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ON EXTREMAL
PARTITIONS OF GRAPHS

by

John A. Mitchem

A Dissertation
Submitted to the
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of the
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CHAPTER I

THE PARTITIONING PROBLEM

Section 1.1

Introduction

The most famous conjecture in graph theory and one of the most famous conjectures in all mathematics is the Four Color Conjecture. Perhaps the best known of several equivalent formulations of this conjecture is:

The chromatic number of any planar graph is at most four.

The chromatic number of a graph G is usually defined as the fewest number of colors needed to color the points of G so that no two adjacent points are colored the same. However, the chromatic number of G may also be defined as the minimum number of subsets in any partition of the point set of G so that each subset induces a totally disconnected subgraph (i.e., a subgraph with no lines). Thus, the chromatic number of a graph G may be described in terms of a minimum partition of the point set of G such that each set induces a graph with a given property; namely, the property of having no lines. By a variation of the property used, other graphical

parameters may be defined in the same way. Indeed, for a parameter f defined in this manner, a dual parameter \bar{f} may be defined as the maximum number of subsets in any partition of the point set of G so that each such subset induces a subgraph not having the prescribed property. Other parameters may be defined in terms of such "extremal" partitions of the line set of G rather than its point set.

The purpose of this thesis is to study minimal partitions of the point set of a graph such that each set in the partition induces a graph with a specified property. More precisely, we investigate parameters arising from the graphical properties "acyclic" and "outerplanar" together with generalizations of these parameters.

Section 1.2

Basic Definitions

We present here some basic definitions and establish some of the notation which will be used in this thesis. Those definitions not given here may be found in [11].

The point set of a graph G is denoted by $V(G)$, while its line set is denoted by $E(G)$. The order of a graph G , denoted $|G|$, is the number of elements

in $V(G)$. Graph G is called empty or totally disconnected if $E(G)$ is the empty set.

The degree, $\deg_G v$, of a point v of G is the number of lines of G incident with v . The subscript G will be dropped when it is apparent which graph is under consideration. The largest degree among the points of G is called the maximum degree of G and is denoted by $\Delta(G)$. Similarly, the minimum degree of G is denoted by $\delta(G)$.

The subgraph induced by a set U of points of G , denoted by $\langle U \rangle$, is that subgraph having U as its point set and whose line set consists of all lines of G incident with two points of U . A subgraph H of a graph G is called induced if $H = \langle U \rangle$ for some set U of points of G . The notation $H < G$ is used to indicate that H is an induced subgraph of G . A set W of points of G is independent if $\langle W \rangle$ is totally disconnected.

There are special classes of graphs which we will frequently encounter. An acyclic graph or forest is a graph with no cycles. A planar graph is one which can be embedded in the plane. A (planar) graph which can be embedded in the plane so that each of its points lies in the boundary of the exterior region is called outerplanar. It is easily shown that any outerplanar graph of order at least

two has two or more points of degree less than three. Also, if G is outerplanar with p points and q lines, then $q \leq 2p-3$.

Two graphs are said to be disjoint if their point sets are disjoint. The union of two graphs G_1 and G_2 is the graph G defined by $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If G_1 and G_2 are disjoint graphs, $\sum_1^2 G_i = G_1 + G_2$ is the graph which consists of $G_1 \cup G_2$ and all lines joining $V(G_1)$ with $V(G_2)$. For $n \geq 3$, let G_1, \dots, G_n be mutually disjoint graphs. Then $\sum_1^3 G_i = (G_1 + G_2) + G_3$ and, in general, $\sum_1^n G_i$ is defined as $\left(\sum_1^{n-1} G_i \right) + G_n$.

The complement \bar{G} of the graph G is a graph which has $V(G)$ as its point set, and two points are adjacent in \bar{G} if and only if these two points are not adjacent in G . The complete graph K_p has every pair of its p points adjacent. A triangle is a subgraph isomorphic to K_3 .

For $n \geq 1$, the complete n-partite graph $K(p_1, \dots, p_n)$ where $1 \leq p_1 \leq \dots \leq p_n$ is the graph $\sum \bar{K}_{p_i}$. The sets $V(\bar{K}_{p_1}), \dots, V(\bar{K}_{p_n})$ are called the partite sets of $K(p_1, \dots, p_n)$.

If v is a point of G , then $G-v$ denotes the graph $\langle V(G) - \{v\} \rangle$ and, in general, if S is a proper subset of $V(G)$, then $G-S$ represents the graph $\langle V(G) - S \rangle$. Let G be connected and S be a set of

points of G . Then S is a cutset of G if $G-S$ is a disconnected graph. A graph G is said to be k -connected if for every integer m , $0 \leq m < k$, the removal of any set of m points of G results in a connected graph with order larger than one.

A partition V_1, \dots, V_n of the point set of a graph G is said to have property P if, for $i = 1, \dots, n$, the subgraph $\langle V_i \rangle$ has property P . Thus, for example, V_1, \dots, V_n is an outerplanar partition of $V(G)$ means that, for $i = 1, \dots, n$, the set V_i induces an outerplanar subgraph of G .

A subdivision of a graph H is a graph G_1 obtained from H by replacing a line uv of H by a new point w and new lines uw and wv . A graph G is said to be homeomorphic from a graph H if $G = H$ or if G can be obtained from H by a sequence of subdivisions.

Finally, throughout the thesis, the symbol $//$ will indicate the end of a proof.

CHAPTER II

ACYCLIC PARTITIONS

In this chapter we consider the concept of point-arboricity and present results dealing with this parameter which are analogues to theorems on chromatic number.

Section 2.1

The Point-Arboricity of a Graph

In [20] Renyi defined the arboricity of a graph as the minimum number of subsets into which the line set of G can be partitioned so that the subgraph induced by each subset is acyclic. Nash-Williams [18] developed a formula which gives the arboricity of any graph.

The point analogue of arboricity is called point-arboricity and is defined as the minimum number of subsets into which the point set of G may be partitioned so that each subset induces an acyclic graph. Equivalently, the point-arboricity of G may be defined as the fewest number of colors needed to color the points of G so that no cycle of G has all of its points colored the same. This term

was introduced by Chartrand, Geller, and Hedetniemi in [4], although the concept was considered by Motzkin in [17]. We denote the point-arboricity of graph G by $f_2(G)$ and observe that the point-arboricity of any graph is the maximum point-arboricity of its components.

As with chromatic number, there is no explicit formula for the point-arboricity of a graph. The value of this parameter has been established for several classes of graphs, however. For example, Chartrand, Kronk, and Wall [7] have determined the point-arboricity of all complete n -partite graphs.

Theorem 2.1A. (Chartrand, Kronk, Wall) Let $G = K(p_1, p_2, \dots, p_n)$ be a complete n -partite graph. Then $f_2(G) = n - \max\{k: \sum_{i=0}^k p_i \leq n-k\}$ where we define $p_0 = 0$.

There are a number of bounds which have been found for the point-arboricity of any graph. In order to state these we define an additional term. A graph G is k -critical with respect to point-arboricity if $f_2(G) = k$ and $f_2(G-v) = k-1$ for all points v in G . A lower bound for the minimum degree of a k -critical graph can now be given.

Theorem 2.1B. (Chartrand, Kronk) If graph G is k -critical with respect to point-arboricity, $k \geq 2$, then $\delta(G) \geq 2(k-1)$.

The preceding result was established in [6] as were the following two theorems, the second of which is a corollary of the first.

Theorem 2.1C. (Chartrand, Kronk) For any graph G , $f_2(G) \leq 1 + \left\lceil \frac{\max \delta(G')}{2} \right\rceil$ where the maximum is taken over all induced subgraphs G' of G .

Theorem 2.1D. (Chartrand, Kronk) For any graph G , $f_2(G) \leq 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

By replacing the word "acyclic" in the definition of point-arboricity by "totally disconnected", we define the more well-known parameter of chromatic number. Formally, we define the chromatic number of a graph G , denoted $f_1(G)$, as the minimum number of subsets into which the point set of G can be partitioned so that each subset induces a totally disconnected graph. The preceding three theorems might all be considered as natural analogues of results on chromatic number.

An upper bound for $f_1(G)$ of a different nature was developed by Wilf [25]. In order to consider this result, we define the adjacency matrix $A = (a_{ij})$ of a graph G

with p points, v_1, \dots, v_p , as the $p \times p$ matrix in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. With this definition Wilf showed that any graph G has chromatic number at most $1 + \epsilon(G)$, where $\epsilon(G)$ is the maximum eigenvalue of the adjacency matrix of G . The proof uses the theorems concerning eigenvalues of the adjacency matrix of a graph which are given in the following table. These results can be developed from properties of eigenvalues given by Wilf in [24] and by Marcus and Minc in [16]. They are specifically listed by Behzad and Chartrand in [1].

TABLE I

Theorem i. The eigenvalues of the adjacency matrix of a graph are real numbers, and $\epsilon(G)$ is non-negative.

Theorem ii. Let G' be a spanning subgraph of G , then $\epsilon(G') \leq \epsilon(G)$.

Theorem iii. If A is the adjacency matrix of a graph G of order p , then
$$\epsilon(G) = \max_{(x,x)} \frac{(Ax,x)}{(x,x)}$$
 where the maximum is taken over all non-zero real vectors x with p entries and (y,z) denotes the inner product of y and z .

* * * * *

Using Theorem 2.1B and the technique employed by Wilf, we verify the following theorem for point-arboricity which serves as an analogue to the theorem of Wilf.

Theorem 2.1E. For any graph G , $f_2(G) \leq 1 + \left\lceil \frac{\epsilon(G)}{2} \right\rceil$.

Proof. If the point-arboricity of G is one, then Theorem 1 implies the required inequality. Let $f_2(G) = k \geq 2$ and G' be any induced k -critical subgraph of G . (For example, we may take G' to be an induced subgraph of minimum order for which $f_2(G') = k$.) Denote by $A = (a_{ij})$ and $A' = (a'_{ij})$ the $p \times p$ and $p' \times p'$ adjacency matrices of G and G' , respectively. Furthermore, let A^* be the $p \times p$ matrix obtained from A by replacing those rows and columns corresponding to points deleted from G in obtaining G' by zero rows and columns, respectively. This implies that the eigenvalues of A^* are those of A' plus an additional $p-p'$ zeros. Theorem 1 implies that $\epsilon(G') = \epsilon(A^*)$. However, A^* is the adjacency matrix of a spanning subgraph consisting of G' along with $p-p'$ isolated points of G . According to Theorem 1i,

$$(2.1) \quad \epsilon(G') = \epsilon(A^*) \leq \epsilon(G).$$

Now consider the p' -vector $x = (1, 1, \dots, 1)$. From Theorem 111, we have

$$(2.2) \quad \epsilon(G') = \max_{y \neq 0} \frac{(A'y, y)}{(y, y)} \geq \frac{(A'x, x)}{p'} = \frac{\sum_{j=1}^{p'} \left(\sum_{i=1}^{p'} a_{ij} \right)}{p'}.$$

The last expression in (2.2) is the average of the row sums of matrix A' . Hence the minimum row sum of A' does not exceed this number. Since G' is k -critical, Theorem 2.1B implies $\delta(G') \geq 2k-2$ and thus the minimum row sum in A' is at least $2k-2$. Hence from (2.1) and (2.2), $\epsilon(G) \geq \epsilon(G') \geq 2k-2$, which gives us the desired inequality. //

Wilf [25] showed that $\epsilon(K_p) = p-1$. This implies that $f_2(K_{2k+1}) = 1+k = 1 + \left\lceil \frac{\epsilon(K_{2k+1})}{2} \right\rceil$ which shows that there is an infinite class of graphs for which the bound in Theorem 2.1E is obtained.

Section 2.2

Point-Arboricity of a Graph and Its Complement

Since there is no formula for the chromatic number of an arbitrary graph, there is no general formula for the chromatic number of the complement of a graph. Nordhaus and Gaddum [19], however, have investigated $f_1(G) + f_1(\bar{G})$ and $f_1(G) \cdot f_1(\bar{G})$ and have obtained sharp bounds in terms of the order of the graph.

Theorem 2.2A. (Nordhaus and Gaddum) If G is a graph with p points, then

$$2\sqrt{p} \leq f_1(G) + f_1(\overline{G}) \leq p+1,$$

$$p \leq f_1(G) \cdot f_1(\overline{G}) \leq \left(\frac{p+1}{2}\right)^2.$$

In this section we obtain an analogous result for point-arboricity. We begin with the following proposition which compares the chromatic number of a graph with its point-arboricity.

Proposition 2.2B. For any graph G ,

$$\frac{f_1(G)}{2} \leq f_2(G) \leq f_1(G).$$

Proof. Since $V(G)$ can be partitioned into $f_1(G)$ sets such that each set induces a totally disconnected graph, each of these sets also induces an acyclic graph. Thus $f_2(G) \leq f_1(G)$.

There is a partition of the point set of G into $f_2(G)$ sets such that each set induces an acyclic subgraph of G . Since the chromatic number of an acyclic graph never exceeds two, $f_1(G) \leq 2f_2(G)$. //

We observe that, for any positive integer m , there exist graphs G and H with point-arboricity m such that

$f_1(G) = m$ and $f_1(H) = 2m$. We may take H to be the graph K_{2m} and G to be the complete m -partite graph $K(2m, 2m, \dots, 2m)$. Clearly the chromatic number of G is m . In order to see that $f_2(G) = m$, we assume $f_2(G) \leq m-1$. Since $(2m+2)(m-1) = 2m^2 - 2$, which is less than the order of G , any partition of $V(G)$ into $m-1$ sets includes a set S with at least $2m+2$ points. But this set induces a graph with a cycle. Thus the bounds in Proposition 2.2B are the best possible in the sense that, for each bound and for any positive integer m , there is a graph G with $f_2(G) = m$ such that equality holds for that bound.

We are now in a position to prove the aforementioned analogue to the Nordhaus-Gaddum Theorem.

Theorem 2.2C. Let G be any graph of order p . Then

$$(2.3) \quad \sqrt{p} \leq f_2(G) + f_2(\overline{G}) \leq \frac{p+3}{2} \quad \text{and}$$

$$(2.4) \quad \frac{p}{4} \leq f_2(G) \cdot f_2(\overline{G}) \leq \left(\frac{p+3}{4} \right)^2.$$

Proof. From Proposition 2.2B we have

$$(2.5) \quad \frac{f_1(G)}{2} \leq f_2(G) \quad \text{and} \quad \frac{f_1(\overline{G})}{2} \leq f_2(\overline{G}).$$

According to Theorem 2.2A

$$(2.6) \quad 2\sqrt{p} \leq f_1(G) + f_1(\bar{G}) \quad \text{and}$$

$$(2.7) \quad p \leq f_1(G) \cdot f_1(\bar{G}).$$

Thus, substituting the inequalities of (2.5) into (2.6) and (2.7), we obtain the left side of (2.3) and (2.4) respectively.

We use induction to verify the right inequality of (2.3). Clearly this inequality holds for $p = 1$ or 2 , so assume that it holds for all graphs with fewer than p points, $p \geq 3$. Let H be a graph of order p . It is easily verified that every graph of order two or more contains two distinct points of equal degree. Let u and v be two such points in H , say with $\deg u = \deg v = d$.

Let $G = H - u - v$ so that $\bar{G} = \bar{H} - u - v$. By the induction hypothesis

$$(2.8) \quad f_2(G) + f_2(\bar{G}) \leq \frac{p+1}{2}.$$

Since any two points of a graph induce an acyclic subgraph, we have

$$(2.9) \quad f_2(H) \leq f_2(G) + 1 \quad \text{and}$$

$$(2.10) \quad f_2(\bar{H}) \leq f_2(\bar{G}) + 1.$$

If strict inequality holds in either (2.9) or (2.10), then

$$f_2(H) + f_2(\bar{H}) < f_2(G) + f_2(\bar{G}) + 2.$$

This, together with (2.8), implies that

$$f_2(H) + f_2(\bar{H}) \leq f_2(G) + f_2(\bar{G}) + 1 \leq \frac{p+3}{2},$$

which is the desired result.

Suppose then that equality holds in both (2.9) and (2.10). Let $f_2(G) = r$ and V_1, \dots, V_r be an acyclic partition of $V(G)$. From (2.9) we conclude that either adding u to G or v to $G+u$ increases the point-arboricity by one. Thus either u is adjacent to two points in each V_i , $i = 1, 2, \dots, r$ or v is adjacent to two points in each set of any acyclic partition of $V(G+u)$ into r sets. Hence,

$$(2.11) \quad d = \deg_H u = \deg_H v \geq 2f_2(G).$$

Similarly (2.10) implies that

$$(2.12) \quad p-d-1 = \deg_{\bar{H}} u = \deg_{\bar{H}} v \geq 2f_2(\bar{G}).$$

Adding (2.11) and (2.12), we obtain

$$p-1 \geq 2f_2(G) + 2f_2(\bar{G}).$$

Thus,

$$f_2(H) + f_2(\bar{H}) = f_2(G) + f_2(\bar{G}) + 2 \leq \frac{p-1}{2} + 2 = \frac{p+3}{2},$$

which is the right inequality of (2.3).

In order to show the right inequality of (2.4), we recall that the geometric mean never exceeds the arithmetic mean. Thus,

$$\left(f_2(H) \cdot f_2(\bar{H}) \right)^{1/2} \leq \frac{f_2(H) + f_2(\bar{H})}{2} = \frac{p+3}{4},$$

which yields the desired result and completes the proof. //

Finck [10] and Stewart [21] have shown independently that for any pair of integers k, k' such that $k+k' \leq p+1$ and $p \leq kk'$, there is a graph G with p points such that $f_1(G) = k$ and $f_1(\bar{G}) = k'$. We note that, if k and k' satisfy these two inequalities, they satisfy all four inequalities of Theorem 2.2A. In order to prove a result for point-arboricity analogous to that of Finck and Stewart, we verify three lemmas.

