A Study of Some Matrix Elements for the Four-Pion Decay Mode of Neutral Bosons

Weller
A STUDY OF SOME MATRIX ELEMENTS
FOR THE FOUR-PION DECAY MODE OF
NEUTRAL BOSONS

by

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INTRODUCTION

In recent years it has been found that certain elementary particles enter into extremely short-lived associations (with lifetimes of the order of $10^{-23}$ seconds) called resonances. It is of considerable interest to determine the quantum numbers of these resonances. Because of the short-lived nature of these phenomena, it is necessary to analyze them by statistical studies of the various possible final states into which they may decay. Such studies necessarily involve the discussion of the relative probabilities of the various final states, since these probabilities are the numbers which one hopes to predict from quantum theory.

One way to relate predicted final-state probabilities to experiment that has been used extensively in the literature is the so-called "scatter diagram", in which the probability of a given final state is proportional to the density of final states in the neighborhood of the coordinates describing that state in some suitably chosen space of dynamic variables. Scatter diagrams thus have meaning (and indeed become useful) only when there is a semi-continuous range of final states, as will be the case in all that follows.

In all cases discussed here, it is possible to describe the final state uniquely by the momenta and energies of the final-state particles. It follows that the coordinates used in the scatter diagrams will be functions of these momenta and energies, at most. Since this is a large number of coordinates in the cases we will discuss, it is necessary to reduce the problem to a two-dimensional...
representation in order to make it graphically manageable. Some methods for doing this will be discussed later.

In the case of three-particle decays of unstable particles, the method introduced by Dalitz\textsuperscript{1} has been quite common in the literature. The form of the Dalitz diagram of interest to the present discussion is the one used to describe three-particle decays. In this form the coordinates of the final states are the kinetic energies of the final-state particles, and they are measured from the sides of an equilateral triangle. A sketch of the Dalitz diagram appears in Figure 1. A relativistic extension of this method also exists in the literature.\textsuperscript{2}

For four-particle final states, the method of Goldhaber\textsuperscript{3} has been widely used. In this approach, the Lorentz-invariant mass of two of the final-state particles is plotted against that of two of the others. The Lorentz-invariant mass is defined by

\[ M_{i...j} = (E_i +...+ E_j)^2 - (\vec{p}_i +...+ \vec{p}_j)^2 \quad (1) \]

where the \( E_i \) and \( \vec{p}_i \) are the energies and momenta, respectively, of the particles, and where the common convention that the speed of light is unity has been followed. A sketch of a Goldhaber diagram appears in Figure 2.

\textsuperscript{1}Dalitz, R. H., "On the Analysis of \( \tau \)-Meson Data and the Nature of the \( \tau \)-Meson" The Philosophical Magazine, 44 (September 1953), 1068-1080.


Figure 1. The Dalitz diagram. A typical event with particle energies $E_1$, $E_2$ and $E_3$ is plotted. The circle is the non-relativistic boundary of the physical region, which becomes distorted into a roughly triangular shape in the relativistic case.

Figure 2. The Goldhaber diagram. The bold lines described by the indicated equations are the boundaries of the physical region. Superscript zero indicates rest mass. A typical point ($M_{12} = A$, $M_{34} = B$) has been plotted.
The purpose of this paper is to calculate in the Golhaber formalism the final-state probability distributions for certain choices of initial state resonances, and to present the results in the form of Goldhaber diagrams. The discussion will be restricted to four-pion decays of electrically neutral resonances with zero isospin and the scatter diagrams will be plotted on the two-dimensional plane of the two-particle linear invariant masses. The mathematical development will follow the method and notation of Nyborg.¹

**GENERAL CONSIDERATIONS**

Beginning with the familiar "Golden Rule" of quantum mechanics:

\[
\text{Transition Probability} = 2\pi |\langle f | T | i \rangle|^2 \cdot \frac{\text{Number of Final States}}{\text{Unit Energy}} \cdot \frac{1}{\text{Unit Time}}.
\]

Feynman² obtains the following for an N-particle decay.

\[
d^{3N}_{N} = 2\pi |\langle f | T | i \rangle|^2 \sum_{f}^{N} \frac{\delta^{3}(\sum_{N}^{f} M_{N} - \sum_{i}^{f} M_{i}) \delta^{3}(\sum_{N}^{f} P_{N} - \sum_{i}^{f} P_{i})}{(2\pi)^{3}} \cdot (2\pi)^{3} \quad (3)
\]


The Dirac delta functions, denoted by $\delta$, express the fact that transitions that do not conserve momentum and energy cannot occur. \[ \prod_{i=1}^{\infty} x_i \] is the continued product $x_1 \cdot x_2 \cdot x_3 \cdots \cdot x_N \cdot d^{3N}R_N$, a differential of order $3N$, is the rate of transition from the initial state of $M$ particles with energies $E_i$ and momenta $p_i$ into the $N$-particle final state in which the $f^{th}$ particle has momentum lying in the volume element $d^3p_f$ about $p_f$. $|\langle f | T | i \rangle|^2$ is the absolute square of the invariant matrix element connecting the initial and final states and will often be written as $M^2$ for the sake of simplicity. The fact that only $M^2$ appears is tantamount to the assumption that the resonance is produced with no net polarization, an assumption which will be made throughout the present work.

It can be seen from the preceding that in order to reduce the problem to one that can be plotted in two dimensions it is necessary to integrate many times. In order to do this, one must make some assumptions or know something about the dependence of $M^2$ on the energies and momenta. Here it is necessary to face the fact of our ignorance; in no case of interest to this discussion is the relevant matrix element known. The detailed nature of the matrix element depends on the interaction causing the decay of the resonance. All cases of interest here involve the strong interaction, which is not well-understood. Nevertheless it is possible, as will be seen, to use symmetry considerations to restrict the form of the matrix element sufficiently to obtain usable results. This is possible because
the matrix element must have the same interchange symmetry in its
dependence on energies and momenta as do the wave functions from which
it is calculated.

It should be noted, however, that even with the use of symmetry
considerations, the results of this paper as well as those of some
others which will be mentioned later rest squarely upon the assumption
that the invariant transition matrix of a decay can at least be approx­
imated by one of its simplest possible forms.

According to equation (3), the probability that a given final
state occurs in a decay is determined by two factors, one depending
purely on relativistic kinematics, the other on the properties of the
resonance. The first factor referred to above, taken alone, gives
rise to what is generally referred to as the "phase space distribu­
tion" of final states.

If the matrix element of the decay is other than a constant
scalar, the resulting distribution will not in general be the same
as the phase space distribution. If this is the case, one would hope
to determine something about the resonance (for example, its spin or
parity) from the experimental distribution. If the decay proceeds by
the strong interaction (a valid assumption for resonances with life­
times of the order of $10^{-23}$ sec.) then one may safely assume that
angular momentum and parity are conserved, since it is generally
agreed that the strong interaction conserves these quantities. If we
now make an arbitrary spin-parity assignment for the resonance, we
fix the general form of the system wave function both before and after the decay. For example, if we assign spin zero of the resonance, the wave function and hence also the matrix element in its energy momentum dependence must have pseudoscalar or scalar form, depending on the parity. This is so because the initial state, being an eigenstate of angular momentum, has the transformation properties of that eigenstate, and since angular momentum is conserved, the final state must have the same properties. If the resulting distribution is not very sensitive to the exact form assumed for the dependence, but only to the assigned spin and parity, one may expect the results to be useful for comparison with experiment. This indeed is the case in some situations, and this approach will be followed here, in hope that it will also prove useful in this case.

The phase space, or kinematical factor mentioned above is also used in simplified form to search for resonances in the first place. In this application one assumes that some of the final-state particles come from a resonance state. The matrix element governing that decay is treated as a constant, and the differential probability is regarded as a function of the total rest-frame energy of those particles suspected of being from a resonance state. Sufficient integrations are then performed to give a one-dimensional smooth curve distribution. When this is compared with an experimental probability distribution, a resonance shows up as a statistically significant deviation from the smooth curve. Of course the theoretical curve must be normalized to the number of events obtained experimentally.
THREE-PION FINAL STATES AND THE DALITZ DIAGRAM

In order to demonstrate how theory may be compared with experiment, we consider the way it was done historically in the case of the \(\omega\)-meson. As mentioned above, symmetry arguments will be used to specify the form of the wave function and the matrix element. To facilitate this, it will be useful to introduce the concept of isospin, or isotopic spin.

In the early days of nuclear physics it became evident that nucleon-nucleon forces were nearly charge-independent if the coulomb forces were ignored, so that it was useful for some purposes to consider a neutron and a proton as merely two manifestations of the same particle. The mathematical formalism that has been developed to exploit this idea is isomorphic to the treatment of angular momentum in quantum mechanics. Thus, for example, where a fermion having spin 1/2 has \(2\cdot\frac{1}{2}+1=2\) possible spin projections, likewise isospin 1/2 is assigned to the nucleon because it has two charge states. It is then possible to make a geometric interpretation similar to that used in the case of spin. One defines a quantization axis, \(I_z\), and says that the \(Z\) component of the isospin may have \(2I+1\) discrete projections, each corresponding to a different charge state. In elementary particle physics the concept of isospin has proved useful for the description of particles that occur in multiplets having the same quantum numbers (spin, parity, etc.) and identical or nearly identical masses, but with different charges. The fundamental utility of the concept,
and the reason that it is introduced in the present work, is that iso-
spin is conserved to a high degree of accuracy in the strong inter-
action.\(^1\) Because it occurs in three charge states, the pion, a parti-
cle of primary interest to this work, is assigned isospin unity.

In the case of the \(\omega\)-meson, which has the short lifetime charac-
teristic of a strong-interaction decay, only one charge state was
observed in the decay pions, that is, the net charge of the final
state was always the same. It was felt to be reasonable to assume
that it had isospin zero because of isospin conservation. It follows
that the isospin part of the wave function must be a zero-isospin
function built up from three pion isospin functions. Källén\(^2\) shows
that the only isospin function that meets this requirement is

\[
|3\pi, 0\rangle = \frac{1}{\sqrt{6}} \left[ |\pi^+, \pi^0, \pi^-\rangle + |\pi^0, \pi^-, \pi^+\rangle + |\pi^-, \pi^+, \pi^0\rangle \right.
\]

\[
- |\pi^+, \pi^-, \pi^0\rangle - |\pi^0, \pi^+, \pi^-\rangle - |\pi^-, \pi^0, \pi^+\rangle \right] . \quad (4)
\]

It is easily verified that this function is antisymmetric under the
interchange of any pair of pions. Since the final state consists of
pions, which are bosons, it follows that the total wave function
must be symmetric under the interchange of any pair of pions. But

\(^1\)Lichtenberg, D. B., *Meson and Baryon Spectroscopy*, Springer-
Verlag, New York, 1965 p. 15.

\(^2\)Källén, G., *Elementary Particle Physics*, Reading, Massachusetts;
since the total wave function is the product of the isospin part by the space part, it follows that the space part must also be antisymmetric under the interchange of any two pions. Since in its dependence on momenta and energies, the transition matrix element $\langle 3\pi | T | \omega \rangle$ has the same symmetry under this interchange as does the space part of the three-pion wave function, it is antisymmetric. The form of the matrix element is thus considerably restricted.

The result obtained by Källén for the rest-frame transition rate is

$$R_{cm} = \frac{1}{(2\pi)^5} \frac{1}{2m_\omega} \int \int \int \int \int \int \int \int \int \delta \left( \sum_{i=1}^{3} k_i \right) \delta \left( \sum_{i=1}^{3} E_i - m_\omega \right) |\langle 3\pi | T | \omega \rangle|^2. \quad (5)$$

Here $m_\omega$ is the rest mass of the $\omega$-meson and the $E_i$ and $k_i$ are the energies and four-momenta, respectively, of the final-state pions. The $k_i$ in the second delta function are three-momenta and the $\Theta$ function is a step function restricting the momenta to positive values. In integrating over the $k_i$ it is shown that $|\langle 3\pi | T | \omega \rangle|^2$ may be considered constant because by using energy and momentum conservation it may be expressed in terms of $E_1$ and $E_2$. His result may be expressed as

$$\frac{\partial^2 R_{cm}}{\partial E_1 \partial E_2} = (\text{constant}) \ |\langle 3\pi | T | \omega \rangle|^2, \quad (6)$$

where the constant does not depend on $E_1$ or $E_2$. In terms of the triangular diagram, this can be put into the form
\[ d^2R_{\text{cm}} = (\text{constant}) |<3\pi|T|\omega>|^2 dS \quad (7) \]

where \( dS \) is an area element on the Dalitz diagram, so that the differential probability of a final state is proportional to the absolute square of the transition matrix element. It may be noted that the probability density on the Dalitz diagram will be symmetric under the interchange of two pions since it is proportional to the absolute square of the antisymmetric matrix element. This makes the Dalitz diagram, which treats all three pions symmetrically, a logical medium in which to express the results.

One can now make spin-parity assignments to the meson and, under the assumption that the matrix elements of the decay can be approximated by their simplest forms, determine possible shapes of the probability distribution on the Dalitz diagram.

It is not necessary to consider the scalar assignment \((0^+)\) since it can be shown\(^1\) from angular momentum and parity conservation that it is impossible for a scalar particle to decay into three pions.

Since the intrinsic parity of the three pions is the product of the geometric parity by the intrinsic parities of the pions (which are odd, or negative) the final state geometric parity will be opposite the initial state parity. Thus for the initial state

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assignments pseudoscalar \((0^+)\), vector \((1^-)\), and pseudovector \((1^+)\), the required forms of the matrix element are scalar, pseudovector, and vector, respectively. If one now writes down the simplest possible functions of momentum and energy that are completely antisymmetric and have the correct transformation properties, one finds that the resulting Dalitz distributions are qualitatively quite different for the different spin-parity assignments. The distribution corresponding to the vector meson has its maximum at the center and vanishes along the boundary, whereas the other two vanish at the center and have maxima along the boundary. Of the latter two, one has threefold rotation symmetry, the other sixfold. If the experimental distribution resembles one of these distributions to the exclusion of the others, one may then make the appropriate spin-parity assignment with a reasonable degree of confidence. This is in fact how the spin and parity of the \(\omega\)-meson were first determined.\(^1\)

FOUR-PION FINAL STATES

Specializing equation (3) to the zero-momentum reference frame for an initial state of total energy \(E\), having a four-particle final state, one obtains

\[
d^{12}R_{4} = \frac{|<f|\mathbf{r}_{i}|i>|^{2}}{(2\pi)^{8/2}E} \frac{d^{3}p_{1}}{E_{1}} \frac{d^{3}p_{2}}{E_{2}} \frac{d^{3}p_{3}}{E_{3}} \frac{d^{3}p_{4}}{E_{4}} \delta^{3}(\sum_{i=1}^{4} \mathbf{p}_{i}) \delta(E-E_{1}-E_{2}-E_{3}-E_{4}) \quad (8)
\]

This result is, within a multiplicative constant, the starting point of Nyborg, et. al.\footnote{Nyborg, et. al., Ames Laboratory Report IS-1214, op. cit.}

It now becomes necessary to make certain assumptions in order to make the problem tractable. Following Nyborg, et. al., we assume at the outset that $M^2$ depends at most upon the invariant masses, and sketch the development of those authors in reducing the twelfth-order differential equation to fourth-order. In this development we use $C$ to denote the aggregation of purely numerical constants which accumulate in the integrations. These constants have no physical significance, since they would be carried away if the final distribution were normalized, and hence will be ignored.

