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Lower Bounds in the Stekloff Problem

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LOWER BOUNDS
IN THE
STEKLOFF PROBLEM

by

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I INTRODUCTION

This project deals with lower-bounds for the first non-zero eigenvalue p_2 of the Stekloff problem. This problem was first stated in a paper by M. W. Stekloff in 1902 [8]. The problem arises from consideration of the vibrations of an elastic membrane with the mass concentrated on the boundary. The mathematical formulation of this physical problem leads to the following eigenvalue problem [5, p.456]:

Let B be a bounded domain (an open, connected subset) of the plane with boundary B^* having a Lipschitz representation. The Stekloff problem for domain B reduces to the determination of the eigenfunctions ϕ_j and the corresponding eigenvalues $p_j (j=1,2,\ldots)$ satisfying

(I.1) $$\Delta\phi_j = 0 \text{ , in } B \text{ , and}$$

(I.2) $$\frac{\partial\phi_j}{\partial n} = p_j\phi_j \text{ , on } B^* \text{ ,}$$

where Δ is the Laplacian operator and $\frac{\partial}{\partial n}$ is differentiation along the outward normal.

The eigenvalues p_j are nonnegative and the least eigenvalue p_1 is zero.

The least positive eigenvalue p_2 is the object of interest in this project. In Chapter II, this eigenvalue is also given a variational characterization. Knowledge of a lower bound for p_2 enables one to compute bounds for a solution of the Neumann problem for the Laplace equation, and also error-bounds in the Neumann problem for the Poisson equation.

In Chapter III, known results on the Stekloff problem are surveyed. Some systems for which the Stekloff problem is explicitly soluble are described. The results are used later to test conjectures for lower bounds for p_2. Lower bounds for p_2 which have been obtained by various workers are briefly described. Except for a recent one due to L. E. Payne [6], these lower bounds are complicated.

In Chapter IV, a first conjecture for a lower bound for p_2 is made by a crude consideration of dimensions. While this agrees with the value of p_2 for the special case of a circular disc, consideration of the case of a rectangle shows that this simple lower bound for p_2 is not possible. A more careful consideration of dimensions suggests a conjecture which is shown to be consistent with the values of p_2 for circular discs and rectangles. Finally, this conjecture is extended to domains in higher dimensions.

II VARIATIONAL FORMULATION

By definition, p_2 is the smallest positive number p satisfying

(II.1) $$\Delta\phi = 0 \text{ , in } B \text{ , and}$$

(II.2) $$\frac{\partial\phi}{\partial n} = p\phi \text{ , on } B^*$$

for some nonzero function ϕ .

This eigenvalue p_2 can also be characterized variationally as the minimum $\bar{p}$ of

$$\frac{\int_B \phi'^2 dx}{\int_{B^*} \phi^2 ds} ,$$

taken over all functions ϕ which are continuous on the closure of B (i.e., $B \cup B^*$) and are C^1 in B and further, are such that $\int_{B^*} \phi ds = 0$ with $\phi\big|_{B^*} \not\equiv 0$. Here throughout, ϕ' stands for the vector-derivative of ϕ .

Equivalence of the Formulations

To show the equivalence of the formulations above, we first prove the following Theorem.

<u>Theorem 1.</u> <u>Let</u> $\bar{\phi}$ <u>be a minimizing function in the definition of</u> $\bar{p}$. <u>Then</u> $\bar{\phi}$ <u>and</u> $\bar{p}$ <u>satisfy</u>

(II.3) $$\Delta\bar{\phi} = 0 \text{ in } B \text{ , and}$$

(II.4) $$\frac{\partial\bar{\phi}}{\partial n} = \overline{p\phi} \text{ on } B^* .$$

Remarks: We assume the existence of a minimizing function $\bar{\phi}$. It is possible to show that under the assumptions made about the boundary B^* , a minimizing function does exist. It can also be proved that the minimizing function $\bar{\phi}$ is C^2 in B . We shall have need to use this fact in our proof below.