Lemma 2.2D. If G is a path of order p , then

$$f_2(\bar{G}) = \left\lceil \frac{p}{4} \right\rceil.$$

Proof. Let G be a path with p points. If $p < 5$, then \bar{G} is acyclic and $f_2(\bar{G}) = 1 = \left\lceil \frac{p}{4} \right\rceil$. Suppose then

that G is the path v_1, v_2, \dots, v_p , $p \geq 5$. Partition $V(\overline{G})$ into $m = \left\lfloor \frac{p}{4} \right\rfloor$ subsets as follows:

$$V_1 = \{v_1, v_2, v_3, v_4\}$$

$$V_2 = \{v_5, v_6, v_7, v_8\}$$

.

.

.

$$V_m = \{v_{4(m-1)+1}, \dots, v_p\}.$$

Since each V_i induces an acyclic subgraph of \overline{G} , $f_2(\overline{G}) \leq \left\lfloor \frac{p}{4} \right\rfloor$. Suppose $f_2(\overline{G}) < \left\lfloor \frac{p}{4} \right\rfloor$, then at least one set S in the acyclic partition of $V(\overline{G})$ has more than four points. Let T be a subset of S with exactly five points. Then $\langle T \rangle$, a subgraph of G , has at most four lines. This implies that $\overline{\langle T \rangle}$, a subgraph of \overline{G} , has at least six lines. Thus $\overline{\langle T \rangle}$ has a cycle which implies $\overline{\langle S \rangle}$ contains a cycle in \overline{G} . From this contradiction it follows that $f_2(\overline{G}) = \left\lfloor \frac{p}{4} \right\rfloor$. //

Lemma 2.2E. Let G_1, G_2, \dots, G_m be mutually disjoint paths, where G_1 has $k \geq 2$ points and each of G_2, \dots, G_m has at least two and at most k points. If $G = \sum_{i=1}^m G_i$, then

$$i. \quad f_2(G) = m \quad \text{and}$$

$$ii. \quad f_2(\overline{G}) = \left\lfloor \frac{k}{4} \right\rfloor.$$

Proof. We first verify i. Since each of the m paths is acyclic, $f_2(G) \leq m$. Let H be a subgraph of G induced by a set which consists of two adjacent points from each of the G_i . Then H is the complete graph on $2m$ points so that $f_2(G) \geq f_2(H) = m$, which yields the desired equality.

In order to prove ii., we see that if $k = 2$ or $k = 3$, then \bar{G} has no cycles. If $k > 3$, then \bar{G} has at least m components. Let H be a component of \bar{G} with a maximum number of lines. Then H is the complement of a path with k points and according to Lemma 2.2D, $f_2(H) = \left\lfloor \frac{k}{4} \right\rfloor$. But the point-arboricity of a graph is the maximum of the point-arboricity of its components, so $f_2(\bar{G}) = \left\lfloor \frac{k}{4} \right\rfloor$. //

Lemma 2.2F. Let a be a fixed positive integer, $g(a') = 2aa' + a'$, and $h(a') = 4a + 2a' - 2$. For all integral values of a' larger than one, $h(a') \leq g(a')$.

Proof. We use induction and observe $h(2) = g(2)$. Assume $h(a') \leq g(a')$ for $a' = q-1$, $q \geq 3$. Thus

$$4a + 2q - 4 = h(q-1) \leq g(q-1) = 2qa - 2a + q - 1.$$

This implies that

$$h(q) = 4a + 2q - 2 \leq 2qa - 2a + 2 + q - 1 = 2qa - 2(a-1) + q - 1.$$

But $(a-1)$ is non-negative, so that

$$2qa-2(a-1)+q-1 \leq 2qa+q-1 < 2qa+q = g(q)$$

which yields the desired inequality. //

Theorem 2.2G. For any triple of positive integers

a, a', p such that

$$\text{i.} \quad a+a' \leq \frac{p+3}{2} \quad \text{and}$$

$$\text{ii.} \quad \frac{p}{4} \leq a \cdot a',$$

there exists a graph G of order p such that

$$f_2(G) = a \quad \text{and} \quad f_2(\overline{G}) = a'.$$

Proof. Without loss of generality we suppose $a' \leq a$ and consider a number of different cases.

Case i. $2a+2a' = p+3$. Let $G_1 = K_{2a-1}$ and $G_i = K_1$ for $i = 2, \dots, 2a'-1$ be mutually disjoint graphs. Denote the union of these $2a'-1$ graphs by G . Then G has $2a+2a'-3 = p$ points, $f_2(G) = f_2(G_1) = f_2(K_{2a-1}) = a$, and $f_2(\overline{G}) = f_2(K_{2a'-1}) = a'$.

Case ii. $2a+2a' = p+2$. Let G be the union of the $2a'-1$ mutually disjoint graphs G_i , where $G_1 = K_{2a}$ and $G_i = K_1$ for $i = 2, \dots, 2a'-1$. Then G has $2a+2a'-2 = p$ points, $f_2(G) = a$, and $f_2(\overline{G}) = a'$.

Case iii. $2a+2a' \leq p+1$ and $p \leq 2aa'+a'$. We form the following mutually disjoint graphs. For $i = 1, 2, \dots, a'$

let $G_1 = K_1$, let $G_{2a'} = K_{2a'}$, and for $i = a'+1, \dots, 2a'-1$ let G_i be a complete graph with at least one point and at most $2a'$ points such that exactly p points are used in these $2a'$ graphs. This is possible since $2a+2a'-1 \leq p$ and $2aa'+a' \geq p$.

Denote the union of these $2a'$ graphs by G . Then $f_2(G) = f_2(G_{2a'}) = a$ and $\bar{G} = K(p_1, p_2, \dots, p_{2a'})$ where $p_i = 1$ for $i = 1, \dots, a'$. Hence, according to Theorem 2.1A, $f_2(\bar{G}) = 2a' - \max\{k: \sum_{i=1}^k p_i \leq 2a' - k\} = 2a' - a' = a'$.

Case iv. $2a+2a' \leq p+1$ and $p > 2aa'+a'$.

Suppose $a' \geq 2$. We form mutually disjoint graphs $G_1, \dots, G_{a'}$, where G_1 is a path with $4a$ points and $G_2, \dots, G_{a'}$ are paths with at most $4a$ points and at least 2 points. In this way we use at most $4aa'$ points, and by inequality ii. in the hypothesis of the theorem $4aa' \geq p$. Also we use at least

$$4a+(a'-1)2 = 4a+2a'-2 = h(a')$$

points and by Lemma 2.2F

$$h(a') \leq g(a') = 2aa'+a' < p,$$

the last inequality being the hypothesis for this case.

Thus $G_1, \dots, G_{a'}$ can be chosen so that exactly p points are used.

Let $\bar{G} = \sum_1^{a'} G_i$, then Lemma 2.2E implies $f_2(\bar{G}) = a'$
 and $f_2(\bar{\bar{G}}) = f_2(G) = \left\{ \frac{4a}{4} \right\} = a$.

Assume $a' = 1$, then we have $2a+2 \leq p \leq 4a$. Let G_1 be a path with four points, G_2, \dots, G_a be paths with two, three, or four points such that the G_i are mutually disjoint. Since this procedure uses at most $4a \geq p$ points and at least $4+(a-1)2 \leq p$ points, we can choose G_i such that exactly p points are used. Denote $\sum_1^a G_i$ by G , then according to Lemma 2.2E $f_2(G) = a$ and $f_2(\bar{G}) = 1 = a'$. Thus, in all cases the theorem is proved. //

This theorem can now be used to show that each of the bounds in Theorem 2.2C is the best possible for infinitely many values of p .

Corollary 2.2H. For any positive integer p there are graphs G and H with p points such that

$$\text{i.} \quad f_2(G) + f_2(\bar{G}) = \left\lfloor \frac{p+3}{2} \right\rfloor \quad \text{and}$$

$$\text{ii.} \quad f_2(H) \cdot f_2(\bar{H}) = \left\lfloor \frac{p}{4} \right\rfloor.$$

Proof. For equation i., we let $a = \left\lfloor \frac{p+3}{2} \right\rfloor - 1$ and $a' = 1$,
 then $a \cdot a' = \left\lfloor \frac{p+1}{2} \right\rfloor \geq \left\lfloor \frac{p}{4} \right\rfloor$. Thus by Theorem 2.2G there is a graph G with p points such that $f_2(G) = a$ and $f_2(\bar{G}) = a'$.

In order to show ii., we let $a = \left\{ \frac{p}{4} \right\}$ and $a' = 1$.
 Then $a+a' = \left\{ \frac{p}{4} \right\} + 1 = \left\{ \frac{p+4}{4} \right\} \leq \left\lceil \frac{p+3}{2} \right\rceil$. Again, by applying
 Theorem 2.2G, there exists a graph H with $f_2(H) = \left\{ \frac{p}{4} \right\}$
 and $f_2(\bar{H}) = 1$, so that ii. is satisfied. //

Corollary 2.2I. There are infinite sets P_1 and P_2 of
 integers with the property that, for every p in P_1
 there is a graph G such that $f_2(G) + f_2(\bar{G}) = \sqrt{p}$, and for
 every p in P_2 there is a graph H such that

$$f_2(H) \cdot f_2(\bar{H}) = \left(\frac{p+3}{4} \right)^2.$$

Proof. Let $P_1 = \{p: p = 4n^2, n = 1, 2, 3, \dots\}$. Then for
 any p in P_1 let $a = a' = \frac{\sqrt{p}}{2}$, which is a positive
 integer. Then a and a' satisfy inequalities i. and ii.
 in Theorem 2.2G, and hence there is a graph G with p
 points such that $f_2(G) = \frac{\sqrt{p}}{2} = f_2(\bar{G})$.

We define P_2 as the set $\{p: p = 4n+1, n = 1, 2, \dots\}$.
 For any p in P_2 let $a = \frac{p+3}{4} = n = a'$. Then a and a'
 satisfy the inequalities of Theorem 2.2G, so there is a
 graph H with p points such that $f_2(H) = \frac{p+3}{4} = f_2(\bar{H})$. //

Section 2.3

Uniquely k -Arborable Graphs

Previously we defined the chromatic number, $f_1(G)$, of a graph G as the minimum number of subsets into which the point set of G can be partitioned so that each subset induces a totally disconnected graph. If $f_1(G) = k$ and there is only one totally disconnected partition of $V(G)$ into k sets, then G is said to be uniquely k -colorable. The properties of uniquely k -colorable graphs have been investigated by Cartwright and Harary in [2], by Harary, Hedetniemi and Robinson in [12], and by Chartrand and Geller in [3].

Analogously we define a graph G to be uniquely k -arborable if $f_2(G) = k$ and there is only one acyclic partition of G into k sets.

We note that every acyclic graph is uniquely 1-arborable. On the other hand, no disconnected graph G with cycles is uniquely k -arborable where $k = f_2(G)$; that is, every uniquely k -arborable graph, $k \geq 2$, is connected. In fact, we will show that a much stronger statement can be made with regard to the connectivity of a uniquely k -arborable graph.

Two additional remarks can be made. Let V_1, \dots, V_k be the acyclic partition of a uniquely k -arborable graph G . Then for each $i = 1, \dots, k$, each point v in V_i is

adjacent to at least two points of V_j , $j \neq i$, because otherwise, joining v with set V_j would form a second acyclic partition of $V(G)$ into k or fewer sets. This implies that $\delta(G) \geq 2k-2$ for a uniquely k -arborable graph G and each V_i , $i = 1, 2, \dots, k$, has at least two points.

We now prove that, to verify a graph G as uniquely k -arborable, it is sufficient to show that there is a unique acyclic partition of the point set of G into k sets.

Proposition 2.3A. Let G be a graph of order p and let $1 \leq k < p$. Then G is uniquely k -arborable if and only if there is a unique acyclic partition of $V(G)$ into k sets.

Proof. The necessity is immediate from the definition of uniquely k -arborable graphs, so we need consider only the sufficiency.

Suppose there is a unique acyclic partition of $V(G)$ into k sets. This implies that $f_2(G) \leq k$. Assume $f_2(G) = m \leq k$, and let V_1, \dots, V_m be an acyclic partition of $V(G)$. At most one of the V_i , say V_m , has less than two points, for otherwise the union of two one-point sets would be acyclic and the point-arboricity would be less than m . We now form an acyclic partition of $V(G)$ into k sets. Form singleton subsets of V_1 until either V_1 has been

completely partitioned into sets with one element or the sets V_i , $i > 1$, together with the sets in the partition of V_1 form exactly k sets. If the former occurs, we partition V_2 into single-element subsets until either V_2 has been completely partitioned into sets with one element or the partitions of V_1 and V_2 together with sets V_3, \dots, V_m form a partition of $V(G)$ into k sets. Continuing in this way we obtain an acyclic partition of $V(G)$ into sets W_1, W_2, \dots, W_k . Since $m < k$, at least one set, say W_1 , contains only a single element. Also, $k < p$ implies that at least one set, say W_2 , contains at least two elements.

From the partition W_1, \dots, W_k , we form another acyclic partition by adding v , an element of W_2 , to set W_1 . Thus we have two distinct acyclic partitions of $V(G)$ into k sets. This contradicts the hypothesis and implies that $f_2(G) = k$. //

Any graph of the form $K(1, n)$ is called a star.

The next proposition shows that uniquely 2-arborable graphs exist. We shall then employ this result to obtain a sufficient condition for a graph to be uniquely k -arborable, $k \geq 2$.

Proposition 2.3B. Let G_1 and G_2 be disjoint, non-empty forests having order at least three. If the graph G_2 with its isolated points deleted is not a star, then $G = G_1 + G_2$ is uniquely 2-arborable.

Proof. We note that $f_2(G) = 2$ because G has a cycle and the partition of $V(G)$ into sets $V(G_1)$ and $V(G_2)$ is an acyclic partition of G . We assume there is another partition of $V(G)$ into sets W_1 and W_2 such that each induces an acyclic graph. Any set with two points in both $V(G_1)$ and $V(G_2)$ induces a graph which contains a cycle on four points. Thus for $i = 1, 2$, W_i does not have two points in both $V(G_1)$ and $V(G_2)$. Furthermore, if one of the W_i , say W_1 , is contained in but not equal to one of the sets $V(G_j)$, $j = 1, 2$, then W_2 contains all of the other set $V(G_j)$ along with at least one other point. Thus W_2 induces a graph which contains a triangle.

Hence, we can now assume, without loss of generality, that for $i = 1, 2$, W_i contains exactly one point v_i of $V(G_i)$. That is, $W_2 = \left(V(G_1) \cup \{v_2\} \right) - \{v_1\}$ and $W_1 = \left(V(G_2) \cup \{v_1\} \right) - \{v_2\}$. However, G_2 does not have all its lines incident with a single point so there are points v and w both different from v_2 which are adjacent in G_2 . Thus the set $\{v_1, v, w\}$ induces a triangle in $\langle W_1 \rangle$ which contradicts the fact that $\langle W_1 \rangle$ is acyclic. //

Theorem 2.3C. For $k \geq 1$, let $G_i = (V_i, E_i)$, $i = 1, 2, \dots, k$, be mutually disjoint non-empty forests of order at least three. If, for $i = 2, 3, \dots, k$, the deletion of the isolated points of G_i does not result in a star, then $G = \sum_{i=1}^k G_i$ is uniquely k -arborable.

Proof. We use induction on k . Proposition 2.3B establishes the result for $k = 2$. Assume that any graph which is the join of $k-1$ forests having the properties given in the hypothesis of the theorem is uniquely $(k-1)$ -arborable. Let G be as given in the theorem; then by the inductive assumption $H_i = G - G_i$ is uniquely $(k-1)$ -arborable for $i = 1, 2, \dots, k$.

We assume that there are two acyclic partitions of $V(G)$ into k sets and show that this leads to a contradiction. One acyclic partition is V_1, \dots, V_k , and let W_1, \dots, W_k be another acyclic partition of $V(G)$. Now, we verify three remarks on the relationship between the sets in each partition and use these observations to complete the proof of the theorem.

Remark A. For every pair of sets V_j, W_i , the set V_j is not contained in W_i and W_i is not contained in V_j . In order to prove this, we assume the contrary and consider three cases.

Case i. There exist integers s and r such that

$V_s = W_r$. Then $\{W_i\}_{i \neq r}$ is a second acyclic partition of H_s , which contradicts our inductive assumption.

Case ii. There exist integers s and r such that V_s is contained in but not equal to W_r . Then there is a point v in $W_r - V_s$. Since $\langle V_s \rangle$ contains a line, $\langle W_r \rangle$ has a cycle, which is impossible.

Case iii. There exist integers s and r such that W_r is contained in but not equal to V_s . In this case, if $\{W_i \cap H_s\}_{i \neq r}$ forms a second acyclic partition of H_s , we have a contradiction to the induction assumption.

Otherwise, $\{W_i \cap H_s\}_{i \neq r} = \{V_j\}_{j \neq s}$ which implies some set V_j is contained in but not equal to some W_i . However, Case ii. shows this is impossible.

In all cases we have a contradiction, and thus Remark A is verified. Each set W_i therefore contains points from two different V_j , and each set V_j contains points from two distinct W_i . Clearly no set W_i has points from three different V_j and no W_i contains two points from each of two distinct V_j . Thus each W_i contains exactly one point from one V_j and all of its other points are from another V_j .

Remark B. No set V_j has points in three different W_i .

In order to establish this, we assume that V_r , for some

integer r , has points in three different W_i . For $j \neq r$, Remark A implies V_j has points in at least two different W_i . Thus there are at least $2k+1$ non-empty intersections of the sets V_j with the W_i as i, j take on all distinct values $1, 2, \dots, k$. However, each W_i intersects exactly two V_j so that there are precisely $2k$ non-empty intersections of the V_j with the W_i . This contradiction proves Remark B.

Remark C. Each W_i consists of exactly one point from one V_j and all but one point of another V_j . In order to show this, we again suppose not. Then there is a set, say W_1 , which contains one point from a V_i , say V_1 , and all of its other points are in another V_i , call it V_2 , but there are at least two points of V_2 which are not in W_1 . In Remark B we proved V_2 does not have points in three different W_i . Thus these remaining points of V_2 together with another point, say v , form another set, say W_2 . The point v is not in V_1 because that would imply that V_1 has points in three different W sets. Thus v is an element of a third V_j , say V_3 . The sets W_3, \dots, W_k contain all points of the sets V_4, \dots, V_k and all but one point from each of the sets V_1 and V_3 . If $k = 3$, then W_3 has two points of each of V_1 and V_3 , which is a contradiction. Thus we suppose $k \geq 4$.