The presence of the momentum delta function makes the integration over one of the momenta trivial: we choose to integrate over $p_4$, and obtain

$$d^9R_4 = C \frac{d\vec{p}_1}{E_1} \frac{d\vec{p}_2}{E_2} \frac{d\vec{p}_3}{E_3} \frac{d\vec{p}_4}{E_4} \delta(E_1 + E_2 + E_3 + E_4 - E) M^2. \quad (9)$$

Taking the $z$-axis to be the direction of $\vec{p}_{12} = \vec{p}_1 + \vec{p}_2$, one may write

$$d\vec{p}_3 = P_3^2 d(\cos \Theta_{(12)3}) d\phi_{(12)3} dP_3. \quad (10)$$

where $\Theta_{(12)3}$ is the angle between the $z$-axis and $\vec{p}_3$, and $\phi_{(12)3}$ is the azimuthal angle of $\vec{p}_3$. The integration over $\Theta_{(12)3}$ may be
replaced by an integration over $E_4$ by the use of the relations

$$E_4^2 = m_4^2 + p_{12}^2 + 2p_{12}p_3 \cos \Theta_{(12)3}$$

$$E_4 dE_4 = p_{12}p_3 d(cos \Theta_{(12)3}) .$$

The result is

$$d^8R_4 = C \frac{d\Phi_1}{E_1} \frac{d\Phi_2}{E_2} \frac{p_3 dP_3 d\phi_{(12)3}}{E_3} \int_{E_4^-}^{E_4^+} dE_4 \delta(E-E_1-E_2-E_3-E_4) M^2$$

(12)

where

$$E_4^\pm = [m_4^2 + (p_{12} \pm p_3)^2]^{1/2} ;$$

because of the energy delta function, no contribution is obtained in the last integration unless the condition

$$E_4^- < E-E_1-E_2-E_3 < E_4^+$$

(13)

is satisfied. If this condition is satisfied, the result is

$$d^8R_4 = C \frac{d\Phi_1}{E_1} \frac{d\Phi_2}{E_2} \frac{dE_3}{P_{12}} d\phi_{(12)3} M^2 .$$

(14)

We have the relation

$$d\Phi_1 d\Phi_2 = p_1^2 dP_1 d(cos \Theta_1) p_2^2 dP_2 d(cos \Theta_2) d\phi_1 d\phi_2 ,$$

which may be substituted into equation (14). Doing so, and integrating over $\phi_1$ and $\phi_2$ leads to
\[ d^6R_4 = C \frac{p_1^2 dP_1}{E_1} \frac{p_2^2 dP_2}{E_2} \frac{d(\cos\theta_1)}{p_{12}} \frac{d(\cos\theta_2)}{p_{12}} \frac{dE_3}{p_{12}} d\phi_{(12)3} M^2. \] (15)

Since the physical situation is described uniquely by the relative orientation of \( \vec{\mathbf{p}}_1 \) and \( \vec{\mathbf{p}}_2 \), and is independent of the orientation of \( \vec{\mathbf{p}}_{12} \), we may assign \( \vec{\mathbf{p}}_1 \) or \( \vec{\mathbf{p}}_2 \) to be the z axis and integrate over all orientations of the other, which is then associated with the angle \( \theta_{12} \). Thus we have

\[ d^5R_4 = C \frac{p_1^2 dP_1}{E_1} \frac{p_2^2 dP_2}{E_2} \frac{d(\cos\theta_{12})}{p_{12}} dE_3 d\phi_{(12)3} M^2. \] (16)

In the two previous integrations, numerical factors that arise have been absorbed into the normalization constant, \( C \). For fixed \( P_1 \) and \( P_2 \) (magnitudes) the variable \( \theta_{12} \) may be replaced by \( P_{12} \) according to

\[ P_{12} \ dP_{12} = P_1 P_2 \ d(\cos\theta_{12}) \]

with the result

\[ d^5R_4 = C \ dE_1 \ dE_2 \ dE_3 \ dP_{12} \ d\phi_{(12)3} M^2. \] (17)

It is possible at this point, by a suitable change of variables to obtain a distribution in invariant masses. In this work however, it will be convenient to integrate first over \( \phi_{(12)3} \). One is left with

\[ d^4R_4 = C \ dE_1 \ dE_2 \ dE_3 \ dP_{12} \int_0^{2\pi} M^2 d\phi_{(12)3}. \] (18)
We now make the transformation to the invariant mass variables, using equation (1):

\[
\begin{align*}
M_{12}^2 &= (E_1 + E_2)^2 - P_{12}^2 \\
M_{34}^2 &= (E - E_1 - E_2)^2 - P_{12}^2 \\
M_{134}^2 &= E^2 - 2E E_2 + M_2^2 \\
M_{124}^2 &= E^2 - 2E E_3 + M_3^2 .
\end{align*}
\] (19)

The Jacobian of the transformation is

\[
\frac{\delta(E_1, E_2, E_3, P_{12})}{\delta(M_{12}^2, M_{34}^2, M_{124}^2, M_{134}^2)} = \frac{1}{16 E^3 P_{12}} , \quad (20)
\]

where the functional form of \( P_{12} \) in terms of the invariant masses is given by

\[
P_{12} = \frac{1}{2E} \left[ M_{12}^2 - M_{34}^2 \right]^2 - 2E^2 \left( M_{12}^2 + M_{34}^2 \right) + E^4 \right]^{1/2} . \quad (21)
\]

Thus equation (18) assumes the form

\[
d^4R_4 = C \frac{1}{P_{12}^3 E^3} \int \int \int \int \frac{2\pi}{0} M^2 d\phi_{(12)3} . \quad (22)
\]

in which the integral is considered a function of the invariant masses and is done first.
SELECTION OF THE MATRIX ELEMENTS

Symmetry Considerations

The development has now been carried as far as possible without specifying the form of the matrix element $M$. To clarify the reasons for the choices of the forms to be studied in the present work it is appropriate to mention certain facts regarding the original motivation for this work.

In 1965, Kernan, Lyon and Crawley\(^1\) reported evidence for a neutral meson of rest mass 1610 MeV which has the prominent decay mode $\pi^+\pi^+\pi^-\pi^-$, among other possible modes. This discovery suggested a theoretical investigation of zero-isospin, four-pion resonances. Since the Kernan article other investigators\(^2\) have seen evidence of the existence of this resonance, but it is now thought to exist in three charge states and so to have isospin one.\(^3\) Hence this particular resonance is not in the class being investigated in this work. Nonetheless, new resonances are continually being discovered, and the method here developed is likely to be of eventual use. It is in any case a theoretical problem of some inherent interest.

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\(^2\)For example, Goldberg, et. al., *Physics Letters* 17 (1965) 354.

In a preliminary attempt to determine the spin-parity assignment of this type of manifestation, Nyborg, Dotson and Good have made an angular correlation study of four-pion resonances. Their results show that while certain useful information may be gained from such studies, it is not possible by this method to distinguish definitely between a scalar \((0^+)\) and a vector \((1^-)\) meson. When this became clear, a search was made for alternate methods of distinguishing between such spin-parity assignments. It was thought likely that the Goldhaber distributions for the two spin-parities might be qualitatively different, as were for example the Dalitz distributions for the \(\omega\)-meson.

Nelson has worked out the Goldhaber distributions for the simplest possible matrix element of several spin-parity choices. He follows a different formalism than will be used here, and his results are plotted using a slightly different format.

The present work will consider only the vector and scalar resonances, but will attempt to discover how sensitive the distribution is to the functional form assumed for the matrix element by considering several forms of increasing complexity for each choice of spin-

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parity. The result will be complementary to the Nelson paper, which investigates a different aspect of the same general question.

As in the case of the $\omega$-meson, symmetry considerations will be used to help specify the form of the matrix element. The chief difference is that the number of pions is now even, so that for example a scalar resonance is now represented by a scalar function, instead of a pseudoscalar function, as was the case for the $\omega$-meson.

The development of the matrix elements which follows will depend, as in the case of the $\omega$-meson, on the use of Bose statistics and isospin conservation. The detailed arguments and notation are largely those of Nyborg, Dotson and Good\textsuperscript{1}.

Bose statistics dictate that the wave function shall be fully symmetric under any permutation of pion labels. Since we have restricted the problem to resonances of zero isospin, we face the problem of constructing zero-isospin eigenfunctions from single-pion eigenfunctions having isospin unity. Since the treatment of isospin is precisely isomorphic to the treatment of angular momentum in this respect, one may write down such wave functions according to the usual rules for the composition of angular momentum, using a table of Clebsch-Gordan coefficients.

\textsuperscript{1}Nyborg, et. al., "A Spin-Parity Test for Four-Pion Resonances", op. cit.
First combining pairs of pions having zero isospin into four-pion eigenfunctions of isospin zero, one finds

\[ x_{00} = \frac{1}{3} \left[ |+++-\rangle + |---+\rangle + |++++\rangle + |-++-\rangle + |+-+-\rangle + |oo+\rangle + |oo-\rangle + |oooo\rangle - |oo--\rangle \right], \quad (23) \]

where \(|+++-\rangle\), for example, denotes a four-pion isospace eigenfunction with the first pion positively charged, the second negatively charged, etc. Next combining pion pairs of isospin unity, one obtains

\[ x_{11} = \frac{1}{2\sqrt{3}} \left[ |+++-\rangle + |---+\rangle - |+++-\rangle - |+-+-\rangle + |+oo-\rangle + |o+o-\rangle + |o-+o\rangle + |-o+o\rangle - |+o-o-\rangle - |o+o-o\rangle - |o-o+\rangle - |-o-o+\rangle \right]. \quad (24) \]

Finally, one may combine pairs of pions of isospin two to obtain

\[ x_{22} = \frac{1}{6\sqrt{5}} \left[ 6|+++-\rangle + 6|---+\rangle + |+++-\rangle + |+-+-\rangle + |+-oo\rangle - 3|o+o-\rangle - 3|0+o-\rangle - 3|o+0-o\rangle - 3|o-o+\rangle - 3|0-o+\rangle - 3|0-o-o\rangle + 2|+oo-\rangle + 2|oo--\rangle + 2|oo+-\rangle + 2|+-oo\rangle + 2|oo-+\rangle + 4|oooo\rangle \right]. \quad (25) \]
It will be convenient to recombine these three functions into three different functions having definite permutation symmetries. To facilitate this, we introduce the Young tableau. This device is of such generality and wide application that a fully detailed treatment cannot be attempted here. Instead, the following conventions will suffice. \[ \begin{array}{c}
1 & 2 & 3 & 4 
\end{array} \]
will denote a function having full permutation symmetry on the particle labels. \[ \begin{array}{c}
1 & 2 
3 & 4 
\end{array} \]
will denote the symmetry obtained by the operation

\[ f' = (1 - p_{13})(1 - p_{24})(1 + p_{12})(1 + p_{34}) f(1, 2, 3, 4), \]

(26)

where \( p_{ij} \) is the permutation operator that interchanges the labels \( i \) and \( j \). Similarly \( f'' = p_{23} f' \) has the permutation symmetry \[ \begin{array}{c}
1 & 3 
2 & 4 
\end{array} \].

One can verify that any permutation applied to either \( f' \) or \( f'' \) leads to a linear combination of \( f' \) and \( f'' \), as for example

\[ p_{13} f' = -f' , \quad p_{34} f' = f' - f'' , \quad \text{etc.} \]

(27)

By assuming a linear combination of the form

\[ x_j = a_j x_{00} + b_j x_{11} + c_j x_{22} \]

and requiring in turn that the interchange properties of \[ \begin{array}{c}
1 & 2 & 3 & 4 
\end{array} \], \[ \begin{array}{c}
1 & 2 
3 & 4 
\end{array} \], and \[ \begin{array}{c}
1 & 3 
2 & 4 
\end{array} \] be obeyed for \( x_s \), \( x_1 \) and \( x_2 \), respectively, one finds without difficulty that
\[ x_s = \frac{1}{2} \sqrt{5} \chi_{00} + \chi_{22} \]
\[ x_1 = -\frac{2}{\sqrt{5}} \chi_{00} - \frac{\sqrt{3}}{\sqrt{5}} \chi_{11} + \chi_{22} \]
\[ x_2 = -2 \frac{\sqrt{3}}{\sqrt{5}} \chi_{11}. \]  

(28)

In each case, the coefficients may be multiplied by an arbitrary common factor which has here been chosen for convenience. The fact that the three new functions are linearly independent implies that no generality has been lost in the change to this set of basis functions.

We now seek to form the complete wave function by considering products of the isospace functions with space functions having the appropriate permutation symmetry in their momentum and energy dependence. It follows from the work of Pais\(^1\) that the most general form for such a wave function is

\[ \psi = A \chi_s \psi_s + B(a \psi_1 \chi_1 + b \psi_2 \chi_2 + c \psi_2 \chi_2 + d \psi_2 \chi_1), \]  

(29)

where \( \psi_s, \psi_1 \) and \( \psi_2 \) have the same permutation symmetry as \( x_s, x_1 \) and \( x_2 \), respectively, and the coefficients are chosen so that \( \psi \) has the full permutation symmetry \([1 \ 2 \ 3 \ 4]\). This is a wave function of considerable complexity, and in order to make the problem tractable, we must consider the full symmetry and partial

symmetry contributions separately; that is, we consider wave functions of the two types:

\[ \psi_A = X_S \psi_S \]  
\[ \psi_B = a x_1 \psi_1 + b x_2 \psi_2 + c x_1 \psi_2 + d x_2 \psi_1 \]  

(30)  
(31)

That we make this further simplification is as much an act of faith as anything else, since its validity can only be tested by experiment. It is not entirely without justification, however, since as Pais\(^1\) has pointed out, the two symmetry types may correspond to physically different couplings. Such a situation would be analogous to that which obtains in atomic physics, where in most atoms of interest either 1-s or j-j coupling is found to dominate.

