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Highly Hamiltonian Graphs and Digraphs

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by

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A cycle that contains every vertex of a graph or digraph is a Hamiltonian cycle. A graph or digraph containing such a cycle is itself called Hamiltonian. This concept is named for the famous Irish physicist and mathematician Sir William Rowan Hamilton. These graphs and digraphs have been the subject of study for over six decades. In this dissertation, we study graphs and digraphs with even stronger Hamiltonian properties, namely highly Hamiltonian graphs and digraphs.

A Hamiltonian graph G of order $n \geq 3$ is k-path Hamiltonian for some positive integer k if for every path P of order k, there exists a Hamiltonian cycle C of G such that P is a path on C. A graph G of order $n \geq 3$ is pancyclic if G contains a cycle of length ℓ for each integer ℓ with $3 \leq \ell \leq n$. These two concepts have been studied extensively. For integers k and n with $2 \leq k \leq n-1$, a graph G of order n is k-path pancyclic if every path P of order k in G lies on a cycle of every length from $k+1$ to n. We present sufficient conditions for graphs to be k-path pancyclic. For a graph G of order $n \geq 3$, we establish sharp lower bounds in terms of n and k for (a) the minimum degree of G, (b) the minimum degree-sum of nonadjacent vertices of G and (c) the size of G such that G is k-path pancyclic

A graph G of order $n \geq 2$ is panconnected if for every two vertices u and v, there is a $u - v$ path of length ℓ for every integer ℓ with $d(u, v) \leq \ell \leq n-1$. For two vertices u and v in a connected graph G, a $u-v$ geodesic is a shortest $u-v$ path in G. A graph G of order n is geodesic-pancyclic if for each pair u, v of vertices of G, every $u-v$ geodesic lies on a cycle of length k for every k with $\max\{2d(u, v), 3\} \leq k \leq n$. For a nontrivial graph G that is not complete, let $\sigma_2(G)$ denote the minimum sum of the degrees of two nonadjacent vertices in G. It is known that if G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-2}{2}$, then G is both panconnected and geodesic-pancyclic. We present improved lower bounds for panconnected and geodesic-pancyclic.

A Hamiltonian digraph D of order $n \geq 3$ is ℓ-path Hamiltonian for some positive integer ℓ with $1 \leq \ell \leq n$ if every (directed) path of order ℓ lies on a (directed) Hamiltonian cycle of D. The Hamiltonian extension number $\mathrm{he}(D)$ of D is the greatest positive integer ℓ such that D is j-path Hamiltonian for every integer j with $1 \leq j \leq \ell$. For a Hamiltonian oriented D graph of order $n \geq 4$, it is shown that (1) if D is $(n-2)$-path Hamiltonian, then D is both $(n-1)$-path Hamiltonian and n-path Hamiltonian and (2) if D is not n-path Hamiltonian, then D is neither $(n-1)$-path Hamiltonian nor $(n-2)$-path Hamiltonian. We also study connections between path Hamiltonian oriented graphs and their underlying graphs. Furthermore, we study path Hamiltonian tournaments, including regular tournaments and almost regular tournaments. For a Hamiltonian graph G, let $\mathcal{H}(G) = \{D : D \text{ is a Hamiltonian orientation of } G\}$. The upper Hamiltonian extension number $\mathrm{he}^+(G)$ of G is the maximum value of $\mathrm{he}(D)$ among all $D \in \mathcal{H}(G)$ and the lower Hamiltonian extension number $\mathrm{he}^-(G)$ is the minimum such value. These two parameters are determined for some classes of Hamiltonian graphs. Results, conjectures and open questions are presented.

A rainbow coloring of a connected graph G is an edge coloring c of G (where adjacent edges may be colored the same) with the property that for every two vertices u and v of G, there exists a $u - v$ rainbow path (no two edges of the path are colored the same). In this case, G is said to be rainbow-connected (with respect to c). The minimum number of colors needed for a rainbow coloring of G is referred to as the rainbow connection number of G. This concept has been studied extensively in the past decade.

A path that contains every vertex of a graph is a Hamiltonian path. A graph G is Hamiltonian-connected if every pair of vertices of G are connected by a Hamiltonian path. A graph is edge-colored if each of its edges is assigned a color (where adjacent edges can be assigned the same color). A path P in an edge-colored graph is a rainbow path if no two edges of P are colored the same. An edge coloring of a Hamiltonian-connected graph G is a Hamiltonian-connected rainbow coloring if every two vertices of G are connected by a rainbow Hamiltonian path. The minimum number of colors required of a Hamiltonian-connected rainbow coloring of G is the rainbow Hamiltonian-connection number $\mathrm{hrc}(G)$ of G. If G has order n and size m, then $n-1 \leq \mathrm{hrc}(G) \leq m$. The rainbow Hamiltonian-connection number is investigated for the Cartesian product of complete graphs and of odd cycles with K_2. As a result of this, both the lower bound $n-1$ and the upper bound m for $\mathrm{hrc}(G)$ are shown to be sharp. We also study Hamiltonian-connected rainbow colorings of the powers of connected graphs and two classes of Hamiltonian-connected graphs having the minimum possible size.

A (directed) path that contains every vertex of a digraph is a Hamiltonian path. A nontrivial digraph D is Hamiltonian-connected if for every pair u, v of distinct vertices of D, there exists both a Hamiltonian $u - v$ path and a Hamiltonian $v - u$ path. For a nontrivial Hamiltonian-connected digraph D, an arc coloring of D is called a Hamiltonian-connected rainbow coloring if for every pair u, v of distinct vertices of D, there is both a rainbow Hamiltonian $u - v$ path and a rainbow Hamiltonian $v - u$ path. The minimum number of colors required of a Hamiltonian-connected rainbow coloring of D is the rainbow Hamiltonian-connection number of D, denoted by $\mathrm{hrc}(D)$. It is shown that (1) if D is a nontrivial Hamiltonian-connected digraph of order n, then $\mathrm{hrc}(D) = n - 1$ or $\mathrm{hrc}(D) = n$ and (2) if G^* is the symmetric digraph of G, then $\mathrm{hrc}(G^*) \leq \mathrm{hrc}(G)$. Consequently, if G is a Hamiltonian-connected digraph $n \geq 4$ and $\mathrm{hrc}(G) = n - 1$, then $\mathrm{hrc}(G^*) = n - 1$. Furthermore, there exist Hamiltonian-connected digraphs D of order n with $\mathrm{hrc}(D) = n$.

An edge coloring of a graph G is proper if every two adjacent edges of G have different colors and the minimum number of colors in a proper coloring of G is the chromatic index of G, denoted by $\chi'(G)$. Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is a proper path in G if no two adjacent edges of P are colored the same. An edge coloring c is a proper-path coloring of a connected graph G if every pair u, v of distinct vertices of G are connected by a proper $u - v$ path in G. The minimum number of colors in a proper-path coloring of G the proper connection number of G. Recently, this topic has been studied by many.

An edge coloring of a Hamiltonian-connected graph G is a proper Hamiltonian-path coloring if every two vertices of G are connected by a properly colored Hamiltonian path. The minimum number of colors in a proper Hamiltonian-path coloring of G is the proper Hamiltonian-connection number of G. Proper Hamiltonian-connection numbers are determined for several classes of Hamiltonian-connected graphs and two classes of Hamiltonian-connected graphs of minimum size. In particular, it is shown that $\mathrm{hpc}(K_n) = 2$ for $n \geq 4$ and $\mathrm{hpc}(C \,\square\, K_2) = 3$ for all prisms $C \,\square\, K_2$, where C is an odd cycle.

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Zhenming Bi

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Chapter 1

Introduction

1.1 Basic Definitions and Notation

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is a *traceable graph*. A *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G and a graph having a Hamiltonian cycle is a *Hamiltonian graph*. A graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u-v$ path for every pair u, v of distinct vertices of G. Among many sufficient conditions for a graph G to be traceable, Hamiltonian or Hamiltonian-connected are those concerning the minimum of the degree sums of two nonadjacent vertices in G and are those concerning the size of G. For a nontrivial graph G that is not complete, let

$$\sigma_2(G) = \min\{\deg u + \deg v : uv \notin E(G)\}.$$

For a connected graph G, let $\mathrm{diam}(G)$ denote the diameter of G (the largest distance between two vertices of G). It is known that if G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n-1$, then G is connected and $\mathrm{diam}(G) \leq 2$. In fact, more can be said. The following result is well known (see [16, p. 152], for example).

Theorem 1.1.1 *If G is a graph of order $n \geq 2$ such that $\sigma_2(G) \geq n-1$, then G is traceable.*

The following two results are due to Ore, the first of which was obtained in 1960 [42] and the second in 1963 [43].

Theorem 1.1.2 (Ore) *If G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then G is Hamiltonian.*

Theorem 1.1.3 (Ore) *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$, then G is Hamiltonian-connected.*

Each of Theorems 1.1.1–1.1.3 has a corollary providing a lower bound on the minimum degree $\delta(G)$ for a graph G to possess the respective property.

Corollary 1.1.4 *If G is a graph of order $n \geq 2$ with $\delta(G) \geq (n-1)/2$, then G is traceable.*

The following corollary is the first theoretical result on Hamiltonian graphs. This result occurred in 1952 and is due to Gabriel Dirac [17].

Corollary 1.1.5 (Dirac) *If G is a graph of order $n \geq 3$ with $\delta(G) \geq n/2$, then G is Hamiltonian.*

Corollary 1.1.6 *If G is a graph of order $n \geq 4$ with $\delta(G) \geq (n+1)/2$, then G is Hamiltonian-connected.*

All bounds stated in Corollaries 1.1.4, 1.1.5 and 1.1.6 are best possible for, in each case, if the respective bound is reduced by $\frac{1}{2}$, then the statement is no longer true.

The following results give other sufficient conditions for a graph to be traceable, Hamiltonian (see [14, p. 136]) or Hamiltonian-connected (see [43]), respectively.

Theorem 1.1.7 *If G is a graph of order $n \geq 3$ and size $m \geq \binom{n-1}{2} + 1$, then G is traceable.*

Theorem 1.1.8 *If G is a graph of order $n \geq 3$ and size $m \geq \binom{n-1}{2} + 2$, then G is Hamiltonian.*

Theorem 1.1.9 (Ore) *If G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2} + 3$, then G is Hamiltonian-connected.*

Some 40–50 years ago, there was a great deal of research activity involving Hamiltonian properties of powers of graphs. For a connected graph G and a positive integer k, the *kth power* G^k of G is the graph whose vertex set is $V(G)$ such that uv is an edge of G^k if $1 \leq d_G(u, v) \leq k$ where $d_G(u, v)$ is the distance between two vertices u and v in G (or the length of a shortest $u - v$ path in G). The graph G^2 is called the *square* of G and G^3 the *cube* of G. In 1960, Sekanina [55] proved that the cube of every connected graph G is Hamiltonian-connected and, consequently, the cube of a connected graph G is Hamiltonian if its order is at least 3. In the 1960s, it was conjectured independently by Nash-Williams [39] and Plummer (see [14, p. 139]) that the square of every 2-connected graph is Hamiltonian. In 1974, Fleischner [20] verified this conjecture. Also, in 1974 and using Fleischner's result, Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [8] proved that the square of every 2-connected graph is Hamiltonian-connected. Thus, the square of every Hamiltonian graph is Hamiltonian-connected.

1.2 Panconnected and Pancyclic Graphs

A graph G of order n is *panconnected* if for every pair u, v of distinct vertices of G, there is a $u - v$ path of length k for every integer k with $d(u, v) \leq k \leq n - 1$. This concept was introduced in the doctoral dissertation [47] of Williamson. It is shown in [1] that if G is a connected graph, then the cube of G is panconnected. The following result of Williamson [48] was obtained in 1977.

Theorem 1.2.1 (Williamson) *If G is a graph of order $n \geq 4$ with $\delta(G) \geq (n+2)/2$, then G is panconnected.*

This bound too is sharp since the statement is no longer true if $(n+2)/2$ is replaced by $(n+1)/2$ [48]. Furthermore, Williamson showed that if G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+2$, then G need not be panconnected. For example, for the graph G of order 8 shown in Figure 1.1, $\sigma_2(G) = 10 = n + 2$ but G is not panconnected since, for example, $d(u, v) = 1$ but G contains no $u - v$ path of length 2 (see [7]).

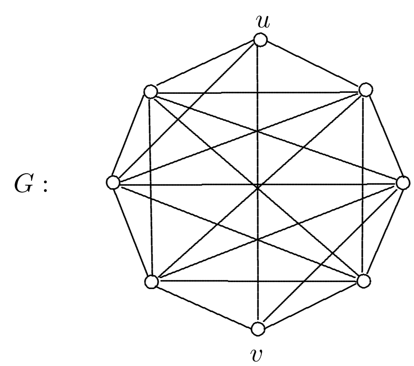


Figure 1.1: A graph G of order n with $\sigma_2(G) = n + 2$
such that G is not panconnected

Williamson [48] not only showed that $\sigma_2(G) \geq n + 2$ does not imply that a graph G of order $n \geq 4$ is panconnected but also $\sigma_2(G) \geq n + c$ does not imply that a graph G of order n is panconnected for any constant c. He did prove the following, however.

Theorem 1.2.2 (Williamson) *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-2}{2}$, then G is panconnected.*

Williamson showed that if n is even, then the lower bound for $\sigma_2(G)$ in Theorem 1.2.2 is best possible, that is, $\frac{3n-2}{2}$ cannot be replaced by $\frac{3n-4}{2}$. Thus, Williamson showed that if the lower bound in Theorem 1.2.2 is reduced by 1, then the result no longer holds. In

Chapter 2, it will be shown that the lower bound $\frac{3n-2}{2}$ in Theorem 1.2.2 can be replaced by $\frac{3n-3}{2}$ when n is odd.

A graph G of order $n \geq 3$ is *pancyclic* if G contains a cycle of every possible length, that is, G contains a cycle of length ℓ for each integer ℓ with $3 \leq \ell \leq n$. The following result was obtained by Bondy [52] in 1971.

Theorem 1.2.3 (Bondy) *If G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then either G is pancyclic or n is even and $G = K_{\frac{n}{2},\frac{n}{2}}$.*

For two vertices u and v in a connected graph G, a $u - v$ *geodesic* is a $u - v$ path of length $d(u, v)$ in G. A graph G of order n is defined in [6] to be *geodesic-pancyclic* if for each pair u, v of vertices of G, every $u - v$ geodesic lies on a cycle of length k for every k with $\max\{2d(u, v), 3\} \leq k \leq n$. In particular, a geodesic-pancyclic graph is *edge-pancyclic*, that is, every edge of G lies on a cycle of each of the lengths $3, 4, \ldots, n$. The following results are due to Chan, Chang, Wang and Horng [6].

Theorem 1.2.4 (Chan, Chang, Wang and Horng) *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-2}{2}$, then G is geodesic-pancyclic.*

Here too, it is known that if n is even, then the lower bound for $\sigma_2(G)$ in Theorem 1.2.4 is best possible, that is, $\frac{3n-2}{2}$ cannot be replaced by $\frac{3n-4}{2}$. In Chapter 2, it will be shown that the lower bound $\frac{3n-2}{2}$ in Theorem 1.2.4 can be replaced by $\frac{3n-3}{2}$ when n is odd.

For a connected graph G of order $n \geq 4$ and an integer k with $1 \leq k \leq n - 3$, the graph G is *k-Hamiltonian* if $G - S$ is Hamiltonian for every set S of k vertices of G and *k-Hamiltonian-connected* if $G - S$ is Hamiltonian-connected for every set S of k vertices of G. If the order of a connected graph G is at least 4, then Chartrand and Kapoor [9] showed that the cube of G is 1-Hamiltonian. Over the years, many other highly Hamiltonian concepts have also been introduced and studied (see [22, 25, 26, 27, 51] for example).

1.3 Path Hamiltonian Graphs

The concepts of Hamiltonian cycles, Hamiltonian paths and Hamiltonian graphs are, of course, named for the famous Irish physicist and mathematician Sir William Rowan Hamilton. Hamilton observed that every path of order 5 on the graph G of the dodecahedron can be extended to a Hamiltonian cycle of G. That is, for every path P of order 5 in G, there exists a Hamiltonian cycle C of G such that P is a path on C.

What Hamilton observed for paths of order 5 on the graph of the dodecahedron does not hold for all paths of order 6 as is illustrated in Figure 1.2 since the path of order 6 (drawn with bold edges) cannot be extended to a Hamiltonian cycle on the graph of the dodecahedron. This led to a concept defined in [54] for all Hamiltonian graphs.

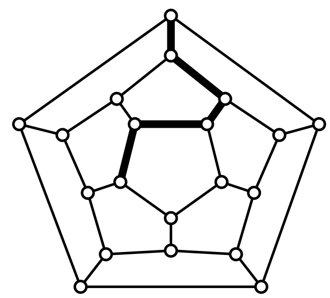


Figure 1.2: The graph G of the dodecahedron

A Hamiltonian graph G of order $n \geq 3$ is *k-path Hamiltonian*, $k \geq 1$, if for every path P of order k, there exists a Hamiltonian cycle C of G such that P is a path on C. The *Hamiltonian cycle extension number* $\mathrm{hce}(G)$ of G is the largest integer k such that G is k-path Hamiltonian. So $1 \leq \mathrm{hce}(G) \leq n$. Therefore, if $\mathrm{hce}(G) = k$, then G is a Hamiltonian graph such that

(1) for every path P of order k, there is a Hamiltonian cycle of G containing P as a subgraph;

(2) for $k \leq n-1$, there is some path Q of order $k+1$ for which there is no Hamiltonian cycle of G containing Q as a subgraph.

Among the results obtained in [54] are the following.