<u>Proof of Theorem 1</u>. For an arbitrary function η which is continuous in $B \cup B^*$, C^1 in B and satisfies $\int_{B^*}\eta ds = 0$, the function

$$Q(t) = \frac{\int_B(\bar{\phi}' + t\eta')^2 dx}{\int_{B^*}(\bar{\phi} + t\eta)^2 ds}$$

has a minimum $\bar{p}$ at $t = 0$.

Therefore,

$$\int_B(\bar{\phi}' + t\eta')^2 dx - \bar{p}\int_{B^*}(\bar{\phi} + t\eta)^2 ds \geq 0.$$

This reduces to

$$\int_B(2t\bar{\phi}'\cdot\eta' + t^2\eta'^2)dx - \bar{p}\int_{B^*}(2t\bar{\phi}\eta + t^2\eta^2)ds \geq 0 ,$$

since $\int_B\bar{\phi}'^2 dx = \bar{p}\int_{B^*}\bar{\phi}^2 ds$.

Thus the function $F(t)$ given by

$$F(t) = \int_B(2t\bar{\phi}'\cdot\eta' + t^2\eta'^2)dx - \bar{p}\int_{B^*}(2t\bar{\phi}\eta + t^2\eta^2)\, ds$$

has a minimum 0 for $t = 0$.

Hence, $F'(0) = 0$, which gives

$$\int_B\bar{\phi}'\cdot\eta' dx - \bar{p}\int_{B^*}\bar{\phi}\eta ds = 0. \qquad \text{(II.5)}$$

By Gauss's divergence theorem,

$$\int_B \overline{\phi}' \cdot \eta' dx = \int_{B^*} (\eta \partial \overline{\phi}/\partial n) ds - \int_B \eta \Delta \overline{\phi} dx .$$

Substitution of this in (II.5) gives

$$\text{(II.6)} \qquad -\int_B \eta \Delta \overline{\phi} dx + \int_{B^*} (\partial \overline{\phi}/\partial n - \overline{p\phi}) \eta ds = 0 .$$

We claim that $\Delta \overline{\phi} = 0$ in B .

For, suppose $\Delta \overline{\phi} \neq 0$ at some point P in B . For definiteness, take $\Delta \overline{\phi}(P) > 0$. Since, as remarked earlier, $\overline{\phi}$ is C^2 in B , we have $\Delta \overline{\phi}$ is continuous in B . Thus there exists a neighbourhood $N(P)$ of P contained in B such that $\Delta \overline{\phi} > 0$ throughout $N(P)$. Then it is possible to construct a C^1 function η which is positive in $N(P)$ and zero elsewhere. (In particular, such an η vanishes on B^* and hence satisfies $\int_{B^*} \eta ds = 0$). For such an η , we have

$$-\int_B \eta \Delta \overline{\phi} dx + \int_{B^*} \eta (\partial \overline{\phi}/\partial n - \overline{p\phi}) ds = -\int_{N(P)} \eta \Delta \overline{\phi} dx < 0 ,$$

in contradiction to (II.6) .

Hence, $\Delta \overline{\phi} = 0$ everywhere in B .

To prove (II.4), we shall first establish a lemma.

<u>Lemma 1</u>. <u>Let f be a continuous function on a rectifiable curve</u> C . <u>If</u> $\int_C \eta f ds = 0$, <u>for all</u> η <u>such that</u> $\int_C \eta ds = 0$, <u>then</u> f <u>is a constant function on</u> C .

<u>Proof</u>. Let $\overline{\eta} = f - \overline{f}$, where $\overline{f} = \int_C f ds / \int_C ds$.

Then $\int_C \overline{\eta} ds = 0$.

Hence, $0 = \int_C \eta f ds = \int_C (f-\overline{f}) f ds = \int_C (f-\overline{f})^2 ds$,

since, $\int_C (f-\overline{f})\overline{f} ds = 0$.

Thus $f = \overline{f}$, a constant, throughout C . This proves the lemma.

Since we have proved that $\Delta\overline{\phi} = 0$ in B , the equation (II.6) now reduces to $\int_{B^*} \eta(\partial\overline{\phi}/\partial n - \overline{p\phi}) ds = 0$, for all η such that η is continuous on $B \cup B^*$, C^1 in B and $\int_{B^*} \eta ds = 0$.