Type A Matrix Elements

We now consider matrix elements of the form

\[ M = \langle R | T | \psi_A \rangle \]  
\[ = \langle R | T | x_S \psi_S \rangle \]  

(32)

where \( \langle R | \) denotes the resonant state. Since the energy and momentum dependence of the matrix element is contained entirely in \( \psi_S \), which has known permutation symmetry properties, we may lump everything that does not depend on energy and momentum into a constant and write

\(^1\)ibid.
\[ M = (\text{constant}) \cdot f_s(E_i, p_i) \]

where \( f_s \) is a function of pion energies and momenta having the
same permutation symmetry as \( \psi_s \).

There remains then only the task of choosing some simple forms
for \( f_s \). As a first guess one might try \( M = E_1 + E_2 + E_3 + E_4 \). This is
easily seen to be trivial, since this quantity is a constant in the
transition, and thus yields only the phase space of the interaction.
The next guess might be \( M = E_1 E_2 E_3 E_4 \). This form is non-trivial,
but the calculation of the Goldhaber distribution may be simplified
further by using the form \( M = \sqrt{E_1 E_2 E_3 E_4} \) so that \( M^2 = E_1 E_2 E_3 E_4 \).
Another relatively simple form is \( M = E_1^2 + E_2^2 + E_3^2 + E_4^2 \). The latter
two matrix elements have been integrated analytically. An analogue
of the latter, \( M^2 = E_1^2 + E_2^2 + E_3^2 + E_4^2 \), has been integrated numerically.
The results are collected in the summary in Section VII.

As a simple vector (1^-) form, one might try \( M = p_1^+ p_2^+ p_3^+ p_4^+ \).
Again, this form is trivial, since it obviously vanishes in the
zero-momentum (center-of-mass) reference frame used here. A similar
but non-trivial form is \( M = E_1 p_1^+ p_2^+ E_3 p_3^+ E_4 p_4^+ \). This form, and
the next logical step, \( M = E_1^2 p_1^+ p_2^+ E_3^2 p_3^+ E_4^2 p_4^+ \), have both been
integrated analytically.

Type B Matrix Elements

We now turn to the combinations of partial symmetries

\[ M = \langle R | T | \psi_B \rangle . \]
We seek first to further simplify the form of the wave function as much as possible. Beginning with equation (31), we may make an arbitrary renormalization without loss of generality, and obtain

$$\psi_B = x_1 \psi_1 + bx_2 \psi_2 + cx_1 \psi_2 + dx_2 \psi_1 .$$  \hspace{1cm} (35)

Requiring that $\psi_B$ have the full symmetry $\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$, one may show that

$$\psi_B = x_1 \psi_1 - 1/2(x_1 \psi_2 + x_2 \psi_1) + x_2 \psi_2 .$$  \hspace{1cm} (36)

At this point, one is guided by the facts of experimental physics in attempting to further simplify the matrix element. As is well known, it is quite difficult to pin down the neutral pion experimentally, since it does not leave a track in a bubble chamber or other ionization detector, and may be detected only indirectly when it happens to decay within the fiducial volume of the detector. If one is measuring pion events, it is thus desirable, if not necessary, to confine the measurements to events which do not include neutral pions. Fortunately, this same restriction brings about considerable simplification of the problem at hand. If one decides to measure only those events which do not contain neutral pions, then the act of measurement selects from the wave function and the matrix element only those components which correspond to eigenfunctions having no neutral pions, so that in calculations intended for such experiments, one need only consider these eigenfunctions. Henceforth, this paper will assume that the experimental
analysis will proceed in such a fashion. A further effect of the experimental method must be considered, namely the choice of labelling systems. Let the labelling system in which pion number 1 is electrically positive, number 2 negative, number 3 positive, and number 4 negative be denoted by \( ++-- \). If such a system is employed, one can see by inspection of equations (28) that the surviving components of \( \chi_1 \) and \( \chi_2 \) may be written as

\[
\chi_1 = \left[ - \frac{2}{\sqrt{5}} \frac{1}{3} - \frac{i}{\sqrt{5}} \frac{1}{2} (-1) + (1) \left( -\frac{1}{\sqrt{5}} \right) \right] |++--\rangle = 0
\]

\[
\chi_2 = -2 \sqrt{\frac{3}{5}} \frac{1}{2} \left( -\frac{1}{\sqrt{5}} \right) |++--\rangle = \frac{1}{\sqrt{5}} |++--\rangle
\]

so that for this charge assignment,

\[
\psi_B = \frac{1}{\sqrt{5}} |++--\rangle \left( -\frac{1}{2} \psi_1 + \psi_2 \right). \quad (38)
\]

Further inspection of \( \chi_1 \) and \( \chi_2 \) reveals that since the states \( |++--\rangle \) and \( |--++\rangle \) appear symmetrically, one obtains the identical result for \( \psi_B \) with the charge assignment corresponding to the latter. In similar fashion one may obtain the following for the charge assignments \( +--+ \) and \( --+- \):

\[
\psi_B = -\frac{1}{2\sqrt{5}} (\psi_1 + \psi_2) |+--+\rangle, \quad (39)
\]

and for the assignments \( +++- \) and \( ---- \),

\[
\psi_B = \frac{1}{\sqrt{5}} (\psi_1 - \frac{1}{2}\psi_2) |+++\rangle. \quad (40)
\]
We may summarize the results to this point and incorporate them in matrix elements as follows.

\[
\begin{align*}
M_1 &= \langle R|T| \frac{1}{\sqrt{5}}(-\frac{1}{2}\psi_1 + \psi_2) |++--\rangle \\
&= (\text{const.})(2F_2 - F_1) \\
M_{II} &= \langle R|T| - \frac{1}{2\sqrt{5}}(\psi_1 + \psi_2) |+--+\rangle \\
&= (\text{const.})(F_1 + F_2) \\
M_{III} &= \langle R|T| \frac{1}{\sqrt{5}}(\psi_1 - \frac{1}{2}\psi_2) |+++-\rangle \\
&= (\text{const.})(2F_1 - F_2)
\end{align*}
\]

As before, the isospin dependence has been lumped into a constant independent of momenta and energies, and \(F_1\) and \(F_2\) are defined to be functions of the pion momenta and energies having the same permutation symmetries as \(\psi_1\) and \(\psi_2\), respectively.

One might now ask whether all three of these matrix elements are distinct, or whether one or more are redundant. The answer to this is that all three have equal significance until one chooses the labelling convention for the variables on the Goldhaber plots. At that point, two of the matrix elements become degenerate, so that one may be ignored. In Section IV, we have already made the choice of \(M_{12}\) and \(M_{34}\) as the Goldhaber variables. For this choice, \(M_I\) and \(M_{II}\) yield identical results, hence \(M_{II}\) will be ignored hereafter. Had
we chosen the Goldhaber variables \( M_{14} \) and \( M_{23} \), \( M_1 \) and \( M_{III} \) would be degenerate. For the remaining choice, \( M_{13} \) and \( M_{24} \), the results of \( M_{II} \) and \( M_{III} \) would be degenerate, so that the overall picture is symmetric, as it ought to be. Exactly how these results come about is shown schematically in Table 1.

The remaining two matrix elements may be shuffled slightly for convenience. We define \( F_3 = F_1 - F_2 \) and note from one of equations (27) that it must have the permutation symmetry \( [123] \). Then one finds easily that

\[
M_1 = F_2 - F_3 \\
M_{III} = F_1 + F_3
\]

(42)

Since the charge assignments now specify these matrix elements unambiguously, we drop the subscripts hereafter. For each of the two charge assignments, five simple algebraic forms of the matrix element were chosen for investigation. In each case, two of the three are vectors and three are scalars. The results for all ten examples are summarized in section VII, and they will not be listed separately here.

SOME INTEGRATIONS

Of the sixteen matrix elements considered in the present work, eleven have been integrated analytically and five have been integrated numerically. Of the eleven analytic integrations, two are the original work of the author and have been checked by others\(^1\), four are

\(^1\)A. C. Dotson, private communication.
Table 1. All labelling schemes and charge assignments for a hypothetical four-pion decay are exhibited, with the data points (AB has been written for $M_{AB}$, etc.) resulting for three choices of Goldhaber variables. The table has been rewritten twice, first eliminating the redundancies due to $M_{AB} = M_{BA}$, second those from points mirror-symmetric about the diagonal of the Goldhaber diagram.
the original work of others\textsuperscript{1} and have been checked by the author and five are entirely the work of others\textsuperscript{2}. The author is solely responsible for the numerical integrations, the numerical evaluation of the analytic results, and all graphical work.

We will not discuss the analytic integrations in detail except in the case of those matrix elements which are the original work of the author. Even in this case, the necessary algebra is so tedious that it will only be outlined in sufficient detail that it can be reproduced by the reader, if he so desires.

The numerical integration technique will be treated only insofar as it is necessary to explain the appropriate computer program (subroutine \textsc{GoldenUM}) in Appendix A. The algorithm is perfectly straightforward, and convergence problems were not encountered. As a check on the numerical technique, all analytic results were reproduced numerically to at least two-digit accuracy at each mesh point on the Goldhaber plot. A level of precision in the numerical integration was established at which the correlation coefficient (to be defined in Section VII) taken between the analytic and numerical evaluations of each matrix element was unity to five-digit accuracy or better. This level was doubled for all distributions which were only evaluated numerically.

The two matrix elements for which the analytic integrations

\textsuperscript{1}\textit{ibid.}

\textsuperscript{2}\textit{ibid.}
will be outlined are

$$M = (\vec{p}_1 - \vec{p}_2) \cdot (\vec{p}_3 - \vec{p}_4) - (\vec{p}_1 - \vec{p}_4) \cdot (\vec{p}_2 - \vec{p}_3),$$

(43)

which corresponds to the charge assignment $+-+-$ and has the permutation symmetry of $F_2 - F_3$, and

$$\vec{M} = (E_2 - E_1)(\vec{p}_3 - \vec{p}_4) + (E_4 - E_3)(\vec{p}_1 - \vec{p}_2) + (E_1 - E_4)(\vec{p}_2 - \vec{p}_3) + (E_2 - E_3)(\vec{p}_1 - \vec{p}_4),$$

(44)

which is of the same charge assignment and symmetry, but which is evidently a vector.

By straightforward vector algebra and the previous definition of $P_{12}$, one may reduce the first of these to

$$M = \vec{p}_{12}^2 + \vec{p}_2^2 + \vec{p}_3^2 - 4\vec{p}_{12} \cdot \vec{p}_2 + 4\vec{p}_{12} \cdot \vec{p}_3 - 6\vec{p}_2 \cdot \vec{p}_3,$$

(45)

and the second, with the relation $E = E_1 + E_2 + E_3 + E_4$, becomes

$$\vec{M} = (E_2 - E_1)(\vec{p}_{12} + 2\vec{p}_3) + (E - E_1 - E_2 - 2E_3)(\vec{p}_{12} - 2\vec{p}_2) + (2E_1 + E_2 + E_3 - E)(\vec{p}_2 - \vec{p}_3) + (E_2 - E_3)(2\vec{p}_{12} - \vec{p}_2 + \vec{p}_3).$$

(46)

At this point it is convenient to introduce a more compact notation and to state some purely algebraic results which will be used in the further consideration of these two matrix elements. We make the following definitions:

$$x = M_{12}^2, \quad y = M_{34}^2, \quad q = M_{124}^2, \quad s = M_{134}^2.$$  

(47)

Using these definitions and the invariant mass relations (19), one can obtain by straightforward algebra the inverse relations;
\[ E_1 = \frac{1}{2E} (s + x - y - m^2), \]
\[ E_2 = \frac{1}{2E} (E^2 + m^2 - s), \]
\[ E_3 = \frac{1}{2E} (E^2 + m^2 - q), \quad \text{and} \]
\[ E_4 = \frac{1}{2E} (-x + y - m^2 + q), \quad (48) \]

where the assumption has been made that the four pions all have mass \( m \).

The general program to be followed is to integrate equation (22) over \( M_{124}^2 \) and \( M_{134}^2 \), and it is appropriate at this point to discuss the limits of the integration. To this end we introduce the notation,

\[ A = \frac{1}{2} (x - y + E^2 + 2m^2) \]
\[ B^2 = E^2 p^2_{12} (1 - \frac{4m^2}{y}) \]
\[ A' = \frac{1}{2} (y - x + E^2 + 2m^2) \]
\[ B'^2 = E^2 p^2_{12} (1 - \frac{4m^2}{x}). \quad (49) \]

The limits on both \( q \) and \( s \) arise essentially from the condition (13) and have been worked out by Nyborg\(^1\). In the notation just described they can be written

\[ q^\pm = A \pm B \quad \text{and} \quad s^\pm = A' \pm B' \quad (50) \]

where + and - refer to the upper and lower limits, respectively.

\(^1\)Nyborg et. al., Ames Laboratory Report IS-1214, op. cit.
Making the definitions
\[ u = (E^2 + m^2)(E^2 + x - y) - 2E^2x \quad \text{and} \]
\[ r = (E^2 + m^2)(E^2 - x) - (E^2 - m^2)y \]
we are able to adapt several more useful results of Nyborg\(^1\).

\[ \hat{p}_{12} \cdot \hat{p}_2 = \frac{1}{4E^2} [u - s(E^2 + x - y)] \]
\[ \hat{p}_{12} \cdot \hat{p}_3 = \frac{1}{4E^2} [(x - y - E^2)(A - q) - 2E^2p_{12}^2] \]
\[ 2\hat{p}_2 \cdot \hat{p}_3 = -\frac{1}{8E^4p_{12}^2} [s(y - x - E^2) + u][q(x - y - E^2) + r] \]
\[ \frac{\pm \cos \phi}{2p_{12}^2} \left\{ \frac{p_{12}^2}{E^2} [(E^2 - m^2)^2 - 2s(E^2 + m^2) + s^2] \right\} \]
\[ - \frac{1}{4E^4} [s(x - y - E^2) + u]^2]^{1/2} \left\{ \frac{p_{12}^2}{E^2} [(E^2 - m^2)^2 - 2q(E^2 + m^2) + q^2] \right\} \]
\[ + q^2 - \frac{1}{4E^2} [q(x - y - E^2) + r]^2]^{1/2} \]

The sign ambiguity in \( \hat{p}_2 \cdot \hat{p}_3 \) is trigonometric in origin and will not affect our results, as will presently be shown.