Theorem 1.3.1 (Chartrand, Fujie and Zhang) *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then*

$$\mathrm{hce}(G) \geq 2\delta(G) - n + 1.$$

The lower bound in Theorem 1.3.1 is sharp.

Theorem 1.3.2 (Chartrand, Fujie and Zhang) *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq rn$ for some rational number r with $1/2 \leq r < 1$, then*

$$\mathrm{hce}(G) \geq (2r-1)n + 1.$$

The lower bound presented in Theorem 1.3.2 for the Hamiltonian cycle extension number of a graph is sharp for every rational number r. The following two theorems are known (see [54], for example).

Theorem 1.3.3 *Let k and n be positive integers such that $n \geq k+2$. If G is a graph of order n such that $\sigma_2(G) \geq n+k-1$, then G is k-path Hamiltonian.*

Theorem 1.3.4 *Let k and n be positive integers such that $n \geq k+2$. If G is a graph of order n and size $m \geq \binom{n-1}{2} + k + 1$, then G is k-path Hamiltonian.*

Again, the lower bounds in both Theorems 1.3.3 and 1.3.4 are best possible for every positive integer k.

1.4 Hamiltonian Digraphs

For a vertex v in a digraph D, the *outdegree* $\operatorname{od} v$ of v is the number of vertices of D to which v is adjacent, while the *indegree* $\operatorname{id} v$ of v is the number of vertices of D from which v is adjacent. The *out-neighborhood* $N^+(v)$ of a vertex v in a digraph D is the set of vertices adjacent from v, while the *in-neighborhood* $N^-(v)$ of v is the set of vertices adjacent to v. Thus, $\operatorname{od} v = |N^+(v)|$ and $\operatorname{id} v = |N^-(v)|$. The *degree* $\deg v$ of a vertex v is defined by $\deg v = \operatorname{od} v + \operatorname{id} v$.

A digraph D is *symmetric* if whenever (u,v) is an arc of D, then (v,u) is an arc of D as well. A digraph D is called an *oriented graph* if whenever (u,v) is an arc of D, then (v,u) is not an arc of D. Thus, an oriented graph D can be obtained from a graph G by assigning a direction to each edge of G. In this case then, the digraph D is called an *orientation* of G. A *tournament* is an orientation of a complete graph.

When discussing digraphs, a path always refers to a directed path and a cycle always refers to a directed cycle.

The *underlying graph* of a digraph D is the graph obtained by replacing each arc (u,v) or symmetric pair (u,v), (v,u) of arcs by the edge uv. A digraph D is *connected* (or *weakly connected*) if the underlying graph of D is a connected graph. A digraph D is *strong* or *strongly connected* if for every pair u, v of vertices, D contains both a $u-v$ path and a $v-u$ path. A digraph D of order at least 3 is *Hamiltonian* if D contains a spanning cycle C. Such a cycle C is then called a *Hamiltonian cycle* of D. Thus, every Hamiltonian digraph is strong. While every Hamiltonian digraph is strong, the converse is not true in general. However, this is not the case for tournaments. The following result is due to Camion [5].

Theorem 1.4.1 (Camion) *A nontrivial tournament T is Hamiltonian if and only if T is strong.*

For a nontrivial digraph D, let $\sigma_2(D)$ denote the minimum value of the degree sums of two nonadjacent vertices in D, namely

$$\sigma_2(D) = \min\{\deg u + \deg v : (u, v), (v, u) \notin E(D)\}.$$

The following result of Meyniel [35] gives a sufficient condition for a digraph to be Hamiltonian.

Theorem 1.4.2 (Meyniel) *If D is a nontrivial strong digraph of order n such that $\sigma_2(D) \geq 2n - 1$, then D is Hamiltonian.*

The following are some known results on Hamiltonian digraphs (see [24, 49]).

Theorem 1.4.3 (Woodall) *If D is a nontrivial digraph of order n such that* $\operatorname{od} u + \operatorname{id} v \geq n$ *whenever u and v are distinct vertices with $(u, v) \notin E(D)$, then D is Hamiltonian.*

Theorem 1.4.4 (Ghouila-Houri) *If D is a strong digraph of order n such that* $\deg v \geq n$ *for every vertex v of D, then D is Hamiltonian.*

The following is a consequence of Theorem 1.4.4.

Corollary 1.4.5 *If D is a digraph of order n such that* $\operatorname{od} v \geq n/2$ *and* $\operatorname{id} v \geq n/2$ *for every vertex v of D, then D is Hamiltonian.*

The following 1934 result of Rédei [40] is the first theoretical result on tournaments. A path in a digraph D containing every vertex of D is a *Hamiltonian path*.

Theorem 1.4.6 (Rédei) *Every tournament contains a Hamiltonian path.*

A tournament T is *transitive* if whenever (u, v) and (v, w) are arcs of T, then (u, w) is also an arc of T. It is known that (1) a tournament is transitive if and only if it contains no cycles, (2) for every positive integer n, there is exactly one transitive tournament of order n and (3) every transitive tournament contains exactly one Hamiltonian path.

A digraph D of order $n \geq 3$ is *pancyclic* if it contains a cycle of length ℓ for each $\ell = 3, 4, \ldots, n$, while D is *vertex-pancyclic* if each vertex of D lies on a cycle of length ℓ for each $\ell = 3, 4, \ldots, n$. Harary and Moser [28] showed the following.

Theorem 1.4.7 (Harary and Moser) *Every nontrivial strong tournament is pancyclic.*

Moon [37] extended this result by showing the following (also see [38]).

Theorem 1.4.8 (Moon) *Every nontrivial strong tournament is vertex-pancyclic.*

While a graph G is Hamiltonian-connected if for every two vertices u and v of G, there is a Hamiltonian $u-v$ path, there are two natural ways to define this concept for digraphs. A digraph D is *weakly Hamiltonian-connected* if for every two vertices u and v of D, there is either a Hamiltonian $u-v$ path or a Hamiltonian $v-u$ path. A digraph D is *strongly Hamiltonian-connected* or, simply, *Hamiltonian-connected* if for every two vertices u and v of D, there is both a Hamiltonian $u-v$ path and a Hamiltonian $v-u$ path. A digraph D of order n is *weakly panconnected* if for each integer ℓ with $3 \leq \ell \leq n-1$ and for every two vertices u and v of D, there is either a $u-v$ path of length ℓ or a $v-u$ path of length ℓ in D. A digraph D of order n is *stronly panconnected* or *panconnected* if for each integer ℓ with $3 \leq \ell \leq n-1$ and for every two vertices u and v of D, there are both a $u-v$ path of length ℓ and a $v-u$ path of length ℓ in D. For a tournament T of order n, the *irregularity* $i(T)$ is defined as

$$i(T) = \max\{|\operatorname{od} x - \operatorname{id} x| : \ x \in V(T)\}.$$

Thus, if $i(T) = 0$, then T is *regular* and if $i(T) = 1$, then T is *almost regular*. Thomassen [45] proved the following result in 1980.

Theorem 1.4.9 *If T is a tournament of order at least $5i(T) + 9$, then T is strongly panconnected. In particular, if T is a regular tournament of order at least 9, then T is strongly panconnected. Furthermore, every almost regular tournament of order at least 10 strongly panconnected.*

The following is a consequence of Theorem 1.4.9

Corollary 1.4.10 *If T is a tournament of order at least $5i(T) + 9$, then T is strongly Hamiltonian-connected. In particular, if T is a regular tournament of order at least 9, then T is strongly Hamiltonian-connected. Furthermore, every almost regular tournament of order at least 10 is strongly Hamiltonian-connected.*

Figure 1.3 shows two regular tournaments T_5 and T_7 of order 5 and 7, respectively. It can be shown that the tournament T_5 in Figure 1.3(a) is not Hamiltonian-connected; while the tournament T_7 in Figure 1.3(b) is Hamiltonian-connected.

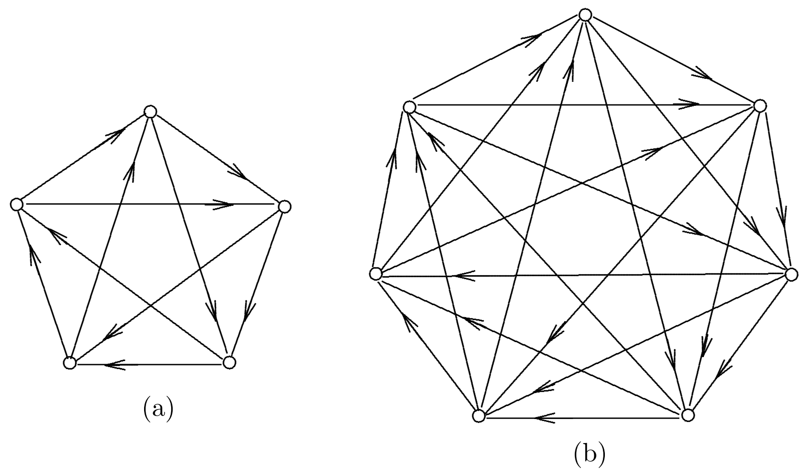


Figure 1.3: Regular tournaments of order 5 and 7

In [45], Thomassen characterized weakly Hamiltonian-connected tournaments and weakly panconnected tournaments and provided a sufficient condition in terms of connectivity for a Hamiltonian path with prescribed initial and terminal vertex. From these results, he verified that every 4-connected tournament is Hamiltonian-connected and that every arc of a 3-connected tournament is contained in a Hamiltonian cycle of the tournament. Furthermore, he described infinite families of tournaments demonstrating that these results are best possible.

We refer to the books [14, 15] for graph theory notation and terminology not described in this dissertation. All graphs and digraphs under consideration here are nontrivial and connected.

Chapter 2

On k-Path Pancyclic Graphs

Inspired by the concept of k-path Hamiltonian graphs, we introduce the concepts of k-path pancyclic graphs and path pancyclic graphs. For integers k and n with $2 \leq k \leq n-1$, a graph G of order n is *k-path pancyclic* if every path P of order k in G lies on a cycle of every length from $k+1$ to n. In particular, a 2-path pancyclic graph G of order n is called an *edge-pancyclic* graph, that is, every edge of G lies on cycles of lengths from 3 to n. A graph G of order $n \geq 3$ is *path pancyclic* if G is k-path pancyclic for each integer k with $2 \leq k \leq n-1$. In this chapter, we present sufficient conditions for a graph to be k-path pancyclic in terms of its order, size, minimum degree as well as the sum of the degrees of every two nonadjacent vertices of the graph.

In addition, we present sharp lower bounds for a graph to be panconnected or geodesic-pancyclic, which improve results by (1) Williamson and (2) by Chan, Chang, Wang and Horng, respectively, presented in Chapter 1.

2.1 Minimum Degree Conditions

In this section, we establish a sufficient condition on the minimum degree of a graph G in terms of its order n and a fixed integer k with $2 \leq k \leq n-1$ such that G is k-path pancyclic. We saw that if $n \geq 4$ and $\delta(G) \geq \frac{n+1}{2}$, then G is Hamiltonian-connected and therefore G is 2-path Hamiltonian. In fact, more can be said. First, we state a result due to Faudree and Schelp [19].

Theorem 2.1.1 (Faudree and Schelp) *If G is a graph of order $n \geq 5$ such that $\sigma_2(G) \geq n+1$, then for every pair u, v of distinct vertices of G, there is a $u-v$ path of length ℓ for every integer ℓ with $4 \leq \ell \leq n-1$.*

With the aid of Theorem 2.1.1, we are able to verify the following.

Theorem 2.1.2 *Let k and n be integers with $n \geq 4$ and $2 \leq k \leq n-1$. If G is a graph of order n such that $\delta(G) \geq \frac{n+k-1}{2}$, then every path of order k lies on a cycle of length ℓ for each integer ℓ with $k+1 \leq \ell \leq n$ except possibly $k+2$.*

Proof. Let P be a path of order k where $2 \leq k \leq n-1$ in G, where say

$$P = (u = v_1, v_2, \ldots, v_k = v)$$

is a $u - v$ path. We consider two cases, according to whether $k = 2$ or $k \geq 3$.

Case 1. $k = 2$. Then $uv \in E(G)$. If $n = 4$, then $G = K_4$ and the result is true trivially. Thus, we may assume that $n \geq 5$. Since $\delta(G) \geq \frac{n+1}{2}$, it follows that $N(u) \cap N(v) \neq \emptyset$ and so there is $w \in V(G)$ such that (u, v, w, u) is a triangle in G. Hence uv lies on a cycle of length $\ell = 3$. Also, since $\delta(G) \geq \frac{n+1}{2}$, it follows by Theorem 2.1.1 that there is a $u - v$ path Q_ℓ of length ℓ for every integer ℓ with $4 \leq \ell \leq n-1$. Thus, uv lies on a cycle of length ℓ for each integer $\ell \in \{5, 6, \ldots, n\}$.

Case 2. $3 \leq k \leq n-1$. If $k = 3$ and $n = 4$, then $G = K_4$ and the result is true trivially. Thus, we may assume that $n \geq 5$. For each integer ℓ with $k+1 \leq \ell \leq n$ and $\ell \neq k+2$, we can write $\ell = (k-2) + \ell'$ for some ℓ' with $3 \leq \ell' \leq n-k+2$ and $\ell' \neq 4$. Then the graph $H = G - \{v_2, v_3, \ldots, v_{k-1}\}$ has order $n_H = n - (k-2)$ and the minimum degree

$$\delta(H) \geq \frac{n+k-1}{2} - (k-2) = \frac{[n-(k-2)]+1}{2} = \frac{n_H + 1}{2}.$$

First, suppose that uv is an edge of H. It then follows by Case 1 that uv lies on a cycle $C_{\ell'}$ of order ℓ' for each integer ℓ' with $3 \leq \ell' \leq n-(k-2)$ and $\ell' \neq 4$. Then the $u - v$ path $C_{\ell'} - uv$ of $C_{\ell'}$ and P form a cycle of order $\ell = \ell' + (k-2)$ in G that contains P. Next, suppose that uv is not an edge of H. Then the graph $H' = H + uv$ has order $n - (k-2)$ and $\delta(H') \geq \delta(H)$. Again by Case 1, the edge uv lies on a cycle $C_{\ell'}$ of order ℓ' in H' for each integer ℓ' with $3 \leq \ell' \leq n-(k-2)$ and $\ell' \neq 4$. Similarly, the $u - v$ path $C_{\ell'} - uv$ of $C_{\ell'}$ and P form a cycle of order $\ell = \ell' + (k-2)$ in G that contains P. ■

The lower bound for the minimum degree of a graph in Theorem 2.1.2 cannot be improved. To see this, let n and k be integers of the same parity such that $n \geq k+2$ and let F_0, F_1, F_2 be three vertex-disjoint graphs where $F_0 = K_k$ is the complete graph of order k and $F_1 = F_2 = K_{(n-k)/2}$ are the complete graph of order $(n-k)/2$. The graph G is then constructed from F_0, F_1, F_2 by joining every vertex of F_0 to every vertex in F_1 and F_2. Then the order of G is n and $\delta(G) = \frac{n+k-2}{2}$. Observe that each path of order k in F_0 does not lie on any cycle of order n in G.

We will deal with the exceptional situation in Theorem 2.1.2 when every path of order k lies on a cycle of length $k+2$ later in this section.

Recall that a graph G of order n is panconnected if for every pair u, v of distinct vertices of G, there is a $u-v$ path of length k for every integer k with $d(u, v) \leq k \leq n-1$. The following result was established by Williamson [48] in 1977, which was also stated in Chapter 1.

Theorem 2.1.3 (Williamson) *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+2)/2$, then G is panconnected.*

With the same minimum degree condition, Randerath, Schiermeyer, Tewes and Nolkmann [44] showed that those graphs are edge-pancyclic in 2002.

Theorem 2.1.4 (Randerath, Schiermeyer, Tewes and Nolkmann) *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+2)/2$, then G is edge-pancyclic.*

For two vertices u and v in a connected graph G, a $u-v$ *geodesic* is a $u-v$ path of length $d(u, v)$ in G. A graph G of order n is defined in [6] to be *geodesic-pancyclic* if for each pair u, v of G, every $u-v$ geodesic lies on a cycle of length k for every k with $\max\{2d_G(u, v), 3\} \leq k \leq n$. In particular, a geodesic-pancyclic graph is edge-pancyclic. The following result is due to Chan, Chang, Wang and Horng (see [6]), which was stated in Chapter 1.

Theorem 2.1.5 (Chan, Chang, Wang and Horng) *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+2)/2$, then G is geodesic-pancyclic.*

Observe that if G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+2)/2$, then $\mathrm{diam}(G) \leq 2$ and so a $u-v$ geodesic in G is either the edge uv or a $u-v$ path of length 2. Therefore, the following result is an extension of Theorems 2.1.4 and 2.1.5.

Theorem 2.1.6 *Let k and n be integers with $n \geq 4$ and $2 \leq k \leq n-1$. If G is a graph of order n such that $\delta(G) \geq \frac{n+k}{2}$, then G is k-path pancyclic.*

Proof. Let P be any path of order $k \geq 2$ in G, say $P = (u = v_1, v_2, \ldots, v_k = v)$ is a $u-v$ path. We consider two cases, according to whether $k = 2$ or $k \geq 3$.

Case 1. $k = 2$. Then $uv \in E(G)$. For each integer ℓ with $3 \leq \ell \leq n$, we can write $\ell = \ell' + 1$ for some $\ell' \geq 2$. Since $\delta(G) \geq (n+2)/2$, it follows by Theorem 2.1.3 that G is panconnected and so G contains a $u-v$ path $Q_{\ell'}$ of length ℓ' for each integer ℓ' with $1 = d_G(u, v) < \ell' \leq n-1$. Then $Q_{\ell'} + uv$ is a cycle of order ℓ.

