Imbedding Graphs in Pseudosurfaces

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IMBEDDING
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CHAPTER I

INTRODUCTION

At the Conference on Graph Theory and Computing in Kingston (Jamaica) in January 1969, A.K. Dewdney proposed the following idea.

Denote the compact orientable 2-manifold of genus $\gamma$ by $S_\gamma$. Identify $2n$ points of $S_\gamma$ in $n$ pairs. The new "surface" you get is denoted by $S(\gamma;n)$, and is called a sphere with $\gamma$ handles and $n$ identifications. One now can ask which graphs are imbeddable in $S(\gamma;n)$ for some fixed $\gamma$ and $n$. It is known that $K_9$ is imbeddable in $S(2;1)$ and the imbedding is a triangulation. This result was generalized by Ringel and Youngs [12] for $K_p$, with $p=9 \pmod{24}$.

In 1969 Dewdney [4] showed that if $G$ is a graph imbeddable in $S(0;1)$, then the chromatic number, $\chi(G)$, is $\leq 5$. Furthermore, this bound is best possible.

At the Proceedings of the Calgary International Conference on Combinatorial Structures and their Applications held at the University of Calgary in June 1969, G. Ringel [13] also mentioned Dewdney's "surface". Ringel also conjectured that the genus of the complete 4-partite graph $K_{p,p,p,p}$ is equal to $(p-1)^2$. The purpose of this project will be to extend Dewdney's
concept of surface and to obtain results related to the
genus of certain graphs. In doing so, a result consistent
with Ringel's conjecture will be obtained.

Definitions of terms that are basic to the study
of imbedding problems are given in Chapter II. In Chapter
III we will present several results on imbedding graphs
and define some additional terms.

In Chapter IV we will define what we mean by a
pseudosurface and present an analogy to the characteristic
of a graph, namely the pseudocharacteristic. In addition
we will derive several other results related to pseudo­
surfaces.

Finally, in Chapter V, we will compute the pseudo-
characteristic of two infinite classes of graphs for
which the regular characteristic is still unknown.
CHAPTER II

BASIC DEFINITIONS AND NOTATION

We present, in this chapter, some basic definitions and establish some of the notation that will be used throughout this project. The definitions that are not given here may be found in Harary [6]. In addition, one may consult Massey [9] for topological terms.

A graph $G$ is a finite nonempty set $V(G)$ of vertices and a set $E(G)$ of unordered pairs of distinct vertices called edges. The order of a graph $G$, denoted by $|G|$, is the number of elements in $V(G)$, and the degree of a vertex is the number of edges to which the vertex belongs.

One may observe that for any graph $G$, it is possible to imbed $G$ on some orientable surface. This can be seen by drawing $G$ in the sphere, and then attaching a handle to the sphere at each crossing of an edge and then allowing one edge to go over the handle and the other under it. We note that, in this situation, one often uses more handles than are actually required.

A 2-manifold is a connected Hausdorff space such that each point has an open neighborhood homeomorphic to the open 2-dimensional disk. In what follows, the term surface will mean a compact orientable 2-manifold.

If $G$ is a graph, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and
E(G) = \{e_1, e_2, \ldots, e_m\}, an imbedding of G in a surface M is a subspace \(G(M)\) of M such that \(G(M) = \bigcup v_i(M) \cup \bigcup e_k(M)\) where we have the following:

1. \(v_1(M), \ldots, v_n(M)\) are \(n\) distinct points of M.
2. \(e_1(M), \ldots, e_m(M)\) are \(m\) open arcs in M, disjoint in pairs.
3. \(e_i(M) \cap v_j(M) = \emptyset\), \(i=1, \ldots, n, j=1, \ldots, m\).
4. If \(e_k = (v_{k_1}, v_{k_2})\), then the open arc \(e_k(M)\) has \(v_{k_1}(M)\) and \(v_{k_2}(M)\) as end points, \(k=1, \ldots, m\).

Now, if the genus of a surface M is \(n\) and a graph G has an imbedding in M but cannot be imbedded in any surface of lower genus, then the imbedding is said to be minimal, and the genus of the graph is defined to be \(n\), denoted by \(\gamma(G) = n\). A graph is said to be planar if \(\gamma(G) = 0\).

Given an imbedding of a graph G in a surface M, each component of the complement of G in M is called a face of the imbedding. If a face is homeomorphic to the open unit disk, it is said to be a 2-cell. A 2-cell imbedding is an imbedding in a surface M such that each face is a 2-cell. The total number of faces for an imbedding will be denoted by \(F\); and for a 2-cell imbedding, we let \(F_i\) denote the number of i-sided faces and \(V_i\) denote the number of vertices of degree i. The imbedding is maximal if no other imbedding of the same graph has more faces.

The maximum genus, \(\gamma_M(G)\), of a connected graph G
is the largest integer $\gamma(N)$ for a surface $N$ in which $G$ has a 2-cell imbedding, where $\gamma(N)$ is the genus of the surface $N$.

The complement $\overline{G}$ of a graph $G$ is the graph with $V(G)$ as its vertex set, and two vertices are adjacent in $\overline{G}$ if and only if these two vertices are not adjacent in $G$.

Denote by $C_p$, the cycle of length $p$, the connected regular graph having $p$ vertices of degree two. The complete graph $K_p$ is a graph with $p$ vertices and all $\binom{p}{2}$ possible edges. A triangle is a subgraph isomorphic to $C_3 (=K_3)$ and, similarly, a quadrilateral is a subgraph isomorphic to $C_4$.

Two graphs $G_1$ and $G_2$ are disjoint if $V(G_1) \cap V(G_2) = \emptyset$; in this case their sum, $G = G_1 + G_2$, is defined by: $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2) \cup E^*$, where $E^*$ consists of all edges joining $V(G_1)$ with $V(G_2)$. The complete $n$-partite graph $K_{p_1,p_2,\ldots,p_n}$, where $p_1 \geq p_2 \geq \ldots \geq p_n \geq 1$, is defined as the iterated sum $K_{p_1} + K_{p_2} + \ldots + K_{p_n}$.

The sets $V(K_{p_1}), \ldots, V(K_{p_n})$ are called the partite sets of $K_{p_1,p_2,\ldots,p_n}$. In particular, $K_{p,q}$ is called the complete bipartite graph and $K_{p,q,r}$ is called the complete tripartite graph.

The least integer greater than or equal to $x$ will be written as $\lceil x \rceil$, and the greatest integer smaller
than or equal to \( x \) will be written as \([x]\).

Finally, the symbol \# will indicate the end of a proof.
CHAPTER III

A SURVEY OF KNOWN RESULTS

In this chapter we survey some of the more well known results concerning imbedding problems in graph theory, and we define some additional concepts that deal directly with this idea of imbedding. We will observe in the next chapter that many of these results have a natural analogue with imbedding graphs in pseudo-surfaces, which will then be defined.

The first theorem that is presented completely classifies compact orientable 2-manifolds [9].

**Theorem 3.1.** Any compact, orientable 2-manifold is homeomorphic to a sphere or to a sphere with handles.

The surface $S_{\gamma}$ may be regarded as a sphere with $\gamma$ handles. If $\gamma(G)=0$, then $G$ may be imbedded in the surface of a sphere. It is a straightforward procedure (using stereographic projection) to show that such a graph may also be imbedded in the plane.

We next present the renowned theorem due to Euler.

**Theorem 3.2.** Let $G$ be a graph having a 2-cell imbedding on $S_{\gamma}$, where $F$ represents the number of faces of the

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imbedding and V and E are the number of vertices and edges of G respectively. Then \( F + V - E = 2 - 2\gamma \).

As a consequence of Euler's Theorem, one says that \( S = S_\gamma \) has Euler Characteristic \( \chi(S) = 2 - 2\gamma \). For example, the sphere and the torus have Euler characteristic two and zero respectively. The Characteristic, \( \chi(G) \), of a graph G is defined in this project to be the largest integer \( \chi(S) \) for all surfaces S in which G can be imbedded. Another useful result, known as the Characterization Theorem, is due to Youngs [18] and is stated next.

**Theorem 3.3.** An imbedding of a connected graph G is minimal if and only if it is a maximal 2-cell imbedding.

We observe at this point that determining the genus of a graph G is equivalent to determining the characteristic of G. To see that not all imbeddings of a connected graph need be 2-cell imbeddings, refer to Figure 3.1, which show the planar graph \( K_4 \) imbedded in the torus, or sphere with one handle. This imbedding of \( K_4 \) has three faces, and face (3) is not a 2-cell. The imbedding is not minimal, and we observe that the Euler Theorem does not apply here as indeed the formula does not hold. We observe further that this does not mean that \( K_4 \) does not have a 2-cell imbedding in the
torus; see Figure 3.2.

![Figure 3.1](image1)

![Figure 3.2](image2)

Two recent developments that deal with the imbedding of connected graphs in surfaces of genus possibly higher than that of the graph are given next. The first theorem is due to Nordhaus, Stewart, and White [11], and the second theorem is due to the three above authors and Ringelisen [10].
Theorem 3.4. \[ \chi_M(K_n) = \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor \]

Theorem 3.5. Let \( G \) be a connected graph. Then \( \chi_M(G) = \chi(G) \) if and only if \( G \) does not contain a subgraph homeomorphic to either \( H \) or \( Q \) of Figure 3.3.

Thus we have, for our example of \( K_4 \), that \( \chi_M(K_4) = 1 \), but \( \chi_M(K_4) \neq \chi(K_4) \) by the above theorem since \( K_4 \) contains a subgraph homeomorphic to \( Q \).

The following two results are due to Battle, Harary, Kodama, and Youngs [1]. These results have frequent applications in determining the genera of certain graphs, as we will see later.