The following may be deduced directly from the invariant mass relations (19).

\[ \hat{p}_2^2 = \frac{1}{4E^2} [(E^2 - m^2)^2 - 2s(E^2 + m^2) + s^2] \]
\[ \hat{p}_3^2 = \frac{1}{4E^2} [(E^2 - m^2)^2 - 2q(E^2 + m^2) + q^2] \]

\(^1\)Ibid.
In the present notation \( P_{12}^2 \) assumes the form

\[
P_{12}^2 = \frac{1}{4E^2} \left[(x - y)^2 - 2E^2(x + y) + E^4\right],
\]

which is independent of \( q \) and \( s \) and is therefore constant for the purpose of the integrations.

It is convenient at this point to postpone further consideration of the second (vector) matrix element and concentrate on the first.

Squaring equation (45), we obtain

\[
M^2 = P_{12}^4 + P_2^4 + P_3^4 + 16(\vec{p}_{12} \cdot \vec{p}_3)^2 + 16(\vec{p}_{12} \cdot \vec{p}_2)^2 + 36(\vec{p}_2 \cdot \vec{p}_3)^2 \\
- 12(\vec{p}_{12} \cdot \vec{p}_3)(P_{12}^2 + P_2^2 + P_3^2 - 4\vec{p}_{12} \cdot \vec{p}_2 + 4\vec{p}_{12} \cdot \vec{p}_3) \\
+ 8(\vec{p}_{12} \cdot \vec{p}_3)(p_{12}^2 + P_2^2 + P_3^2 - 4\vec{p}_{12} \cdot \vec{p}_2) \\
- 8(\vec{p}_{12} \cdot \vec{p}_2)(p_{12}^2 + P_2^2 + P_3^2) + 2P_{12}^2(P_{12}^2 + P_2^2 + P_3^2) + 2P_{12}^2P_2^2.
\]

(57)

Because of the \( \phi(12)_3 \) dependence in \( \vec{p}_{12} \cdot \vec{p}_3 \), we must first consider the integration over \( \phi(12)_3 \) implied in equation (22). If we write \( \vec{p}_{12} \cdot \vec{p}_3 \) in accordance with equation (54) in the form

\[
\vec{p}_{12} \cdot \vec{p}_3 = a \pm b \cos \phi(12)_3,
\]

then the appropriate integrations are as follows:

\[
\int_0^{2\pi} \vec{p}_{12} \cdot \vec{p}_3 \, d\phi(12)_3 = 2\pi a \pm b(0) = 2\pi a ; \\
\int_0^{2\pi} d\phi(12)_3 = 2\pi ; \\
\int_0^{2\pi} (\vec{p}_{12} \cdot \vec{p}_3)^2 \, d\phi(12)_3 = 2\pi(a^2 + 1/2 b^2).
\]

One can see that the sign ambiguity is of no consequence since it
involves only the terms which are linear in \(\cos \phi(12)_3\), which vanish under integration. The factor of \(2\pi\), which appears in all integrations, will be absorbed in the normalization constant.

From the preceding, we are able to write the effective contribution of the term \(36(P_2\cdot P_3)^2\) to the integral as

\[
a^2 + \frac{1}{2} b^2 = \frac{9}{64E^4 p_{12}^4} [s(y - x - E^2) + u]^2 [q(x - y - E^2) + r]^2
\]

\[
+ \frac{9}{8p_{12}^4} \left\{ \frac{p_{12}^2}{E^2} \left[ (E^2 - m^2)^2 - 2s(E^2 + m^2) + s^2 \right] - \frac{1}{4} \left[ s(y - x - E^2) + u \right]^2 \right\} \left\{ \frac{p_{12}^2}{E^2} \left[ (E^2 - m^2)^2 - 2q(E^2 + m^2) + q^2 \right] - \frac{1}{4} \left[ q(x - y - E^2) + r \right]^2 \right\}
\]

(59)

A corresponding expression may be easily written down for the term that involves \(P_2\cdot P_3\) linearly.

One may use the expressions of equations (52) and (53) for the remaining terms and obtain the following result:

\[
16E^4M^2 = 16E^4 p_{12}^4 + [ (E^2 - m^2)^2 - 2s(E^2 + m^2) + s^2 ]^2
\]

\[
+ [(E^2 - m^2)^2 - 2q(E^2 + m^2) + q^2]^2 + 16[u - s(E^2 + x - y)]^2
\]

\[
+ 16[(x - y - E^2)(A - q) - 2E^2 p_{12}^2]^2 + \frac{9}{4E^4 p_{12}^4} [s(y - x - E^2) + u]^2
\]

\[
x [q(x - y - E^2) + r]^2 + \frac{18E^4}{p_{12}^4} \left\{ \frac{p_{12}^2}{E^2} \left[ (E^2 - m^2)^2 - 2s(E^2 + m^2) 
\right.\right.
\]

\[
+ s^2 \left.\left. - \frac{1}{4} \left[ s(y - x - E^2) + u \right]^2 \right\} x
\]
This expression may now be integrated by perfectly straightforward techniques to obtain the Goldhaber distribution in x and y. The algebra involved in collecting terms, etc. is tedious and not very edifying and thus it will be omitted. For the aid of any who would attempt to reproduce the result, a table of integrals and useful identities has been included in Appendix C. The result of the integration may be written

\[ d^2R_4 = \frac{C}{p_{12}E^3} \, dx \, dy \int \int M^2 \, dq \, ds \]
or,

\[\begin{align*}
\frac{d^2 R_4}{dx dy} &= \frac{\partial x \partial y \Delta q \Delta s}{p_{12}^2 E^3} \left\{ [4E^2 p_{12}^2 - E^2(x + y) + 8E^2 m^2]^2 \\
- \frac{2}{3} B^2 f^2 [(E^2 + x - y)^2 + 8E^2 m^2 - 5E^2 x - E^2 y] \\
- \frac{2}{3} B^2 [(E^2 + y - x)^2 + 8E^2 m^2 - 5E^2 y - E^2 x] \\
+ \frac{1}{5} B^4 + \frac{1}{5} B^4 + \frac{2}{9} B^2 B^4 + 4B^2 B^4 (1 + \frac{x}{p_{12}^2} + \frac{y}{p_{12}^2} + \frac{3xy}{p_{12}^2}) \right\}.
\end{align*}\]

where

\[\Delta q \Delta s = (q^+ - q^-)(s^+ - s^-)\]

\[= \frac{4E^2 p_{12}^2}{xy} \left[(x^2 - 4m^2 x)(y^2 - 4m^2 y)\right]^{1/2}.\]

Turning to the matrix element of equation (44), we find it convenient to substitute for the various energies the expressions (48), to obtain after some cancellation,

\[M = \frac{1}{E} \left\{ P_{12}^- [ -2s + 2q - (x - y)] + P_2^+ [2(x - y) + (2m^2 + E^2) + s - 3q] + P_3^- [ - 2(x - y) + (2m^2 + E^2) + q - 3s] \right\}.
\]

By squaring this result and applying the expressions (52),(53) and (58) one may obtain after considerable shuffling of terms,

\[4E^4 M^2 = 4E^2 p_{12}^2 [(q - s)^2 + (A - q)^2 + (A^1 - s)^2 - 2q(A - q) \]

\[+ 2s(A - q) + 2q(A^1 - s) - 2s(A^1 - s) - 2(A - q)(A^1 - s)] \]

\[+ [(E^2 - m^2)^2 - 2s(E^2 + m^2) + s^2][9(A - q)^2 + (A^1 - s)^2] \]
\[-6(A - q)(A' - s)] + [(E^2 - m^2)^2 - 2q(E^2 + m^2) + q^2][9(A' - s)^2
\]
\[+ (A - q)^2 - 6(A - q)(A' - s)] + 2[u - s(E^2 + x - y)][3(A' - q)
\]
\[X(q - s) - (A' - s)(q - s) - 3(A - q)^2 + 4(A - q)(A' - s)
\]
\[-(A' - s)^2] + 2[(A - q)(x - y - E^2) - 2E^2p^2_{12}][3(A' - s)(q - s)
\]
\[-(A - q)(q - s) - 4(A' - s)(A - q) + (A - q)^2 + 3(A' - s)^2]
\[-\frac{1}{2E^2p^2_{12}} [s(y - x - E^2) + u][q(x - y - E^2) + r][10(A - q)(A' - s)
\]
\[-3(A - q)^2 - 3(A' - s)^2]. \tag{63}

In this expression, the only $\phi_{(12)3}$ dependence is linear in $\cos\phi_{(12)3}$
so that the coefficients of such terms may be dropped, in accordance
with equation (58). The following result may be obtained using
standard integration techniques:

\[d^2R_4 = C \frac{dx \space dy}{P_{12}E^3} \int \int \hat{\sigma}^2 dq \space ds
\]
\[= C \frac{dx \space dy}{P_{12}E^3} \frac{dq \space ds}{4E^4} \left\{ 4E^2p^2_{12}(x - y)^2
\]
\[+ \frac{1}{3} B^2[ - \frac{3}{2} E^4 + 2E^2(x - y) - \frac{7}{2}(x - y)^2 + 16E^2x - 40E^2m^2]
\]
\[+ \frac{1}{3} B^2[ - \frac{3}{2} E^4 + 2E^2(x - y) - \frac{7}{2}(x - y)^2 + 16E^2y - 40E^2m^2]
\]+ 2B^2B + \frac{1}{5} [B^4 + B^4] - \frac{5}{E^2p^2_{12}} (1B^2B^2)(E^2 + x - y)(E^2+y-x)
\]+ 2E^2p^2_{12} [B^2 + B^2]. \tag{64}
SUMMARIZED RESULTS

A CDC 3600 Fortran program consisting of several subroutines and functions has been written to produce the numerical and graphical results summarized in this section. Only a brief description of the program will be given here; detailed notes, listings, and a sample of the output appear in Appendix A. The main program, Leader, reads from cards a list of matrix elements to be calculated, compared, and (on an input option) plotted. For each matrix element in the list, Leader calls Genratr, which either evaluates the distribution analytically through a call to Goldhabr, integrates it numerically through a call to Goldnum, or picks up the results of previous evaluations from the card reader. At this stage, printed output is supplied by Printer, and (on input option) punched output is provided for subsequent runs. The distribution is evaluated on a mesh of points covering the Goldhaber triangle in such a way that the legs of the triangle are divided into N equal segments, where N may be specified in the program. For each matrix element, Leader now calls Projectr, which performs what amounts to a further numerical integration by adding the values of the distribution down each column of points. For each matrix element for which the input plot option has been activated, Leader calls Plotter, which produces a skewed isometric drawing of the distribution by generating magnetic tape output for the Calcomp 570 plotter. Finally, with all the distributions stored, Leader calls Corco, which evaluates and prints out the point-by-point numerical correlation coefficient according
to the formula
\[ C(P, P') = \left\{ \sum_{i=1}^{N} (P_i - \bar{P})(P'_i - \bar{P}') \right\} \]
\[ \times \left\{ \sum_{i=1}^{N} (P_i - \bar{P})^2 \times \sum_{i=1}^{N} (P'_i - \bar{P}')^2 \right\}^{-1/2} \]

where \( P \) and \( P' \) are any pair of distributions and \( \bar{P} \) is the mean value of the distribution, taken over the \( N \) points \( P_i \) and \( P'_i \) on the Goldhaber triangle at which the function was evaluated. No attempt will be made in the present work to justify this extension of the usual use of the correlation coefficient, since its use here is intended only to give a semi-quantitative measure of the degree of similarity between two distributions, or the extent to which they may be distinguished, a purpose it serves better than any other tool of which the author is aware. The methods necessary to establish whether or not a given experimental distribution fitted one or another of the theoretical distributions given here would have to be somewhat more sophisticated and could be made considerably more powerful.

In what follows we give for each matrix element the algebraic dependence on energies and momenta, the symmetry type and charge assignment (if any), the method of integration used, a label for reference to the table of correlation coefficients (table 2.), and a drawing of the distribution. The drawings show only the physical region of the Goldhaber triangle and the projected distribution;

they are collected at the end of this section. In constructing the labels, \((2A, 1^-A)\) for example, the following rules have been used: for the leading subscript, 1 denotes the full symmetry \([1234]\), 2A denotes the partial symmetry of the type B matrix elements with charge assignment ++--, and 2B the type B symmetry with charge assignment +--+; then follows the spin-parity assignment \(J^P\); following the comma, different algebraic forms are assigned ordinal letters, A, B, C, etc., in approximate order of increasing complexity. In the cases where it has been obtained, the integrated distribution is given also.