Case 2. $k \geq 3$. For each integer ℓ with $k+1 \leq \ell \leq n$, we can write $\ell = \ell' + (k-1)$ for some integer ℓ' with $2 \leq \ell' \leq n-k+1$. Then the graph $H = G - \{v_2, v_3, \ldots, v_{k-1}\}$ has order $n_H = n-(k-2)$ and

$$\delta(H) \geq \frac{n+k}{2} - (k-2) = \frac{[n-(k-2)]+2}{2} = \frac{n_H+2}{2}.$$

Thus H is panconnected and furthermore $d_H(u,v) \leq 2$. Therefore, H contains a $u-v$ path $Q_{\ell'}$ of length ℓ' for each integer ℓ' with $2 \leq \ell' \leq n_H - 1$. Then $Q_{\ell'}$ and P form a cycle of order $\ell = \ell' + 1 + (k-2) = \ell' + (k-1)$ in G that contains P. ■

The lower bound for the minimum degree of a graph in Theorem 2.1.6 cannot be improved. To see this, let $k \geq 2$ be an integer and let $G = kK_k \vee \overline{K}_{k^2-k+1}$ be the join of kK_k and $\overline{K}_{k^2-k+1}$, where kK_k is the union of k vertex-disjoint copies of K_k. Then G is a k^2-regular graph of order $n = 2k^2-k+1$. Observe that $\delta(G) = \frac{n+k-1}{2} = k^2$. However, each path of order k in any subgraph K_k in G does not lie on a cycle of order $k+2$ in G.

We now determine the exceptional situation in Theorem 2.1.2 when every path of order k lies on a cycle of length $k+2$. By Theorem 2.1.6, if G is a graph of order $n \geq 4$ with $\delta(G) \geq \frac{n+k}{2}$, then G is k-path pancyclic for $2 \leq k \leq n-1$ and so every path of order k lies on a cycle of length $k+2$. Thus, it remains to consider the case when $\delta(G) = \frac{n+k-1}{2}$ and k and n are of opposite parity.

Theorem 2.1.7 *Let k and n be integers with $n \geq 5$ and $2 \leq k \leq n-3$, where k and n are of opposite parity, and let G be a graph of order n such that $\delta(G) = \frac{n+k-1}{2}$. Then every path of order k in G lies on a cycle of length $k+2$ if and only if (i) $n = 5, 6$ or (ii) $n \geq 7$ and $k = n-3$.*

Proof. First suppose that $n = 5, 6$. If $n = 5$, then $k = 2$, $\delta(G) = 3$ and $G = K_5 - M$ where M is a matching of size 1 or 2. If $n = 6$, then $k = 3$, $\delta(G) = 4$ and $G = K_6 - M$ where M is a matching of size 1, 2 or 3. In each case, it is easy to see that every path of order k lies on a cycle of length $k+2$ in G. Next, suppose that $n \geq 7$ and $k = n-3$. Then $\delta(G) = n-2$ and we show that every path of order $n-3$ lies on a cycle of length $n-1$ in G. Let $P = (u_1, u_2, \ldots, u_{n-3})$ be a path of order $n-3$ in G and let $V(G) - V(P) = \{v_1, v_2, v_3\}$. Since $\delta(G) = n-2$, each of v_1, v_2, v_3 is adjacent to at least one vertex in $\{v_1, v_2, v_3\}$ and to at least one vertex in $\{u_1, u_{n-3}\}$. Hence we may assume that (v_1, v_2, v_3) is a path in G and $v_2u_{n-3} \in E(G)$. Also, u_1 is adjacent to at least two of v_1, v_2, v_3 and so u_1 is adjacent to at least one of v_1 and v_3, say $u_1v_1 \in E(G)$. Hence, P lies on the cycle $(u_1, u_2, \ldots, u_{n-3}, v_2, v_1, u_1)$ of length $n-1$ in G.

For the converse, we show that for each pair k, n of integers with $n \geq 7$ and $2 \leq k \leq n-5$ where k and n are of opposite parity, there is a graph G of order n with $\delta(G) = \frac{n+k-1}{2}$ such that G contains a k-path that does not lie on any cycle of length $k+2$ in G. We start with three graphs G_1, G_2 and G_3. Let $G_1 = K_k$ be the complete graph of order k and u and v are two distinct vertices of G_1, let $G_2 = \overline{K}_{\frac{n-k-1}{2}}$ be the empty graph of order $\frac{n-k-1}{2}$ and let $G_3 = K_{\frac{n-k-1}{2}}$ be the complete graph of order $\frac{n-k-1}{2}$. The graph G of order n is constructed from G_i $(1 \leq i \leq 3)$ by adding a new vertex w and (i) joining w to each vertex in both G_1 and G_3, (ii) joining each vertex of G_1 to every vertex in G_2 and (iii) joining each vertex in both $V(G_1) - \{u, v\}$ (if $k \geq 3$) and G_2 to every vertex of G_3. The graph G is shown in Figure 2.1. If $x \in \{u, v, w\} \cup V(G_2)$, then $\deg_G x = \frac{n+k-1}{2}$, if $x \in V(G_1) - \{u, v\}$, then $\deg_G x = n-1 > \frac{n+k-1}{2}$ (since $k < n-3$) and if $x \in V(G_3)$, then $\deg_G x = n-3 \geq \frac{n+k-1}{2}$ (since $k \leq n-5$). Thus, $\delta(G) = \frac{n+k-1}{2}$. Let P_k be a $u-v$ Hamiltonian path of order k in G_1. Since $N_G(u) - V(P_k) = N_G(v) - V(P_k) = V(G_2) \cup \{w\}$, which is an independent set of vertices in G, it follows that if u is adjacent u' and v is adjacent to v' where $u' \neq v'$, then $u'v' \notin E(G)$. Thus P_k does not lie on any cycle of length $k+2$ in G. ■

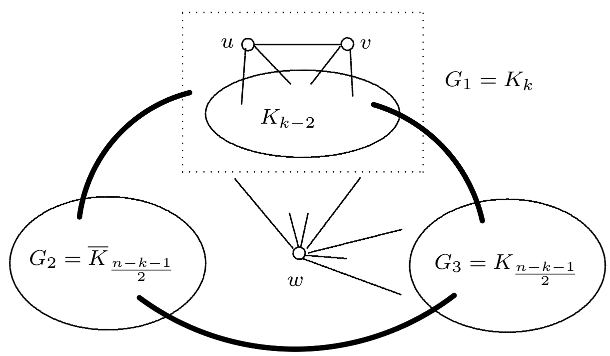


Figure 2.1: The graph G in the proof of Theorem 2.1.7

The following is a consequence of Theorems 2.1.2 and 2.1.7.

Corollary 2.1.8 *Let k and n be integers with $n \geq 4$ and $2 \leq k \leq n-1$ and let G be a graph of order n with $\delta(G) = \frac{n+k-1}{2}$. Then G is k-path pancyclic if and only if (i) $4 \leq n \leq 6$ or (ii) $n \geq 7$ and $k \in \{n-3, n-1\}$.*

2.2 Degree-Sum and Size Conditions

In this section, we establish sufficient conditions on the degree-sum of nonadjacent vertices and the size of a graph G (in terms of its order and a fixed integer k) such that G

is k-path pancyclic. We begin with the degree-sum condition. We saw in Theorem 1.1.3 that if G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$, then G is Hamiltonian-connected. It is known, however, that there are non-panconnected graphs G of order n such that $\sigma_2(G) \geq n+2$ (see [14, p. 133]). We saw such a graph of order 8 in Chapter 1. Next, we illustrate this fact with the following more general example, which will provide information on k-path pancyclic graphs.

Let $n = 2p+2$, where $p \geq 3$, and let $H = K_{2p}$ be the complete graph of order $2p$. Partition $V(H)$ into V_1 and V_2 with $|V_1| = |V_2| = p$. Define G to be the graph obtained by adding two adjacent vertices x and y to H and joining (1) x to every vertex in V_1 and (2) y to every vertex in V_2. Then $\deg x = \deg y = p+1$ and $\deg u = 2p$ for all $u \in V(G) - \{x, y\}$. Thus if u and v are two nonadjacent vertices in G, then $\deg u + \deg v = 2p + p + 1 = (2p+2) + (p-1) \geq n+2$ since $p \geq 3$. However, there is no $x-y$ path of length 2 in the graph G. Therefore, G is not 2-path pancyclic since xy does not lie on a cycle of order 3 in G. In addition, if P is an $x-y$ path of order k for some integer k with $4 \leq k \leq n-1$, then P does not lie on a cycle of order $k+1$ in G. Thus, G is not k-path pancyclic. This example also illustrates the fact that there is no constant c such that if G is a graph of order n with $\sigma_2(G) \geq n+c$, then G is panconnected. Similarly, this example provides the following.

Proposition 2.2.1 *For any two integers k and n with $n \geq 4$ and $2 \leq k \leq n-1$, there is no constant c such that if G is a graph of order n with $\sigma_2(G) \geq n+c$, then G is k-path pancyclic.*

The following two results provide sufficient conditions on $\sigma_2(G)$ in terms of the order of a graph G such that G is panconnected and geodesic-pancyclic, respectively (see [6, 48]), which were also stated in Chapter 1.

Theorem 2.2.2 (Williamson) *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-2}{2}$, then G is panconnected.*

Theorem 2.2.3 (Chan, Chang, Wang and Horng) *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-2}{2}$, then G is geodesic-pancyclic.*

It can be shown that if G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-2}{2}$, then $\operatorname{diam}(G) \leq 2$. Therefore, the following result is an extension of these two theorems above.

Theorem 2.2.4 *Let k and n be integers with $n \geq 4$ and $2 \leq k \leq n-1$. If G is a graph of order n such that $\sigma_2(G) \geq \frac{3n+k-4}{2}$, then G is k-path pancyclic.*

Proof. By Theorem 2.2.2, the statement is true for $k = 2$. Thus, we may assume that $k \geq 3$. Let P be a path of order k in G, say $P = (x = v_1, v_2, \ldots, v_k = y)$ is an $x - y$ path. Let $H = G - \{v_2, v_3, \ldots, v_{k-1}\}$. The order of H is $n_H = n - (k-2) = n - k + 2$. If u and v are any two nonadjacent vertices of H, then

$$\deg_H u + \deg_H v \geq \frac{3n + k - 4}{2} - 2(k-2) = \frac{3(n-k+2) - 2}{2} = \frac{3n_H - 2}{2}.$$

Thus, H is panconnected by Theorem 2.2.2 and, furthermore, $d_H(x, y) \leq 2$. Therefore, H contains an $x - y$ path $Q_{\ell'}$ of length ℓ' for each integer ℓ' with

$$2 \leq \ell' \leq n_H - 1 = n - k + 1.$$

Then $Q_{\ell'}$ and P form a cycle of order

$$\ell = \ell' + 1 + (k-2) = \ell' + (k-1)$$

in G that contains P for each ℓ with $k + 1 \leq \ell \leq n$. ■

If $2 \leq k \leq n-2$, then Theorem 2.2.4 can also be verified with the aid of Theorem 2.1.6 as follows. Assume, to the contrary, that G is not k-path pancyclic. It then follows by Theorem 2.1.6 that there is a vertex u in G such that $\deg u < \frac{n+k}{2} \leq n - 1$ (since $k \leq n - 2$). Thus, there is a vertex v in G such that u and v are nonadjacent and so $\deg_G v \leq n - 2$. However then,

$$\deg_G u + \deg_G v < \tfrac{n+k}{2} + (n-2) = \tfrac{3n+k-4}{2},$$

which is a contradiction.

The lower bound $(3n + k - 4)/2$ in Theorem 2.2.4 for the sum of the degrees of two nonadjacent vertices of a graph cannot be replaced by $(3n + k - 6)/2$. For example, let $H = K_{2p}$ be the complete graph of order $2p$ for some integer $p \geq 3$, let $F = K_k$ be the complete graph of order $k \geq 3$ and let $P = (x = v_1, v_2, \ldots, v_k = y)$ is an $x - y$ Hamiltonian path in F. Partition $V(H)$ into V_1 and V_2 with $|V_1| = |V_2| = p$. Define G to be the graph obtained by (1) joining x to every vertex in V_1, (2) joining y to every vertex in V_2 and (3) joining each vertex v_i $(2 \leq i \leq k-1)$ to every vertex in H. Then the order of G is $n = 2p + k$. Furthermore,

$$\deg_G x = \deg_G y = p + (k-1) \text{ and } \deg z = (2p-1) + (k-1) = 2p + k - 2$$

for all $z \in V(G) - V(F)$. Thus, if u and v are two nonadjacent vertices in G, then

$$\deg_G u + \deg_G v = (2p + k - 2) + (p + k - 1) = 3p + 2k - 3 = \frac{3n + k - 6}{2}.$$

Observe that the path P of order k does not lie on a cycle of order $k+1$ in G and so G is not k-path pancyclic. On the other hand, the lower bound $(3n+k-4)/2$ in Theorem 2.2.4 for the sum of the degrees of two nonadjacent vertices of a graph can be replaced by $(3n+k-5)/2$ when n and k are of opposite parity. In order to show this, we first present a useful lemma.

Lemma 2.2.5 *Let k and n be integers with $n \geq 4$ and $2 \leq k \leq n-1$. If G be a graph of order n such that $\delta(G) \geq \frac{n+k-1}{2}$, then $|N(u) \cap N(v)| \geq k-1$ for every two distinct vertices u and v of G.*

Proof. For two distinct vertices u and v of G, let $A = N(u) - N(v)$, $B = N(v) - N(u)$ and $C = N(u) \cap N(v)$. Since $\deg_G u = |A| + |C| \geq \frac{n+k-1}{2}$ and $\deg_G v = |B| + |C| \geq \frac{n+k-1}{2}$, it follows that $|A| \geq \frac{n+k-1}{2} - |C|$ and $|B| \geq \frac{n+k-1}{2} - |C|$. If $uv \notin E(G)$, then $|A| + |B| + |C| \leq n-2$; while if $uv \in E(G)$, then $|A| + |B| + |C| \leq n$. In either case,

$$|C| \leq n - (|A| + |B|) \leq n - 2\left(\tfrac{n+k-1}{2} - |C|\right)$$

and so $|N(u) \cap N(v)| = |C| \geq k-1$ as desired. ■

We are now prepared to present a lower bound for a graph to be k-path pancyclic. which is an improvement of Theorem 2.2.4.

Theorem 2.2.6 *Let k and n be integers with $n \geq 4$ and $2 \leq k \leq n-1$. If G is a graph of order n with $\sigma_2(G) \geq \frac{3n+k-5}{2}$, then G is k-path pancyclic.*

Proof. First, we show that if $\sigma_2(G) \geq \frac{3n+k-5}{2}$, then $\delta(G) \geq \frac{n+k-1}{2}$. Let $v \in V(G)$ such that $\deg_G v = \delta(G)$. Since G is not complete, $\delta(G) \leq n-2$ and so there is a vertex u in G such that $uv \notin E(G)$ and $\deg_G u \leq n-2$. Because $\deg_G u + \deg_G v \geq \sigma_2(G) \geq \frac{3n+k-5}{2}$, it follows that

$$\deg_G v \geq \frac{3n+k-5}{2} - \deg_G u \geq \frac{3n+k-5}{2} - (n-2) = \frac{n+k-1}{2}$$

and so $\delta(G) \geq \frac{n+k-1}{2}$. If $\delta(G) \geq \frac{n+k}{2}$, then G is k-path pancyclic by Theorem 2.1.6. Thus, we may assume that $\delta(G) = \frac{n+k-1}{2}$ and so k and n are of opposite parity. We can further assume that $2 \leq k \leq n-5$ by Corollary 2.1.8. Now by Theorem 2.1.2, it suffices to verify the following:

> For integers n and k with $n \geq 7$ and $2 \leq k \leq n-5$, if G is a graph of order n such that $\sigma_2(G) \geq \frac{3n+k-5}{2}$ and $\delta(G) = \frac{n+k-1}{2}$, then every k-path in G lies on a cycle of length $k+2$.

Assume, to the contrary, that there is a path $P = (u_1, u_2, \ldots, u_k)$ of order k for some integer k with $2 \leq k \leq n-5$ that does not lie on any cycle of length $k+2$ in G. Since $\delta(G) = \frac{n+k-1}{2}$, it follows by Theorem 2.1.2 that P lies on a cycle of length $k+1$ in G. Let $(u_1, u_2, \ldots, u_k, w, u_1)$ be such a cycle. If there is $v \in V(G) - (V(P) \cup \{w\})$ such that v is adjacent to both u_1 and w or v is adjacent to both u_k and w, then P lies on the cycle $(u_1, u_2, \ldots, u_k, w, v, u_1)$ or on the cycle $(u_1, u_2, \ldots, u_k, v, w, u_1)$ of length $k+2$. Thus, we may assume that no vertex in $V(G) - (V(P) \cup \{w\})$ is adjacent to both u_1 and w or to both u_k and w.

Since $\delta(G) \geq \frac{n+k-1}{2}$, it follows by Lemma 2.2.5 that

$$|N(u_1) \cap N(w)| \geq k-1 \text{ and } |N(u_k) \cap N(w)| \geq k-1.$$

This implies that

$$N(u_1) \cap N(w) = \{u_2, u_3, \ldots, u_k\} \text{ and } N(u_k) \cap N(w) = \{u_1, u_2, \ldots, u_{k-1}\}.$$

Hence $wu_i \in E(G)$ for $1 \leq i \leq k$, $u_1u_i \in E(G)$ for $2 \leq i \leq k$ and $u_ku_i \in E(G)$ for $1 \leq i \leq k-1$. Let

$$X = V(G) - (V(P) \cup \{w\}) = \{x_1, x_2, \ldots, x_{n-k-1}\}.$$

Since $\delta(G) = \frac{n+k-1}{2}$ and u_1 is adjacent to exactly k vertices in $V(P) \cup \{w\}$, it follows that u_1 is adjacent to at least $\frac{n-k-1}{2}$ vertices in X. We may assume, without loss of generality, that u_1 is adjacent to x_i for $1 \leq i \leq \frac{n-k-1}{2}$. Similarly, w is adjacent to the k vertices of P and so w is adjacent to at least $\frac{n-k-1}{2}$ vertices in X. Since there is no vertex in X that is adjacent to both u_1 and w or to both u_k and w and $|X| = n-k-1$, it follows that

(i) u_1 is adjacent to exactly $\frac{n-k-1}{2}$ vertices in X, namely x_i for $1 \leq i \leq \frac{n-k-1}{2}$,

(ii) w is not adjacent to any x_i ($1 \leq i \leq \frac{n-k-1}{2}$) and so w is adjacent to all of the $\frac{n-k-1}{2}$ remaining vertices in X, namely x_j for $\frac{n-k+1}{2} \leq j \leq n-k-1$ and

(iii) u_k and u_1 have exactly the same neighbors in X.