The remaining points of V_1 must all be in a W_i , and the remaining points of V_3 must be in a different W_i .

Thus, without loss of generality, we may suppose $W_3 = V_3 \cup \{v_4\}$ where $v_4 \in V_4$. If $k = 4$, then W_4 has at least two points of each of V_1 and V_4 , which again is a contradiction. Otherwise, the remaining points of V_1 must be in one W_i and the remaining points of V_4 must be in a different W_i . Thus we can assume $W_4 = V_4 \cup \{v_5\}$ where v_5 is an element of V_5 . Continuing in this way, we see that W_k must have two points from each of V_1 and V_k . This is impossible, and so Remark C is proved.

We now select r such that $V_1 \cap W_r$ contains a single point v . Then there is an integer $s \neq 1$ such that $w \in V_s$ and $W_r = \{v\} \cup (V_s - \{w\})$. By hypothesis, not all lines of G_s are incident with a single point. Thus $\langle V_s - \{w\} \rangle$ has a line and $\langle W_r \rangle$ contains a triangle. This contradiction implies there is a unique acyclic partition of $V(G)$ into k sets. So by Proposition 2.3A, G is uniquely k -arborable. //

Previously it was observed that, for $k > 1$, any uniquely k -arborable graph is connected. In order to give a stronger result, we present the following observation.

Proposition 2.3D. In any uniquely k -arborable graph, $k > 1$, the union of any two (of the k) sets of the acyclic partition induces a connected graph which contains a cycle.

Proof. The cyclic property is true because otherwise the point-arboricity of G would be at most $(k-1)$.

Let V_1, V_2 be two sets of the partition. Since G is uniquely k -arborable, $\langle V_1 \cup V_2 \rangle$ is uniquely 2-arborable and by the previous remark $\langle V_1 \cup V_2 \rangle$ is connected. //

Proposition 2.3E. If graph G is uniquely k -arborable, $k \geq 2$, then G is $(k-1)$ -connected.

Proof. If $k = 2$, the desired result is implied by the aforementioned observation that G is connected. Thus, let $k > 2$, and suppose G is not $(k-1)$ -connected. Then there is a set S of $k-2$ points whose removal disconnects G . There are two sets V_1 and V_2 of the partition of $V(G)$ into k acyclic sets which have no points in S . According to Proposition 2.3D, the union of V_1 and V_2 induces a connected graph. That is, all points of V_1 and V_2 are in the same component of $G-S$. Another acyclic partition of $V(G)$ into k sets can be obtained by adding any point v in another component of $G-S$ to set V_1 . Then V_1 together with point v induces the graph $\langle V_1 \rangle \cup K_1$ which is acyclic. This contradicts the fact that G is uniquely k -arborable. //

Since, for $2 \leq m \leq k$, the union of any m of the k acyclic sets of a uniquely k -arborable graph induces a uniquely m -arborable graph, we have the following result.

Corollary 2.3F. Let G be a uniquely k -arborable graph with acyclic partition V_1, \dots, V_k . For $2 \leq m \leq k$, the union of any m sets of the partition induces an $(m-1)$ -connected graph.

The next theorem shows that the conclusion of Proposition 2.3E cannot be improved.

Theorem 2.3G. For every $k \geq 2$, there is a uniquely k -arborable graph which is not k -connected.

Proof. For each $k \geq 2$, we define a graph $G^{(k)}$ which has a cut set of $k-1$ points and is uniquely k -arborable.

Let G_1, G_2, \dots, G_k denote mutually disjoint graphs such that, for $i = 1, 2, \dots, k-1$, G_i is the path $v_1^{(i)}, v_2^{(i)}, \dots, v_9^{(i)}$ and G_k consists of two disjoint paths $P = v_1^{(k)}, v_2^{(k)}, v_3^{(k)}, v_4^{(k)}$ and $P' = v_5^{(k)}, v_6^{(k)}, v_7^{(k)}, v_8^{(k)}$.

For $i = 1, 2, \dots, k-1$, let

$$L^{(i)} = \{v_j^{(i)} : j = 1, 2, 3, 4, 5\} \quad \text{and}$$

$$R^{(i)} = \{v_j^{(i)} : j = 5, 6, 7, 8, 9\}.$$

Also, let

$$L^{(k)} = \{v_j^{(k)} : j = 1, 2, 3, 4\} \quad \text{and}$$

$$R^{(k)} = \{v_j^{(k)} : j = 5, 6, 7, 8\}.$$

Further, define

$$L = \bigcup_1^k L^{(i)} \quad \text{and} \quad R = \bigcup_1^k R^{(i)}.$$

Now denote by $G^{(k)}$ the graph which consists of the union of the k mutually disjoint graphs G_i together with all possible lines joining points in distinct $L^{(i)}$ and all possible lines joining points in distinct $R^{(i)}$. The graph $G^{(2)}$ is shown in Fig. 2.1. Furthermore, if S is a subset of $V(G^{(k)})$, then we call $S \cap L$ and $S \cap R$ the left side of S and right side of S respectively.

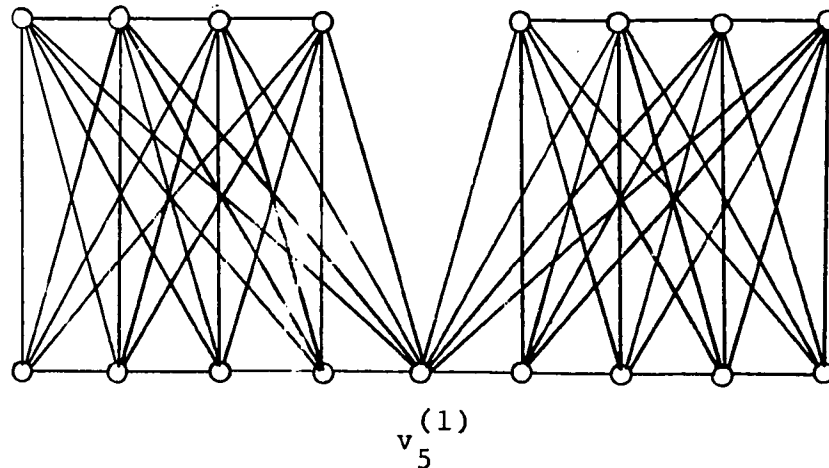


Fig. 2.1

We first consider the graph $G^{(2)}$. Since the removal of $v_5^{(1)}$ disconnects the graph, $G^{(2)}$ is not 2-connected. Clearly $f_2(G) = 2$, so that it remains to show that the only acyclic partition of $V(G^{(2)})$ is $V(G_1), V(G_2)$.

Suppose W_1, W_2 is another acyclic partition of $V(G^{(2)})$. Then one of the sets, say W_1 , has at least nine points and neither W_1 nor W_2 contains all of $V(G_j)$ for $j = 1$ or 2 . Thus W_1 contains points from both $V(G_1)$ and $V(G_2)$.

Since W_1 has at least nine points, one side of W_1 has at least five points. If these five points are in both $V(G_1)$ and $V(G_2)$, they induce a cycle which contradicts that $\langle W_1 \rangle$ is a forest. Thus we suppose all five of these points are in $V(G_1)$. Then since $v_5^{(1)}$ is in both sets L and R , the other side of W_1 must have five points not all in $V(G_1)$. These five points induce a cycle, which again contradicts that $\langle W_1 \rangle$ is acyclic. Hence $G^{(2)}$ is uniquely 2-arborable.

Induction on k is now used to show that, for $k > 2$, $G^{(k)}$ is uniquely k -arborable. We have already shown that $G^{(2)}$ is uniquely 2-arborable. Assume $G^{(k-1)}$ is uniquely $(k-1)$ -arborable and consider $G^{(k)}$.

The sets $V(G_1), \dots, V(G_k)$ form an acyclic partition of $V(G^{(k)})$ into k sets. According to Proposition 2.3A, it suffices to show that this partition is unique. Suppose there is a partition W_1, \dots, W_k of $V(G^{(k)})$ into acyclic sets which is different from $V(G_1), \dots, V(G_k)$.

Assume that one of these sets, say W_i , is equal to $V(G_j)$ for some $j = 1, 2, \dots, k-1$. Then $W_1, \dots, W_{i-1}, W_{i+1}, \dots, W_k$ is an acyclic partition of

$G^{(k)} - G_j = G^{(k-1)}$ different from

$$V(G_1), \dots, V(G_{j-1}), V(G_{j+1}), \dots, V(G_k).$$

However, by the inductive assumption this is impossible.

Thus one set, call it W_1 , must contain $m \geq 9$ points and $W_1 \neq V(G_j)$ for $j = 1, \dots, k$. These m points must come from at least two different $V(G_j)$. But $m \geq 9$ implies that one side of W_1 contains at least five points. Furthermore, using the argument given for $G^{(2)}$, we can say that one side of W_1 contains five points in two or more different $V(G_j)$. These five points induce a cycle which implies W_1 does not induce an acyclic graph. This contradiction implies the partition $V(G_1), \dots, V(G_k)$ is unique and thus $G^{(k)}$ is uniquely k -arborable.

The graph $G^{(k)} - \{v_5^{(i)} : i = 1, \dots, k-1\}$ is disconnected, which implies the connectivity of $G^{(k)}$ does not exceed $k-1$. This completes the proof. //

We now present a number of necessary conditions for a graph to be uniquely k -arborable. This section is concluded by showing that, for $k \geq 2$, any uniquely k -arborable graph is non-outerplanar and, for $k \geq 3$, any uniquely k -arborable graph is non-planar.

Proposition 2.3H. If G is a uniquely k -arborable graph, $k \geq 2$, and V_1, \dots, V_k is the acyclic partition of G into k sets, then $|V_i| \geq 3$ for $i = 1, 2, \dots, k$.

Proof. Assume one set, say V_1 , has exactly two points and let u, v be the points of V_1 . Since $\langle V_2 \rangle$ is acyclic, V_2 can be partitioned into sets S_1 and S_2 such that each S_i induces a totally disconnected graph. Hence $S_1 \cup \{v\}$ and $S_2 \cup \{u\}$ both induce acyclic graphs. This contradicts the fact that $\langle V_1 \cup V_2 \rangle$ is uniquely 2-arborable. //

Proposition 2.3I. If G is a uniquely k -arborable graph, $k \geq 2$, and V_1, \dots, V_k is the acyclic partition of $V(G)$ such that $|V_1| \leq |V_2| \leq \dots \leq |V_k|$, then $|V_2| \geq 4$.

Proof. Suppose V_2 has exactly three points v_1, v_2 , and v_3 , and let $V_1 = \{u_1, u_2, u_3\}$. Then two points of V_1 , say u_1 and u_2 , are non-adjacent as are two points, say v_1 and v_2 , of V_2 . Hence the sets $\{u_1, u_2, v_3\}$ and $\{v_1, v_2, u_3\}$ induce acyclic sets and we have two different acyclic partitions of $V_1 \cup V_2$ into two sets. This is impossible, and thus $|V_2| > 3$. //

Proposition 2.3J. Let G be a uniquely k -arborable graph where $k \geq 2$. If v is a point of G with $\deg v \leq 2k-1$, then $G-v$ is uniquely k -arborable.

Proof. Let V_1, \dots, V_k be the partition of $V(G)$ into k acyclic sets, and suppose $v \in V_1$. The set $V_1 - v$ is non-empty, and $V_1 - v, V_2, \dots, V_k$ is an acyclic partition of

$G-v$. Assume W_1, W_2, \dots, W_k is an acyclic partition of $G-v$. Since $\deg_G v \leq 2k-1$, there is a set W_i , say W_1 , which has at most one point adjacent in G to v . Thus $W_1' = \{v\} \cup W_1, W_2, \dots, W_k$ is an acyclic partition of G . There is only one acyclic partition of $V(G)$ into k sets. Thus $W_1' = V_1$ since both sets contain v . Without loss of generality we may say $W_i = V_i$, $i = 2, 3, \dots, k$. Then $W_1 = V_1 - v$, $W_2 = V_2, \dots, W_k = V_k$ is the only acyclic partition of $V(G-v)$ into k sets. Hence, by Proposition 2.3A, $G-v$ is uniquely k -arborable. //

Proposition 2.3K. Let G_0 be a uniquely k -arborable graph, $k \geq 2$, with p points and q lines. Then $p \geq 4k-1$ and $q \geq kp$.

Proof. Propositions 2.3H and 2.3I imply that $p \geq 4k-1$.

If $\delta(G_0) \geq 2k$, then the sum of the degrees of the points of G_0 is at least $2kp$, which implies that $q \geq kp$. Thus we need only consider the case where there is a point v_0 of G_0 such that $\deg v_0 \leq 2k-1$. Proposition 2.3J implies that $G_1 = G_0 - v_0$ is uniquely k -arborable. Either the graph G_1 has a minimum degree at least $2k$ or there is a point v_1 in G_1 such that $\deg_{G_1} v_1 \leq 2k-1$. In the former case we stop deleting points of G . In the latter case $G_2 = G_1 - v_1$ is uniquely

k -arborable. If $\delta(G_2) \geq 2k$, we stop deleting points of G ; otherwise, there exists a point $v_2 \in V(G_2)$ such that $G_3 = G_2 - v_2$ is uniquely k -arborable. Proposition 2.3J implies that we can continue deleting points until we obtain a graph G_m which is uniquely k -arborable, has minimum degree at least $2k$, and has been obtained by subtracting m points, v_0, \dots, v_{m-1} , from G_0 . The number of lines in G_m is at least $k(p-m)$. We recall that uniquely k -arborable graphs have minimum degree at least $2k-2$. For $i = 0, 1, \dots, m-1$, G_i is uniquely k -arborable, and thus each v_i has degree in G_i not less than $2k-2$. In the removal of each point, we also removed at least $2k-2$ lines which implies that $(2k-2)m \geq km$ lines were deleted from G_0 in obtaining G_m . Thus G_0 has not less than $k(p-m) + km = kp$ lines. //

The preceding proposition implies that, for $k \geq 3$, any uniquely k -arborable graph with p points has at least $3p$ lines. Since every planar graph of order $p(\geq 3)$ has at most $3p-6$ lines, we arrive at the following:

Corollary 2.3L. If G is uniquely k -arborable, $k \geq 3$, then G is non-planar.

An outerplanar graph of order $p(\geq 2)$ has at most $2p-3$ lines, which enables us to state another corollary.

Corollary 2.3M. For $k \geq 2$, no uniquely k -arborable graph is outerplanar.

According to Corollary 2.3M, no uniquely 2-arborable graph is outerplanar. The author has been unable, however, to find a planar, uniquely 2-arborable graph. This leads us to the following:

Conjecture 2.3N. If $k \geq 2$ and G is uniquely k -arborable, then G is non-planar.

Section 2.4

On Graphs with Prescribed Point-Arboricity and Clique Number

In 1947 B. Descartes [8] asked whether there exists a graph with chromatic number four which contains no triangles. The proposer answered the question affirmatively in [9]. The following year, the Russian graph theorist A. A. Zykov [26] proved a more general result. Before stating Zykov's theorem, we define the clique number of a graph G , denoted $\omega(G)$, as the maximum number of points in any complete subgraph of G .

Theorem 2.4A. (Zykov) For any integers k, d with $2 \leq d \leq k$, there exists a graph G such that $\omega(G) = d$ and $f_1(G) = k$.

For the special case $d = 2$, Theorem 2.4A can be stated: For any integer $k \geq 2$, there exists a graph with chromatic number k which has no triangles. In this section, we present an analogue of Zykov's theorem for point-arboricity. In Chapter IV, a generalization of both theorems will be given.

Theorem 2.4B. For each positive integer m there exists a graph G_m with no triangles such that $f_2(G_m) = m$.

Proof. Let m be a given positive integer, and denote by W_1 a set of $2m^2$ points. We define the graph H_1 to be the totally disconnected graph with $V(H_1) = W_1$.

If $m = 1$, we set $G_1 = H_1$ and observe that G_1 has the desired properties. If $m > 1$, we define a graph H_2 . For the purpose of doing this, we first define a number of different sets. Let

$$B_2 = \{S : S \text{ is an independent set of points of } H_1 \text{ and } |S| \geq 2\}.$$

(Hence B_2 contains $2^{2m^2} - 2m^2 - 1$ elements.) For each $S \in B_2$, we let V_S be a set consisting of $2m^2$ points such that V_S is disjoint from H_1 . Furthermore, for distinct sets S and S' of B_2 , the sets V_S and $V_{S'}$ are disjoint. Sets W_2 , E_S , and E_2 are now defined as follows:

$$W_2 = \bigcup \{V_S : S \in B_2\},$$

$$E_S = \{uv: u \in S, v \in V_S\}, \quad \text{and}$$

$$E_2 = \bigcup \{E_S: S \in B_2\}.$$

Finally, we define H_2 by letting $V(H_2) = W_2 \cup V(H_1)$ and $E(H_2) = E_2 \cup E(H_1)$.

Suppose graph H_k , $2 \leq k < m$, has been formed. We now define graph H_{k+1} . Let

$$B_{k+1} = \{T: T \text{ is an independent set of points of } H_k \text{ and } |T| \geq 2\}.$$

For each set T in B_{k+1} , let V_T be a set with $2m^2$ points with the properties that $V_T \cap V_{T'} = \emptyset$ for $T \neq T'$ and $V_T \cap V(H_k) = \emptyset$. We let

$$W_{k+1} = \bigcup \{V_T: T \in B_{k+1}\},$$

$$E_T = \{uv: u \in T, v \in V_T\},$$

$$E_{k+1} = \bigcup \{E_T: T \in B_{k+1}\},$$

$$V(H_{k+1}) = W_{k+1} \cup V(H_k), \quad \text{and}$$

$$E(H_{k+1}) = E_{k+1} \cup E(H_k).$$

The m th step of this construction forms a graph H_m , and we let $G_m = H_m$.