Hence, by Lemma 1,

$\partial\overline{\phi}/\partial n - \overline{p\phi} = c$, a constant, on B^* .

By the divergence theorem,

$\int_{B^*} c ds = \int_{B^*} (\partial\overline{\phi}/\partial n - \overline{p\phi}) ds = \int_B \Delta\overline{\phi} dx - \overline{p}\int_{B^*} \overline{\phi} ds = 0$,

since $\Delta\overline{\phi} = 0$ and $\int_{B^*} \overline{\phi} ds = 0$.

Therefore, $c = 0$.

Thus, $\partial\overline{\phi}/\partial n = \overline{p\phi}$ on B^* , which completes the proof of (II.4) and Theorem 1.

To prove that $\overline{p} = p_2$, we also need the following lemma.

<u>Lemma 2</u>. $\overline{p} > 0$.

<u>Proof</u>. Suppose $\overline{p} = 0$. Let $\overline{\phi}$ be the corresponding minimizing function.

Then, $\int_B \overline{\phi}'^2 dx = 0$

Hence, $\overline{\phi}' = 0$ everywhere in B .

Therefore, $\overline{\phi}$ is constant in B and so on $B \cup B^*$.

Then $\int_{B^*}\bar{\phi}ds = 0$ gives $\bar{\phi} = 0$.

This , however, violates the conditions on $\bar{\phi}$ in the definition of $\bar{p}$.

Now we shall prove the equivalence of the eigenvalue and the variational formulations of p_2 .

<u>Theorem 2</u>. $\bar{p} = p_2$.

<u>Proof</u>. First we shall prove that if ϕ is an eigenfunction corresponding to p_2 , then

$$\int_{B^*}\phi ds = 0 \text{ and } \frac{\int_B \phi'^2 dx}{\int_{B^*}\phi^2 ds} = p_2 .$$

We have, since $\Delta\phi = 0$ in B and $\partial\phi/\partial n = p_2\phi$ on B^* ,

$$\begin{aligned} p_2\int_{B^*}\phi ds &= \int_{B^*}(\partial\phi/\partial n)ds \\ &= \int_B \Delta\phi dx \\ &= 0 . \end{aligned}$$

Therefore, $\int_{B^*}\phi ds = 0$, since $p_2 \neq 0$.

Also, we have, multiplying $\partial\phi/\partial n = p_2\phi$ through by ϕ and integrating on B^* ,

$$p_2\int_{B^*}\phi^2 ds = \int_{B^*}\phi(\partial\phi/\partial n)ds$$
$$= \int_B \phi'^2 dx + \int_B \phi\Delta\phi dx$$
$$= \int_B \phi'^2 dx$$

by Green's Theorem.

Note that $\phi|_{B^*} \not\equiv 0$, since otherwise it would follow that $\phi \equiv 0$ in $B \cup B^*$.

Hence, $\int_{B^*}\phi ds = 0$ and $p_2 = \dfrac{\int_B \phi'^2 dx}{\int_{B^*}\phi^2 ds}$.

Since $\bar{p}$ is the minimum of the values of p satisfying the latter relations, we have

$$\bar{p} \le p_2 .$$

Next, as shown in Theorem 1, $\bar{p}$ and the minimizing function $\bar{\phi}$ corresponding to it satisfy the differential equations (II.1) and (II.2).

Also, by Lemma 2, $\bar{p} \not\equiv 0$.

Hence, $p_2 \le \bar{p}$.

This proves that $\bar{p} = p_2$ and completes the proof of the equivalence of the two formulations.

Uses for the Variational Formulation

The variational formulation for p_2 allows us to demonstrate some uses for p_2 (or its lower bounds) in the Neumann problem.

If ϕ is any twice differentiable function in B

which is such that $\Delta\phi = 0$ in B and $\int_{B^*}\phi ds = 0$, then we have

$$\int_{B^*}\phi^2 ds \leq p_2^{-1}\int_B \phi'^2 dx .$$

Now, by Green's theorem,

$$\int_{B^*}\phi(\partial\phi/\partial n)ds = \int_B \phi'^2 dx + \int_B \phi\Delta\phi dx$$
$$= \int_B \phi'^2 dx$$

since $\Delta\phi = 0$ in B .