This situation is illustrated in Figure 2.2, where

$$X_1 = \{x_1, x_2, \ldots, x_{\frac{n-k-1}{2}}\} \text{ and } X_2 = X - X_1.$$

Therefore,

$$\deg_G u_1 = \deg_G u_k = \deg_G w = \frac{n+k-1}{2}. \tag{2.1}$$

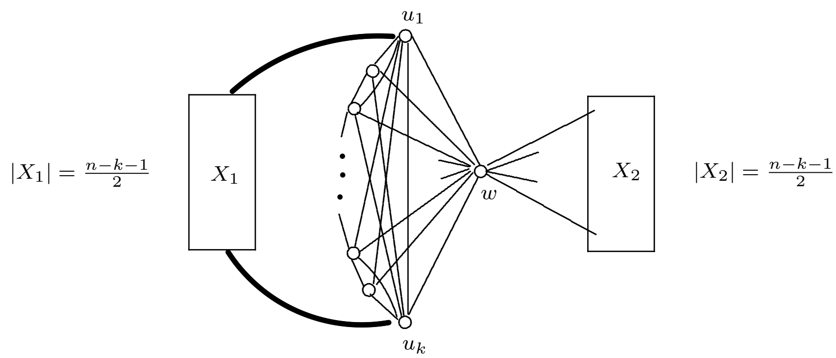


Figure 2.2: A step in the proof of Theorem 2.2.6

Now let $x \in X_2$. Since x is adjacent to neither u_1 nor u_k, it follows that $\deg_G x \le n-3$. Because $\sigma_2(G) \ge \frac{3n+k-5}{2}$ and $u_1x \notin E(G)$, it follows that

$$\deg_G u_1 + \deg_G x \ge \frac{3n+k-5}{2}.$$

However then,

$$\deg_G u_1 \ge \frac{3n+k-5}{2} - \deg_G x \ge \frac{3n+k-5}{2} - (n-3) = \frac{n+k+1}{2},$$

which, however, contradicts (2.1). ■

The lower bound $\frac{3n+k-5}{2}$ in Theorem 2.2.6 is best possible. In fact, for every pair k, n of integers with $n \ge 7$, $2 \le k \le n-1$ and n and k are of opposite parity, there exists a graph G of order n such that $\sigma_2(G) = \frac{3n+k-7}{2}$ and G is not k-path pancyclic. To see this, let $G_1 = K_k$, where u and v are two distinct vertices of G_1, and $G_2 = K_{n-k}$. Partition the vertex set $V(G_2)$ into two sets X and Y where $|X| = \frac{n-k-1}{2}$ and $|Y| = \frac{n-k+1}{2}$. (Note that (1) if $k = 2$, then $V(G_1) - \{u, v\} = \emptyset$ and (2) if $k = n-1$, then $X = \emptyset$.) The graph G of order n is constructed from G_1 and G_2 by

(i) joining u to every vertex in X and v to every vertex in Y and

(ii) joining each vertex in $V(G_1) - \{u, v\}$ to every vertex in G_2.

The graph G is shown in Figure 2.3. Then $\deg_G u = \frac{n+k-3}{2}$, $\deg_G v = \frac{n+k-1}{2}$ and $\deg_G w = n-1$ for each $w \in V(G_1) - \{u, v\}$ (if $V(G_1) - \{u, v\} \neq \emptyset$), $\deg_G w = n-2$ for $w \in V(G_2)$. Thus,

$$\sigma_2(G) = \deg_G u + \deg_G y = \frac{3n+k-7}{2}$$

for each $y \in Y$. Let P_k be a $u-v$ Hamiltonian path of order k in G_1. Since

$$(N_G(u) - V(P_k)) \cap (N_G(v) - V(P_k)) = \emptyset,$$

it follows that P_k does not lie on any cycle of length $k+1$ in G. Hence, G is not k-path pancyclic.

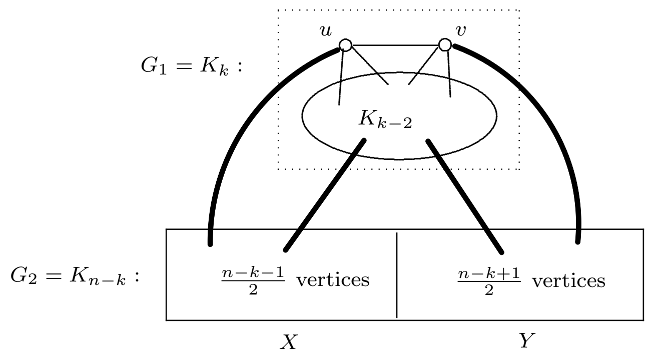


Figure 2.3: Illustrating that the lower bound $\frac{3n+k-5}{2}$ in Theorem 2.2.6 is sharp

The following two results provide sufficient conditions on the size of a graph G for G to be panconnected or geodesic-pancyclic, respectively (see [6, 48]).

Theorem 2.2.7 (Williamson) *If G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2} + 3$, then G is panconnected.*

Theorem 2.2.8 (Chan, Chang, Wang and Horng) *If G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2} + 3$, then G is geodesic-pancyclic.*

Since the diameter of a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2} + 3$ is at most 2, the following is an extension of Theorem 2.2.8.

Theorem 2.2.9 *Let k and n be positive integers such that $n \geq k+2$. If G is a graph of order n and size $m \geq \binom{n-1}{2} + k + 1$, then G is k-path pancyclic.*

Proof. By Theorem 2.2.7, the statement is true for $k = 2$. Thus, we may assume that $k \geq 3$. Let G be a graph of order $n \geq k+2$ and size $m \geq \binom{n-1}{2} + k + 1$ and let $P = (u = v_1, v_2, \ldots, v_k = v)$ be a path of order k in G. Let $H = G - \{v_2, v_3, \ldots, v_{k-1}\}$. Thus, H has order $n_H = n - k + 2$ and size

$$\begin{aligned} m_H &\geq \binom{n-1}{2} + k + 1 - [(n-1) + (n-2) + \cdots + (n-k+2)] \\ &= \binom{n-k+1}{2} + 3 = \binom{n_H - 1}{2} + 3. \end{aligned}$$

Thus, H is panconnected by Theorem 2.2.7 and, furthermore, $d_H(x, y) \leq 2$. Therefore, H contains an $u-v$ path $Q_{\ell'}$ of length ℓ' for each integer ℓ' with $2 \leq \ell' \leq n_H - 1 = n-k+1$. Then $Q_{\ell'}$ and P form a cycle of order $\ell = \ell' + 1 + (k-2) = \ell' + (k-1)$ in G that contains P for each ℓ with $k+1 \leq \ell \leq n$. ■

The bound on the size m of a graph in Theorem 2.2.9 cannot be improved. To see this, let G be a graph of order $n \geq k+2 \geq 4$ consisting of a complete subgraph G' of order $n-1$, where $V(G') = \{v_1, v_2, \ldots, v_{n-1}\}$ and another vertex v adjacent to $v_1, v_2, \ldots, v_k$. Then the size of G is $m = \binom{n-1}{2} + k$. However, the path $P = (v_1, v_2, \ldots, v_k)$ of order k lies on no Hamiltonian cycle of G. Hence P cannot be extended to a cycle of order n in G. Thus, G is not k-path pancyclic.

2.3 An Improved Bound for Panconnected Graphs

As we indicated in Chapter 1, it is known that if n is even, then the lower bound for $\sigma_2(G)$ presented in Theorem 2.2.2 for a graph G to be panconnected is best possible; that is, $\frac{3n-2}{2}$ cannot be replaced by $\frac{3n-4}{2}$. With the aid of Theorem 2.2.6, we next show that $\frac{3n-2}{2}$ in Theorem 2.2.2 can be replaced by $\frac{3n-3}{2}$ when n is odd.

Theorem 2.3.1 *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq \frac{3n-3}{2}$, then G is panconnected.*

Proof. If n is even, then $\sigma_2(G) \geq \left\lceil \frac{3n-3}{2} \right\rceil = \frac{3n-2}{2}$ and so the result follows by Theorem 2.2.2. Thus, we may assume that n is odd and so $n \geq 5$. Since $\sigma_2(G) \geq \frac{3n-3}{2}$ and $n \geq 5$, it follows that $\delta(G) \geq \frac{n+1}{2}$. By Lemma 2.2.5, the diameter diam(G) of G (the largest distance between two vertices of G) is at most 2. Let u and v be any two vertices of G. Thus $d(u, v) = 1$ or $d(u, v) = 2$. We show that there is a $u-v$ path of length ℓ for every integer ℓ with $d(u, v) \leq \ell \leq n-1$.

First, suppose that $d_G(u, v) = 1$ or $uv \in E(G)$. Then (u, v) is a $u-v$ path of length $\ell = 1$. For $2 \leq \ell \leq n-1$, we apply Theorem 2.2.6 to the path (u, v) of order 2 (that is, $k = 2$ in Theorem 2.2.6). Since $\sigma_2(G) \geq \frac{3n-3}{2} = \frac{3n+k-5}{2}$, it follows that G is 2-path pancyclic and so (u, v) lies on a cycle $C_{\ell'}$ of length ℓ' in G for $3 \leq \ell' \leq n$. Then $C_{\ell'} - uv$ is a $u-v$ path of length $\ell = \ell' - 1$ in G for $2 \leq \ell \leq n-1$.

Next, suppose that $d_G(u, v) = 2$. Since $n \geq 5$, it follows that

$$\sigma_2(G) \geq \frac{3n-3}{2} = (n+1) + \frac{n-5}{2} \geq n+1.$$

By Theorem 2.1.1, there is a $u-v$ path of length ℓ for every integer ℓ with $4 \leq \ell \leq n-1$. Since $d_G(u, v) = 2$, there is a $u-v$ path of length $\ell = 2$. Thus, it remains to show that there is a $u - v$ path of length $\ell = 3$. Let (u, w, v) be a $u - v$ path in G. Because $\delta(G) \geq \frac{n+1}{2}$ it follows that $N_G(v) \cap N_G(w) \neq \emptyset$ by Lemma 2.2.5. Let $x \in N_G(v) \cap N_G(w)$. Since $u \notin N_G(v)$ and so $u \notin N_G(v) \cap N_G(w)$, it follows that $x \neq u$. Hence, (u, w, x, v) is a $u - v$ path of length 3. Therefore, there is a $u - v$ path of length ℓ for every integer ℓ with $d(u, v) = 2 \leq \ell \leq n - 1$. ■

The lower bound $\frac{3n-3}{2}$ in Theorem 2.3.1 is best possible; that is, $\frac{3n-3}{2}$cannot be replaced by $\frac{3n-5}{2}$ when n is odd. To see this, let G be the graph constructed from the complete graph K_{n-3} where $n \geq 7$ is odd and the path (u, v, w) of order 3 by

(i) joining u to exactly $\frac{n-3}{2}$ vertices of K_{n-3},

(ii) joining v to the remaining $\frac{n-3}{2}$ vertices of K_{n-3} and

(iii) joining w to every vertex in K_{n-3}.

See Figure 2.4. Then $\deg_G u = \frac{n-1}{2}$, $\deg_G v = \frac{n+1}{2}$, $\deg_G w = \deg_G x = n - 2$ for each vertex x of K_{n-3}. Hence, $\sigma_2(G) = \frac{3n-5}{2}$. Since $N_G(u) \cap N_G(v) = \emptyset$, it follows that G contains no $u - v$ path of length 2. Therefore, G is not panconnected.

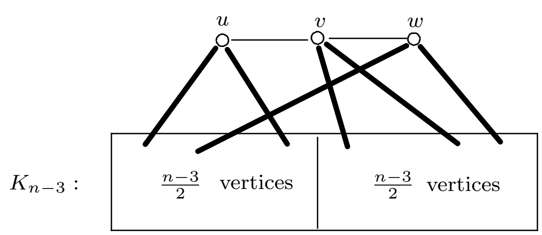


Figure 2.4: Illustrating that the lower bound $\frac{3n-3}{2}$ for $\sigma_2(G)$ in Theorem 2.3.1 is sharp

2.4 An Improved Bound for Geodesic-Pancyclic Graphs

As we indicated in Chapter 1, it is known that if n is even, then the lower bound for $\sigma_2(G)$ presented in Theorem 2.2.3 for a graph G to be geodesic-pancyclic is best possible; that is, $\frac{3n-2}{2}$ cannot be replaced by $\frac{3n-4}{2}$. However, it is not known whether $\frac{3n-2}{2}$ can be replaced by $\frac{3n-3}{2}$ when n is odd. First, observe that Theorem 2.2.3 is best possible for $n = 5$. For example, let $G = C_4 \vee K_1$ be the wheel of order 5 (the join of a 4-cycle C_4 and K_1) where $C_4 = (v_1, v_2, v_3, v_4, v_1)$ and $V(K_1) = \{v\}$. Then $\sigma_2(G) = 6 = \frac{3 \cdot 5 - 3}{2}$. Since the $v_1 - v_3$ geodesic (v_1, v, v_3) of order 3 (or the $v_2 - v_4$ geodesic (v_2, v, v_4) of order 3) does not lies on any 5-cycle in G, it follows that G is not geodesic-pancyclic. On the other hand, for an odd integer $n \geq 7$, the lower bound for $\sigma_2(G)$ in Theorem 2.2.3 can be improved, as we show next.

Theorem 2.4.1 *If G is a graph of order $n \geq 4$ and $n \neq 5$ such that $\sigma_2(G) \geq \frac{3n-3}{2}$, then G is geodesic-pancyclic.*

Proof. If n is even, then $\sigma_2(G) \geq \lceil \frac{3n-3}{2} \rceil = \frac{3n-2}{2}$ and so the result follows by Theorem 2.2.3. Thus, we may assume that $n \geq 7$ is odd. Since $\sigma_2(G) \geq \frac{3n-3}{2}$, it follows that $\delta(G) \geq \frac{n+1}{2}$ and so $\mathrm{diam}(G) \leq 2$. Let u and v be any two vertices of G. Hence, either $d(u,v) = 1$ or $d(u,v) = 2$. We show that every $u - v$ geodesic lies on a cycle of length ℓ for every integer ℓ with $\max\{2d(u,v), 3\} \leq \ell \leq n$.

First, suppose that $d(u,v) = 1$ or $uv \in E(G)$. We apply Theorem 2.2.6 to the path (u,v) of order 2 (that is, $k = 2$ in Theorem 2.2.6). Since $\sigma_2(G) \geq \frac{3n-3}{2} = \frac{3n+k-5}{2}$, it follows that G is 2-path pancyclic and so (u,v) lies on a cycle of each length $\ell = 3, 4, \ldots, n$.

Next, suppose that $d(u,v) = 2$ and so $uv \notin E(G)$. Since $\delta(G) \geq \frac{n+1}{2}$ and $uv \notin E(G)$, it follows that $|N_G(u) \cap N_G(v)| \geq 3$. Hence, every $u - v$ geodesic lies on a cycle of length $\ell = 4$. For $5 \leq \ell \leq n$, let (u, w, v) be an arbitrary $u - v$ geodesic for some $w \in N_G(u) \cap N_G(v)$ and let $H = G - w$. Then the order of H is $n_H = n - 1$. To show that (u, w, v) lies on a cycle of length ℓ in G for each $\ell = 5, 6, \ldots, n$, it suffices to show that H contains a $u - v$ path of length k for each $k = 3, 4, \ldots, n-2$, as these paths together with the path (u, w, v) produce the cycles in G with the desired property.

Since $|N_G(u) \cap N_G(v)| \geq 3$, it follows that $|N_H(u) \cap N_H(v)| \geq 2$ and so $d_H(u,v) = 2$. First, we show that H contains a $u - v$ path of length 3. If this were not the case, then no vertex in $N_H(u)$ is adjacent to any vertex in $N_H(v)$; for otherwise, let $u' \in N_H(u)$ and $v' \in N_H(v)$ such that $u'v' \in E(H)$ and then (u, u', v', v) is a $u - v$ path of length 3. Let $x \in N_H(u)$. Then $xy \notin E(H)$ for each $y \in N_H(v)$ and so $xy \notin E(G)$ for each $y \in N_H(v)$. Since $|N_H(v)| = \deg_H v = \deg_G v - 1 \geq \frac{n-1}{2}$, it follows that

$$\deg_G x \leq (n-1) - \tfrac{n-1}{2} = \tfrac{n-1}{2},$$

which contradicts the fact that $\delta(G) \geq \frac{n+1}{2}$. Therefore, H contains a $u - v$ path of length 3. For $4 \leq k \leq n-2$, since $\sigma_2(G) \geq \frac{3n-3}{2}$ and $n \geq 7$, it follows that

$$\sigma_2(H) \geq \frac{3n-3}{2} - 2 = \frac{3n-7}{2} = n + \frac{n-7}{2} \geq n_H + 1.$$

By Theorem 2.1.1, there is a $u - v$ path of length k in H for every integer k with $4 \leq k \leq n-2$.

