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The Constructive Theory of Distributions

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THE CONSTRUCTIVE THEORY
OF DISTRIBUTIONS

by

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of the
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1. HISTORY

With the advent of operational calculus at the end of the nineteenth century, several formulas were developed which were not, in a satisfactory manner, justified mathematically. Among the most famous examples are the Heaviside and the Dirac delta functions. These were used successfully even though the Dirac function is, by no means, a function in the ordinary sense. In this respect the problem was similar to the one of the algebraists of the 15th and 16th centuries who dealt with solving the equation \( x^2 + 1 = 0 \). The real number system was of course inadequate, yet, extending the reals by defining the imaginary number \( i \), they produced a solution to the equation. In this manner mathematicians dealt with the above problem utilizing functional analysis. To this end they sought some kind of extension of the concept of the ordinary function which would resolve the discrepancies. This new concept of the generalization of functions, referred to as distributions, first appeared in 1936 in a paper concerning weak solution and weak convergence by Sobolev.
It was Laurent Schwartz, however, who developed the theory with rigor in the late 1940's. His approach was to define distributions as continuous functionals on certain spaces. Through his work, distributions came to include continuous functions and Lebesgue locally summable functions, as well as "functions" like that of Dirac. Also, the concept of derivative was extended to all such functions. One of the beautiful features of this new theory is that it can be formulated for a function of several variables as easily as for a function of one variable. Consequently, the gaps in operational calculus were filled, and other mathematical branches, such as Fourier series and integrals, convolutions, and partial differential equations, became more complete.

In the late 1950's Mikusinski and Sikorski took a different approach to the problem. They developed the theory of distributions taking the sequential approach. Although the methods of approach are quite different, it can be shown that the basic definitions of distribution are equivalent, and the end results in both developments coincide. The methods of functional analysis employed by Schwartz are not necessary here, since distributions are not defined as functionals; rather they are defined as equivalence classes of fundamental sequences in an open set.
One of the notions which prompted the generalization of functions of classical analysis was that of impulse, an important concept from physics.

Assume a ball is rolled with the acceleration of \(1\text{ m/sec}^2\) for 1 second. Therefore, its velocity is \(1\text{ m/sec}\). One can achieve the same final velocity by rolling the ball at \(\alpha \text{ m/sec}^2\) for \(\frac{1}{\alpha}\) seconds. For that matter, the final velocity will always be \(1\text{ m/sec}\) if the ball is rolled at \(\alpha(t)\text{ m/sec}^2\) for \(\frac{1}{\alpha(t)}\) seconds, where \(\alpha(t)\) is a function of time. Another way of expressing this is to roll the ball at \(\alpha(t)\text{ m/sec}^2\) with \(\int \alpha(t) dt = 1\). The same result is obtained by generating a large acceleration over a short interval of time, as in the case of striking the ball with a hammer. Here the concept is referred to as acceleration impulse and is denoted by \(\delta(t)\text{ m/sec}^2\), where \(\delta(t)\) is the well known Dirac delta "function" defined by Dirac as

\[
\delta(t) = \begin{cases} 
\infty, & t = 0 \\
0, & t \neq 0
\end{cases}
\]

with \(\int \delta(t) dt = 1\).
Dirac was aware that the above $\delta(t)$ was not a function in the ordinary sense but that it could be approximated by functions in the following way. Let $\delta_n(t)$ be a sequence of continuous or integrable functions with the following properties:

i.) There exists a sequence $a_n$ of positive numbers converging to zero such that $\delta_n(t) = 0$ for $|t| \geq a_n$,

ii.) $\int_{-\infty}^{\infty} \delta_n(t) dt = 1$,

iii.) $\delta_n \geq 0$.

We see that $\delta_n(t) \to 0$ for $t \neq 0$ and that $\delta_n(t) \to +\infty$ for $t = 0$. In addition, there exists a uniformly convergent sequence of continuous functions $F_n(t)$ with the property that $F_n''(t) = \delta_n(t)$. That this is the case follows easily by taking

$$F_n(t) = \int_{-\infty}^{t} d\tau \int_{-\infty}^{\tau} \delta_n(t_1) dt_1.$$

It is this property that leads to the generalization of functions and the theory of distributions.
3. FUNDAMENTAL SEQUENCES

First, we give the following definition:

**Definition 3.1.** Let $I$ be any subset of $\mathbb{R}^q$. A sequence of functions $f_n$ converges uniformly on $I$ to $f$ if and only if the function $f$ is defined on $I$, and, for every $\epsilon > 0$, there exists an index $n_0$ such that, for every $n > n_0$, $f_n$ is defined on $I$ and $|f_n(x) - f(x)| < \epsilon$ for every $x \in I$. We denote this by $f_n \rightarrow f$. Note that the $f_n$ are not necessarily all defined on $I$.

**Definition 3.2.** A function $c$ defined on an open $I \subset \mathbb{R}^q$ is said to be smooth if it is continuous on $I$, as well as its partial derivatives of all orders. We denote the derivative of order $k$ of $c$ by $c^{(k)}$, where

$$c^{(k)} = \frac{\partial^{\nu_1+\ldots+\nu_q} c}{\partial \xi_1^{\nu_1} \ldots \partial \xi_q^{\nu_q}}$$

with $k = (\nu_1, \ldots, \nu_q)$ and $x = (\xi_1, \ldots, \xi_q)$.

**Definition 3.3.** Let $a = (\alpha_1, \ldots, \alpha_q)$ and $b = (\beta_1, \ldots, \beta_q)$. Then $I = (a, b)$ is an interval in $\mathbb{R}^q$ if and only if $I = \{x = (\xi_1, \ldots, \xi_q) : \alpha_i < \xi_i < \beta_i, i = 1, \ldots, q\}$. The interval $I$ is open and bounded.
**Definition 3.4.** The interval $I$ is in a set $O$ if and only if $\overline{I}$ is in $O$.

We now define the most basic concept in distribution theory, namely, that of the fundamental sequence.

**Definition 3.5.** A sequence of functions $\phi_n$ is fundamental, or d-convergent, in an open set $O \subset \mathbb{R}^d$, if and only if, given any interval $I$ in $O$, there is a uniformly convergent sequence of smooth functions $\phi_n$ in $I$ and an order $k$ such that $\phi_n^{(k)} = \phi_n$. Here, the sequence $\phi_n$ and $k$ depend on the choice of $I$, and the $\phi_n$, which are also smooth functions, need not be defined on all of $O$.

We now have the following results concerning fundamental sequences:

**Theorem 3.6.** If a sequence $\phi_n$ is fundamental, then the sequence of derivatives $\phi_n^{(m)}$ is fundamental, where $m$ is any order.

**Proof:** If $\phi_n$ is fundamental, then, given any $I$ in $O$, there exist smooth functions $\phi_n$ and an order $k$ such that $\phi_n^{(k)} = \phi_n$, and the $\phi_n$ converge uniformly in $I$. Thus,
Theorem 3.7. If a sequence \( v_n \) is fundamental in every interval \( I \) in \( 0 \), then it is fundamental in \( 0 \).

Proof: It follows from our definition of fundamental sequences.

Definition 3.8. A sequence of smooth functions \( v_n \) converges almost uniformly in \( 0 \) if, for every interval \( I \) in \( 0 \), \( v_n \) converges uniformly in \( I \).

Lemma 3.9. If \( v_n \) is almost uniformly convergent in \( 0 \), then \( v_n \) is fundamental in \( 0 \).

Proof: If \( v_n \) is almost uniformly convergent in \( 0 \), then, by definition, \( v_n \) is uniformly convergent in every \( I \) in \( 0 \). Take \( I \) to be arbitrary in \( 0 \), and let
\[
\psi_n = v_n \quad \text{and} \quad k = 0.
\]
Then, \( v_n \) is fundamental in interval \( I \) by Definition 3.5. Hence, \( v_n \) is fundamental in \( 0 \) by Theorem 3.7.

Lemma 3.10. If a sequence of smooth functions \( v_n \) converges almost uniformly in \( 0 \), then the pointwise limit of \( v_n \) is a continuous function in \( 0 \).
Proof: It follows from Definitions 3.8 and 3.2.