Theorem 3.6. If \( G \) is a connected graph having \( k \) blocks \( B_1, B_2, \ldots, B_k \), then \( \chi(G) = \sum_{i=1}^{k} \chi(B_i) \).
Corollary 3.7. The genus of any graph is the sum of the genera of its components.

As a consequence of the above theorem and corollary, one may say that the genus of any graph is the sum of the genera of its blocks.

We next state the celebrated theorem due to Kuratowski [7], which completely characterizes planar graphs. At this time no similar characterization of graphs of genus one is known.

Theorem 3.8. A graph $G$ is planar if and only if $G$ does not contain a subgraph isomorphic to (within vertices of degree two) either $K_5$ or $K_{3,3}$.

A convenient way to represent imbeddings of a graph $G$ is by the following method. Let $G$ be a connected graph with $n$ vertices, and define $V(i) = \{k: (i,k) \in E(G)\}$, for $i=1,2,...,n$ (where following Youngs, we use $(i,k)$ for unordered pairs and $[i,k]$ for ordered pairs). Let $p_i: V(i) \to V(i)$ be cyclic permutation of $V(i)$ of length $n_i = |V(i)|$, for $i=1,2,...,n$. We now present a theorem due to Edmonds [5] (see also Youngs [18]) which gives us a correspondence between 2-cell imbeddings and choices of the $p_i$. 

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Theorem 3.9. Each choice \((p_1, \ldots, p_n)\) determines a 2-cell imbedding \(G(M)\) of \(G\) in an orientable 2-manifold \(M\), such that there is an orientation on \(M\) which induces a cyclic ordering of the edges \((i, k)\) at \(i\) in which the immediate successor to \((i, k)\) is \((i, p_i(k))\), \(i = 1, 2, \ldots, n\). Conversely, for any 2-cell imbedding \(G(M)\) of \(G\) in an orientable 2-manifold \(M\) with a given orientation, there is a \((p_1, \ldots, p_n)\) which determines the particular imbedding. Now, in fact, given \((p_1, \ldots, p_n)\) there is an algorithm which produces the determined imbedding.

The algorithm is as follows: Let \(D = \{(a, b) : (a, b) \in E(G)\}\), and define \(P : D \rightarrow D\) by \(P([a, b]) = [b, p_b(a)]\). Then \(P\) is a permutation on the set \(D\) of directed edges of \(G\) (where each edge of \(G\) is associated with two oppositely-directed directed edges). The orbits under \(P\) determine the faces of the corresponding imbedding.

This is a very useful result, but one notes that it is usually a lengthy procedure to select a suitable permutation \(P\) from the large number of possible permutations in order to find the genus of the graph \(G\).

As an illustration of these concepts, we consider an imbedding of the complete bipartite graph \(K_{3, 3}\) in the surface \(S_1\), as shown in Figure 3.4.
Here we have:

\[ V(K_{3,3}) = \{1, 2, 3, 4, 5, 6\} \]
\[ V(1) = V(2) = V(3) = \{4, 5, 6\} \]
\[ V(4) = V(5) = V(6) = \{1, 2, 3\} \]
\[ n(i) = 3, \ i = 1, 2, 3, 4, 5, 6. \]

The vertex permutations are seen to be:

\[ P_1 : (4, 6, 5) \]
\[ P_2 : (4, 6, 5) \]
\[ P_3 : (4, 5, 6) \]
\[ P_4 : (1, 2, 3) \]
\[ P_5 : (1, 3, 2) \]
\[ P_6 : (1, 3, 2). \]

This imbedding is a 2-cell imbedding, by Edmond's Theorem. In the imbedding, as shown in Figure 3.4, the handle is attached to a sphere and the handle carries one edge.
Figure 3.4.

For the imbedding of $K_{3,3}$, in Figure 3.4, we find that the orbits under $P$ determine the faces of the imbedding. There are three faces, one being 10-sided and two being 4-sided.
(1) $[1,4],[4,2],[2,6],[6,1],[1,5],[5,3],[3,6],
[6,2],[2,5],[5,1].$

(2) $[1,6],[6,3],[3,4],[4,1].$

(3) $[2,4],[4,3],[3,5],[5,2].$

Now, for an orbit of length $m$ beginning with some directed edge $[a,b]$, we must have that $P^m([a,b])=[a,b]$, where $P^n=P(P^{n-1})$ recursively and $m$ is minimal. For the above example, the first orbit has $P^{10}([1,4])=P([5,1])=[1,4]$, and this equation holds for no positive value of $m$ less than 10; thus we have an orbit of length 10, corresponding to a 10-sided face of the imbedding. We observe that each edge of $G$ appears as two directed edges in $D$; thus the sum of the lengths of the orbits is equal to $2E$. The above imbedding of $K_{3,3}$ is 2-cell; therefore we may verify the Euler formula $F+V-E=2-2\gamma$, where $\gamma=1$ is the genus of $S_1$, and also in this case $\gamma$ is the genus of $K_{3,3}$.

From this point on an orbit will be represented in the abbreviated form 1-6-3-4, instead of the more cumbersome notation $[1,6],[6,3],[3,4],[4,1]$. Observe that when we write 1-6-3-4 for an orbit of length four, it is implied that $P_4(3)=1$ and $P_4(4)=6$.

For completeness, we next list some of the known genus formulae. In 1965 Ringel [12] showed that
$\gamma(K_{p,q})=\lfloor\frac{(p-2)(q-2)}{4}\rfloor$ for $p \geq q \geq 2$, where in each minimal imbedding that he found, all of the faces are quadrilateral with at most one exception. In 1968 Ringel and Youngs [15] established the Heawood map-coloring conjecture by showing that $\gamma(K_p)=\lfloor\frac{(p-3)(p-4)}{12}\rfloor$ for $p \geq 3$.

Here, all faces in a minimal imbedding are triangular, with at most five exceptions. In proving this result, Ringel and Youngs used the theory of current graphs (introduced by Gustin) and vortex theory, depending upon the residue of $p$ modulo 12. (See Youngs [19].)

Just recently, (1970) Ringel and Youngs [14] showed that $\gamma(K_{mp,p,p})=\frac{(p-1)(p-2)}{2}$, producing minimal imbeddings in which every face is a triangle. Also, in 1969, White [17] showed that $\gamma(K_{mp,p,p})=\frac{(mp-2)(p-1)}{2}$, producing $2p^2$ triangular faces and $(mp^2-p^2)$ quadrilateral faces in a minimal imbedding of the graph.

We now present two theorems, due to White [16], which will be used in Chapter V.

**Theorem 3.10.** The genus of the graph $K_{p,q,r}$, $p \geq q \geq r \geq 1$, is bounded below by: $\gamma(K_{p,q,r}) \geq \lfloor\frac{(p-2)(q+r-2)}{4}\rfloor$.

**Theorem 3.11.** If $F_3=2qr$ in a 2-cell imbedding of $K_{p,q,r}$ (in a surface $M$), then $\gamma(K_{p,q,r}) \leq \gamma(M)=\frac{(p-2)(q+r-2)}{4} + \sum_{i=3}^{\lfloor\frac{1}{4}\rfloor} (i-2)F_{2i}$, where $p \geq q \geq r \geq 2$. 

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Before going on to Chapter IV, we consider two additional topological parameters of a graph $G$, and say something about printed circuits. Define the **thickness**, $t(G)$, of a graph $G$ to be the minimum number of planar subgraphs whose union is $G$, and the **crossing number**, $cr(G)$, of $G$ as the minimum number of pairwise intersections of its edges when $G$ is drawn in the plane. One can think of a printed circuit [3] as an electrical network which is formed by printing components and connecting wires on one or more surfaces of a nonconducting material. Since the printed wires are not insulated, no two of them should cross except where a junction is intended.

It is clear that such a network can be printed on a single plane surface if and only if it corresponds to a planar graph, that is, the genus of the graph is zero. Now, suppose that we want to print an arbitrary nonplanar network on two or more plane surfaces, in such a way that the interconnections may occur only at the original junction points (vertices) of the network (graph). We may print lines on each surface, but we may not create additional holes in the plate. The minimum number of surfaces that are needed for this is given by the thickness of the network (graph).

Now, suppose we draw the network in a plane in
such a way that at most two wires intersect at any point which is not an intended junction. We can then print the entire network in one plane, except that at each intersection, not an intended junction, one of the wires would bypass the other, by passing temporarily to the other side of the plate; that is, a segment of this wire would be printed on the other side. The minimum number of such intersections is given by the crossing number of the network (graph).

Suppose that we again have a nonplanar network and all vertices are printed on both sides of a plane surface in a manner that they are opposite each other and each corresponding pair is joined, through the plate, by a conducting wire. Again, we may print some lines on one side of the surface, and some on the other side, and we may create additional holes in the plate. In performing this type of operation on a plate, we obtain a special case of a pseudosurface (which will be defined in Chapter IV) and we observe that any nonplanar network can be so printed.

Thus we observe that there are possible applications of the topological parameters to printed circuit theory.
CHAPTER IV

RESULTS ON PSEUDOSURFACES

In this chapter we define what we mean by a pseudosurface; in addition, we will define terms that are related to pseudosurfaces, which will enable us to develop a theory for the imbedding of graphs in pseudosurfaces. We also will establish several other results that are relative to the study of pseudosurfaces.