Therefore, H contains a $u - v$ path of length k for each $k = 3, 4, \ldots, n-2$ and so (u, w, v) lies on a cycle of length ℓ in G for every integer ℓ with $5 \leq \ell \leq n$. ■

The lower bound $\frac{3n-3}{2}$ in Theorem 2.4.1 is best possible; that is, $\frac{3n-3}{2}$ cannot be replaced by $\frac{3n-5}{2}$ when n is odd. To see this, let G be the graph constructed from the complete graph K_{n-3} where $n \geq 7$ is odd and the path (u, w, v) of order 3 by

(i) joining u to exactly $\frac{n-3}{2}$ vertices of K_{n-3},

(ii) joining v to the remaining $\frac{n-3}{2}$ vertices of K_{n-3} and

(iii) joining w to every vertex in K_{n-3}.

See Figure 2.5. Then $\deg_G u = \deg v = \frac{n-1}{2}$, $\deg_G w = n-1$ and $\deg_G x = n-2$ for each vertex x of K_{n-3}. Hence, $\sigma_2(G) = \frac{3n-5}{2}$. Since $d(u, v) = 2$ and the $u-v$ geodesic (u, w, v) does not lie on any cycle of length 4 in G, it follows that G is not geodesic-pancyclic.

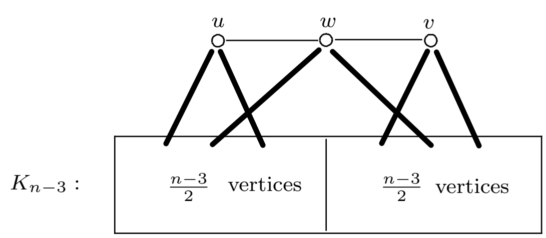


Figure 2.5: Showing that the lower bound $\frac{3n-3}{2}$ for $\sigma_2(G)$ in Theorem 2.4.1 is sharp

Chapter 3

On k-Path Hamiltonian Digraphs

3.1 Introduction

A digraph D is *Hamiltonian* if D contains a spanning (directed) cycle. Such a cycle is called a *Hamiltonian cycle* of D. A Hamiltonian digraph D of order $n \geq 3$ is *ℓ-path Hamiltonian* for some positive integer ℓ with $1 \leq \ell \leq n$ if every (directed) path of order ℓ lies on a (directed) Hamiltonian cycle of D. In this chapter, all paths and cycles refer to directed paths and cycles. Thus, every Hamiltonian digraph D is 1-path Hamiltonian and a Hamiltonian digraph D is 2-path Hamiltonian if each arc of D lies on a Hamiltonian cycle in D. The *Hamiltonian extension number* $\text{he}(D)$ of D is the greatest positive integer ℓ such that D is *j-path Hamiltonian* for every integer j with $1 \leq j \leq \ell$. Therefore, if D is a Hamiltonian digraph of order $n \geq 3$, then

$$1 \leq \text{he}(D) \leq n$$

and furthermore,

(i) $\text{he}(D) = 1$ if and only if there is an arc in D that does not lie on any Hamiltonian cycle of D and

(ii) $\text{he}(D) = n$ if and only if every path in D lies on some Hamiltonian cycle of D; that is, for each integer j with $1 \leq j \leq n$, every path of order j lies on some Hamiltonian cycle of D.

First, we describe some classes of Hamiltonian digraphs D of order $n \geq 4$ such that $\text{he}(D) = 1$. For a pair n, r of integers with $3 \leq r \leq n-1$, let $C_{n,r}$ be the Hamiltonian digraph of order n obtained from the directed n-cycle $(v_1, v_2, \ldots, v_n, v_1)$ by adding the arc (v_1, v_r). The digraphs $C_{7,3}$ and $C_{7,4}$ are shown in Figure 3.1. Since the arc (v_1, v_r) does not lie on any Hamiltonian cycle of D, it follows that $\text{he}(C_{n,r}) = 1$. In fact, $C_{n,r}$ is

ℓ-path Hamiltonian if and only if either $\ell = 1$ or $n - r + 3 \leq \ell \leq n$. For example, $C_{n,3}$ of order $n \geq 4$ is ℓ-path Hamiltonian if and only if $\ell \in \{1, n\}$ and $C_{n,4}$ of order $n \geq 5$ is ℓ-path Hamiltonian if and only if $\ell \in \{1, n-1, n\}$. While every path of order 7 lies on a Hamiltonian cycle in the digraph $C_{7,3}$ of Figure 3.1, the path $(v_1, v_3, v_4, v_5, v_6, v_7)$ of order 6 lies on no Hamiltonian cycle in the digraph $C_{7,3}$.

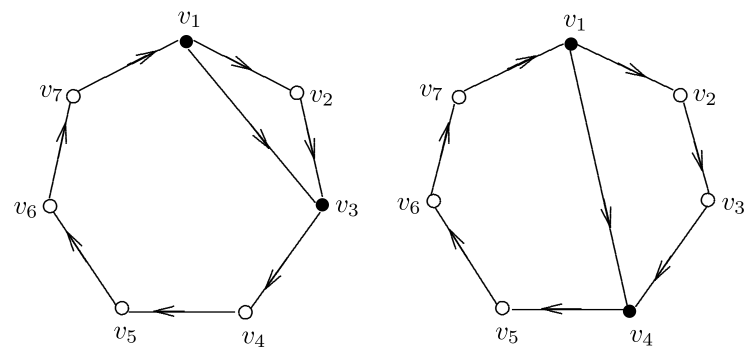


Figure 3.1: The digraphs $C_{7,3}$ and $C_{7,4}$

There are ℓ-path Hamiltonian digraphs for some integers $\ell \geq 3$ that are not $(\ell - 1)$-path Hamiltonian. We have seen this for the class of Hamiltonian digraphs $C_{n,r}$ $(3 \leq r \leq n-1)$.

We now describe another class of Hamiltonian digraphs D_n of order $n \geq 4$ such that $\text{he}(D_n) = 1$. Let D_n be the Hamiltonian digraph of order $n \geq 4$ and size $n + 2$ with Hamiltonian cycle $(v_1, v_2, \ldots, v_n, v_1)$ such that $e_1 = (v_3, v_1)$ and $e_2 = (v_2, v_4)$ are arcs of D_n. Figure 3.2 shows the digraph D_n for $n = 7$.

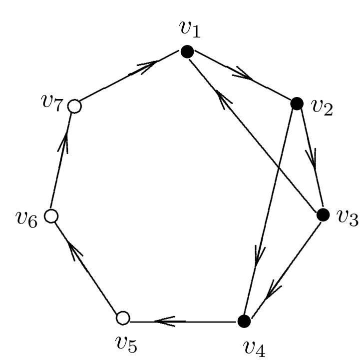


Figure 3.2: A Hamiltonian digraph D_7 of order 7

No Hamiltonian cycle in D_n contains e_1 (or e_2) and so $\text{he}(D_n) = 1$. In fact, by considering subpaths of the n-path $(v_3, v_1, v_2, v_4, v_5, \ldots, v_n)$, it follows that D_n is ℓ-path Hamiltonian if and only if $\ell = 1$. Consequently, the Hamiltonian tournament of order 4 is ℓ-path Hamiltonian if and only if $\ell = 1$.

Next, we describe two classes of Hamiltonian digraphs D of order $n \geq 4$ such that $\text{he}(D) = n$. For a graph G, the symmetric digraph G^* is obtained from G by replacing each edge uv of G by the two arcs (u, v) and (v, u). The symmetric digraphs K_4^* and C_5^* are shown in Figure 3.3.

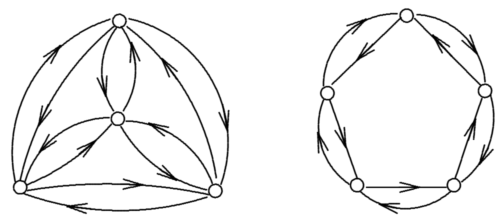


Figure 3.3: Symmetric digraphs K_4^* and C_5^*

For integers k and r where $k, r \geq 2$, the digraph $D_{k,r}$ has vertex set V that can be expressed as the disjoint union $\cup_{i=1}^{k} V_i$ of k subsets V_i of V where $|V_i| = r$ for $1 \leq i \leq k$ such that (u, v) is an arc of $D_{k,r}$ if and only if $u \in V_i$ and $v \in V_{i+1 (\text{mod } k)}$. If $k \geq 3$, then $D_{k,1}$ is the directed k-cycle $\vec{C}_k$. The graph $D_{3,2}$ is shown in Figure 3.4.

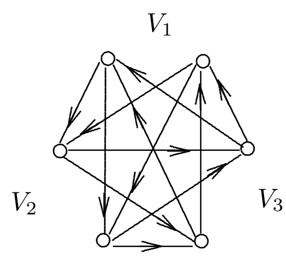


Figure 3.4: The digraph $D_{3,2}$

The following result was obtained by Chartrand, Kronk and Lick [13] in 1969.

Theorem 3.1.1 *A Hamiltonian digraph D of order $n \geq 3$ has Hamiltonian extension number n if and only if D is either K_n^*, C_n^* or $D_{k,r}$ for some positive integers k and r where $k \geq 2$ with $n = kr$.*

Thus, the Hamiltonian digraphs $K_4^*, C_5^*, D_{3,2}$ in Figures 3.3 and 3.4, whose orders are $n = 4, 5, 6$, have Hamiltonian extension number n.

3.2 On $(n-1)$-Path Hamiltonian Digraphs

We saw that all those Hamiltonian digraphs of order n in Theorem 3.1.1 have Hamiltonian extension number n. In this section, we describe some results concerning $(n-1)$-path

Hamiltonian digraphs as well as $(n-2)$-path Hamiltonian digraphs.

Figure 3.5 shows all Hamiltonian digraphs of order 3. By Theorem 3.1.1, $\mathrm{he}(D_1) = \mathrm{he}(D_2) = 3$. For the remaining digraphs of Figure 3.5, $\mathrm{he}(D_3) = \mathrm{he}(D_4) = 1$.

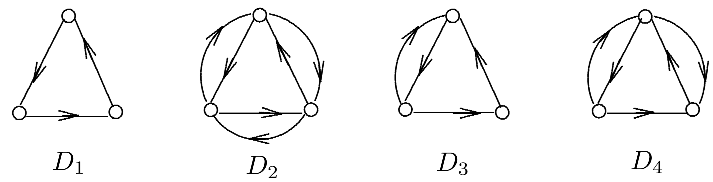


Figure 3.5: Hamiltonian digraphs of order 3

Again by Theorem 3.1.1, the symmetric digraphs K_4^* and C_4^* as well as $\vec{C_4}$ are the only Hamiltonian digraphs of order 4 having Hamiltonian extension number 4. The Hamiltonian digraph of order 4 in Figure 3.6 is not a symmetric digraph. Since the path (w, x, u) of order 3 lies on no Hamiltonian cycle of D, it follows that $\mathrm{he}(D) \leq 2$. Let $C = (u, x, w, v, u)$ be a Hamiltonian cycle of D. The table accompanying Figure 3.6 shows that each arc of D lies on a Hamiltonian cycle of D. Hence, $\mathrm{he}(D) = 2$.

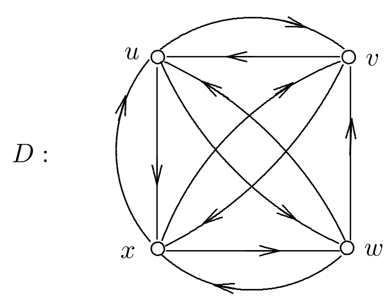


Figure 3.6: A Hamiltonian digraph of order 4

arcs e of D	Hamiltonian cycle containing arcs e
(v, u), (u, x), (x, w), (w, v):	C
(u, v), (v, x), (w, u):	(u, v, x, w, u)
(x, u), (u, w):	(x, u, w, v, u)
(w, x), (x, v):	(w, x, v, u, w)

The digraph in Figure 3.6 shows that if D is a Hamiltonian digraph of order n that is $(n-2)$-path Hamiltonian, then D need not be $(n-1)$-path Hamiltonian. This digraph is not an oriented graph however.

Theorem 3.2.1 *If D is a Hamiltonian oriented graph of order $n \geq 4$ such that D is $(n-2)$-path Hamiltonian, then D is also $(n-1)$-path Hamiltonian.*

Proof. Since the result holds if D itself is a cycle, suppose that this is not the case. Note that a digraph D of order 4 is 2-path Hamiltonian if and only if D itself is a cycle. Hence, suppose that D is $(n-2)$-path Hamiltonian, where $n \geq 5$. Consider an $(n-1)$-path $P = (v_1, v_2, \ldots, v_{n-1})$ with the vertex v_n not on P. Since $P' = P - v_{n-1}$ is an $(n-2)$-path, there is an arc between v_{n-1} and v_n in D. We show that (v_{n-1}, v_n) must be an arc of D.

Assume, to the contrary, that (v_n, v_{n-1}) is an arc of D. Since P' lies on a Hamiltonian cycle of D, it follows that $(v_{n-2}, v_n), (v_{n-1}, v_1) \in E(D)$. Next, we consider the $(n-2)$-path $P'' = P - v_1$. Since P'' lies on a Hamiltonian cycle of D, it follows that $(v_1, v_n), (v_n, v_2) \in E(D)$. Hence, D contains the $(n-2)$-cycle $C = (v_2, v_3, \ldots, v_{n-2}, v_n, v_2)$ and so $C - (v_2, v_3)$ is the $(n-2)$-path $(v_3, v_4, \ldots, v_{n-2}, v_n, v_2)$. Since $C - (v_2, v_3)$ lies on a Hamiltonian cycle of D, it follows that $(v_1, v_3), (v_2, v_{n-1}) \in E(D)$. If $n = 5$, then the 3-path (v_1, v_2, v_4) lies on no Hamiltonian cycle, which is a contradiction.

We may therefore assume that $n \geq 6$. Since the $(n-2)$-path $C - (v_3, v_4)$ lies on a Hamiltonian cycle of D, it follows (v_1, v_4) and (v_3, v_{n-1}) must be arcs in D. However, the $(n-2)$-path $(v_1, v_4, v_5, \ldots, v_{n-2}, v_n, v_{n-1})$ lies on no Hamiltonian cycle, a contradiction.

Hence, as claimed, (v_{n-1}, v_n) is an arc of D. Then (v_n, v_1) must be also an arc in order for P' to be on a Hamiltonian cycle. Consequently, P is on a Hamiltonian cycle in D and so D is $(n-1)$-path Hamiltonian. ■

On the other hand, if D is a Hamiltonian digraph of order n that is $(n-1)$-path Hamiltonian (whether D is an oriented graph or not), then D is also n-path Hamiltonian.

Theorem 3.2.2 *If D is a Hamiltonian digraph of order $n \geq 4$ that is $(n-1)$-path Hamiltonian, then D is also n-path Hamiltonian.*

Proof. Let D be an $(n-1)$-path Hamiltonian digraph of order $n \geq 4$ and let $P = (v_1, v_2, \ldots, v_n)$ be an n-path of D. Then $P' = (v_1, v_2, \ldots, v_{n-1})$ is an $(n-1)$-path of D. Since D is $(n-1)$-path Hamiltonian, it follows that P' lies on a Hamiltonian cycle of D and so $(v_{n-1}, v_n), (v_n, v_1) \in E(D)$. Therefore, $C = (v_1, v_2, \ldots, v_n, v_1)$ is a Hamiltonian cycle of D containing P. ■

As a consequence of Theorem 3.2.1, we have the following useful fact.

Corollary 3.2.3 *Let D be a Hamiltonian oriented graph of order $n \geq 4$.*

(a) *If D is $(n-2)$-path Hamiltonian, then D is both $(n-1)$-path Hamiltonian and n-path Hamiltonian.*

(*b*) *If D is not n-path Hamiltonian, then D is neither $(n-1)$-path Hamiltonian nor $(n-2)$-path Hamiltonian.*

The converses of both Theorems 3.2.1 and 3.2.2 are false. To see this, recall that for integers n and r with $3 \leq r \leq n-1$, the Hamiltonian digraph $C_{n,r}$ of order n (obtained from a directed n-cycle $(v_1, v_2, \ldots, v_n, v_1)$ by adding the arc (v_1, v_r)) is ℓ-path Hamiltonian if and only if either $\ell = 1$ or $n-r+3 \leq \ell \leq n$. As we mentioned, the digraph $C_{n,3}$ of order $n \geq 4$ is n-path Hamiltonian but not $(n-1)$-path Hamiltonian and the digraph $C_{n,4}$ of order $n \geq 5$ is both n-path Hamiltonian and $(n-1)$-path Hamiltonian but not $(n-2)$-path Hamiltonian.

Next, we provide sufficient conditions under which a Hamiltonian oriented graph D of order n is neither $(n-1)$-path Hamiltonian nor $(n-2)$-path Hamiltonian.

