4-1971

The $*_s$-Product of Arithmetic Functions

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by

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A Project Report
Submitted to the
Faculty of The Graduate College
In partial fulfillment
of the
Specialist in Arts Degree

Western Michigan University
Kalamazoo, Michigan
April, 1971

ACKNOWLEDGEMENTS

I am extremely grateful to Professors Anthony Gioia and Erik Schreiner for their patience and guidance in helping me prepare my thesis. I am also very grateful to the Department of Mathematics of Western Michigan University, not only for the financial support which enabled me to obtain my degree, but also for the personal encouragement on the part of numerous faculty members.

Kathryn D. Kopec

MASTERS THESIS M-2759

KOPEC, Kathryn Diane, 1946-
THE $*_s$-PRODUCT OF ARITHMETIC FUNCTIONS.

Western Michigan University, Sp.A., 1971
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

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INTRODUCTION

An arithmetic function is a function whose domain is the set of positive integers Z^+ and whose range is a subset of the complex numbers C. The set of all arithmetic functions shall be denoted by A. If f and g are arithmetic functions, the Dirichlet product of f and g, denoted by f o g, is defined by

$$(f \circ g)(n) = \sum_{d|n} f(n/d)g(d),$$

where the summation extends over all positive divisors d of n.

The Dirichlet product is the most widely known binary operation on A. It has proven to be a valuable tool in the study of arithmetic functions. As a generalization of the Dirichlet product, we introduce the $*_s$-product of arithmetic functions f and g, defined by

$$(0.1) \qquad (f *_s g)(n) = \sum_{d|(n,s)} f(n/d)g(d).$$

Here, (n,s) denotes the greatest common divisor of n and s, which for n and s not both zero is defined to be the largest positive integer dividing both n and s. In (0.1), s is some arbitrary but fixed non-negative integer, and the summation extends over all positive divisors d of (n,s). We note that since $(n,0) = n$ for any positive integer n, $(f *_0 g)(n) = (f \circ g)(n)$, so that the $*_s$-product is indeed a generalization of the Dirichlet product.

For another specific example, consider the following two special arithmetic functions. For each $r \in C$, the iota function i_r is defined by

(0.2) $$i_r(n) = n^r.$$

The Mobius function μ is defined by

(0.3) $$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^k & n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

It is known [1, p. 44] that Ramanujan's trigonometric sum, defined by

$$c_s(n) = \sum_{\substack{(m,s)=1 \\ 0 \le m < s}} \cos(2\pi ms/n) \qquad [1, p. 39]$$

can be expressed as

$$c_s(n) = \sum_{d|(n,s)} \mu(n/d)\cdot d.$$

That is, $c_s(n) = (\mu *_s i_1)(n)$. The $*_s$-product therefore encompasses more than just a generalization of the Dirichlet product.

In this paper, we shall first examine the algebraic properties of the system $(A, *_s)$. Second, we shall look at series of the type $\sum_{n=1}^{\infty} \frac{(f *_s g)(n)}{n^t}$, with the major result (Theorem 5) being a generalization of a noted theorem for series involving the Dirichlet product.

THE SET OF ARITHMETIC FUNCTIONS AND THE BINARY OPERATION $*_s$

From the definition of the $*_s$-product, whenever f and g are arithmetic functions and n is a positive integer, $(f *_s g)(n)$ is a complex number, and so $f *_s g$ is also an arithmetic function. Consequently, the $*_s$-product is a binary operation on A.

<u>THEOREM</u> <u>1</u>: (Associativity) $f *_s (g *_s h) = (f *_s g) *_s h$ for all $f, g, h \in A$ if and only if $s = 0$ or $s = 1$.

<u>PROOF</u>: If $s = 0$, we have the Dirichlet product, which is known to be associative [1, p. 14]. If $s = 1$,

$$\begin{aligned}(f *_1 (g *_1 h))(n) &= \sum_{d|(n,1)} f(n/d)(g *_1 h)(d) \\ &= f(n)(g *_1 h)(1) \\ &= f(n)g(1)h(1),\end{aligned}$$

while

$$\begin{aligned}((f *_1 g) *_1 h)(n) &= \sum_{d|(n,1)} (f *_1 g)(n/d)h(d) \\ &= (f *_1 g)(n)h(1) \\ &= f(n)g(1)h(1).\end{aligned}$$