We first verify that G_m has no triangles. Suppose the contrary. Any triangle in G_m has its points, call them v_1, v_2 , and v_3 , in three different W_i , say

W_{i_1} , W_{i_2} , and W_{i_3} respectively, since each $\langle W_i \rangle$ is a totally disconnected subgraph. Thus $m \geq 3$, and we may assume without loss of generality that $i_1 < i_2 < i_3$. Since v_1 and v_2 are adjacent in $G_m = H_m$ and $i_1 < i_2 < i_3$, we have that v_1 and v_2 are adjacent in H_{i_3-1} . This implies that v_3 is not adjacent to both v_1 and v_2 . Thus we have a contradiction and G_m has no triangles.

Next we show that $f_2(G_m) = m$. Since W_1, W_2, \dots, W_m is a partition of $V(G_m)$ and each $\langle W_i \rangle$ is totally disconnected, $f_2(G_m) \leq m$. Assume $f_2(G_m) < m$. Then there exists an acyclic partition of $V(G_m)$ into sets U_1, U_2, \dots, U_{m-1} . Since W_1 has $2m^2$ points, at least one set of the partition, say U_1 , has $2m$ or more points of W_1 . Let

$$\{u_1^{(1)}, u_2^{(1)}, \dots, u_{2m}^{(1)}\} \subset U_1 \cap W_1.$$

Since $u_1^{(1)}$ and $u_2^{(1)}$ are independent in $V(H_1)$, there is a set S_2 of $2m^2$ points of W_2 which are adjacent to only the points $u_1^{(1)}$ and $u_2^{(1)}$. No two of the points of S_2 are in U_1 because this would imply that $\langle U_1 \rangle$ has a cycle. Thus, at least $2m$ of these points are in some U_i different from U_1 , say U_2 . Let

$$\{u_1^{(2)}, u_2^{(2)}, \dots, u_{2m}^{(2)}\} \subset S_2 \cap U_2.$$

Since every element of $S_2 \cap U_2$ is adjacent in H_2 to exactly $u_1^{(1)}$ and $u_2^{(1)}$, the set $\{u_3^{(2)}, u_4^{(2)}, u_3^{(1)}, u_4^{(1)}\}$ is an independent set in H_2 . Thus, in W_3 there exists a set S_3 of $2m^2$ points adjacent to exactly the points $u_3^{(2)}, u_4^{(2)}, u_3^{(1)}, u_4^{(1)}$. No two of the points of S_3 can be in U_i for $i = 1, 2$. Thus, there is a set U_j with $j \geq 3$ such that U_j has at least $2m$ points of S_3 . Without loss of generality, we suppose this set is U_3 . Let

$$\{u_1^{(3)}, \dots, u_{2m}^{(3)}\} \subset S_3 \cap U_3.$$

Since every point in $S_3 \cap U_3$ is adjacent in H_3 to only the points $u_3^{(2)}, u_4^{(2)}, u_3^{(1)}, u_4^{(1)}$, the set

$$\{u_5^{(3)}, u_6^{(3)}, u_5^{(2)}, u_6^{(2)}, u_5^{(1)}, u_6^{(1)}\} \text{ is independent in } H_3.$$

Assume now that, for $j < m-1$, the graph H_j has an independent set of points

$$T = \{u_{2j-1}^{(j)}, u_{2j}^{(j)}, u_{2j-1}^{(j-1)}, u_{2j}^{(j-1)}, \dots, u_{2j-1}^{(1)}, u_{2j}^{(1)}\}$$

such that, for $i = 1, \dots, j$, $u_{2j-1}^{(i)}, u_{2j}^{(i)} \in U_i \cap S_i$

where $S_1 = W_1$ and, for $i = 2, \dots, j$, S_i is a set of $2m^2$ points each adjacent in H_j to exactly the points of

$$\{u_{2(i-1)-1}^{(i-1)}, u_{2(i-1)}^{(i-1)}, u_{2(i-1)-1}^{(i-2)}, u_{2(i-1)}^{(i-2)}, \dots, u_{2(i-1)-1}^{(1)}, u_{2(i-1)}^{(1)}\}$$

and

$$\{u_1^{(i)}, u_2^{(i)}, \dots, u_{2m}^{(i)}\} \subset U_i \cap S_i.$$

Since the points of T are independent in H_j , there exists a subset S_{j+1} of W_{j+1} with $2m^2$ points which are adjacent to exactly the points of T . However, for $k \leq j$, no two points of S_{j+1} are in U_k because otherwise, these two points together with $u_{2j-1}^{(k)}$ and $u_{2j}^{(k)}$ would induce a cycle in U_k . Thus, there are at least $2m$ points of S_{j+1} in some U_i , $i \geq j+1$, say U_{j+1} . Let

$$\{u_1^{(j+1)}, u_2^{(j+1)}, \dots, u_{2m}^{(j+1)}\} \subset S_{j+1} \cap U_{j+1}.$$

Since the points of $S_{j+1} \cap U_{j+1}$ are adjacent to exactly the points of T , the set

$$\{u_{2j+1}^{(j+1)}, u_{2j+2}^{(j+1)}, u_{2j+1}^{(j)}, u_{2j+2}^{(j)}, \dots, u_{2j+1}^{(1)}, u_{2j+2}^{(1)}\}$$

is independent.

Thus, taking $j = m-2$, we have formed an independent set T' of points of H_{m-1} where

$$T' = \{u_{2m-3}^{(m-1)}, u_{2m-2}^{(m-1)}, u_{2m-3}^{(m-2)}, u_{2m-2}^{(m-2)}, \dots, u_{2m-3}^{(1)}, u_{2m-2}^{(1)}\}$$

and $u_{2m-3}^{(i)}, u_{2m-2}^{(i)}$ are in U_i for $i = 1, \dots, m-1$.

There exists a set S_m of $2m^2$ points of W_m which are adjacent to the points of T' . However, at least two

points v_1, v_2 of S_m are in U_i for some $i = 1, \dots, k$

and thus, the set $\{v_1, v_2, u_{2m-3}^{(i)}, u_{2m-2}^{(i)}\}$ induces a cycle.

This contradicts the assumption that U_1, \dots, U_{m-1} is an acyclic partition of $V(G_m)$. Hence, $f_2(G_m) = m$. //

The preceding result may also be stated in the following terms: For every positive integer k , there exists a graph G_k with point-arboricity k and clique number 2. A generalization of this is now immediate.

Corollary 2.4C. For any integers k and d such that $d \geq 2$ and $k \geq \begin{Bmatrix} d \\ - \\ 2 \end{Bmatrix}$, there exists a graph G with $f_2(G) = k$ and $\omega(G) = d$.

Proof. Let G_k be as in Theorem 2.4B, and define G as $G_k \cup K_d$. Since $f_2(K_d) = \begin{Bmatrix} d \\ - \\ 2 \end{Bmatrix} \leq k$ and the point-arboricity of a graph is the maximum point-arboricity of its components, we have $f_2(G) = f_2(G_k) = k$. Clearly $\omega(G) = d$ since G_k has no triangles. //

As a final remark, we note that the graph in Corollary 2.4C may be connected by introducing a line from a point of K_d to a point of G_k .

CHAPTER III

OUTERPLANAR PARTITIONS

We have seen that both the chromatic number and the point-arboricity of a graph G may be described in terms of partitioning $V(G)$ into the minimum number of sets such that each set induces a graph with a given property, namely that of being totally disconnected and acyclic respectively. Chartrand, Geller, and Hedetniemi [4] have introduced a generalization of these parameters, which we present.

A graph G is said to have property F_n , $n \geq 1$, if G has no subgraph homeomorphic from K_{n+1} or $K\left(\left[\frac{n+2}{2}\right], \left\{\frac{n+2}{2}\right\}\right)$. Kuratowski's well-known theorem, given in [14], states that a graph is planar if and only if it has no subgraph homeomorphic from K_5 or $K(3,3)$. Similarly, Chartrand and Harary [5] have characterized outerplanar graphs as those having no subgraphs homeomorphic from K_4 or $K(2,3)$. Thus, graphs with property F_4 and F_3 are planar and outerplanar graphs respectively. Graphs which contain no subgraphs homeomorphic from K_3 or $K(2,2)$ are the graphs with no cycles, and a graph with no subgraph homeomorphic from K_2 or $K(1,2)$ has no lines. Thus, empty graphs and acyclic graphs are graphs with property

F_1 and F_2 respectively. Graphs with property F_n , $n \geq 5$, have not been given special names.

The aforementioned generalization of chromatic number and point-arboricity can now be given. The point-partition number $f_n(G)$, $n \geq 1$, of a graph G is defined as the minimum number of subsets into which the point set of G can be partitioned so that the subgraph induced by each subset has property F_n . In addition to $f_1(G)$ and $f_2(G)$ being the chromatic number and point-arboricity respectively of G , $f_3(G)$ is called the point-outerthickness of graph G , and $f_4(G)$ is its point-thickness. In this chapter, we investigate properties of the parameter f_3 .

Section 3.1

Hypo-Outerplanar Graphs

There are often varying degrees to which a graph may fail to possess a given property P . For example, a graph G may fail to have property P , but each subgraph $G-v$ obtained by the deletion of a point v of G may actually have property P . If this occurs, then G is said to have hypo-property P . Thus, a graph G is hypo-outerplanar if G is not outerplanar but, for each $v \in V(G)$, $G-v$ is outerplanar. In this section we investigate hypo-outerplanar graphs.

Wagner [23] has given a characterization of hypo-planar graphs. Among the graphs which he has shown to be hypo-planar are graphs homeomorphic from K_5 , certain non-planar graphs which are embeddable in the Möbius strip, and various non-planar graphs which consist of a path P , a cycle disjoint from P , and lines joining endpoints of the path with points on the cycle.

Earlier we defined a graph G to be k -critical with respect to point-arboricity if $f_2(G) = k$ and $f_2(G-v) = k-1$ for each point v of G . Using the parameter f_n , we now make a more general definition: a graph G is k -critical with respect to f_n if $f_n(G) = k$ and $f_n(G-v) = k-1$ for every point v in G . For example, to say that a graph G is 2-critical with respect to f_4 means that $f_4(G) = 2$ and $f_4(G-v) = 1$ for each point v of G . Hypo-planar graphs are therefore graphs which are 2-critical with respect to f_4 . Graph G is hypo-outerplanar if and only if G is 2-critical with respect to f_3 .

We present a characterization of hypo-outerplanar graphs after first proving an elementary result.

Proposition 3.1A. Let G be a hypo-outerplanar graph with p points and q lines. Then

- i. G has at least three points of degree not exceeding 3 and
- ii. $q \leq 2p-2$.

Proof. To prove i., we let v be a point of G . Since $G-v$ is outerplanar, it has two points v_1 and v_2 of degree at most 2. Thus the degree of v_1 and v_2 in G does not exceed 3. The graph $G-v_1$ is outerplanar, and it contains a point $v_3 \neq v_2$ such that in $G-v_1$ $\deg v_3 \leq 2$. Hence, $\deg_G v_3 \leq 3$ and we have three points v_1 , v_2 , and v_3 of degree at most 3.

In proving ii., we let q_v be the number of lines in $G-v$. Since $G-v$ is outerplanar, $q_v \leq 2(p-1)-3 = 2p-5$ for all points v in G . Part i. implies G has a point u of degree not more than 3. However, $q_u \leq 2p-5$ and thus $q \leq q_u+3 \leq 2p-2$. //

We note that, for the hypo-outerplanar graph K_4 , $q = 2p-2$.

Our characterization of hypo-outerplanar graphs requires a few additional terms.

A graph G homeomorphic from $K(2,3)$ is called a theta-graph. The two points in G of degree 3 are called primary points and are denoted v_1 and v_2 . A point u

adjacent to a primary point v_i is called a secondary point, and we say u is secondary to v_i .

A super-theta-graph is a theta-graph together with exactly one of the following:

- i. a line joining the primary points,
- ii. a line joining two points secondary to v_1 and/or a line joining two points secondary to v_2 .

A graph G is called a quasi-wheel if G is a cycle together with a new point u adjacent with at least three points of the cycle. The point u is called the hub of G .

We are now ready to commence the characterization of hypo-outerplanar graphs. Since a graph is outerplanar if and only if it has no subgraph homeomorphic from K_4 or $K(2,3)$, it follows that theta-graphs, super-theta-graphs, and quasi-wheels are not outerplanar. We verify that each graph in these three classes is hypo-outerplanar.

Proposition 3.1B. Every theta-graph is hypo-outerplanar.

Proof. Let G be homeomorphic from $K(2,3)$, and let v_1 and v_2 be its primary points. For $i = 1, 2$, $G - v_i$ is acyclic and therefore outerplanar.

Graph G consists of three paths from v_1 to v_2 . Let u be a point of G which is not a primary point.

Then the removal of u from G disconnects one of the paths from v_1 to v_2 and $G-u$ is therefore outerplanar.

Thus, since G is not outerplanar, it is hypo-outerplanar. //

Proposition 3.1C. If G is a super-theta-graph, G is hypo-outerplanar.

Proof. We observe that super-theta-graphs are planar, so that we may suppose G is embedded in the plane. Since G is not outerplanar, we need only show that $G-v$ is outerplanar for each $v \in V(G)$. In order to do this, we distinguish three cases.

Case i. Assume G has a line joining primary points v_1 and v_2 . The graph $G-v_i$, $i = 1, 2$, is acyclic and hence outerplanar. If u is a non-primary point of G , then $G-u$ contains three paths from v_1 to v_2 and one of these paths contains only one line. All points and lines of $G-u$ not on one of these paths do not lie on any cycle. It follows that $G-u$ is outerplanar.

Case ii. Suppose G has a line joining points secondary to v_1 and a line joining points secondary to v_2 . For $i = 1, 2$, $G-v_i$ contains a cycle with one chord. Since all other points and lines of $G-v_i$ are in no cycle, $G-v_i$ is outerplanar.

Let u be a point of G different from v_1 and v_2 . If $\deg u = 2$ (i.e., u is not incident with one of the extra lines), then $G-u$ consists of a cycle with two chords, and all other points and lines belong to no cycle. Thus $G-u$ is outerplanar. If $\deg u \geq 3$ (i.e., u is incident with at least one of the two extra lines), then $G-u$ has one cycle with at most one chord, and all other points and lines belong to no cycle. This implies that $G-u$ is outerplanar. Hence, for any $v \in V(G)$, $G-v$ is outerplanar.

Case iii. Assume G consists of a theta-graph and one line joining points secondary to the same primary point, say v_1 . In this case G is a subgraph of the super-theta-graph G' , which consists of G together with a line joining points secondary to v_2 . For any $u \in V(G) = V(G')$, $G-u$ is outerplanar since it is a subgraph of the outerplanar graph $G'-u$.

Hence, in all three cases the removal of any point of G results in an outerplanar graph. This implies that G is hypo-outerplanar. //

Proposition 3.1D. Every quasi-wheel is hypo-outerplanar.

Proof. Let G be a quasi-wheel with hub u . Then $G-u$ is a cycle, say $v_1, v_2, \dots, v_n, v_1$, and is outerplanar.

Define integers k and m as follows:

$$k = \max\{i: v_i \text{ is adjacent to } u \text{ and } i > 1\},$$

$$m = \min\{i: v_i \text{ is adjacent to } u \text{ and } i > 1\}.$$

Since at least two of the points v_2, v_3, \dots, v_n are adjacent to u , $k \neq m$. The graph $G - v_1$ contains a cycle and possibly chords incident with u . Any other points and lines of $G - v_1$ are not in any cycle (See Fig. 3.1). Thus, $G - v_1$ is outerplanar. Since G is not outerplanar, it is hypo-outerplanar. //

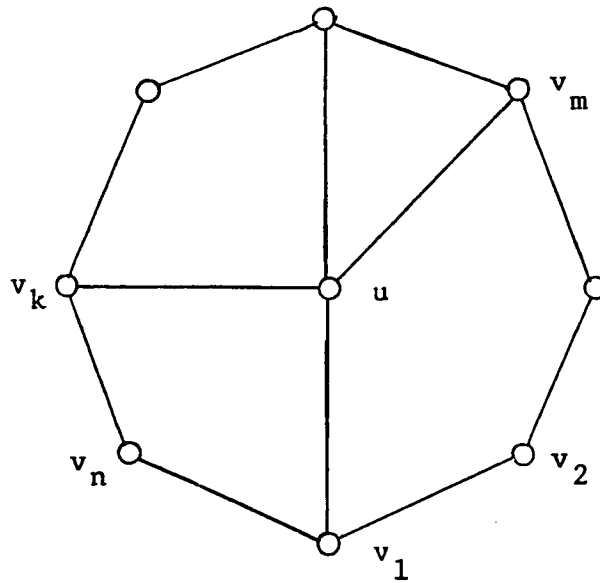


Fig. 3.1

Next we consider some additional lemmas.

Lemma 3.1E. If a graph G has a subgraph H which is homeomorphic from K_4 or $K(2,3)$ and $V(G) \neq V(H)$, then G is not hypo-outerplanar.

Proof. There is a point v in G which is not in H .
 The graph $G-v$ is not outerplanar since H is not.
 Thus, G is not hypo-outerplanar. //

Lemma 3.1F. Let G be homeomorphic from K_4 . If there are two non-adjacent lines u_1u_2 and u_3u_4 of K_4 which are subdivided to obtain G , then G is not hypo-outerplanar.

Proof. Let u be the point introduced by the subdivision of u_1u_2 . Then $G-u$ has a subgraph homeomorphic from $K(2,3)$ and hence is not outerplanar. //

Let G be a theta-graph H together with various additional lines. We denote the primary points of H by v_1 and v_2 and observe that these extra lines are of the following types:

1. Lines incident with v_1 and v_2 .
2. Lines incident with one primary and one non-primary point.
3. Lines incident with two non-secondary, non-primary points.
4. Lines incident with one secondary point and one non-secondary, non-primary point.
5. Lines incident with two secondary points.

We observe that lines of types 4. and 5. may be further classified as follows:

- 4a. Lines incident with one non-secondary, non-primary point and one point which is secondary to only one primary point.
- 4b. Lines incident with one non-secondary, non-primary point and one point which is secondary to both primary points.
- 5a. Lines incident with two points which are secondary to the same primary point.
- 5b. Lines incident with one point which is secondary only to v_1 and one point which is secondary only to v_2 .