However, there exist fundamental sequences which do not converge at any point. For example, let \( c_n(x) = n^3 \cos nx, \ x \in \mathbb{R} \). Define \( \xi_n(x) = \frac{\cos nx}{n} \) and \( k = 4 \). Clearly, \( \xi_n \to 0 \) and \( \xi_n(k) = c_n \). Also, there exist fundamental sequences which diverge at exactly one point. For example, let

\[
\nu_n(x) = \frac{n}{(1+e^{nx})(1+e^{-nx})}, \ x \in \mathbb{R}.
\]

After careful observation, \( \nu_n \to 0 \) if \( x \neq 0 \), but \( \nu_n \to +\infty \) if \( x = 0 \). Yet, the sequence \( \nu_n \) is fundamental with

\[
\nu_n(x) = \int_{-\infty}^{x} \frac{dx}{1+e^{-nx}} \L(x) = \begin{cases} 
0, & x \leq 0 \\
x, & x > 0
\end{cases}
\]

Lemma 3.11. If \( \varsigma_{1n}, \ldots, \varsigma_{qn} \) are fundamental sequences in \( \mathbb{R} \), then \( \varsigma_n(x) = \varsigma_{1n}(x_1) \cdots \varsigma_{qn}(x_q) \) is a fundamental sequence in \( \mathbb{R}^q \).

Proof: \( \varsigma_{1n} \) is fundamental in \( \mathbb{R} = \mathbb{R}^{x_{1n}} \) such that

\[
\varsigma_{1n} = \varsigma_{1n}. \varsigma_{2n} \text{ is fundamental in } \mathbb{R} = \mathbb{R}^{x_{2n}} \text{ such that}
\]

\[
\varsigma_{2n} = \varsigma_{2n}.
\]

\[
\vdots
\]

\[
\vdots
\]
\( q_n \) is fundamental in \( R = \mathbb{R}^q \) such that \( \hat{q}_n = q_n \).

Therefore,

\[
\hat{\xi}_n(x) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi_2)\cdots\hat{\psi}_n(\xi_q)
\]

\[
= (\hat{\psi}_{1n}(\xi_1)\hat{\psi}_{2n}(\xi_2)\cdots\hat{\psi}_{qn}(\xi_q))^{(k)}(\xi_1, \xi_2, \ldots, \xi_q)
\]

\[
= \hat{\xi}_n(x), \text{ where } x = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q
\]

and

\[
\hat{\xi}_n = \hat{\psi}_{1n} \cdots \hat{\psi}_{qn}.
\]

4. EQUIVALENT SEQUENCES, DISTRIBUTIONS

Definition 4.1. The sequences \( \hat{\xi}_n \) and \( \hat{\psi}_n \) are said to be equivalent if and only if the interlaced sequence

\( \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_2, \ldots \) is fundamental. This we denote by

\( \hat{\psi}_n \sim \hat{\psi}_n \).

We now show that a necessary and sufficient condition for \( \hat{\psi}_n \sim \hat{\psi}_n \) is: For any interval \( I \subset \mathbb{R} \), there exist sequences of smooth functions \( \hat{\psi}_n \) and \( \hat{\psi}_n \) which converge uniformly to the same limit function in \( I \) and

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satisfy for some $k$, $\xi_n^{(k)} = \varphi_n$ and $\psi_n^{(k)} = \psi_n$ in $I$.

Let $\xi_n \Rightarrow \varphi_n$ denote that $\xi_n$ and $\varphi_n$ converge uniformly to the same limit function. Now, if $\varphi_n \sim \psi_n$, then the interlaced sequence $\zeta_n = \{\varphi_1, \psi_1, \varphi_2, \psi_2, \ldots\}$ is fundamental. Therefore, there exist $k$ and $\varphi_n$ such that

$$\varphi_n^{(k)} = \begin{cases} 
\varphi_n', & n = 2j + 1 \\
\psi_n, & n = 2j + 2 
\end{cases}$$

for $j = 0, 1, 2, \ldots$. This implies that there exists $\xi_n$ such that $\xi_n^{(k_1)} = \varphi_n$, $n = 2j + 1$ and there exists $\psi_n$ such that $\psi_n^{(k_2)} = \psi_n'$, $n = 2j + 2$. Hence, there exist $\xi_n$ and $\psi_n$ such that $\xi_n^{(k)} = \varphi_n$ and $\psi_n^{(k)} = \psi_n'$, where $\xi_n \Rightarrow \varphi_n$.

In order to give some important properties of equivalent sequences, we now develop the concept of repeated integrals. Let $f$ be a continuous function in $I$, $x = (\xi_1, \ldots, \xi_q)$, $x_0 = (\xi_{o1}, \ldots, \xi_{oq}) \in I$, $k = (\kappa_1, \ldots, \kappa_q)$, where $\kappa_i$ are nonnegative integers, and $t = (t_1, \ldots, t_q)$. We use induction to define

$$J_k(x) = \int_{x_0}^{x} f(x) dt^k.$$
Let

\[ J_{o}(x) = \int_{x_{o}}^{x} f(t) \, dt = f(x), \]

and

\[ J_{k+e_{j}}(x) = \int_{x_{o}}^{x} J_{k}(x) \, dt_{j} \]

\[ = \int_{x_{o}}^{x} \left[ \int_{x_{o}}^{x} f(t) \, dt_{k} \right] \, dt_{j} \]

\[ = \int_{x_{o}}^{x} \left[ \int_{x_{o}}^{x_{1}} \ldots \int_{x_{o q}}^{x_{q}} f(t_{1}, \ldots, t_{q}) \, dt_{q} \ldots dt_{1} \right] \, dt_{j} \]

\[ = \int_{x_{o}}^{x_{1}} \ldots \int_{x_{o q}}^{x_{q}} f(t_{1}, \ldots, t_{q}) \, dt_{q} \ldots dt_{1} \]

Therefore,

\[ J_{k+e_{j}}(x) = J_{k_{1}}(x) = \int_{x_{o}}^{x} f(t) \, dt_{k_{1}}, \quad \text{where} \]

\[ k_{1} = (\kappa_{1}, \ldots, \kappa_{j+1}, \ldots, \kappa_{q}). \]

Here the \( k + e_{j} \) represents the point whose j-th coordinate is \( \kappa_{j} + 1 \) and whose other coordinates are the same as those for \( k \). In fact, \( J_{k}(x) \) is a repeated
integral where $f$ is integrated $\xi_1$ times with respect to $\xi_1$, $\xi_2$ times with respect to $\xi_2$, ..., $\xi_q$ times with respect to $\xi_q$, and the order of the integration makes no difference since $f$ is continuous. The following properties hold with $f, x, x_o, \text{ and } k$ as given above:

i.) $\int_{x_o}^{x} \lambda f(t) dt^k = \int_{x_o}^{x} f(t) dt^k$, $\lambda$ is a number,

ii.) $\int_{x_o}^{x} \left[ f(t) + g(t) \right] dt^k = \int_{x_o}^{x} f(t) dt^k + \int_{x_o}^{x} g(t) dt^k$,

iii.) $\left( \int_{x_o}^{x} f(t) dt^k \right)^{(k)} = f(x)$.

We now prove two theorems before defining distributions.

**Theorem 4.2.** If for the given functions $\omega_n$, there is a uniformly convergent sequence of smooth functions $\omega_n$ in $I$ such that $\omega_n^{(k)} = \omega_n$ for some $k$, then for every $\ell \geq k$ there is a uniformly convergent sequence of smooth functions $\omega_n$ in $I$ such that $\omega_n^{(\ell)} = \omega_n$.

**Proof:** Define $\omega_n(x) = \int_{x_o}^{x} \omega_n(t) dt^{\ell-k}$, $x_o \in I$. Then
The uniform convergence of \( \tilde{v}_n \) follows from that of \( \tilde{t}_n \) and the boundedness of the interval \( I \).

\textbf{Theorem 4.3.} The relation "\( \sim \)" is an equivalence relation.

\textbf{Proof:} The reflexive and symmetric properties follow trivially from the fact that the sequences are fundamental. We prove the transitive property. Assume \( \tilde{v}_n \sim \tilde{v}_n \) and \( \tilde{v}_n \sim \tilde{v}_n \). By Lemma 3.5, there exist \( \tilde{v}_n \) and \( \tilde{v}_n \) such that \( \tilde{t}_n = \tilde{v}_n \) and \( \tilde{t}_n = \tilde{v}_n \), where \( \tilde{t}_n \sim \tilde{v}_n \) in \( I \), and there exist \( \tilde{v}_n \) and \( \tilde{v}_n \) such that \( \tilde{v}_n \) and \( \tilde{v}_n \) in \( I \). By Theorem 4.2, assume \( \ell = k \). Now, define

\[
\tilde{v}_n = \tilde{v}_n - \tilde{v}_n + \tilde{v}_n.
\]

Therefore,
Thus, there exist \( \hat{\nu}_n \) and \( \tilde{\nu}_n \) such that \( \hat{\nu}_n = \nu_n \) and \( \tilde{\nu}_n = \nu_n \), where \( \hat{\nu}_n \rightarrow \tilde{\nu}_n \) in \( I \). Hence, \( \nu_n \sim \nu_n \).