Hence, $\int_{B^*}\phi^2 ds \leq p_2^{-1}\int_{B^*}\phi(\partial\phi/\partial n)ds$

so that $\left[\int_{B^*}\phi^2 ds\right]^2 \leq p_2^{-2}\left[\int_{B^*}\phi(\partial\phi/\partial n)ds\right]^2$

$$\leq p_2^{-2}\left(\int_{B^*}\phi^2 ds\right)\left(\int_{B^*}(\partial\phi/\partial n)^2 ds\right),$$

by Schwarz's inequality.

Thus, we have,

$$(11.7) \qquad \int_{B^*}\phi^2 ds \leq p_2^{-2}\int_{B^*}(\partial\phi/\partial n)^2 ds .$$

In the Neumann problem for the Laplace-equation on the domain B , we consider a function ϕ satisfying

$$\Delta\phi = 0 \text{ , in } B$$

and $\partial\phi/\partial n = g$, on B^* , where g is a specified function. We can take $\int_{B^*}\phi ds = 0$.

The inequality (II.7) then gives an L_2-bound for ϕ on B^* in terms of p_2 (or any lower bound for p_2) .

In the Neumann problem for the Poisson-equation on the domain B we consider a function G satisfying

$$\Delta G = f \text{ , in } B$$

and

$$\partial G/\partial n = g \ , \text{ on } \ B^* \ ,$$

where f and g are specified functions.

We approximate G by a twice differentiable function Ψ on B such that

$$\Delta\Psi = f \ , \text{ in } \ B$$

and

$$\int_{B^*}\Psi ds = \int_{B^*}G ds \ .$$

G can be adjusted so that the later condition is satisfied.

With $\phi = G - \Psi$ in (II.7), we have, since $\Delta\phi = 0$ in B and $\int_{B^*}\phi ds = 0$,

$$\int_{B^*}(G-\Psi)^2 ds \le p_2^{-2}\int_{B^*}(\partial\phi/\partial n)^2 ds = p_2^{-2}\int_{B^*}(g-\frac{\partial\Psi}{\partial n})^2 ds$$

This gives an L_2 error-bound on B^* in terms of p_2 (or any lower-bound for p_2) .

III A SURVEY OF KNOWN RESULTS

Explicitly Soluble Systems

In this chapter, we first survey a few particular cases in which the eigenvalue p_2 of the Stekloff problem can be computed explicitly.

<u>A circular disc</u>: For a circular disc of radius r, the eigenvalues p of the Stekloff problem are given by

$$p = \frac{n}{r}\ , \ n = 0,1,2,3,\ \ldots$$

These can be computed by the method of separation of variables in polar coordinates.

In particular,

$$p_2 = \frac{1}{r}\ .$$

<u>An annulus</u>: Let B be an annulus with inner and outer radii r_1 and r_2 respectively. The eigenvalues p of the Stekloff problem are given by 0, $\dfrac{r_1 + r_2}{r_1 r_2 \log(r_2/r_1)}$,

and the roots of the quadratic equations

$$p^2 r_1 r_2 (r_2^{2n} - r_1^{2n}) - np(r_1+r_2)(r_1^{2n}+r_2^{2n}) + n^2(r_2^{2n}-r_1^{2n}) = 0\ ,$$

$n = 1,2,3,\ \ldots$

These also can be obtained by the method of separation of variables in polar coordinates.

<u>A rectangle</u>: Let B be a rectangle with sides of length $2a$, $2b$. The eigenvalues p of the Stekloff problem are given by

$\omega \cot \omega b$ where ω is a solution of $\tanh \omega a = \tan \omega b$ or $\cot \omega b$,

$-\omega \tan \omega b$ where ω is a solution of $\tanh \omega a = -\tan \omega b$ or $-\cot \omega b$,

and the similar forms obtained by interchange of a and b. For a square with side $2a$, $\frac{1}{a}$ is also an eigenvalue. These eigenvalues can be obtained by the method of separation of variables in Cartesian coordinates.