Matrix Element Number 1

Form: \(M = 1\) (phase space only)
Symmetry type: \([1234]\)
Charge assignment: None
Method of integration: Analytic
Label: \((10^+,A)\)
Integrated distribution:
\[
\int\int M^2 \, dq \, ds = \int\int dq \, ds = \Delta q \Delta s
\]

Matrix Element Number 2

Form: \(M = [E_1E_2E_3E_4]^{1/2}\)
Symmetry type: \([1234]\)
Charge assignment: None
Method of integration: Analytic
Label: \((10^+,B)\)
Integrated distribution:

\[
\frac{1}{\Delta q \Delta s} \int \int M^2 \, dq \, ds = \left[ \frac{1}{4} (x - y - E^2)^2 - \frac{1}{3} \left( E^2 p_{12}^2 \right)(1 - \frac{4m^2}{y}) \right] \\
\times \left[ \frac{1}{4} (x - y + E^2)^2 - \frac{1}{3} \left( E^2 p_{12}^2 \right)(1 - \frac{4m^2}{x}) \right]
\]

Matrix Element Number 3

Form: \(M = E_1^2 + E_2^2 + E_3^2 + E_4^2\)
Symmetry type: \(\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}\)
Charge assignment: None
Method of integration: Analytic
Label: \((10^+,C)\)
Integrated distribution:

\[
\frac{4E^4}{\Delta q \Delta s} \int \int M^2 \, dq \, ds = \frac{1}{4} [(x - y)^2 + E^4]^2 + \frac{1}{2} [(x - y)^2 + E^4]E^2 p_{12}^2 \\
\times \left( 2 - \frac{4m^2}{x} - \frac{4m^2}{y} \right) + \frac{1}{5} E^4 p_{12}^4 \left[ (1 - \frac{4m^2}{x})^2 + (1 - \frac{4m^2}{y})^2 \right] \\
+ \frac{2}{9} E^4 p_{12}^4 (1 - \frac{4m^2}{x})(1 - \frac{4m^2}{y})
\]

Matrix Element Number 4

Form: \(M = p_1^2 + p_2^2 + p_3^2 + p_4^2\)
Symmetry type: \(\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}\)
Charge assignment: None

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Method of integration: Numerical
Label: \((\,1\,0^+, D)\)

Matrix Element Number 5

Form: \(M = E_1^+P_1 + E_2^+P_2 + E_3^+P_3 + E_4^+P_4\)
Symmetry type: \([1\,2\,3\,4]\)
Charge assignment: None
Method of integration: Analytic
Label: \((\,1\,1^-, A)\)

Integrated distribution:
\[
\frac{4E^2}{\Delta q\Delta s} \int \int M^2 \, dq \, ds = p_{12}^2 (x - y)^2 + \frac{1}{3} \left[ \frac{(x - y)^2}{E^2} - 4m^2 - x + 2y \right]
\]
\[
\frac{E^2}{2}P_{12}^2 (y - 4m^2) + \frac{1}{3} \left[ \frac{(y - x)^2}{E^2} - 4m^2 - y + 2x \right] \frac{E^2}{x}P_{12}^2 (x - 4m^2)
\]
\[
+ \frac{(x - y)^2 - E^4}{18E^4P_{12}} \left[ \frac{E^4}{4y} \right]_{12} (x - 4m^2)(y - 4m^2) + \frac{1}{5E^2} \left[ \frac{E^4}{y^2} \right]_{12} (x - 4m^2)^2
\]
\[
x(y - 4m^2)^2 + \frac{E^4}{x^2}P_{12}^4 (x - 4m^2)^2
\]

Matrix Element Number 6

Form: \(M = E_1^+P_1 + E_2^+P_2 + E_3^+P_3 + E_4^+P_4\)
Symmetry type: \([1\,2\,3\,4]\)
Charge assignment: None
Method of integration: Analytic
Label: \((\,1\,1^-, B)\)
Integrated distribution:

\[
\frac{64E^6}{\Delta s \Delta q} \iint M^2 \, dq \, ds = 4E^6 p_{12}^2 (x - y)^2 + \frac{1}{3} E^4 p_{12}^2 (1 - \frac{4m^2}{y}) \\
\cdot [2(E^2 + y - x)^2 (5y - 3x - 8m^2) + 8E^2 y (x - y)] + \frac{1}{3} E^4 p_{12}^2 \\
\cdot (1 - \frac{4m^2}{x}) [2(E^2 + x - y)^2 (5x - 3y - 8m^2) - 8E^2 x (x - y)] \\
- \frac{2}{9} E^4 p_{12}^2 (1 - \frac{4m^2}{x}) (1 - \frac{4m^2}{y}) \left\{ 4E^2 p_{12}^2 + 4[E^4 + (x - y)^2] \\
+ \frac{(E^2 + x - y)^2 (E^2 + y - x)^2}{E^2 p_{12}^2} \right\} + \frac{1}{5} E^4 p_{12}^2 (1 - \frac{4m^2}{y})^2 [4E^2 p_{12}^2 \\
+ 8(E^2 + y - x)^2] + \frac{1}{5} E^4 p_{12}^2 (1 - \frac{4m^2}{x}) [4E^2 p_{12}^2 + (E^2 + x - y)^2]
\]

Matrix Element Number 7

Form: \( \hat{M} = (E_1 - E_3)(\hat{p}_2 - \hat{p}_4) + (E_2 - E_4)(\hat{p}_1 - \hat{p}_3) + (E_1 - E_4) \cdot (\hat{p}_2 - \hat{p}_3) + (E_2 - E_3)(\hat{p}_1 - \hat{p}_4) \)

Symmetry type: \[ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} + \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \]

Charge assignment: ++--

Method of integration: Analytic

Label: \((2A^{1-}, A)\)

Integrated distribution:

\[
\frac{16E^4}{\Delta s \Delta q} \iint M^2 \, dq \, ds = 4E^2 p_{12}^2 (x - y)^2 + \frac{1}{6} E^2 p_{12}^2 (1 - \frac{4m^2}{x}) [E^2 (x + y) \\
- 8E^2 m^2 + 3E^2 (x - y) - 4(x - y)^2] + \frac{1}{6} E^2 p_{12}^2 (1 - \frac{4m^2}{y}) [E^2 (x + y)]
\]
\[-8E^2m^2 - 3E^2(x - y) - 4(x - y)^2 \] - \frac{1}{18} E^2p^2_{12}(1 - \frac{4m^2}{x})(1 - \frac{4m^2}{y})

\[X(x - y - E^2)(y - x - E^2) + \frac{1}{5} E^4p^4_{12}(1 - \frac{4m^2}{x})^2 + \frac{1}{5} E^4p^4_{12}(1 - \frac{4m^2}{y})^2\]

Matrix Element Number 8

Form: \[\hat{M} = (E_1^2 - E_3^2)(P_2 - P_4) + (E_2^2 - E_4^2)(P_1 - P_3) + (E_1^2 - E_4^2)\]

\[X(P_2 - P_3) + (E_2^2 - E_3^2)(P_1 - P_4)\]

Symmetry type: \[\begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}, \begin{array}{cc}
1 & 2 \\
4 & 3 \\
\end{array}\]

Charge assignment: ++--

Method of integration: Numerical

Label: \((2A_1^-, B)\)

Matrix Element Number 9

Form: \[M = (E_1 - E_3)(E_2 - E_4) + (E_1 - E_4)(E_2 - E_3)\]

Symmetry type: \[\begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}, \begin{array}{cc}
1 & 2 \\
4 & 3 \\
\end{array}\]

Charge assignment: ++--

Label: \((2A_0^+, A)\)

Integrated distribution:

\[\frac{4E^4}{\Delta q \Delta s} \int \int M^2 dq \, ds = (x - y)^4 - \frac{1}{6} (x - y)^2 [\frac{f(x,y)}{x} + \frac{f(y,x)}{y}]\]

\[+ \frac{1}{8} \frac{f^2(x,y)}{10x^2} + \frac{f^2(y,x)}{10y^2} + \frac{f(x,y)f(y,x)}{9xy}]\]

where \(f(x,y) = (x - 4m^2)(4E^2p^2_{12})\)
Matrix Element Number 10

Form: \[ M = (E_1^2 - E_2^2)(E_3^2 - E_4^2) + (E_1^2 - E_3^2)(E_2^2 - E_4^2) \]

Symmetry type: \[ \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} + \begin{array}{cc} 1 & 2 \\ 4 & 3 \end{array} \]

Charge assignment: ++--

Method of integration: Numerical

Label: \((2A^0, B)\)

---

Matrix Element Number 11

Form: \[ M = (P_1 - P_3) \cdot (P_2 - P_4) + (P_1 - P_4) \cdot (P_2 - P_3) \]

Symmetry type: \[ \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} + \begin{array}{cc} 1 & 2 \\ 4 & 3 \end{array} \]

Charge assignment: ++--

Method of integration: Analytic

Label: \((2A^0, C)\)

Integrated distribution:

\[
\frac{E^4}{\Delta q \Delta s} \int \int M^2 \, dq \, ds = \{ \ 3E^2P^2_{12} + 4m^2E^2 - \frac{1}{4} [(x - y)^2 + E^4] \}^2 \\
- \frac{1}{3} \left\{ 3E^2P^2_{12} - \frac{1}{4} [(x - y)^2 + E^4] \right\} E^2P^2_{12}(2 - \frac{4m^2}{x} - \frac{4m^2}{y}) \\
+ \frac{1}{18} E^2P^2_{12}(1 - \frac{4m^2}{x})(1 - \frac{4m^2}{y}) + \frac{1}{20} E^4P^4_{12}[(1 - \frac{4m^2}{x})^2 + (1 - \frac{4m^2}{y})^2] \\
\]

Matrix Element Number 12

Form: \[ M = (E_1 - E_2)(P_3 - P_4) + (E_3 - E_4)(P_1 - P_2) - (E_1 - E_4)X \]
\[ x(P_2 - P_3) - (E_2 - E_3)(P_1 - P_4) \]

Symmetry type: \[ \begin{array}{c|c|c|c}
1 & 3 & 1 & 3 \\
2 & 4 & 2 & 4 \\
4 & 3 & 4 & 3 \\
\end{array} \]

Charge assignment: +-+-

Method of integration: Analytic

Label: \( (2B_1^1,A) \)

Integrated distribution:

\[
\begin{align*}
\frac{4E^4}{\Delta q\Delta s} \int \int M^2 \, dq \, ds &= 4E^2P^2_{12}(x - y)^2 \\
+ \frac{1}{3} E^2P^2_{12}(1 - \frac{4m^2}{y}) &\left[ - \frac{3}{2} E^4 - 2E^2(x-y) - \frac{7}{2} (x-y)^2 + 16E^2x - 40E^2m^2 \right] \\
+ \frac{1}{3} E^2P^2_{12}(1 - \frac{4m^2}{x}) &\left[ - \frac{3}{2} E^4 + 2E^2(x-y) - \frac{7}{2} (x-y)^2 + 16E^2y - 40E^2m^2 \right] \\
+ 2E^4P^4_{12}(1 - \frac{4m^2}{x})(1 - \frac{4m^2}{y}) &+ \frac{1}{5} E^4P^4_{12}(1 - \frac{4m^2}{x})^2 + \frac{1}{5} E^4P^4_{12}(1 - \frac{4m^2}{y})^2 \\
- \frac{5}{9} E^2P^2_{12}(1 - \frac{4m^2}{y})(1 - \frac{4m^2}{x}) &\left( E^2 + x - y \right) \left( E^2 + y - x \right) \\
+ 2E^4P^4_{12} \left( 2 - \frac{4m^2}{x} - \frac{4m^2}{y} \right) & \\
\end{align*}
\]

Matrix Element Number 13

Form:

\[ M = (E^2_1 - E^2_2)(P_3 - P_4) + (E^2_3 - E^2_4)(P_1 - P_2) - (E^2_1 - E^2_4) \]

\[ + x(P_2 - P_3) - (E^2_2 - E^2_3)(P_1 - P_4) \]

Symmetry type: \[ \begin{array}{c|c|c|c}
1 & 3 & 1 & 3 \\
2 & 4 & 2 & 4 \\
4 & 3 & 4 & 3 \\
\end{array} \]

Charge assignment: +-+-

Method of integration: Numerical
Label: \((_{2B}^{-},B)\)

**Matrix Element Number 14**

Form: \( M = -(E_1 - E_2)(E_3 - E_4) + (E_1 - E_4)(E_2 - E_3) \)

Symmetry type: \( \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \)

Charge assignment: \(+--\)

Method of integration: Analytic

Label: \((_{2B}^{0+},A)\)

Integrated distribution:

\[
\frac{16E^4}{\Delta q ds} \iint M^2 \, dq \, ds = (x - y)^4 - \frac{(x - y)}{6} \left[ \frac{f(x,y)}{x} + \frac{f(y,x)}{y} \right] \\
+ \frac{1}{8} \left[ \frac{1}{10x^2} f^2(x,y) + \frac{1}{10y^2} f^2(y,x) + \frac{19}{9} \frac{f(x,y)f(y,x)}{xy} \right]
\]

where \( f(x,y) = (x - 4m^2)(4E^2p_{12}^2) \)

**Matrix Element Number 15**

Form: \( M = -(E_1^2 - E_2^2)(E_3^2 - E_4^2) + (E_1^2 - E_4^2)(E_2^2 - E_3^2) \)

Symmetry type: \( \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \)

Charge assignment: \(+--\)

Method of integration: Numerical

Label: \((_{2B}^{0+},B)\)

**Matrix Element Number 16**

Form: \( M = -( \vec{P}_1 - \vec{P}_2) \cdot (\vec{P}_3 - \vec{P}_4) + (\vec{P}_1 + \vec{P}_4) \cdot (\vec{P}_2 - \vec{P}_3) \)
Symmetry type: \( \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \)

Charge assignment: ++--

Method of integration: Analytic

Label: \((2B^0{+}, C)\)

Integrated distribution:

\[
\frac{16E^4}{\Delta q \Delta s} \int \int M^2 \, dq \, ds = \left[ 4E^2p^2_{12} - E^2(x + y) + 8E^2m^2 \right]^2 \\
- \frac{2}{3} E^2p^2_{12} \left( 1 - \frac{4m^2}{x} \right) \left( (E^2 + x - y)^2 + 8E^2m^2 - 5E^2x - E^2y \right) \\
- \frac{2}{3} E^2p^2_{12} \left( 1 - \frac{4m^2}{y} \right) \left( (E^2 + y - x)^2 + 8E^2m^2 - 5E^2y - E^2x \right) \\
+ \frac{1}{5} E^4p^4_{12} \left( 1 - \frac{4m^2}{x} \right)^2 + \frac{1}{5} E^4p^4_{12} \left( 1 - \frac{4m^2}{y} \right)^2 + \frac{2}{9} E^4p^4_{12} \left( 1 - \frac{4m^2}{x} \right) \\
x \left( 1 - \frac{4m^2}{y} \right) + 4E^4p^4_{12} \left( 1 - \frac{4m^2}{x} \right) \left( 1 - \frac{4m^2}{y} \right) \left( 1 + \frac{x}{p^2_{12}} + \frac{y}{p^2_{12}} + \frac{3xy}{p^4_{12}} \right)
\]

DISCUSSION AND CONCLUSIONS

In order for the results we have found to be useful, the following must be true: for a given charge assignment and its corresponding symmetry type the Goldhaber distributions should allow one to distinguish clearly between alternate spin-parity choices, and for a given spin-parity choice and charge assignment the shape of the distribution must not depend very strongly on the precise form used for the matrix element.

