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CHARACTERIZATIONS OF SOME
FUNCTORS OF CATEGORIES
OF BANACH SPACES

by

Kenneth L. Pothoven

A Dissertation
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TABLE OF CONTENTS

SECTION		PAGE
I	PRELIMINARIES	1
II	THE HOM AND TENSOR FUNCTORS AND EXACT SEQUENCES	14
III	NATURAL TRANSFORMATIONS AND THEIR DUALS . .	37
IV	CHARACTERIZATIONS OF COMPACT FUNCTORS . . .	49

I PRELIMINARIES

In [11] Mityagin and Shvarts list many problems concerning functors and dual functors in categories of Banach spaces. Included in these problems are the following questions: (1) What properties characterize compact functors? and (2) If a functor is compact is its dual functor compact? The motivation for this present paper is to answer these questions. Precisely, the purpose of this paper is threefold:

- (1) to investigate when the hom functor Ω_X and the tensor functor Σ_X take (strictly) normal exact sequences to normal exact sequences.
- (2) to answer the question: If a functor takes compact operators to compact operators, does its dual do the same?
- (3) to find when Σ_X and Ω_X take compact operators to compact operators. More generally, the purpose is to find a characterization of all functors that take compact operators to compact operators.

This section consists of a discussion of preliminary concepts needed for these investigations. Functional analysis concepts may generally be found in [3]. Categorical terms may be found in [10]. In the following discussion, \mathcal{B} will denote the category in which the objects are Banach spaces over the real scalar field and the morphisms

are continuous linear functions, sometimes called mappings or operators. Elements of the scalar field I are generally denoted by letters such as α, β , and γ . The objects of \mathbb{B} will be designated by letters such as A, B, C, X, Y , or Z ; the morphisms will be denoted by letters like f, g, h , and k . However, in a later section of the discussion, these letters will also be used to denote functions that are not morphisms in \mathbb{B} . The set $\mathbb{B}(A, B)$ is the set of all morphisms from A to B . The notation $f: A \longrightarrow B$ is used to mean f is a morphism in $\mathbb{B}(A, B)$. $\mathbb{B}(A, B)$ is a Banach space with norm given by $|f| = \sup_{|a| \leq 1} |f(a)|$.

The concept of functor will be fundamental to what follows.

Definition 1.1 A (covariant) functor $F: \mathbb{B} \longrightarrow \mathbb{B}$ is an assignment of each object A in \mathbb{B} to an object $F(A)$ (or FA) in \mathbb{B} and of each morphism $f: A \longrightarrow B$ in \mathbb{B} to a morphism $F(f): F(A) \longrightarrow F(B)$ (or $Ff: FA \longrightarrow FB$) in \mathbb{B} subject to the following conditions:

- (1) If the composition $g \circ f: A \longrightarrow C$ of $f: A \longrightarrow B$ and $g: B \longrightarrow C$ is defined in \mathbb{B} , then $F(g \circ f) = F(g) \circ F(f): F(A) \longrightarrow F(C)$ in \mathbb{B} .
- (2) If $i_A: A \longrightarrow A$ is the identity of A in \mathbb{B} , then $F i_A = i_{FA}$, the identity of FA in \mathbb{B} .
- (3) For each A and B in \mathbb{B} , $F: \mathbb{B}(A, B) \longrightarrow \mathbb{B}(FA, FB)$ is a linear contraction. This means

$$F(f+g) = Ff + Fg \qquad f, g \in \mathbb{B}(A, B)$$

$$F(\alpha f) = \alpha F(f) \quad \alpha \in I, \text{ and}$$

$$|F(f)| \leq |f|.$$

Note: The concept of a contravariant functor is defined similarly. The difference between a covariant and contravariant functor lies in the fact that if $F:B \rightarrow B$ is contravariant and $f:A \rightarrow B$ is a morphism in B , then Ff is a morphism from FB to FA . Thus for each A and B in B , F induces a map from $B(A,B)$ to $B(FB,FA)$, and (1) in (1.1) becomes $F(g \circ f) = F(f) \circ F(g): F(C) \rightarrow F(A)$. An important example of a contravariant functor is the functor $*$ which assigns to each A in B its conjugate space A^* and to each $f:A \rightarrow B$ its adjoint $f^*:B^* \rightarrow A^*$.

Remark 1.2 Condition (3) in (1.1) is generally not part of the definition of a functor. All functors in the following are assumed to satisfy this condition.

Functors will be denoted by letters such as F , G , and H .

Definition 1.3 Let F, G be functors from B to B . A natural transformation $\eta:F \rightarrow G$ is an assignment to each object X in B of a morphism $\eta_X:FX \rightarrow GX$ in B such that:

(1) for any morphism $f:A \rightarrow B$ in B the following diagram is commutative.

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \eta_A \downarrow & & \downarrow \eta_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

(2) $\sup \{|\eta_X| : X \text{ in } \mathbb{B}\}$ is finite.

If for each X in \mathbb{B} , $\eta_X : FX \rightarrow GX$ is an isomorphism (one-to-one, onto), then η is called a natural isomorphism, and F and G are naturally isomorphic. If for each X in \mathbb{B} , $\eta_X : FX \rightarrow GX$ is an equivalence, that is, an isometric isomorphism, then F and G are called naturally equivalent.

Remark 1.4 Although condition (2) is generally not part of the definition of a natural transformation, all natural transformations in this paper will be assumed to satisfy it. Natural transformations will be designated by letters like θ, η, τ , and λ .

Basic to the following study will be the concept of the projective tensor product of two Banach spaces. A brief description of this concept is given below. A more detailed description can be found in [14] and [16]. Let A and B be two Banach spaces and let $I^{A \times B}$ be the vector space over I consisting of all functions $f : A \times B \rightarrow I$. For each (a, b) in $A \times B$, let $a * b$ be the element in $I^{A \times B}$ defined by

$$\begin{aligned} a * b(p, q) &= 1 && \text{if } (p, q) = (a, b), \text{ and} \\ a * b(p, q) &= 0 && \text{if } (p, q) \neq (a, b). \end{aligned}$$

Let $I^{(A \times B)}$ be the subspace of $I^{A \times B}$ spanned by the elements of the type $a * b$. Define a relation \sim on $I^{(A \times B)}$ by the following rules:

$$\begin{aligned} (1) \quad & (a_1 + a'_1) * b_1 + a_2 * b_2 + \dots + a_n * b_n \sim a_1 * b_1 + \\ & a'_1 * b_1 + a_2 * b_2 + \dots + a_n * b_n. \end{aligned}$$

$$(2) \quad a_1*(b_1+b'_1)+a_2*b_2+ \dots +a_n*b_n \sim a_1*b_1+ \\ a_1*b'_1+a_2*b_2+ \dots +a_n*b_n.$$

$$(3) \quad \alpha_1(a_1*b_1)+\alpha_2(a_2*b_2)+ \dots +\alpha_n(a_n*b_n) \sim \\ (\alpha_1 a_1)*b_1+(\alpha_2 a_2)*b_2+ \dots +(\alpha_n a_n)*b_n.$$

$$(4) \quad (\alpha_1 a_1)*b_1+(\alpha_2 a_2)*b_2+ \dots +(\alpha_n a_n)*b_n \sim \\ a_1*(\alpha_1 b_1)+a_2*(\alpha_2 b_2)+ \dots +a_n*(\alpha_n b_n).$$

Now define the equivalence relation \sim on $I(A \times B)$ by $\sum \alpha_i(a_i*b_i) \sim \sum \beta_i(c_i*d_i)$ if and only if $\sum \alpha_i(a_i*b_i)$ can be transformed into $\sum \beta_i(c_i*d_i)$ by a finite number of applications of the rules (1) - (4). The algebraic tensor product $A \otimes B$ of A and B is the quotient space $I(A \times B)/\sim$, and the equivalence class of $\sum \alpha_i(a_i*b_i)$ is designated by $\sum \alpha_i a_i \otimes b_i$.

The linear space $A \otimes B$ can be made into a normed linear space with the norm given by

$$|u| = \inf\{\sum |\alpha_i| |a_i| |b_i| : u = \sum \alpha_i a_i \otimes b_i\}.$$

It can be shown that this defines a crossnorm on $A \otimes B$, that is, a norm with the additional property that

$$|a \otimes b| = |a| |b| \text{ for } a \in A \text{ and } b \in B. \text{ The completion of}$$

$A \otimes B$ with this norm, denoted by $\hat{A \otimes B}$, is called the projective tensor product of A and B , or simply the tensor product of A and B .

If $f:A \rightarrow C$ and $g:B \rightarrow D$ are morphisms in \mathcal{B} , $f \otimes g: A \otimes B \rightarrow C \otimes D$ (considered as normed spaces) is the continuous linear map given by

$$f \otimes g(\sum a_i \otimes b_i) = \sum f(a_i) \otimes g(b_i).$$

It can be checked that $|f \circ g| = |f| |g|$. By definition $f \circ g : A \hat{\otimes} B \longrightarrow C \hat{\otimes} D$ is the unique extension of $f \circ g$ to $A \hat{\otimes} B$ of the same norm.

The following proposition is useful in dealing with the projective tensor product.

Proposition 1.5 If A , B , and C are Banach spaces, then $B(A \hat{\otimes} B, C)$ and $B(B, B(A, C))$ are isometrically isomorphic (equivalent).

Sketch of Proof. A complete proof can be found in [14]. Define $\xi : B(A \hat{\otimes} B, C) \longrightarrow B(B, B(A, C))$ by $[(\xi f)b] = f(a \otimes b)$ for $f \in B(A \hat{\otimes} B, C)$, $b \in B$, and $a \in A$. The function ξ is linear and $|\xi(f)| \leq |f|$. Define $\hat{\mu} : B(B, B(A, C)) \longrightarrow B(A \otimes B, C)$ by $\hat{\mu}(g)(a \otimes b) = (g(b))a$ for $g \in B(B, B(A, C))$ and extend linearly to $A \otimes B$. For each g in $B(B, B(A, C))$, $|\hat{\mu}(g)| \leq |g|$, which means that for each g , $\hat{\mu}(g) : A \otimes B \longrightarrow C$ is uniformly continuous. By the Principle of Extension by Continuity [3, p.25], $\hat{\mu}(g) : A \otimes B \longrightarrow C$ has a unique uniformly continuous extension $\mu(g) : A \hat{\otimes} B \longrightarrow C$. Let $\mu : B(B, B(A, C)) \longrightarrow B(A \hat{\otimes} B, C)$ map g to the unique extension $\mu(g)$. It can be shown μ is linear and also $|\mu(g)| \leq |g|$. Because μ is an inverse for ξ , the proposition follows.

Note: The equals symbol "=" may sometimes be used to mean two Banach spaces are equivalent. Thus, " $B(A \hat{\otimes} B, C) = B(B, B(A, C))$ " means these spaces are isometrically isomorphic.

Two functors that will play a prominent role in the

following discussion are the hom and tensor functors, designated for an X in \mathbb{B} by Ω_X and Σ_X respectively. For each X in \mathbb{B} , Ω_X is defined by the following assignments:

- (1) If A is in \mathbb{B} , $\Omega_X(A) = \mathbb{B}(X, A)$.
- (2) If $f: A \rightarrow B$ is a morphism in \mathbb{B} , then $\Omega_X(f): \Omega_X A \rightarrow \Omega_X B$ is given by $\Omega_X(f)g = f \circ g$ for $g \in \mathbb{B}(X, A)$.

Since

$$|\Omega_X f| = \sup_{|g| \leq 1} |\Omega_X(f)g| \leq \sup_{|g| \leq 1} |f| |g| = |f|,$$

Ω_X can easily be seen to be a functor.

For each X in \mathbb{B} the functor Σ_X is given by the following assignments:

- (1) If A is in \mathbb{B} , $\Sigma_X A = X \hat{\otimes} A$.
- (2) If $f: A \rightarrow B$ is in \mathbb{B} , $\Sigma_X f: X \hat{\otimes} A \rightarrow X \hat{\otimes} B$ is the morphism $i_X \hat{\otimes} f$.

Remark 1.6 If F and G are functors from \mathbb{B} to \mathbb{B} , then the compositions $F \circ G$ and $G \circ F$ are also functors from \mathbb{B} to \mathbb{B} . Also to each functor $F: \mathbb{B} \rightarrow \mathbb{B}$ can be associated a functor $F_*^*: \mathbb{B} \rightarrow \mathbb{B}$, defined by the following rules:

- (1) $F_*^*(A) = F(A^*)^*$ for each A in \mathbb{B} where $F(A^*)^*$ is the conjugate (dual) space of $F(A^*)$.
- (2) If $f: A \rightarrow B$ is in \mathbb{B} , $F_*^*(f) = F(f^*)^*: F(A^*)^* \rightarrow F(B^*)^*$, the adjoint of $F(f^*)$.

Another useful functor is the "double-dual" functor from \mathbb{B} to \mathbb{B} designated by $**$. It assigns to each A in \mathbb{B} its second conjugate space A^{**} and to each $f: A \rightarrow B$

the second adjoint $f^{**}:A^{**} \longrightarrow B^{**}$.

Proposition 1.7 The functors Ω_X^{**} and $(\Sigma_X)^*$ are naturally equivalent.

Proof: The fact that for each A in \mathbb{B} , $\mathbb{B}(X, A^{**})$ is isometrically isomorphic to $(X \hat{\otimes} A^*)^*$ is given by (1.5) as $\mathbb{B}(X, A^{**}) = \mathbb{B}(X, \mathbb{B}(A^*, I))$ and $(X \hat{\otimes} A^*)^* = \mathbb{B}(X \hat{\otimes} A^*, I)$. Let $f:A \longrightarrow B$ be any morphism in \mathbb{B} . It must be shown that the diagram

$$\begin{array}{ccc}
 (X \hat{\otimes} A^*)^* & \xrightarrow{(i_X \hat{\otimes} f^*)^*} & (X \hat{\otimes} B^*)^* \\
 \downarrow \xi_A & & \downarrow \xi_B \\
 \mathbb{B}(X, A^{**}) & \xrightarrow{\Omega_X(f^{**})} & \mathbb{B}(X, B^{**})
 \end{array}$$

commutes where ξ_A and ξ_B are the equivalences given in (1.5). Let $g \in (X \hat{\otimes} A^*)^*$, $x \in X$, and $b^* \in B^*$. Then

$$\begin{aligned}
 [(\Omega_X f^{**}(\xi_A(g)))x]b^* &= [(f^{**} \circ \xi_A(g))x]b^* \\
 &= [f^{**}(\xi_A(g)(x))]b^* \\
 &= [\xi_A(g)(x) \circ f^*]b^* \\
 &= g(x \circ f^*(b^*)).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 ([\xi_B((i_X \hat{\otimes} f^*)^*(g))]x)b^* &= ([\xi_B(g \circ (i_X \hat{\otimes} f^*))]x)b^* \\
 &= [g \circ (i_X \hat{\otimes} f^*)](x \circ b^*) \\
 &= g(x \circ f^*(b^*)).
 \end{aligned}$$

Proposition 1.8 The contravariant functors $^{**}\Sigma_X$ and

$\Omega_X \circ *$ are also naturally equivalent. This means if $f:A \rightarrow B$ is in \mathbb{B} , the diagram

$$\begin{array}{ccc}
 \mathbb{B}(X, B^*) & \xrightarrow{\Omega_X f^*} & \mathbb{B}(X, A^*) \\
 \mu_B \downarrow & & \downarrow \mu_A \\
 (X \hat{\otimes} B)^* & \xrightarrow{(i_X \hat{\otimes} f)^*} & (X \hat{\otimes} A)^*
 \end{array}$$

commutes where μ_B and μ_A are given by (1.5).