Proposition 3.2.4 *Let D be a Hamiltonian oriented graph of order $n \geq 4$ containing a Hamiltonian cycle C.*

(a) *If there exists a 3-path (x, y, z) in C such that $(x, z) \in E(D)$, then D is neither $(n-1)$-path Hamiltonian nor $(n-2)$-path Hamiltonian.*

(b) *If there exists a 3-path (x, y, z) in D such that $(x, z) \in E(C)$, then D is neither $(n-1)$-path Hamiltonian nor $(n-2)$-path Hamiltonian.*

Proof. Let $C = (v_1, v_2, \ldots, v_n, v_1)$. For (a), we may assume, without loss of generality, that $(v_1, v_3) \in E(D)$. Since $(v_2, v_1) \notin E(D)$, the $(n-1)$-path $(v_1, v_3, v_4, \ldots, v_n)$ does not lie on any Hamiltonian cycle in D. Therefore, D is not $(n-1)$-path Hamiltonian. For (b), we may assume that $(v_\alpha, v_1, v_{\alpha+1})$ is a 3-path in D for some integer α with $3 \leq \alpha \leq n-2$. Then D contains the $(n-1)$-path $P = (v_3, v_4, \ldots, v_\alpha,\ v_1, v_{\alpha+1}, v_{\alpha+2}, \ldots, v_n)$. If D is $(n-1)$-Hamiltonian, then P lies on a Hamiltonian cycle in D, which implies that $(v_n, v_2) \in E(D)$. This contradicts (a) however. By Theorem 3.2.1, it follows that D is also not $(n-2)$-path Hamiltonian. ■

We now consider connections between ℓ-path Hamiltonian oriented graphs and their underlying graphs. The underlying graph of a Hamiltonian digraph is certainly Hamiltonian. While an orientation of a Hamiltonian graph G may or may not be Hamiltonian, the graph G has at least one Hamiltonian orientations. If a given Hamiltonian graph G contains vertices of high degree, then any Hamiltonian orientation of G cannot be ℓ-path Hamiltonian for certain values of ℓ, which we show next.

Theorem 3.2.5 *Let G be a Hamiltonian graph of order n such that $\Delta(G) = n-1 \geq 4$ or $\delta(G) \geq n-2 \geq 5$. If D is a Hamiltonian orientation of G, then D is neither $(n-1)$-path Hamiltonian nor $(n-2)$-path Hamiltonian.*

Proof. By Theorem 3.2.1, it suffices to show that D is not $(n-1)$-path Hamiltonian. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in D. First, suppose that $\Delta(G) = n-1$ and that $\deg v_1 = \Delta(G)$. By Proposition 3.2.4(a), if either (v_{n-1}, v_1) or (v_1, v_3) is an arc of D, then D is not $(n-1)$-path Hamiltonian. Hence, we may assume that (v_3, v_1, v_{n-1}) is a 3-path in D. This implies that there is a vertex v_t with $3 \leq t \leq n-2$ such that (v_t, v_1) and (v_1, v_{t+1}) are both arcs of D and so (v_t, v_1, v_{t+1}) is a 3-path in D. Thus, by Proposition 3.2.4(b), D is not $(n-1)$-path Hamiltonian.

Next, suppose that $\Delta(G) = \delta(G) = n-2 \geq 5$. That is, G is an $(n-2)$-regular graph of even order $n \geq 8$. Assume, to the contrary, that there is a Hamiltonian orientation D of G such that D is $(n-1)$-path Hamiltonian. Again, let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in D. Since $\deg_G v_1 = n-2$, there is exactly one vertex v_i $(3 \leq i \leq n-2)$ that is not adjacent to v_1 in G. We now consider two cases, according to whether v_1v_3 and v_1v_{n-1} are edges of G.

Case 1. *$v_1v_3 \notin E(G)$ or $v_1v_{n-1} \notin E(G)$.* We may assume that $v_1v_3 \notin E(G)$ since the argument for the case when $v_1v_{n-1} \notin E(G)$ is similar. Thus, v_1v_4 and v_3v_5 are edges of G. We claim that (v_1, v_4) and (v_5, v_3) are arcs in D. Assume, to the contrary, that (v_4, v_1) is an arc in D. By Proposition 3.2.4(a), the arc (v_1, v_{n-1}) must belong to D. Thus, both (v_4, v_1) and (v_1, v_{n-1}) are arcs in D, which then implies that (v_t, v_1, v_{t+1}) is a 3-path in D for some t with $4 \leq t \leq n-2$. However then, D is not $(n-1)$-path Hamiltonian by Proposition 3.2.4(b), which is a contradiction. Therefore, (v_1, v_4) is an arc in D. The fact that (v_5, v_3) is an arc of D follows from Proposition 3.2.4(a).

Since (v_1, v_4) and (v_5, v_3) are arcs in D, it follows that

$$P = (v_6, v_7, \ldots, v_n, v_1, v_4, v_5, v_3)$$

is an $(n-1)$-path in D. Because (v_2, v_3) is an arc of D, it follows that P cannot be extended to a Hamiltonian cycle in D, which is a contradiction.

Case 2. *$v_1v_3, v_1v_{n-1} \in E(G)$.* By Proposition 3.2.4(a), (v_3, v_1, v_{n-1}) is a 3-path in D. Then $v_1v_\alpha \notin E(G)$ for some integer α with $4 \leq \alpha \leq n-2$.

We now make two claims, the first of which is the following:

$$\text{For each integer } j \text{ with } \alpha+1 \leq j \leq n-2, \text{ the arc } (v_1, v_j) \text{ belongs to } D. \qquad (3.1)$$

Assume, to the contrary, that (v_j, v_1) is an arc in D for some j with $\alpha + 1 \leq j \leq n-2$. Since (v_1, v_{n-1}) is also an arc in D, it follows that (v_t, v_1, v_{t+1}) is a 3-path in D for some t with $\alpha + 1 \le t \le n - 2$. However then, D is not $(n-1)$-path -Hamiltonian by Proposition 3.2.4(b), which is a contradiction. Therefore, (v_1, v_j) must be an arc of D for each j with $\alpha + 1 \le j \le n - 2$, verifying the claim described in (3.1).

The second claim is the following:

$$\text{For each integer } j \text{ with } 3 \le j \le \alpha - 1, \text{ the arc } (v_j, v_1) \text{ belongs to } D. \tag{3.2}$$

Assume, to the contrary, that (v_1, v_j) is an arc in D for some j with $3 \le j \le \alpha - 1$. Let $t \in \{3, 4, \ldots, \alpha - 1\}$ be the smallest integer such that $(v_1, v_t) \in E(D)$. Thus, $t \ge 4$ and $(v_j, v_1) \in E(D)$ for $3 \le j \le t - 1$. Since $(v_{t-1}, v_1) \in E(D)$ and $(v_1, v_t) \in E(D)$, it follows that (v_{t-1}, v_1, v_t) is a 3-path in D. However then, since $(v_{t-1}, v_t) \in E(D)$, it follows by Proposition 3.2.4(b) that D is not $(n-1)$-path -Hamiltonian, which is a contradiction. Therefore, (v_j, v_1) belongs to D for each j with $3 \le j \le \alpha - 1$, verifying the claim described in (3.2).

First, suppose that $\alpha = 4$. Then (v_1, v_6) is an arc in D by (3.1). It follows by Proposition 3.2.4(a) that (v_4, v_2) is also an arc in D. Then $P = (v_4, v_2, v_3, v_1, v_6, v_7, \ldots, v_n)$ is an $(n-1)$-path in D. Since (v_5, v_4) is not an arc of D, it follows that P does not lie on a Hamiltonian cycle in D, a contradiction.

Next, suppose that $5 \le \alpha \le n - 2$. By (3.1) then, $(v_1, v_{\alpha+1})$ is an arc in D. Since $v_1 v_\alpha \notin E(G)$ and $\delta(G) = n - 2$, it follows that $v_n v_\alpha,\ v_\alpha v_3 \in E(G)$. By Proposition 3.2.4(a), both arcs (v_n, v_{n-2}) and $(v_\alpha, v_{\alpha-2})$ belong to D. Applying (3.1) and (3.2) to the vertex v_α, we see that

(1) if $2 \le j \le \alpha - 2$, then $(v_\alpha, v_j) \in E(D)$ and

(2) if $\alpha + 2 \le j \le n$, then $(v_j, v_\alpha) \in E(D)$.

In particular, (v_n, v_α) and (v_α, v_3) are arcs in D. Hence,

$$P = (v_1, v_{\alpha+1}, v_{\alpha+2}, \ldots, v_n, v_\alpha, v_3, v_4, \ldots, v_{\alpha-1})$$

is an $(n-1)$-path in D. Since (v_1, v_2) is an arc of D, it follows that P does not lie on a Hamiltonian cycle in D. ■

Let G be a Hamiltonian graph of order $n \ge 7$ containing a pair x, y of nonadjacent vertices with $\deg x = \deg y = n - 2$. If D is a Hamiltonian orientation of G and D is ℓ-path Hamiltonian, then $1 \le \ell \le n - 3$ or $\ell = n$. This fact is a consequence of the proof of Theorem 3.2.5, which we state next.

Corollary 3.2.6 *Let G be a Hamiltonian graph of order $n \ge 7$ that is the join of a graph of order $n - 2$ and a graph of order 2 (so that $K_{2,n-2} \subseteq G$.) If D is a Hamiltonian orientation of G and D is ℓ-path Hamiltonian, then $1 \le \ell \le n - 3$ or $\ell = n$.*

The condition stated in Corollary 3.2.6 that the order of a Hamiltonian graph is at least 7 is necessary. To see this, consider a Hamiltonian orientation D of $G = K_{2,2,2}$ of order 6. Let $C = (v_1, v_2, \ldots, v_6, v_1)$ be a Hamiltonian cycle in D. If there are two vertices x and y such that $xy \notin E(G)$ and $\vec{d}_C(x, y) = 2$, say $v_1v_3 \notin E(G)$, then either at least one of $(v_3, v_5), (v_4, v_1), (v_5, v_1)$ is an arc in D or $(v_6, v_1, v_4, v_5, v_3)$ is a 5-path in D. In either case, D is not 5-path Hamiltonian. Hence, suppose that the sets $V_i = \{v_i, v_{i+3}\}$, $1 \le i \le 3$, are the three partite sets of G. By Proposition 3.2.4(a), the orientation D shown in Figure 3.7 is the only Hamiltonian orientation of G that is 5-path Hamiltonian. In fact, $D = D_{3,2}$, which is shown in Figure 3.4 and we have seen that $D_{3,2}$ is ℓ-path Hamiltonian for all $\ell \in [6] = \{1, 2, \ldots, 6\}$. These observations together with Theorem 3.2.5 yield the following result.

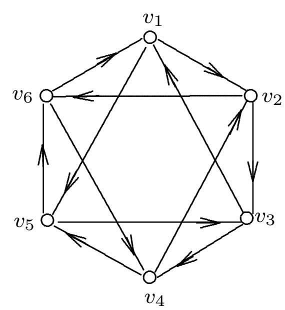


Figure 3.7: A Hamiltonian orientation D of $K_{2,2,2}$ for which $\text{he}(D) = 6$

Corollary 3.2.7 *For an even integer $n \ge 4$, let $G = K_n - M$ where M is a perfect matching in G and let D be a Hamiltonian orientation of G. Then D is $(n-1)$-path Hamiltonian if and only if either* (i) *$n = 4$ and $D = \vec{C}_4$ or* (ii) *$n = 6$ and $D = D_{3,2}$. Furthermore,* $\text{he}(D) = n$ *for each $D \in \{\vec{C}_4, D_{3,2}\}$.*

3.3 Preliminary Results on Tournaments

It is well known that there are only four non-isomorphic tournaments of order 4, only one of which is strong. Consequently, by Theorem 1.4.1, there is only one Hamiltonian tournament of order 4. This tournament can be constructed from the 4-cycle $C = (v_1, v_2, v_3, v_4, v_1)$ by adding the two arcs (v_1, v_3) and (v_2, v_4). The cycle C is the only Hamiltonian cycle in T. Since the arc (v_2, v_4) does not lie on C, the tournament T is not 2-path Hamiltonian. Consequently, the paths (v_2, v_4, v_1) and (v_2, v_4, v_1, v_3) do not lie on C either. Thus, T is not ℓ-path Hamiltonian for any $\ell \in \{2, 3, 4\}$. Since T is obviously 1-path Hamiltonian, T is ℓ-path Hamiltonian if and only if $\ell = 1$. By Theorem 3.2.5, if a Hamiltonian tournament of order 5 is ℓ-path Hamiltonian, then $\ell \neq 3, 4$.

In the remainder of this section, we therefore restrict our attention to Hamiltonian tournaments of order 5 or more. For each integer $n \geq 5$, we next describe a Hamiltonian tournament D_n of order n with $V(D_n) = \{v_1, v_2, \ldots, v_n\}$. Let T_n be the transitive tournament of order $n \geq 5$ with vertex set $V(T_n) = \{v_1, v_2, \ldots, v_n\}$ and arc set

$$E(T_n) = \{(v_i, v_j) : \; 1 \leq i < j \leq n\}.$$

The tournament D_n is then obtained from T_n by changing the directions of the two arcs (v_1, v_n) and (v_2, v_4). This is shown in Figure 3.8. Hence, $C = (v_1, v_2, v_3, \ldots, v_n, v_1)$ is a Hamiltonian cycle in D_n.

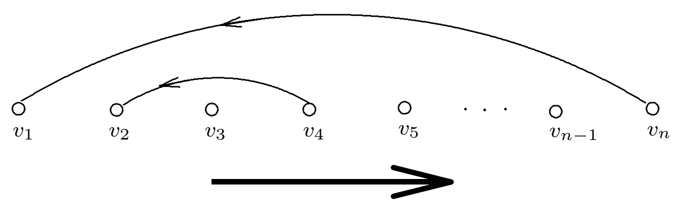


Figure 3.8: The tournament D_n of order $n \geq 5$

We now make some observations concerning the Hamiltonian tournaments D_n for $n \geq 5$.

Observation 3.3.1 *Let $n \geq 5$. The Hamiltonian tournament D_n of order $n \geq 5$ is ℓ-path Hamiltonian if and only if*

$$\ell \in \begin{cases} \{1, 2, n\} & \text{if } n = 5 \\ \{1, n\} & \text{if } n \geq 6. \end{cases}$$

In particular, the arc (v_2, v_n) lies on no Hamiltonian cycle in D_n, and the 3-path (v_2, v_n, v_1) lies on no Hamiltonian cycle in D_n. Furthermore, the path $(v_k, v_{k+1}, \ldots, v_1, v_3, v_5)$ lies on no Hamiltonian cycle in D_n where $6 \leq k \leq n$.

Observation 3.3.2 *Let D be a Hamiltonian tournament of order $n = 3, 4, 5$.*

(a) *For $n = 3$, D is ℓ-path Hamiltonian for $\ell = 1, 2, 3$.*

(b) *For $n = 4$, D is ℓ-path Hamiltonian only if $\ell = 1$.*

(c) *For $n = 5$, D is ℓ-path Hamiltonian for $\ell = 1, 2, 5$. Furthermore, D is 5-path Hamiltonian if and only if $D = D_5$.*

Combining Theorem 3.2.5 and Observation 3.3.2, we obtain the following results for tournaments.

Corollary 3.3.3 *If D is an ℓ-path Hamiltonian tournament of order $n \geq 4$, then $1 \leq \ell \leq n-3$ or $\ell = n$.*

We saw that the Hamiltonian digraph $C_{n,r}$ of order n, where n and r satisfy $3 \leq r \leq n-1$, is ℓ-path Hamiltonian if and only if $\ell = 1$ or $n - r + 3 \leq \ell \leq n$. Certainly, the *underlying graph* G of $C_{n,r}$ is the n-cycle $(v_1, v_2, \ldots, v_n, v_1)$ with the chord $e = (v_1, v_r)$. Since e does not lie on any Hamiltonian cycle in G, it follows that G is ℓ-path Hamiltonian if and only if $\ell = 1$. On the other hand, while a complete graph of order $n \geq 3$ is clearly ℓ-path Hamiltonian for $1 \leq \ell \leq n$, Corollary 3.3.3 states that every tournament of order $n \geq 4$ is not ℓ-path Hamiltonian for $\ell \in \{n-2, n-1\}$. These examples show that

(1) there is an ℓ-path Hamiltonian graph G for some integer ℓ such that G has an orientation D that is not ℓ-path Hamiltonian and

(2) there is a graph G' that is not ℓ'-path Hamiltonian for some integer ℓ' but G has an orientation D' that is ℓ'-path Hamiltonian.

In order to present some preliminary results on tournaments, we first establish some additional definitions and notation. For a digraph D, the *maximum outdegree* $\Delta^+(D)$ and the *minimum outdegree* $\delta^+(D)$ of D are defined, respectively, by

$$\begin{aligned} \Delta^+(D) &= \max\{\operatorname{od} v : v \in V(D)\} \\ \delta^+(D) &= \min\{\operatorname{od} v : v \in V(D)\}. \end{aligned}$$

Similarly, the *maximum indegree* $\Delta^+(D)$ and the *minimum indegree* $\delta^+(D)$ of D are defined, respectively, by

$$\begin{aligned} \Delta^-(D) &= \max\{\operatorname{id} v : v \in V(D)\} \\ \delta^-(D) &= \min\{\operatorname{id} v : v \in V(D)\}. \end{aligned}$$

The *maximum semi-degree* $\Delta^0(D)$ and the *minimum semi-degree* $\delta^0(D)$ of D are defined, respectively, by

$$\begin{aligned} \Delta^0(D) &= \max\{\Delta^+(D), \Delta^-(D)\} \\ \delta^0(D) &= \min\{\delta^+(D), \delta^-(D)\}. \end{aligned}$$

Hence, for every digraph D, it follows that

$$\begin{aligned} \delta^0(D) \le \delta^+(D) \le \Delta^+(D) \le \Delta^0(D) \\ \delta^0(D) \le \delta^-(D) \le \Delta^-(D) \le \Delta^0(D). \end{aligned} \tag{3.3}$$

Furthermore, if D is a tournament of order n, then there exist vertices u and v of D such that $\operatorname{od} u = \Delta^+(D)$, $\operatorname{id} u = \delta^-(D)$ and $\operatorname{od} v = \Delta^-(D)$ and $\operatorname{id} v = \delta^+(D)$. Thus,

$$\Delta^+(D) + \delta^-(D) = \Delta^-(D) + \delta^+(D) = \Delta^0(D) + \delta^0(D) = n - 1. \tag{3.4}$$

The following three lemmas are well-known facts (see [14, 38]).