Thus, we have established associativity for the cases $s = 0$ and $s = 1$. On the other hand, suppose $s > 1$. Let $s = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $k \geq 1$, $\alpha_i \geq 1$ for $i = 1, 2, \ldots, k$, and the p_i are distinct primes. We must find some positive integer n such that $(f *_s (g *_s h))(n) \neq ((f *_s g) *_s h)(n)$ for some $f, g, h \in A$. Let $n = p_1^{\alpha_1+1}$. Then

$$(1.1) \qquad (f *_s (g *_s h))(n) = \sum_{d|p_1^{\alpha_1}} f(n/d)(g *_s h)(d)$$

$$= f(p_1^{\alpha_1+1})(g *_s h)(1) + f(p_1^{\alpha_1})(g *_s h)(p_1) + \ldots$$

$$+ f(p_1)(g *_s h)(p_1^{\alpha_1}).$$

The important thing to note is that 1 does not appear as an argument of f in (1.1). However,

$$(1.2) \qquad ((f *_s g) *_s h)(n) = \sum_{d|p_1^{\alpha_1}} (f *_s g)(n/d)h(d)$$

$$= (f *_s g)(p_1^{\alpha_1+1})h(1) + \ldots + (f *_s g)(p_1)h(p_1^{\alpha_1}).$$

The last term in this sum can be expanded as

$$[f(p_1)g(1) + f(1)g(p_1)]h(p_1^{\alpha_1}),$$

which contains a term involving $f(1)$. In order to prove the theorem, it is thus sufficient to choose f, g and h such that the right hand sides of (1.1) and (1.2) are not equal--for example, let $f(1) = 1$, $f(n) = 0$ if $n \geq 2$ and let g and h be the constant function 1. Thus the $*_s$-product is not associative when $s > 1$.

<u>THEOREM</u> <u>2</u>: (Commutativity) $f *_s g = g *_s f$ for all $f, g \in A$ if and only if $s = 0$.

<u>PROOF</u>: If $s = 0$, we have the Dirichlet product, which is known to be commutative [1, p. 15]. For a fixed $s \neq 0$, let $n \neq 1$ be chosen such that $(n,s) = 1$. Then $(f *_s g)(n) = f(n)g(1)$, whereas $(g *_s f)(n) = g(n)f(1)$. As in Theorem 1, f and g can be chosen so that $f(n)g(1) \neq g(n)f(1)$.

<u>THEOREM</u> <u>3</u>: An arithmetic function g is a right identity for the $*_s$-product if and only if

$$g(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \text{ and } n \mid s. \end{cases}$$

Before proceeding with the proof, note that due to the arbitrary choice of the value $g(n)$ when $n \nmid s$, one may conclude that there are an infinite number of right identities, and hence that no left identities exist.

PROOF: Suppose that g satisfies the conditions of the theorem. Then for any $f \in A$,

$$(1.3) \qquad (f *_s g)(n) = \sum_{d|(n,s)} f(n/d)g(d)$$

$$= f(n)g(1) + \sum_{\substack{d|(n,s) \\ d > 1}} f(n/d)g(d).$$

But if $d|(n,s)$, $d|s$, and so $g(d) = 0$ for all d in the above summation. So (1.3) becomes

$$(f *_s g)(n) = f(n)g(1)$$

$$= f(n).$$

On the other hand, suppose g is a right identity for the $*_s$-product. Then for every $f \in A$, $(f *_s g)(n) = f(n)$ for all $n \in Z^+$. In particular, $(f *_s g)(1) = f(1)$, which is true if and only if $f(1)g(1) = f(1)$. Since this must hold for all $f \in A$, the last equality holds if and only if $g(1) = 1$. If $n \nmid s$, say $(n,s) = k$, where $k < n$, then n does not appear as an argument of g in $(f *_s g)(n)$, since

$$(f *_s g)(n) = \sum_{d|(n,s)} f(n/d)g(d)$$

$$= \sum_{d|k} f(n/d)g(d).$$

Therefore, $g(n)$ can be chosen arbitrarily for $n \nmid s$. Now suppose there exists $n > 1$ such that $n|s$ and $g(n) \neq 0$. Let k be the smallest such n. Then $(k,s) = k$ and

$$(f *_s g)(k) = f(k) + \sum_{\substack{d|(k,s) \\ 1 < d < k}} f(k/d)g(d) + f(1)g(k).$$

Since g is a right identity, we must have

$$\sum_{\substack{d|(k,s) \\ 1 < d < k}} f(k/d)g(d) + f(1)g(k) = 0.$$

In particular, when $f(1) \neq 0$, there must exist d, $1 < d < k$, such that $g(d) \neq 0$, which is a contradiction.

