during numerical integration. There is no precise way of determining a bound on the truncation error when applying integration formulas to multiple integrals. A good estimate of the accuracy of the integration can be made by comparing successive answers for the integral as the number of increments used in the integration formula is doubled (see appendix Numerical Double Integral). This technique is only applicable in cases where the integrand is especially well-behaved, does not oscillate, and does not have narrow peaks. The solid angle function is such a function; therefore, it can be analyzed in the above manner. The following graphs, figures 11 through 13, verify the well-behavedness of the integrands needed in determining the average solid angle. Round off error was minimized by using double precision arithmetic (keeping 16 significant figures instead of 8) and did not appear to be significant as will be seen in the sample calculation.
SAMPLE CALCULATION

The dimensions used for the sample calculation, along with output taken from the computer, are given in the table below. The values for $\tilde{\Omega}_{II}$ times area$_{II}$ and $\tilde{\Omega}_{I}$ times area$_{I}$ converged very quickly, as was expected, since the integrands (shown in figures 11 through 13) were very smooth and well-behaved. The column denoted by N gives the number of points at which the integrands were evaluated. In the case of $\tilde{\Omega}_{II}$, since it was a

| radius of source | 1.27 cm |
| radius of detector | 1.90 cm |
| distance between detectors | 30.0 cm |

<table>
<thead>
<tr>
<th>$\tilde{\Omega}<em>{II}$X(area$</em>{II}$)</th>
<th>$\tilde{\Omega}<em>{I}$X(area$</em>{I}$)</th>
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<td>0.004578513 sr-cm$^2$</td>
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</tr>
<tr>
<td>0.035972017</td>
<td>0.004578511</td>
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<td>64</td>
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<tr>
<td>0.036112105</td>
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<td>128</td>
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</table>

$\tilde{\Omega} = 0.032121583$ sr

Upper bound on error introduced by expansions of omega = 0.4162 %
Upper bound on error due to numerical integrations = 0.0044 %
Upper bound on total error = 0.4206 %
double integral, the number \( N \) should be squared.

The values for the solid angle as a function of \( y \) 
\((x = 0)\) can be calculated directly without using
numerical approximations. Table 2 shows how these
true values compare with the values obtained from the
approximated solid angle function.

<table>
<thead>
<tr>
<th>y coordinate</th>
<th>true value</th>
<th>approximated value</th>
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<td>0.0468704</td>
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<tr>
<td>-0.635</td>
<td>0.0461256</td>
<td>0.0461205</td>
</tr>
<tr>
<td>-0.762</td>
<td>0.0453933</td>
<td>0.0453884</td>
</tr>
<tr>
<td>-0.889</td>
<td>0.0446782</td>
<td>0.0446736</td>
</tr>
<tr>
<td>-1.016</td>
<td>0.0439798</td>
<td>0.0439755</td>
</tr>
<tr>
<td>-1.143</td>
<td>0.0432977</td>
<td>0.0432935</td>
</tr>
</tbody>
</table>

It is of interest to compare the average value of
the solid angle to the value of the solid angle sub-
tended at the center of the source. This is the value
that would be used if the source was approximated to be
a point. Figure 14 shows both the average value of the
solid angle and the point source approximation value as
a function of the separation distance between detectors.
It is important to notice that even for large distances there is a substantial difference between these values (detector and source of dimensions given in table 1).

The results from the sample calculation are in good agreement with what was expected, and in all cases are well within the theoretical error bound limits. This indicates there were no significant round off errors occurring during computer computations.
First integral of $\Omega I \cdot ds I$ in sr-cm

$x$ in cm

figure 11
\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{Graph showing \( y = -1.0707 \text{ cm} \) and \( y = -0.7558 \text{ cm} \) for different values of \( x \) in cm.}
\label{fig:figure12}
\end{figure}

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\( y = -0.3149 \text{ cm} \)

**Figure 13**

\[
\text{\( \Omega_{II} \text{ in steradians} \)}
\]
One half separation distance between detectors in cm (d)

figure 14
APPENDIX A

Computer Program

The following program consists of a main subroutine, five subprogram subroutines, and two function subroutines all written in FORTRAN using double precision arithmetic.