Proof: By (1.5) μ_B and μ_A are equivalences. It remains to be shown that the diagram commutes. Let $g \in \mathbb{B}(X, B^*)$. Then for $x \in X$ and $a \in A$,

$$\begin{aligned}
 [((i_X \hat{\otimes} f)^* \circ \mu_B)g](x \otimes a) &= [\mu_B(g) \circ (i_X \hat{\otimes} f)](x \otimes a) \\
 &= [\mu_B(g)](x \otimes f(a)) \\
 &= g(x)(f(a)).
 \end{aligned}$$

Also

$$\begin{aligned}
 [(\mu_A \circ \Omega_X f^*)g](x \otimes a) &= [\mu_A(f^* \circ g)](x \otimes a) \\
 &= [(f^* \circ g)x](a) \\
 &= [g(x) \circ f](a) = g(x)(f(a)).
 \end{aligned}$$

Notation and Definition 1.9 If F and G are two functors, $(F \rightarrow G)$ will denote the "class" of all natural transformations from F to G . When $(F \rightarrow G)$ is a set, it is a Banach space with the norm given by

$$|\tau| = \sup\{|\tau_X| : X \in B\} \quad \tau \in (F \rightarrow G),$$

and addition and scalar multiplication given by

$$\begin{aligned}
 (\tau + \eta)_X &= \tau_X + \eta_X & X \text{ in } B, \text{ and} \\
 (\alpha\tau)_X &= \alpha\tau_X & X \text{ in } B \text{ and } \alpha \in I.
 \end{aligned}$$

Definition 1.10 Let $F:\mathcal{B}\longrightarrow\mathcal{B}$ be a functor. The dual functor to F , denoted as DF , is the functor given by the following assignments:

- (1) If X is in \mathcal{B} , $DFX = (F\longrightarrow\Sigma_X)$. (It will be established below that this is a set.)
- (2) If $f:X\longrightarrow Y$ is a morphism in \mathcal{B} , $DFf:DFX\longrightarrow DFY$ is given by the equation

$$(DFf(\tau))_A = (f\hat{\otimes}i_A)\circ\tau_A \quad \tau\in DFX, \text{ and } A \text{ in } \mathcal{B}.$$

This is depicted in the following diagram.

$$\begin{array}{ccc}
 FA & \xrightarrow{\tau_A} & \Sigma_X A \\
 & \searrow (DFf(\tau))_A & \downarrow f\hat{\otimes}i_A \\
 & & \Sigma_Y A
 \end{array}$$

Lemma 1.11 If A and B are any two Banach spaces, then $\mathcal{B}(A,B)$ and $(\Sigma_A\longrightarrow\Sigma_B)$ are equivalent.

Proof: Let $f\in\mathcal{B}(A,B)$. Define $\tau_f\in(\Sigma_A\longrightarrow\Sigma_B)$ by the formula

$$(\tau_f)_X = f\hat{\otimes}i_X:\Sigma_A X\longrightarrow\Sigma_B X \text{ for } X \text{ in } \mathcal{B}.$$

Then $|\tau_f| = |f|$ since for each X in \mathcal{B} , $|(\tau_f)_X| = |f\hat{\otimes}i_X| = |f|$. Let $g:X\longrightarrow Y$ be in \mathcal{B} . It is easily checked that the diagram

$$\begin{array}{ccc}
 \Sigma_A X & \xrightarrow{\Sigma_A g} & \Sigma_A Y \\
 (\tau_f)_X \downarrow & & \downarrow (\tau_f)_Y \\
 \Sigma_B X & \xrightarrow{\Sigma_B g} & \Sigma_B Y
 \end{array}$$

commutes so that τ_f is a natural transformation. Now let τ be any element in $(\Sigma_A \longrightarrow \Sigma_B)$. Define $f:A \longrightarrow B$ as $\tau_I : \Sigma_A I (=A) \longrightarrow \Sigma_B I (=B)$. It must be shown that for each X in \mathbb{B} , $\tau_X = f \hat{\circ} i_X$. Let $\hat{x}:I \longrightarrow X$ be given by $\hat{x}(e) = x$. By the commutativity of the diagram

$$\begin{array}{ccc}
 A = \Sigma_A I & \xrightarrow{\Sigma_A \hat{x}} & \Sigma_A X \\
 \downarrow f & & \downarrow \tau_X \\
 B = \Sigma_B I & \xrightarrow{\Sigma_B \hat{x}} & \Sigma_B X
 \end{array}$$

$$\tau_X(a \otimes x) = (\tau_X \circ (i_A \hat{\otimes} \hat{x})) a \otimes e = (i_B \hat{\otimes} \hat{x})(f(a)) = f(a) \otimes x = f \hat{\otimes} i_X(a \otimes x).$$

Lemma 1.12 If $F:B \longrightarrow B$ is a functor and A is in \mathbb{B} , then

$(\Omega_A \longrightarrow F)$ is isometric to FA .

Proof: For $a \in FA$ define τ_a in $(\Omega_A \longrightarrow F)$ by the formula

$$(\tau_a)_X f = F(f)a \quad \text{for } f \in \Omega_A X = \mathbb{B}(A, X).$$

It can be checked that τ_a is a natural linear transformation. It is in $(\Omega_A \longrightarrow F)$, since

$$\begin{aligned}
 |\tau_a| &= \sup_{X \text{ in } \mathbb{B}} |(\tau_a)_X| = \sup_{X \text{ in } \mathbb{B}} \sup_{|f| \leq 1} |(\tau_a)_X f| \\
 &= \sup_{X \text{ in } \mathbb{B}} \sup_{|f| \leq 1} |F(f)a| \leq \sup_{X \text{ in } \mathbb{B}} |a| = |a|.
 \end{aligned}$$

Now if τ is any member of $(\Omega_A \longrightarrow F)$, put $a = \tau_A(i_A) \in FA$.

Since the diagram

$$\begin{array}{ccc}
 \Omega_A A & \xrightarrow{\Omega_A f} & \Omega_A X \\
 \downarrow \tau_A & & \downarrow \tau_X \\
 FA & \xrightarrow{Ff} & FX
 \end{array}$$

commutes for any $f:A \longrightarrow X$, $\tau_X(f) = F(f)a$. This means the correspondence above is onto, that is $\tau = \tau_a$. Since

$$|a| = |\tau_A i_A| = |(\tau_a)_A i_A| \leq |\tau_a| \leq |a|,$$

the correspondence is isometric.

Examples 1.13

(1) The functors Σ_A and Ω_A are dual to each other for each A in \mathcal{B} . That $D\Omega_A = \Sigma_A$ (that is, is equivalent to) follows from the equation

$$D\Omega_A X = (\Omega_A \longrightarrow \Sigma_X) = \Sigma_X A = \Sigma_A X$$

using (1.12). That $D\Sigma_A = \Omega_A$ follows from (1.11) and the equation

$$D\Sigma_A X = (\Sigma_A \longrightarrow \Sigma_X) = \mathcal{B}(A, X) = \Omega_A X.$$

It remains to be shown that the relations are natural, that is, in the first case if $f:X \longrightarrow Y$, then the diagram

$$\begin{array}{ccc} D\Omega_A X & \xrightarrow{D\Omega_A f} & D\Omega_A Y \\ \downarrow & & \downarrow \\ \Sigma_A X & \xrightarrow{\Sigma_A f} & \Sigma_A Y \end{array}$$

commutes. Let τ be in $D\Omega_A X = (\Omega_A \longrightarrow \Sigma_X)$. Using the isometry established in (1.12) between $D\Omega_A X$ and $\Sigma_X A$, the lower left half of the diagram takes τ to $i_A \hat{\circ} f(\tau_A(i_A))$. Using the isometry of (1.12) again, the upper right half takes τ to the same element. The naturality in the second case is shown in a similar manner.

(2) The dual of the identity functor $I_{\mathcal{B}}:\mathcal{B} \longrightarrow \mathcal{B}$ is

itself. Using (1.11), that $DI_{\mathbb{B}} = I_{\mathbb{B}}$ follows from

$$DI_{\mathbb{B}}X = (I_{\mathbb{B}} \longrightarrow \Sigma_X) = (\Sigma_I \longrightarrow \Sigma_X) = \mathbb{B}(I, X) = X = I_{\mathbb{B}}X.$$

That these isometries for each X in \mathbb{B} give a natural transformation can be checked as in (1).

Definition 1.14 A morphism $f:A \longrightarrow B$ in \mathbb{B} is a normal morphism if the induced continuous linear function from $A/\text{Ker } f \longrightarrow f(A)$ is an isometry. In addition, it is a strictly normal morphism if for each b in $f(A)$, there is an $a \in A$ such that $f(a) = b$ and $|a| = |b|$.

Definition 1.15 A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in \mathbb{B} is a normal exact sequence if it is exact (f is a monomorphism, g is an epimorphism, and $\text{Ker } g = \text{Im } f$) and each morphism is normal. It is strictly normal exact if it is exact and each morphism is strictly normal.

II THE HOM AND TENSOR FUNCTORS AND EXACT SEQUENCES

This section is devoted to the investigation of when the hom and tensor functors take (strictly) normal exact sequences to normal exact sequences. In order to proceed with this investigation, preliminary knowledge of the spaces $L^1(\mu)$ is needed. This is given below. A more detailed account may be found in [1].

In the following discussion, E will always denote a locally compact topological space. $K(E)$ will designate the vector space of all real-valued continuous functions f on E with compact support.

Definition 2.1 A positive (Radon) measure on E is a linear functional $\mu:K(E) \rightarrow \mathbb{R}$ which satisfies the following conditions:

- (1) If $f(x) \geq 0$, $x \in E$, then $\mu(f) \geq 0$.
- (2) For each compact subset K of E , there exists a number $M(K, \mu) \geq 0$ such that for each f in $K(E)$ with support contained in K ,

$$|\mu(f)| \leq M(K, \mu) \cdot \|f\|_{\infty} \quad \text{where} \quad \|f\|_{\infty} = \sup_{x \in E} |f(x)|.$$

Note: Each space L^1 defined below depends on some locally compact space E and a measure μ . In the discussion that follows, some fixed space E and measure μ are understood.

Remarks and Definitions 2.2 To each positive extended

real valued f defined on E , one can associate a positive number $\mu^*(f)$ (perhaps $+\infty$), designated sometimes by

$\int^* f d\mu$, called the upper integral of f . If f is in $K(E)$, then $\mu^*(f) = \mu(f)$; that is, μ^* is an extension of μ .

The exterior measure $\mu^*(A)$ of $A \subset E$ is defined to be $\mu^*(\chi_A)$.

$A \subset E$ is negligible if $\mu^*(A) = 0$. An extended real-valued

positive function f is negligible if $f(x) = 0$, $x \in E$,

almost everywhere; that is, $\{x \in E : f(x) \neq 0\}$ is negligible.

Definition 2.3 Let S be any set. Two functions $f, g: E \rightarrow S$

are equivalent (with respect to μ) if $f(x) = g(x)$ almost

everywhere on E . A function $f: A \rightarrow S$, $A \subset E$, is said to be

defined almost everywhere if $E \setminus A$ is negligible. The

relation " f is related to g if f is equivalent to g " is

an equivalence relation on the set S^E of functions from

E to S . The equivalence class of f is denoted by \tilde{f} . The

equivalence class of a function $f: A \rightarrow S$, $A \subset E$, defined

almost everywhere on E , can still be considered and

contains, among other functions, all functions defined

everywhere on E and equal to f at all points in A . If

the set S is a vector space over I , by defining

$$\tilde{f} + \tilde{g} = \widetilde{f + g} \quad f, g \in S^E, \text{ and}$$

$$\alpha \tilde{f} = \widetilde{\alpha f} \quad \alpha \in I,$$

a vector space structure on the set of equivalence classes is obtained.

Now let B be a Banach space over I .

Definition 2.4 For each function $f: E \rightarrow B$ and for each

integer $1 \leq p < +\infty$, define $N_p(f, \mu)$ or $N_p(f)$ to be the finite or infinite positive number,

$$N_p(f) = (\int^* |f(\cdot)|^p d\mu)^{1/p},$$

where $|f(\cdot)|$ is the positive function $E \rightarrow I$ given by $x \mapsto |f(x)|$.

Remark 2.5 N_p is a semi-norm on B^E . Consider now the set of equivalence classes of elements of B^E formed by the above relation. It can be shown that the function $N_p(\tilde{f}) = N_p(f)$ is a well-defined function from this set of equivalence classes to I ; that is, if $\tilde{f} = \tilde{g}$ then $N_p(f) = N_p(g)$. Moreover, this function defines a norm on the set of equivalence classes. This means

$$N_p(\alpha \tilde{f}) = |\alpha| N_p(\tilde{f}) \quad \alpha \in I,$$

$$N_p(\tilde{f} + \tilde{g}) \leq N_p(\tilde{f}) + N_p(\tilde{g}), \text{ and}$$

$$N_p(\tilde{f}) = 0 \text{ if and only if } \tilde{f} = \tilde{0}.$$

Definition 2.6 For $1 \leq p < +\infty$, let $\mathcal{F}_B^p(E, \mu)$ or \mathcal{F}_B^p (E and μ being understood) be the semi-normed vector space of all elements f in B^E such that $N_p(f) < +\infty$. Let $K_B(E)$ denote the vector space of all continuous functions $f: E \rightarrow B$ with compact support. ($K_B(E)$ can be shown to be a subspace of \mathcal{F}_B^p .) Define $\mathcal{L}_B^p(E, \mu)$ or \mathcal{L}_B^p to be the closure in the space \mathcal{F}_B^p of $K_B(E)$. Define L_B^p to be the normed linear space of equivalence classes of functions in \mathcal{L}_B^p . Elements of \mathcal{L}_B^p are called pth power integrable functions.