As far as right inverses are concerned (with respect to a particular right identity), we have that if $f \in A$ has a right inverse, say h, then f is a Dirichlet unit (i.e. $f(1) \neq 0$). For suppose g is a right identity. Then $(f *_s h)(1) = g(1)$ if and only if $f(1)h(1) = 1$. However, not all Dirichlet units have right inverses with respect to every right identity g. Consider α, the identity for the Dirichlet product, defined by

$$\alpha(n) = \begin{cases} 1 & n = 1 \\ 0 & n \geq 2. \end{cases}$$

Using this particular identity, choosing $n \in Z^+$, $n > 1$, such that $(n,s) = 1$, and selecting a Dirichlet unit f such that $f(n) \neq 0$, we must have h such that $(f *_s h)(n) = \alpha(n) = 0$. But $(f *_s h)(n) = f(n)h(1) \neq 0$. So this particular function f has no right inverse with respect to α.

For the case when $s = 0$, it is known [1, p. 16] that an arithmetic function f has an inverse if and only if f is a Dirichlet unit. When $s = 1$, we are able to characterize those functions having a right inverse with respect to a particular right identity.

THEOREM 4: Let $s = 1$. Then $f \in A$ has a right inverse h with respect to the identity g if and only if $f = k \cdot g$, where $k \in C$, $k \neq 0$.

PROOF: If $f = k \cdot g$, let $h = 1/k \cdot g$. Then

$$\begin{aligned} (f *_1 h)(n) &= \sum_{d|1} f(n/d)h(d) \\ &= f(n)h(1) \\ &= (k \cdot g(n))(1/k \cdot g(1)) \\ &= g(n). \end{aligned}$$

Suppose there exists h such that $f *_1 h = g$. Then for all $n \in Z^+$,

$$(f *_1 h)(n) = f(n)h(1)$$
$$= g(n),$$

so that $f(n) = 1/h(1) \cdot g(n)$.

DIRICHLET SERIES

In this chapter, we shall look at the $*_s$-product from a different point of view. A fruitful area of study in number theory has been that of Dirichlet series--series of the type $\sum_{n=1}^{\infty} \frac{f(n)}{n^t}$, where f is an arithmetic function and t is either real or complex. If f and g are arithmetic functions, $f \circ g$ is an arithmetic function, and thus $\sum_{n=1}^{\infty} \frac{(f \circ g)(n)}{n^t}$ is also a Dirichlet series. An important result concerning this last series is that when $\sum_{n=1}^{\infty} \frac{f(n)}{n^t}$ and $\sum_{n=1}^{\infty} \frac{g(n)}{n^t}$ converge absolutely, then

$$\sum_{n=1}^{\infty} \frac{(f \circ g)(n)}{n^t} = \sum_{n=1}^{\infty} \frac{f(n)}{n^t} \sum_{n=1}^{\infty} \frac{g(n)}{n^t} . \tag{2.1}$$

Recalling how the Dirichlet product is just the $*_0$-product and noting that $\sum_{n=1}^{\infty} \frac{g(n)}{n^t}$ can be written as $\sum_{d|0} \frac{g(d)}{d^t}$, $d \in Z^+$, (2.1) becomes

$$\sum_{n=1}^{\infty} \frac{(f *_0 g)(n)}{n^t} = \sum_{n=1}^{\infty} \frac{f(n)}{n^t} \sum_{d|0} \frac{g(d)}{d^t}, \quad d \in Z^+. \tag{2.2}$$

Naturally, the question arises as to whether we can generalize the result in (2.2) by replacing 0 by s, for any positive integer s. The affirmative answer will be found in Theorem 5. Before proving this theorem, we first examine some specific examples of Dirichlet series of the type $\sum_{n=1}^{\infty} \frac{(f *_s g)(n)}{n^t}$ in order to acquire some feeling for them.

For some choices of f and g, $\sum_{n=1}^{\infty} \frac{(f *_s g)(n)}{n^t}$ is finite.