A line uv of G which is not a line in H is called a basic line of G if it is of type 2., 3., 4a., or 5b.

A point which is secondary to both primary points is called a double-secondary point. We now present the final lemma.

Lemma 3.1G. Let G consist of a theta-graph H together with some additional lines. If G is hypo-outerplanar, then it contains no basic lines.

Proof. Suppose uv is a basic line of G . We consider three cases depending on uv .

Case i. Line uv is of type 4a. or type 3. One point of line uv , say u , is neither a primary nor a secondary point of H . The other point v is not primary and is not a double-secondary point. The points u and v may lie on the same or on different v_1 - v_2 paths in H (See Fig. 3.2). In either case there is a point u' on

the same v_1-v_2 path as u such that $G-u'$ contains a theta-graph and therefore is not outerplanar.

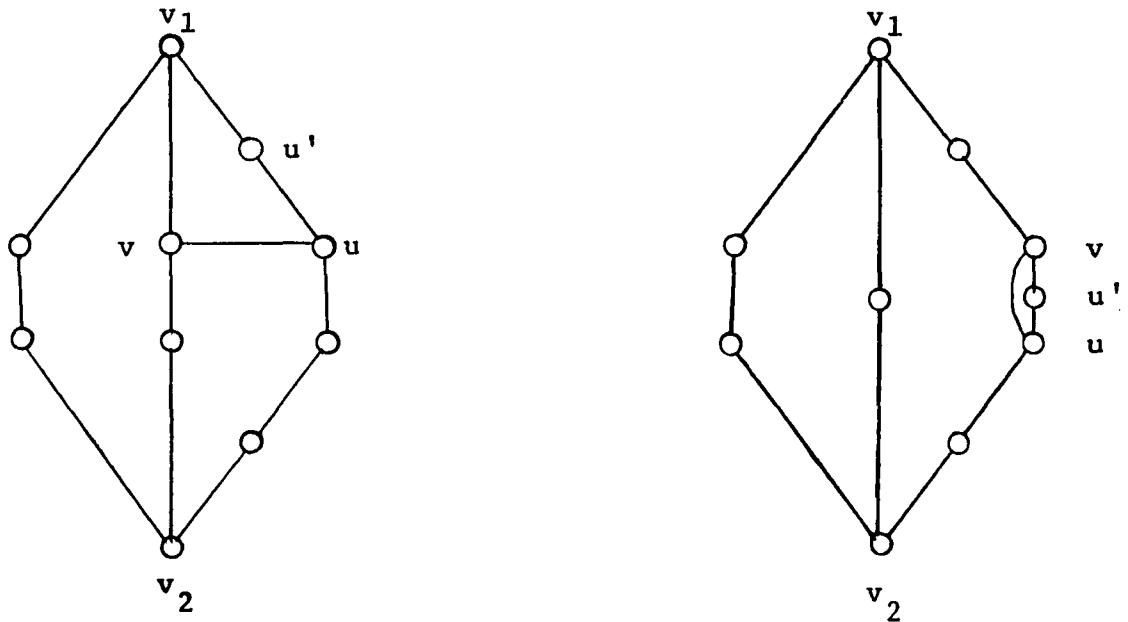


Fig. 3.2

Case ii. Line uv is of type 5b. We let u be secondary to v_2 and v be secondary to v_1 (See Fig. 3.3a). There is a point u' in G such that $G-u'$ contains a theta-graph with primary points v and v_2 . Thus, $G-u'$ is not outerplanar.

Case iii. Line uv is of type 2. Let point v be a primary point and u be a non-primary point (See Fig. 3.3b). There is a point u' such that $G-u'$ contains a theta-graph with the same primary points as H . Hence, $G-u'$ is not outerplanar.

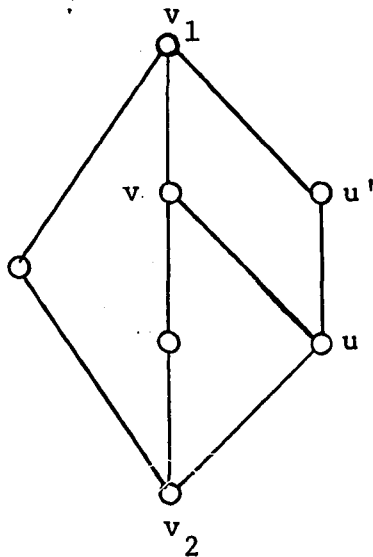


Fig. 3.3a

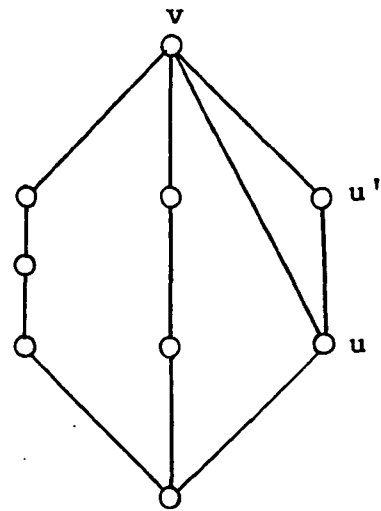


Fig. 3.3b

In all three cases we have shown that graph G has a point whose removal does not result in an outerplanar graph. This contradicts that G is hypo-outerplanar and implies that G contains no basic lines. //

We continue our characterization of hypo-outerplanar graphs. Since such graphs are not outerplanar, we know that any graph which is hypo-outerplanar has a subgraph homeomorphic from K_4 or $K(2,3)$.

Proposition 3.1H. If G is hypo-outerplanar and has a subgraph homeomorphic from $K(2,3)$, then G is a theta-graph, a quasi-wheel, or a super-theta-graph.

Proof. Let G be hypo-outerplanar and have a subgraph H which is homeomorphic from $K(2,3)$. Lemma 3.1E implies

that $V(H) = V(G)$. If G is itself homeomorphic from $K(2,3)$, then G is a theta-graph. Thus, we need only consider a graph G which has a subgraph H homeomorphic from $K(2,3)$ where $V(H) = V(G)$ and $H \neq G$ (i.e., G consists of H together with some extra lines). We let v_1 and v_2 be the primary points of H . Lemma 3.1G implies that G has no basic lines.

Non-basic lines are of types 1., 4b., and 5a. That is, non-basic lines are those incident with both primary points, those incident with a double-secondary point, and those incident with two points which are both secondary to the same primary point. We now distinguish two cases.

Case i. One of the non-basic lines is $v_1 v_2$. If $v_1 v_2$ is the only extra line, then G is a super-theta-graph. Suppose uv is a non-basic line of G different from $v_1 v_2$. Then u and v must lie on different $v_1 v_2$ paths in H . There is a point w different from u and v such that w lies on the third $v_1 v_2$ path in H . $G-w$ is not outerplanar because it contains a subgraph homeomorphic from K_4 . This contradicts the hypothesis that G is hypo-outerplanar.

Case ii. All non-basic lines are of type 4b. or 5a. If there is just one non-basic line, then G is a quasi-wheel or a super-theta-graph. If all of the non-basic

lines are incident with a double-secondary point, then G is a quasi-wheel. We thus assume that G has at least two non-basic lines and not all non-basic lines are incident with a double-secondary point.

We observe now that none of the non-basic lines are of type 4b. To verify this we suppose the contrary and let uv be a line with u secondary to both v_1 and v_2 and v neither secondary nor primary. Then there is another non-basic line w_1w_2 in G such that $w_i \neq u$ for $i = 1, 2$ (See Fig. 3.4). Thus the deletion of one of the primary points results in a non-outerplanar graph. This contradiction implies that none of the non-basic lines are of type 4b.

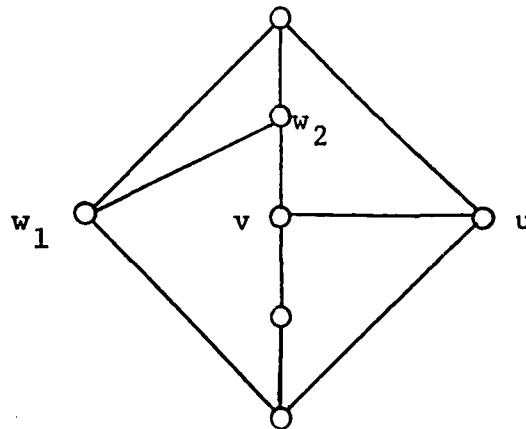


Fig. 3.4

Hence, we can suppose that G has at least two non-basic lines, all of type 5a., and that not all non-basic lines are incident with a double-secondary point. With

these restrictions we now show: The graph G does not have three non-basic lines. To prove this we suppose the contrary. If these three lines are all incident with points secondary to the same primary point, say v_1 , then $G-v_1$ contains a subgraph homeomorphic from K_4 and thus is not outerplanar. Now we can suppose that two non-basic lines join points secondary to v_1 and another line, call it vw , joins points secondary to v_2 (See Fig. 3.5), where one of v and w , say v , is not secondary to v_1 . Also recall that not all three lines are incident with a double-secondary point. Thus, $G-v$ contains a subgraph homeomorphic from $K(2,3)$ where u and w are primary points.

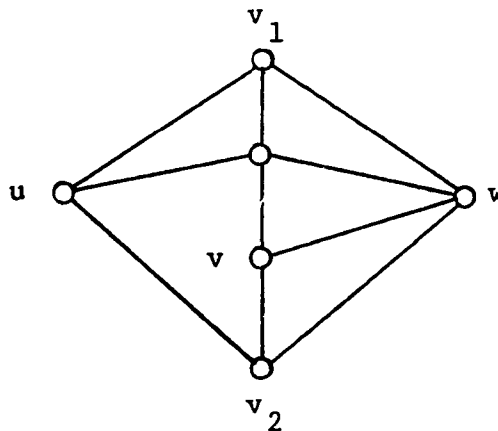


Fig. 3.5

There are therefore exactly two non-basic lines, both of type 5a., and these two lines are not both incident with a double-secondary point. If one line joins points

secondary to v_1 and the other joins points secondary to v_2 , then G is a super-theta-graph. If both lines join points secondary to the same primary point, say v_1 , then these two lines are adjacent at a point w . Since w is not a double-secondary point, there exists a point $v \neq w$ which is on the same $v_1 v_2$ path in H as w (See Fig. 3.6). Graph $G-v$ is a theta-graph, and hence G is not hypo-outerplanar.

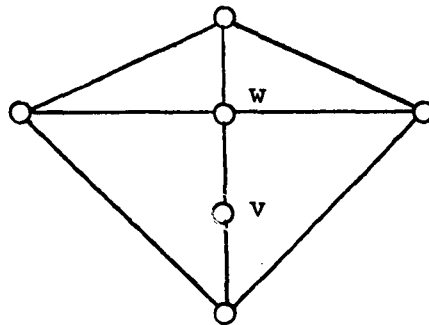


Fig. 3.6

We have now shown that, if G is hypo-outerplanar with a subgraph homeomorphic from $K(2,3)$, then G is a theta-graph, a super-theta-graph, or a quasi-wheel. //

Proposition 3.1I. If graph G is hypo-outerplanar and has a subgraph homeomorphic from K_4 , then G is a quasi-wheel or a super-theta-graph.

Proof. Let H be a subgraph of G which is homeomorphic from K_4 . Lemma 3.1E implies $V(H) = V(G)$. Therefore,

G consists of H together with a (possibly empty) set of extra lines.

The graph H can be formed by a sequence of subdivisions of lines of K_4 . If the sequence is empty, $H = K_4$ and G is necessarily isomorphic to H . Thus, G is a quasi-wheel. Assume then that the sequence of subdivisions is non-empty. If the subdivisions occur on four or more lines of K_4 , at least two of these lines must be non-adjacent. Lemma 3.1F implies that H is not hypo-outerplanar. From this, it follows that the subdivisions can occur on at most three lines of K_4 . We consider two cases.

Case 1. There is a point v_4 of K_4 not incident with any line of K_4 which must be subdivided to form H .

Since H is not isomorphic to K_4 , H must be obtainable from K_4 by at least one subdivision of a line not incident with v_4 . Let this line be v_1v_2 , and denote the other point of K_4 by v_3 . Let the graph formed by a single subdivision of line v_1v_2 be labeled H' , and observe that $H' - v_3v_4$ is a copy of $K(2,3)$ with primary points v_1 and v_2 (See Fig. 3.7). Graph H can now be formed by a (perhaps empty) sequence of subdivisions of H' . By hypothesis, line v_3v_4 is not subdivided to obtain H , so that $H - v_3v_4$ is a theta-graph. Thus, H consists of a graph homeomorphic from $K(2,3)$ together

with an extra line. However, G consists of H together with a set (possibly empty) of extra lines. This implies that G consists of $H-v_3v_4$ which is homeomorphic from $K(2,3)$ along with extra lines. From Proposition 3.1H we know all such hypo-outerplanar graphs are quasi-wheels or super-theta-graphs.

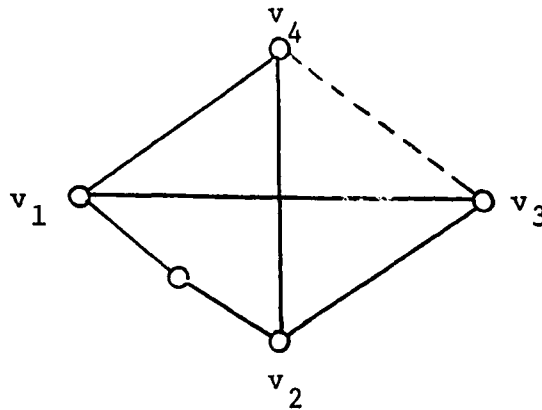


Fig. 3.7

Case ii. Every point of K_4 is incident with at least one line which must be subdivided to obtain H . Suppose every point of K_4 is incident with at least one line which is subdivided and at least one line which is not subdivided to obtain H . Then two non-adjacent lines of K_4 are subdivided in obtaining H . Lemma 3.1F shows that H is not hypo-outerplanar, which implies that G is not hypo-outerplanar.

We thus know that all lines incident with one point, say v_1 , of K_4 must be subdivided in order to obtain H . Label the other points of K_4 by v_2 , v_3 , and v_4 . The

graph $H-v_3v_4$ is then homeomorphic from $K(2,3)$ because it consists of three disjoint v_1-v_2 paths of length two or more. Graph G is therefore hypo-outerplanar and consists of a theta-graph together with additional lines. Proposition 3.1H implies that such graphs are quasi-wheels or super-theta-graphs. //

From Propositions 3.1B, 3.1C, and 3.1D, we know that every theta-graph, super-theta-graph, and quasi-wheel is hypo-outerplanar. Since each hypo-outerplanar graph contains a subgraph homeomorphic from K_4 or $K(2,3)$, Propositions 3.1H and 3.1I imply that every hypo-outerplanar graph is a quasi-wheel, a super-theta-graph, or a theta-graph. Thus, we obtain the following characterization of hypo-outerplanar graphs.

Theorem 3.1J. A graph G is hypo-outerplanar if and only if G is a theta-graph, a super-theta-graph, or a quasi-wheel.

This characterization yields a number of corollaries.

Corollary 3.1K. Any hypo-outerplanar graph has at most two points of degree greater than 3 and at most one point of degree greater than 4.

Corollary 3.1L. Hypo-outerplanar graphs are planar.

Fig. 3.8 shows that the converse of Corollary 3.1L does not hold.



Fig. 3.8

Before considering the final corollary, we define a wheel as a quasi-wheel with p points and maximum degree equal to $p-1$.

Wagner [23] proved that every hypo-planar graph G has chromatic number at most 5 and that G has chromatic number 5 if and only if it consists of K_2 and a cycle C_n , disjoint from K_2 , where n is odd, such that every point of K_2 is adjacent to every point of C_n . We now present an analogous result for hypo-outerplanar graphs.

Corollary 3.1M. If G is hypo-outerplanar, then the chromatic number of G does not exceed 4 and G has chromatic number 4 if and only if G is a wheel with an even number of points.

Proof. Observe that a wheel is a cycle together with one point u not on the cycle such that u is adjacent with every point of the cycle. If wheel G has an even number of points, the cycle has an odd number of points and hence any coloring of G can be done with four colors. It is apparent that any coloring of G requires four colors. If wheel G has an odd number of points, then G can be colored with three colors.

A quasi-wheel which is not a wheel can be 3-colored by coloring the hub u and one point not adjacent to u with color 1 and then alternating colors 2 and 3 among the other points.

A super-theta-graph G with line v_1v_2 can be 3-colored by coloring v_1 with color 1, v_2 with color 2, the three points secondary with v_2 with color 3, and alternating colors 2 and 3 among the remaining points on each path in the theta-subgraph of G .

If G is a super-theta-graph with two lines joining secondary points and if G is not a quasi-wheel, then G consists of two triangles along with three mutually disjoint paths joining points in different triangles. Fig. 3.9 gives a coloring for the triangles so that G can be 3-colored.

Since any subgraph of a 3-colorable graph is 3-colorable, theta-graphs and super-theta-graphs with one line joining secondary points are 3-colorable. //

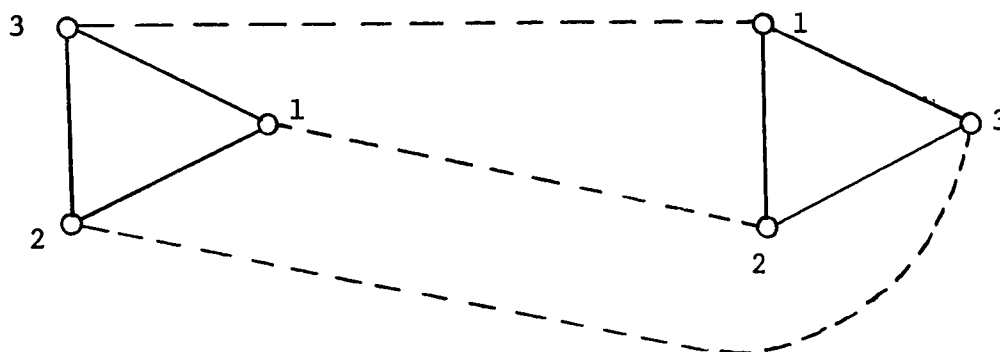


Fig. 3.9

Section 3.2

The Point-Outerthickness of Complete n-Partite Graphs

In this section we consider the minimal point-partition number with respect to the property of being outerplanar, i.e., the parameter f_3 . It would be desirable to have a formula for the value of $f_3(G)$ for any graph G . This seems to be extremely difficult, as there is no formula for f_1 or f_2 . In view of the futile attempts by numerous outstanding mathematicians to prove the Four Color Conjecture, it is unlikely that a formula for $f_1(G)$ for an arbitrary graph G will be found.