Proposition 2.7 L_B^p is a Banach space.

Proof: See [1, p.133].

Remark 2.8 Although the elements of L_B^p are equivalence classes of functions, they are generally thought of as functions that are p^{th} power integrable. It must be kept in mind that two functions are considered the same if they are equivalent.

Definition 2.9 A function $f:E \rightarrow B$ is measurable if for each compact subset K of E there exists a negligible set $N \subset K$ and a partition of $K \setminus N$, formed from a sequence (K_n) of compact sets, such that the restriction of f to each K_n is continuous. A subset A of E is measurable if its characteristic function χ_A is measurable.

Proposition 2.10 In order that a function $f:E \rightarrow B$ be measurable, it is necessary and sufficient that it satisfy the conditions:

- (1) the set $f^{-1}(U)$ is measurable where U is any closed ball in B ; and
- (2) for each compact set $K \subset E$, there is a countable subset H of B such that $f(x) \in \bar{H}$ (closure of H) for almost all $x \in K$.

Proof: See [1, p.191].

Proposition 2.11 In order that a function $f:E \rightarrow B$ be measurable, it is necessary and sufficient that

- (1) the set $f^{-1}(U)$ is measurable where U is any closed ball in B ; and
- (2) for each compact set $K \subset E$, there is a negligible set S in K such that $f(K \setminus S)$ is

separable.

Proof: It must be established that (2) is equivalent to (2) of (2.10). Clearly (2) of (2.10) is equivalent to the statement:

(3) For each compact set $K \subseteq E$ there is a countable subset H of B such that $f(K \setminus S) \subset \bar{H}$ for some negligible set S in K .

It must still be shown that (2) is equivalent to (3).

Suppose (3) is true. Then for each compact subset K of E , $f(K \setminus S)$ is contained in a separable metric space for some negligible set S in K . Since a subset of a separable metric space is separable, $f(K \setminus S)$ is separable. Now suppose (2) is true. Then for each compact subset K of E , $f(K \setminus S)$ is separable for some negligible set S in K . Let H be a countable set in $f(K \setminus S)$ so that \bar{H} (closure in $f(K \setminus S)$) = $f(K \setminus S)$. Then $f(K \setminus S) \subset \bar{H}$ (closure in B).

Proposition 2.12 In order that $f: E \rightarrow B$ be p^{th} power integrable for $1 \leq p < +\infty$, it is necessary and sufficient that f be measurable and $N_p(f)$ be finite.

Proof: See [1, p.194].

Proposition 2.13 Suppose A and B are Banach spaces and $h: A \rightarrow B$ is in B . For each $f \in \mathcal{I}_B^p$, $h \circ f$ belongs to \mathcal{I}_B^p ($1 \leq p < +\infty$). Moreover, if $\tilde{f} = \tilde{g}$, then $\widetilde{h \circ f} = \widetilde{h \circ g}$.

Proof: Since $f \in \mathcal{I}_B^p$, for each $\epsilon > 0$ there exists a function $g \in K_A(E)$ such that $N_p(f-g) \leq \epsilon$. Since

$$|h \circ f - h \circ g(\cdot)| = |h \circ (f-g)(\cdot)| \leq |h| \cdot |f-g(\cdot)|,$$

$$N_p(h \circ f - h \circ g) \leq |h| \cdot N_p(f - g) \leq \varepsilon \cdot |h|.$$

Because $h \circ g$ is continuous and has compact support, $h \circ f$ is in \mathcal{L}_B^p , being in the closure of $K_B(E)$. The last statement of the proposition is true because subsets of negligible sets are negligible.

Example 2.14 Take the special case where E is any space with the discrete topology. Then $K(E) = \{f: E \rightarrow I \mid f(x) = 0 \text{ for all but finitely many } x \in E\}$. Define $\mu: K(E) \rightarrow I$ by $\mu(f) = \sum_{x \in E} f(x)$, a finite sum. The function μ satisfies the condition of (2.1). If B is a Banach space and $f \in K_B(E)$, $N_1(f) = \sum_{x \in E} |f(x)|$, a finite sum. A function $f: E \rightarrow B$ is in \mathcal{L}_B^1 if and only if $\sum_{x \in E} |f(x)|$ is finite by (2.6). The space L_B^1 in this case is denoted by \mathcal{L}_B^1 , and the measure μ is called a discrete measure.

Note: Since the spaces L_A^p , A in \mathcal{B} , are objects in \mathcal{B} , its elements will be denoted by letters like a, b, x , and y ; letters like f, g , and h will now be reserved for morphisms in \mathcal{B} .

Proposition (2.13) helps to make the next definition valid.

Definition 2.15 Let $L^p: \mathcal{B} \rightarrow \mathcal{B}$ ($1 \leq p < \infty$) be the functor given by the rules:

- (1) $L^p A = L_A^p$, and
- (2) If $f: A \rightarrow B$ is in \mathcal{B} , $L^p f: L_A^p \rightarrow L_B^p$ is given by $L^p f(\tilde{x}) = f \circ \tilde{x}$ for $\tilde{x} \in L_A^p$.

The following result is proved by Grothendieck [5, p.59].

Proposition 2.16 The functors $\Sigma_{L_I}^1$ and L^1 are naturally equivalent.

Proposition 2.17 If

$$0 \longrightarrow B \xrightarrow{f} B \xrightarrow{g} C$$

is an exact sequence in \mathcal{B} with f normal, then

$$0 \longrightarrow L^p A \xrightarrow{L^p f} L^p B \xrightarrow{L^p g} L^p C$$

is exact for $1 \leq p < \infty$ with $L^p f$ normal.

Proof: Since f is isometric, for $x \in L^p A$,

$$|f \circ x(\cdot)| = |f(x(\cdot))| = |x(\cdot)|.$$

$$\begin{aligned} \text{Therefore, } N_p(L^p f(\tilde{x})) &= N_p(\widetilde{f \circ x}) = N_p(f \circ x) = \\ &= (\int^* |f(x(\cdot))|^p)^{1/p} = (\int^* |x(\cdot)|^p)^{1/p} = \\ &= N_p(x) = N_p(\tilde{x}). \end{aligned}$$

This means $L^p(f)$ is isometric.

Now suppose $\tilde{y} \in L_B^p$ such that $L^p g(\tilde{y}) = \widetilde{g \circ y} = \tilde{0}$. Then $g \circ y = 0$ almost everywhere on E so that $y(t) \in \text{Ker } g = \text{Im } f$ almost everywhere for $t \in E$. It can be supposed that $y(t) \in \text{Ker } g$ for all $t \in E$. Define \tilde{x} in L_A^p to be the equivalence class of $x: E \rightarrow A$ defined by the rule: if $t \in E$, $x(t)$ is the unique element in A so that $f(x(t)) = y(t)$, and $|x(t)| = |y(t)|$. It must be shown that $x \in L_A^p$. By (2.12) it suffices to show x is measurable and $N_p(x)$ is finite. Since $|x(\cdot)| = |y(\cdot)|$, $N_p(x) = N_p(y)$ is finite. To show x is measurable the conditions of (2.11) must be verified. Let U be any closed ball in A . Then $x^{-1}(U) = \tilde{y}^{-1}(f(U))$ since f is isometric. Suppose $U = \{a: |a - a_0| \leq r\}$. Then $f(U) = \{f(a): |f(a) - f(a_0)| \leq r\}$.

Let $V = \{b : b \in B \text{ and } |b - f(a_0)| \leq r\}$. Then $f(U) = V \cap f(A)$.

$$\begin{aligned} \text{Hence, } y^{-1}(f(U)) &= y^{-1}(V \cap f(A)) = y^{-1}(V) \cap y^{-1}(f(A)) \\ &= y^{-1}(V) \cap E = y^{-1}(V). \end{aligned}$$

Since y is measurable, $y^{-1}(V)$ is a measurable set and hence $x^{-1}(U)$ is measurable.

It must still be shown that for any compact set $K \subset E$, there is a negligible set S so that $x(K \setminus S)$ is separable. Since $y : E \rightarrow B$ is measurable, a negligible set S does exist so that $y(K \setminus S)$ is separable. Let H be a countable dense subset of $y(K \setminus S)$. Let H' be the subset of $x(K \setminus S)$ in a one-to-one correspondence with H via f . H' is countable. Also,

$$\begin{aligned} \bar{H}(\text{closure in } y(K \setminus S)) &= y(K \setminus S) \cap \bar{H}(\text{closure in } B) \\ &= y(K \setminus S). \end{aligned}$$

$$\begin{aligned} \text{Hence, } x(K \setminus S) &= f^{-1}(y(K \setminus S)) = f^{-1}(y(K \setminus S) \cap \bar{H}) \\ &= x(K \setminus S) \cap f^{-1}(\bar{H}) = x(K \setminus S) \cap \bar{H}'(\text{closure in } A) \\ &= \bar{H}'(\text{closure in } x(K \setminus S)). \end{aligned}$$

This means H' is dense in $x(K \setminus S)$ so that $x(K \setminus S)$ is separable.

Proposition 2.18 If $B \xrightarrow{g} C \rightarrow 0$ is strictly normal exact

in \mathcal{B} , then for any X in \mathcal{B} ,

$$X \hat{\otimes} B \xrightarrow{i_X \hat{\otimes} g} X \hat{\otimes} C \rightarrow 0$$

is normal exact.

Proof: Since $(i_X \otimes g) : X \otimes B \rightarrow X \otimes C$ is a surjection, $i_X \hat{\otimes} g(X \hat{\otimes} B)$

is dense in $X \hat{\otimes} C$. To show $i_X \hat{\otimes} g$ is a surjection, it will

be shown that the induced map from $X \hat{\otimes} B / \text{Ker}(i_X \hat{\otimes} g)$ to $X \hat{\otimes} C$

is an isometric map. Let $\sum_{i=1}^n x_i \otimes c_i$ be in $X \otimes C$. By assumption,

for each $i = 1, \dots, n$; there is a $b_i \in B$ such that $|b_i| = |c_i|$ and

$$i_X \hat{\otimes} g \left(\sum_{i=1}^n x_i \otimes b_i \right) = \sum_{i=1}^n x_i \otimes g(b_i) = \sum_{i=1}^n x_i \otimes c_i.$$

Therefore, $\sum_{i=1}^n |x_i| |b_i| = \sum_{i=1}^n |x_i| |c_i|$. Considering

$\left[\sum_{i=1}^n x_i \otimes b_i \right]$ as an element of $X \hat{\otimes} B / \text{Ker}(i_X \hat{\otimes} g)$,

$$\left| \left[\sum_{i=1}^n x_i \otimes b_i \right] \right| = \inf_{u \in \left[\sum_{i=1}^n x_i \otimes b_i \right]} |u| \geq \left| \sum_{i=1}^n x_i \otimes c_i \right|.$$

Now let $\sum_{j=1}^m x'_j \otimes c'_j = \sum_{i=1}^n x_i \otimes c_i$ in $X \hat{\otimes} C$. By assumption, for

each $j = 1, \dots, m$ there exists b'_j such that $|b'_j| = |c'_j|$

and $g(b'_j) = c'_j$. Therefore,

$$i_X \hat{\otimes} g \left(\sum_{j=1}^m x'_j \otimes b'_j \right) = \sum_{j=1}^m x'_j \otimes c'_j = \sum_{i=1}^n x_i \otimes c_i, \text{ and}$$

$$\left| \sum_{j=1}^m x'_j \otimes b'_j \right| \leq \sum_{j=1}^m |b'_j| |x'_j| = \sum_{j=1}^m |x'_j| |c'_j|.$$

Since this is true for each element of $X \hat{\otimes} C$ equal to

$$\sum_{i=1}^n x_i \otimes c_i, \quad \left| \left[\sum_{i=1}^n x_i \otimes b_i \right] \right| \leq \left| \sum_{i=1}^n x_i \otimes c_i \right|.$$

Now let y be any element of $i_X \hat{\otimes} g(X \hat{\otimes} B)$. Then there is a sequence of elements y_i in $X \hat{\otimes} C$ such that $y_i \rightarrow y$ in $X \hat{\otimes} C$. By the preceding argument, there exist z_i in $X \hat{\otimes} B$ such that $|[z_i]| = |y_i|$ and $i_X \hat{\otimes} g(z_i) = y_i$. Since

$$|[z_i] - [z_j]| = |[z_i - z_j]| = |y_i - y_j|,$$

$[z_i]$ is a Cauchy sequence in $X \hat{\otimes} B / \text{Ker}(i_X \hat{\otimes} g)$ converging to some $[z]$ and $i_X \hat{\otimes} g(z) = y$. Also,

$$|[z]| = \lim |[z_i]| = \lim |y_i| = |y|.$$

Thus $X \hat{\otimes} B / \text{Ker}(i_X \hat{\otimes} g) \rightarrow X \hat{\otimes} C$ is an isometric map.

Proposition 2.19 If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$

is a strictly normal exact sequence in \mathbb{B} , then

$$0 \longrightarrow L_I^1 \hat{\otimes} A \longrightarrow L_I^1 \hat{\otimes} B \longrightarrow L_I^1 \hat{\otimes} C \longrightarrow 0$$

is normal exact in \mathbb{B} .

Proof: By (2.18), $L_I^1 \hat{\otimes} B \longrightarrow L_I^1 \hat{\otimes} C \longrightarrow 0$ is normal exact. By (2.16), the functors $\Sigma_{L_I^1}$ and L^1 are naturally equivalent by a natural equivalence τ . Hence the diagram

$$\begin{array}{ccccccc}
 (*) & 0 & \longrightarrow & L_I^1 \hat{\otimes} A & \longrightarrow & L_I^1 \hat{\otimes} B & \longrightarrow & L_I^1 \hat{\otimes} C & \longrightarrow & 0 \\
 & & & \downarrow \tau_A & & \downarrow \tau_B & & \downarrow \tau_C & & \\
 (**) & 0 & \longrightarrow & L_A^1 & \xrightarrow{L^1 f} & L_B^1 & \xrightarrow{L^1 g} & L_C^1 & \longrightarrow & 0
 \end{array}$$

commutes. Since $L^1 f$ is isometric and $(**)$ is exact at L_B^1 by (2.17), sequence $(*)$ is normal exact.