The main subroutine calls and supplies input to the subprogram subroutines: The parameters S, D, and R represent the detector radius, one half the distance between detectors, and the radius of the source of radiation. The other parameters will be defined when the subprogram subroutines are described.

The first subprogram subroutine called by the main subroutine is Double. It performs the double integration needed in determining \( \bar{\mu} \) times area\( \Xi \). The input parameters A and B are the lower and upper limits of integration in the y direction. The integration limits in the x direction are functions of y and are contained within the subroutine Double. They are denoted by \( p(y) \) and \( q(y) \). The integration technique used by Double is an iterative process (described
in appendix B, Numerical Double Integration) which requires an error tolerance (denoted by ERR) to stop the iterations. ANS, NY, and NM are output parameters which are respectively the value of the double integral, the number of points the function had to be evaluated at, and the number of iterations required to achieve the desired accuracy.

The next subprogram subroutine called is Single. This subroutine performs the single integration needed to determine \( \int \), times area. The input parameters X0 and X1 are the lower and upper integration limits. The output parameters AN, NY, and NM are analogous to the output parameters of Double.

The third subprogram subroutine, Avg, uses the output of Double and Single to determine the average value of the solid angle, \( \overline{\Omega} \), denoted by AVE in the program. Avg also outputs E1, E2, and ET which are respectively the upper error bound due to truncation of expansions of \( \overline{\Omega} \), the upper bound on error introduced in numerical integrations, and the upper bound on the error in the final value of the average solid angle.

The last two subprogram subroutines called are Plot1 and Plot2. These subroutines evaluate the inte-
grands of the single and double integral at various points. Plots from their outputs indicate how well-behaved and smooth the integrands are, which is extremely important when applying certain numerical techniques.

The two function subroutines \( F \) and \( U \) represent the two integrands. \( F \) represents \( \int_{II} \), which is the integrand of the double integral, and \( U \) is the first integral of \( \int_{I} \), which is the integrand of the single integral.