**Definition of distributions:** This last result partitions the set of all fundamental sequences in an open set 0 into disjoint equivalence classes called distributions in 0. Thus, the concept of distributions is obtained by identification of equivalent fundamental sequences.

For any smooth function \( \varphi \) in 0, we can consider a fundamental sequence \( \varphi_n \) by letting \( \varphi_n = \varphi \) for every \( n \). Since the class of all fundamental sequences which are equivalent to \( \varphi_n \) is a distribution, it is quite natural to identify this distribution with a smooth function. In this manner, distributions can be considered as a generalization of smooth functions, i.e., smooth functions are particular distributions.

5. CONTINUOUS AND LOCALLY INTEGRABLE FUNCTIONS

We begin with the following:

**Lemma 5.1.** If a sequence of smooth functions \( \varphi_n \) is almost uniformly convergent in 0, then it is fundamental.

**Proof:** Define \( \check{\varphi}_n = \varphi_n \) with \( k = 0 \).

We give the following definitions:
**Definition 5.2.** A sequence of smooth functions \( \varphi_n \) which is almost uniformly convergent in \( 0 \) is called \( C \)-fundamental.

**Definition 5.3.** Two sequences of smooth functions \( \varphi_n \) and \( \chi_n \) are \( C \)-equivalent in \( 0 \) if and only if they converge almost uniformly to the same limit.

The \( C \)-equivalence is an equivalence relation. It follows that all sequences converging to the same continuous function belong to the same \( C \)-equivalence class. Hence, each class is represented by a continuous function. But, because every \( C \)-equivalence class is a unique subclass of a distribution, continuous functions are represented by distributions.

**Definition 5.4.** A function \( f \) is called locally integrable in an open set \( 0 \) if it is integrable over every interval \( I \) in \( 0 \).

**Definition 5.5.** A sequence of functions \( \varphi_n \) converges locally to a locally integrable function \( f \) in \( 0 \) if and only if, for every interval \( I \) in \( 0 \), and, for every \( \epsilon > 0 \), there exists an index \( n_0 \) such that, for \( n > n_0 \), the function \( \varphi_n \) is integrable on \( I \) and

\[
\int_I |\varphi_n - f| < \epsilon.
\]

We now give the following theorems:
Theorem 5.6. Every locally convergent sequence of smooth functions is fundamental.

**Proof:** Assume \( \varphi_n \) is locally convergent to \( f \) which is locally integrable in \( O \). Thus, \( f \) is integrable over every interval \( I \) in \( O \). Hence, for every \( I \) in \( O \) and for every \( \epsilon > 0 \), there exists \( n_0 \) such that, for \( n > n_0 \), the function \( \varphi_n \) is integrable on \( I \), and

\[
\int_I |\varphi_n - f| < \epsilon.
\]

If \( I = (x_0, x) \) in \( O \), \( x_0 \in O \),

\[
\xi_n(x) = \int_{x_0}^x \varphi_n (t) dt + F(x) = \int_{x_0}^x f(t) dt,
\]

since

\[
|\xi_n(x) - F(x)| \leq \int_{x_0}^x |\varphi_n - f| < \epsilon.
\]

Also,

\[
\int_{x_0}^x \varphi_n = \int_{x_0}^x \varphi_n (t) dt^k,
\]

where \( k = (1, \ldots, 1) \). But

\[
\xi_n^{(k)}(x) = \left( \int_{x_0}^x \varphi_n (t) dt^k \right)^{(k)} = \varphi_n(x).
\]

**Lemma 5.7.** If a sequence of smooth functions \( \varphi_n \) is locally convergent in \( O \), then it is fundamental.
Proof: Define \( \psi_n = \varphi_n \) with \( k = 0 \).

**Definition 5.8.** A sequence of smooth functions \( \varphi_n \) which is locally convergent in \( O \) is called L-fundamental.

**Definition 5.9.** Two sequences of smooth functions \( \varphi_n \) and \( \psi_n \) are L-equivalent in \( O \) if and only if they converge locally to the same limit.

L-equivalence is an equivalence relation. As before in the discussions of the different types of equivalence, we now can consider L-equivalence classes. All sequences converging to the same locally integrable function belong to the same L-equivalence class. Thus, each class can be identified with a locally integrable function. Again, because every L-equivalence class is a unique subclass of a distribution, locally integrable functions are represented by distributions. Thus, locally integrable functions are certain distributions.

**Remark 5.10.** Although we have been showing that smooth functions, continuous functions, and locally integrable functions are distributions, most distributions are not functions. For example, earlier the sequence

\[
\psi_n = \frac{n}{(1+e^{nx})(1+e^{-nx})}
\]

was considered. It was shown to be fundamental, and it
can be shown to belong to the famous Dirac delta distribution, which is not a function.

6. REGULAR OPERATIONS

As may be expected, many of the operations defined on functions can be easily extended to distributions.

Let $O_1, \ldots, O_n$ be open subsets of Euclidean spaces which are not necessarily of the same dimension. We define the smooth functions $\varphi_1, \ldots, \varphi_n$ in $O_1, \ldots, O_n$, respectively. Assume that $A$ is a well defined operation on the $\varphi_i$, $i = 1, \ldots, n$, and that the result of the operation is a smooth function in a fixed subset $O$ of some Euclidean space. We let $A(\varphi_1, \ldots, \varphi_n)$ denote an operation on $\varphi_i$, $i = 1, \ldots, n$.

**Definition 6.1.** The operation $A$ is regular if and only if, for any sequences $\varphi_{1n}, \ldots, \varphi_{rn}$ fundamental in $0_{1n}, \ldots, 0_{rn}$, respectively, the sequence $\varphi_n = A(\varphi_{1n}, \ldots, \varphi_{rn})$ is fundamental in $O = O_{1n} \cap \ldots \cap O_{rn}$.

Let the sequences $\varphi_{1n}, \ldots, \varphi_{rn}$ belong to the distributions $f_1, \ldots, f_r$, respectively. The sequence $\varphi_n$ belongs to a distribution $f$. We denote this by $A(f_1, \ldots, f_r) = f$.

We have the following lemmas:
Lemma 6.2. If \( \varphi_{1n} \sim \varphi_{21}, \ldots, \varphi_{rn} \sim \varphi_{2n} \), then \( A(\varphi_{1n}, \ldots, \varphi_{rn}) \sim A(\varphi_{21}, \ldots, \varphi_{2n}) \).

**Proof:** \( \varphi_{1n} \sim \varphi_{21}, \varphi_{11}, \varphi_{12}, \ldots \) is fundamental.

\( \varphi_{2n} \sim \varphi_{21}, \varphi_{22}, \varphi_{23}, \ldots \) is fundamental.

\( \vdots \)

\( \varphi_{rn} \sim \varphi_{rl}, \varphi_{r1}, \varphi_{r2}, \varphi_{r3}, \ldots \) is fundamental. Applying a regular operation \( A \), we get

\[
A(\varphi_{11}, \varphi_{21}, \ldots, \varphi_{rl}) = \varphi_{1l},
\]

\[
A(\varphi_{11}, \varphi_{21}, \ldots, \varphi_{rl}) = \varphi_{1l},
\]

\[
A(\varphi_{12}, \varphi_{22}, \ldots, \varphi_{r2}) = \varphi_{2l},
\]

\[
A(\varphi_{12}, \varphi_{22}, \ldots, \varphi_{r2}) = \varphi_{2l},
\]

\[
A(\varphi_{1n}, \varphi_{2n}, \ldots, \varphi_{rn}) = \varphi_{nl},
\]

\[
A(\varphi_{1n}, \varphi_{2n}, \ldots, \varphi_{rn}) = \varphi_{nl}.
\]

We now have a new sequence:

\( \varphi_{1l}, \varphi_{1l}, \varphi_{2l}, \varphi_{2l}, \ldots, \varphi_{nl}, \varphi_{nl} \).

Since \( A \) is regular, the resulting sequence is fundamental. This lemma shows that the operation \( A \) is well defined.
Lemma 6.3. If \( \varphi \) is smooth in \( O \), then \( \lambda \varphi \) is smooth in \( O \), for every fixed number \( \lambda \).

Proof: This result follows directly from Definition 3.2.

Lemma 6.4. If \( \varphi_n \) is a fundamental sequence in \( O \), then \( \lambda \varphi_n \) is fundamental in \( O \), where \( \lambda \) is a fixed number.

Proof: Since \( \varphi_n \) is fundamental, then there exist \( \hat{\varphi}_n \) and \( k \) such that \( \hat{\varphi}_n^{(k)} = \varphi_n \). Then,

\[
(\lambda \hat{\varphi}_n)^{(k)} = \lambda \hat{\varphi}_n^{(k)} = \lambda \varphi_n.
\]

Now, if the distribution \( \varphi \) represents the equivalence class of the fundamental sequence \( \varphi_n \), then \( \lambda \varphi \) is a regular operation.