Corollary 2.20 If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is strictly normal exact in \mathbb{B} , then $0 \longrightarrow L_A^1 \longrightarrow L_B^1 \longrightarrow L_C^1 \longrightarrow 0$ is normal exact in \mathbb{B} .

Proposition 2.20 Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a strictly normal exact sequence in \mathbb{B} . If X is a retract of L_I^1 (meaning there exists $f: X \longrightarrow L_I^1$ with $|f| \leq 1$ and $g: L_I^1 \longrightarrow X$ with $|g| \leq 1$ such that $g \circ f = i_X$), then

$$0 \longrightarrow X \hat{\otimes} A \longrightarrow X \hat{\otimes} B \longrightarrow X \hat{\otimes} C \longrightarrow 0$$

is normal exact in \mathbb{B} .

Proof: Since $g \circ f = i_X$ and $|g| \leq 1$, f is an isometric map. Since for any Banach space Y , $f \hat{\otimes} i_Y: X \hat{\otimes} Y \longrightarrow L_I^1 \hat{\otimes} Y$ and $g \hat{\otimes} i_Y: L_I^1 \hat{\otimes} Y \longrightarrow X \hat{\otimes} Y$ are in \mathbb{B} so that $(g \hat{\otimes} i_Y) \circ (f \hat{\otimes} i_Y) = i_{X \hat{\otimes} Y}$, $|f \hat{\otimes} i_Y| \leq 1$, and $|(g \hat{\otimes} i_Y)| \leq 1$; $f \hat{\otimes} i_Y$ is an isometric map. Since the following diagram is commutative with the

middle row normal exact by (2.19), the top (or bottom) row is normal exact.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X \hat{\otimes} A & \longrightarrow & X \hat{\otimes} B & \longrightarrow & X \hat{\otimes} C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L \hat{\otimes} A & \longrightarrow & L \hat{\otimes} B & \longrightarrow & L \hat{\otimes} C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X \hat{\otimes} A & \longrightarrow & X \hat{\otimes} B & \longrightarrow & X \hat{\otimes} C \longrightarrow 0
 \end{array}$$

Lemma 2.22 If

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is normal exact in \mathbb{B} , then

$$0 \longrightarrow C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \longrightarrow 0$$

is strictly normal exact.

Proof: $B^* \xrightarrow{f^*} A^* \longrightarrow 0$ is strictly normal exact due to the Hahn-Banach theorem [3, p.63]. To show $0 \longrightarrow C^* \xrightarrow{g^*} B^*$ is strictly normal exact, it must be shown $|g^*(c^*)| = |c^*|$ for all $c^* \in C^*$. Since $|g^*| \leq 1$, $|g^*(c^*)| \leq |c^*|$. Let $\epsilon > 0$ be arbitrary. For each c in C , there exists b_ϵ in B so that $g(b_\epsilon) = c$ and $|b_\epsilon| \leq |c| + \epsilon$. Hence,

$$\begin{aligned}
 |c^*(c)| &= |c^*(g(b_\epsilon))| = |g^*(c^*)(b_\epsilon)| \\
 &\leq |g^*(c^*)| |b_\epsilon| \leq |g^*(c^*)| (|c| + \epsilon) \\
 &= |g^*(c^*)| |c| + |g^*(c^*)| \epsilon.
 \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $|c^*(c)| \leq |g^*(c^*)| \cdot |c|$ so that $|c^*| \leq |g^*(c^*)|$.

Exactness at B^* must yet be shown. Suppose $f^*(b^*) = b^* \circ f = 0$.

$$\begin{array}{ccccc}
 & & I & & \\
 & & \uparrow & \swarrow & \\
 & & b^* & c^* & \\
 & f & \downarrow & g & \\
 0 & \longrightarrow & A & \longrightarrow & B \longrightarrow C \longrightarrow 0
 \end{array}$$

This means $\text{Ker } b^* \supset \text{Im } f = \text{Ker } g$. Define $c^*: C \rightarrow I$ by $c^*(c) = b^*(b)$ where $g(b) = c$. This is well-defined, since if $b' \in B$ so that $g(b') = c$, then $g(b-b') = 0$ so that $b^*(b-b') = 0$. By an argument similar to that above, $|c^*| = |b^*|$ and clearly $g^*(c^*) = b^*$.

Corollary 2.23 If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a normal exact sequence with B reflexive, then it is strictly normal exact.

The proof of the following proposition can be found in [7].

Proposition 2.24 X is equivalent to a space L_I^1 (for some Radon measure μ on a locally compact space E) if and only if for each Banach space B and closed linear subspace A of B , the map $i_X \hat{\otimes} f: X \hat{\otimes} A \rightarrow X \hat{\otimes} B$ is an isometric (into) map where $f: A \rightarrow B$ is the insertion map.

Lemma 2.25 If X is a Banach space such that X^{**} is (equivalent to) a space L_I^1 , then X is also a space L_I^1 .

Proof: Let A be any Banach space, $B \subset A$ any closed subspace, and $f: A \rightarrow B$ the insertion map. By (2.24) it suffices to show $i_X \hat{\otimes} f$ is isometric. Let $\eta_X: X \rightarrow X^{**}$ be the natural embedding. It suffices to show $\eta_X \hat{\otimes} i_A$ in the diagram

$$\begin{array}{ccc} X \hat{\otimes} A & \xrightarrow{i_X \hat{\otimes} f} & X \hat{\otimes} B \\ \downarrow \eta_X \hat{\otimes} i_A & & \downarrow \eta_X \hat{\otimes} i_B \\ X^{**} \hat{\otimes} A & \xrightarrow{i_{X^{**}} \hat{\otimes} f} & X^{**} \hat{\otimes} B \end{array}$$

is isometric. By (2.22) it is the same to show $(\eta_X \hat{\otimes} i_A)^*: (X^{**} \hat{\otimes} A)^* \rightarrow (X \hat{\otimes} A)^*$ is a (strictly) normal epimorphism.

However, $(X^{**}\hat{\otimes} A)^*$ and $(X\hat{\otimes} A)^*$ are $BL(X^{**}, A)$ and $BL(X, A)$ respectively, the spaces of continuous bilinear forms on $X^{**}\times A$ and $X\times A$. Also, $BL(X, A)$ is equivalent to $\mathcal{B}(A, X^*)$ by the map $H \longrightarrow G_H$, where $(G_H(a))x = H(x, a)$. For each H in $BL(X, A)$, associate to G_H the element $\eta_{X^*} \circ G_H$ in $\mathcal{B}(A, X^{***})$. Since $\mathcal{B}(A, X^{***})$ is equivalent to $BL(X^{**}, A)$, to $\eta_{X^*} \circ G_H$ is associated the form \bar{H} in $BL(X^{**}, A)$, given by

$$\bar{H}(x^{**}, a) = [\eta_{X^*} \circ G_H(a)]x^{**}.$$

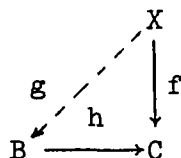
Now \bar{H} "restricted" to $X \times A$ is equal to H . Indeed, if \hat{x} is in $X \subset X^{**}$ (consider X as subspace of X^{**}), then

$$\begin{aligned}\bar{H}(\hat{x}, a) &= [\eta_{X^*} \circ G_H(a)]\hat{x} = [\eta_{X^*}(G_H(a))]\hat{x} \\ &= \hat{x}(G_H(a)) = [G_H(a)]x = H(x, a).\end{aligned}$$

Also, $|H| = |\eta_{X^*} \circ G_H| = |G_H| = |H|$.

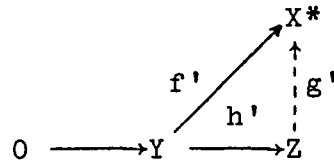
Hence, $(\eta_{X^*} \hat{\otimes} i_A)^*$ is a strictly normal epimorphism or $\eta_{X^*} \hat{\otimes} i_A$ is an isometric map. It follows that $i_X \hat{\otimes} f$ is isometric.

Lemma 2.26 Suppose X is a Banach space satisfying the conditions illustrated in the following diagram.

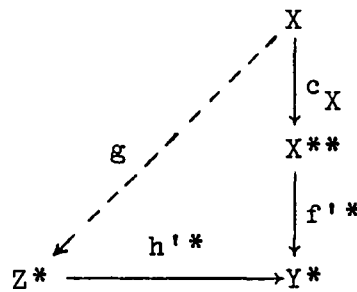


The morphism h is a normal epimorphism, while $\dim B = 3$ and $\dim C = 2$. It is assumed that for all such B, C , and $h: B \longrightarrow C$, for any $f \in \mathcal{B}(X, C)$, the diagram can be filled in with $g \in \mathcal{B}(X, B)$ so that $h \circ g = f$ and $|g| = |f|$. Then X^* satisfies the condition that for any Banach space $Y \subset Z$

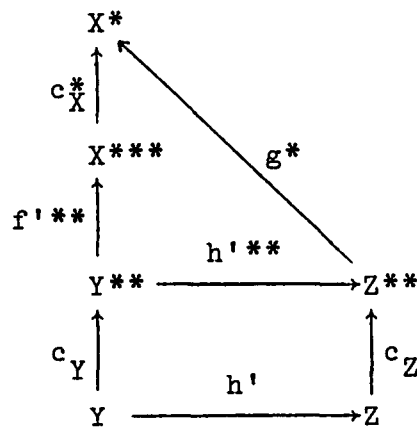
with $\dim Z = 3$ and $\dim Y = 2$, every element in $B(Y, X^*)$ has a norm preserving extension from Z to X^* .



Proof: A morphism $g' \in B(Z, X^*)$ must be found so that $|g'| = |f'|$ and $g' \circ h' = f'$ where h' is the insertion map. Consider the dual diagram where $c_X: X \rightarrow X^{**}$ is the canonical embedding.



By assumption, $g: X \rightarrow Z^*$ exists so that $h'^* \circ g = f'^* \circ c_X$ and $|f'^* \circ c_X| = |g|$.



If f' can be shown to be $c_X^* \circ f'^{**} \circ c_Y$, then letting $g' = g^* \circ c_Z$, the proposition is proved. Let $y \in Y$ and $x \in X$. Then if \hat{y} denotes $c_Y(y)$ and \hat{x} denotes $c_X(x)$,

$$\begin{aligned}
[(c_X^* \circ f'^* \circ c_Y)y]x &= [c_X^*(f'^*(\hat{y}))]x \\
&= [f'^*(\hat{y}) \circ c_X]x \\
&= [\hat{y} \circ f'^* \circ c_X]x = [\hat{y} \circ f'^*]\hat{x} \\
&= \hat{y}(f'^*(\hat{x})) = \hat{y}(\hat{x} \circ f') \\
&= (\hat{x} \circ f')y = \hat{x}(f'(y)) \\
&= f'(y)(x).
\end{aligned}$$

This means $c_X^* \circ f'^* \circ c_Y = f'$.

The next lemma will also be used below.

Lemma 2.27 If X is a Banach space, $S_X(x_0, r_0)$ denotes $\{x \in X : |x - x_0| \leq r_0\}$. If two balls $S_X(x_1, r_1)$ and $S_X(x_2, r_2)$ intersect in X , then $S_X(x_1, r_1) \cap S_X(x_2, r_2) \cap A \neq \emptyset$ where A is any two-dimensional subspace of X containing x_1 and x_2 .

Proof: Let $C = (1-\alpha)x_1 + \alpha x_2$, $0 \leq \alpha \leq 1$, be the curve in X connecting x_1 and x_2 . C is homeomorphic to $[0, 1]$.

Hence there exists a maximum α_0 in $[0, 1]$ so that

$$\begin{aligned}
(1-\alpha_0)x_1 + \alpha_0 x_2 &\text{ is in } S_X(x_1, r_1). \text{ It can be shown that} \\
|(1-\alpha_0)x_1 + \alpha_0 x_2 - x_1| &= r_1 \text{ and } |(1-\alpha_0)x_1 + \alpha_0 x_2 - x_2| \leq r_2.
\end{aligned}$$

The following facts, given here without proof, will be needed for the next proposition.

Facts 2.28 (1) If X is a Banach space, let $\{S_X(x_\alpha, r_\alpha)\}$ be a collection of mutually intersecting balls in X .

Then there is a Banach space $Z \supset X$ with $\dim Z/X = 1$ such that $\bigcap_{\alpha} S_Z(x_\alpha, r_\alpha) \neq \emptyset$. For proof see [8, p.51] and [12].

(2) Let X be a Banach space such that $S_X(0, 1)$ has at least one extreme point, and such that X has the following property:

For every collection of four mutually intersecting balls $\{S_X(x_i, r_i) : i = 1, 2, 3, 4\}$ such that $\{x_i : i = 1, 2, 3, 4\}$ span a two-dimensional subspace of X ,

$$\bigcap_{i=1}^4 S(x_i, r_i + \varepsilon) \neq \emptyset \text{ for every } \varepsilon > 0.$$

Then X^* is a space L_1^1 . For proof see [8, p.71].

Theorem 2.29 The following statements are equivalent.

(1) X is equivalent to a space L_1^1 .

(2) If B is reflexive and

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is any normal exact sequence in B , then

$$0 \longrightarrow B(X, A) \longrightarrow B(X, B) \longrightarrow B(X, C) \longrightarrow 0$$

is strictly normal exact.

(3) Same as (2) with B finite dimensional.

(4) Same as (2) with $\dim B = 3$ and $\dim C = 2$.

Proof: Statement (1) implies (2). By (2.22)

$$0 \longrightarrow C^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0$$

is strictly normal exact. By (2.19)

$$0 \longrightarrow (X \hat{\otimes} C^*) \longrightarrow (X \hat{\otimes} B^*) \longrightarrow (X \hat{\otimes} A^*) \longrightarrow 0$$

is normal exact. Again by (2.22)

$$0 \longrightarrow (X \hat{\otimes} A^*)^* \longrightarrow (X \hat{\otimes} B^*)^* \longrightarrow (X \hat{\otimes} C^*)^* \longrightarrow 0$$

is strictly normal exact. By (1.7) the following diagram

is commutative with ξ_A, ξ_B , and ξ_C equivalences.

$$\begin{array}{ccccccc} (*) & 0 & \longrightarrow & (X \hat{\otimes} A^*)^* & \longrightarrow & (X \hat{\otimes} B^*)^* & \longrightarrow & (X \hat{\otimes} C^*)^* & \longrightarrow & 0 \\ & & & \downarrow \xi_A & & \downarrow \xi_B & & \downarrow \xi_C & & \\ (**) & 0 & \longrightarrow & B(X, A^{**}) & \longrightarrow & B(X, B^{**}) & \longrightarrow & B(X, C^{**}) & \longrightarrow & 0 \end{array}$$

This means sequence (**) is strictly normal exact, or

since B is reflexive (hence A and C are reflexive),

$$0 \longrightarrow B(X,A) \longrightarrow B(X,B) \longrightarrow B(X,C) \longrightarrow 0$$

is strictly normal exact.