<u>EXAMPLE 1</u>: $\sum_{n=1}^{\infty} \frac{(\alpha *_s f)(n)}{n^t} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n^t} \sum_{d|(n,s)} \alpha(n/d) f(d) \right\}.$

Since $\alpha(n/d) = \begin{cases} 1 & d = n \\ 0 & d \neq n \end{cases}$, and $d = n$ in the above summation if and only if $n|s$, we have

$$\sum_{n=1}^{\infty} \frac{(\alpha *_s f)(n)}{n^t} = \sum_{n|s} \frac{f(n)}{n^t}.$$

We have seen that the $*_s$-product is not commutative when $s > 0$. In examples 2 and 3 we shall find different and concise ways of expressing the series $\sum_{n=1}^{\infty} \frac{(\mu *_s i_0)(n)}{n^t}$ and $\sum_{n=1}^{\infty} \frac{(i_0 *_s \mu)(n)}{n^t}$

respectively, the iota function i_0 and the Mobius function μ having been defined in (0.2) and (0.3). Before beginning example 2, we need a lemma, in which we shall use the notation $s \ll n$ if $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ and $(n,s) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}$.

<u>LEMMA</u>: If $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, then

$$\sum_{d|(n,s)} \mu(n/d) = \begin{cases} 1 & n = 1 \\ (-1)^k & s \ll n \\ 0 & \text{otherwise.} \end{cases}$$

<u>PROOF</u>: If $n = 1$,

$$\sum_{d|(n,s)} \mu(n/d) = \mu(1)$$
$$= 1.$$

If $s \ll n$,

$$\sum_{d|(n,s)} \mu(n/d) = \mu\left(\prod_{i=1}^{k} p_i\right)$$
$$= (-1)^k.$$

If neither $n = 1$, nor $s \ll n$, we consider three cases.

Case 1: Suppose $n = \prod_{i=1}^{k} p_i$, and $(n,s) = \prod_{i=1}^{j} p_i$, $1 \le j \le k$. Then

$$\sum_{d|(n,s)} \mu(n/d) = \sum_{d|\prod_{i=1}^{j} p_i} \mu(\prod_{i=1}^{k} p_i/d)$$

$$= \mu(\prod_{i=1}^{k} p_i) + \sum_{m=1}^{j} \mu(\prod_{\substack{i=1 \\ i\neq m}}^{k} p_i) + \ldots + \mu(\prod_{i=j+1}^{k} p_i)$$

$$= (-1)^k\binom{j}{0} + (-1)^{k-1}\binom{j}{1} + \ldots + (-1)^{k-j}\binom{j}{j}$$

$$= 0.$$

Case 2: Suppose $n = \prod_{i=1}^{m} p_i^{\alpha_i} \prod_{i=m+1}^{k} p_i$, $1 \le m \le k$, where $\alpha_i \ge 2$ for $i = 1, 2, \ldots, m$, and $(n,s) = \prod_{i=1}^{t} p_i^{\alpha_i - 1} \prod_{i=t+1}^{m} p_i^{\alpha_i} \prod_{i=m+1}^{r} p_i$, $0 \le t < k$, $t \le m \le k$, and $m+1 \le r \le k$ (there may be no r). Then

$$\sum_{d|(n,s)} \mu(n/d) = \mu(\prod_{i=1}^{k} p_i) + \sum_{j=t+1}^{r} \mu(\prod_{i\neq j}^{k} p_i) + \sum_{q>j}^{r} \sum_{j=t+1}^{r} \mu(\prod_{i\neq j,q}^{k} p_i)$$

$$+ \ldots + \mu(\prod_{i=1}^{t} p_i \prod_{i=r+1}^{k} p_i)$$

$$= (-1)^k + (-1)^{k-1}\binom{r-(t+1)+1}{1} + (-1)^{k-2}\binom{r-(t+1)+1}{2} + \ldots$$

$$+ (-1)^{t+(k-(r+1)+1)}$$

$$= (-1)^k\binom{r-t}{0} + (-1)^{k-1}\binom{r-t}{1} + \ldots + (-1)^{k-(r-t)}\binom{r-t}{r-t}$$

$$= 0.$$

Case 3: Suppose n is the same as in case 2, but now we have

$$(n,s) = \prod_{i=1}^{t} p_i^{\alpha_i - q_i} \prod_{i=t+1}^{m} p_i^{\alpha_i} \prod_{i=m+1}^{r} p_i, \quad 0 \le t < k, \quad t \le m \le k,$$

$m+1 \le r \le k$ (there may be no r), where $q_i \ge 2$ for at least one i, $1 \le i \le t$. But then in $\sum_{d|(n,s)} \mu(n/d)$, n/d will always contain the factor $p_i^{q_i}$, no matter what divisor d is selected. Hence each term in $\sum_{d|(n,s)} \mu(n/d)$ is 0.