Lemma 6.5. If \( \varphi \) is smooth in \( O \), then \( \varphi^{(k)} \) is smooth in \( O \), for any order \( k \).

Proof: This result follows from the definition of smooth functions.

Lemma 6.6. If \( \varphi_n \) is fundamental in \( O \), then \( \varphi_n^{(k)} \) is fundamental in \( O \).

Proof: Since \( \varphi_n \) is fundamental, then there exist \( \hat{\varphi}_n \) and \( t \) such that \( \hat{\varphi}_n(t) = \varphi_n \). Hence,
\[ \varphi^{(k)}_{n} = \left[ \frac{\varphi(t)}{n^j} \right] = \varphi(t+k). \]

Again, let the distribution \( \varphi \) represent the equivalence class of the fundamental sequences \( \varphi_{n} \). Also, let the equivalence class of the fundamental sequences \( \varphi^{(k)}_{n} \) be denoted by \( \varphi^{(k)} \). Then \( \varphi^{(k)} \) is called the k-th order distributional derivative of the distribution \( \varphi \).

**Lemma 6.7.** If \( \varphi \) is a smooth function in \( O \), then it is smooth in every open subset \( O_{1} \) of \( O \).

**Proof:** This result follows from Definition 3.2.

**Lemma 6.8.** Every fundamental sequence in \( O \) is fundamental in every subset \( O_{1} \) of \( O \).

Hence, we can conclude that the restriction of the domain to an open subset of \( O \) is a regular operation. Subsequently, if a distribution is defined in an open set \( O \), then it can be restricted to \( O_{1} \), an open subset of \( O \). Here it is pertinent to note that we do not usually restrict a distribution to a point because this is not a regular operation.

If \( x = (\xi_{1}, \ldots, \xi_{q}) \) and \( y = (\eta_{1}, \ldots, \eta_{q}) \) are points of \( R^{q} \), then the sum \( x + y = (\xi_{1} + \eta_{1}, \ldots, \xi_{q} + \eta_{q}) \) is also a point of \( R^{q} \).
Definition 6.9. The function \( \psi(x) = \varphi(x+\xi_0) \) is called the translation of a function \( \varphi(x) \) by \( \xi_0 \).

Let \( 0_{\xi_0} = \{ x : x + \xi_0 \in 0 \} \). Then, if \( \varphi \) is defined in \( 0 \), \( \psi \) is defined in \( 0_{\xi_0} \).

Lemma 6.10. If \( \varphi(x) \) is a smooth function in an open set \( 0 \), then \( \varphi(x+\xi_0) \) is a smooth function in \( 0_{\xi_0} \).

Proof: This result follows from the definition of smooth functions.

Lemma 6.11. If \( \varphi_n(x) \) is a fundamental sequence in \( 0 \), then \( \varphi_n(x+\xi_0) \) is a fundamental sequence in \( 0_{\xi_0} \).

Proof: Since \( \varphi_n \) is fundamental in \( 0 \), there exist \( \varphi_n \) and \( k \) such that \( \varphi_n(x+\xi_0) = \varphi_n^{(k)}(x+\xi_0) \) where \( x + \xi_0 \in 0 \), and the sequence \( \varphi_n \) converges uniformly in every \( I \subset 0 \).

Thus, translation is a regular operation. Also, it follows that, if \( f(x) \) is a distribution in \( 0 \), then \( f(x+\xi_0) \) is a distribution in \( 0_{\xi_0} \). Here, \( f(x) \) does not mean that distributions have values at points. It is used as a symbolic notation.

Definition 6.12. Two distributions are said to be equivalent at a point \( x \) if and only if there exists some neighborhood of \( x \) such that their restrictions to that
neighborhood are identical.

**Definition 6.13.** The germ of a distribution \( f \) at \( x \) is the equivalence class of all distributions which are equivalent to \( f \) at \( x \). This germ is considered to be a value of \( f \) at \( x \).

Now, if a distribution is defined in an open set \( O \), then it takes on a value at each \( x \in O \), but this value is a germ. In this terminology every distribution is a germ-valued function.

**Lemma 6.14.** If \( \omega \) is smooth in \( O_1 \) and \( \psi \) is smooth in \( O_2 \), then the sum \( \omega + \psi \) is smooth in the intersection \( O = O_1 \cap O_2 \).

**Proof:** This follows from Definition 3.2.

**Lemma 6.15.** If the sequences \( \varphi_n \) and \( \psi_n \) are fundamental in \( O_1 \) and \( O_2 \), respectively, then \( \varphi_n + \psi_n \) is fundamental in \( O = O_1 \cap O_2 \).

**Proof:** Since \( \varphi_n \) is fundamental in \( O_1 \), \( \varphi_n \) is fundamental in \( O \), and there exist \( \varphi_n \) and \( k_1 \) such that \( \varphi_n = \varphi_n \). Similarly, since \( \psi_n \) is fundamental in \( O_2 \), \( \psi_n \) is fundamental in \( O \), and there exist \( \psi_n \) and \( k_2 \) such that \( \psi_n = \psi_n \). Let \( k_1 = (\lambda_1, \ldots, \lambda_q) \), \( k_2 = (\lambda_1, \ldots, \lambda_q) \), \( k = k_1 + k_2 \).
Both $\tau_n$ and $\nu_n$ are uniformly convergent. Hence,
$\tau_n + \nu_n$ is uniformly convergent, and

$$(\tau_n + \nu_n)(k) = \tau_n(k) + \nu_n(k) = \varphi_1(k) + \varphi_2(k) = \varphi + \varphi_n.$$  

Consequently, addition is a regular operation on two functions. Thus, if $f$ and $g$ are distributions defined in $O_1$ and $O_2$, respectively, then the sum $f + g$ is a distribution defined in $O = O_1 \cap O_2$.

**Lemma 6.16.** If $\varphi$ is smooth in $O_1$ and $\psi$ is smooth in $O_2$, then the difference $\varphi - \psi$ is smooth in the intersection $O = O_1 \cap O_2$.

**Proof:** This result follows immediately from the operations of addition and multiplication by a number $-1$.

**Lemma 6.17.** If the sequences $\varphi_n$ and $\psi_n$ are fundamental in $O_1$ and $O_2$, respectively, then $\varphi_n - \psi_n$ is fundamental in $O = O_1 \cap O_2$.
Proof: The proof is similar to that of Lemma 6.15.

Therefore, subtraction is a regular operation on two functions. Consequently, if $f$ and $g$ are distributions defined in $O_1$ and $C_2$, respectively, then the difference $f - g$ is a distribution defined in $O = O_1 \cap O_2$.

**Lemma 6.18.** If $\psi_{1n} \sim \psi_{1n}', \ldots, \psi_{rn} \sim \psi_{rn}'$, then $A(\psi_{1n}', \ldots, \psi_{rn}') \sim A(\psi_{1n}, \ldots, \psi_{rn})$.

**Proof:** $\psi_{1n} \sim \psi_{1n}'; \psi_{11}', \ldots, \psi_{12}', \ldots$ is fundamental. $\psi_{2n} \sim \psi_{2n}'; \psi_{21}', \ldots, \psi_{22}', \ldots$ is fundamental.

$\vdots$

$\psi_{rn} \sim \psi_{rn}'; \psi_{r1}', \ldots, \psi_{r2}', \ldots$ is fundamental. Now, applying a regular operation $A$, we get

\[
A(\psi_{11}', \psi_{21}', \ldots, \psi_{r1}') = \psi_1',
\]
\[
A(\psi_{11}', \psi_{21}', \ldots, \psi_{r1}') = \psi_1',
\]
\[
A(\psi_{12}', \psi_{22}', \ldots, \psi_{r2}') = \psi_2',
\]
\[
A(\psi_{12}', \psi_{22}', \ldots, \psi_{r2}') = \psi_2',
\]
$\vdots$

\[
A(\psi_{1n}', \psi_{2n}', \ldots, \psi_{rn}') = \psi_n
\]
\[
A(\psi_{1n}', \psi_{2n}', \ldots, \psi_{rn}') = \psi_n
\]
Thus, \( A \) applied to a finite number of functions results in another function. We now have a new sequence:

\[ c_1', \psi_1', c_2', \psi_2', \ldots, c_n', \psi_n'. \]

Since \( A \) is regular, this resulting sequence is fundamental. Hence,

\[ A(c_1'n', \ldots, c_nn') \sim A(\psi_1'n', \ldots, \psi nn'). \]

This lemma shows that the operation \( A \) is well defined.

Now, if multiplication is considered, we find that it is not a regular operation. In other words, there exist two fundamental sequences \( c_n \) and \( \psi_n \) such that the product \( c_n \psi_n \) is not a fundamental sequence.