The proofs that (2) implies (3) and (3) implies (4) are trivial.

Statement (4) implies (1). From the hypothesis of (4), X satisfies the conditions of (2.26). It will be established that X^* (in which the unit ball always has an extreme point by the Krein-Milman theorem [3, p.440]) satisfies (2) of (2.28) so that X^{**} will be a space L_1^1 . By (2.25) X will then be a space L_1^1 . It is sufficient to show that for every collection of four mutually intersecting balls $\{S_{X^*}(x_i, r_i) : i = 1, 2, 3, 4\}$ such that the centers span a two-dimensional subspace of X^* , there exists a x in X^* such that $|x - x_i| \leq r_i$ for $i = 1, \dots, 4$. Let Y be the two-dimensional subspace spanned by the set $\{x_i : i = 1, 2, 3, 4\}$. By (2.27) the balls in $\{S_Y(x_i, r_i) : i = 1, 2, 3, 4\}$ are mutually intersecting. By (1) of (2.28) there exists $Z \supset Y$ with $\dim Z/Y = 1$ and a point z in Z such that $|z - x_i| = r_i$ for $i = 1, \dots, 4$. Let $g: Z \rightarrow X^*$ be the operator whose restriction to Y is the insertion $f: Y \rightarrow X^*$ and for which $|g| = |f|$ (see (2.26)). Then $x = g(z)$ satisfies for each $i = 1, \dots, 4$,

$$|x - x_i| = |g(z - x_i)| \leq |z - x_i| \leq r_i. \quad \text{Q.E.D.}$$

One might ask whether the condition that B be reflexive in (2) of (2.29) may be dropped. The answer

is contained in the following theorem. In order to prove this theorem, several observations from the literature are needed, and they are given here without proof.

Facts 2.30 (1) Let X be a space L^1_I where the measure is not discrete. This means that L^1_I is not a space ℓ^1_I (see (2.14)). Then X contains a subspace isometric to $L^1(0,1)$. ($L^1(0,1)$ is the classical space of Lebesgue integrable functions from $(0,1)$ to I .) See [6, p.159]. Hence, X has a subspace isometric to the two-dimensional inner product space [9, p.493].

(2) A Banach space is called smooth if the norm is Gateaux differentiable at every point on the boundary of its unit ball. The space ℓ^1_I does not contain any smooth subspace [9, p.498]. The two-dimensional inner product space is smooth [2, p.119].

Theorem 2.31 The following statements are equivalent.

(1) X is equivalent to a space ℓ^1_I .

(2) If

$$(*) \quad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is any normal exact sequence in \mathbb{B} , then

$$(**) \quad 0 \longrightarrow \mathbb{B}(X, A) \longrightarrow \mathbb{B}(X, B) \longrightarrow \mathbb{B}(X, C) \longrightarrow 0$$

is normal exact.

(3) If $(*)$ is strictly normal exact in \mathbb{B} , then

$(**)$ is strictly normal exact.

(4) If $0 \longrightarrow A \longrightarrow B \longrightarrow X \longrightarrow 0$

is strictly normal exact in \mathbb{B} , then for every

Banach space Y ,

$$0 \longrightarrow \mathcal{B}(Y, A) \longrightarrow \mathcal{B}(Y, B) \longrightarrow \mathcal{B}(Y, X) \longrightarrow 0$$

is strictly normal exact.

Proof: Statement (3) implies (1). Let A be a closed linear subspace of the Banach space B and let $f: A \longrightarrow B$ be the insertion map. The sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

is a normal exact sequence. By (2.22), the sequence

$$0 \longrightarrow (B/A)^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0$$

is strictly normal exact. Therefore by (3),

$$0 \longrightarrow \mathcal{B}(X, (B/A)^*) \longrightarrow \mathcal{B}(X, B^*) \longrightarrow \mathcal{B}(X, A^*) \longrightarrow 0$$

is strictly normal exact. Therefore by (1.8), the following diagram is commutative.

$$\begin{array}{ccccccc} (*) & 0 & \longrightarrow & \mathcal{B}(X, (B/A)^*) & \longrightarrow & \mathcal{B}(X, B^*) & \longrightarrow \mathcal{B}(X, A^*) \longrightarrow 0 \\ & & & \downarrow \mu_{B/A} & & \downarrow \mu_B & \downarrow \mu_A \\ (**) & 0 & \longrightarrow & (X \hat{\otimes} B/A)^* & \longrightarrow & (X \hat{\otimes} B)^* & \longrightarrow (X \hat{\otimes} A)^* \longrightarrow 0 \end{array}$$

Therefore (**) is strictly normal exact which implies,

$$0 \longrightarrow (X \hat{\otimes} A)^* \longrightarrow (X \hat{\otimes} B)^* \longrightarrow (X \hat{\otimes} C)^* \longrightarrow 0$$

is strictly normal exact. This means $X \hat{\otimes} A \longrightarrow X \hat{\otimes} B$ is an isometric map, so that by (2.24), X is equivalent to a space L^1_I . It must still be established that the measure μ is discrete, that is, X is a space λ^1_I . If not, by (1) of (2.30), X has a subspace isometric to the two-dimensional inner product space. Consider the space $\lambda^1_I = \lambda^1_I(S_X, \mu)$ as in (2.14) where S_X is the unit ball of X with the discrete topology. If $t \in \lambda^1_I$, t can be written

as $\sum_{x_i \in S_X} \lambda_i x_{x_i}$ where $t(x_i) = \lambda_i$ and $\sum |\lambda_i| < \infty$. Define

$h: \ell_I^1 \rightarrow X$ by $h(\sum_{x_i \in S_X} \lambda_i x_{x_i}) = \sum \lambda_i x_i$. Then the sequence

$$0 \rightarrow \text{Ker } h \rightarrow \ell_I^1 \xrightarrow{h} X \rightarrow 0$$

is strictly normal exact. If

$$0 \rightarrow \mathcal{B}(X, \text{Ker } h) \rightarrow \mathcal{B}(X, \ell_I^1) \rightarrow \mathcal{B}(X, X) \rightarrow 0$$

is strictly normal exact, the map $i_X: X \rightarrow X$ has a norm preserving lifting $X \rightarrow \ell_I^1$. This means the diagram

$$\begin{array}{ccc} & & X \\ & \swarrow f & \downarrow i_X \\ \ell_I^1 & \xrightarrow{h} & X \end{array}$$

commutes for some $f: X \rightarrow \ell_I^1$ where $|f| = 1$. Moreover, f is an isometric map as $|x| = |h(f(x))| \leq |f(x)| \leq |x|$ for all x in X . Therefore, ℓ_I^1 contains a smooth subspace, the space isometric to the inner product subspace of X . By (2) of (2.30), this is impossible.

Statement (1) implies (2). Let Z be any Banach space. The space $\mathcal{B}(\ell_I^1(E, \mu), Z)$ is isometrically isomorphic to ℓ_Z^∞ , the space of bounded sequence (z_α) , $\alpha \in E$, of elements of Z . This isomorphism is given by

$$(z_\alpha) \in \ell_Z^\infty \mapsto ((\lambda_\alpha) \in \ell_I^1 \mapsto \sum_{\alpha \in E} \lambda_\alpha z_\alpha).$$

Let $k: \ell_I^1 \rightarrow C$ be in \mathcal{B} and let $\varepsilon > 0$ be arbitrary.

$$\begin{array}{ccc} & \ell_I^1 & \\ & \downarrow k & \\ B & \xrightarrow{g} & C \rightarrow 0 \end{array}$$

There exists, therefore, a unique element $c = (c_\alpha)$ in ℓ_C^∞

corresponding to k as above; and $|c| = \sup_{\alpha \in E} |c_\alpha| = |k|$.

For each $\alpha \in E$, let $b_\alpha \in B$ be such that $g(b_\alpha) = c_\alpha$ and

$|c_\alpha| \geq |b_\alpha| - \epsilon$. Set $b = (b_\alpha)_{\alpha \in E}$. Then $b \in \ell_B^\infty$ and

$$|b| = \sup_{\alpha \in E} |b_\alpha| \leq \sup_{\alpha \in E} (|c_\alpha| + \epsilon) \leq |c| + \epsilon.$$

The element b corresponds to a continuous linear map

$h: \ell_I^1 \rightarrow B$ with $|h| = |b| \leq |k| + \epsilon$. It is immediate that

$g \circ h = k$. Hence,

$$\mathcal{B}(\ell_I^1, B) \rightarrow \mathcal{B}(\ell_I^1, C) \rightarrow 0$$

is normal exact. It is easy to show that for any space

X in \mathcal{B} the sequence

$$0 \rightarrow \mathcal{B}(X, A) \rightarrow \mathcal{B}(X, B) \rightarrow \mathcal{B}(X, C)$$

is normal exact. Hence (1) implies (2).

Statement (1) implies (3). If

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is strictly normal exact, in the proof of (1) implies (2),

$|c_\alpha| = |b_\alpha|$ with $g(b_\alpha) = c_\alpha$ so that $|b| = |c|$. Therefore

$|k| = |h|$.

Statement (2) implies (1). By a demonstration

similar to that in the proof that (3) implies (1), it can

be shown that X must be equivalent to a space L_I^1 . It

must still be shown that this is a space ℓ_I^1 . Consider

again the space $\ell_I^1(S_X, \mu)$ as in (2.14) where S_X is the

unit ball of X . As shown in the proof of (3) implies (1),

X is isomorphic to a quotient space $\ell_I^1(S_X, \mu)/H$. By

assumption, since

$$0 \rightarrow H \rightarrow \ell_I^1(S_X, \mu) \xrightarrow{h} \ell_I^1(S_X, \mu)/H \rightarrow 0$$

is normal exact,

$$0 \longrightarrow B(X, H) \longrightarrow B(X, \ell_I^1) \longrightarrow B(X, \ell_I^1/H) \longrightarrow 0$$

is normal exact. If $g \in B(X, \ell_I^1/H)$ is the isomorphism mentioned above, then $k: X \longrightarrow \ell_I^1$ exists so that $h \circ k = g$ or $(g^{-1} \circ h) \circ k = i_X$. Therefore, X is a retract of ℓ_I^1 . In ℓ_I^1 the weakly compact subsets are compact [3, p.295].

Therefore, the same is true for X . However, if X is a space L_I^1 where the measure is nondiscrete, X contains $L^1(0,1)$ by (1) of (2.30). But in $L^1(0,1)$ the sequence $f_n = \sin nx$ for $n = 1, 2, \dots$, for example, converges weakly but not in the norm topology. See [18, pp.336 + 377]. Therefore X must be a space ℓ_I^1 .

Statement (4) is equivalent to (1). The proof can be found in [9, p.498].

Theorem 2.30 The following statements are equivalent.

(1) X is equivalent to L_I^1 .

(2) If

$$(*) \quad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is strictly normal exact, then

$$(**) \quad 0 \longrightarrow X \hat{\otimes} A \longrightarrow X \hat{\otimes} B \longrightarrow X \hat{\otimes} C \longrightarrow 0$$

is normal exact.

(3) If $(*)$ is normal exact with B reflexive, then $(**)$ is normal exact.

(4) If $(*)$ is normal exact with B finite dimensional, then $(**)$ is normal exact.

(5) If $(*)$ is normal exact with B of dimension

3 and C of dimension 1, then $(**)$ is normal exact.

Proof: Statement (1) implies (2) by (2.19) Using (2.23), statement (2) implies (3). It is easy to see that (3) implies (4) and (4) implies (5). It must now be shown that (5) implies (1). Let

$$0 \rightarrow W \rightarrow Y \rightarrow Z \rightarrow 0$$

be any normal exact sequence in \mathcal{B} with $\dim Y = 3$ and $\dim Z = 2$. By (2.22),

$$0 \rightarrow Z^* \rightarrow Y^* \rightarrow W^* \rightarrow 0$$

is strictly normal exact with $\dim Y^* = 3$ and $\dim W^* = 1$. By assumption,

$$0 \rightarrow X \hat{\otimes} Z^* \rightarrow X \hat{\otimes} Y^* \rightarrow X \hat{\otimes} W^* \rightarrow 0$$

is normal exact. By (2.22),

$$0 \rightarrow (X \hat{\otimes} W^*)^* \rightarrow (X \hat{\otimes} Y^*)^* \rightarrow (X \hat{\otimes} Z^*)^* \rightarrow 0$$

is strictly normal exact. As in the proof of (2.29), this means that

$$0 \rightarrow \mathcal{B}(X, W^{**}) \rightarrow \mathcal{B}(X, Y^{**}) \rightarrow \mathcal{B}(X, Z^{**}) \rightarrow 0$$

is strictly normal exact. Hence,

$$0 \rightarrow \mathcal{B}(X, W) \rightarrow \mathcal{B}(X, Y) \rightarrow \mathcal{B}(X, Z) \rightarrow 0$$

is strictly normal exact. By (2.29), X is equivalent to L_I^1 .

III NATURAL TRANSFORMATIONS AND THEIR DUALS

Definition 3.1 Let $\tau:F \rightarrow G$ be a natural transformation from functor F to G . The dual transformation $D\tau:DG \rightarrow DF$ (refer to (1.10)) to τ is the transformation which assigns to each object X in \mathcal{B} the morphism $(D\tau)_X:DGX \rightarrow DFX$ defined by

$$((D\tau)_X \eta)_A = \eta_A \circ \tau_A \text{ for } \eta \in DGX, A \text{ in } \mathcal{B}.$$

Definition 3.2 Let $\tau:F \rightarrow G$ be a natural transformation. To each morphism $f:X \rightarrow Y$ in \mathcal{B} associate the morphism $\tau(f)$ defined as $\tau_Y \circ F(f)$ or its equal $Gf \circ \tau_X$.

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \downarrow \tau_X & & \downarrow \tau_Y \\ GX & \xrightarrow{Gf} & GY \end{array}$$

The transformation τ is called a compact natural transformation if whenever $f:X \rightarrow Y$ is a compact operator [3, p.485], $\tau(f)$ is also a compact operator.

Definition 3.3 Similar to (3.2), the concepts of weakly compact, epimorphic, and monomorphic natural transformations are defined. Thus, for example, τ is a monomorphic transformation if whenever f is a monomorphism, $\tau(f)$ is a monomorphism.