EXAMPLE 2: $$\sum_{n=1}^{\infty} \frac{(\mu *_s i_0)(n)}{n^t} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n^t} \sum_{d|(n,s)} \mu(n/d) \right\}$$

$$= 1 + \sum_{s \ll n} \frac{(-1)^{p(n)}}{n^t},$$

where $p(n)$ denotes the number of prime factors of n.

EXAMPLE 3: In this, our last example, we examine $\sum_{n=1}^{\infty} \frac{(i_0 *_s \mu)(n)}{n^t}$.

From [2, p. 235], $\sum_{d|n} \mu(d) = \alpha(n)$. So that

$$\sum_{d|(n,s)} \mu(d) = \alpha((n,s))$$

$$= \begin{cases} 1 & (n,s) = 1 \\ 0 & (n,s) \neq 1. \end{cases}$$

Thus

$$\sum_{n=1}^{\infty} \frac{(i_0 *_s \mu)(n)}{n^t} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n^t} \sum_{d|(n,s)} \mu(d) \right\}$$

$$= \sum_{(n,s)=1} \frac{1}{n^t} .$$

This last sum is reminiscent of the Riemann Zeta function $\zeta(t)$, defined at complex t where $Re(t) > 1$ by

$$\zeta(t) = \sum_{n=1}^{\infty} \frac{1}{n^t} ,$$

for $\sum_{n=1}^{\infty} \frac{1}{n^t}$ can be thought of as $\sum_{(n,0)=n} \frac{1}{n^t}$. One may follow up this observation and define, for each $s \in Z^+$, new functions $\zeta_s(t)$ by

$$(2.3) \qquad \zeta_s(t) = \sum_{(n,s)=1} \frac{1}{n^t}, \quad s \in Z^+, \quad t \in C, \quad Re(t) > 1.$$

It is known 2, p. 246] and 3, p. 34] that

$$(2.4) \qquad \zeta(t) = \prod_p \frac{1}{1-p^{-t}}, \quad t \in C, \quad Re(t) > 1,$$

where the product is taken over all primes p. Using (2.3), one may obtain

PROPOSITION 1:

$$\zeta_s(t) = \prod_{(p,s)=1} \frac{1}{1-p^{-t}}, \quad t \in C, \quad Re(t) > 1,$$

where the product is taken over all primes p relatively prime to s. The proof of the proposition is almost a word-for-word repetition of the proof of (2.4) appearing in 2, p. 246], so it is omitted here. Hardy and Wright use (2.4) to obtain many results about the Zeta function. It is possible to use Proposition 1 to obtain analogs about the ζ_s functions. Again, the proofs are only slight modifications of the analogous results and will thus be omitted.

PROPOSITION 2: If $f(1) = 1$, f is multiplicative (i.e. $f \in A$ and if $(a,b) = 1$, then $f(ab) = f(a)f(b)$) and $\sum_{n=1}^{\infty} \frac{f(n)}{n^t}$ is absolutely convergent, then

$$\sum_{(n,s)=1} \frac{f(n)}{n^t} = \prod_{(p,s)=1} \left\{ 1 + \frac{f(p)}{p^t} + \frac{f(p^2)}{p^{2t}} + \ldots \right\}.$$

<u>PROPOSITION 3</u>: $\dfrac{1}{\zeta_s(t)} = \displaystyle\sum_{(n,s)=1} \frac{\mu(n)}{n^t}$, $t \in C$, $Re(t) > 1$.