**Lemma 6.19.** If \( c_1n' \sim \psi_1n', \ldots, c_nn' \sim \psi nn' \) holds for some fundamental sequences \( c_1n', \psi_1n', \ldots, c_nn', \psi nn' \), but \( A(c_1n', \ldots, c_nn') \sim A(\psi_1n', \ldots, \psi nn') \) does not hold, then operation \( A \) is not regular.

**Proof:** This result follows directly from Lemma 6.18.

Now, let \( A(\varphi_1, \varphi_2) = \varphi_1 \varphi_2 \). Define \( \varphi_1n = \varphi_2n = 0 \) and \( \varphi_1n = \varphi_2n = \cos nx \). Then, \( \varphi_1n \sim \varphi_1n \) and \( \varphi_2n \sim \varphi_2n' \)

since \( 0 \sim \cos nx \). Thus, \( \varphi_1n \varphi_2n = \frac{1}{2} + \frac{1}{2} \cos 2nx \sim \frac{1}{2} \).

Since \( \varphi_1n \varphi_2n = 0 \), \( \varphi_1n \varphi_2n \sim \varphi_1n \varphi_2n \) does not hold. Therefore, multiplication is not a regular operation.
Multiplication, however, can be considered as an operation on one function if the second factor is held fixed. Let $x$ be the fixed factor.

Lemma 6.20. If $x$ is a smooth function in an open set $O_1$, and $\psi_n$ is a fundamental sequence in another open set $O_2$, then the sequence $x \psi_n$ is fundamental in the intersection $O = O_1 \cap O_2$.

Proof: Since $\psi_n$ is fundamental in $O_2$ and $0 \subset O_2$, there exists, for any given interval $I \subset O$, a uniformly convergent sequence of smooth functions $\psi[k]_n$ such that $\psi[k]_n = \psi_n$, for some $k$.

Now, for every order $m$ and every smooth function $x$ in $O$, the sequence $x \psi^{(m)}_n$ is fundamental in $I$. This can be shown by induction. For if $m = 0$, then $x \psi^{(0)}_n = x \psi_n$ is fundamental in $I$. Now, assume that the sequence is fundamental in $I$ for some $m$. It is fundamental for $m + e_j$, since

$$x^{(m+e_j)}_n = x^{(m)}_n \psi^e_j - x^e_j \psi^{(m)}_n.$$

But, by Theorem 3.6 and the hypothesis, the right side of the equation is a difference of two sequences which are fundamental. In particular, if $m = k$, $x \psi_n$ is fundamental in $I$. Since the interval $I$ is arbitrary in $O$, the sequence $x \psi_n$ is fundamental in $O$. 

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Thus, multiplication of a fundamental sequence by a smooth function is a regular operation. Subsequently, if \( \omega \) is a smooth function in \( O_1 \), and \( f \) is a distribution in \( O_2 \), then the product \( \omega f \) is well defined and is a distribution in \( O = O_1 \cap O_2 \). If \( \omega \) were a constant function, then it reduces to the case considered in Lemma 6.4.

Now, let the product \( \varphi(x) \psi(y) \) represent the product of two functions \( \varphi(x_1, \ldots, x_q) \) and \( \psi(y_1, \ldots, y_r) \) with separated variables.

**Lemma 6.21.** If \( \varphi \) is a smooth function in an open set \( O_1 \subset \mathbb{R}^q \), and \( \psi \) is a smooth function in an open set \( O_2 \subset \mathbb{R}^r \), then the product \( \varphi(x) \psi(y) \) is a smooth function in the Cartesian product \( O = O_1 \times O_2 \subset \mathbb{R}^{q+r} \).

**Proof:** Let \( 1 \leq i \leq q \) and \( 1 \leq j \leq r \). Hence

\[
[\varphi(x) \psi(y)]^{(e_i)} = [\varphi(x)]^{(e_i)} \psi(y),
\]

and

\[
[\varphi(x) \psi(y)]^{(e_j)} = \varphi(x) [\psi(y)]^{(e_j)}.
\]

Thus,

\[
[\varphi(x) \psi(y)]^{(e_i+e_j)} = \left[ \left[ \varphi(x)^{(e_i)} \psi(y) \right]^{(e_j)} \right]^{(e_i)} = \left[ \varphi(x) \right]^{(e_i)} \left[ \psi(y) \right]^{(e_j)}.
\]
Lemma 6.22. If \( \varphi \) is a smooth function in \( O_1 \subseteq \mathbb{R}^q \), and 
\( \psi \) is a smooth function in \( O_2 \subseteq \mathbb{R}^r \), then the multiplica-
tion \( \varphi(x)\psi(y) \) in \( O = O_1 \times O_2 \) is a regular operation.

Proof: Assume that \( \varphi_n \) and \( \psi_n \) are fundamental sequences in \( O_1 \) and \( O_2 \), respectively. Thus, there exist smooth functions \( \hat{\varphi}_n, \hat{\psi}_n \), such that \( \hat{\varphi}_n(k) = \varphi_n \) and \( \hat{\psi}_n(j) = \psi_n \). Note that \( k = (\lambda_1, \ldots, \lambda_q) \) and \( j = (\lambda_1, \ldots, \lambda_r) \). Now,

\[
\varphi_n(x)\psi_n(y) = \hat{\varphi}_n(k)(x)\hat{\psi}_n(j)(y)
\]

\[
= \left[ \hat{\varphi}_n(x)\hat{\psi}_n(y) \right]^{(k,j)}
\]

\[
= \left[ \hat{\varphi}_n(x)\hat{\psi}_n(y) \right]^{(\lambda_1, \ldots, \lambda_q, \lambda_1, \ldots, \lambda_r)}
\]

Hence, there exist smooth functions \( \hat{\varphi}_n \hat{\psi}_n \) such that

\[
\varphi_n\psi_n = (\hat{\varphi}_n \hat{\psi}_n)^{(m)}, \text{ where } m = (k,j) = (\lambda_1, \ldots, \lambda_q, \lambda_1, \ldots, \lambda_r).
\]

Thus, \( \varphi_n \psi_n \) is a fundamental sequence.

Consequently, if \( f \) and \( g \) are distributions defined in open sets \( O_1 \) and \( O_2 \), respectively, then the 
product \( f(x)g(y) \) is a distribution in \( O = O_1 \times O_2 \).  

7. CONVOLUTION WITH A SMOOTH FUNCTION 
of BOUNDED CARRIER

Definition 7.1. Let \( f \) be a continuous or locally in-
tegrable function in an open set \( O \), and let \( \omega \) be a 
smooth function vanishing everywhere outside a bounded
open set $B$ and $w(x) \neq 0$ for $x \in B$. Then $B$ is called the carrier of $w$.

**Definition 7.2.** The convolution $f * w$ is defined as the integral

$$\int_B f(x-t)w(t)dt$$

$$= \int_B f(\xi_1-\tau_1, \ldots, \xi_q-\tau_q)x(\tau_1, \ldots, -\tau_q)d\tau_1 \ldots d\tau_q,$$

where $x = (\xi_1, \ldots, \xi_q)$ and $t = (\tau_1, \ldots, -\tau_q)$. The integral is defined in the open set

$$0_B = \{x \in \Omega: (x-t \in \Omega, \forall t \in B) \subset \Omega\}.$$

The integral is extended over $\mathbb{R}^q$. If we assume that the integrand $f(x-t)w(t)$ is equal to zero whenever one of its factors is zero, even if the second factor is undetermined, the extension has meaning. We can write

$$\int f(x-t)w(t)dt$$

instead of $\int_B f(x-t)w(t)dt$, whenever the integral is extended over $\mathbb{R}^q$. We first show convolution is commutative.

**Lemma 7.3.** $f * w = w * f$.

**Proof:**

$$f * w = \int f(x-t)w(t)dt, \quad x, t \in \mathbb{R}^q$$

$$= \int f(u)w(x-u)du, \quad (\text{where})$$

$$u = x-t$$
= \nu \ast f(x).

The following lemma is a direct consequence of the definition:

**Lemma 7.4.** The convolution \( f \ast \nu \) is a smooth function in \( \Omega_B \).

**Definition 7.5.** If \( f \) has a continuous partial derivative of order \( m \), then \( f \) is said to be of class \( C^m \).