Our goal here will be to establish a formula for the point-outerthickness of an important and large class of

graphs, namely the complete n -partite graphs. As was stated in Chapter II, Chartrand, Kronk, and Wall have developed a formula for the point-arboricity of any complete n -partite graph.

We begin with a number of observations.

Remark 3.2A. For every positive integer p , $f_3(K_p) = \left\{ \begin{matrix} p \\ - \\ 3 \end{matrix} \right\}$.

Proof. This formula follows from the fact that a set of three points of a complete graph induces K_3 , which has property F_3 , and a set of four points induces K_4 , which does not have property F_3 . //

Remark 3.2B. A complete n -partite graph G , $n \geq 2$, is outerplanar if and only if G is isomorphic to one of the following: $K(1,1,2)$, $K(2,2)$, $K(1,1,1)$, or $K(1,m)$ where m is any positive integer.

Proof. Let G be isomorphic to one of the graphs $K(1,1,2)$, $K(2,2)$, $K(1,1,1)$, or $K(1,m)$. Then G has no subgraph homeomorphic from K_4 or $K(2,3)$ and is therefore outerplanar.

For $n \geq 2$, we suppose that G is a complete n -partite graph. If $n \geq 4$, then G contains K_4 as a subgraph and is not outerplanar. Thus we need only consider $n = 2$ or 3 . If G has five points and $n = 3$, then G contains $K(2,3)$ as a subgraph which contradicts the

hypothesis that G is outerplanar. This implies that $K(1,1,2)$ and $K(1,1,1)$ are the only outerplanar complete tripartite graphs. Any complete bipartite graph with five points and at least two points in each partite set contains $K(2,3)$. Hence, any outerplanar bipartite graph is of the form $K(1,m)$ or $K(2,2)$. //

Remark 3.2C. Let S be a set of at least five points of a complete n -partite graph G . If $\langle S \rangle$ is outerplanar, then $\langle S \rangle$ is either empty or a star and S has all but possibly one point from a single partite set.

Proof. Since G is a complete n -partite graph, any set S of points of G induces a complete m -partite graph for some $m \leq n$. Thus, if $m = 1$, the graph $\langle S \rangle$ is empty. For $m > 1$, the fact that $\langle S \rangle$ is outerplanar with at least five points, together with Remark 3.2B, implies that $\langle S \rangle$ is a star. //

Denote the partite sets of the complete n -partite graph $K(p_1, p_2, \dots, p_n)$ by V_1, V_2, \dots, V_n where $|V_i| = p_i$, $1 \leq i \leq n$, and recall $p_1 \leq p_2 \leq \dots \leq p_n$. We begin the development of a formula for point-outerthickness of any complete n -partite graph with a special case.

Proposition 3.2D. Let $G = K(p_1, p_2, \dots, p_n)$ and $p_{a+1} \geq 3$, where a is the least integer such that $\sum_{i=1}^a p_i \geq n-a$. Then

- i. $f_3(G) = n-a$ if $\sum_1^a p_i = n-a$ and
- ii. $f_3(G) = n-a+1$ if $\sum_1^a p_i > n-a$.

Proof. We consider two cases and in each case show that the desired result is an upper bound for the point-outerthickness of G . Then, combining the two cases, we verify that statements i. and ii. are indeed correct.

Case i. Suppose $\sum_1^a p_i = n-a$. Then the number of points

in $\bigcup_1^a V_i$ is equal to the number of sets in the

collection $\{V_{a+1}, V_{a+2}, \dots, V_n\}$. Thus, we can partition

$V(G)$ into $n-a$ sets S_1, S_2, \dots, S_{n-a} , where

$S_j = V_{n+1-j} \cup \{v_j\}$, $1 \leq j \leq n-a$, and each v_j is an

element of $\bigcup_1^a V_i$. Each S_j induces a star with at least

four points and is outerplanar. This implies that

$$f_3(G) \leq n-a.$$

Case ii. Assume $\sum_1^a p_i > n-a$. Since a was defined as

the least positive integer such that $\sum_1^a p_i \geq n-a$, we have

$\sum_1^{a-1} p_i < n-a+1$. This implies that the number of elements

in $\bigcup_1^{a-1} V_i$ is less than the number of sets in the

collection V_a, V_{a+1}, \dots, V_n . Thus, we can form $r = \sum_{i=1}^{a-1} p_i$

mutually disjoint subsets S_1, S_2, \dots, S_r of $V(G)$, with

$S_j = V_{n+1-j} \cup \{v_j\}$, $1 \leq j \leq r$, and where each v_j is

an element of $\bigcup_{i=1}^{a-1} V_i$. Next, form mutually disjoint point

sets S_{r+1}, \dots, S_{n-a} where, for $k = r+1, \dots, n-a$,

$S_k = V_{n+1-k} \cup \{v_k\}$ and the v_k are distinct elements of

V_a . Since $\sum_{i=1}^a p_i > n-a$, we have some points of V_a which

are not in any S_j , $j = 1, \dots, n-a$. Call this set of

points S_{n-a+1} . The sets S_1, \dots, S_{n-a} each induce a star

(which is, of course, outerplanar) with at least four

points. The set S_{n-a+1} induces a totally disconnected

graph which is obviously outerplanar. From this it follows

that $f_3(G) \leq n-a+1$.

These two cases are now considered together. In each

case, denote the aforementioned upper bound by s and

suppose $f_3(G) = t < s$. Then $V(G)$ can be partitioned

into sets T_1, T_2, \dots, T_t where $|T_i| \geq |T_{i+1}|$ and each set

induces an outerplanar graph. Since $t < s$, there is an

integer j , $1 \leq j \leq t$, such that $|T_j| > |S_j|$. Let h

be the largest such integer. Then, for any i such that

$h < i \leq t$, $|T_i| \leq |S_i|$. If $h < t$, then this observation,

together with the fact that the set $\{T_{h+1}, \dots, T_t\}$ has fewer elements than the set $\{S_{h+1}, \dots, S_s\}$, implies that

$$\left| \bigcup_{h+1}^t T_i \right| < \left| \bigcup_{h+1}^s S_i \right|.$$

Thus, in any event,

$$\left| \bigcup_1^h T_i \right| > \left| \bigcup_1^h S_i \right|,$$

and

$$\left| \bigcup_1^h T_i \right| - h > \left| \bigcup_1^h S_i \right| - h.$$

We observe that, in Case i, every set in the partition S_1, \dots, S_{n-a} has at least four points and, in Case ii, every set of the partition S_1, \dots, S_{n-a+1} has at least four points, except possibly S_{n-a+1} . Thus, the cardinality of S_h is at least four. Since, for $i < h$, $|T_i| \geq |T_h| > |S_h| \geq 4$, Remark 3.2C implies that each T_i , $i \leq h$, has all but at most one point from a single partite set. If such a point exists for a given T_i , denote it by w_i . Then, for $i \leq h$, define $T_i' = T_i - \{w_i\}$ for all i for which w_i exists and $T_i' = T_i$, otherwise. This implies that the set $\bigcup_1^h T_i'$ has all of its points in h

or fewer partite sets. However, the fact that, for $i \leq h$, $S_i = V_{n+1-i} \cup \{v_i\}$ implies that

$$\left| \bigcup_{i=n-h+1}^n V_i \right| = \left| \bigcup_{i=1}^h S_i \right|^{-h}.$$

Hence, the union of any h partite sets has at most

$$\left| \bigcup_{i=1}^h S_i \right|^{-h} \text{ points, but}$$

$$\left| \bigcup_{i=1}^h S_i \right|^{-h} < \left| \bigcup_{i=1}^h T_i \right|^{-h} \leq \left| \bigcup_{i=1}^h T_i' \right|$$

implies that $\bigcup_{i=1}^h T_i'$ cannot have all of its points in h or fewer partite sets. Thus, we have a contradiction and $f_3(G) = s$ in both cases. //

Corollary 3.2E. Let $G = K(p_1, p_2, \dots, p_n)$ with $p_{a+1} \geq 3$ where a is the least integer such that $\sum_{i=1}^a p_i \geq n-a$. Then,

$$f_3(G) = n - \max\{b: \sum_{i=1}^b p_i \leq n-b\}.$$

Proof. If $\sum_{i=1}^a p_i = n-a$, then $a = \max\{b: \sum_{i=1}^b p_i \leq n-b\}$

and Proposition 3.2D implies $f_3(G) = n-a$. If $\sum_{i=1}^a p_i > n-a$,

then $\sum_{i=1}^{a-1} p_i < n-(a-1)$ and $a-1 = \max\{b: \sum_{i=1}^b p_i \leq n-b\}$.

Proposition 3.2D implies that $f_3(G) = n-a+1 = n-(a-1)$. //

We now consider the point-outerthickness of $K(p_1, \dots, p_n)$ if $p_{a+1} \leq 2$.

Proposition 3.2F. Let $G = K(p_1, p_2, \dots, p_n)$, $n \geq 2$, and suppose $p_{a+1} \leq 2$, where a is the least integer such that $\sum_{i=1}^a p_i \geq n-a$. Also define

$$p_0 = 0,$$

$$r = \max\{i: p_i \leq 2\}, \quad \text{and}$$

$$k = \max\{i: p_i \leq 1\}.$$

$$\text{If } k+r-n \leq \frac{2}{3}(2r-n), \quad \text{then } f_3(G) = \left\{ \frac{\sum_{i=1}^r p_i + 3(n-r)}{4} \right\}.$$

$$\text{If } k+r-n > \frac{2}{3}(2r-n), \quad \text{then } f_3(G) = \left\{ \frac{2n-r}{3} \right\}.$$

Proof. In the proof we exhibit a partition of $V(G)$ into the desired number, say s , of subsets, each of which induces an outerplanar graph. Then we show that there is no outerplanar partition into fewer than s sets.

The inequality $r \geq a+1 > a$ implies that

$$\sum_{i=1}^a p_i \geq n-a > n-r. \quad \text{Thus there are more elements in the}$$

set $\bigcup_{i=1}^a V_i$ than sets in the collection $\{V_{r+1}, V_{r+2}, \dots, V_n\}$.

Hence we can form $n-r$ mutually disjoint sets

S_1, S_2, \dots, S_{n-r} where $S_j = V_{n+1-j} \cup \{v_j\}$, $1 \leq j \leq n-r$,

and $v_j \in \bigcup_1^a V_i$. Moreover, the points v_j are always selected successively from the set V_i with i minimum such that V_i has points remaining.

Each of the sets S_1, S_2, \dots, S_{n-r} induces a star with at least four points. Also, there are $\sum_1^r p_i - (n-r) > 0$ points of G not in any set S_i , $1 \leq i \leq n-r$. Each of these points is contained in a partite set of G which consists of at most two elements.

Case i. Suppose $k+r-n \leq \frac{2}{3}(2r-n)$. We observe that k is the number of one-point partite sets of G and $n-r$ is the number of sets formed thus far in the partition. If $k+r-n = k-(n-r)$ is positive, we have $k+r-n$ unused one-point partite sets of G . In defining the sets

S_1, S_2, \dots, S_{n-r} , we used points from at most $2(n-r)$ partite sets of G . Thus, there are at least

$n-2(n-r) = 2r-n$ partite sets of G which are disjoint from each S_i , $i = 1, \dots, n-r$. Since $k+r-n \leq \frac{2}{3}(2r-n)$,

we know that the number of unused one-point partite sets

of G is no greater than two-thirds of the number of

unused partite sets. Thus, we can form mutually disjoint

sets S_{n-r+1}, \dots, S_q , each consisting of two one-point

partite sets and one two-point partite set until we have

at most one unused partite set with one point. All remaining partite sets have precisely two points. If $k+r-n$ is not positive, then there are only two-point partite sets of G remaining and perhaps one more point which is an element of a two-point partite set from which the other point has been used. Thus, in either case, we have only two-point partite sets remaining, and possibly one extra point. With the remaining points, we may form mutually disjoint sets which consist of the union of two of the remaining two-point partite sets until at most one partite set with two points is unused. There are at most three points remaining. These points form a set which induces an outerplanar graph. Thus, we have partitioned $V(G)$ into

$$n-r + \left\{ \frac{\sum_{i=1}^r p_i - (n-r)}{4} \right\} = \left\{ \frac{\sum_{i=1}^r p_i + 3(n-r)}{4} \right\} = s$$

sets. Each set induces an outerplanar graph; and each set, with at most one exception, has at least four points.

This construction shows that $f_3(G) \leq s$. Suppose $f_3(G) < s$. Then $V(G)$ can be partitioned into sets T_1, T_2, \dots, T_t , $t < s$, such that each T_i induces an outerplanar subgraph and $|T_i| \geq |T_{i+1}|$ for $i = 1, 2, \dots, t-1$. There is a largest positive integer h

with the property that $|T_h| > |S_h|$. Then

$$\left| \bigcup_{h+1}^t T_i \right| < \left| \bigcup_{h+1}^s S_i \right|,$$

which implies that

$$\left| \bigcup_1^h T_i \right|^{-h} > \left| \bigcup_1^h S_i \right|^{-h}.$$

Since all S_i , except possibly S_s , have at least four points, T_h must have at least five points. Thus, for $i \leq h$, T_i has five or more points and Remark 3.2C implies that each such T_i has all but possibly one point from a single partite set. If such a point exists for a given T_i , denote it by w_i . Then, for $i \leq h$, define $T_i' = T_i - \{w_i\}$ for all i for which w_i exists and $T_i' = T_i$, otherwise. This implies that the set $\bigcup_1^h T_i'$ has all of its points in h or fewer partite sets. We now consider two subcases depending upon h .

Subcase a. $h \leq n-r$. In this subcase, $S_i = V_{n+1-i} \cup \{v_i\}$ for $i = 1, \dots, h$. This implies that

$$\left| \bigcup_{n-h+1}^n V_i \right| = \left| \bigcup_1^h S_i \right|^{-h}.$$

Hence, the union of any h partite sets has at most

$\left| \bigcup_1^h S_i \right|^{-h}$ points. However,

$$\left| \bigcup_1^h S_i \right|^{-h} < \left| \bigcup_1^h T_i \right|^{-h} \leq \left| \bigcup_1^h T_i' \right|.$$

Thus, $\left| \bigcup_1^h T_i' \right|$ cannot have all of its points in h or fewer partite sets. This is a contradiction.

Subcase b. $h > n-r$. The sets S_1, \dots, S_{n-r} exhaust all partite sets with three or more points. Since h is necessarily less than s , the sets S_{n-r+1}, \dots, S_h each use at least one partite set with two points. Without loss of generality, we may assume that these are the partite sets $V_{n+1-(n-r+1)}, \dots, V_{n+1-h}$. This implies that

$$\left| \bigcup_{n-h+1}^n V_i \right| < \left| \bigcup_1^h S_i \right|^{-h}.$$

The union of any h partite sets has at most $\left| \bigcup_{n-h+1}^n V_i \right|$ points. However, the fact that

$$\left| \bigcup_{n-h+1}^n V_i \right| < \left| \bigcup_1^h S_i \right|^{-h} < \left| \bigcup_1^h T_i \right|^{-h} \leq \left| \bigcup_1^h T_i' \right|$$

implies that $\bigcup_1^h T_i'$ cannot have all of its points in h or fewer partite sets. This is a contradiction.

In both subcases we have a contradiction, and thus $f_3(G) = s$.

Case ii. Suppose $k+r-n > \frac{2}{3}(2r-n)$. In this case, $2r-n$ is non-negative. In order to verify this, we suppose otherwise; then,

$$(3.2) \quad k \leq r < \frac{1}{2}n.$$

However, the hypothesis for this case implies that

$$3k > r+n.$$

Since $k \leq r$, we obtain

$$2k = 3k - k > r + n - r = n,$$

which contradicts (3.2).

Since $2r-n$ is non-negative, $k+r-n$ is positive. Thus, after forming sets S_1, S_2, \dots, S_{n-r} , the remaining partite sets include $k-(n-r) = k+r-n > 0$ sets with a single point. This implies that

$$S_i = v_i \cup v_{n+1-i}, \quad i = 1, 2, \dots, n-r,$$

and we have precisely $n-2(n-r) = 2r-n$ unused partite sets of G . Since $k+r-n > \frac{2}{3}(2r-n)$, there are more than twice as many unused partite sets with one point as unused partite sets with two points. It follows that we can form disjoint sets $S_{n-r+1}, \dots, S_{n-k}$ in such a way that each set consists

of four points, two from two-point partite sets and two from distinct one-point partite sets. When this is done, there are $k-(n-r)-2(r-k) = 3k-r-n$ points remaining in G . These points induce a complete subgraph and have an outerplanar partition into $\left\lfloor \frac{3k-r-n}{3} \right\rfloor$ sets. Let the sets in this partition be denoted by S_{n-k+1}, \dots, S_s . Here,

$$s = n-k + \left\lfloor \frac{3k-r-n}{3} \right\rfloor = \left\lfloor \frac{2n-r}{3} \right\rfloor, \quad \text{and} \quad f_3(G) \leq \left\lfloor \frac{2n-r}{3} \right\rfloor.$$

In order to show that $f_3(G) = s$, we suppose that $f_3(G) = t < s$. Then there exists an outerplanar partition of $V(G)$ into sets T_1, \dots, T_t such that $|T_i| \geq |T_{i+1}|$ for $i = 1, \dots, t-1$. Let h be the largest integer such that $|T_h| > |S_h|$.