Let \( S \) denote the surface of genus \( \gamma \). Denote, by \( A \), a set consisting of \( \sum_{i=1}^{t} n_i m_i \geq 0 \) distinct points of \( S \), with \( 1 < m_1 < m_2 < \ldots < m_t \). Partition \( A \) into \( n_i \) sets of \( m_i \) points each, \( i=1,2,\ldots,t \). For each set of the partition, identify all the points of that set. The resulting topological space is called a pseudosurface, and is designated by \( S(\gamma; n_1(m_1), n_2(m_2), \ldots, n_t(m_t)) \). Each point of \( S(\gamma; g) \) that results from the identifications of \( m_i \) points of \( S_\gamma(i=1,2,\ldots,t) \) is called a singular point; therefore, the pseudosurface \( S^1=S(\gamma; n_1(m_1), n_2(m_2), \ldots, n_t(m_t)) \) has \( \sum_{i=1}^{t} n_i \) singular points. If \( G \) is a graph imbedded in \( S^1 \) such that a vertex \( v \) of \( G \) lies at a singular point of \( S^1 \), then \( v \) is called a singular vertex relative to the imbedding. Throughout this project (unless otherwise stated) if \( G \) is imbedded in \( S^1 \), we
will assume that every singular point is a singular vertex relative to the imbedding, and thus we must have that \( |G| \geq \sum_{i=1}^{t} n_i \). For convenience of notation, we let \( E_l = \{ \gamma_1; \{ n_i \}_{i=2} \} \) denote the pseudosurface \( S^1 = S(\gamma_1; n_2(2), n_3(3), n_4(4), \ldots) \), where there are only a finite number of \( n_i \neq 0 \), and each \( n_i \) is a non-negative integer. We will use this later notation when we are considering an arbitrary pseudosurface as opposed to the notation that was given in the definition of pseudosurface when we have a specific pseudosurface under consideration. Now, for \( S^1 \), define the subscript corresponding to each \( n_i \) as the label corresponding to each singular point counted by \( n_i \); the degree of each singular point \( (i) \) is the natural number \( i \), denoted by \( d(i) \). Call \( n_i \) the coefficient of the singular point \( (i) \). As usual, two sequences \( \{ n_i \}_{i=2} \) and \( \{ p_i \}_{i=2} \) will be called equal if \( n_i = p_i \), \( i = 1, 2, 3, \ldots \).

As an illustration of the above terms, an example of \( S^1 = S(1; 2(2), 1(3)) = S(1; 4) \) is shown in Figure 4.1. The three singular points are \( (2), (2), (3) \), each having degree 2, 2, and 3 respectively, and the coefficients of \( (2) \) and \( (3) \) are 2 and 1 respectively. If we would imbed a graph \( G \) on \( S^1 \), \( |G| \geq 3 \), then the three singular points would be singular vertices.

Given an imbedding of a graph \( G \) in a pseudosurface
S, each component of the complement of G in S is called a face of the imbedding, where an imbedding of G in S is defined in the same manner as an imbedding of G in a surface. (See Chapter II.) An imbedding in a pseudosurface S is a 2-cell imbedding if each face is homeomorphic to the open unit disk. We say that an imbedding, of a graph, is triangular if each face of the imbedding is a triangle, and an imbedding is quadrilateral if each face of the imbedding is a quadrilateral.

Figure 4.1.
We now present an idea that will aid us in proving several results (later on) in this chapter. Let $G$ be a graph having a 2-cell imbedding in $S^1 = S(Y; n_1(m_1), n_2(m_2), ..., n_t(m_t)) = S(Y; g)$, where by definition $g = \sum_{i=1}^{t} n_i(m_i - 1)$. We now carry out an operation on $S^1$ which will reverse the identifications which were made on $S_Y$ in order to obtain the pseudosurface $S^1$. Now, for each singular vertex, corresponding to a singular point labeled $m_i$, $1 \leq i \leq t$, of the imbedding, reverse the identification and thus obtain $m_i$ new vertices, which in the aggregate are adjacent to the same vertices of $G$ as was the singular vertex. That is, for each $m_i (i = 1, 2, ..., t)$ we obtain $m_i$ split vertices, $n_i$ distinct times. We thus obtain a new graph, say $G^*$, having a 2-cell imbedding on $S_Y$ (the faces are unchanged by this process). We now observe the following facts about $G^*$: $|G^*| + \sum_{i=1}^{t} n_i(m_i - 1) = |G| + g$, and the number of edges and faces of $G^*$ remains the same as the number of edges and faces of $G$. From this point on in this project, $G^*$ will have the above meaning.

At this point we develop a result that is a natural analog to Theorem 3.3. We first present a lemma, in which we use the graph $G^*$ that was defined earlier in this chapter.
Lemma 4.1. Let $G$ be a connected graph minimally imbedded in $S(\gamma;g)$. Then the graph $G^*$ is minimally imbedded in $S_\gamma$, and $G^*$ is connected.

Proof: We assume that $g \geq 1$, for if $g=0$, then $G^*=G$ and thus the result holds. We now show that $G^*$ is minimally imbedded in $S_\gamma$. Suppose to the contrary that $G^*$ is not minimally imbedded in $S_\gamma$. Since there are no singular points on $S_\gamma$, we must be able to remove at least one handle and re-imbed $G$ in the resulting surface. Suppose we remove only one handle. Now, we assume that we have an imbedding of $G^*$ in $S_\gamma$. We now reidentify the same points as before in order to obtain $G$ from $G^*$. Thus, we have an imbedding of $G$ in $S(\gamma-1;g)$, which is a contradiction. Therefore $G^*$ is minimally imbedded in $S_\gamma$.

We show now that $G^*$ is connected. Again suppose to the contrary that $G^*$ is not connected. Since $G^*$ is not connected, let $B_1, B_2, \ldots, B_r$ ($r \geq 2$) be the components of $G^*$. Let $F_1=B_1$ and $F_2=B_2 \cup B_3 \cup \ldots \cup B_r$, so we have that $G^*=F_1 \cup F_2$. By a result of Battle, Harary, Kodama, and Youngs [1], we have that $\gamma(G^*)=\gamma(F_1)+\gamma(F_2)=\gamma$. Consider $G^*$ to be imbedded, not on one surface of genus $\gamma(G^*)$, but on two surfaces $M_1$ and $M_2$, such that $\gamma(F_1)=\gamma(M_1)$ and $\gamma(F_2)=\gamma(M_2)$. Now, let $v$ be a singular vertex for
G, such that now we have vertex $v_1$ associated with $F_1$ and $v_2$ associated with $F_2$, where $v_1$ and $v_2$ were formed as a result of reversing the identifications on the pseudosurface. We may have more than just $v_1$ and $v_2$, but this does not affect the argument. We now regain $G^*$ imbedded in one surface in the following manner. There are open 2-cells $C_i$ in $M_i (i=1,2)$, with simple closed boundary curves $J_i$ such that $(C_i \cup J_i) \cap F_i = v_i (i=1,2)$. Identify $J_1$ of $(M_1 - C_1)$ with $J_2$ of $(M_2 - C_2)$ so that $v_1$ is identified with $v_2$. (See Figure 4.2.) This produces a surface $(M_1 - C_1) \cup (M_2 - C_2)$ containing $G^*$. Now, re-identify all the remaining singular vertices on our surface in order to obtain $G$. But now we have an imbedding of $G$ in $S(v; g-1)$, which is a contradiction. Thus $G^*$ must be connected.

![Figure 4.2.](image-url)
Theorem 4.2. Let $G$ be a connected graph minimally imbedded in $S(\gamma;g)$; then the imbedding must be 2-cell.

Proof: Again, obtain the graph $G^*$ from $G$. By Lemma 4.1 $G^*$ is a connected graph, and is minimally imbedded in $S_\gamma$. Thus, as a consequence of Youngs' Characterization Theorem, this imbedding of $G^*$ is a 2-cell imbedding in $S_\gamma$. Now, re-identify all the singular vertices on our surface in order to obtain $G$ again; no faces are affected by this procedure, so that we still have a 2-cell imbedding. Hence, the minimal imbedding in $S(\gamma;g)$ is a 2-cell imbedding. #

The next result that will be presented is particularly useful in obtaining lower bounds for the genera of graphs.

Theorem 4.3. Let $G$ be a graph having a 2-cell imbedding on $S(\gamma;g)$, where $F$ represents the number of faces of the imbedding and $V$ and $E$ are the number of vertices and edges of $G$ respectively. Then $F+V-E=2-2\gamma-g$.

Proof: Let $S^1=S(\gamma;n_1(m_1),n_2(m_2),...,n_t(m_t))$ represent the pseudosurface on which $G$ has a 2-cell imbedding, where by definition $g=\sum_{i=1}^{t} n_i (m_i-1)$. We now obtain $G^*$
from G, and thus by Euler's Theorem, we have that
\[ F + V + \sum_{i=1}^{t} n_i(m_i - 1) - E = 2 - 2\gamma. \]
Now, reverse the operation, on G, to regain \( S^1 \), in which G has a 2-cell imbedding, and recall that \( g = \sum_{i=1}^{t} n_i(m_i - 1) \). Hence we have our desired result that \( F + V - E = 2 - 2\gamma - g \). 

Now, as a consequence of Theorem 4.3, we say that a pseudosurface \( S^1 = S(\gamma; g) \) has Euler Characteristic \( \chi^1(S^1) = 2 - 2\gamma - g \). For example, \( S^1 \) displayed in Figure 4.1 has Euler Characteristic \(-4\). The pseudocharacteristic, \( \chi^1(G) \), of a graph G is defined to be the largest integer \( \chi^1(S) \) for all pseudosurfaces S in which G can be imbedded.

We now present two corollaries that can be used in obtaining bounds for the pseudocharacteristic of different classes of graphs.

**Corollary 4.4.** If G is a connected graph with \( V \) vertices and \( E \) edges, then \( \chi^1(G) \leq V - \frac{E}{3} \), and equality holds if and only if G has a triangular imbedding in a pseudosurface.