By a comparison of these eigenvalues, we obtain for $b > a$,

$$p_2 = \frac{\mu}{a} \cot \frac{\mu b}{a} ,$$

where μ is the smallest positive root of the equation

$$\tanh \mu = \cot \frac{\mu b}{a}$$

[3, p. 158; note that the statement contains a misprint]

Known Lower Bounds

The following results on lower bounds for the Stekloff eigenvalue p_2 have been obtained for certain types of domains.

<u>J. H. Bramble and L. E. Payne [1]</u>: Let B be simply-connected (the results extend to higher dimensions). Let

S be the interior of a sphere of radius a contained in B. Denote by R the region $B - \overline{S}$, where $\overline{S}$ is the closure of S.

Let u be any sufficiently smooth function in $B \cup B^*$ and let $\overline{f}$ be a sufficiently smooth vector-field defined in $\overline{R}$. Suppose that α is a positive function in R and $\overline{f}$ is such that

$$\overline{f} \cdot \overline{n} \geq K_1 > 0 \quad \text{on} \quad B^*$$

$$-\overline{f} \cdot \overline{n} \leq K_2 \quad \text{on} \quad S^*$$

and

$$\text{div.}\ \overline{f} + \frac{1}{\alpha}|\overline{f}|^2 \leq 0 \quad \text{in} \quad R$$

where $\overline{n}$ is the component of the unit normal directed outward on R^* and K_1 and K_2 are constants.

Let $K_3 = \max.\ (aK_2\ , \overline{\alpha})$, where $\overline{\alpha}$ is an upper bound for α in R.

Under these conditions,

$$p_2 \geq \frac{K_1}{K_3}\ .$$

<u>J. R. Kuttler and V. G. Sigillito [2]</u>: If B is star-shaped with respect to a point P which is taken as the origin for polar coordinates, then B^* is expressed by $r = R(\theta)$, $0 \leq \theta \leq 2\pi$.

In this case

$$p_{2k+1} \geq p_{2k} \geq \frac{k\left[1 - 2/\left(1 + \sqrt{1 + 4 \text{ min. } (R/R')^2}\right)\right]}{\text{max. } \sqrt{R^2 + R'^2}}$$

$k = 1,2,3, \ldots ,$

where $R' = \frac{dR}{d\theta}$.

Thus in particular,

$$p_2 \geq \frac{1 - 2/\left(1 + \sqrt{1 + 4 \text{ min. } (R/R')^2}\right)}{\text{max. } \sqrt{R^2 + R'^2}}$$

J. R. Kuttler and V. G. Sigillito [3]: If B is star-shaped with respect to the origin O , then

$$p_2 \geq \frac{\mu_2 h_{\text{min.}}}{2\left(1 + \mu_2^{1/2} \cdot r_{\text{max.}}\right)} ,$$

where $r(P)$ is the distance from P in B^* to the origin, $h(P)$ is the distance from the origin to the line tangent to B^* at P and μ_2 is the first non-zero eigenvalue of the free membrane problem for B given by

$$\Delta u + \mu u = 0 \text{ , in } B$$

and

$$\partial u/\partial n = 0 \text{ , on } B^* .$$

J. R. Kuttler and V. G. Sigillito [4]: Suppose B has two distinct perpendicular axes of symmetry, which are taken to be the x_1 and x_2 - axes.

Then,

$$p_2^{-1} \le \max_{i=1,2} \left[\max_{x_i} f_i(x_i)\left(1 + [f_i'(x_i)]^2\right)^{1/2} \right] ,$$

where $x_2 = \pm f_1(x_1)$ and $x_1 = \pm f_2(x_2)$ are representations for B^* .

<u>L. E. Payne [6]</u>: Let B be simply-connected. Then,

$$K_{max.} \ge p_2 \ge K_{min.} ,$$

where $K_{max.}$ and $K_{min.}$ are respectively the maximum and the minimum curvatures of B^* .

The equality signs hold if and only if the domain is the interior of a circle.