The most compact statement of the results is the matrix of
Figure 7: Goldhaber distribution and projected distribution for matrix element number 5.

\[ M = E_1 P_1 + E_2 P_2 + E_3 P_3 + E_4 P_4 \]

\[ (\xi, \mu) \]
Figure 8: Goldhaber distribution and projected distribution for matrix element number 6.

\[ M = E_1 P_1 + E_2 P_2 + E_3 P_3 + E_4 P_4 \]

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\[ \mathbf{M} = (E_1 - E_3)(\mathbf{P}_2 - \mathbf{P}_4) + (E_2 - E_4)(\mathbf{P}_1 - \mathbf{P}_3) \\
+ (E_1 - E_4)(\mathbf{P}_2 - \mathbf{P}_3) + (E_2 - E_3)(\mathbf{P}_1 - \mathbf{P}_4) \]

Figure 9. Goldhaber distribution and projected distribution for matrix element number 7.
\[ \mathbf{M} = (E_1^2 - E_3^2)(\mathbf{P}_2 - \mathbf{P}_4) + (E_2^2 - E_4^2)(\mathbf{P}_1 - \mathbf{P}_3) + (E_1^2 - E_4^2)(\mathbf{P}_2 - \mathbf{P}_3) + (E_2^2 - E_3^2)(\mathbf{P}_1 - \mathbf{P}_4) \]

\((2A, B)\)

Figure 10. Goldhaber distribution and projected distribution for matrix element number B.
\[ M = (E_1 - E_3)(E_2 - E_4) + (E_1 - E_4)(E_2 - E_3) \]

Fig. 11. Goldhaber distribution and projected distribution for matrix element number 9.
\[ M = (E_1^2 - E_3^2)(E_2^2 - E_4^2) + (E_1^2 - E_4^2)(E_2^2 - E_3^2) \]
\[ M = (\vec{P}_1 - \vec{P}_3)(\vec{P}_2 - \vec{P}_4) + (\vec{P}_1 - \vec{P}_4)(\vec{P}_2 - \vec{P}_3) \]

Figure 13. Goldhaber distribution and projected distribution for matrix element number 11.
Figure 14. Goldhaber distribution and projected distribution for matrix element number 12.

\[
\mathbf{M} = (E_1 - E_2)(\mathbf{P}_3 - \mathbf{P}_4) + (E_3 - E_4)(\mathbf{P}_2 - \mathbf{P}_1) - (E_2 - E_3)(\mathbf{P}_1 - \mathbf{P}_4)
\]
\[ \vec{M} = (E_1^2 - E_2^2)(\vec{P}_3 - \vec{P}_4) + (E_3^2 - E_4^2)(\vec{P}_1 - \vec{P}_2) - (E_1^2 - E_4^2)(\vec{P}_2 - \vec{P}_3) - (E_2^2 - E_3^2)(\vec{P}_1 - \vec{P}_4) \]

\( (2B^1, B) \)

Figure 15. Goldhaber distribution and projected distribution for matrix element number 13.
Figure 16. Goldhaber distribution and projected distribution for matrix element number 14.

\[ M = -(E_1 - E_2)(E_3 - E_4) + (E_1 - E_4)(E_2 - E_3) \]

\((2_B O^+, A)\)
\[ M = -(E_1^2 - E_2^2)(E_3^2 - E_4^2) + (E_1^2 - E_4^2)(E_2^2 - E_3^2) \]

\[ (2B O', B) \]

Figure 17. Goldhaber distribution and projected distribution for matrix element number 15.
\[ M = - (\vec{P}_1 - \vec{P}_2) \cdot (\vec{P}_3 - \vec{P}_4) + (\vec{P}_1 - \vec{P}_4) \cdot (\vec{P}_2 - \vec{P}_3) \]

Figure 18. Goldhaber distribution and projected distribution for matrix element number 16.
correlation coefficients, which is shown in Table 2. We will express our conclusions in terms of these coefficients for the sake of brevity, denoting the correlation between matrix elements numbered \(N\) and \(M\) by \((N,M)\).

For the type A matrix elements one can see that the results are precisely as had been hoped. That is, all the vector-scalar correlations are much lower than the vector-vector and scalar-scalar correlations. Furthermore, all correlations of the latter two types are so high that there seems to be virtually no sensitivity to the exact form of the matrix element.

The results are somewhat more ambiguous for the type B matrix elements. Consider as an example the charge assignment +++-- (matrix elements numbered 7-11). We see that the vector-scalar correlations are low, as had been hoped. However, the correlations \((9,11)\) and \((10,11)\) are quite low so that there is evidently strong dependence on the exact form assumed for the matrix element. Thus even though one might by chance find that an experimental distribution did match number 9,10 or 11 well, there is no clear interpretation of such an occurrence.

For the charge assignment ++-- the same disappointing conclusion must be reached. Again the vector-scalar correlations are low, but the correlations \((14,16)\) and \((15,16)\) are also quite low, although not as low as the corresponding ones for the charge assignment +++--.

Certain other interesting facts may be learned by inspection of Table 2. One can see for example that the \(0^+\) matrix elements of
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<tr>
<td>(10^+,D)</td>
<td>4.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>.730</td>
<td>.703</td>
<td>.722</td>
<td>.590</td>
<td>.350</td>
<td>.340</td>
<td>.433</td>
<td>.907</td>
<td>.901</td>
<td>.814</td>
<td>.806</td>
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<tr>
<td>(11^-,A)</td>
<td>5.00</td>
<td>1.00</td>
<td>.997</td>
<td>.970</td>
<td>.954</td>
<td>.202</td>
<td>.197</td>
<td>.045</td>
<td>.743</td>
<td>.683</td>
<td>.421</td>
<td>.402</td>
<td>.263</td>
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<tr>
<td>(11^-,B)</td>
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<td>1.00</td>
<td>.962</td>
<td>.952</td>
<td>.216</td>
<td>.210</td>
<td>.073</td>
<td>.698</td>
<td>.634</td>
<td>.408</td>
<td>.390</td>
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<tr>
<td>(2A1^-,A)</td>
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<td>1.00</td>
<td>.980</td>
<td>.248</td>
<td>.245</td>
<td>-.03</td>
<td>.778</td>
<td>.709</td>
<td>.381</td>
<td>.362</td>
<td>.275</td>
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<tr>
<td>(2A1^-,B)</td>
<td>8.00</td>
<td>1.00</td>
<td>.185</td>
<td>.184</td>
<td>-.14</td>
<td>.663</td>
<td>.585</td>
<td>.203</td>
<td>.184</td>
<td>.141</td>
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<tr>
<td>(2A0^+,A)</td>
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<td>1.00</td>
<td>1.00</td>
<td>.373</td>
<td>.175</td>
<td>.162</td>
<td>.515</td>
<td>.515</td>
<td>.234</td>
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<tr>
<td>(2A0^+,B)</td>
<td>10.00</td>
<td>1.00</td>
<td>.359</td>
<td>.170</td>
<td>.156</td>
<td>.503</td>
<td>.502</td>
<td>.225</td>
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<tr>
<td>(2A0^+,C)</td>
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<td>1.00</td>
<td>.043</td>
<td>.054</td>
<td>.614</td>
<td>.618</td>
<td>.605</td>
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<tr>
<td>(2B1^-,A)</td>
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<td>.993</td>
<td>.657</td>
<td>.647</td>
<td>.617</td>
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<tr>
<td>(2B1^-,B)</td>
<td>13.00</td>
<td>1.00</td>
<td>.681</td>
<td>.673</td>
<td>.643</td>
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<tr>
<td>(2B0^+,A)</td>
<td>14.00</td>
<td>1.00</td>
<td>1.00</td>
<td>.641</td>
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<tr>
<td>(2B0^+,B)</td>
<td>15.00</td>
<td>1.00</td>
<td>.643</td>
<td></td>
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<tr>
<td>(2B0^+,C)</td>
<td>16.00</td>
<td>1.00</td>
<td></td>
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</tbody>
</table>

**Table 2.** The matrix of correlation coefficients.
full symmetry (numbers 2-4) can scarcely be distinguished from the phase space (number 1). This means that this class would be difficult to distinguish experimentally from a non-resonant reaction by analysis of Goldhaber distributions. This is not really alarming though, since this method is not proposed for use in detecting a resonance, but only for gaining information about one already known to exist.

Among the interesting (if not particularly significant) facts that may be gotten from the graphs in conjunction with the correlation coefficients is the following: the correlation of the $0^+$ full symmetry scalars with numbers 12 and 13 are all quite high. What the pattern is may be seen by referring to the graphs of the appropriate matrix elements. Except for numbers 12 and 13, which anomalously resemble the phase space, the vector matrix elements all tend to peak along the edges at the legs of the Goldhaber triangle. That 12 is anomalous is purely an effect of the symmetry and charge assignment because that matrix element is otherwise identical to number 7, another vector. The analytic reason for this behavior is probably the absence of terms of the form $(1 - 4m_1^2/m_{ij}^2)$ in the integrated distribution of number 12. Such terms appear in all other vectors. A similar result would probably be found for number 13 if an analytic result were obtained.

In conclusion one can be quite optimistic about the possibilities for the use of the type A matrix elements for comparison with experiment. The type B elements will probably not prove so useful.
because of the strong form dependence of the scalars and the odd behavior of numbers 12 and 13.
APPENDIX A

On the following pages are listings of the CDC 3600 Fortran programs which were used to generate the output presented in this paper.

Some words of caution are in order for the reader who would attempt to run the programs on other machines. The version of Fortran used here is not readily compatible with machines using less than a 48-bit word. One particular case that will likely cause trouble is the use of identifiers of more than six characters. Most machines allow a maximum of six characters in an identifier. Some forms of syntax allowable in 3600 Fortran will cause diagnostics in other machines, as for instance the use of * to define a literal Hollerith expression.

Subroutine Plotter makes use of several library programs which in their precise form are probably unique to the University of Wisconsin Computing Center (UWCC). These are Plot, Symbol, Number and Spline1. The first three of these are supplied in basic form by the California Computer Products Company (Calcomp) and are used to generate magnetic tape output for the Calcomp 570 plotter. In the event that the user does not have access to such a machine, Subroutine Plotter is essentially useless without considerable rewriting. If the user does have access to such a machine, the routine should be useable with little change, as Calcomp supplies software for most major brands of computers with essentially the same calling sequences as used here. Spline1 is a cubic spline interpolation function.
which is available at many large-scale computer installations. If it is not available to the reader, he should experience little difficulty in writing a suitable interpolation function. Its purpose, in any case is simply to interpolate for the plotter a smooth curve between actual calculated values of the Goldhaber distributions.

Some notes on the operation of the programs may prove helpful. Note that in Subroutine Goldhabr, E and m as carried in the analytic expressions of Section VII have been replaced by the ratio $Z = \frac{E}{m}$. The m has been factored out and ignored, in effect. This is of no physical consequence, as the appropriate power of m would be absorbed in the normalization constant in any case. In Goldnum, on the other hand, E and m have been carried explicitly in GeV. This seeming inconsistency is actually an intentional check on the arithmetic. The numerical integration in Goldnum has been done by the relatively unsophisticated trapezoidal rule. This simplified writing and debugging the program, at the expense of running time. In the event that a much slower computer is used, it might be worthwhile to use a more efficient method here, as even the 3600 required about two to three minutes to evaluate one distribution by this method. Note that a double numerical integration is involved. The input specifications are given in terms of the Hollerith analogue of the labels used in Section VII, and another labelling scheme peculiar to the author and not otherwise mentioned here. The conversion to another labelling scheme would be easily accomplished.
PROGRAM LEADER
COMMON/HLOCK/E,NP
COMMON/HLOCK2/K!
DIMENSION MONSTER(21,21,25)
TYPE_REAL MONSTER
DIMENSION TEMP(21,21)
DIMENSION LAB1(25),LAB2(25),METHOD(25),KPLCT(25),KPC(25)
DATA(F=1.0)
DATA(NP=1)
DATA(K=1)
DATA(I=D)
DATA(J=M)
NINT=70
NPTS=1
JMC=1
READ 900,LAB2(JMC),LAB1(JMC),METHOD(JMC),KPLCT(JMC),KPC(JMC)
IF(LAH(JMC).EQ.8H) GO TO 20
PRINT 902,LAB(JMC),LAB2(JMC),METHOD(JMC),KPLCT(JMC),KPC(JMC)
JMC=JMC+1
GO TO 10
NMAT=JMC-1
DO 100 JN=1,NMAT
CALL GENPAIR(JN,METHOD(JN),NPTS,TEM1,LAB1(JN),LAB2(JN),SJM,KPC(JN),F)
CALL PROJECTR(JN,METHOD(JN),NPTS,TEM1,LAB1(JN),LAB2(JN),F)
IF(KPLCT(JN)) CALL PLOTTER(JN,METHOD(JN),NPTS,TEM1,LAB1(JN),LAB2(JN),F)
DO 50 JJ=1,NPTS
DO 50 JJ=1,NPTS
50 CONTINUE
CALL CURCO(MONSTER,NPTS,NMAT,LAB1,LAB2,METHOD)
END
SUBROUTINE GENRATR(JM, METHOD, NPTS, PD, LAB1, LAB2, KPCH, FACTOR)

DIMENSION PD(NPTS), LGLUN(16)

DATA (LGLUN=4H10+R, 8H10+C, 4H10+U, 4H11-4A, 8H11-4A, 8H1A1-4A, 8H21-4A, 8H2A1-4U, 8H2A0+4U, 8H2B1-4A, 8H2B1-4U, 8H2B0+4U, 8H2B0+4U, 8H2B0+4U, 8H28+4U)

DATA (LGLUH=4H1, 8H2, 8H9, 8H9, 8H10, 4H10) GO TO (100, 200, 300) * METHOD

100 IFNDO=0
   DO 11 J=1,11
   IF (LAB1 .EQ. LGLUN(J)) IFNDO=J
   IF (IFNDO .NE. 0) GO TO 112

110 PRINT 900, JM, LAB1, METHOD

900 FORMAT (5 GENRATR CANNOT FIND SUBSCRIPT FOR NUMBER*, 13, 4 WITH LABEL *, AB, * METHOD *, I12)

RETURN

112 CALL GOLUHAHR(IFNDO, NPTS, PD, LAB1, LAB2, KPCH, FACTOR)

RETURN

200 IFNDO=0
   DO 21 J=1,11
   IF (LAB2 .EQ. LGLUN(J)) IFNDO=J
   IF (IFNDO .NE. 0) GO TO 212

210 PRINT 900, JM, LAB1, METHOD

RETURN

212 CALL GOLUHNUM(IFNDO, NPTS, PD, LAB1, LAB2, KPCH, FACTOR)

RETURN

300 CONTINUE
   N1=NPTS
   N2=N1-1
   READ 902, FACTOR, LAB1, LAB2

902 FORMAT (30X, 12, 10X, 5X, 10X, 15X, AB)

   NLC=1
   NHC=11
   DO 22 J=NLC, N1
   J=N+2-J
   READ 22, (PD(IR), IR=NLC, NRC)