I:PLICIT REAL*8(A-H,Z-O)
DIMENSION AHS(10),HY(10),NY(10),AN(10)
S=.1.90500
D=.15.00
R=.1.2700
COMMON S,D,R
A=-R/D/SQRT((S/D)**2+1.00)+.000270
B=.9990100
X0=0.00
XI=R/D/SQRT((D/S)**2+1.00)
ERR=.0000400
CALL DOBLE(S,D,R,A,B,ERR,ANS,MY,H)
PRINT 1
2 FORMAT(1H1,18X,'OMEGA-I AVERAGE)2AREA=IT)/22X,1'FOR N SQUARE ITERR.'/
PRINT 5, (AN(I),HY(I),I=1,N)
3 FORMAT(15X,F12.5,17X,13/)
CALL SINGLE(S,D,R,X0,X1,ERR,AN,H,Y,1)
PRINT 4
4 FORMAT(/ 16X,'OMEGA-I AVERAGE)2AREA=IT)/24X,1'FOR N ITERR.'/
PRINT 5, (AN(I),HY(I),I=1,N)
CALL AVG(AN(I),AN(I),ERR,S,D,R,AVE,E1,E2,ET)
PRINT 5,AVE,E1,E2,ET
5 FORMAT(/ 10X,'THE AVERAGE VALUE OF OMEGA=',F12.0,/
1'UPPER BOUND ON ERROR DUE TO EXPANSIONS OF OMEGA=',/
1F7.4,1X,'%'/'UPPER BOUND ON TOTAL ERROR DUE TO NUMERICAL ',/
1'INTEGRATION =',F7.4,1X,'%'/'UPPER BOUND ON TOTAL ',/
1'ERROR =',F7.4,1X,'%')
PRINT 8
8 FORMAT(1H1,16X,'VALUES OF FIRST INTEGRAL OF OMEGA-I'1/12X,'X'10000)
CALL PLOT1(S,D,R,X0,X1)
PRINT 5
6 FORMAT(1H1,13X,'VALUES OF OMEGA-I'//8X,'X',8X,'Y',/
12X,'OMEGA-I',5X/
CALL PLOT2(S,D,R,A,B)
STOP
END
SUBROUTINE DOUBLE(S, D, A, B, ERR, ANS, NY, NJ)
IMPLICIT REAL*8(A=1, I=1, D=2)
DIMENSION ANS(10), NY(10)
P(Y)=S*Y/D+.C0001D0
Q(Y)=S/R(Y-R-Y)
H(Y)=(Q(Y)-P(Y))/Y:
DO 10 I=1,7
I=I+2:
D=I/2:
NY(I)=I:
C1=0.D0:
C2=0.D0:
AA=H(A)*(F(P(A),A)-F(Q(A),A))-
1H(B)*(F(P(B),B)-F(Q(B),B)):
DO 20 J=1,N:
T=2.D0*I-1.D0:
V=2.D0*J:
A=B+V*DX:
Z=A+V*DX:
AB=H(A)*(4.*F(P(A)+T*H(A),A)+2.*F(P(A)+V*H(A),A))-
AC=H(B)*(4.*F(P(B)+T*H(B),B)+2.*F(P(B)+V*H(A),B))
AD=4.D0*H(G)*(F(P(G),G)-F(Q(G),G)):
RT=2.D0*H(Z)*(F(P(Z),Z)-F(Q(Z),Z)):
C1=C1+AB+AC+AD+RT:
DO 50 J=1,N:
T=2.D0*J-1.D0:
V=2.D0*J:
AE=4.*H(G)*(4.*F(P(G)+T*H(G),G)+2.*F(P(G)+V*H(G),G)):
AF=2.*H(Z)*(4.*F(P(Z)+T*H(Z),Z)+2.*F(P(Z)+V*H(Z),Z)):
C2=C2+AE+AF:
CONTINUE
30 CONTINUE
ANS(NJ)=(AA+C1+C2)*DX/3.D0:
Z1=DA3S(ANS(NJ)-ANS(NJ-1)):
Z2=DA3S(ANS(NJ)):
Z3=Z1/Z2:
IF(Z3-ERR)3,5,10:
10 CONTINUE
5 RETURN
END
SUBROUTINE SINGLE(S,D,R,X0,X1,ERR,AN,NH,NK)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION NY(10),NH(10)
ERR=ERR/200.0
DO 10 NH=1,NH
I=2*2**NH
N=I/2
H=I
DX=(X1-X0)/N
NY(NH)=H
CI=0.0
DO 20 I=1,N
T=2.0*I-1.0
G=2.0*I*1
C=4.0*U(X0+T*DX)+2.0*U(X0+G*DX)
CI=CI+C
20 CONTINUE
AN(NH)=DX*(U(X0)-U(X1)+CI)/3.0
Z1=DABS(AN(NH)-AN(NH-1))
Z2=DABS(AN(NH))
Z3=Z1/Z2
10 IF(Z3-ERR)5,5,10
CONTINUE
RETURN
END