IV CONJECTURES FOR LOWER BOUNDS

First Conjecture

Since p_2 is given by

$$p_2 = \min. \frac{\int_B \phi'^2 dx}{\int_{B^*} \phi^2 ds}$$

where the minimum is taken over all functions ϕ continuous in $B \cup B^*$, C^1 in B and such that $\int_{B^*} \phi ds = 0$, considerations of dimensions suggest a lower bound for p_2 of the form $\frac{k}{D}$, where k is a constant and D is the diameter of the domain B.

Such a conjecture would be analogous to the following for μ_2, the first non-zero eigenvalue in the free membrane problem. This eigenvalue μ_2 has the variational characterization given by

$$\mu_2 = \min. \frac{\int_B \phi'^2 dx}{\int_B \phi^2 dx}$$

where the minimum is taken over all functions ϕ which are C^1 in B and are such that $\int_B \phi dx = 0$. Here, similar dimensional considerations suggest a lower bound of the form $\frac{k}{D^2}$ where k is a constant and D is the diameter of B. The lower bound π^2/D^2 was obtained by L. E. Payne and H. F. Weinberger [7] for convex n - dimensional domains.

The conjecture $p_2 \geq \frac{k}{D}$ is consistent with the value of p_2 for a circular disc which, as shown before, is given by $p_2 = \frac{2}{D}$.

Consider, however, this conjecture for a rectangle with sides $2a$ and $2b$ with $b \geq a$. Let $\theta = \tan^{-1}(\frac{b}{a})$.

As stated before, p_2 in this case is given by

$p_2 = \frac{\mu}{a} \cot\left(\frac{\mu b}{a}\right)$, where μ is the smallest positive root of the equation $\tanh\mu - \cot(\frac{\mu b}{a}) = 0$. Here, $D = 2a\sec\theta$, and hence $p_2 D = 2\mu\sec\theta\cot(\mu\tan\theta)$, where

$$\tanh\mu - \cot(\mu\tan\theta) = 0 .$$

Since $b \geq a$, we have, $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$.

Clearly, $\frac{\pi}{4\tan\theta} \leq \mu \leq \frac{\pi}{2\tan\theta}$.

Therefore, as $\theta \to \frac{\pi}{2} - 0$, $\mu \to 0$.

Hence, $\lim\limits_{\theta\to\pi/2 - 0} (p_2 D)^2$

$$= \lim_{\mu\to 0} \{2\mu\sec\theta\cot(\mu\tan\theta)\}^2$$

$$= \lim_{\mu\to 0} 4(\mu^2 + \mu^2\tan^2\theta)\tanh^2\mu \text{ , since}$$

$\cot(\mu\tan\theta) = \tanh\mu$.

Now, since $\frac{\pi}{4} \leq \mu\tan\theta \leq \frac{\pi}{2}$,

$$0 \leq 4(\mu^2 + \mu^2\tan^2\theta)\tanh^2\mu \leq 4(\mu^2 + \frac{\pi^2}{4}) \cdot \tanh^2\mu .$$

Hence, $\lim\limits_{\theta\to\pi/2-0} (p_2 D) = 0$.

This shows that a lower bound for p_2 of the form $\frac{k}{D}$ with k a positive constant is not possible.

Second Conjecture

A closer consideration of dimensions in the variational formulation for p_2 suggests a lower bound for p_2 of the form $\frac{kA}{D^2L}$, where k is a positive constant, A the area of the region B, and L the length of B^*.

This conjecture for a lower bound for p_2 agrees with the value of p_2 for a circular disc given before since in that case, we have

$$p_2 = \frac{2}{D} = \frac{8A}{D^2L} .$$

This requires that $k \leq 8$.

Next, consider the case of a rectangle with sides $2a$ and $2b$ with $b \geq a$. Again, if $\theta = \tan^{-1}(\frac{b}{a})$, we have, $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ and $D = 2a\sec\theta$, $A = D^2\sin\theta\cos\theta$ and $L = 2D\cdot(\sin\theta + \cos\theta)$. p_2 is, as before, given by $\frac{\mu}{a}\cot(\frac{\mu b}{a})$ where μ is the smallest positive root of $\tanh\mu - \cot(\frac{\mu b}{a}) = 0$.

We show that $\lim\limits_{\theta\to\pi/2-0} \frac{p_2 D^2 L}{A} = \pi^2$.