In the partition of $V(G)$ into s subsets, each of the sets S_1, \dots, S_{n-r} induces a tree with at least four points and each of the sets $S_{n-r+1}, \dots, S_{n-k}$ induces the outerplanar graph $K(1,1,2)$. Now, each of $\langle S_{n-k+1} \rangle, \dots, \langle S_{s-1} \rangle$ is isomorphic to $K(1,1,1) = K_3$ and $\langle S_s \rangle$ is the complete graph on one, two, or three points. We observe that $|T_h| \geq 4$ and distinguish two subcases depending upon the number of points in T_h .

Subcase a. The cardinality of T_h is at least five.

Using the same argument as in Case i, we have that

$$\left| \bigcup_{i=1}^h T_i \right|^{-h} > \left| \bigcup_{i=1}^h S_i \right|^{-h}.$$

From the fact that T_h has at least five points, we know that, for $i \leq h$, $|T_i| \geq 5$ and, according to Remark 3.2C, all but perhaps one point of T_i are from a single partite set. As before, if such a point exists for a given T_i , denote it by w_i . Then, for $i \leq h$, define $T_i' = T_i - \{w_i\}$ for all i for which w_i exists and $T_i' = T_i$, otherwise. This implies that the set $\bigcup_{i=1}^h T_i'$ has all of its points in h or fewer partite sets of G .

The sets S_1, \dots, S_{n-r} exhaust all partite sets with at least three points and, for $i \leq n-r$, $S_i = V_{n+1-i} \cup \{v_i\}$. If $r \neq k$, then $S_{n-r+1}, \dots, S_{n-k}$ use all partite sets with two points and we can suppose, without loss of generality, that $S_i = V_{n+1-i} \cup \{u_i, w_i\}$, for $n-r \leq i \leq n-k$. This implies that, if $h \leq n-r$, then

$$\left| \bigcup_{i=n-h+1}^n V_i \right| = \left| \bigcup_{i=1}^h S_i \right|^{-h}$$

and, if $h > n-r$, then

$$\left| \bigcup_{i=n-h+1}^n V_i \right| < \left| \bigcup_{i=1}^h S_i \right|^{-h}.$$

In either case,

$$\left| \bigcup_{n-h+1}^n v_i \right| \leq \left| \bigcup_1^h s_i \right|^{-h} < \left| \bigcup_1^h T_i \right|^{-h} \leq \left| \bigcup_1^h T_i' \right|.$$

Thus, $\bigcup_1^h T_i'$ cannot have all of its points in h or fewer partite sets, which is a contradiction.

Subcase b. The cardinality of T_h is four. For $i \leq h$, T_i has four or more points and at least two of these points must be in a single partite set of G . Since $|T_h| = 4$, S_h has three points. In fact, S_h is the union of three partite sets, each of which contains only one point. Recall that $|T_1| \geq |T_2| \geq \dots \geq |T_t|$. Further, order the collection T_i , $i = 1, \dots, t$, so that, if $|T_i| = |T_j|$, then $i < j$ if T_i has more points from some partite set than T_j has from any partite set.

Let $m = \max\{i: p_i \leq 3\}$. The partite sets

v_1, \dots, v_k each contain exactly one point.

If $r \neq k$, then

v_{k+1}, \dots, v_r each contain exactly two points.

If $m \neq r$, then

v_{r+1}, \dots, v_m each contain exactly three points.

If $n \neq m$, then

v_{m+1}, \dots, v_n each contain four or more points.

Suppose each of the sets V_{m+1}, \dots, V_n is contained in some T_i . From the ordering on T_i we may assume, without loss of generality, that $V_i \subset T_{n+1-i}$, for $i = m+1, \dots, n$. Also recall that, for $i = 1, \dots, n-k$, $S_i = V_{n+1-i} \cup W_i$ where W_i consists of one or two points. From the fact that S_h consists of three points from three different partite sets, we have that

$$(3.1) \quad h > n-k.$$

The sets $T_{n-m+1}, T_{n-m+2}, \dots, T_h$ each have at least four points and therefore at least two points from one partite set. However, all partite sets with at least four points are used in sets T_1, \dots, T_{n-m} . Thus, we need $h - (n-m+1) + 1$ partite sets with two or three points, and there are only $(n-k) - (n-m+1) + 1 = m-k$ such partite sets. Hence, using inequality (3.1), we have a contradiction.

Therefore, we assume that at least one of the partite sets with four or more points, say V_{i_0} , has points in two or more of the sets T_i . With this assumption, we show that we can obtain a new outerplanar partition of $V(G)$ into t sets such that V_{i_0} is contained in some set of the new partition.

If V_{i_0} has three or more points in one T_i , say T_b , then T_b has at most one point which is not in V_{i_0} . By adding all other points of V_{i_0} to T_b , we obtain

a new outerplanar partition of $V(G)$ into t sets such that V_{i_0} is contained in some set of the new partition.

Thus, we suppose V_{i_0} does not have three or more points in any T_i . If V_{i_0} has two points in T_b and one or two points in T_c , $c \neq b$, we can add the points of $T_c \cap V_{i_0}$ to T_b and one point of $T_b - V_{i_0}$ (if such a point exists) to T_c . Call these new sets T_b' and T_c' . Both induce an outerplanar graph, and $T_b' \cup T_c' = T_b \cup T_c$. We now have an outerplanar partition of $V(G)$ into t sets such that V_{i_0} has three or more points in one T_i . From the preceding paragraph, we know we can now obtain an outerplanar partition of $V(G)$ into t sets such that V_{i_0} is contained in one of the sets.

If V_{i_0} has each point in a different T_i , then T_h has at most one point of V_{i_0} . Let w_1, w_2 , and w_3 be points in $T_h - V_{i_0}$. Add all points of V_{i_0} to T_h . Since V_{i_0} has at least four points, three of these points must have been in three distinct T_i different from T_h , say T_{i_1}, T_{i_2} , and T_{i_3} . For $k = 1, 2, 3$, insert w_k into T_{i_k} . Then T_h' consists of V_{i_0} plus possibly one other point. We have a new outerplanar partition of $V(G)$ into t sets with all points of V_{i_0} in a single set.

Thus, we have shown that, if $m \neq n$, there exists an outerplanar partition of $V(G)$ into t sets such that

one partite set V_{i_0} , $m < i_0 \leq n$, is contained in some set of the partition. Among the outerplanar partitions of $V(G)$ into t sets, select one which has a maximum number, say M , of V_i , $m < i \leq n$, each of which is contained in some set of the partition. Call this partition T_1, \dots, T_t , and order the sets of this partition in the same manner as the previous partition. Again, we let h be the largest integer such that $|T_h| > |S_h|$. If $|T_h| \geq 5$, we apply subcase a, and we obtain a contradiction. If $|T_h| = 4$ and each V_i , $m < i \leq n$ is contained in some T_j , we have the contradiction developed in the first part of this subcase. Thus, we suppose $|T_h| = 4$ and there exists a partite set V_{i_0} , $m < i_0 \leq n$, which is not contained in any T_j .

If V_{i_0} has at least three points in one T_j , say T_b , we add all other points of V_{i_0} to T_b . Thus, a new outerplanar partition of $V(G)$ into t sets is obtained. In removing points of V_{i_0} from the various T_j , we did not move any points from V_i , $i \neq i_0$. We now have an outerplanar partition of $V(G)$ into t sets such that $M+1$ partite sets with at least four points are contained in various T_j . This is a contradiction.

If V_{i_0} has exactly two points in some T_i , say T_b , then V_{i_0} has one or two points in T_c , $c \neq b$. We add

the points of $T_c \cap V_{i_0}$ to T_b and add one point of $T_b - V_{i_0}$ (if such a point exists) to T_c . Call these new sets T_b' and T_c' , respectively. Clearly, T_b did not contain a V_i , $m < i \leq n$, and any V_i contained in T_c is still contained in T_c' . Thus, we have an outerplanar partition of $V(G)$ into t sets such that V_{i_0} has three or more points in one set, and M partite sets V_i , $m < i \leq n$, are each contained in some T_j . According to the previous paragraph, this leads to a contradiction.

Finally, before considering the possibility that V_{i_0} has its points distributed among $|V_{i_0}|$ different T_j , we show that T_h does not contain V_i , for $i = m+1, \dots, n$. To do this, we suppose there exists V_{i_1} , $m < i_1 \leq n$, which is contained in T_h . Since $|T_h| = 4$ and $|V_{i_1}| \geq 4$, we know that $T_h = V_{i_1}$. From our ordering on the partition T_1, \dots, T_t , we know that the sets T_1, \dots, T_h have at most $h-1$ points from one-point partite sets of G . The sets T_{h+1}, \dots, T_t have at most $\left| \bigcup_{i=h+1}^t T_i \right|$ points from one-point partite sets of G . The partition T_1, \dots, T_t uses all one-point partite sets of G , and the number used must be not more

than $h-1 + \left| \bigcup_{h+1}^t T_i \right|$. Thus,

$$(3.2) \quad h-1 + \left| \bigcup_{h+1}^t T_i \right| \geq k.$$

The set S_h is the union of three one-point partite sets of G , and thus the sets S_{h+1}, \dots, S_s each consist of only points from one-point partite sets; that is, the sets S_{h+1}, \dots, S_s contain $\left| \bigcup_{h+1}^s S_i \right|$ points from one-point partite sets. However, each of the sets S_1, \dots, S_h contains at least one point from a one-point partite set. Thus, the partition S_1, \dots, S_s contains at least $h + \left| \bigcup_{h+1}^s S_i \right|$ points from one-point partite sets. It follows that

$$(3.3) \quad k \geq h + \left| \bigcup_{h+1}^s S_i \right|.$$

The fact, previously observed, that $\left| \bigcup_{h+1}^s S_i \right| > \left| \bigcup_{h+1}^t T_i \right|$,

together with (3.2) and (3.3), implies that

$$k \geq h + \left| \bigcup_{h+1}^s S_i \right| > h-1 + \left| \bigcup_{h+1}^t T_i \right| \geq k.$$

This is clearly impossible, and hence T_h does not contain a V_i , $m < i \leq n$. The set T_h therefore has points from at least two different partite sets.

Keeping in mind that T_h must have points from at least two different partite sets, we suppose V_{i_0} has each point in a different T_j . Then T_h has at most one point of V_{i_0} . Let w_1 , w_2 , and w_3 be points in $T_h - V_{i_0}$. Add all points of V_{i_0} to T_h . Since V_{i_0} has at least four points, three of these points must be in distinct T_j different from T_h , say T_{i_1} , T_{i_2} , and T_{i_3} . For $k = 1, 2, 3$, insert w_k into T_{i_k} . As before, this yields a new outerplanar partition of $V(G)$ into t sets. Since T_h did not contain any partite sets with four or more points, this new partition has $M+1$ sets, each of which contains a V_i , $m < i \leq n$. This is a contradiction.

In all cases where $|T_h| = 4$, we have a contradiction. Hence, in both subcases, we have a contradiction and $f_3(G) = s$. //

We now combine Corollary 3.2E and Proposition 3.2F in order to give a complete formula for the point-outerthickness of a complete n -partite graph.

Theorem 3.2H. Let $G = K(p_1, p_2, \dots, p_n)$, $n \geq 2$,
 $p_0 = 0$, and define a as the least integer such that

$$\sum_{i=1}^a p_i \geq n-a. \text{ Also define}$$

$$r = \max\{i: p_i \leq 2\} \quad \text{and}$$

$$k = \max\{i: p_i \leq 1\}.$$

Then,

$$f_3(G) = \begin{cases} n - \max\{b: \sum_{i=1}^b p_i \leq n-b\} & \text{if } p_{a+1} \geq 3, \\ \left\{ \frac{\sum_{i=1}^r p_i + 3(n-r)}{4} \right\} & \text{if } p_{a+1} \leq 2 \\ & \text{and } k+r-n \leq \frac{2}{3}(2r-n), \\ \left\{ \frac{2n-r}{3} \right\} & \text{if } p_{a+1} \leq 2 \\ & \text{and } k+r-n > \frac{2}{3}(2r-n). \end{cases}$$

CHAPTER IV

POINT-PARTITION NUMBERS

In the preceding chapter, the point-partition number $f_n(G)$, $n \geq 1$, of a graph G is defined as the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a graph with property F_n . As we have noted previously, the chromatic number and point-arboricity of G are then $f_1(G)$ and $f_2(G)$ respectively. We now consider the parameter f_n , $n \geq 3$, and investigate possible generalizations of theorems dealing with chromatic number and point-arboricity to the parameters f_n for all positive integers n .

Section 4.1

Generalizations of Chromatic Number and Point-Arboricity

Proposition 2.2B shows that, for each graph G

$$\frac{f_1(G)}{2} \leq f_2(G) \leq f_1(G).$$

In [4] Chartrand, Geller, and Hedetniemi show that the chromatic number of any outerplanar graph does not exceed

3. This theorem is used to establish a result, analogous to Proposition 2.2B, dealing with point-outerthickness.

Proposition 4.1A. For any graph G ,

$$(4.1) \quad \frac{f_1(G)}{3} \leq f_3(G) \quad \text{and}$$

$$(4.2) \quad f_3(G) \leq f_1(G).$$

Proof. Inequality (4.2) follows immediately from the fact that any totally disconnected graph is outerplanar.

To establish (4.1) we observe that there is a partition of $V(G)$ into $f_3(G)$ subsets such that the graph induced by each subset is outerplanar. Since any outerplanar graph has chromatic number at most 3, $f_1(G) \leq 3f_3(G)$, which yields the desired inequality. //

Clearly, graphs with property F_1 also have property F_n for $n \geq 1$. This implies the following:

Proposition 4.1B. For any graph G and any positive integer n ,

$$(4.3) \quad f_n(G) \leq f_1(G).$$

Unfortunately, the corresponding generalization of inequality (4.2) is not so easily established. In fact, after stating this generalization as a conjecture, we

show that, for $n = 4$, this statement is equivalent to the Four Color Conjecture.

Conjecture 4.1C. For every graph G and any positive integer n ,

$$(4.4) \quad \frac{f_1(G)}{n} \leq f_n(G).$$

Proposition 4.1D. For $n = 4$, inequality (4.4) is equivalent to the Four Color Conjecture.

Proof. Suppose that the Four Color Conjecture is true so that every planar graph has chromatic number at most 4. For a graph G , the set $V(G)$ may be partitioned into $f_4(G)$ subsets, each of which induces a planar graph. It then follows that $f_1(G) \leq 4f_4(G)$.

Conversely, assume $\frac{f_1(G)}{4} \leq f_4(G)$ for any graph G .

If H is any planar graph, then $f_4(H) = 1$. It follows that $f_1(H) \leq 4f_4(H) = 4$ which implies the Four Color Conjecture. //

For the complete graph K_{np} , $f_n(K_{np}) = p$ and $f_1(K_{np}) = np$. Thus, for any positive integer n , there is a graph such that equality holds in (4.4). The following proposition shows that, for $n \geq 1$, there is an infinite class of graphs for which equality holds in (4.3).

Proposition 4.1E. For any positive integers n and k , there exists a graph H such that $f_n(H) = f_1(H) = k$.

Proof. If $n = 1$, let $H = K_k$. For $k = 1$ and for any natural number n , any totally disconnected graph is the required example.

Suppose $n \geq 2$ and $k \geq 2$. Let H be the complete k -partite graph $K(p_1, \dots, p_k)$ where $p_i = kn$ for $i = 1, \dots, k$. Certainly $f_1(H) = k$, and thus $f_n(H) \leq k$.

Assume $f_n(H) < k$, and W_1, \dots, W_{k-1} is a partition of $V(H)$ such that each W_i induces a graph with property F_n . Since $(nk+n)(k-1) = nk^2 - n < nk^2 = |H|$, one of the sets, say W_1 , contains more than $nk+n$ points. The fact that graph H has exactly k partite sets implies that W_1 must contain at least $n+1$ points from one partite set, say V_1 . The set V_1 contains precisely kn points, and thus W_1 must have at least $n+1$ points which are not in V_1 . It follows that $K(n+1, n+1)$ is a subgraph of $\langle W_1 \rangle$. However,

$$\left\lfloor \frac{n+2}{2} \right\rfloor \leq \left\{ \frac{n+2}{2} \right\} \leq n+1, \text{ and thus } \langle W_1 \rangle \text{ contains } K\left(\left\lfloor \frac{n+2}{2} \right\rfloor, \left\{ \frac{n+2}{2} \right\}\right). \text{ This implies that the graph induced by}$$

W_1 does not have property F_n . This is a contradiction, and hence $f_n(H) = k = f_1(H)$. //

Proposition 4.1F. Let m , n , and k be positive integers and $m < n$. Then, for any graph G , $f_n(G) \leq f_m(G)$. Furthermore, there exists a graph H such that $f_n(H) = f_m(H) = k$.

Proof. For $m < n$, a graph with property F_m also has property F_n . There is a partition of $V(G)$ into $f_m(G)$ subsets such that each subset induces a graph with property F_m . Thus, each induced subgraph has property F_n and this implies that $f_n(G) \leq f_m(G)$.

According to Proposition 4.1E, there exists a graph H with $f_n(H) = f_1(H) = k$. However, the first part of this proposition implies that $f_n(H) \leq f_m(H) \leq f_1(H)$, which establishes the last part of this proposition. //

If G is a graph which is k -critical with respect to f_1 , then $\delta(G) \geq k-1$ where $\delta(G)$ denotes the minimum degree of G . This result can be used to establish the fact, first proved by Szekeres and Wilf [22], that for any graph G , $f_1(G) \leq 1 + \max \delta(G')$ where the maximum is taken over all induced subgraphs G' of G . The analogues for point-arboricity to these two theorems have already been stated in Chapter II. We combine these four results in the following two statements:

Theorem 4.1G. If G is k -critical with respect to f_n , for $n = 1, 2$, then

$$(4.5) \quad \delta(G) \geq nk - n.$$

Theorem 4.1H. For $n = 1, 2$ and for any graph G ,

$$(4.6) \quad f_n(G) \leq 1 + \left\lfloor \frac{\max \delta(G')}{n} \right\rfloor$$

where the maximum is taken over all induced subgraphs G' of G .

Contrary to what one might hope, the inequalities in Propositions 4.1G and 4.1H are not valid for $n \geq 3$. We first show this for $n = 3$.