SUBROUTINE AVG(A1,A2,ERR,S,D,R,AVE,E1,E2,ET)
IMPLICIT REAL*8(A-H,O-Z)
PI=3.141592650
AVE=4.0*(A1+A2)/(PI*R*R)
Z=(2.0*S/R)**2
E1=Z*Z*100.0
E2=(ERR+ERR/10.0)*100.0
ET=E1+E2
RETURN
END

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SUBROUTINE PLOT1(S,D,R,X0,X1)
IMPLICIT REAL*8(A-H,O-Z)
DX=(X1-X0)/20,DO
DO 10 I=1,20
ARG=X0+(I-1)*DX
UT=U(ARG)
PRINT 9,ARG,UT
10 CONTINUE
RETURN
END

SUBROUTINE PLOT2(S,D,R,A,B)
IMPLICIT REAL*8(A-H,O-Z)
P(Y)=-S*Y/D+.00000
Q(Y)=DSQRT(R*R-Y*Y)
H(Y)=(Q(Y)-P(Y))/20.0C
DX=(B-A)/20.0C
DO 10 I=1,20
Y=H+(I-1)*DX
DO 10 J=1,20
X=P(Y)+(J-1)*H(Y)
AF=F(X,Y)
PRINT 7,X,Y,AF
10 CONTINUE
RETURN
END
FUNCTION F(X, Y)
IMPLICIT REAL*8(A-Z)
COMMON S, D, R
Z=X*X
J=S+S
N=(Z*D+J*Y)/(X*(D+Y))
M=2.0*D*X*D/(D+Y)
P=3.0*(D-Y)/(D+Y)
Q=2.0*(Z+J)
D=Z+Z+Z+Z+J
D=(D+Y)**2
E=(D-Y)**2
S2=2.500*{(D-D+1)/B+.5D0*1/(D*3)}/E
S1=D*E**22/3
S4=2.500*(2.0*D-G/2)/(E**E).
C2=2.500*(4.0*D-1)/E+.5D0*1/(E*E))/E
C1=J*C2
C4=2.500*(2.0*D-G/E)/(E**E)
L=J-N*!
SQT=DSQRT(L)
ARC=(N+I)/P
T=I/S
AS1=DATAN(ARG/DSQRT(1.00-ARG**ARG))
AS1'=DATAN(T/DSQRT(1.00-T**T))
C=(Z+J)*G+(3.00*D*G+2.00*Y*Y)
XY=4.00*X*Y
SC5=XY*(-3.00+C/(R*X))/3.00*E*E*(D+Y))
SC5=-2.00*XY/(E**E*E*(D+Y))
P12=1.57073530
F=31*(P12+AS1)+C1*(P12-AS1)+A(*1-1)-C2*N)*SQT
L=(SC5+B4*(1.0-N)-C4*N)*SQT*L+SC5*SQT*L+L
RETURN
END

FUNCTION U(X)
IMPLICIT REAL*8(A-Z)
COMMON A, D, R
P=-DSQRT(R**R-X**X)
Q=D**X/A
D1=D-Q
D2=D-P
U=4.00**I**4+9.00*A**A**A*X**A**A**4
Z=(2.00*X*A+A**A)
P1=3.14159260
U=2.500*P1*A**A*X*(4.00/D1+1.00*/W/D1**5+Z/D1**5-4.00/D2
L=1.00*W/D2**5-Z/D2**5)
RETURN
END
APPENDIX B

Numerical Double Integration

Due to the complexity of the integrand of the double integral needed in the averaging of \( \Omega_{II} \) over the source, it was felt impractical to use a higher order integration formula than Simpson's Rule. The solid angle function is quite well-behaved and is ideally suited to Simpson's Rule. The following equations define and adapt to a double integral Simpson's 1/3 Rule:

\[
\int_{a}^{b} f(x) \, dx = \frac{h}{3} \left\{ f(a) - f(b) + 4 \sum_{i=1}^{N} f(a + (2i - 1)h) + 2f(a + 2ih) \right\},
\]

where \( h = \frac{b-a}{n} \) and \( n \) is an even integer;

\[
\int_{a}^{b} \int_{p(y)}^{q(y)} f(x, y) \, dx \, dy = \frac{h}{3} \left\{ \int_{p(a)}^{q(a)} \int_{p(a)+hi}^{q(a)+h} f(x, a) \, dx \, dy - \int_{p(a)}^{q(a)} \int_{p(a)+2hi}^{q(a)+2h} f(x, a+2ih) \, dx \right\}
+ \frac{hH(a)}{9} \left\{ f(p(a), a) - f(q(a), a) + \sum_{j=1}^{N} 4f(p(a) + (2j-1)h, a) + 2f(p(a) + 2jh, a) \right\}
\]

41

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-h H(b) \{ f(p(b), b) - f(q(b), b) \\
+ \sum_{i=1}^{N} 4f(p(b) + (2j-1)H(b), b) + 2f(p(b) + 2jH(b), b) \\
+ 4h \sum_{i=1}^{N} \left[ H(a+(2i-1)h) \left[ f(p(a+(2i-1)h), a+(2i-1)h) \\
+ f(q(a+(2i-1)h), a+(2i-1)h) \right] \\
+ \sum_{j=1}^{N} 4f(p(a+(2i-1)h) + (2j-1)H(a+(2i-1)h), a+(2i-1)h) \\
+ 2f(p(a+(2i-1)h) + 2jH(a+(2i-1)h), a+(2i-1)h) \right] \}
+ 2h \sum_{i=1}^{N} \left[ H(a+2ih) \left[ f(p(a+2ih), a+2ih) \\
+ f(q(a+2ih), a+2ih) \right] \\
+ \sum_{j=1}^{N} 4f(p(a+2ih) + (2j-1)H(a+2ih), a+2ih) \\
+ 2f(p(a+2ih) + 2jH(a+2ih), a+2ih) \right] \},

where H(x) = q(x) - p(x).

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