Lemma 3.3.4 *If D is a tournament of order $n \ge 3$ with $\Delta^+(D) - \delta^+(D) < n/2$, then D is Hamiltonian.*

Equivalently, Lemma 3.3.4 can be stated as follows:

> *If D is a tournament of order $n \ge 3$ with $\operatorname{od} u - \operatorname{od} v < n/2$ for every pair u, v of vertices of D, then D is Hamiltonian.*

Lemma 3.3.5 *If x is a vertex of maximum outdegree $\Delta^+(D)$ in a tournament D, then $\vec{d}(x, v) \le 2$ for every vertex v of D.*

Lemma 3.3.6 *Each vertex x of a nontrivial tournament D with $\operatorname{od} x = \Delta^+(D)$ is the initial vertex of some Hamiltonian path in D.*

Lemma 3.3.6 can be extended as follows.

Lemma 3.3.7 *Every nontrivial tournament D contains a Hamiltonian $x-y$ path where $\operatorname{od} x = \Delta^+(D)$ and $\operatorname{od} y = \delta^+(D)$.*

Proof. Since the statement holds for tournaments of order at most 3, we may assume that D is a tournament of order $n \ge 4$. Let

$$X = \{v \in V(D) : \operatorname{od} v = \Delta^+(D)\} \text{ and } Y = \{v \in V(D) : \operatorname{od} v = \delta^+(D)\}.$$

If $\Delta^+(D) = \delta^+(D)$, then $X = Y = V(D)$ and the result is immediate by Lemma 3.3.6. Hence, we may assume that $\delta^+(D) < \Delta^+(D)$ and so $X \cap Y = \emptyset$. For each $v \in X$, let $\ell(v)$ be the order of a longest $v - u$ path where $u \in Y$ and let

$$\ell = \max\{\ell(v) : v \in X\}.$$

We claim that $\ell \geq 3$. Assume, to the contrary, that $\ell = 2$. Then $|Y| = 1$, say $Y = \{y\}$. By Lemma 3.3.6, each vertex in X is the initial vertex of some Hamiltonian path in D. This implies that $(x, y) \in E(D)$ for each $x \in X$. Since $\operatorname{od} y = \delta^+(D) < \Delta^+(D)$, it follows that $N^-(y) \neq \emptyset$. Let $x \in X$ and $z \in N^-(y)$. We now consider $\operatorname{od} z$. Since $(z, y) \in E(D)$, it follows that $(z, x) \in E(D)$; for otherwise, (x, z, y) has order 3, which contradicts the fact that $\ell = 2$. Furthermore, if $w \in N^+(x)$, then $(z, w) \in E(D)$, for otherwise, (x, w, z, y) has order 4, which is impossible. Hence, $N^+(x) \cup \{x\} \subseteq N^+(z)$. However then, $\operatorname{od} z \geq \operatorname{od} x + 1 = \Delta^+(D) + 1$, a contradiction. Therefore, $\ell \geq 3$, as claimed.

Let $x \in X$ such that $\ell(x) = \ell$ and let $P = (x = v_1, v_2, \ldots, v_\ell = y)$ be an $x - y$ path of order ℓ where $y \in Y$. We claim that P is a Hamiltonian path.

Assume, to the contrary, that this is not the case. Thus, $3 \leq \ell \leq n - 1$. Let

$$Z_1 = N^-(y) - V(P) \text{ and } Z_2 = N^+(y) - V(P).$$

Then

$$Z = Z_1 \cup Z_2 = V(D) - V(P) \neq \emptyset.$$

If $z \in Z$ and (z, v_i) is an arc for some $i \geq 2$, then so is (z, v_{i-1}); for otherwise, z can be inserted between v_{i-1} and v_i in P to form an $x - y$ path that is longer than P, which cannot occur.

Next, let z be an arbitrary element of Z. Then exactly one of the following three situations occurs:

(i) $V(P) \subseteq N^+(z)$,

(ii) $V(P) \subseteq N^-(z)$,

(iii) there exists an integer α $(1 \leq \alpha \leq \ell - 1)$ such that $v_i \in N^+(z)$ if and only if $1 \leq i \leq \alpha$.

Obviously, $z \in Z_1$ if and only if (i) occurs. We consider two cases, according to whether $Z_1 \neq \emptyset$ or $Z_1 = \emptyset$.

Case 1. $Z_1 \neq \emptyset$. Let $z \in Z_1$. Then (i) occurs and so $\vec{d}(x, z) = 2$ by Lemma 3.3.5. Hence, there exists $z' \in Z_2$ such that (x, z', z) is a path. However then,

$$(x, z', z, v_2, v_3, \ldots, v_{\ell-1}, y)$$

is an $x - y$ path that is longer than P, which is impossible. Therefore, this case cannot occur.

Case 2. $Z_1 = \emptyset$. Thus, $Z = Z_2 \neq \emptyset$. Let $z \in Z$. Since (P, z) is a path longer than P, it follows that $z \notin Y$, that is,

$$|Z| \leq \operatorname{od} y < \operatorname{od} z. \tag{3.5}$$

The condition in (3.5) implies that (ii) cannot occur or $V(P) \not\subseteq N^-(z)$. To show this, assume, to the contrary, that $V(P) \subseteq N^-(z)$. Since (1) $N^+(y) = Z = V(D) - V(P)$, (2) $N^+(z) \cap N^-(z) = \emptyset$ and (3) $V(P) \subseteq N^-(z)$, it follows that $N^+(z) \cap V(P) = \emptyset$ and so $N^+(z) \subseteq V(D) - V(P) = N^+(y)$. However then, $\operatorname{od} z \leq \operatorname{od} y$, which is impossible by (3.5).

Next, we claim that $N^+(x) \subseteq V(P)$. Assume, to the contrary, that $N^+(x) - V(P) \neq \emptyset$. Then $N^+(x) - V(P) \subseteq Z$. Let $z' \in N^+(x) - V(P) \subseteq Z$. Since $V(P) \not\subseteq N^-(z')$, there is $v_j \in V(P)$, where $2 \leq j \leq \ell - 1$, such that $(z', v_j) \in E(D)$. Then $(z', v_i) \in E(D)$ for each integer i with $1 \leq i \leq j$ (for otherwise, there is an $x - y$ path longer than P). In particular, $(z', v_1) = (z', x) \in E(D)$, which is impossible. Hence, $N^+(x) \subseteq V(P)$, as claimed.

Now, let $\alpha = \max\{i : v_i \in N^+(z)\}$. Since $N^+(x) \subseteq V(P)$, we have $(z, x) \in E(D)$ and so $1 \leq \alpha \leq \ell - 1$. Since (z, P) is also a path longer than P, it follows that $z \notin X$ and so

$$\operatorname{od} z < \operatorname{od} x. \tag{3.6}$$

If $\operatorname{od} z = 1$, then by (3.5) $\operatorname{od} y = 0$ and so $Z_2 = \emptyset$, which is impossible. Hence, $2 \leq \operatorname{od} z < \operatorname{od} x \leq |V(P)| - 1$. Then $2 \leq \ell - 2$ and so $\ell \geq 4$ and the four arcs (z, x), (z, v_2), $(v_{\ell-1}, z)$, (y, z) are in D.

If (x, v_i) is an arc for some i $(3 \leq i \leq \ell - 1)$, then so is (y, v_{i-1}) to avoid an $x - y$ path longer than P (for otherwise, $(x, v_i, v_{i+1}, \ldots, v_\ell = y, z, v_2, \ldots, v_{i-1}, y)$ is an $x - y$ path longer than P). Hence, if we let $N = N^+(x) - \{v_2, y\}$, then $\operatorname{od} x \leq |N| + 2$ and $\operatorname{od} y \geq |N| + |Z| \geq |N| + 1$. Hence, $\operatorname{od} x - \operatorname{od} y = 1$. However, this contradicts the fact that $\operatorname{od} y < \operatorname{od} z < \operatorname{od} x$ by (3.5) and (3.6). Thus, Case 2 cannot occur. ■

With the aid of Lemma 3.3.7, we have the following result.

Proposition 3.3.8 *Let D be a tournament of order at least 3. If $e = (x, y)$ is an arc in D such that $\operatorname{od} x = \delta^+(D)$ and $\operatorname{od} y = \Delta^+(D)$, then e lies on a Hamiltonian cycle in D.*

Proof. Let D' be the tournament obtained from D by reversing the direction of e, that is, (y, x) is now an arc of D'. Then $\Delta^+(D') = \Delta^+(D) + 1$ and $\delta^+(D') = \delta^+(D) - 1$. Furthermore, $\operatorname{od}_{D'} v = \Delta^+(D')$ if and only if $v = y$ and $\operatorname{od}_{D'} v = \delta^+(D')$ if and only if

$v = x$. By Lemma 3.3.7 then, there exists a Hamiltonian $y - x$ path P in D' and so in D as well. The path P together with the arc e forms a Hamiltonian cycle in D. ∎

3.4 Two Classes of Tournaments

We are now prepared to consider two classes of well-known tournaments. A digraph D is *regular* if there is a nonnegative integer r such that $\operatorname{od} v = \operatorname{id} v = r$ for every vertex v in D. In this case, D is said to be *r-regular* and the order of D is $2r + 1$.

A digraph D is *almost regular* if there is a nonnegative integer r such that D contains exactly r vertices of out-degree r and the remaining r vertices have out-degree $r - 1$. In this case, D is said to be *almost r-regular* and the order of D is $2r$. An almost regular tournament can be obtained from a regular tournament by deleting a vertex. The following two results are consequences of Lemma 3.3.4.

Proposition 3.4.1 *Every regular tournament of order at least* 3 *is Hamiltonian.*

Proposition 3.4.2 *Every almost regular tournament of order at least* 4 *is Hamiltonian.*

The following is a consequence Proposition 3.3.8.

Corollary 3.4.3 *Every nontrivial regular tournament is* 2*-path Hamiltonian.*

While every arc in a nontrivial regular tournament lies on a Hamiltonian cycle, it turns out that every arc in such a tournament of order $n \geq 5$ lies on a Hamiltonian path that *cannot* be extended to a Hamiltonian cycle. Since a 2-regular tournament of order 5 is unique, this can be verified readily. For $n \geq 7$, we have a stronger fact, namely that every arc is the initial arc of a Hamiltonian path that cannot be extended to a Hamiltonian cycle. In order to show this fact, we first verify the following.

Lemma 3.4.4 *Let D be an almost regular tournament of order $n \geq 6$. Then for every $x \in V(D)$ with $\operatorname{od} x = \Delta^+(D)$, there exists a Hamiltonian $x - y$ path in D with $\operatorname{od} y = \Delta^+(D)$.*

Proof. Suppose that D is an almost r-regular tournament of order $n = 2r$ for some integer $r \geq 3$. Let

$$X = \{v \in V(D) : \operatorname{od} v = r\}.$$

Thus, $|X| = r$. Let x be an arbitrary vertex in X. For each $v \in V(D)$, let $\ell(v)$ be the order of a longest $x - v$ path in D. Let $y \in X$ such that

$$\ell = \ell(y) = \max\{\ell(v) : v \in X\}.$$

If $\ell = 2$, then $|X| = 2$. This implies that $r = 2$, which is impossible since $n \geq 6$. Hence, $\ell \geq 3$. Let $P = (x = v_1, v_2, \ldots, v_\ell = y)$ be an $x - y$ path of order ℓ.

It suffices to show that P is a Hamiltonian path in D, namely $\ell = 2r$. Assume, to the contrary, that this is not the case. Let $Z = V(D) - V(P)$, which is then nonempty. Observe that, for each $z \in Z$, exactly one of the following three situations occurs:

(i) $V(P) \subseteq N^-(z)$,

(ii) $V(P) \subseteq N^+(z)$ and

(iii) there exists an integer α ($1 \leq \alpha \leq \ell - 1$) such that $v_i \in N^+(z)$ if and only if $1 \leq i \leq \alpha$.

Hence, if we let $Z = Z_1 \cup Z_2$, where $Z_1 = N^+(x) - V(P)$ and $Z_2 = N^-(x) - V(P)$, then $z \in Z_1$ if and only if (i) occurs. We consider the following two cases, according to whether $Z_1 \neq \emptyset$ or $Z_1 = \emptyset$.

Case 1. $Z_1 \neq \emptyset$. Let $z \in Z_1$. Then z satisfies (i) and so $N^+(z) \subseteq Z$. If $z' \in N^+(z)$, then $z' \in N^+(v_2)$ since otherwise an $x - y$ path that is longer than P is formed by inserting (z, z') between x and v_2 in P. However then, $N^+(z) \cup \{z, v_3\} \subseteq N^+(v_2)$ and so $\operatorname{od} v_2 - \operatorname{od} z \geq 2$, which is impossible. Thus, this case never occurs.

Case 2. $Z_1 = \emptyset$. Then $Z = Z_2$ and so every vertex in Z satisfies either (ii) or (iii). Hence, $r = \operatorname{od} x \leq |V(P)| - 1$ (or, equivalently, $r - 1 = \operatorname{id} x \geq |Z|$). Let $z \in Z$. If z satisfies (ii), then $|V(P)| \leq \operatorname{od} z \leq r$, which contradicts the fact that $r \leq |V(P)| - 1$. Therefore, for every $z \in Z$, (iii) occurs and so (y, z, x) is a path. Since (P, z) is an $x - z$ path longer than P, it follows that $\operatorname{od} z = r - 1$. In other words, $X \cap Z = \emptyset$.

If $|Z| \geq 2$, then there exists a vertex $z' \in Z$ such that (z', v_2) is an arc since $|Z| \leq r - 1$. Now let $N = N^+(x) - \{v_2\}$, which must be a subset of $V(P)$. If $v_i \in N$, then $(x, v_i, v_{i+1}, \ldots, v_{\ell-1}, y, z', v_2, v_3, \ldots, v_{i-1})$ is an $x - v_{i-1}$ path that is longer than P and so $\operatorname{od} v_{i-1} = r - 1$. Hence, P contains at least $|N|$ $(= r - 1)$ vertices not belonging to X. However then, $|X| \leq 2r - |N \cup Z| < r$, which cannot occur. Therefore, we may assume that $Z = \{z\}$ and $\ell = 2r - 1$. Moreover, $N^+(z) = \{x, v_2, \ldots, v_{r-1}\}$ and $N^-(z) = \{v_r, v_{r+1}, \ldots, v_{2r-2}, y\}$.

If $v_\beta \in N^+(x)$ for some β ($3 \leq \beta \leq r$), then $v_\gamma \in N^-(v_{\beta-1})$ for $r + 1 \leq \gamma \leq 2r - 1$ since the $x - y$ path $(x, v_\beta, v_{\beta+1}, \ldots, v_{\gamma-1}, z, v_2, v_3, \ldots, v_{\beta-1}, v_\gamma, v_{\gamma+1}, \ldots, v_{2r-2}, y)$ cannot exist. However then, $\operatorname{od} v_{\beta-1} \leq r - 2$, a contradiction. Therefore, $N^+(x) = \{v_2, v_{r+1}, v_{r+2}, \ldots, v_{2r-2}, y\}$, which then implies that $\operatorname{od} v_i = r - 1$ for $r \leq i \leq 2r - 2$ since

$(x, v_{i+1}, v_{i+2}, \ldots, v_{2r-2}, y, z, v_2, v_3, \ldots, v_i)$ is a Hamiltonian $x - v_i$ path for $r \leq i \leq 2r-2$. Thus, $X = \{x, v_2, \ldots, v_{r-1}, y\}$. Now, depending on the direction of the arc between v_r and y, exactly one of the two paths

$$(x, y, v_r, v_{r+1}, \ldots, v_{2r-2}, z, v_2, v_3, \ldots, v_{r-1})$$
$$(x, v_{r+1}, v_{r+2}, \ldots, v_{2r-2}, z, v_2, v_3, \ldots, v_r, y)$$

is a Hamiltonian path in D with x as the initial vertex. However, this is again a contradiction as both v_{r-1} and y belong to X. ∎

We are prepared to show the following result.

Proposition 3.4.5 *If D is an r-regular tournament where $r \geq 3$, then every arc in D is the initial arc of a Hamiltonian path that cannot be extended to a Hamiltonian cycle in D.*

Proof. Let $e = (x, y)$ be an arc in D. Then the tournament $D' = D - x$ of order $2r \geq 6$ is almost regular and $\mathrm{od}_{D'}\, v = r$ if and only if $v \in N^+(x)$. By Lemma 3.4.4, the existence of a Hamiltonian $y - z$ path P' in D' with $z \in N^+(x)$ is guaranteed. Then $P = (x, P')$ is a Hamiltonian $x - z$ path in D whose initial arc is e. Furthermore, P cannot be extended to a Hamiltonian cycle as $z \in N^+(x)$. ∎

In summary, we have the following.

Corollary 3.4.6 *Let D be a regular tournament of odd order $n \geq 3$.*

- $\star$ *For $n \geq 5$, every vertex is the initial vertex of a Hamiltonian path that cannot be extended to a Hamiltonian cycle.*
- $\star$ *For $n \geq 7$, every arc is the initial arc of a Hamiltonian path that cannot be extended to a Hamiltonian cycle.*

Furthermore,

(1) *D is 2-path Hamiltonian for every odd $n \geq 3$ and*

(2) *D is n-path Hamiltonian if and only if $n = 3$.*

We now turn our attention to almost regular tournaments. There is exactly one Hamiltonian tournament D of order 4, which is also the only tournament D of order 4 that is almost regular. We have already seen that D is not 2-path Hamiltonian. There are exactly two arcs not belonging to any Hamiltonian cycle, which are shown in Figure 3.9 as dashed line segments.