**Proof:** Let G be minimally imbedded in a pseudosurface, and let \( F \) represent the number of faces of the imbedding. We observe that \( 2E = \sum_{i \geq 3} iF_i \). But
\[ \sum_{i \geq 3} iF_i \geq 3 \sum_{i \geq 3} F_i = 3F, \]
so that \( F \leq \frac{2}{3}E \). Since \( \chi^1(G) = F + V - E \), we have by substitution, that \( \chi^1(G) \leq V - \frac{E}{3} \). We now observe
Corollary 4.5. If $G$ is a connected graph having no odd cycles, where $V$ and $E$ represent the number of vertices and edges of $G$ respectively, then $\chi^1(G) \leq V - \frac{E}{2}$, and equality holds if and only if $G$ has a quadrilateral imbedding in a pseudosurface.

Proof: Imbed $G$ minimally in a pseudosurface, letting $F$ represent the number of faces of the imbedding. Again we have that $2E = \sum_{i \geq 3} iF_i$, but since $G$ has no odd cycles, we note that $\sum_{i \geq 3} iF_i = \sum_{i \geq 4} iF_i$. But $\sum_{i \geq 4} iF_i \geq 4\sum_{i \geq 4} F_i = 4F$, so that $F \leq \frac{E}{2}$. In this case, we obtain $\chi^1(G) \leq V - \frac{E}{2}$. We now observe that $2E = \sum_{i \geq 4} F_i = 4\sum_{i \geq 4} F_i = 4F$, if and only if $G$ has a quadrilateral imbedding in a pseudosurface. Thus equality is obtained. #

We now present a result that gives us a relation between pseudosurfaces and surfaces.

Proposition 4.6. A pseudosurface $E = \{y; \{n_i\}_{i=2}^\infty\}$ is a surface if and only if $n_i = 0$, for $i = 2, 3, \ldots$. 

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Proof: For the necessity of the condition, suppose to the contrary that there is an \( n_i \neq 0 \). Therefore, there exists a singular point \( (i) \) on \( E \) of degree \( d(i) \). By definition, \( E \) is a surface if and only if every point of \( E \) lies in an open sphere contained entirely in \( E \). But there is no open disk that contains \( (i) \) which lies entirely in \( E \). Thus we must have that \( n_i = 0 \) for all \( i = 1, 2, 3, \ldots \). For the sufficiency of the condition, we assume that \( n_i = 0 \) for all \( i \). Thus \( E \) contains no singular points, and is therefore a sphere or a sphere with handles. Hence \( E \) is a surface.

At this point we observe that according to Edmonds' Theorem, for each choice of \( (p_1, p_2, \ldots, p_n) \), for a connected graph \( G \), we get a 2-cell imbedding in a compact orientable 2-manifold. By Proposition 4.6, a pseudosurface is not necessarily a 2-manifold, but Edmonds' Theorem, with minor modifications, will carry through to 2-cell imbeddings of connected graphs in compact orientable pseudosurfaces. We will now present this modification of Edmonds' Theorem.

Let \( G \) be a connected graph with \( n \) vertices, and define \( V(i) = \{k: (i, k) \in E(G)\} \), for \( i = 1, 2, 3, \ldots, n \). Let \( p_i: V(i) \rightarrow V(i) \) be a permutation of \( V(i) \) (not necessarily cyclic) for \( i = 1, 2, 3, \ldots, n \). We will assume also that
the graph $G^*$ (obtained from $G$ by the manner described earlier) is connected.

**Theorem 4.7.** Each choice $(p_1, p_2, \ldots, p_n)$ determines a 2-cell imbedding $G(S^1)$ of $G$ in an orientable pseudosurface $S^1$, such that there is an orientation on $S^1$ which induces an ordering of the edges $(i, k)$ at $i$ in which the immediate successor to $(i, k)$ is $(i, p_i(k))$, $i=1, 2, \ldots, n$. Conversely, for any 2-cell imbedding $G(S^1)$ of $G$ in a pseudosurface $S^1$ with a given orientation, there is a $(p_1, p_2, \ldots, p_n)$ which determines the particular imbedding.

**Proof:** We first prove the converse. Let $S^1 = S(Y; n_1(m_1), n_2(m_2), \ldots, n_t(m_t))$, and let $G$ be a connected graph with a 2-cell imbedding, $G(S^1)$, in $S^1$ with a given orientation. Now, let $v_i, i=1, 2, 3, \ldots, r$, be all the vertices of $G$ which are not singular vertices, and let $v(j_{m_i}), j=1, 2, 3, \ldots, n_i$, represent the singular vertices for a fixed $n_i(m_i)$, where $1 \leq i \leq t$. Thus, we have all the singular vertices of $G$ labeled as we let $n_i(m_i)$ take on all its values for $i=1, 2, 3, \ldots, t$. Now, for each singular vertex $v(j_{m_i})$ in $G$, $1 \leq j \leq n$, we label the vertices, in $G^*$, that were formed by reversing the identifications, by $v(j_{m_i}, k), k=1, 2, 3, \ldots, m_i$; therefore, $|V(G^*)| = r + t\sum_{i=1}^{t} n_i m_i$. The number of faces and edges for $G^*$ in $S_Y$
remain the same as for \( G \) in \( S^1 \). We thus have a 2-cell imbedding of \( G^* \) in \( S_v \). Therefore, by Edmonds' Theorem, we obtain \((p_{v_1}, p_{v_2}, \ldots, p_{v_r}, p_{v}(j, m_i, k))\) where

\[
p_{v_i} : V(i) \to V(i), i = 1, 2, \ldots, r
\]

and

\[
p_{v}(j, m_i, k) : V\setminus v(j, m_i, k) \to V\setminus v(j, m_i, k),
\]

where

\[
j = 1, 2, \ldots, n_i
\]

\[
k = 1, 2, \ldots, m_i
\]

are cyclic permutations of length \(|V(i)|\) and \(|V(v(j, m_i, k))|\) respectively.

We now reverse the operation, obtaining again \( S^1 \), in which \( G \) has a 2-cell imbedding \( G(S^1) \). Now, for each fixed singular vertex \( v(j, m_i), 1 \leq j \leq n_i \), the permutation at this vertex becomes the following:

\[
p_{v}(j, m_i) : V(v(j, m_i)) \to V(v(j, m_i)), \text{ where}
\]

then \( p_{v}(j, m_i) \) is a one-to-one map from \( V(v(j, m_i)) \) into itself, represented as \( m_i \) (disjoint) cycles, and the \( p_i(i = 1, 2, 3, \ldots, r) \) remain the same. We thus have the converse.

To establish the theorem in the other direction, assume that we have \((p_1, p_2, \ldots, p_r)\), and now we must
show that this determines a 2-cell imbedding $G(S^1)$ of $G$ in $S^1$. The proof of this is somewhat similar to the converse. Using the same notation that was developed above; therefore, we have $(p_1, p_2, \ldots, p_r, p_v(j, m_i))$ for $j=1, 2, 3, \ldots, n_i$ and $1 \leq i \leq t$. We now form $G^*$ from $G$, and thus obtain $(p_1, p_2, \ldots, p_r, p_v(j, m_i, k))$, for $j=1, 2, \ldots, n_i$, $k=1, 2, \ldots, m_i$ and $1 \leq i \leq t$. Therefore, from Edmond's Theorem we have that this determines a 2-cell imbedding of $G^*$ in $S^1$. Now we again obtain $G$ from $G^*$ and thus obtain our desired result. #

Now, given $(p_1, p_2, \ldots, p_n)$ there is an algorithm which produces the determined imbedding. The algorithm is the same as in Chapter III. We will also use the shorter notation for exhibiting the orbits (See Chapter III).

In proving Theorem 4.7, we assumed that the graph $G^*$ was connected. We observe that under this assumption, the theorem is still applicable in finding a minimal imbedding of any connected graph $G$; this is so because of Lemma 4.1. The obvious difficulty arising is that of selecting a suitable permutation $P$ from the large number of possible permutations.

At this time, we will present three rather interesting observations, and in doing so, we will illustrate one of the uses of Theorem 4.7.
I. There exist a connected graph $G$, and two pseudo-surfaces $S^1$ and $S^2$, such that $\chi^1(S^1) = \chi^1(S^2)$ with the property that $G$ has a 2-cell imbedding on $S^1$ but not on $S^2$.

**Example**

Let $S^1 = S(0;2(2))$ and $S^2 = S(0;1(3))$, therefore $\chi^1(S^1) = \chi^1(S^2) = 0$. Let $G = K_6$. We now show that $K_6$ has a 2-cell imbedding on $S^1$ but not on $S^2$. Let $V(K_6) = \{1,2,3,4,5,6\}$. If the permutations of $K_6$ are as follows

\[
\begin{align*}
P_1 &: (34)(526) \\
P_2 &: (615)(43) \\
P_3 &: (16524) \\
P_4 &: (13256) \\
P_5 &: (16432) \\
P_6 &: (12345),
\end{align*}
\]

then the orbits, that determine the face distributions, are:

\[
\begin{align*}
1-2-5 & , 1-3-6-4 \\
1-4-3 & , 2-4-5-3 \\
1-5-6 & , 2-6-3-5. \\
1-6-2 \\
2-3-4 \\
4-6-5
\end{align*}
\]
We therefore have a 2-cell imbedding of $K_6$ on $S^2$, with $F_3=6$, $F_4=3$, and with singular vertices 1 and 2. To show that $K_6$ does not have a 2-cell imbedding on $S^2$, we suppose that it does. We now remove the singular vertex and all its incident edges, and then we reverse the identification that led to the singular point of $S^2$; we now have $K_5$ imbedded on $S_0$, the sphere, which is a contradiction to Kuratowski's Theorem. Thus $K_6$ does not have a 2-cell imbedding on $S^2$.

II. It is known for the two Kuratowski graphs, $K_5$ and $K_3,3$, that $\chi(K_5)=\chi(K_3,3)=0$. We show here that the pseudocharacteristic of these two graphs is greater than zero.