22 FORMAT (12, (PD(IR), IR=NLC, NRC)

   IF (NRC=N) 25, 28, 28

25 NLC=NLC+11
   NHC=NHC+11
   GO TO 22

26 CONTINUE

   PRINT 919, JM, LAB1, LAB2

919 FORMAT (13, 10X, 5X, 10X, 15X, AB)
SUBROUTINE GOLDBR(MATNUM, N1, PD, LAB1, LAB2, KPCH, FAC, STOR)

DIMENSION PD(N1, N1)
COMMON/BLOCK/E, N2
COMMON/BLOCK2/K, L

EXTERNAL FINT
Z=E/1.958
N=N-1

PRINT 6, MATNUM
FORMAT(*GOLDBR DISTRIBUTION FOR MATRIX ELEMENT*)
5, * No. * I13
GO TO (7*9), K

7 PRINT 0
8 FORMAT(27H IN LINEAR INVARIANT MASSES)
GO TO 11

10 FORMAT(28H IN SQUARED INVARIANT MASSES)
11 GO TO (12*14), L

12 PRINT 13
13 FORMAT (19H WITHOUT L FUNCTION)
GO TO 16

14 PRINT 15
15 FORMAT (16H WITH L FUNCTION)
16 CONTINUE

PRINT 900, LAB1, LAB2
900 FORMAT(* WELLER DESIGNATION * +, A10, * ELLER DESIGNATION * +, A8, * )

DO 300 I=1, N1
300 DO 100 J=1, N1
E=1
FJ=J

IF (I+J-M-2. > 17*17*1.6)
20 GO TO (20*30), L

RX=2.+((E-1.)*Z4.)/EN
RY=2.+((FJ-1.)*Z4.)/FN
X=X*X*X
Y=Y*Y

GO TO 40

30 X=4.*((E-1.)*Z2.)*Z2.*Z2.)/EN
Y=4.*((FJ-1.)*Z2.)*Z2.*Z2.)/EN

40 P12SQ=(X+Y)**2.**Z2.**Z2.**(X+Y)
P12SO=(P12SQ+Z2.*Z2.*Z2.)/(4.*Z2.)
IF (P12SQ.LT.0.. AND.ASF(P12SQ).LT.000001) P12SO=-P12SQ

GO TO (50, 60), L

50 ELLFN=1.

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GO TO 70
ELLFN=4*SQRZ*P12SQ/(X*Y)
ELLFN=ELLFN*SQR((X*X-4*X)*(Y*Y-4*Y))/SQR(PI)  
7 CONTINUE
PD(I,J)=ELLFN*FACT(MATNUM,X,Y,Z,P12SQ)
100 CONTINUE
CALL Printer(N1,PD,FACT)
C NorH1/e
PRINT 136,MATNUM
130 FORMAT(*)NORMALIZED DISTRIBUTION FOR NO*,I3)
SUM=0
DO 135 I=1,N1
IF (I+J-N-2) 131,131,135
131 SUM=SUM+PD(I,J)
135 CONTINUE
FACT=1000./SUM
DO 140 I=1,N1
DO 140 J=1,N1
IF (I+J-N-2) 130,130,140
136 PD(I,J)=PD(I,J)*FACT
140 CONTINUE
CALL Printer(N1,PD,FACT)
IF (KPCH .EQ. 0) GO TO 117
PUNCH 000,MATNUM,FACTOR,LAB1,LAB2
909 FORMAT(*)ANAL RESULTS NO*,I3, MULT BY ***10.5,
*** WELLEr ***AB *** WELLEr/OUTSON ***AR)
PUNCH 919
919 FORMAT(*B0X)
NLC=1
NRC=1
113 DO 114 J=NLC,N1
NRC=NRC+1
PUNCH 115, (PD(IR,JH), IH=NLC,NRC)
114 PUNCH 919
115 FORMAT(11F7.4)
IF (N=N-C-N) 116,117,117
116 NLC=NLC+1
NRC=NRC+1
GO TO 113
117 CONTINUE
END
FUNCTION FINT(\text{MATNUM}, X, Y, Z, P1, 2, SQ)
GO TO (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11) * \text{MATNUM}

1 CONTINUE

C \text{NO. 1} \ (10 + d)
C \text{MATNUM} = 1 + E(1) * E(2) * E(3) * E(4)

\text{FINT1} = \frac{2}{3} * (X - Y - Z) * (X - Y - Z) - (Z * P1 * 2 * SQ) * (1.4 / X)
\text{FINT2} = \frac{2}{3} * (X - Y + Z) * (X - Y + Z) - (Z * P1 * 2 * SQ) * (1.4 / X)
\text{FINT} = \text{FINT1} + \text{FINT2}
RETURN

2 CONTINUE

C \text{NO. 2} \ (10 + d + c)
C \text{MATNUM} = 1 + E(1) * E(2)
\text{FINT1} = \frac{1}{3} * \frac{(X - Y)}{(Z * P1 * 2 * SQ) * (2.4 / X - 4.2 / Y)}
\text{FINT2} = \frac{1}{3} * \frac{(X - Y)}{(Z * P1 * 2 * SQ) * (1.4 / X)}
\text{FINT} = \text{FINT1} + \text{FINT2}
RETURN

3 CONTINUE

C \text{NO. 3} \ (11 + a)
C \text{MATNUM} = 1 + E(1) * E(2)
\text{FINT1} = \frac{1}{3} * \frac{(X - Y)}{(Z * P1 * 2 * SQ) * (2.4 / X - 4.2 / Y)}
\text{FINT2} = \frac{1}{3} * \frac{(X - Y)}{(Z * P1 * 2 * SQ) * (1.4 / X)}
\text{FINT} = \text{FINT1} + \text{FINT2}
RETURN

4 CONTINUE

C \text{NO. 4} \ (11 - 3)
C \text{MATNUM} = 1 + E(1) * E(2)
\text{FINT1} = \frac{1}{3} * \frac{(X - Y)}{(Z * P1 * 2 * SQ) * (2.4 / X - 4.2 / Y)}
\text{FINT2} = \frac{1}{3} * \frac{(X - Y)}{(Z * P1 * 2 * SQ) * (1.4 / X)}
\text{FINT} = \text{FINT1} + \text{FINT2}
RETURN

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FINTE = MENU (-2, 9, 4, Z, P12SQ, P12SU, 1, 4, Y) 
FIN1 = Z*Z*P12SQ*P12SU*1, 4, /Y) 
FIN7 = FINT5* (4, Z*Z*P12SU*8, (Z*Z+Y-y) *(Z*Z+X-x)) 
FIN6 = Z*Z*P12SQ*P12SU*1, 4, /X) *(1, 4, X) 
FIN6 = FINT6* (4, Z*Z*P12SU*1, 4, /X) *(Z+Y) *(Z+X-y) 
FIN1 = FINT1+FINT2+FINT3+FINT4+FINT5+FINT6
RETURN
C CONTINUE
C NO. 8 (2A1-2A) (2A1-2A) 
C M = SUM (DELTA*DELTA) 
H = Z*Z*P12SQ*1, 4, /Y) 
HSQ = Z*Z*P12SU*1, 4, /X) 
FIN1 = Z*Z*P12SQ* (x-y) *(x-y) 
FIN2 = (1, 3, a, (x-y) *(x-y) *(x-y) *(x-y) 
FIN3 = (1, 3, a, (x-y) *(x-y) *(x-y) *(x-y) 
FIN4 = Z*Z*P12SQ*1, 4, /X) *(BSQ) 
FIN5 = Z*Z*P12SQ* (RSU) *(RSU) 
FIN6 = (5, Z*Z*P12SQ) *(1, 2, a, Z*Z*P12SU) 
FIN1 = FINT1+FINT2+FINT3+FINT4+FINT5+FINT6
RETURN
C CONTINUE
C NO. 7 (2A2-2A) 
C M = SUM (DELTA*DELTA) 
FX = (x-4, a, 4, Z*Z*P12SQ) 
FY = (y-4, a, 4, Z*Z*P12SQ) 
FIN1 = (x-y) *(x-y) *(x-y) *(x-y) 
FIN2 = (1, 7, a, (FX+FY) *(x-y) *(x-y) 
FIN3 = (1, 7, a, (FX+FY) *(x-y) *(x-y) 
FIN4 = (5, Z*Z*P12SQ) *(1, 7, a, Z*Z*P12SU) 
FIN1 = FINT1+FINT2+FINT3
RETURN
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CONTINUE

C M = SUM (DELTA E # DELTA E)  

C m = SUM (DELTA P # DELTA P)  

M = SUM (DELTA P # DELTA P)  

C B = SUM (DELTA P # DELTA P)  

C C = SUM (DELTA P # DELTA P)  


C FINT2 = (X - Y) * (X - Y) * (1.5) * (FX/FX/FX/Y)  

C FINT3 = (1.5) * (FX/FX/10.5*FX*FY/FY/10.5*FY*FY*10.5*FY*FY)  

C FINT4 = FINT1 + FINT2 + FINT3  

RETURN

CONTINUE

C M = SUM (DELTA P # DELTA P)  

C FINT1 = (3.5) * Z * Z * P12SQ * (X - Y) * (X - Y) * (X - Y)  

C FINT2 = -(X - Y) * (1.5) * FINT1 + (X - Y)  

C FINT3 = (1.5) * Z * Z * P12SQ * P12SQ * (1.5) * Z * Z * P12SQ  

C FINT4 = FINT1 + FINT2 + FINT3  

RETURN

CONTINUE

C M = SUM (DELTA P # DELTA P)  

C FINT1 = (2.5) * Z * Z * P12SQ * (X - Y) * (X - Y) * (X - Y)  

C FINT2 = -(2.5) * BPSQ * (Z * Z * P12SQ * (X - Y) * (X - Y) * (X - Y)  

C FINT3 = -(2.5) * BPSQ * (Z * Z * P12SQ * (X - Y) * (X - Y) * (X - Y)  

C FINT4 = FINT1 + FINT2 + FINT3  

RETURN

CONTINUE

C M = SUM (DELTA P # DELTA P)  

C FINT1 = 1.0  

C RETURN

END
SUBROUTINE GOLUNUM(MATNUM,N1,PD,LAB1,LAB2,IP,FACTOR)  
COMMON/LOCK/E*N2  
DIMENSION PD(N1,N1)  
COMMON/TRANSFER/X,Y,Q,S,A,P12SQ,ESQ,M,MSQ  
TYPE REAL MSQ,ESQ*INCR  
EXTERNAL EMSQ  
DATA (A=13958)  
DATA (ESQ=.0194A25764)  
N=N1-1  
PRINT 902*MATNUM,LAB1,LAB2,E*N2  
  FORMAT (*1NUMBER OF INTEGRATION INTERVAL*,I3,/* WELLER DESIGNATION *,A/* CENTER OF MASS ENERGY,*,F10.6,/* GEV,*,5X,PLOT/G*,/* RIJU NUMBER IS *,I3,/* NO. OF INTEGRATION INTERVAL,*)  
ESQ=ESQ*2  
INCR=(E=4.*M)/N  
DO 100 I=1,N1  
DO 100 J=1,N1  
IF ((I+J),GT,N1) GO TO 100  
IF (I,L), J) GO TO 100  
X=2.*2*(I-1)*INCR  
X=X**2  
Y=2.*2*(J-1)*INCR  
Y=Y**2  
P12SQ=.25*1/ESQ*2*(X-Y)**2*ESQ*(X+Y)+ESQ**2  
A=.5*(X-Y+ESQ+2.*MSQ)  
8SQ=ESQ*P12SQ*(I-4.*MSQ/Y)  
B=SQRTF(A)+SF(BSQ)  
AP=.5*(Y-X+ESQ+2.*MSQ)  
BPSQ=ESQ*P12SQ*(I-4.*MSQ/X)  
BP=SQRTF(AP+BPSQ)  
PD(I,J)=AP-BP  
QLO=A-B  
SLO=AP-BP  
DELTAQ=2.*B/N2  
DELTAS=2.*BP/N2  
C=(1./(SQRTF(AP+BPSQ)))  
DO 100 K=1,N2  
DO 100 L=1,N2  
FK=K $  
FL=L  
Q=(FK-5.)*DELTAQ+QLO  
S=(FL-5.)*DELTAS+SLO  
P(I,J)=PD(J,I)=PD(I,J)+C*EMSQ(MATNUM,E)*DELTAQ*DEL  
CSTOP  
100 CONTINUE  
CALL PRINTER(N1,PD,FACTOR)  
PRINT 130,MATNUM
FORMAT(*1NORMALIZED DISTRIBUTION FOR NO.*,I3)
SUM=0.
DO 135 I=1,N1
DO 135 J=1,N1
IF (I+J=N-2) 131,131,135
131 SUM=SUM+PD(I,J)
135 CONTINUE
FACTOR=1000./SUM
DO 140 I=1,N1
DO 140 J=1,N1
IF (I+J=N-2) 136,136,140
136 PD(I,J)=PD(I,J)*FACTOR
140 CONTINUE
CALL PRINTER(N1,PD,FACTOR)
IF (KPUNCH.EQ.0) GO TO 117
PUNCH 909,*MATNUM,FACTOR,LAB1,LAB2
909 FORMAT(* NUM RESULTS NO.*,I3,* MULT BY *,F10.5,*HEW,*,F10.5)
$* WELLER *,AH, * WELLER/DOTSON *,AH)
PUNCH 919
919 FORMAT(*BOX)
NLC=1
NRC=1
113 DO 114 J=NLC,N1
114 JR=N+J-J
PUNCH 115, (PD(IR,JR), IR=NLC,NRC)
115 PUNCH 919
116 NLC=NLC+1
IF (NRC=N) 116,117,117
NRC=NRC+1
117 CONTINUE
F(N)
FUNCTION EMSQ(MATNUM, E)
COMMON/TRANSFER/X, Y, Q, S, A, P12SQ, ESQ, MSQ
TYPE REAL M, MSQ
IF(MATNUM .EQ. 14) GO TO 14
IF(MATNUM .EQ. 9 .OR. MATNUM .EQ. 13 ) GO TO 200
RECE=1./(2.*E)
E1=RECE*(S+X-Y-MSQ)
E2=RECE*(ESQ-MSQ-5)
E3=RECE*(ESQ+MSQ-5)
E4=RECE*(Y-X-MSQ+Q)
GO TO (300,300,300,200,200,200,300,200,200,200)
$300*200*200*300*300*300 MATNUM
CONTINUE
FRONT=25./ESQ
U=(ESQ+MSQ)*(ESQ+X-Y)-2.*ESQ*X
R=(ESQ+MSQ)*(ESQ-Y)-(ESQ-MSQ)*Y
P12P2=FRONT*(U-S*(ESQ+X-Y))
P12P3=FRONT*((X-Y-ESQ)*(A-Q)-2.*ESQ*P12SQ)
P2P2=FRONT*((ESQ-MSQ)**2-2.*S*(ESQ+MSQ)*S)**2)
P3P3=FRONT*((ESQ-MSQ)**2-2.*Q*(ESQ+MSQ)*Q)**2)
P2P3=1./(16.*P12SQ**4)*(S*(Y-X-ESQ)+U)*(Q*(X-Y-ESQ)+S)
CONTINUE
GO TO (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17)
$MATNUM
1 EMSQ=E1*E2*E3*E4
C WELLER 1 WELLER/DOTSON 10+B
C M=SQRT(E(1)**2+E(2)**2+E(3)**2+E(4)**2)
RETURN
2 CONTINUE
C WELLER 2 WELLER/DOTSON 10+C
C M=SUM(E(I)**2)
EMSQ=E1**2+E2**2+E3**2+E4**2
EMSQ=EMSQ**2
RETURN
3 CONTINUE
C WELLER 2A WELLER/DOTSON 10+D
C M=SUM(P(I)**2)
EMSQ=E1**2+E2**2+E3**2+E4**2-4.*MSQ
EMSQ=EMSQ**2
RETURN
4 CONTINUE
C WELLER 3 WELLER/DOTSON 11+A
C M=SUM(E(I)**P(I))
A1=(E1-E4) $ A2=(E2-E1) $ A3=(E3-E4)
GO TO b
E=E1=E1*#2
L2=E2**2
£3=E3*#2
r4=EA**2