**Lemma 7.6.** If \( \nu \) is of class \( C^m \), then \( (f \ast \nu)(m) = f \ast \nu(m) \), for every order \( m \).

**Proof:** Let \( x, t \in \mathbb{R}^d \). Now,

\[
(f \ast \nu)(m)(x) = \int f(t) \nu(x-t) dt \bigg|_{(m)}
\]

\[
= \int f(t) \nu(x-t)(m) dt
\]

\[
= \int f(t) \nu(m)(x-t) dt
\]

\[
= [f \ast \nu(m)](x).
\]

In the same way, we can prove the following:

**Lemma 7.7.** If \( f \) is of class \( C^m \), then \( (f \ast \nu)(m) = f(m) \ast \nu \), for every order \( m \).

We now show that the convolution is associative.
Lemma 7.8. If \( \omega_1 \) and \( \omega_2 \) are smooth and have bounded carrier \( B_1 \) and \( B_2 \), respectively, then

\[ (f \ast \omega_1) \ast \omega_2 = f \ast (\omega_1 \ast \omega_2) \]

in \( (O_{B_1})_{B_2} \).

**Proof:** Let \( x, t \in \mathbb{R}^q \).

\[
(f \ast \omega_1) \ast \omega_2)(x) = \int (f \ast \omega_1)(x-t) \omega_2(t) dt
\]

\[
= \int \int f(x-t-s) \omega_1(s) \omega_2(t) ds dt
\]

\[
= \int \int f(x-t-s) \omega_1(s) \omega_2(t) ds dt
\]

\[
= \int \int f(x-u) \omega_1(u-t) \omega_2(t) du dt,
\]

(where \( u = t + s \) for fixed \( t \))

\[
= \int f(x-u) \int \omega_1(u-t) \omega_2(t) dt du
\]

\[
= \int f(x-u) (\omega_1 \ast \omega_2)(u) du
\]

\[
= [f \ast (\omega_1 \ast \omega_2)](x).
\]

In what follows we show that the convolution of a continuous or locally integrable function \( f \) with a fixed smooth function \( \omega \) is a regular operation, and it can be extended to a distribution \( f \).
Theorem 7.9. If \( \psi_n \) is a fundamental sequence in \( O \), and \( w \) is a smooth function of bounded carrier \( B \), then the sequence \( \psi_n \ast w \) converges almost uniformly in \( O_B \). Therefore, it is a fundamental sequence in \( O_B \).

Proof: Let \( I \) be an interval inside \( O_B \). Then there is a uniformly convergent sequence of smooth functions \( \psi_n \) such that \( \psi_n^{(k)} = \psi_n \) in \( I \) for some \( k \). Then,

\[
\psi_n \ast w = \psi_n^{(k)} \ast w = \psi_n \ast w^{(k)} = \int \psi_n(x-t)w^{(k)}(t)dt.
\]

Now, the sequence \( \psi_n(x-t) \) converges uniformly for \( x \in I \) and \( t \in B \). Since \( w^{(k)} \) is a smooth function of carrier \( B \), this implies that the integral converges uniformly for \( x \in I \). Therefore, \( \psi_n \ast w \) is a fundamental sequence in \( O_B \).

Theorem 7.10. If \( f \) is a distribution in \( O \), and \( w \) is a smooth function of bounded carrier \( B \), then \( f \ast w \) is a smooth function in \( O_B \).

Proof: Let \( \psi_n \) be a fundamental sequence for \( f \). By Theorem 7.9, \( f_k = \psi_n \ast w^{(k)} \) converges almost uniformly in \( O_B \). Since \( \psi_n \ast w^{(k)} = (\psi_n \ast w)^{(k)} \), then \( f_k \) is a smooth function in \( O_B \). But, the sequence \( \psi_n \ast w \) is equivalent to the constant sequence \( \psi_n = f = \lim_{n \to \infty} \psi_n \ast w \).

Since, by definition, the sequence \( \psi_n \ast w \) is fundamental for \( f \ast w \), it follows that \( f \ast w \) is smooth in \( O_B \).
3. CALCULATIONS WITH DISTRIBUTIONS

The following identities used in calculations with functions can be easily extended to distributions:

\[ \varphi_1 + \varphi_2 = \varphi_2 + \varphi_1 \]
\[ (\varphi_1 + \varphi_2) + \varphi_3 = \varphi_1 + (\varphi_2 + \varphi_3) \]
\[ (\varphi_1 - \varphi_2) + \varphi_2 = \varphi_1 \]
\[ \lambda \varphi = \varphi \]
\[ (\lambda_1 \lambda_2) \varphi = \lambda_1 (\lambda_2 \varphi) \]
\[ (\lambda_1 + \lambda_2) \varphi = \lambda_1 \varphi + \lambda_2 \varphi \]
\[ \lambda (\varphi_1 + \varphi_2) = \lambda \varphi_1 + \lambda \varphi_2 \]
\[ (\varphi_1 + \varphi_2)(m) = \varphi_1(m) + \varphi_2(m) \]
\[ (\varphi \ast \varphi)(m) = \varphi(m) \ast \varphi = \varphi \ast \varphi(m) \]
\[ (\varphi \varphi_j) = \varphi \varphi_j + \varphi \varphi_j \]

It is not necessary to give direct proofs for these formulas in the case of distributions. All the operations involved here are regular operations and a finite number of regular operations is again a regular operation. Hence, instead of proving any of these identities, we can give a general rule:
If the members of the equality consist of a finite number of regular operations, and the equality holds for smooth functions, then it also holds for distributions.

We recall that the product of a distribution by a smooth function $\omega$ is a regular operation, but not the product of two distributions. Hence, in the above identities the smooth functions $\omega$ cannot be replaced by distributions, and, thus, they play the role of parameters when the functions $\omega$ are replaced by distributions.

9. DELTA SEQUENCES AND THE DIRAC DELTA DISTRIBUTION

In this section we define the Dirac delta distribution. Suppose $r$ is a real-valued nonnegative smooth function in $\mathbb{R}^d$ such that $r(x) = 0$ for $|x| > 1$ and $\int r = 1$. Let $\alpha_n$ be a null sequence of positive numbers.

We define a sequence $\delta_n(x)$ by

$$\delta_n(x) = \alpha_n^{-q} r \left( \frac{x}{\alpha_n} \right).$$

We show that this sequence has the following properties:

i.) $\delta_n(x) = 0$ for $|x| > \alpha_n$,

ii.) $\int \delta_n = 1$. 
iii.) \( \frac{k}{n} \left| \delta_n^{(k)} \right| \leq M_k \), where \( M_k \) is independent of \( n \).

To show i.) we consider \( |x| > \alpha_n \). Then

\[
\frac{|x|}{\alpha_n} > 1 = \frac{|x|}{|\alpha_n|} > 1
\]

\[
= r\left( \frac{x}{\alpha_n} \right) = 0.
\]

Thus, \( \delta_n(x) = 0 \) for \( |x| > \alpha_n \).

For ii.), we have

\[
\int \delta_n(x) \, dx = \int \alpha_n^{-q} r\left( \frac{x}{\alpha_n} \right) \, dx
\]

\[
= \alpha_n^{-q} \int \cdots \int r\left( \frac{\xi_1}{\alpha_n}, \cdots, \frac{\xi_q}{\alpha_n} \right) d\xi_1 \cdots d\xi_q
\]

\[
= \alpha_n^{-q} \int \cdots \int r(\tau_1, \ldots, \tau_q) \alpha_n^q d\tau_1 \cdots d\tau_q,
\]

(\text{where } \tau_i = \frac{\xi_i}{\alpha_n} \text{, } i = 1, \ldots, q)

\[
= \alpha_n^{-q} \int \cdots \int r(\tau_1, \ldots, \tau_q) (\alpha_n^q) d\tau_1 \cdots d\tau_q
\]

\[
= \int \cdots \int r(\tau_1, \ldots, \tau_q) d\tau_1 \cdots d\tau_q
\]

\[
= 1.
\]

To show iii.), we have
\[ a_n \int_{\mathcal{D}_n} |\delta_n(x)| \, dx = \alpha_n^{-q} \int_{\mathcal{D}_1} \frac{\partial^k}{\partial t^k} \delta \, dt \, \tau_1^{k_1} \cdots \tau_q^{k_q} \]

\[ = \alpha_n^{-q} \int_{\mathcal{D}_1} \frac{\partial^k}{\partial t^k} \delta \, dt \, \tau_1^{k_1} \cdots \tau_q^{k_q}, \]

(where \( \tau_i = \frac{\xi_i}{\alpha_n} \), \( i = 1, \ldots, q \))

\[ = \alpha_n^{-q} r(t) \]

\[ = \alpha_n^{-q} r \left( \frac{x}{\alpha_n} \right) \]

\[ \leq M_k, \text{ where } M_k \text{ does not depend on } n \]

since \( r \) is bounded for each \( k \).

We now define delta sequence.

**Definition 9.1.** Each sequence of real-valued nonnegative functions possessing the three above properties is called a delta sequence.

The following lemmas lead to the definition of the Dirac delta distribution.