In fact we will show that $\chi^1(K_5)=\chi^1(K_3,3)=1$. First, we show that $K_5$ goes on $S^1=S(0,1(2))$. The permutations are as follows:

$$p_1: (23)(45) \quad p_4: (1325)$$
$$p_2: (1543) \quad p_5: (1423),$$
$$p_3: (1245)$$

where $V(K_5)=[1,2,3,4,5]$. Then the orbits, that determine the face distribution are as follows:
We therefore have a 2-cell imbedding of $K_5$ on $S^1$, with $F_3 = 4$ and $F_4 = 2$, and we have that $\chi(S^1) = 1$. Second, we also show that $K_{3,3}$ goes on $S^1$. Let $V(K_{3,3}) = \{1, 2, 3, 4, 5, 6\}$, such that the vertices $\{1, 2, 3\}$ and $\{4, 5, 6\}$ constitute the two partite sets. The permutations are:

\begin{align*}
p_1 &: (65)(4) \\
p_2 &: (654) \\
p_3 &: (645) \\
p_4 &: (132) \\
p_5 &: (132) \\
p_6 &: (123)
\end{align*}

Then the orbits are:

\begin{align*}
1-5-3-6 \\
1-6-2-5 \\
2-6-3-4
\end{align*}

We therefore have a 2-cell imbedding of $K_{3,3}$, also on $S^1$, with $F_4 = 3$ and $F_6 = 1$.

III. For any positive integer $n$, it is possible to
find a connected graph $G$, such that $\chi(G) - \chi(G) = n$.

Let $G = nK_5$, where $1K_5 = K_5$, $2K_5$ is two copies of $K_5$ with only one vertex in common, and then $nK_5 = (n-1)K_5$ with the additional $K_5$ having one vertex in common with $(n-1)K_5$, (See Figure 4.3.).

\[ \begin{array}{c}
1 & 2 & 3 \\
K_5 & K_5 & K_5 \\
\ldots & & \ldots \\
(n-1) & & K_5
\end{array} \]

Figure 4.3.

Now, by a result due to Battle, Harary, Kodoma, and Youngs, the genus of any graph is the sum of the genera of its blocks. Thus we have that $\gamma(nK_5) = n\gamma(K_5) = n$; therefore $\chi(nK_5) = 2-2n$. Clearly $G = nK_5$ can be imbedded on $S^1 = S(0; n(2))$, so that $\chi^1(G) = \chi^1(S^1) = 2-n$. Thus we have that $\chi^1(G) - \chi(G) = \chi^1(G) - (2-2n) = 2 - n - 2 + n = n$.

**Theorem 4.9.** Let $E_1 = \{ y_i; \{ n_i \}_{i=1}^\infty \}$ and $E_2 = \{ y_2; \{ p_i \}_{i=1}^\infty \}$ represent two pseudosurfaces $S^1$ and $S^2$ respectively. If $\{ n_i \}_{i=1}^\infty \neq \{ p_i \}_{i=1}^\infty$, then $S^1$ and $S^2$ are not homeomorphic.

**Proof:** Assume to the contrary that $S^1$ and $S^2$ are homeomorphic. Let $F$ be a homeomorphism from $S^1$ onto
$S^2$, denoted by $F: S^1 \to S^2$. We first claim that under $F$, singular points must be mapped to singular points. For, suppose that (i) is a singular point of $S^1$, such that $F(i) = a$, where $a$ is not a singular point of $S^2$. Now, a deleted neighborhood, $N_i$ of (i), is mapped into a deleted neighborhood, $N_a$ of $a$, under $F$. For $N_i$ sufficiently small, $N_i$ is the union of $i \geq 2$ disconnected open sets, and $N_a$ is an open disk with the point $a$ deleted. But then $N_i$ must be homeomorphic to $N_a$, which is certainly not true. Thus, under the assumption that $S^1$ and $S^2$ are homeomorphic, the number of singular points for $S^1$ and $S^2$ must be equal, for otherwise $F$ will not be one-to-one.

Now, by the hypothesis of the theorem, there exist two coefficients $n_t$ and $p_t$, such that $n_t \neq p_t$. Let $t$ be the smallest subscript, where $2 \leq t < \infty$. We also assume that the positive integer $n_t$ is greater than the non-negative integer $p_t$. We have $n_t (\geq 1)$ singular points of degree $d(t)$ in $S^1$ and $p_t (0 \leq p_t < n_t)$ singular points of degree $d(t) = t$ in $S^2$. Now, when the singular points of $S^1$ are put into a one-to-one correspondence with the singular points of $S^2$ (as induced by $F$), we have at least one singular point of $S^2$, say $r_2$ of degree $d(r_2)$, matched with a singular point of $S^1$, say $r_1$ of degree $d(r_1)$, such that $d(r_2) \neq d(r_1)$. A sufficiently small
deleted neighborhood of \( r_1 \) is then not homeomorphic to the image under \( F \), since the number of components do not coincide, as \( d(r_1) \neq d(r_2) \). Hence under our assumption that \( S^1 \) and \( S^2 \) are homeomorphic, there is a contradiction. Thus \( S^1 \) and \( S^2 \) are not homeomorphic.

We now present a corollary that will be of use in establishing later results.

**Corollary 4.10.** If \( S^1 \) and \( S^2 \) are homeomorphic, then
\[
\chi^1(S^1) = \chi^1(S^2).
\]

**Proof:** As a consequence of Theorem 4.3, \( \chi^1(S^1) = 2 - 2\gamma_1 - g_1 \) and \( \chi^1(S^2) = 2 - 2\gamma_2 - g_2 \), where \( g_1 = \sum_{i=2}^{\infty} n_i (i-1) \) and \( g_2 = \sum_{i=2}^{\infty} p_i (i-1) \) respectively. In order to obtain our desired result, we must show that \( 2\gamma_1 + g_1 = 2\gamma_2 + g_2 \). Since \( S^1 \) and \( S^2 \) are homeomorphic, we obtain, from Theorem 4.9, that \( \{n_i\}_{i=2}^{\infty} = \{p_i\}_{i=2}^{\infty} \), and thus we see that \( g_1 = g_2 \). Now, since \( S^1 \) and \( S^2 \) are obtained from \( S_{\gamma_1} \) and \( S_{\gamma_2} \) respectively, and \( S^1 \) is homeomorphic to \( S^2 \), thus \( \gamma_1 \) must equal \( \gamma_2 \), for otherwise \( S^1 \) and \( S^2 \) would not be homeomorphic. Hence \( \chi^1(S^1) = \chi^1(S^2) \).

We now give an example to show that the converse of the above corollary does not hold. Let \( S^1 = S(0;1(3)) \)
and $S^2=S(0;2(2))$. Here we have that $\chi^1(S^1)=\chi^1(S^2)=0$, but by Theorem 4.9, $S^1$ is not homeomorphic to $S^2$. The next corollary will give us a characterization of when two pseudosurfaces are homeomorphic.

**Corollary 4.11.** Two pseudosurfaces $S^1$ and $S^2$ are homeomorphic if and only if $E_1=E_2$.

**Proof:** The sufficient condition is immediate, so we need only to consider the necessary condition. Let $E_1=\{\gamma_1;\{n_i\}_{i=2}^\infty\}$ and $E_2=\{\gamma_2;\{p_i\}_{i=2}^\infty\}$; therefore, we must show two things: (a) $\{n_i\}=\{p_i\}$, and (b) $\gamma_1=\gamma_2$. We obtain (a) from the contrapositive of Theorem 4.9. From Corollary 4.10 we have that $\chi^1(S^1)=\chi^1(S^2)$. Therefore, $2\gamma_1+g_1=2\gamma_2+g_2$; but from part (a) above, $g_1=g_2$, so that $2\gamma_1=2\gamma_2$, or $\gamma_1=\gamma_2$. Hence we have that $E_1=E_2$. #

Before proceeding any further, let us define a **partition** of a positive integer $x$ as any collection of positive integers (repetition allowed) whose sum is equal to $x$. The total number of partitions of $x$ is designated by $P(x)$. In this project we define $P(0)=1$. We now present a lemma that will assist us in proving the next theorem.
Lemma 4.12. Let $S^1 = S(\gamma; 1(m_1))$, where $m_1 \geq 1$, be a pseudo-surface. Then there are, including $S^1$, $P(m_1-1)$ non-homeomorphic pseudosurfaces obtainable from $S_\gamma$ having the same characteristic.

Proof: The characteristic of $S^1$ is given by

$$\chi^1(S^1) = 2 - 2\gamma - g_1$$

where $g_1 = m_1 - 1$. Since $\gamma$ remains fixed, pseudosurfaces have the same characteristic if and only if $g_1$ remains invariant.

Let $n_1 + n_2 + \ldots + n_k (k \geq 2)$ represent an arbitrary partition of $g_1$, and let $A = \{n_1, n_2, \ldots, n_k\}$. Partition $A$ under the equivalence relation of equality, obtaining the equivalence classes $B_1, B_2, \ldots, B_t, t < k$. Let $b_i$ equal the cardinality of $B_i$, for each $i = 1, 2, \ldots, t$, and select a representative $c_i$ from each $B_i$.

Now, form the pseudosurface $S^2$, where $S^2 = S(\gamma; b_1(c_1 + 1), b_2(c_2 + 1), \ldots, b_t(c_t + 1))$. Therefore, $\chi^1(S^2) = 2 - 2\gamma - g_2$,

where $g_2 = \sum_{i=1}^t b_i c_i$. Since $\bigcup_{i=1}^t B_i = A$, and the $B_i$ $\bigcup_{i=1}^t B_i = A$, and the $B_i$ $(i = 1, 2, 3, \ldots, t)$ are pairwise disjoint, $b_i$ equals the number of $n_j$ such that $n_j = c_i + 1$, and $c_i$ is a representative from each $B_i$, we have that $g_1 = g_2$. Thus $\chi^1(S^1) = \chi^1(S^2)$, and note that $S^1$ is not homeomorphic to $S^2$ because of Theorem 4.9 (since $k \geq 2$). Since we picked just one arbitrary partition of $g_1 (= m_1 - 1)$, for $k \geq 2$, therefore we have $P(g_1) = P(m_1 - 1)$ non-homeomorphic pseudosurfaces, including $S^1$ with the
same characteristic.