Definition 3.4 A functor $F:\mathcal{B} \rightarrow \mathcal{B}$ is compact(weakly compact, epimorphic, or monomorphic) if the identity natural transformation $1_F:F \rightarrow F$ is compact (weakly compact, epimorphic, or monomorphic).

The purpose of this section is to prove, among other results, that if a transformation is compact, its dual is compact.

Definitions and Notations 3.5 Let $F: \mathbb{B} \rightarrow \mathbb{B}$ be a functor and $F_*^*: \mathbb{B} \rightarrow \mathbb{B}$ the functor given in (1.6). For each A in \mathbb{B} define $\tilde{\Sigma}_A$ as the compositions of the functors $**$ and Σ_A . Using the functors $\tilde{\Sigma}_A$, define functor $\tilde{D}F$ by the rules:

(1) if X is in \mathbb{B} , $\tilde{D}FX = (F \rightarrow \tilde{\Sigma}_X)$, the natural transformations from F to $\tilde{\Sigma}_X$.

(2) if $f: X \rightarrow Y$ is a morphism in \mathbb{B} , then $\tilde{D}Ff:$

$\tilde{D}FX \rightarrow \tilde{D}FY$ is defined by the commutative diagram

$$\begin{array}{ccc}
 FA & \xrightarrow{\tau_A} & \tilde{\Sigma}_X A \\
 & \searrow (\tilde{D}Ff(\tau))_A & \downarrow (f \hat{\circ} i_A)** \\
 & & \tilde{\Sigma}_Y A
 \end{array}$$

where $\tau \in (F \rightarrow \tilde{\Sigma}_X) = \tilde{D}FX$.

For each A in \mathbb{B} and X in \mathbb{B} , let η_X^A be the natural embedding of $A \hat{\otimes} X$ into $(A \hat{\otimes} X)**$. It is easy to verify that for each A in \mathbb{B} the maps η_X^A generate a natural transformation $\eta^A: \Sigma_A \rightarrow \tilde{\Sigma}_A$. Using the natural transformations η^A , define a natural transformation $\rho: DF \rightarrow \tilde{D}F$ by the assignments:

$$\rho_A(\theta) = \eta^A \circ \theta \quad \theta \in DFA, A \text{ in } \mathbb{B}.$$

These assignments are depicted by the following commutative diagram.

$$\begin{array}{ccc}
 FX & \xrightarrow{\theta_X} & \Sigma_A X \\
 & \searrow (\rho_A \theta)_X & \downarrow \eta_X^A \\
 & & \tilde{\Sigma}_A X
 \end{array}$$

Lemma 3.6 The assignment ρ is indeed a natural transformation.

Proof: Let $f:A \rightarrow B$ be in \mathbb{B} . It must be verified that the following diagram commutes.

$$\begin{array}{ccc}
 DFA & \xrightarrow{DFf} & DFB \\
 \rho_A \downarrow & & \downarrow \rho_B \\
 \tilde{DFA} & \xrightarrow{\tilde{D}Ff} & \tilde{DFB}
 \end{array}$$

Let $\theta \in DFA = (F \rightarrow \Sigma_A)$. Then if X is in \mathbb{B} ,

$$[(\tilde{D}Ff \circ \rho_A) \theta]_X = [\tilde{D}Ff(\eta_X^A \circ \theta)]_X = (f \hat{\otimes} i_X)^{**} \circ \eta_X^A \circ \theta_X.$$

Also,

$$\begin{aligned}
 [(\rho_B \circ DFf) \theta]_X &= [\rho_B(DFf(\theta))]_X = \eta_X^B \circ DFf(\theta)_X \\
 &= \eta_X^B \circ (f \hat{\otimes} i_X) \circ \theta_X.
 \end{aligned}$$

In the diagram below, $(f \hat{\otimes} i_X)^{**} \circ \eta_X^A = \eta_X^B \circ (f \hat{\otimes} i_X)$.

$$\begin{array}{ccc}
 A \hat{\otimes} X & \xrightarrow{f \hat{\otimes} i_X} & B \hat{\otimes} X \\
 \eta_X^A \downarrow & & \downarrow \eta_X^B \\
 (A \hat{\otimes} X)^{**} & \xrightarrow{(f \hat{\otimes} i_X)^{**}} & (B \hat{\otimes} X)^{**}
 \end{array}$$

This means

$$(f \hat{\otimes} i_X)^{**} \circ \eta_X^A \circ \theta_X = \eta_X^B \circ (f \hat{\otimes} i_X) \circ \theta_X.$$

Since for each A in \mathbb{B} , $|\rho_A(\theta)| = |\eta_X^A \circ \theta|$ and η_X^A is an isometric map for each X in \mathbb{B} , $|\rho_A(\theta)| = |\theta|$; that is, ρ_A is an isometric map.

Definition 3.7 For each A in \mathbb{B} , let $\text{Tr}: A^* \hat{\otimes} A \rightarrow I$ be the continuous linear operator which is the unique extension of the map from $A^* \hat{\otimes} A \rightarrow I$ given by

$$\sum_{i=1}^n a_i^* \hat{\otimes} a_i \mapsto \sum_{i=1}^n a_i^*(a_i).$$

(It can be checked that $|\text{Tr}| = 1$.) Also for each A in \mathbb{B} , define the linear mapping $\lambda_A: \text{DFA} \rightarrow F^* A$ by the rule $\lambda_A(\theta) = \text{Tr} \circ \theta_{A^*}$ for $\theta \in \text{DFA}$; this is shown by the following commutative diagram.

$$\begin{array}{ccc} F(A^*) & \xrightarrow{\theta_{A^*}} & A^* \hat{\otimes} A \\ & \searrow \lambda_A(\theta) & \swarrow \text{Tr} \\ & I & \end{array}$$

Since $|\lambda_A(\theta)| = |\text{Tr} \circ \theta_{A^*}| \leq |\theta_{A^*}| \leq |\theta|$, $|\lambda_A| \leq 1$.

Lemma 3.8 The mappings λ_A generate a natural transformation $\lambda: \text{DF} \rightarrow F^*$.

Proof: Since $|\lambda| \leq 1$, condition (1) of (1.3) remains to be shown. Let $f: A \rightarrow B$ be in \mathbb{B} and $\theta \in \text{DFA}$. It must be demonstrated that $((Ff^*)^* \circ \lambda_A)\theta = (\lambda_B \circ \text{DF}f)\theta$.

$$\begin{array}{ccc} \text{DFA} & \xrightarrow{\text{DF}f} & \text{DFB} \\ \lambda_A \downarrow & & \downarrow \lambda_B \\ (FA^*)^* & \xrightarrow{(Ff^*)^*} & (FB^*)^* \end{array}$$

Now

$$\begin{aligned} ((Ff^*)^* \circ \lambda_A)\theta &= \lambda_A(\theta) \circ Ff^* = \text{Tr} \circ \theta_{A^*} \circ Ff^*, \text{ and} \\ (\lambda_B \circ \text{DF}f)\theta &= \lambda_B(\text{DF}f(\theta)) = \text{Tr} \circ (f \hat{\otimes} i_{B^*}) \circ \theta_{B^*}. \end{aligned}$$

Since $\theta \in (F \rightarrow \Sigma_A)$, the following diagram commutes.

$$\begin{array}{ccc}
 FB^* & \xrightarrow{\theta_{B^*}} & \Sigma_A B^* \\
 Ff^* \downarrow & & \downarrow i_A \hat{\otimes} f^* \\
 FA^* & \xrightarrow{\theta_{A^*}} & \Sigma_A A^*
 \end{array}$$

Therefore to prove the proposition, it suffices to show that $\text{Tr} \circ (i_A \hat{\otimes} f^*) = \text{Tr} \circ (f \hat{\otimes} i_{B^*})$. Let $\sum_{i=1}^n a_i \otimes b_i^*$ be an element in $A \hat{\otimes} B^*$. Then

$$\begin{aligned}
 [\text{Tr} \circ (i_A \hat{\otimes} f^*)] \left(\sum_{i=1}^n a_i \otimes b_i^* \right) &= \text{Tr} \left(\sum_{i=1}^n a_i \otimes (b_i^* \circ f) \right) \\
 &= \sum_{i=1}^n b_i^* (f(a_i)),
 \end{aligned}$$

and

$$\begin{aligned}
 [\text{Tr} \circ (f \hat{\otimes} i_{B^*})] \left(\sum_{i=1}^n a_i \otimes b_i^* \right) &= \text{Tr} \left(\sum_{i=1}^n f(a_i) \otimes b_i^* \right) \\
 &= \sum_{i=1}^n b_i^* (f(a_i)).
 \end{aligned}$$

Therefore, $\text{Tr} \circ (i_A \hat{\otimes} f^*) = \text{Tr} \circ (f \hat{\otimes} i_{B^*}) : A \hat{\otimes} B^* \rightarrow I$.

The next important lemma is proved in [11, p.82].

Lemma 3.9 There exists a natural transformation $\phi : F_*^* \rightarrow \tilde{D}F$ such that $\phi \circ \lambda = \rho$ and $|\phi| \leq 1$.

Proposition 3.10 For each A in \mathcal{B} , λ_A is an isometric linear function of DFA into $(FA^*)^*$.

Proof: Let $\theta \in \text{DFA}$. Using (3.6) and (3.9),

$$|\theta| = |\rho_A(\theta)| = |(\theta_A \circ \lambda_A)\theta| \leq |\lambda_A \theta| \leq |\theta|;$$

which implies $|\theta| = |\lambda_A(\theta)|$.

Corollary 3.11 Let X in \mathcal{B} be finite dimensional. If FX is finite dimensional, DFX is finite dimensional. Let X in \mathcal{B} be reflexive. If FX is reflexive, DFX is reflexive.

Definition 3.12 Let $F : \mathcal{B} \rightarrow \mathcal{B}$ be a functor. A subfunctor

$G:B \longrightarrow B$ of F is a functor so that,

(1) for each X in B , GX is a closed subspace of FX ; and

(2) if $f:X \longrightarrow Y$ is in B , $Ff(GX) \subset GY$ and $Gf = Ff$ on GX .

Proposition 3.13 For any functor $F:B \longrightarrow B$, DF is naturally equivalent to a subfunctor R of F_*^* .

Proof: For each A in B , let RA be the subspace of $(FA^*)^*$ equivalent to DFA by (3.10). Since the diagram

$$\begin{array}{ccc} DFA & \xrightarrow{DFf} & DFB \\ \lambda_A \downarrow & & \downarrow \lambda_B \\ (FA^*)^* & \xrightarrow{(Ff^*)^*} & (FB^*)^* \end{array}$$

commutes for $f:A \longrightarrow B$ in B , and each $y \in RA$ equals $\lambda_A(\theta)$ for a unique $\theta \in DFA$; $(Ff^*)^*(y) = (\lambda_B \circ DFf)\theta$, an element in RB . This means $(Ff^*)^*(RA) \subset RB$. Therefore, by defining Rf to be $(Ff^*)^*$ restricted to RA , these assignments make R a subfunctor of F_*^* naturally equivalent to DF .

Lemma 3.14 Let $\tau:F \longrightarrow G$ be a natural transformation.

For each A in B , let τ_{*A}^* be the morphism $(\tau_A^*)^*:(GA^*)^* \longrightarrow (FA^*)^*$. The morphisms τ_{*A}^* generate a natural transformation $\tau_*^*:G_*^* \longrightarrow F_*^*$.

Proof: Let $f:X \longrightarrow Y$ be in B . Since τ is a natural transformation, the following diagram commutes.

$$\begin{array}{ccc} FY^* & \xrightarrow{Ff^*} & FX^* \\ \tau_{Y^*}^* \downarrow & & \downarrow \tau_{X^*}^* \\ GY^* & \xrightarrow{Gf^*} & GX^* \end{array}$$

Hence, the following diagram commutes.

$$\begin{array}{ccc}
 (GX^*)^* & \xrightarrow{(Gf^*)^*} & (GY^*)^* \\
 \tau_{*X}^* \downarrow & & \downarrow \tau_{*Y}^* \\
 (FX^*)^* & \xrightarrow{(Ff^*)^*} & (FY^*)^*
 \end{array}$$

Since $|\tau_{*A}^*| = |(\tau_A^*)^*| = |\tau_A^*|$, $|\tau_{*}^*| < \infty$ so that τ_{*}^* is a natural transformation.

Definition 3.15 Let $\tau:F \longrightarrow G$ be a natural transformation, F' a subfunctor of F , and G' a subfunctor of G . Then F' and G' are compatible subfunctors with respect to τ if for each X in B , the restriction $\bar{\tau}_X$ of τ_X to $F'X \subset FX$ maps $F'X$ into $G'X$.

Lemma 3.16 The mappings $\bar{\tau}_X$ defined in (3.15) generate a natural transformation $\bar{\tau}:F' \longrightarrow G'$.

Proof: Let $f:X \longrightarrow Y$ be in B . Since $\bar{\tau}_X$, $\bar{\tau}_Y$, $F'f$, and $G'f$ are the restrictions of τ_X , τ_Y , Ff , and Gf ; the inner square below commutes.

$$\begin{array}{ccccc}
 & & Ff & & \\
 & & \xrightarrow{\quad} & & \\
 FX & & & & FY \\
 & \nwarrow & & \nearrow & \\
 & F'X & \xrightarrow{F'f} & F'Y & \\
 & \downarrow \bar{\tau}_X & & \downarrow \bar{\tau}_Y & \\
 & G'X & \xrightarrow{G'f} & G'Y & \\
 & \nwarrow & & \nearrow & \\
 & GX & \xrightarrow{Gf} & & GY \\
 & \downarrow \tau_X & & \downarrow \tau_Y &
 \end{array}$$

Proposition 3.17 Let $\tau:F \longrightarrow G$ be a natural transformation

and $\tau_*^*: G_*^* \rightarrow F_*^*$ the generated transformation defined in (3.14). Then DG and DF are naturally equivalent to compatible subfunctors of G_*^* and F_*^* respectively (compatible with respect to τ_*^*).