**Lemma 9.2.** Every delta sequence is a fundamental sequence.
Proof: Let \( x = (x_1, \ldots, x_q), \ t = (\tau_1, \ldots, \tau_q), \ s = (s_1, \ldots, s_q) \in \mathbb{R}^q \). Define

\[
\Delta_n(x) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{t} \xi_n(s) \, ds \right) dt.
\]

Then,

\[
\Delta'_n(x) = \int_{-\infty}^{x} \xi_n(t) \, dt, \quad \text{where} \quad \xi'_n = \frac{\partial \xi_n}{\partial x_1 \cdots \partial x_q}.
\]

and \( \Delta''_n(x) = \xi_n(x) \). Assume that \( n > m \), and let \( a_m = (a_{m1}, \ldots, a_{mq}) \). Now

\[
|\Delta_n - \Delta_m| = \left| \int_{-\infty}^{x} \left[ \int_{-\infty}^{t} (\xi_n - \xi_m)(s) \, ds \right] dt \right|,
\]

(\text{where } q = (1, \ldots, l))

\[
\leq \int_{-\infty}^{x} \left[ \int_{-\infty}^{t} \xi_n(s) \, ds + \int_{-\infty}^{t} \xi_m(s) \, ds \right] dt \leq 2 q \int_{-\infty}^{x} dt \leq 2 q \alpha_{mi}, \quad \text{(where } \alpha_{mi} = \max\{a_{m1}, \ldots, a_{mq}\})
\]

\[< \varepsilon \text{ for } m \text{ sufficiently large.} \]
Hence \( z_n \in \mathbb{R}^q \).

Lemma 9.3. If \( z_{1n} \) and \( z_{2n} \) are delta sequences, then the interlaced sequence

\[
\delta_{11}, \delta_{21}, \delta_{12}, \delta_{22}, \ldots,
\]

is a delta sequence.

Proof: Let \( z_{1n} \) and \( z_{2n} \) be delta sequences. Thus,

1. \( z_{1n} = 0 \) for \( |x| > \alpha_{1n} \),
   \( z_{2n} = 0 \) for \( |x| > \alpha_{2n} \),

2. \( z_{1n} = 1 \),

3. \( z_{2n} = 1 \)

\( \alpha_{1n} \) is independent of \( n \), where \( M_{k_1} \) is independent of \( n \),

\( \alpha_{2n} \) is independent of \( n \), where \( M_{k_2} \) is independent of \( n \).
Consider $\delta_k = \{\delta_{11}', \delta_{21}', \delta_{12}', \delta_{22}', \ldots\}$. Now, we need to show that $\delta_k$ satisfies the three properties.

i.) $\delta_k = 0$ for $|x| > a = \max \{a_{1n}, a_{2n}\}$

ii.) $\int \delta_k = 1$, (since $\int \delta_{1n} = \int \delta_{2n}$).

iii.) $\alpha_k \int |\delta^{(m)}| \leq M_m = \max \{M_{k_1}, M_{k_2}\}$.

By this lemma, all delta sequences belong to the same equivalence class and consequently determine a distribution. We denote this distribution by $\delta$ and refer to it as the Dirac delta distribution. It should be noted that the class of all delta sequences is a subclass of all the fundamental sequences belonging to $\delta_k$.

To illustrate this point consider an earlier example.

Let $\psi_n = \frac{n}{(1+e^{nx})(1+e^{-nx})}$, and let $\delta_n$ be a delta sequence. $\psi_n$ is fundamental by the earlier observation, and $\delta_n$ is fundamental by Lemma 9.2. Now, consider the interlaced sequence $\varphi_n \{\psi_{1}, \delta_{1}, \psi_{2}, \delta_{2}, \ldots\}$. From the previous work $\varphi_n = \psi_n = \psi^{(k)}$, where

$\psi_n = \int_{-\infty}^{x} \frac{dx}{1+e^{-nx}}$ with $k = 2$,

and
\[ \gamma_n : L = \begin{cases} 
0, & x \leq 0 \\
, & x > 0 
\end{cases} \]

considering the odd entries in the sequence. Thus, in the

even entries \( \varphi_n \), let \( \varphi_n = \delta_n = \delta_n^{(i)} \), where \( \delta_n = \delta_n^i \)

with \( i = 0 \), since \( \delta_n \downarrow 0 \). By Theorem 4.2, for

2 - k > i = 0, there exists a uniformly convergent se-

quence of smooth functions \( \delta_n^i \) such that \( \delta_n^i(k) = \varphi_n \) in I. Therefore, there exist \( \delta_n^i \) and \( \gamma_n \) such that

\[ \varphi_n = \delta_n^i(k) = \gamma_n(k) \]

where \( \delta_n^i \uparrow \gamma_n \), and it follows

that \( \gamma_n \sim \delta_n \).

We now prove the following:

Lemma 9.4. If \( \delta_{1n} \) and \( \delta_{2n} \) are delta sequences, then

\( \delta_{3n} = \delta_{1n} \ast \delta_{2n} \) is a delta sequence.

Proof: Observe that

\[ \delta_{3n}(x) = (\delta_{1n} \ast \delta_{2n})(x) = \int \delta_{1n}(x-t) \delta_{2n}(t) dt. \]

Now verify the three properties.

i.) \( \delta_{3n}(x) = 0 \) for \( |x| > \alpha_{3n} = \min\{\alpha_{1n}, \alpha_{2n}\} \).

ii.) \[ \int \delta_{3n}(x) dx = \int \int \delta_{1n}(x-t) \delta_{2n}(t) dx dt \]

\[ = \int_{\delta_{2n}} \int \delta_{1n}(x-t) dx dt \]
ii.) \( \int \delta_{2n}(t) dt \), (since \( \int \delta_{1n} = 1 \))
\( = 1 \), (since \( \int \delta_{2n} = 1 \)).

iii.) \( \frac{k_3}{\alpha_{3n}} \int \left| \hat{\delta}_{3n}(x) \right| dx \)
\( = \frac{k_3}{\alpha_{3n}} \int \left| \left( \hat{\delta}_{1n} \ast \hat{\delta}_{2n} \right)^{\left(k_1 + k_2\right)} \right| dx, \)

(where \( k_3 = k_1 + k_2 \))
\( = \frac{k_3}{\alpha_{3n}} \int \left| \left( \hat{\delta}_{1n} \ast \hat{\delta}_{2n} \right)^{\left(k_1\right)} \right| \left( \hat{\delta}_{2n}^{\left(k_2\right)} \right) dx \)
\( = \frac{k_3}{\alpha_{3n}} \int \left| \delta_{1n} (x-t) \right| \left| \delta_{2n}^{\left(k_2\right)} (t) \right| dt dx \)
\( = \frac{k_1 k_2}{\alpha_{1n} \alpha_{2n}} \int \int \left| \delta_{1n} (x-t) \right| \left| \delta_{2n}^{\left(k_2\right)} (t) \right| dx dt, \)

(where \( \frac{k_3}{\alpha_{3n}} = \frac{k_1 k_2}{\alpha_{1n} \alpha_{2n}} \))
\( = \frac{k_2}{\alpha_{2n}} \int \left| \delta_{2n}^{\left(k_2\right)} (t) \right| dt, \left( \int \left| \alpha_{1n} \int \left| \delta_{1n} (x-t) \right| dx \right| \right) \)
\( \leq M_k \frac{k_2}{\alpha_{2n}} \int \left| \delta_{2n}^{\left(k_2\right)} (t) \right| dt, \) (since
\( \frac{k_1}{\alpha_{1n}} \int \left| \delta_{1n} \right| \leq M_k, \) where \( M_k \)

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does not depend on \( n \))

\[
\leq M_{k_1 M_{k_2}}, \quad \text{since } \sum_{2n}^{k_2} \left| \delta_{2n}^{(k_2)} \right| \leq M_{k_2},
\]

where \( M_{k_2} \) does not depend on \( n \).

Hence, let \( M_{k_3} = M_{k_1 M_{k_2}} \).

**Lemma 9.5.** Let \( \delta_n \) be a delta sequence, and let \( B_n \) be carriers of \( \delta_n \). If \( f \) is a continuous function in an open set \( 0 \subseteq \mathbb{R}^q \), then the sequence \( f_n = f * \delta_n \) consists of smooth functions in open subsets \( O_{B_n} \subseteq 0 \) such that \( O_{B_n} \to 0 \).

**Proof:** Let \( O_{B_n} = \{ x \in 0 : \overbar{x-t} \subseteq 0 \ \text{for all} \ t \in B_n \} \), and let \( B_n = (-\alpha_n, \alpha_n) \). Thus, \( \lim_{t \to 0} O_{B_n} = 0 \).