<u>PROPOSITION 4</u>: $\dfrac{\zeta_s(t-1)}{\zeta_s(t)} = \displaystyle\sum_{(n,s)=1} \frac{\varphi(n)}{n^t}$, $t \in C$, $Re(t) > 2$,

where the Euler φ-function $\varphi(n)$ is defined to be the number of positive integers less than n and relatively prime to n. $\varphi(1)$ is defined to be 1. Finally,

<u>PROPOSITION 5</u>:

$$\zeta_s(t)\zeta_s(t-k) = \sum_{(n,s)=1} \frac{\sigma_k(n)}{n^t}, \quad Re(t) > 2, \quad Re(t) > k+1,$$

where the sigma function σ_k is defined by

$$\sigma_k(n) = \sum_{d|n} d^k, \quad k \in Z^+ \cup \{0\},$$

d ranging over the positive divisors of n.

We now come to the major result of this chapter. It is known

[1, p. 80] that if $\sum_{n=1}^{\infty} \frac{f(n)}{n^t}$ and $\sum_{n=1}^{\infty} \frac{g(n)}{n^t}$ converge absolutely, then

$$\sum_{n=1}^{\infty} \frac{(f \circ g)(n)}{n^t} = \sum_{n=1}^{\infty} \frac{f(n)}{n^t} \sum_{n=1}^{\infty} \frac{g(n)}{n^t} .$$

This conclusion may also be expressed as

$$\sum_{n=1}^{\infty} \frac{(f *_0 g)(n)}{n^t} = \sum_{n=1}^{\infty} \frac{f(n)}{n^t} \sum_{d|0} \frac{g(d)}{d^t} .$$

We now generalize this result.

<u>THEOREM 5</u>: If $\sum_{n=1}^{\infty} \frac{f(n)}{n^t}$ is absolutely convergent, and $s > 0$,

then

$$\sum_{n=1}^{\infty} \frac{(f *_s g)(n)}{n^t} = \sum_{n=1}^{\infty} \frac{f(n)}{n^t} \sum_{d|s} \frac{g(d)}{d^t} , \quad d \in Z^+.$$

<u>PROOF</u>: We first extend f by defining it at rational arguments by setting $f(n/d) = 0$ when $d \nmid n$. Then

$$(f *_s g)(n) = \sum_{d|(n,s)} f(n/d)g(d) = \sum_{d|s} f(n/d)g(d) \quad \forall \ n \in Z^+.$$

Let $d_1, d_2, \ldots, d_r$ denote the positive divisors of s. Then

$$\sum_{n=1}^{\infty} \frac{(f *_s g)(n)}{n^t} = \sum_{n=1}^{\infty} \frac{1}{n^t} \left\{ \sum_{d|s} f(n/d)g(d) \right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^t} \left\{ \sum_{\substack{i=1 \\ d_i|s}}^{r} f(n/d_i)g(d_i) \right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^t} \left\{ f(n/d_1)g(d_1) + \ldots + f(n/d_r)g(d_r) \right\}$$

$$= \sum_{n=1}^{\infty} \frac{f(n/d_1)g(d_1)}{n^t} + \ldots + \sum_{n=1}^{\infty} \frac{f(n/d_r)g(d_r)}{n^t} .$$

If in each sum we set $n/d_i = k$, we obtain $n = kd_i$ and

$$\sum_{n=1}^{\infty} \frac{f(n/d_i)g(d_i)}{n^t} = \sum_{k=1}^{\infty} \frac{f(k)g(d_i)}{k^t d_i^{\,t}} .$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{f(n/d_1)g(d_1)}{n^t} + \ldots + \sum_{n=1}^{\infty} \frac{f(n/d_r)g(d_r)}{n^t} =$$

$$\sum_{k=1}^{\infty} \frac{f(k)g(d_1)}{k^t d_1^{\,t}} + \ldots + \sum_{k=1}^{\infty} \frac{f(k)g(d_r)}{k^t d_r^{\,t}}$$

$$= \sum_{k=1}^{\infty} \frac{f(k)}{k^t} \left\{ \frac{g(d_1)}{d_1{}^t} + \cdots + \frac{g(d_r)}{d_r{}^t} \right\}$$

$$= \sum_{k=1}^{\infty} \frac{f(k)}{k^t} \sum_{d|s} \frac{g(d)}{d^t} .$$

Thus we have

$$\sum_{n=1}^{\infty} \frac{(f *_s g)(n)}{n^t} = \sum_{n=1}^{\infty} \frac{f(n)}{n^t} \sum_{d|s} \frac{g(d)}{d^t} .$$

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