Proof: Let R and S be the subfunctors of F_*^* and G_*^* which by virtue of (3.13) are naturally equivalent to DF and DG respectively. Let $\lambda: DF \rightarrow F_*^*$ and $\lambda': DG \rightarrow G_*^*$ be these equivalences. Let $f: X \rightarrow Y$ be a morphism in \mathbb{B} . Consider the following diagram.

$$(3.18) \quad \begin{array}{ccccc} & & (Gf^*)^* & & \\ & & \xrightarrow{\quad} & & \\ (GX^*)^* & \xrightarrow{\quad Sf \quad} & (GY^*)^* & & \\ \downarrow \tau_{*X}^* & \nearrow i_X & \downarrow \tau_{*Y}^* & \nearrow j_Y & \\ SX & \xrightarrow{\quad D\tau_X \quad} & SY & \xrightarrow{\quad D\tau_Y \quad} & \\ \downarrow \tau_{*X}^* & \nearrow \lambda_X^{-1} & \downarrow \tau_{*Y}^* & \nearrow \lambda_Y^{-1} & \\ DGX & \xrightarrow{\quad DGf \quad} & DGY & \xrightarrow{\quad D\tau_Y \quad} & \\ \downarrow D\tau_X & \nearrow \lambda_X & \downarrow D\tau_Y & \nearrow \lambda_Y & \\ DFX & \xrightarrow{\quad DFf \quad} & DFY & \xrightarrow{\quad Rf \quad} & \\ \downarrow \tau_{*X}^* & \nearrow j_X & \downarrow \tau_{*Y}^* & \nearrow j_Y & \\ RX & \xrightarrow{\quad (Ff^*)^* \quad} & RY & \xrightarrow{\quad} & \\ \downarrow \tau_{*X}^* & \nearrow j_X & \downarrow \tau_{*Y}^* & \nearrow j_Y & \\ (FX^*)^* & \xrightarrow{\quad} & (FY^*)^* & & \end{array}$$

Define $\bar{\tau}_{*X}^*$ to be the morphism $\lambda_X \circ D\tau_X \circ \lambda_X^{-1}$. It must be shown $j_X \circ \bar{\tau}_{*X}^* = \tau_{*X}^* \circ i_X$ where i_X and j_X are insertion maps. Let $s \in SX$. Then $s = Tr \circ \theta_X$ for a unique $\theta_X \in DGX$. Also $(D\tau_X(\theta))_A = \theta_A \circ \tau_A$ for each A in \mathbb{B} and $D\tau_X(\theta) \in DFX$. There-

fore, $\bar{\tau}_{*X}^*(s)$ in RX is the element $\text{Tr} \circ \theta_{X*} \circ \tau_{X*}^*: F(X^*) \longrightarrow I$. Now consider $s = \text{Tr} \circ \theta_{X*}$ as an element of $(GX^*)^*$ via i_X . By definition $\tau_{*X}^* = (\tau_{X*})^*: (GX^*)^* \longrightarrow (FX^*)^*$. Therefore, $\tau_{*X}^*(\text{Tr} \circ \theta_{X*}) = \text{Tr} \circ \theta_{X*} \circ \tau_{X*}^*$. Hence, $j_X \circ \bar{\tau}_{*X}^* = \tau_{*X}^* \circ i_X$ and $\bar{\tau}_{*X}^*$ is the restriction of τ_{*X}^* to SX . Similarly $\bar{\tau}_{*Y}^*$ is the restriction of τ_{*Y}^* to SY .

Proposition 3.19 If $\tau: F \longrightarrow G$ is a compact natural transformation, then $\tau_{*}^*: G_{*}^* \longrightarrow F_{*}^*$ is also.

Proof: Let $f: X \longrightarrow Y$ be compact. It must be shown that any one of the equal morphisms

$$(\tau_{Y*})^* \circ (Gf^*)^* = \tau_{*Y}^* \circ G_{*}^* f = F_{*}^* f \circ \tau_{*X}^* = (Ff^*)^* \circ \tau_{X*}^*$$

is compact. Since $f^*: Y^* \longrightarrow X^*$ is compact [3, p.485] and τ is a compact transformation, $Gf^* \circ \tau_{Y*}$ is compact. Hence $(Gf^* \circ \tau_{Y*})^* = (\tau_{Y*})^* \circ (Gf^*)^*$ is compact.

Lemma 3.20 Let $\tau: F \longrightarrow G$ be a natural transformation, F' and G' compatible subfunctors with respect to τ of F and G , and $\bar{\tau}: F' \longrightarrow G'$ the transformation given by (3.16). If τ is a compact transformation so is $\bar{\tau}: F' \longrightarrow G'$.

Proof: Let $f: X \longrightarrow Y$ be a compact mapping. The morphism $G'f \circ \bar{\tau}_X$ is the restriction of the morphism $Gf \circ \tau_X$ to $F'X$ (refer to (3.16)). Let S' be the closed unit ball of $F'X$ and S the closed unit ball of FX . It suffices to show $G'f \circ \bar{\tau}_X(S')$ is sequentially compact in $G'Y$ (see [3, p.22]). Since $S' \subset S$ and $G'f \circ \bar{\tau}_X(S') = Gf \circ \tau_X(S') \subset Gf \circ \tau_X(S)$, every sequence of points in $G'f \circ \bar{\tau}_X(S')$ is a sequence in $Gf \circ \tau_X(S)$. Since $Gf \circ \tau_X(S)$ is sequentially compact, every

sequence of points in $Gf' \circ \bar{\tau}_X(S')$ has a subsequence converging to a point of GY . However since $G'Y$ is closed in GY , this point is in $G'Y$; and $Gf' \circ \bar{\tau}_X(S')$ is sequentially compact.

Lemma 3.21 Let $\tau: F \rightarrow H$ be a natural transformation which is the composition of the natural transformations $\eta: F \rightarrow G$ and $\theta: G \rightarrow H$; that is, $\tau = \theta \circ \eta$. If θ or η is compact, τ is compact.

Proof: Let $f: X \rightarrow Y$ be a compact map. In the commutative diagram

$$\begin{array}{ccc}
 FX & \xrightarrow{Ff} & FY \\
 \eta_X \downarrow & & \downarrow \eta_Y \\
 GX & \xrightarrow{Gf} & GY \\
 \theta_X \downarrow & & \downarrow \theta_Y \\
 HX & \xrightarrow{Hf} & HY
 \end{array}$$

if $Hf \circ \theta_X$ is compact, then $Hf \circ \tau_X = Hf \circ \theta_X \circ \eta_X$ is compact (see [3, p.486]). If $\eta_Y \circ Ff$ is compact, then $\tau_Y \circ Ff = \theta_Y \circ \eta_Y \circ Ff$ is compact.

Theorem 3.22 If $\tau: F \rightarrow G$ is a compact natural transformation, then $D\tau: DG \rightarrow DF$ is a compact natural transformation.

Proof: Since $\tau: F \rightarrow G$ is compact, by (3.19) $\tau_*^*: G_*^* \rightarrow F_*^*$ is compact. By (3.17), DG and DF are naturally equivalent to compatible subfunctors S and R of G_*^* and F_*^* . By (3.20),

the transformation $\bar{\tau}_*^*: S \longrightarrow R$ is compact. Referring to diagram (3.18), $D\tau_X = \lambda_X^{-1} \circ \bar{\tau}_*^* \circ \lambda_X'$ for each X in \mathcal{B} , or $D\tau = \lambda^{-1} \circ \bar{\tau}_*^* \circ \lambda'$. By (3.21) $D\tau$ is compact.

Theorem 3.23 If $\tau: F \longrightarrow G$ is a weakly compact natural transformation, then $D\tau: DG \longrightarrow DF$ is a weakly compact natural transformation.

Proof: By similar proofs, it can be shown that with the words "weakly compact" inserted for the word "compact" in (3.19), (3.20), and (3.21) the corresponding statements are true. Using these altered statements, the proof of this theorem is similar to the proof of (3.22).

Theorem 3.24 If $\tau: F \longrightarrow G$ is an epimorphic natural transformation, then $D\tau: DG \longrightarrow DF$ is a monomorphic natural transformation.

Proof: Let $f: A \longrightarrow B$ be a monomorphism. Then $f^*: B^* \longrightarrow A^*$ is an epimorphism. Hence, $\tau_{A*} \circ Ff^*: FB^* \longrightarrow GA^*$ is an epimorphism since τ is epimorphic. Therefore,

$$(\tau_{A*} \circ Ff^*)^* = (Ff^*)^* \circ \tau_{A*}^* = F_{*}^* f \circ \tau_{*A}^* = \tau_{*}^* f$$

is a monomorphism, or $\tau_{*}^*: G_{*}^* \longrightarrow F_{*}^*$ is monomorphic. Using diagram (3.18), it can be seen that $DFf \circ D\tau_X$ is a monomorphism. Hence $D\tau$ is monomorphic.

Theorem 3.25 Let $F: \mathcal{B} \longrightarrow \mathcal{B}$ be a functor. Then

- (1) if F is compact, DF is compact;
- (2) if F is weakly compact, DF is weakly compact; and
- (3) if F is epimorphic, DF is monomorphic.

Proof: Let $\iota_F: F \rightarrow F$ be the identity transformation. Then $D\iota_F = \iota_F$. Therefore if ι_F is compact, weakly compact, or epimorphic, by (3.22), (3.23), and (3.24), $D\iota_F = \iota_F$ is respectively compact, weakly compact, or monomorphic. Q.E.D.

Results (3.22) through (3.25) have been proved independently in a similar way by Evans in [4].

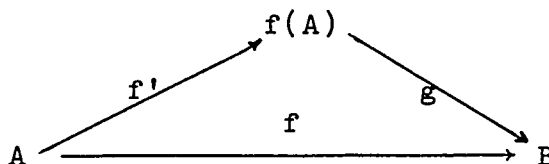
IV CHARACTERIZATIONS OF COMPACT FUNCTORS

In this section, necessary and sufficient conditions will be given for the functors Σ_X and Ω_X to be compact. More generally, conditions will be given to insure that certain functors $F:\mathbb{B} \rightarrow \mathbb{B}$ are compact.

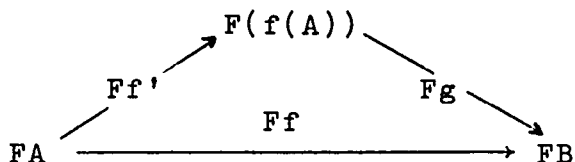
Definition 4.1 A functor $F:\mathbb{B} \rightarrow \mathbb{B}$ has finite rank (or is a finite rank functor) if whenever a morphism $f:A \rightarrow B$ has finite rank ($\dim f(A)$ is finite), $Ff:FA \rightarrow FB$ has finite rank.

Lemma 4.2 A functor $F:\mathbb{B} \rightarrow \mathbb{B}$ has finite rank if and only if FX is finite dimensional whenever X is finite dimensional.

Proof: Let $f:A \rightarrow B$ in \mathbb{B} have finite rank. Since $f(A)$ is finite dimensional, it is a Banach space and the following is a commutative diagram in \mathbb{B} , where g and f' are the obvious maps.



Therefore the following diagram commutes.



Therefore, $Ff(FA)$ is contained in a finite dimensional Banach space $Fg(F(f(A)))$ and Ff has finite rank.

Conversely, let A be a finite dimensional Banach

space. Then $i_A: A \rightarrow A$ has finite rank. Hence $i_{FA}(FA) = FA$ has finite dimension.

Definition 4.3 Let X and Y be in \mathcal{B} . An element f in $\mathcal{B}(X, Y)$ is a Fredholm operator if

- (1) $\text{Ker } f = f^{-1}(0)$ is finite dimensional, and
- (2) $\text{Coker } f = Y/f(X)$ is finite dimensional.

The proof of the following lemma can be found in [13, p.120].

Lemma 4.4 Let X and Y be in \mathcal{B} . If $f \in \mathcal{B}(X, Y)$ and there exists h and h' in $\mathcal{B}(Y, X)$ such that $h \circ f - i_X$ and $f \circ h' - i_Y$ are compact, then f is a Fredholm operator. Conversely, if f is a Fredholm operator, then there exists a g in $\mathcal{B}(Y, X)$ so that $g \circ f - i_X$ and $f \circ g - i_Y$ have finite rank and hence are compact.

Definition 4.5 A functor $F: \mathcal{B} \rightarrow \mathcal{B}$ is a Fredholm functor if whenever $f: A \rightarrow B$ is a Fredholm operator, $Ff: FA \rightarrow FB$ is a Fredholm operator.

Proposition 4.6 A functor $F: \mathcal{B} \rightarrow \mathcal{B}$ is a Fredholm functor if and only if it has finite rank.

Proof: Let A be a finite dimensional Banach space. Then $0: A \rightarrow A$ (the zero morphism) is a Fredholm operator.

Hence $F(0) = 0: FA \rightarrow FA$ is Fredholm, which implies $\text{Ker } F(0) = FA$ is finite dimensional.

Conversely, let F have finite rank and let $f: A \rightarrow B$ be a Fredholm operator. Using (4.4), there is a $g: B \rightarrow A$ in \mathcal{B} such that $f \circ g - i_B$ and $g \circ f - i_A$ have finite rank. Hence,

$Ff \circ Fg - i_{FB}$ and $Fg \circ Ff - i_{FA}$ have finite rank and hence are compact. By (4.4), Ff is a Fredholm operator.

Corollary 4.7 If $F:B \rightarrow B$ is a Fredholm functor so is $DF:B \rightarrow B$.

Proof: If F has finite rank so does DF by (3.11) and (4.2).

Lemma 4.8 If $F:B \rightarrow B$ is a compact functor, then F has finite rank and hence is a Fredholm functor.

Proof: Let A be a Banach space. Then $i_A:A \rightarrow A$ is compact if and only if A is finite dimensional. Let A be finite dimensional. Then, by assumption, $i_{FA}:FA \rightarrow FA$ is compact so that FA is finite dimensional. The result follows from (4.2).

Corollary 4.9 If Σ_X is a compact functor, X must be finite dimensional.

Proof: By (4.8), if A is a finite dimensional Banach space, then $X \hat{\otimes} A$ must be finite dimensional. In particular, $X \hat{\otimes} I = X$ must be finite dimensional. Q.E.D.

Let $N = \{1, 2, \dots, n\}$ with the discrete topology. The space $\ell_A^1(N, \mu)$, A in \mathcal{B} , is the space of all n -tuples $a = (a_1, \dots, a_n)$ made up of elements of A with $\mu(a) = \sum_{i=1}^n |a_i|$. Define the functor $\ell_n^1:B \rightarrow B$ by $\ell_n^1 A = \ell_A^1(N, \mu)$.

Proposition 4.10 For each integer n , $\ell_n^1:B \rightarrow B$ is a compact functor.