**Definition 9.6.** The sequence \( f_n \) in Lemma 9.5 is called a regular sequence for \( f \).

**Theorem 9.7.** If \( f \) is a continuous function in an open set \( 0 \), then its regular sequence \( f * \delta_n \) converges to \( f \) almost uniformly in \( 0 \).

**Proof:** Let \( I \) be an interval inside \( 0 \). For every \( \epsilon > 0 \), there is an index \( n_\epsilon \) such that for \( n > n_\epsilon \),

\[
|f(x-t) - f(x)| < \epsilon \quad \text{for } x \in I \text{ and } |t| < \alpha_n.
\]
Subsequently,

\[
| (f \ast \delta_n)(x) - f(x) | = \left| \int f(x-t) \delta_n(t) dt - \int f(x) \delta_n(t) dt \right|
\]

(since \( \int \delta_n(t) dt = 1 \))

\[
\leq \int |f(x-t) - f(x)| \delta_n(t) dt
\]

\[
\leq \epsilon M_0 \quad \text{for } x \in I \text{ and } n > n_0.
\]

**Corollary 9.8.** If \( f \) is a continuous function in an open set \( O \), and, for every smooth function \( u \) of bounded carrier \( B \), \( f \ast u = 0 \) in \( O_B \), then \( f = 0 \) in \( O \).

**Proof:** If \( \delta_n \) is a delta sequence with carriers \( B_n \), then \( f \ast \delta_n = f \ast u = 0 \) in \( O_B \). Also, \( f \ast \delta_n \) is a regular sequence. Thus, by Theorem 9.7, \( f \ast \delta_n \) converges to \( f \) almost uniformly in \( O \). Therefore, \( f = 0 \) in \( O \).

**Theorem 9.9.** If \( f \) is a locally integrable function in an open set \( O \), then its regular sequence \( f \ast \delta_n \) converges to \( f \) locally in \( O \). In other words, given any interval \( I \) inside \( O \),

\[
\int_I |f \ast \delta_n - f| \to 0.
\]
Proof: Let $I$ be any given interval inside $O$. By the Lebesgue Dominated Convergence Theorem, \( \int_I |f(x-t) - f(x)| \, dx \to 0 \) as $t \to 0$. Thus, if $|t| < \alpha_n$, and $n$ is large enough, say $n > n_0$, then \( \int_I |f(x-t) - f(x)| \, dx < \epsilon_n' \) with $\epsilon_n' \to 0$. For $x \in I$ and $n > n_0$, we have

\[
|f \ast \delta_n - f| = \left| \int f(x-t) \delta_n(t) \, dt - \int f(x) \delta_n(t) \, dt \right|
\]

\[
\leq \int \left| f(x-t) - f(x) \right| \delta_n(t) \, dt.
\]

Hence,

\[
\int_I |f \ast \delta_n - f| \leq \int_I \left| f(x-t) - f(x) \right| \delta_n(t) \, dt \, dx
\]

\[
= \int_I \left| f(x-t) - f(x) \right| \delta_n(t) \, dt \, dx \n\)

\[
= \int \left| \delta_n(t) \right| \int_I \left| f(x-t) - f(x) \right| \, dx \, dt
\]

\[
\leq M \int \left| \delta_n(t) \right| dt \int_I \left| f(x-t) - f(x) \right| \, dx
\]

\[
\leq M_0 \epsilon_n.
\]

Corollary 9.10. If $f$ is a locally integrable function in an open set $O$, and, for every smooth function of bounded carrier $B$, $f \ast w = 0$ in $O_B$, then $f = 0$ almost everywhere in $O$. 

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**Proof:** If $\delta_n$ is a delta sequence with carriers $B_n$, then $f * \delta_n = f * \omega = 0$ in $O_B$. By Theorem 9.9, $f * \delta_n$, the regular sequence of $f$, converges to $f$ locally in norm in $O$. Since the limit of a locally in norm convergent sequence is determined up to a null set, it follows that $f = 0$ almost everywhere in $O$.

Corollary 9.8 is an immediate consequence of Corollary 9.10.

10. PROPERTIES OF THE DIRAC DELTA DISTRIBUTION

In this last section we state, for the most part without proofs, several properties of the Dirac delta distribution.

**Property 10.1.** If $\omega$ is a smooth function, then $\omega \delta$ is a distribution.

**Property 10.2.** If $\omega$ is a smooth function, then $\omega(x) \delta_n(x-x_0) \sim \omega(x_0) \delta_n(x-x_0)$.

**Property 10.3.** If $\omega$ is a smooth function, then $\omega \star \delta_n$ is a fundamental sequence for $\omega \star \delta$.

**Property 10.4.** If $\omega$ is a smooth function, then $\omega \star \delta_n$ is a fundamental sequence for $\omega$. 

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Property 10.5. If \( x \) is a smooth function, then
\[
6(x) \delta(x-x_0) = x(x_0) \delta(x-x_0).
\]

Property 10.5 follows from Property 10.2, and the following Property 10.6 follows from Properties 10.3 and 10.4.

Property 10.6. If \( x \) is a smooth function, then
\[
x \ast \delta = x.
\]

Property 10.7. \( \delta \) is not a locally integrable function.

It is easily established that \( \delta(x) = \delta(x_1) \cdots \delta(x_q) \) for \( x = (x_1, \ldots, x_q) \in \mathbb{R}^q \) by considering \( \delta(x) \) as a product of separated variables, which is a regular operation, and then it is sufficient to show that, if \( \delta_n \) is a delta sequence in \( \mathbb{R} \), then \( \delta_n(x_1) \cdots \delta_n(x_q) \) is a delta sequence in \( \mathbb{R}^q \). Consequently, all cases concerning the delta distribution can be reduced to the one-dimensional case. We let \( \xi \in \mathbb{R} \) and define the Heaviside function \( H \):

\[
H(\xi) = \begin{cases} 
0 & \text{for } \xi < 0 \\
1 & \text{for } \xi \geq 0
\end{cases}
\]

We now state and prove the following property:
Property 10.8. Let $\delta_n$ be a delta sequence. The sequence

$$\tau_n(\xi) = \int_{-\infty}^{\xi} \delta_n$$

converges locally to $H(\xi)$.

Proof: For the proof we first note that $\tau_n(\xi) = 0$ if $\xi < \alpha_n$, $\tau_n(\xi) = 1$ if $\xi > \alpha_n$, and $\tau_n(\xi) \leq 1$ if $-\alpha_n \leq \xi \leq \alpha_n$. Then,

$$\int_I |\tau_n(\xi) - H(\xi)| d\xi \leq \alpha_n$$

for all $\xi$.

The following is an important property of the Dirac delta distribution:

Property 10.9. $H'(\xi) = \delta(\xi)$, where $H'$ denotes the distributional derivative of $H$.

To prove this we let $\delta_n$ be a delta sequence. By the preceding property, $\tau_n(\xi) = \int_{-\infty}^{\xi} \delta_n$ converges locally to $H(\xi)$. Then by Theorem 5.6 $\tau_n$ is a fundamental sequence for $H$. Thus, $\tau_n' = \delta_n$ is a fundamental sequence for $H'$. 

Property 10.10. If $\omega$ is a smooth function of a real variable $\xi$, then

$$(\omega f)' = \omega f' + \omega f'$$
for every distribution $f$ of $\xi$.

We point out that all the operations involved above are regular.

Now, since $H$ is a locally integrable function from the above property with $f = H$, we obtain, by induction, the property:

**Property 10.11.** \[ \left[ \frac{\partial^\mu}{\partial \xi^\mu} H(\xi) \right] = H(\xi) \] for positive integers.

This, in turn, leads directly to the following property:

**Property 10.12.** \[ \left[ \frac{\partial^{m+1}}{\partial \xi^m} H(\xi) \right] = \xi(\xi) \] for positive integers.

We restate the above property for the case of $q$-dimensional space.

**Property 10.13.** Let $x = (\xi_1, ..., \xi_q)$, $m = (m_1, ..., m_q)$, $x^m = \xi_1^{m_1} \cdots \xi_q^{m_q}$, $m! = m_1! \cdots m_q!$, and $H(x) = H(\xi_1) \cdots H(\xi_q)$. Then

\[ \left[ \frac{\partial^m}{\partial \xi^m} H(x) \right]^{(m+1)} = \xi(x) \]

for positive integers $\xi_i$, $i = 1, ..., q$, where $m + 1$ denotes the vector whose all coordinates are by 1 greater than the corresponding coordinates of $m$. 

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In conclusion, we have shown that every distribution is differentiable in our sequential approach to the theory of distributions. There is a more complicated definition of distribution than the one presented here. It was formulated by Schwartz and based on much functional analysis and topology. Essentially he defined a distribution as a continuous linear functional on a function space. The two definitions appear very different, yet it is possible to prove that they are indeed equivalent.
I. BOOKS