We now are in a position to present the last theorem of this chapter, which will give us the number of non-homeomorphic pseudosurfaces with the same characteristic.

Theorem 4.13. Let $n$ be an integer, where $n \geq -2$, then there are

$$\binom{n+2}{2} \sum_{i=0} p\left(2i+\frac{1+(-1)^n}{2}\right)$$

non-homeomorphic pseudosurfaces with characteristic $-n$.

Proof: For $n=-2$ and $n=-1$ we obtain from the formula, in the theorem, $P(1)$ and $P(0)$ respectively, where $P(1)=P(0)=1$ (by definition). For $n=-2$, it is clear that there is exactly one pseudosurface $S^1$ with $\chi^1(S^1)=2$, namely the sphere, and for $n=-1$, it is again clear that there is exactly one pseudosurface $S^2$ with $\chi^2(S^2)=1$, namely $S^2=S(0;1(2))$. Thus the theorem holds for $n=-2$ and $n=-1$.

Now, let $S^i=S(\gamma_i;1(m_i))$, where $\gamma_i=\left\lfloor \frac{n+2-2i}{2} \right\rfloor$ and $m_i=2i+\frac{3-(-1)^n}{2}$ for $i=0,1,2,\ldots,\left\lfloor \frac{n+2}{2} \right\rfloor$ and $n \geq 0$. We observe that if $n$ is even and $i=0$, then $m_0=1$; but this corresponds to $S_0$ with no singular points.
We now proceed to show that $\chi^i(S^i) = -n$, for $i=0,1,2,\ldots,\left[\frac{n+2}{2}\right]$. We have that $\chi^i(S^i) = 2 - 2\gamma_i - g_i$, where $g_i = m_i - 1$. Let $i$ be arbitrary but fixed; then

$$\chi^i(S^i) = 2 - 2\left[\frac{n+2-2i}{2}\right] - (2i + \left(\frac{3-(-1)^n}{2}\right) - 1).$$

We now consider two cases.

**Case (a).** Let $n$ be even. Then we have by the above that $\chi^i(S^i) = 2 - (n+2-2i) - (2i+1) + 1 = -n$.

**Case (b).** Let $n$ be odd. Then again we have that $\chi^i(S^i) = 2 - (n+1-2i) - (2i+2) + 1 = -n$.

Now observe that for a fixed $n \geq 0$, and each $i \neq j$, $S^i$ is not homeomorphic to $S^j(i, j=0,1,\ldots,\left[\frac{n+2}{2}\right])$ because of Theorem 4.9, and that we have accounted for all $S^i$ having one fixed singular point and pseudocharacteristic $-n$. We now apply Lemma 4.12 to each $S^i$, $i=0,1,\ldots,\left[\frac{n+2}{2}\right]$.

Again we have two cases.

**Case (1).** Suppose $n$ is even. For each $S^i$, we have $P(m_i - 1) = P(2i + 1 - 1) = P(2i)$ non-homeomorphic pseudosurfaces. We therefore obtain a total of

$$\sum_{i=0}^{\left[\frac{n+2}{2}\right]} P(2i) = \sum_{i=0}^{\left[\frac{n+2}{2}\right]} P\left(2i + \left(\frac{1+(-1)^n}{2}\right)\right)$$

non-homeomorphic pseudosurfaces.
Case (2). Suppose $n$ is odd. Then for each $S^i$, we have

$$P(m_i-1) = P(2i+2-1) = P(2i+1) = P\left(2i + \left(\frac{1+(-1)^n}{2}\right)\right)$$
/nginx-homeomorphic pseudosurfaces. Thus, in this case, we also obtain

a total of

$$\sum_{i=0}^{\left\lfloor \frac{n+2}{2} \right\rfloor} P\left(2i + \left(\frac{1+(-1)^n}{2}\right)\right)$$

ginx-homeomorphic pseudosurfaces.

With the aid of the above theorem we observe that there are many non-homeomorphic pseudosurfaces with the same characteristic; but perhaps only one of these pseudosurfaces will give us minimal imbedding of a graph. So, one of the tasks in determining the pseudocharacteristic of a graph is then to select the most efficient pseudosurface (for a fixed pseudocharacteristic) for the imbedding of the graph. We saw earlier in this chapter an example of selecting the correct pseudosurface for $K_6$.

We are now prepared to construct minimal imbeddings for different classes of graphs, and will thus be able to determine their pseudocharacteristic exactly.
CHAPTER V

PSEUDOCHARACTERISTICS

In this chapter we present some of the main results of this project. First, we will compute the pseudocharacteristic of the regular complete 4-partite graph $K_{p,p,p,p}$; we then will generalize this result. Secondly, we will determine the pseudocharacteristic for $K_{2m,2n,r}$.

As was stated in Chapter III, Ringel and Youngs [14] have shown that $\gamma(K_{p,p,p}) = \frac{(p-1)(p-2)}{2}$, producing a (minimal) triangular imbedding. The orbits of a particular minimal imbedding of $K_{p,p,p}$ are given by White [17], and they are as follows: We designate the vertex set of $K_{p,p,p}$ by $V(K_{p,p,p}) = \{1,2,3,...,3p\}$, with adjacencies at vertex $i$ given by:

- $V(i) = \{2,4,6,...,2p,2p+1,2p+2,...,3p\}, i=1,3,5,...,2p-1$
- $V(i) = \{1,3,5,...,2p-1,2p+1,2p+2,...,3p\}, i=2,4,6,...,2p$
- $V(i) = \{1,2,3,...,2p\}, i=2p+1,2p+2,2p+3,...,3p$

(a): $(2j-1)-(2p+1)-(2i-2j+2), \ 1 \leq j \leq p, 1 \leq i \leq p$
(b): $(2j-1)-(2i-2j+2)-(2p+i-1), \ 1 \leq j \leq p, 2 \leq i \leq p$
(c): $(2j-1)-(2i-2j+2)-(3p), \ 1 \leq j \leq p, \ i=p$

(we do our arithmetic modulo $2p$, writing $2p$ for 0)

Observe that each orbit corresponds to a triangular
face, where by (a) we get $p$, by (b) we get $p(p-1)$, and by (c) we get $p$ triangular faces. Therefore, in all, we have $2p$ triangular faces.

We now prove a lemma that will aid us in accomplishing our first objective of this chapter.

**Lemma 5.1.** For the minimal imbedding of $K_{p,p,p}$ given above, the set of $2p^2$ triangular faces contains $2p$ mutually disjoint subsets of $p$ faces each, so that each subset of $p$ faces contains all $3p$ vertices of $K_{p,p,p}$.

**Proof:** The proof of this lemma will have two main cases. Each case will produce $p$ subsets, so that combining the two cases, we will obtain $2p$ mutually disjoint subsets with the properties of the lemma. We also point out at this time, that when we exhibit a scheme for determining the subsets, it will be understood that we will reduce the $j$'s (mod $p$) and write $p$ instead of 0. We now consider the cases and assign these $2p^2$ faces to parts of the mutually disjoint subsets.

**Case 1.** We will subdivide this case into two subcases.

**Subcase (i)** Suppose that $p$ is odd. For a fixed $k (k=1,2,\ldots,p)$, the $p$ faces of part $k$ are determined, where we only use the orbits under (a), as follows:
Observe here that for any part $k$, we get all $3p$ vertices and all the parts are mutually disjoint.

**Subcase (ii).** Suppose that $p$ is even. Now, for a fixed $k(k=1,2,...,p)$ we will obtain \( \frac{p}{2} \) faces for each part $k$ under the orbits of (a), and the other \( \frac{p}{2} \) faces for the corresponding part $k$ under the orbits of (b).

First, the \( \frac{p}{2} \) faces of each part $k$, under (a), are determined as follows:

\[
\begin{align*}
  j &= k, \quad i = 1 \\
  j &= k + 1, \quad i = p \\
  j &= k + 2, \quad i = p - 1 \\
  j &= k + 3, \quad i = p - 2 \\
  &\vdots \\
  j &= k + \left(\frac{p}{2}-1\right), \quad i = p - \left(\frac{p}{2}-2\right) = \frac{p}{2} + 2.
\end{align*}
\]

Second, the \( \frac{p}{2} \) faces of each part $k$, under (b), are
determined as follows:

\[
\begin{align*}
  j &= k + \frac{p}{2}, \quad i = \frac{p}{2} + 2 \\
  j &= k + \left(\frac{p}{2} + 1\right), \quad i = \frac{p}{2} + 1 \\
  j &= k + \left(\frac{p}{2} + 2\right), \quad i = \frac{p}{2} \\
  \quad & \quad \quad \vdots \\
  j &= k + (p - 2), \quad i = 4 \\
  j &= k + (p - 1), \quad i = 3.
\end{align*}
\]

Thus, for \( p \) even, we have for any part \( k \) all \( 3p \) vertices and all the parts are mutually disjoint.

We therefore have \( p \) mutually disjoint subsets of \( p \) faces each, so that each subset of \( p \) faces contains all \( 3p \) vertices of \( K_{p,p,p} \). For the other \( p \) subsets, we have the second case.