C
^EELER d A
aELLER/DOTSON 2B0+B

MsSUM(6ELr/(E#*2)*OELTfl(F**2) )

RETURN

EMS=PI2SO*A1*2 +P2P2*A2*2 +P3P3*A3*2
+ 2.*P1P2P3*A1*A2*A3

RETURN

E=

End

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SUBROUTINE PROJECTR(JM, METHOD, N1, PD, LAB1, LAB2, FACTOR)

DIMENSION SUM(100)
DIMENSION LMTH(3)
DATA(LMTH=8HANALYTIC, 8HNUMERIC, 8HREADER)
DIMENSION PD(N1, N1)
N=N1 - 1
DO 29 J=1, N1
DO 100 I=1, N1
29 SUM(J)=0.
DO 100 J=1, N1
IF(I+J-N=2) 35, 35, 100
35 SUM(J)=SUM(J)*PD(I, J)*FACTOR
100 CONTINUE
PRINT 125
125 FORMAT(1H1)
PRINT 126, LAB1, LAB2, LMTH(METHOD)
126 FORMAT(* PROJECTION OF WELLER **AR*
WELLER/DOOTSON **AB** GENERATE METHOD, **AB**/*)
DO 150 J=1, N1
PRINT 160, J, SUM(J)
160 FORMAT(* SUM(*, I2,* )=*F10.5)
150 CONTINUE
SUBROUTINE PRINTER(N1,PD,PWR)
DIMENSION PD(N1,N1)
N=N1-1
BIGA=PD(1,1)
DO 102 I=1,N1
    DO 102 J=1,N1
        IF (1+J-N-2) 302,302,102
302 IF (BIGA-PD(I,J)) 101,102,102
101 BIGA=PD(I,J)
102 CONTINUE
PWR=0.000001
103 P*P=P*P*10.
    IF (BIGA*P*P) 104,104,103
104 PWR=P*P/10.
    PRINT 105,PWR
105 FORMAT(/** ARRAY DIVIDED BY**E15.8/)
    DO 106 I=1,N1
    DO 106 J=1,N1
        PD(I,J)=PD(I,J)/PWR
    NLC=1
    NRC=15
107 IF (NRC .GT. N1) NRC=N1
    DO 104 J=NLC,N1
    JR=N+2-J
108 PRINT 109,(PD(1R,JR),1R=NLC,NRC)
109 FORMAT(15F8.4)
    IF (NRC .GE. N1) GO TO 112
110 NLC=NLC+15
    NRC=NRC+15
    GO TO 107
112 CONTINUE
END
SUBROUTINE PLOTTER(JM,METHOD,N1,P0,LAB1,LAB2,FACTOR)

DIMENSION LMTH(3),PD(N1,N1)
DIMENSION XS(31),YS(31),YST(130),C(35),O(35),XST(130)

DATA(CONV=.01735278)
DATA(LMTH=8HANALYTIC,6HNUMERIC,dHR EADER)
DATA(XQ=.5.)
DATA(YO=2.)
DATA(EDGE=.14675)
DATA(THETA=1.0.)
DATA(NMULT=4)
N=N1-1
SCALE=.019676
SCALE=SCALE*FACTOR
CALL PLOT(0.,0.,2.)
ECOSPL=EDGE*COSF((30.*THETA)*CONV)
ECOSMN=EDGE*COSF((30.*THETA)*CONV)
ESINPL=EDGE*SINF((30.*THETA)*CONV)
ESINMN=EDGE*SINF((30.*THETA)*CONV)
DO 100 I=1,N1
100 IF(I+J=N+2)35,35,100
35 X=X0+(I-1)*ECOSPL-(J-1)*ECOSMN
Y=Y0+(I-1)*ESINPL+(J-1)*ESINMN
CALL PLOT(X*Y,3)
Y=Y+PD(I+J)*SCALE
CALL PLOT(X*Y,2)
DO 200 I=1,N1
200 JSTOP=N1-I+1
NSPL=(N1-1)*NMULT+1
DO 120 J=1,JSTOP
120 XST(J)=J-D
YST(J)=PD(I,J)
K=1
DO 140 JF=1,NSPL
XST(JF)=(JF-1)/NMULT
YST(JF)=SPLINEI(XS,YS,XST(JF),JSTOP,C,D,K)
140 K=2
X=X0+(I-1)*ECOSPL
Y=Y0+(I-1)*ESINPL
CALL PLOT(X*Y,3)
DO 160 JF=1,NSPL
X=X0+(I-1)*ECOSPL-XST(JF)*ECOSMN
Y=Y0+(I-1)*ESINPL+XST(JF)*ESINMN+YST(JF)*SCALE
160 K=2
CALL PLOT(X*Y,2)
X=X0-(I-1)*ECOSMN
Y=Y0-(I-1)*ESINMN
CALL PLOT(X*Y,3)

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DO 180 JF=1,NSPL
X=X0 - (I-1)*ECOSMN * XST(JF)*ECOSPL
Y=Y0 + (I-1)*ESINMN + XST(JF)*ESINPL + YST(JF)*SCALE
IF
180 CALL PLOT(X,Y,2)

200 CONTINUE
X=X0 - N*ECOSMN
Y=Y0 + N*ESINMN + S*EDGE
H=S*EDGE
IF(H.LT.0.7)H=0.7
CALL SYMBOL(X,Y,H,LAB1,0.,H)
Y=Y - EDGE
CALL SYMBOL(X,Y,H,LAB2,0.,H)
CALL SYMBOL(X,Y,H,LMTH(METH0),0.,H)
XOP=X0 + 10.*ECOSMN
YOP=Y0 -10.*ESINMN
YS=YMP + 2.
CALL PLOT(XOP,YS,3)
CALL PLOT(XUP,YUP,2)
XF=XOP + N*ECOSPL
YF=YOP + N*ESINPL
CALL PLOT(XF,YF,2)

DJ 100 I=1,N1
SUM=*
DO 210 J=1,N1
IF(1+J-I-2)200,208,210
200 CONTINUE
X=XOP + (I-1)*ECOSPL
Y=YOP + (I-1)*ESINPL
CALL SYMBOL(X,Y,14,0.,-1)
Y=Y+SUM*SCALE*.5
300 CONTINUE
CALL SYMBOL(X,Y,14,0.,-1)
Y=0.
X=X0 + 20.*EDGE*1.3
CALL PLOT(X,Y,-3)
RETURN
END
SUBROUTINE CORCU(PD,NM1,NMAT,LAB1,LAB2,METHOD)
DIMENSION PD(NM1,NM1,25)
DIMENSION LAB1(NMAT),LAB2(NMAT),METHOD(NMAT)
DIMENSION SUMA(25),SUMAA(25),SUMAB(25,25),CC(25,25)
DIMENSION LMTH(3)

DATA(LMTH=BMANALYT,NMNUMER,8HREADER)
DIMENSION MST(25)
N=M1-1
EN1=M1*(N+1)/2
PRINT 800,EN1

100 FORMAT(6H,EN1=,E12.5)
DO 50 J=1,NMAT
MST(J)=LMTH(METHOD(J))
DO 60 LA=1,NMAT
SUM(N)=0.0
SUMAA(LA)=0.0
DO 60 LB=1,NMAT
SUMAB(LA,LA)=0.0
DO 100 I=1,NM1
DO 100 J=1,NM1
IF (I+J<=2) 60 65 100
65 SUMA(LA)=SUMA(LA)+PD(I,J)*LA
DO 100 LA=1,NMAT
50 SUMA(LA)=SUMA(LA)+PD(I,J)*LA
SUMAA(LA)=SUMAA(LA)+PD(I,J)*LA
DO 70 LB=1,NMAT
SUMAB(LA,LA)=SUMAB(LA,LA)+PD(I,J,LA)*PD(I,J,LA)
70 SUMAB(LA,LA)=SUMAB(LA,LA)+PD(I,J,LA)*PD(I,J,LA)
CONTINUE
100 CONTINUE
DO 120 LA=1,NMAT
DO 120 LB=1,NMAT
CC(LA,LB)=0.0
120 TEMPI=SUMAB(LA,LB)-SUMAA(LA)*SUMAB(LB)/EN1
TEMP2=SUMAA(LA)-SUMAA(LA)*SUMAA(LA)/EN1
TEMP3=SUMAA(LA)+SUMAB(LA,LB)/EN1
TEMP4=SQRT(TEMP2*TEMP3)
CC(LA,LB)=TEMPI/TEMP4
CONTINUE
120 CONTINUE
DO 150 IDUMY=1,3
PRINT 131
131 FORMAT(12F0.4)
JLC=1
JRC=14
IF (NM1.LT.JRC) JRC=NM1
PRINT 132,(IDUMY,IDUMY=JLC,JRC)
PRINT 133,(LAB1(I),I=JLC,JRC)
PRINT 133,(LAB2(I),I=JLC,JRC)
PRINT 134,(CC(I,J),I=JLC,JRC)
134 FORMAT(5X,14(E15.8))
```
PRINT 134, (MST(I), I=JLC, JRC)
DO 140 J=1, NMAT
PRINT 134
140 PRINT 137, (J*(CC(I,J), I=JLC, JRC))
137 FORMAT (1X, 12*2X, 14*(F7.4, 1X))
138 FORMAT (1X)
139 IF (JRC .GE. NMAT) GO TO 150
JLC=JLC + 14
JRC=JRC + 14
GO TO 124
150 CONTINUE
END
```
On the following page an actual sample of the Calcomp plotter output from a typical run of the computer program is given. The labels "2A" and "10+,D" are Hollerith analogues of the two labels mentioned earlier in this appendix. The symbol "READER" originates in subroutine Genratr, and indicates that the data for this plot was input from the card reader. The other two possibilities for this label would be "ANALYTIC" and "NUMERIC", with obvious meanings.

The rest of the graph is self-explanatory, except for the fact that the projected distribution is plotted at half scale, relative to the scale of the Goldhaber triangle.
APPENDIX B

We explain briefly how the skewed isometric drawings are made. The usual isometric drawing of a cube looks like this.

If one makes an isometric drawing of a transparent cube the result is confusing because of the superposition of successive vertical lines.

This problem can be avoided by skewing the drawing as shown below.

The skew angle in the above example is ten degrees. All the graphs of Goldhaber distributions in the present work are made with this same skew angle.

The effect resembles that achieved by the use of perspective drawing but retains the advantage of the isometric drawing in that all lines are actual scale length.
APPENDIX C

All of the following are easy to show, and the proofs will not be given.

\[ \int_{q^-}^{q^+} dq = 2B = \Delta q \]
\[ \int_{q^-}^{q+} dq = 2AB = \Delta q \]

\[ \int_{q^-}^{q+} q^2 \ dq = \Delta q (A^2 + \frac{1}{3} B^2) \]
\[ \int_{q^-}^{q+} q^3 \ dq = \Delta q A (A^2 + B^2) \]

\[ \int_{q^-}^{q+} q^4 \ dq = \Delta q (A^4 + 2A^2B^2 + \frac{1}{5} B^4) \]

\[ \int_{q^-}^{q+} (A - q)^2 (F + F_1 q + F_2 q^2) \ dq = \Delta q \left[ \frac{1}{3} B^2 (F + F_1 A + F_2 A^2) + \frac{1}{5} F_2 B^4 \right] \]

\[ \int_{q^-}^{q+} (A - q) (F + F_1 q + F_2 q^2 + F_3 q^3) \ dq = \Delta q \left[ - \frac{1}{3} B^2 (F_1 + 2AF_2 + 3A^2F_3) - \frac{1}{5} F_3 B^4 \right] \]

In the last two expressions the \( F_i \) are independent of \( q \). Expressions analogous to those given for \( q \) hold for \( s \) also. They may be obtained from the \( q \) expressions by priming \( A \) and \( B \) everywhere and replacing \( q \) with \( s \) everywhere. The following identities are also useful.

\( (x - y - E^2)A + r = 2E^2p_{12} \)
\( (y - x - E^2)A' + u = 2E^2p_{12} \)

\[ [(E^2 - m^2)^2 - 2A'(E^2 + m^2) + A'^2] = E^2(p_{12}^2 + x - 4m^2) \]

\[ [(E^2 - m^2)^2 - 2A(E^2 + m^2) + A^2] = E^2(p_{12}^2 + y - 4m^2) \]
BIBLIOGRAPHY