Proof: Let $f:A \rightarrow B$ be a compact mapping. It suffices to show that if $(a_1, \dots, a_n)_j = (a_{1j}, a_{2j}, \dots, a_{nj})$ for $j = 1, 2, \dots$

is a sequence in $\prod_n^1 A$ so that for each j , $\sum_{i=1}^n |a_{ij}| \leq 1$, then $\prod_n^1 f((a_{1j}, \dots, a_{nj})) = (f(a_{1j}), f(a_{2j}), \dots, f(a_{nj}))$ for $j = 1, 2, \dots$ has a convergent subsequence. Since (a_{1j}) is a sequence in A such that $|a_{1j}| \leq 1$ for $j = 1, 2, \dots$ and f is compact, the sequence $f(a_{1j})$ has a convergent subsequence $f(a_{1j_k})$. Thus a subsequence $(f(a_{1j_k}), f(a_{2j_k}), \dots, f(a_{nj_k}))$ of the original sequence $(f(a_{1j}), \dots, f(a_{nj}))$ is obtained where $f(a_{1j_k})$ converges to a point b_1 of B . In a similar manner, obtain from the sequence $f(a_{2j_k})$ a subsequence converging to a point b_2 of B . Using this subsequence, a subsequence of $(f(a_{1j_k}), \dots, f(a_{nj_k}))$ is obtained in which the two sequences formed by the first two components on each n -tuple each converges to a point of B . Continuing in this manner, eventually a subsequence of $(f(a_{1j}), \dots, f(a_{nj}))$ for $j = 1, 2, \dots$ is obtained in which the sequence formed by each component converges to a point of B . Clearly, this subsequence will converge to the element of $\prod_n^1 B$ that has for its components these n -limits.

Proposition 4.11 The functor $\Sigma \prod_I^1(N, \mu)$ is a compact functor.

Proof: By (2.16), the functors $\Sigma \prod_I^1(N, \mu)$ and \prod_n^1 are naturally equivalent by a transformation τ . Let $f: A \rightarrow B$ be a compact mapping. By naturality, the following diagram commutes.

$$\begin{array}{ccc}
\ell_I^1(N, \mu) \hat{\otimes} A & \xrightarrow{i \ell_I^1 \hat{\otimes} f} & \ell_I^1(N, \mu) \hat{\otimes} B \\
\tau_A \downarrow & & \uparrow \tau_B^{-1} \\
\ell_n^1 A & \xrightarrow{\ell_n^1 f} & \ell_n^1 B
\end{array}$$

By (4.10), $\ell_n^1 f$ is compact. Therefore $i \ell_I^1 \hat{\otimes} f$ is compact.

Proposition 4.12 If X is a finite dimensional Banach space, then $\Sigma_X: B \rightarrow B$ is a compact functor.

Proof: Let the dimension of X be n . Since $\ell_I^1(N, \mu)$ has dimension n if $N = \{1, 2, \dots, n\}$, $\ell_I^1(N, \mu)$ and X are isomorphic via an isomorphism f . Therefore when $g: A \rightarrow B$ is in B , the following diagram commutes.

$$\begin{array}{ccc}
X \hat{\otimes} A & \xrightarrow{i_X \hat{\otimes} g} & X \hat{\otimes} B \\
f \hat{\otimes} i_A \downarrow & & \uparrow f^{-1} \hat{\otimes} i_B \\
\ell_I^1(N, \mu) \hat{\otimes} A & \xrightarrow{i \ell_I^1 \hat{\otimes} g} & \ell_I^1(N, \mu) \hat{\otimes} B
\end{array}$$

If g is compact, then by (4.11) $i \ell_I^1 \hat{\otimes} g$ is compact. Therefore $i_X \hat{\otimes} g$ is compact.

The following theorem summarizes the above discussion.

Theorem 4.13 The following statements are equivalent.

- (1) X is a finite dimensional Banach space.
- (2) $\Sigma_X: B \rightarrow B$ is a compact functor.
- (3) $\Sigma_X: B \rightarrow B$ is a Fredholm functor.
- (4) $\Omega_X: B \rightarrow B$ is a compact functor.
- (5) $\Omega_X: B \rightarrow B$ is a Fredholm functor.

The question remains whether each finite rank functor is a compact functor. The following discussion shows that

in all known cases the answer is in the affirmative.

Definition 4.14 A Banach space X is said to satisfy the approximation property if for each $\epsilon > 0$ and for each relatively compact subset C of X , there exists a map $f: X \rightarrow X$ of finite rank so that

$$|f(x) - x| < \epsilon \quad \text{for all } x \text{ in } C.$$

Note: No Banach space is known that does not satisfy (4.14). See [5, p.135] and [17, p.521].

The following lemma is contained in a theorem by Grothendieck [5, p.168].

Lemma 4.15 Let X be a Banach space that satisfies the approximation property. Then for any Banach space Y , for any compact mapping $f: Y \rightarrow X$, there exists for each $\epsilon > 0$ a mapping $f_\epsilon: Y \rightarrow X$ of finite rank such that $|f - f_\epsilon| < \epsilon$.

Proof: Since $f: Y \rightarrow X$ is compact, $f(S)$ is relatively compact in X if S is the unit ball of Y . Since X satisfies (4.14), given $\epsilon > 0$, there exists $f_0: X \rightarrow X$ of finite rank such that $|f_0(x) - x| < \epsilon$ for x in $f(S)$. Let $f_\epsilon = f_0 \circ f: Y \rightarrow X$. Then f_ϵ has finite rank and

$$|f_\epsilon - f| = \sup_{x \in S} |f_0(f(x)) - f(x)| < \epsilon.$$

Proposition 4.16 Let $\overline{\mathcal{B}}$ denote the full subcategory of \mathcal{B} consisting of Banach spaces which satisfy the approximation property. Then a functor $F: \overline{\mathcal{B}} \rightarrow \overline{\mathcal{B}}$ is compact if and only if F is a finite rank functor.

Proof: Necessity follows from (4.8). To prove the converse, let $f: A \rightarrow B$ be a compact operator in $\overline{\mathcal{B}}$. By (4.15),

there exists a sequence $f_n: A \rightarrow B$ of operators with finite dimensional range in B converging to f in the uniform operator topology. By the hypothesis, $Ff_n: FA \rightarrow FB$ are operators of finite rank and hence are compact. Since

$$\|Ff_n - Ff\| \leq \|f_n - f\| \quad \text{for } n = 1, 2, \dots,$$

the sequence Ff_n converges to Ff in the uniform operator topology. Hence Ff is compact [3, p.486].

Theorem 4.17 Let $f: A \rightarrow B$ be any compact mapping with B satisfying the approximation property. If $F: B \rightarrow B$ is a functor, then $Ff: FA \rightarrow FB$ is compact if and only if F is a finite rank functor.

Proof: The proof is identical to that of (4.16).

Corollary 4.18 Let $f: A \rightarrow B$ be any compact operator with B a space with a Schauder basis. If $F: B \rightarrow B$ is a functor, then Ff is compact if and only if F is a finite rank functor.

Proof: Every Banach space with a Schauder basis satisfies the approximation property. Indeed, let $\{b_n\}$ be a normalized basis. Each b in B has a unique representation $b = \sum_{i=1}^{\infty} f_i(b)b_i$ where $f_i \in B^*$ and the series converges uniformly on every compact subset of B .

Example 4.19 Consider the functor $\ell_n^1: B \rightarrow B$ given after (4.6) where n is some positive integer. The functor ℓ_n^1 can also be considered as a functor from \bar{B} to \bar{B} . Indeed, let A be in \bar{B} . It must be shown that $\ell_n^1 A$ is in \bar{B} . Let C be any relatively compact set in $\ell_n^1 A$ and $\epsilon > 0$ be arbitrary.

For each $i = 1, \dots, n$ let $p_i: \ell_n^1 A \rightarrow A$ be the i^{th} projection: $p_i((a_1, \dots, a_n)) = a_i$. For each i , $p_i(\bar{C})$ is compact in A . Let $f_i: A \rightarrow A$ be a map in \mathcal{B} of finite rank so that for each x in $p_i(\bar{C})$, $|f_i(x) - x| \leq \epsilon/n$. Define $f_\epsilon: \ell_n^1 A \rightarrow \ell_n^1 A$ by the formula $f_\epsilon((a_1, \dots, a_n)) = (f_1(a_1), \dots, f_n(a_n))$. Then f_ϵ is in \mathcal{B} and has finite rank. Also for each (a_1, \dots, a_n) in $C \subset \ell_n^1 A$,

$$|f_\epsilon((a_1, \dots, a_n)) - (a_1, \dots, a_n)| = \sum_{i=1}^n |f_i(a_i) - a_i| \leq \epsilon.$$

Therefore $\ell_n^1 A$ is in $\bar{\mathcal{B}}$. This example reaffirms the validity of (4.10).

Now let G and $F: \mathcal{B} \rightarrow \mathcal{B}$ be functors such that for each A in \mathcal{B} , GA is isomorphic to FA . Then if F can be considered as a functor from $\bar{\mathcal{B}}$ to $\bar{\mathcal{B}}$, so can G . If X is a finite dimensional space, $\Sigma_X A$ and $\Omega_X A$ are isomorphic to $\ell_n^1 A$ where n is the dimension of X (see (2.15) and [17, p.522]). Therefore Σ_X and Ω_X map $\bar{\mathcal{B}}$ into $\bar{\mathcal{B}}$ if X has finite dimension.

The following lemma is similar to (4.15) and is contained in a result by Grothendieck [5, p.168].

Lemma 4.20 If Y is in \mathcal{B} so that Y^* satisfies the approximation property, then for any Banach space X , for any compact mapping $f: Y \rightarrow X$, there exists for each $\epsilon > 0$ a mapping $f_\epsilon: Y \rightarrow X$ in \mathcal{B} of finite rank so that $|f_\epsilon - f| < \epsilon$.

Proof: Since $f: Y \rightarrow X$ is compact, it is weakly compact. Therefore, $f^{**}(Y^{**}) \subset \eta_X(X) \subset X^{**}$ where $\eta_X: X \rightarrow X^{**}$ is the natural embedding [3, p.482]. Since $f^*: X^* \rightarrow Y^*$ is also

compact, the image of the unit ball of X^* by f^* is contained in a relatively compact subset C of Y^* . By assumption, for each $\epsilon > 0$ there is a mapping $f'_\epsilon: Y^* \rightarrow Y^*$ in \mathcal{B} of finite rank so that $|f'_\epsilon(y^*) - y^*| < \epsilon$ for all y^* in C . Now represent f'_ϵ by the sum $f'_\epsilon(y^*) = \sum f_i(y^*)y_i^*$, $y^* \in Y^*$, $f_i \in Y^{**}$, and where y_i^* are basis elements for the range of f'_ϵ in Y^* . Then $f'_\epsilon \circ f^* = \sum (f_i \circ f^*)(\cdot)y_i^*$, and by the above argument, $f_i \circ f^* = f^{**}(f_i) = x_i \in \eta_X(X) \subset X^{**}$. This implies

$$\begin{aligned} |f^* - \sum x_i(\cdot)y_i^*| &= \sup_{x^* \in X^*, |x^*| \leq 1} |f^*(x^*) - \sum x_i(x^*)y_i^*| \\ &= \sup_{|x^*| \leq 1} |f^*(x^*) - \sum f_i(f^*(x^*))y_i^*| \\ &= \sup_{|x^*| \leq 1} |f^*(x^*) - f'_\epsilon(f^*(x^*))| < \epsilon \end{aligned}$$

since $f^*(x^*) \in C$. Since $f^* - \sum x_i(\cdot)y_i^*$ is the adjoint of $f - \sum y_i^*(\cdot)x_i$ (treating the x_i as elements of X), the inequality

$$|f - \sum y_i^*(\cdot)x_i| < \epsilon$$

is true. Letting $f_\epsilon = \sum y_i^*(\cdot)x_i$, the lemma is proved.

Theorem 4.21 Let $f: Y \rightarrow X$ be any compact map with Y^* satisfying the approximation property. If $F: \mathcal{B} \rightarrow \mathcal{B}$ is a functor, then Ff is compact if and only if F is a finite rank functor.

Lemma 4.22 Let $h: C \rightarrow A$ be a normal epimorphism and $g: B \rightarrow D$ a normal monomorphism in \mathcal{B} . Then

- (1) $f: A \rightarrow B$ is compact if and only if $f \cdot h: C \rightarrow B$ is compact, and

(2) $f:A \longrightarrow B$ is compact if and only if $g \circ f:A \longrightarrow D$ is compact.

Proof: (1) If f is compact, $f \circ h$ is compact. Suppose $f \circ h$ is compact. Since $h:C \longrightarrow A$ is a normal epimorphism, there is a bounded set U in C such that $h(U) = S$, the unit ball in A . By assumption, $f(S) = f(h(U))$ is relatively compact in B .

(2) If f is compact, $g \circ f$ is compact. Now suppose $g \circ f$ is compact. Then $f^* \circ g^*$ is compact with g^* a normal epimorphism. Hence by (1), f^* is compact. Therefore f is compact.

The following proposition gives sufficient conditions for a functor to be compact.

Proposition 4.23 Let $F:B \longrightarrow B$ be a functor.

(1) If F is a finite rank functor and preserves normal epimorphisms, then F is compact.

(2) If F is a finite rank functor and preserves normal monomorphisms, then F is compact.

Proof: (1) Let $f:A \longrightarrow B$ be a compact map. Consider the space $\ell_I^1 = \ell_I^1(S, \mu)$ as in (2.14) where S is the unit ball of A with the discrete topology. Then the map $h:\ell_I^1 \longrightarrow A$ given by $h(x) = \sum_{s \in S} x(s)s$ is a normal epimorphism. The operator $f \circ h:\ell_I^1 \longrightarrow B$ is compact. According to Grothendieck [5, p.185], the space $(\ell_I^1)^*$ satisfies the approximation property. By (4.21), $F(f \circ h) = Ff \circ Fh$ is compact. Since Fh is a normal epimorphism, by (4.22), Ff is compact.

(2) Again, let $f:A \longrightarrow B$ be a compact map. Let S^* be the unit ball of B^* . The dual of $\ell_I^1(S^*, \mu)$ is the space $\ell_I^\infty(S^*, \mu)$, the Banach space of all functions $x:S^* \longrightarrow I$ such that $\|x\| = \sup_{s^* \in S^*} |x(s^*)|$ is finite. According to Grothendieck [5, p.185], $\ell_I^\infty(S^*, \mu)$ satisfies the approximation property. The map $g:B \longrightarrow \ell_I^\infty(S^*, \mu)$ given by $g(b) = x_b:S^* \longrightarrow I$, where $x_b(s^*) = s^*(b)$, is an isometric map. The map $g \circ f$ is compact. By (4.17), $F(g \circ f) = F(g) \circ F(f)$ is a compact map. Since $F(g)$ is an isometric map, $F(f)$ is compact by (4.22).

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