We have, as before

$$p_2 = \frac{2\mu\sec\theta\cot(\mu\tan\theta)}{D} ,$$

where μ is the smallest positive root of

$$\tanh\mu = \cot(\mu\tan\theta) \text{ , with } \frac{\pi}{4} \le \theta \le \frac{\pi}{2} \text{ and}$$

$$\frac{\pi}{4\tan\theta} \le \mu \le \frac{\pi}{2\tan\theta} .$$

Thus $\frac{\pi}{4} \le \cot^{-1}(\tanh\mu) \le \frac{\pi}{2}$ and $\mu \to 0$ as $\theta \to \frac{\pi}{2} - 0$.

Now, $\lim\limits_{\theta\to\pi/2-0} \frac{p_2 D^2 L}{A}$

$$= \lim_{\theta\to\pi/2-0} \frac{p_2 D^2 . 2D(\cos\theta + \sin\theta)}{A}$$

$$= 2 \lim_{\theta\to\pi/2-0} \frac{p_2 D^3}{A}$$

$$= 2 \lim_{\theta\to\pi/2-0} \frac{2\mu D^3 \sec\theta \cot(\mu\tan\theta)}{D\cdot D^2 \sin\theta\cos\theta}$$

$$= 4 \lim_{\theta\to\pi/2-0} \frac{\mu^3 \sec^3\theta \cdot \cot(\mu\tan\theta)}{\mu^2 \tan\theta}$$

$$= 4 \lim_{\mu\to 0} \frac{\left[\mu^2 + \{\cot^{-1}(\tanh\mu)\}^2\right]^{3/2}}{\cot^{-1}(\tanh\mu)} \cdot \left(\frac{\tanh\mu}{\mu}\right) ,$$

since $\mu\tan\theta = \cot^{-1}(\tanh\mu)$.

Since $\frac{\pi}{4} \le \cot^{-1}(\tanh\mu) \le \frac{\pi}{2}$, we have, as $\mu \to 0$, $\cot^{-1}(\tanh\mu) \to \frac{\pi}{2}$.

Also, $\lim\limits_{\mu\to 0} \left(\frac{\tanh\mu}{\mu}\right) = 1$.

Therefore, $\lim\limits_{\theta\to\pi/2-0} \frac{p_2 D^2 L}{A}$

$$= 4 \cdot \frac{\left[0 + \frac{\pi}{4}^2\right]^{3/2}}{\left(\frac{\pi}{2}\right)} . 1 = 4 . \left(\frac{\pi}{2}\right)^2 = \pi^2 ,$$

which is approximately 9.869.

The approximate values of $\frac{p_2 D^2 L}{A}$ can be computed for different values of θ between 45° and 90° and tabulated as follows:

θ	45°	60°	65°	67°	$67^\circ 30'$	68°	70°	80°	90°
$\frac{p_2 D^2 L}{A}$	11.01	8.39	8.083	8.076	8.074	8.113	8.225	8.54	9.869

All these values are consistent with the conjecture

$$p_2 \geq \frac{8A}{D^2 L} .$$

Conjecture for Multidimensional Domains

Analogous to the above conjecture for a lower bound for p_2 for two-dimensional domains, we can make a conjecture for a lower bound for p_2 for an n-dimensional domain B with boundary B^* .

Suggested again by considerations of dimensions in a variational formulation for p_2 , the analogous conjecture for an n-dimensional domain B would be of the form

$$p_2 \geq \frac{KV}{D^2 S} ,$$

where K is a positive constant, V the n-dimensional volume of B , S the measure of the hyper-surface B^* and D the diameter of the domain B.

The value of p_2 for an n-dimensional

spherical domain B of diameter D is given by

$$p_2 = \frac{2}{D}\text{ , [1, p. 825] .}$$

Since the volume V and the surface-area S of an n-dimensional sphere are connected by the relation

$$V = \frac{SD}{2n}$$

where D is the diameter of the sphere, we have,

$$p_2 = \frac{2}{D} = \frac{4nV}{D^2 S}\ ,$$

which agrees with the above conjecture and further, requires that $K \le 4n$.

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