**Case 2.** We again subdivide this case according as \( p \) is odd or even.

**Subcase (i).** Suppose \( p \) is odd. Now, for a fixed \( k(k=1,2,\ldots,p) \), the \( p \) faces of part \( (p + k) \), where we will use (c) for \( j = k, i = 1 \) and (b) for all the rest, are determined as follows:
\[ j = k, \quad i = 1 \]
\[ j = k + 1, \quad i = 3 \]
\[ j = k + 2, \quad i = 5 \]
\[ \vdots \]
\[ j = k + (\frac{p-3}{2}), \quad i = p - 2 \]
\[ j = k + (\frac{p-1}{2}), \quad i = p \]
\[ j = k + (\frac{p+1}{2}), \quad i = 2 \]
\[ j = k + (\frac{p+3}{2}), \quad i = 4 \]
\[ \vdots \]
\[ j = p - 1, \quad i = (p - 1) - 2k \]
\[ j = p, \quad i = (p + 1) - 2k \]
\[ j = 1, \quad i = (p + 3) - 2k \]
\[ j = 2, \quad i = (p + 5) - 2k \]
\[ \vdots \]
\[ j = k - 2, \quad i = p - 3 \]
\[ j = k - 1, \quad i = p - 1. \]

Observe again that for any part \( k \), we get all \( 3p \) vertices and the parts are mutually disjoint.

**Subcase (ii).** Suppose \( p \) is even. Under this assumption, that \( p \) is even, the situation becomes somewhat more complicated; therefore, first we assume that
\( p \equiv 0 \pmod{4} \), and secondly we assume that \( p \equiv 2 \pmod{4} \).

First, suppose \( p \equiv 0 \pmod{4} \), then for a fixed \( k(k=1,2,...,p) \), the \( p \) faces of part \( (p + k) \) are determined as follows:

**under (a)**

\[
\begin{align*}
  j &= k + 1, i = 3 \\
  j &= k + 2, i = 2 \\
  j &= k + 5, i = 4 \\
  j &= k + 7, i = 5 \\
  j &= k + 9, i = 6 \\
  j &= k + (p - 3), i = \frac{p}{2} \\
  j &= k + (p - 1), i = \frac{p}{2} + 1
\end{align*}
\]

**under (b)**

\[
\begin{align*}
  j &= k + 3, i = \frac{p-8}{2} \\
  j &= k + 4, i = \frac{p-6}{2} \\
  j &= k + 5, i = \frac{p-12}{2} \\
  j &= k + 8, i = \frac{p-10}{2}
\end{align*}
\]

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j = k + (p - 10), i = p - 2
j = k + (p - 8), i = p - 3
j = k + (p - 6), i = p
j = k + (p - 4), i = p - 1
j = k + (p - 2), i = 2

and for j = k, i = 1 we use (c).

Thus, for p ≡ 0 (mod 4), we have, for any part (p + 4), all 3p vertices and all the parts are mutually disjoint.

Second, suppose p=2 (mod 4), then for a fixed k(k=1,2,...,P), the p faces of part (p + k) are determined as follows:

\[ \text{under (a)} \]

\[ j = k + 1, \quad i = 3 \]
\[ j = k + 2, \quad i = 2 \]
\[ j = k + 5, \quad i = 4 \]
\[ j = k + 7, \quad i = 5 \]
\[ j = k + 9, \quad i = 6 \]
\[ j = k + 11, \quad i = 7 \]
\[ \vdots \]
\[ j = k + (p - 3), \quad i = \frac{p}{2} \]
\[ j = k + (p - 1), \quad i = \frac{p}{2} + 1 \]
under (b)

\[
\begin{align*}
    j &= k + 3  \quad i = \frac{p+6}{2} \\
    j &= k + 4  \quad i = \frac{p+10}{2} \\
    j &= k + 6  \quad i = \frac{p+8}{2} \\
    j &= k + 8  \quad i = \frac{p+14}{2} \\
    j &= k + 10 \quad i = \frac{p+12}{2} \\
    j &= k + 12 \quad i = \frac{p+18}{2} \\
    j &= k + (p - 10) \quad i = p - 2 \\
    j &= k + (p - 8) \quad i = p - 3 \\
    j &= k + (p - 6) \quad i = p \\
    j &= k + (p - 4) \quad i = p - 1 \\
    j &= k + (p - 2) \quad i = 2 \\
\end{align*}
\]

and for \( j = k, \ i = 1 \) we use (c).

Now, for \( p \equiv 2 \ (\text{mod} \ 4) \), we have again, for any part \((t+k)\), all \(3p\) vertices and all the parts are mutually disjoint.

Thus, by combining the two main cases above, we have \(2p\) sets of \(p\) faces each, the sets being mutually disjoint by the manner in which they were selected.

Furthermore, each set of \(p\) faces contains all \(3p\) vertices of the graph \(K_{p,p,p}\).  #
We are now in a position to prove the following theorem.

**Theorem 5.2.** For the regular complete 4-partite graph $K_{p,p,p,p}$ ($p \geq 1$); we have that $\chi^1(K_{p,p,p,p}) = 2p(2-p)$.

**Proof:** The method of proof will be to imbed $K_{p,p,p,p}$ in a pseudosurface in such a way that each face of the imbedding is a triangle.

We first imbed $K_{p,p,p}$ triangularly in $S_y$, where $y = \frac{(p-1)(p-2)}{2}$, in the manner described in [17], which is consistent with the result of Ringel and Youngs [14].

By Lemma 5.1, the set of $2p^2$ triangular faces for the imbedding of $K_{p,p,p}$ contains $2p$ mutually disjoint subsets of $p$ faces each so that each subset contains all $3p$ vertices of $K_{p,p,p}$. Now, for this theorem, we need only take $p$ of these subsets; and for each subset $j$, $1 \leq j \leq p$, add one vertex in the interior of each of the $p$ faces (thus we have added $p^2$ additional vertices). From each added vertex, we insert 3-edges joining the 3-vertices that determine the particular face within which the vertex was added. We now note that we have destroyed $p$ triangular faces of the original imbedding of $K_{p,p,p,p}$ for each of the $p$ subsets; but each time, in doing so, we have produced $3p$ new triangular faces.
Thus, we have obtained a different graph, say $G$, which has a triangular imbedding in $S_\gamma$. Now, for each subset $j$, $1 \leq j \leq p$, we identify the $p$ new vertices that were added as one singular vertex. After performing this identification process for each of the subsets, we obtain the graph $K_{p,p,p,p}$ (from $G$), having a triangular imbedding in the pseudosurface $S(\gamma;p(p))$. We may then compute the pseudocharacteristic of $K_{p,p,p,p}$ by using Corollary 4.4. As $V=4p$ and $E=6p^2$, we see that $\chi^1(K_{p,p,p,p})=2p(2-p)$.

We now observe that if Ringel's conjecture is true for the genus of $K_{p,p,p,p}$, then the characteristic of this graph is $2p(2-p)$, which is the same as in the above theorem.

At this point we present a theorem that is a generalization of Theorem 5.2.

**Theorem 5.3.** For the graph $K_{p,p,p,p,m}$ ($1 \leq m \leq 2p$), we have that $\chi^1(K_{p,p,p,p,m})=-p^2-mp+3p+m$.

**Proof:** As in the proof of Theorem 5.2, we imbed $K_{p,p,p,p}$ triangularly in $S_\gamma$, where $\gamma=\frac{(p-1)(p-2)}{2}$. Now, from Lemma 5.1, we have $2p$ subsets with the appropriate properties given in the lemma. Therefore, for each subset $j$, $j=1,2,\ldots,m \leq 2p$, we add one vertex in the
interior of each of the p faces (thus totally, we have added pm additional vertices). Now proceed as we did in the proof of the above theorem; obtaining a triangular imbedding of $K_{p, p, p, m}$ in the pseudosurface $S(\gamma; m(p))$. Then by Corollary 4.4, with $V=3p+m$ and $E=3p^2+3pm$, we have that $\chi^1(K_{p, p, p, m})=-p^2-mp+3p+m$. #

**Theorem 5.4.** The characteristic of the graph $K_{p, q, r}$ ($p \geq q \geq r \geq 2$) is bounded above by:

$$\chi(K_{p, q, r}) \leq 2 - 2\left\lceil \frac{(p-2)(q+r-2)}{4} \right\rceil.$$  

**Proof:** We have, as a consequence of Euler's Theorem, that $\chi(K_{p, q, r}) = 2 - 2\gamma(K_{p, q, r})$. By Theorem 3.10.

$$\gamma(K_{p, q, r}) \geq \left\lceil \frac{(p-2)(q+r-2)}{4} \right\rceil;$$  

therefore, by substitution we obtain our desired bound. #

**Theorem 5.5.** If $F = 2qr$ in a 2-cell imbedding of $K_{p, q, r}$ ($p \geq q \geq r \geq 2$) in a surface $M$, then

$$\chi(K_{p, q, r}) \geq \chi(M) = 2 - 2\left\lceil \frac{(p-2)(q+r-2)}{4} \right\rceil \sum_{i \geq 3} (i-2)F_{2i}.$$  

**Proof:** Again, we have that $\chi(K_{p, q, r}) = 2 - 2\gamma(K_{p, q, r})$. 

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By Theorem 3.11, \( \gamma(K_p,q,r) \leq \gamma(M) = \frac{(p-2)(q+r-2)}{4} + \sum_{i=3}^{\infty} (i-2)F_{2i} \).

Therefore, by substitution we again obtain our desired result.