Proposition 4.1I. For any integer $k > 1$, there exists a graph G which is k -critical with respect to f_3 and $\delta(G) < 3k - 3$. Furthermore, $f_3(G) > 1 + \frac{\max \delta(G')}{3}$ where the maximum is taken over all induced subgraphs of G .

Proof. Let $G = K_{3k-4} + \overline{K}_3$. Then G is a subgraph of K_{3k-1} and $f_3(G) \leq f_3(K_{3k-1}) = k$. Since any induced subgraph of G with more than four points contains $K(2,3)$, it follows that induced outerplanar subgraphs have at most four points. We assume $f_3(G) \leq k-1$ and let V_1, \dots, V_{k-1} be an outerplanar partition of $V(G)$. Each

V_1 has at most four points. From the fact that $3(k-1)+2 = 3k-1$ is the number of points of G , we know that at least two of the sets of the partition must have four points. However, any set with four points from K_{3k-4} or three points from K_{3k-4} and one point from \overline{K}_3 induces the graph K_4 . This implies that at most one of the sets of the partition has four points. This is a contradiction, and hence $f_3(G) = k$.

In order to show that G is k -critical with respect to f_3 , we consider $H = G - v$ where $v \in V(G)$. We note that H has order $3k-2$. Let S be a subset of $V(H)$ consisting of two points from K_{3k-4} and two points from \overline{K}_3 . (If $k = 2$ and $v \in K_{3k-4}$, then let S consist of the one remaining point of K_2 and all points of \overline{K}_3 .) The set S induces an outerplanar graph, $|V(H) - S| = 3k-6$, and $H - S$ is the complete graph on $3k-6$ points. Thus, $f_3(H - S) = k-2$, which implies $f_3(H) = k-1$, and G is k -critical with respect to f_3 .

It is clear that $\delta(G) = 3k-4 < 3k-3$. In order to establish the second part of the proposition, let G' be an induced subgraph of G . If G' contains a point v of \overline{K}_3 , then $\deg v \leq 3k-4$ and $\delta(G') \leq 3k-4$. If G

contains no point of \overline{K}_3 , then it has at most $3k-4$ points and $\delta(G') \leq 3k-4$. Hence,

$$1 + \frac{\max \delta(G')}{3} = 1 + \frac{3k-4}{3} = k - \frac{1}{3} < k = f_3(G),$$

which completes the proof. //

Combining Theorem 2.1D with the well-known result that, for any graph G ,

$$f_1(G) \leq 1 + \Delta(G),$$

we arrive at the following:

Theorem 4.1J. For $n = 1, 2$ and for any graph G ,

$$(4.7) \quad f_n(G) \leq 1 + \left\lceil \frac{\Delta(G)}{n} \right\rceil.$$

For $n \geq 4$, the graph $G = K \left(\left\lfloor \frac{n+2}{2} \right\rfloor, \left\{ \frac{n+2}{2} \right\} \right)$ shows

that inequality (4.7) is not valid. This follows from

the fact that $1 + \frac{\Delta(G)}{n} \leq 1 + \frac{n+3}{2n} < 2 = f_n(G)$. This

observation can be used to show that inequalities (4.5) and (4.6) do not hold in general for $n \geq 4$. To do this, we let $n \geq 4$ and suppose that, for any graph G ,

$$f_n(G) \leq 1 + \frac{\max \delta(G')}{n} \quad \text{where the maximum is taken over all}$$

induced subgraphs G' of G . Since $\delta(G') \leq \Delta(G)$ for all induced subgraphs G' of G , we conclude that

$$f_n(G) \leq 1 + \frac{\Delta(G)}{n} \quad \text{for any graph } G. \text{ This is impossible, and}$$

hence the conclusion of Theorem 4.1H is not correct for $n \geq 4$.

Assume that, for $n \geq 4$, $\delta(H) \geq nk - n$ where H is a k -critical graph with respect to f_n . We show that this assumption implies Theorem 4.1H for any graph G and $n \geq 4$. Let G be a graph such that $f_n(G) = k \geq 2$. Let H be any induced subgraph of G with minimum order such that $f_n(H) = k$. Therefore, H is k -critical with respect to f_n . Also,

$$\delta(H) \leq \max_{G' < H} \delta(G')$$

because H itself is an induced subgraph of H . Also, each induced subgraph of H is an induced subgraph of G so that

$$\max_{G' < H} \delta(G') \leq \max_{G' < G} \delta(G').$$

According to our assumption, $\delta(H) \geq nk - n$ so that

$$\max_{G' < G} \delta(G') \geq nk - n = n(f_n(G) - 1).$$

However, we have already shown that this is not necessarily true for $n \geq 4$. From this contradiction, it follows that the conclusion of Theorem 4.1G is not valid for $n \geq 4$.

In summary, we have shown that the conclusions in Theorem 4.1G and 4.1H are not valid for $n \geq 3$ and the conclusion of Theorem 4.1J is not valid for $n \geq 4$.

Whether or not $f_3(G) \leq 1 + \frac{\Delta(G)}{3}$ for every graph G

remains an open question.

Lick and White [15] have introduced another generalization of chromatic number and point-arboricity and have obtained results which suggest the direction that one probably must take in order to generalize Theorems 4.1G, 4.1H, and 4.1J completely.

A graph G is called n -degenerate for $n \geq 0$ if, for each induced subgraph H of G , $\delta(H) \leq n$. The 0-degenerate graphs and the 1-degenerate graphs are the totally disconnected and acyclic graphs respectively. Lick and White have then defined the point-partition number $g_n(G)$, $n \geq 0$, of a graph G as the minimum number of sets into which the point set $V(G)$ can be partitioned so that each set induces an n -degenerate graph. Thus, the chromatic number and point-arboricity of a graph G are $g_0(G)$ and $g_1(G)$ respectively. Using parameter g_n ,

the following generalizations of Theorems 4.1G, 4.1H, and 4.1J can be established.

Theorem 4.1K. (Lick, White) If graph G is k -critical with respect to g_n , then

$$\delta(G) \geq (k-1)(n+1).$$

Theorem 4.1L. (Lick, White) For any graph G ,

$$g_n(G) \leq 1 + \left\lceil \frac{\max \delta(H)}{n+1} \right\rceil$$

where the maximum is taken over all induced subgraphs H of G .

Theorem 4.1M. (Lick, White) For any graph G ,

$$g_n(G) \leq 1 + \left\lceil \frac{\Delta(G)}{n+1} \right\rceil.$$

Other results which Lick and White have generalized by using parameter g_n , $n \geq 0$, include Theorems 2.1A and 2.1E.

Section 4.2

Graphs with Prescribed Clique Number
and Point-Partition Number

In this, the concluding section, we reconsider several results established in Chapters II and III and discuss their possible generalizations to the parameters mentioned in Section 4.1. We include here a generalization of Theorem 2.4A.

We have remarked that a number of results for chromatic number and point-arboricity which do not readily generalize to the parameters f_n have been generalized to the parameters g_n . In Section 2.2 we stated the Nordhaus-Gaddum Theorem concerning the chromatic number of complementary graphs and proved a corresponding result for point-arboricity. Similar theorems for parameters f_n , $n \geq 3$, have not been established. Lick and White have succeeded in developing a generalization of the Nordhaus-Gaddum Theorem for the parameters g_n , $n \geq 0$. However, for some values of n , the lower bounds which they obtained are not best possible.

A graph G is called uniquely k -partitionable with respect to property F_n , $n \geq 1$, if $f_n(G) = k$ and there is only one partition of $V(G)$ into k sets such that each set induces a graph with property F_n . In

Section 2.3 we investigated uniquely k -partitionable graphs with respect to property F_2 (i.e., uniquely k -arborable graphs). For $n \geq 3$, uniquely k -partitionable graphs with respect to property F_n have not been investigated.

In this thesis, we defined a graph G to have hypo-property P if G does not have property P but $G-v$ has property P for every $v \in V(G)$. A characterization was given for hypo-outerplanar graphs (i.e., graphs with hypo-property F_3) which was similar to Wagner's characterization of hypo-planar graphs (i.e., graphs with hypo-property F_4). Graphs with hypo-property F_n , $n \geq 5$, have not been studied; however, a complete characterization of such graphs would appear to be extraordinarily complicated.

In Section 2.4 we stated a theorem due to Zykov; namely, for any integers k, d such that $k \geq d > 1$, there exists a graph G with chromatic number k and clique number d . We then proved a corresponding result for point-arboricity. A generalization of Zykov's Theorem for parameter f_n , $n \geq 1$, is now given. The proof closely parallels that given in Chapter II, with minor alterations to accommodate the general parameter f_n . Again, we commence with the special case $d = 2$.

Theorem 4.2A. For every pair n, m of positive integers, there exists a graph G with no triangles such that $f_n(G) = m$.

Proof. For $n = 1$, this theorem was proved by Zykov [26]. We let $n \geq 2$ and $m \geq 1$ be integers. Define H_1 as the totally disconnected graph with point set W_1 , where W_1 consists of nm^2 points.

If $m = 1$, H_1 is the required graph. We therefore suppose that $m > 1$ and define a graph H_2 . For the purpose of doing this, we introduce a number of different sets. Let

$$B_2 = \{S : S \text{ is a set of independent points of } H_1 \text{ and } |S| \geq n\}.$$

For each set S in B_2 , we let V_S be a set of nm^2 points disjoint from $V(H_1)$. Moreover, for $S \neq S'$, the sets V_S and $V_{S'}$ are disjoint. The sets W_2 , E_S , and E_2 are defined as follows:

$$W_2 = \bigcup \{V_S : S \in B_2\},$$

$$E_S = \{uv : u \in S, v \in V_S\},$$

$$E_2 = \bigcup \{E_S : S \in B_2\}.$$

Finally, we define H_2 by letting $V(H_2) = W_2 \bigcup V(H_1)$ and $E(H_2) = E_2 \bigcup E(H_1)$.

Suppose that, for $2 \leq k < m$, graph H_k has been defined. We now form the graph H_{k+1} . Let

$$B_{k+1} = \{T: T \text{ is an independent subset of } V(H_k) \text{ and } |T| \geq n\}.$$

For each set T in B_{k+1} , let V_T be a set with nm^2 points with the properties that $V_T \cap V_{T'} = \emptyset$ for $T \neq T'$ and $V_T \cap V(H_k) = \emptyset$. We let

$$W_{k+1} = \bigcup \{V_T: T \in B_{k+1}\},$$

$$E_T = \{uv: u \in T, v \in V_T\},$$

$$E_{k+1} = \bigcup \{E_T: T \in B_{k+1}\},$$

$$V(H_{k+1}) = W_{k+1} \bigcup V(H_k), \quad \text{and}$$

$$E(H_{k+1}) = E_{k+1} \bigcup E(H_k).$$

The m th step of this construction forms a graph H_m , and we let $G = H_m$.

In order to show that G has no triangles, we assume that the set $\{v_1, v_2, v_3\}$ induces a triangle in G . The fact that $\langle W_i \rangle$, $i = 1, \dots, m$, is totally disconnected in G implies that v_1 , v_2 , and v_3 are in three different W_i , say W_{i_1} , W_{i_2} , and W_{i_3} respectively. Without loss of generality, we may assume that $i_1 < i_2 < i_3$. This implies that v_1 and v_2 are adjacent in H_{i_3-1} . Thus,

v_3 is not adjacent to both v_1 and v_2 . We have a contradiction, and hence G has no triangles.

It remains only to show that $f_n(G) = m$. For $i = 1, 2, \dots, m$, the graph $\langle W_i \rangle$ is totally disconnected and therefore has property F_n . Thus $f_n(G) \leq m$. Assume $f_n(G) < m$. Then there is a partition of $V(G)$ into $m-1$ sets U_1, U_2, \dots, U_{m-1} such that the subgraph of G induced by each U_i has no subgraph homeomorphic from $K\left(\left[\frac{n+2}{2}\right], \left\{\frac{n+2}{2}\right\}\right)$. Since $n > 1$, it follows that each $\langle U_i \rangle$ does not contain $K(n, n)$.

The set W_1 has nm^2 points. Therefore, one set of the partition, say U_1 , contains more than nm points of W_1 . Let

$$\{u_1^{(1)}, u_2^{(1)}, \dots, u_{nm}^{(1)}\} \subset U_1 \cap W_1.$$

The fact that the set

$$M_1 = \{u_1^{(1)}, \dots, u_n^{(1)}\}$$

is independent in $V(H_1)$ implies there is a set S_2 of nm^2 points of W_2 such that each point of S_2 is adjacent to exactly the points of M_1 . The average number of points of S_2 in each of U_1, \dots, U_{m-1} exceeds nm . The set U_1 has less than n points of S_2 because,

otherwise, $\langle U_1 \rangle$ would contain $K(n, n)$. Thus, at least nm points of S_2 are in some U_i different from U_1 , say U_2 . Let

$$\{u_1^{(2)}, u_2^{(2)}, \dots, u_{nm}^{(2)}\} \subset S_2 \cap U_2.$$

Since every element of $S_2 \cap U_2$ is adjacent to exactly the points of M_1 , the set

$$M_2 = \{u_{n+1}^{(2)}, \dots, u_{2n}^{(2)}, u_{n+1}^{(1)}, \dots, u_{2n}^{(1)}\}$$

is independent in H_2 . In W_3 there is a set S_3 of nm^2 points adjacent to exactly the points of M_2 . No set of n points of S_3 can be in U_1 nor in U_2 . However, these nm^2 points must be distributed among the sets U_1, \dots, U_{m-1} . Thus, there is a set U_j , $j > 2$, say U_3 , such that U_j has at least nm points of S_3 . We let

$$\{u_1^{(3)}, \dots, u_{nm}^{(3)}\} \subset S_3 \cap U_3.$$

Since every point in $S_3 \cap U_3$ is adjacent to exactly the points in M_2 , the set

$$M_3 = \{u_{2n+1}^{(3)}, \dots, u_{3n}^{(3)}, u_{2n+1}^{(2)}, \dots, u_{3n}^{(2)}, u_{2n+1}^{(1)}, \dots, u_{3n}^{(1)}\}$$

is independent in H_3 .

Assume now that, for $j < m-1$, the graph H_j has an independent set of points

$$M_j = \{u_{(j-1)n+1}^{(j)}, \dots, u_{jn}^{(j)}; \dots; u_{(j-1)n+1}^{(1)}, \dots, u_{jn}^{(1)}\}$$

such that, for $i = 1, \dots, j$,

$$\{u_{(j-1)n+1}^{(i)}, \dots, u_{jn}^{(i)}\} \subset U_i \cap S_i$$

where $S_1 = W_1$ and, for $i = 2, \dots, j$, S_i is a set of nm^2 points each adjacent to exactly the points of M_{i-1} and $U_i \cap S_i$ contains the points $\{u_1^{(i)}, u_2^{(i)}, \dots, u_{nm}^{(i)}\}$.

Since the points of M_j are independent in H_j , there is a set S_{j+1} of nm^2 points which are adjacent to exactly the points of M_j . No set of n points of S_{j+1} can be in U_k for $k \leq j$ because, otherwise, these n points, together with $u_{(j-1)n+1}^{(k)}, \dots, u_{jn}^{(k)}$ would induce $K(n, n)$. Also, the nm^2 points must be distributed among U_1, \dots, U_{m-1} . Thus, there are at least nm points of S_{j+1} in some U_i , $i \geq j+1$, say U_{j+1} . Let

$$\{u_1^{(j+1)}, \dots, u_{nm}^{(j+1)}\} \subset S_{j+1} \cap U_{j+1}.$$

Since the points of $S_{j+1} \cap U_{j+1}$ are adjacent to exactly the points of M_j , the set

$$M_{j+1} = \{u_{jn+1}^{(j+1)}, \dots, u_{(j+1)n}^{(j+1)}; \dots; u_{jn+1}^{(1)}, \dots, u_{(j+1)n}^{(1)}\}.$$

is independent in H_{j+1} .

Thus, for $j = m-2$, we have formed an independent set $M_{j+1} = M_{m-1}$ of points of G which contain n points from each of U_1, \dots, U_{m-1} . Thus, there exists a set S_m of nm^2 points of W_m such that the points of S_m are adjacent to exactly the points of M_{m-1} . For some $i = 1, \dots, m-1$, the set $S_m \cap U_i$ must have more than nm points, and therefore $\langle U_i \rangle$ contains $K(n, n)$. This contradiction implies $f_n(G) = m$. //

As with the special case for point-arboricity, the analogue of Zykov's result is immediate.

Corollary 4.2B. For any integers n , k , and d such that $n \geq 1$, $d \geq 2$, and $k \geq \left\lceil \frac{d}{n} \right\rceil$, there exists a graph H such that $f_n(H) = k$ and $\omega(H) = d$.

Proof. Let G be the graph given in Theorem 4.2A determined by k and n . Define H as $G \cup K_d$. The fact that G has no triangles implies that $\omega(H) = \omega(K_d) = d$. Since $f_n(H) = \max\{f_n(G), f_n(K_d)\}$ and $f_n(K_d) = \left\lceil \frac{d}{n} \right\rceil \leq k$, it follows that $f_n(H) = k$. //

As with Corollary 2.4B, the graph H given above can be made connected by adding a line joining a point in G with a point in K_d .

In [13] House showed that, for any integers k and d such that $1 < d \leq k$, there exists a graph G such that $\omega(G) = d$ and G is k -critical with respect to f_1 .

For $n \geq 1$, graph H , given in Corollary 4.2B, contains a subgraph H' which is k -critical with respect to f_n . For example, take H' as any smallest induced subgraph of H such that $f_n(H') = k$. If $d = 2$, graph H' still has clique number d . However, if $d \geq 3$, the maximal complete subgraph of H may have been destroyed in obtaining H' . Thus, H has a subgraph which is k -critical with respect to f_n . But all such subgraphs may have clique number less than d . Hence, we can only conjecture that a result analogous to that obtained by House exists for f_n , $n \geq 2$.

Conjecture 4.2C. For any integers n , d , and k such that $n \geq 2$, $d \geq 3$, and $k \geq \left\lceil \frac{d}{n} \right\rceil$, there exists a graph H which is k -critical with respect to f_n and $\omega(H) = d$.

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