Recall that in 1965 Ringel [12] showed that \( \gamma(K_p,q) = \left\{ \frac{(p-2)(q-2)}{4} \right\} \) (for \( p \geq q \geq 2 \)). We now want to consider the case where \( p \) and \( q \) are even; therefore, let \( p = 2m, q = 2n, m \geq n \), where \( m \) and \( n \) are positive integers. Now, a particular minimal imbedding for \( K_{2m,2n} \), with \( F = F_4 = 2mn \), is given by Ringel [12] and is as follows:

\[
V(i) = \begin{cases} 
\{ j : 2m + 1 \leq j \leq 2(m + n), 1 \leq i \leq 2m \\
\{ j : 1 \leq j \leq 2m, 2m + 1 \leq i \leq 2(m + n) \}
\end{cases}
\]

\[
P_1, P_2, \ldots, P_{2m-1} : (2m+1,2m+2,\ldots,2(m+n)-1,2(m+n))
P_2, P_4, \ldots, P_{2m} : (2(m+n),2(m+n)-1,\ldots,2m+2,2m+1)
P_{2m+1}, P_{2m+3}, \ldots, P_{2(m+n)-1} : (1,2,3,\ldots,2m-1,2m)
P_{2m+2}, P_{2m+4}, \ldots, P_{2(m+n)} : (2m,2m-1,\ldots,3,2,1)
\]

The following lemma will be used to compute the pseudocharacteristic of \( K_{2m,2n}, r(1 \leq r \leq 2n) \).

**Lemma 5.6.** For the (minimal) quadrilateral imbedding of \( K_{2m,2n} \) given above (\( m \geq 1 \)), the set of \( 2mn \) quadrilateral faces may be partitioned into \( 2n \) subsets of \( m \) faces each,
so that each subset of $m$ faces has the following properties:

i) There are $n$ mutually disjoint faces.

ii) There are $(m-n)$ faces containing the remaining $(2m-2n)$ vertices of $K_{2m,2n}$.

Proof: We write out the orbits (each corresponding to a quadrilateral face) determined by the permutation $P$ as defined by the permutation $p_i, 1 \leq i \leq 2(m+n)$ (see Chapter 3), given above:

(a): $(2g-1) - (2h-1) - 2g - (2h-2), 1 \leq g \leq m; m+1 \leq h \leq m+n$

(b): $(2g-1) - (2h-1) - 2g - (2m+2n), 1 \leq g \leq m; h = m+1$

(c): $2j - (2k-1) - (2j+1) - 2k, 1 \leq j \leq m; m+1 \leq k \leq m+n$

(d): $2j - (2k-1) - 1 - 2k, j = m; m+1 \leq k \leq m+n$

In the proof we will consider two cases. Each case produces $n$ subsets, so that combining the two cases, we will obtain $2n$ subsets of $m$ faces each so that each subset satisfies the two conditions (i) and (ii). We now consider the two cases and assign these $2mn$ faces to parts of the partition.

Case 1. First, for each part $t (t=1,2,3,...,n)$, we produce $n$ faces satisfying (i), and secondly, for each part $t$ we produce $(m-n)$ faces satisfying (ii).
First, we use the following scheme:

\[
\begin{align*}
g &= t, & h &= m + 1 \\
g &= t + 1, & h &= m + 2 \\
g &= t + 2, & h &= m + 3 \\
\vdots & & \vdots \\
g &= t + (n - 3) = m + (n - 2), & h &= m + (n - 2) \\
g &= t + (n - 2), & h &= m + (n - 1) \\
g &= t + (n - 1), & h &= m + n
\end{align*}
\]

where we reduce the g’s (mod n), write n instead of 0, and where all of the above are determined under (a) except for \(g = t, h = m + 1\) which is determined under (b).

Second, we use the following scheme:

\[
\begin{align*}
g &= n + 1, & h &= m + t \\
g &= n + 2, & h &= m + t \\
g &= n + 3, & h &= m + t \\
\vdots & & \vdots \\
g &= m, & h &= m + t
\end{align*}
\]

where for \(t = 1\) we use the orbits under (b), and for \(t = 2, 3, \ldots, n\) we use (a).
Case 2. We first produce $n$ faces satisfying (i), for each part $s (1, 2, 3, \ldots, n)$, and secondly, for each part $s$, we produce $(m-n)$ faces satisfying (ii).

First, we use the following scheme:

\[
\begin{align*}
    j &= s & k &= m + 1 \\
    j &= s + 1 & k &= m + 2 \\
    j &= s + 2 & k &= m + 3 \\
    \vdots & \quad \vdots \\
    j &= s + (n - 2) & k &= m + (n - 1) \\
    j &= s + (n - 1) & k &= m + n
\end{align*}
\]

where we reduce the $j$'s (mod $n$), and write $n$ instead of 0. Now, for the above scheme, if $m = n$, then use the orbits under (c) except for $j = n$ we use (d). If $m \neq n$, then determine all of the above under (c).

Secondly, we use the following scheme:

\[
\begin{align*}
    j &= n + 1 & k &= m + s \\
    j &= n + 2 & k &= m + s \\
    j &= n + 3 & k &= m + s \\
    \vdots & \quad \vdots \\
    \vdots & \quad \vdots
\end{align*}
\]
\[ j = m - 1, \quad k = m + s \]

\[ j = m, \quad k = m + s \]

where \( j = m, k = m + s \) is determined under (d) and all the rest under (c).

Now, observe that \( n \) faces include 4 distinct vertices each, and that \((m-n)\) faces include 2 additional distinct vertices each, so that each of the \(2n\) subsets contain

\[ 4n + 2(m-n) = 2n + 2m \]

vertices, covering all of the vertices of \( K_{2m,2n} \). Hence we have our desired result, combining the above two cases. \( \neq \)

We are now in a position to prove the following theorem.

**Theorem 5.7.** For the complete tripartite graph \( K_{2m,2n,r} \)

where \( 2m \geq 2n \geq r \geq 1 \), we have that

\[ x_{K_{2m,2n,r}} = 2(m + n - mn) - r(m - 1) \]

**Proof:** First observe that, since \( \left\lfloor \frac{r(m-1)}{2} \right\rfloor \geq \frac{r(m-1)}{2} \), we have

\[ 2\left\lfloor \frac{(2m-2)(2n+r-2)}{4} \right\rfloor \geq 2\left\lfloor \frac{(2m-2)(2n-2)}{4} \right\rfloor + r(m - 1). \]
But then
\[ 2 - 2\left(\frac{2m-2)(2n+r-2)}{4}\right) \leq 2 - 2(m - 1)(n - 1) - r(m - 1). \]

Now, by Theorem 5.4, with \( p = 2m \) and \( q = 2n \), we obtain the inequality:
\[ x(K_{2m,2n,r}) \leq 2 - 2(m - 1)(n - 1) - r(m - 1) = 2(m + n - mn) - r(m - 1). \]

From Theorem 5.5, with \( p = 2m \) and \( q = 2n \), if \( F_3 = 4nr \) we obtain the inequality:
\[ x(K_{2m,2n,r}) \geq 2 - 2(m - 1)(n - 1) - r(m - 1) - \frac{1}{2}(i - 2)F_2 - \frac{1}{6}F_6 - \frac{1}{8}F_8 - \frac{1}{10}F_{10} - \ldots. \]

As a result of the second inequality, it is possible to show that equality is achieved in the first inequality above, provided we produce a 2-cell imbedding of \( K_{2m,2n,r} \) for which \( F_3 = 4nr \) and \( F_4 = F - F_3 \). With the aid of Lemma 5.6, we will now produce the desired imbedding.

We first imbed \( K_{2m,2n} \) in \( S^r \) in such a way that all \( 2mn \) faces of the imbedding are quadrilaterals, where \( r = (m - 1)(n - 1) \), in the manner described in [12], and given above.
Now, for each subset $j$, $j = 1, 2, 3, \ldots, r^2n$, we add one vertex in the interior of each of the $m$ faces (thus we have introduced a total of $rm$ new vertices). For a fixed subset $j$ ($1 \leq j \leq r$), we do the following: we first add 4 edges in the interior of each of the $n$ (or $m$) mutually disjoint faces, connecting the new vertex to the 4 vertices determining the face, which is possible by Lemma 5.6. We then insert 2 edges in the interior of each of the $m - n$ ($\geq 0$) remaining faces in such a way that the new vertex, in each of these faces, is adjacent to two nonadjacent vertices (as described by property (ii) of Lemma 5.6) in the boundary of the face. We therefore have obtained $4n$ triangles, and $2(m - n)$ quadrilaterals for the subset $j$. Now, carry out this procedure for each of the $r$ subsets. Thus, we have obtained a new graph, say $G$, which has an imbedding in $S_\gamma$ with $F = 2n(r+m) + rm$, where $F_3 = 4nr$ and $F_4 = 2r(m-n) + m(2n-r) = F - F_3$.

Now, for each subset $j$, $1 \leq j \leq r$, we identify the $m$ new vertices that we added as one singular vertex, and thus we obtain the graph $K_{2m, 2n, r}$ (from $G$), having an imbedding in the pseudosurface $S(\gamma; r(m))$, such that $F_3 = 4nr$ and $F_4 = F - F_3$. We thus have that $\chi(K_{2m, 2n, r}) = 2(m + n + mn) - r(m - 1)$. #
In 1969 White [16] made the following conjecture:

\[ \gamma(K_{p,q,r}) = \frac{(p-2)(q+r-2)}{4} \quad (p \geq q \geq r). \]

Equivalently, the conjecture states that

\[ x(K_{p,q,r}) = 2 - 2\gamma = 2 - 2\frac{(p-2)(q+r-2)}{4}. \]

If we let \( p = 2m, q = 2n \) (\( m \geq n \)), we then obtain

\[ x(K_{2m,2n,r}) = 2 - 2\frac{(m-1)(2n+r-2)}{2}. \]

We now observe, with the aid of Theorem 5.7, that \( x(K_{2m,2n,r}) \leq x^!(K_{2m,2n,r}) \) (assuming the truth of the conjecture) and equality holds if and only if \( r(m-1) \) is even; otherwise the difference is 1.
BIBLIOGRAPHY