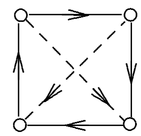


Figure 3.9: The almost regular tournament of order 4

Now, let D be an almost regular tournament of order 6. There are exactly five such tournament D, two of which contain an arc that cannot be extended to a Hamiltonian cycle, which we denoted by D_1 and D_2 as shown in Figure 3.10. The tournament D_2 is obtained from D_1 by by reversing the direction of the arc (x, y). In each of D_1 and D_2, the arc, denoted as a dashed line, is the only arc that does not lie on any Hamiltonian cycle.

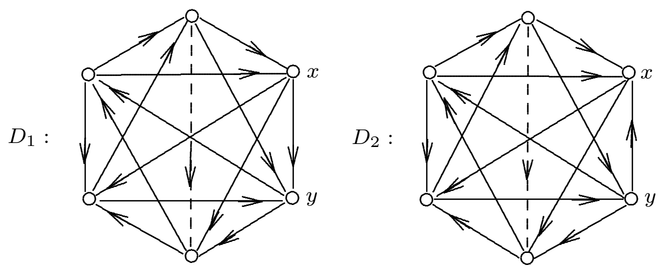


Figure 3.10: Two almost regular tournaments of order 6

Proposition 3.4.7 *An almost regular tournament D of order* 6 *is not* 2*-path Hamiltonian if and only if it is isomorphic to one of the two tournaments D_1 or D_2 shown in* Figure 3.10. *Furthermore, there is exactly one arc that belongs to no Hamiltonian cycle in D_i for each for $i = 1, 2$.*

Proof. Let D be an almost regular tournament of order 6 with

$$V(D) = \{v_1, v_2, \ldots, v_6\}.$$

Thus, $\Delta^+(D) = \delta^+(D) + 1 = 3$. Since neither D_1 nor D_2 is 2-path Hamiltonian, it remains to verify the converse. Suppose that D is not 2-path Hamiltonian. We show that $D = D_1$ or $D = D_2$. Assume, without loss of generality, that $e = (v_5, v_6)$ is an arc that does not lie on any Hamiltonian cycle in D. Let D' be the tournament of order 4 obtained from D by deleting v_5 and v_6. It can be verified that if $\Delta^+(D') = 3$ or $\delta^+(D') = 0$, then the arc e does belong to a Hamiltonian cycle in D. Hence, D' must be isomorphic to the tournament shown in Figure 3.9. In particular, D' contains a Hamiltonian cycle, say $(v_1, v_2, v_3, v_4, v_1)$. If (v_6, v_1) is an arc, say, then so is (v_5, v_4) in order to avoid a Hamiltonian cycle containing e. Hence, it must be that $\operatorname{od} v_6 < \operatorname{od} v_5$. If either (v_6, v_2) or (v_6, v_4) is also an arc in addition to the arc (v_6, v_1), say the former, then both (v_5, v_1) and (v_5, v_4) are arcs. Then exactly one of the 6-cycles

$$(v_5, v_6, v_2, v_4, v_1, v_3, v_5) \text{ and } (v_5, v_6, v_1, v_3, v_4, v_2, v_5)$$

is in D depending on the direction of the arc between v_2 and v_4, both of which contains the arc e.

It then follows that, if $e = (v_5, v_6)$ is an arc that does not lie on a Hamiltonian cycle in D, then we may assume that

(a) $\operatorname{od} v_5 = \operatorname{od} v_6 + 1 = 3$,

(b) D' contains a 4-cycle $C' = (v_1, v_2, v_3, v_4, v_1)$ and

(c) $N^+(v_6) = \{v_1, v_3\}$.

Since the tournament with (v_1, v_3) as an arc in D is isomorphic to the tournament with (v_3, v_1) as an arc in D, we may suppose that (v_1, v_3) is an arc. By (c), it also must be that $N^+(v_5) = \{v_2, v_4, v_6\}$. Hence, D must contain the digraph in Figure 3.11 as a spanning subgraph. Therefore, D is isomorphic to one of the two tournaments D_1 or D_2 shown in Figure 3.10.

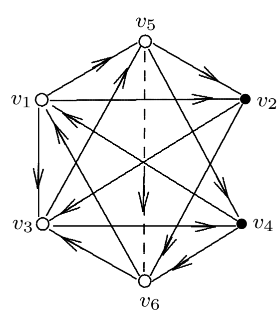


Figure 3.11: A subgraph in a tournament of order 6 that is not 2-path Hamiltonian

Finally, we show that every arc in $E(D) - \{(v_5, v_6)\}$ lies on a Hamiltonian cycle. Observe that if $(x, y) \in E(D)$ with $\operatorname{od} x = \operatorname{od} y + 1 = 3$ and $(x, y) \neq (v_5, v_6)$, then (x, y) belongs to exactly one of the three 6-cycles

$$(v_1, v_2, v_3, v_5, v_4, v_6, v_1),\ (v_1, v_5, v_2, v_6, v_3, v_4, v_1),\ (v_1, v_3, v_5, u, w, v_6, v_1),$$

where $\{u, w\} = \{v_2, v_4\}$. By (a), the desired result follows. ■

Using the fact that an almost regular tournament of order 6 contains at most one arc that cannot be extended to a Hamiltonian cycle, we are able to establish the following general result.

Theorem 3.4.8 *Every almost regular tournament of order at least* 8 *is* 2*-path Hamiltonian.*

Proof. Let D be a tournament of even order $2r \geq 8$ with $\Delta^+(D) = \delta^+(D) + 1 = r$. Hence, $\{\mathrm{od}\, v, \mathrm{id}\, v\} = \{r-1, r\}$ for each vertex v. Assume, to the contrary, that $e = (x, y)$ is an arc in D that lies on no Hamiltonian cycle in D. If we let $N_1 = N^-(x)$ and $N_2 = N^+(y)$, then $|N_1| = \mathrm{id}\, x$ and $|N_2| = \mathrm{od}\, y$. We consider the tournament D' of order $2r - 2 \geq 6$ obtained from D by deleting x and y. Note that $e = (x, y)$ lies on a Hamiltonian cycle in D if and only if D' contains a Hamiltonian $y' - x'$ path with $x' \in N_1$ and $y' \in N_2$.

Observe that, for each $v \in V(D')$,

$$\mathrm{od}_{D'} v = \begin{cases} \mathrm{od}_D\, v - 2 & \text{if } v \in N_1 - N_2 \\ \mathrm{od}_D\, v & \text{if } v \in N_2 - N_1 \\ \mathrm{od}_D\, v - 1 & \text{otherwise.} \end{cases}$$

Hence, $r - 3 \leq \delta^+(D') \leq r - 2$ and $r - 1 \leq \Delta^+(D') \leq r$. Furthermore, $N_1 - N_2 \neq \emptyset$ if $\delta^+(D') = r - 3$ and $N_2 - N_1 \neq \emptyset$ if $\Delta^+(D') = r$.

Case 1. $\Delta^+(D') - \delta^+(D') = 3$. Then there exists a Hamiltonian $y' - x'$ path P' in D' with $\mathrm{od}_{D'}\, x' = r - 3$ and $\mathrm{od}_{D'}\, y' = r$. Since $x' \in N_1 - N_2$ and $y' \in N_2 - N_1$, a Hamiltonian cycle (P', x, y, y') in D containing the arc (x, y) is produced, which is a contradiction. Hence, this case never occurs.

Case 2. $\Delta^+(D') - \delta^+(D') \leq 2$. Then D' is Hamiltonian by Lemma 3.3.4, say $C' = (v_1, v_2, \ldots, v_{2r-2}, v_1)$ is a Hamiltonian cycle in D'. If $v_1 \in N_2$, say, then $v_{2r-2} \notin N_1$ in order to avoid a Hamiltonian cycle in D containing e. Hence, it must be that $\mathrm{od}\, y < \mathrm{od}\, x$ and so $|N_1| = |N_2| = r - 1$. Since D is almost regular, $\mathrm{od}_D\, x = r$ and $\mathrm{od}_D\, y = r - 1$. Since $V(D') = 2r - 2$ and $|N_1| = |N_2| = r - 1$, it then follows that $N_1 \cap N_2 = \emptyset$. If $\mathrm{od}_D\, z = r$ for some $z \in N_2$, then $\mathrm{od}_{D'}\, z = \mathrm{od}_D\, z = r$. Since $\Delta^+(D') - \delta^+(D') \leq 2$, it follows that $\mathrm{od}_D\, v = r$ for each vertex $v \in N_1$. Thus, each vertex in $N_1 \cup \{x, z\}$ has outdegree r in D, that is, D contains at least $r + 1$ vertices of outdegree r in D, which is impossible.

So we may assume that $\mathrm{od}_D\, z = r - 1$ for each vertex $z \in N_2$. Since D is almost regular and $\mathrm{od}_D\, y = r - 1$, it then follows that $\mathrm{od}_D\, u = r$ for each $u \in N_1 \cup \{x\}$ and $\mathrm{od}_D\, v = r - 1$ for each $v \in N_2 \cup \{y\}$. Thus, $\mathrm{od}_{D'}\, u = r - 2$ for each $u \in N_1$ and $\mathrm{od}_{D'}\, v = r - 1$ for each $v \in N_2$. Hence, D' is an almost regular tournament of order $2r - 2$. By Lemma 3.3.7 D' contains a Hamiltonian $y' - x'$ path P with $\mathrm{od}_{D'}\, y' = r - 1$

and $\mathrm{od}_{D'}\, x' = r - 2$. Then (P, x, y, y') is a Hamiltonian cycle in D containing $e = (x, y)$, a contradiction. ■

Corollary 3.4.9 *Let D be a regular or almost regular tournament of order $n \geq 3$. If $n \neq 4, 6$, then D is 2-path Hamiltonian.*

In a regular tournament of order $n \geq 5$, a 3-path (x, y, z) with $(x, z) \in E(D)$ lies on a Hamiltonian cycle in D. This is because the tournament $D - y$ contains a Hamiltonian $z - x$ path by Proposition 3.3.8. If D is a regular tournament of order $n \geq 9$, then *every* 3-path lies on a Hamiltonian cycle, that is, D is 3-path Hamiltonian. To see this, proving the following is sufficient.

Lemma 3.4.10 *Let D be an almost regular tournament of order $n \geq 8$. For every pair x, y of vertices with $\mathrm{od}\, x > \mathrm{od}\, y$, there exists a Hamiltonian $x - y$ path.*

Proof. If (y, x) is an arc, then the result is immediate by the fact that D is 2-path Hamiltonian. If (x, y) is an arc, then let D' be the tournament obtained from D by replacing (x, y) by $e' = (y, x)$. Then D' is also an almost regular tournament and so e' lies on a Hamiltonian cycle in D', that is, there exists a Hamiltonian $x - y$ path in D'. Since this path exists in D as well, the result now follows. ■

Corollary 3.4.11 *Every regular tournament of order $n \geq 9$ is 3-path Hamiltonian.*

A consequence of Corollary 3.4.11 is that, if D is an almost regular tournament of order $n \geq 8$, then every 3-path in D lies on a Hamiltonian path. By examining almost regular tournaments of order less than 8, we obtain the following result.

Corollary 3.4.12 *Let D be an almost regular tournament of order $n \geq 4$. Then D is 2-path Hamiltonian if and only if every 3-path lies on a Hamiltonian path in D.*

Figure 3.12 shows the two almost regular tournaments of order 6 that are not 2-path Hamiltonian, which we already saw in Figure 3.10. In each of these two tournaments, the 3-path expressed by dashed line segments is the unique 3-path that does not belongs to any Hamiltonian path in the tournament.

Problem 3.4.13 *Is every almost regular tournament of sufficiently large even order 3-path-Hamiltonian?*

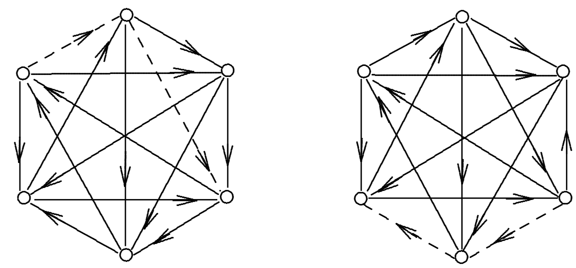


Figure 3.12: Two almost regular tournaments of order 6

No almost regular tournament of order at most 6 is 3-path Hamiltonian and there is an almost regular tournament of order 8 that is not 3-path Hamiltonian. Thus, if there is an even integer N such that an almost regular tournament of order $n \geq N$ is 3-path Hamiltonian, then $N \geq 10$. In fact, there is a more general question.

Problem 3.4.14 *For a fixed positive integer k, does there exist a positive integer $N(k)$ such that every regular or every almost regular tournament tournament of order $n \geq N(k)$ is k-path Hamiltonian?*

3.5 Upper and Lower Hamiltonian Extension Numbers

For a Hamiltonian graph G, let

$$\mathcal{H}(G) = \{D : D \text{ is a Hamiltonian orientation of } G\}.$$

The *upper Hamiltonian extension number* $\mathrm{he}^+(G)$ of G is the maximum value of $\mathrm{he}(D)$ among all $D \in \mathcal{H}(G)$ and the *lower Hamiltonian extension number* $\mathrm{he}^-(G)$ is the minimum such value. That is,

$$\mathrm{he}^+(G) = \max\{\mathrm{he}(D) : D \in \mathcal{H}(G)\} \quad \text{and} \quad \mathrm{he}^-(G) = \min\{\mathrm{he}(D) : D \in \mathcal{H}(G)\}.$$

Therefore, if G is a Hamiltonian graph of order $n \geq 3$, then

$$1 \leq \mathrm{he}^-(G) \leq \mathrm{he}^+(G) \leq n.$$

If $G = C_n$, $n \geq 3$, then there is only one Hamiltonian orientation of G, while we saw that this is also the case if $G = K_4$. Consequently,

$$\mathrm{he}^+(C_n) = \mathrm{he}^-(C_n) = n \text{ for all } n \geq 3 \text{ and } \mathrm{he}^+(K_4) = \mathrm{he}^-(K_4) = 1.$$

This suggests the following question:

What other graphs G have the property that $\mathrm{he}^+(G) = \mathrm{he}^-(G)$?

The result below shows that for any other graphs G with $\mathrm{he}^+(G) = \mathrm{he}^-(G)$, it is necessary that $\mathrm{he}^-(G) = 1$.

Proposition 3.5.1 *For a Hamiltonian graph G of order $n \geq 3$,*

$$\mathrm{he}^-(G) = \begin{cases} n = \mathrm{he}^+(G) & \textit{if } G = C_n \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. Let $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a Hamiltonian cycle in G. Direct the n edges $v_i v_{i+1}$ $(1 \leq i \leq n)$ on C such that (v_i, v_{i+1}) are arcs to obtain a directed n-cycle $\vec{C}$. If $G \neq C$, then orient each edge $v_i v_j \in E(G) - E(C)$ as (v_i, v_j) is an arc if and only if $1 \leq i < j \leq n$. In the resulting orientation of G, observe that $\vec{C}$ is the only directed n-cycle and so each of the arcs not belonging to $\vec{C}$ cannot lie on any Hamiltonian cycle. Hence, $\mathrm{he}^-(G) = 1$. ∎

To further illustrate these concepts, we determine the numbers $\mathrm{he}^+(W_n)$ and $\mathrm{he}^-(W_n)$ for the wheel $W_n = C_n \vee K_1$ of order $n + 1$.

Proposition 3.5.2 *For each integer $n \geq 3$,*

$$\mathrm{he}^+(W_n) = \begin{cases} 1 & \textit{if } n \textit{ is odd} \\ 2 & \textit{if } n \textit{ is even} \end{cases}$$

$$\mathrm{he}^-(W_n) = 1.$$

Proof. By Proposition 3.5.1, $\mathrm{he}^-(W_n) = 1$ for all $n \geq 3$. It therefore remains only to determine $\mathrm{he}^+(W_n)$. Let W_n be constructed from the n-cycle $C = (v_1, v_2, \ldots, v_n, v_1)$ by adding the vertex v, which is joined to each vertex of C. Let D be an arbitrary Hamiltonian orientation of W_n. In any Hamiltonian cycle in the graph W_n, the vertex v must be adjacent to two adjacent vertices v_i and v_j of C, where then $|j - i| \equiv 1 \pmod{n}$ for $i, j \in \{1, 2, \ldots, n\}$. Hence, we may assume that D contains the arcs (v_n, v) and (v, v_1) and the (directed) path $(v_1, v_2, \ldots, v_n)$. This is illustrated in Figure 3.13.

We consider two cases, according to whether n is odd or n is even.

Case 1. *$n \geq 3$ is odd.* We have already seen that $\mathrm{he}^+(K_4) = \mathrm{he}^-(K_4) = 1$. Since $W_3 = K_4$, we may assume that $n \geq 5$. Because n is odd, either D contains two arcs (v, v_i), (v, v_{i+1}) or two arcs (v_i, v) and (v_{i+1}, v) for some $i \in \{1, 2, \ldots, n-1\}$. If D contains two arcs (v, v_i), (v, v_{i+1}), then (v, v_{i+1}) lies on the unique Hamiltonian path $P = (v, v_{i+1}, v_{i+2}, \ldots, v_{i-1}, v_i)$ in D. Since (v_i, v) is not an arc of D, it follows that (v, v_{i+1})

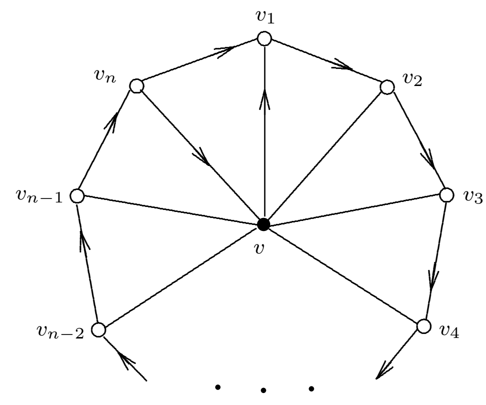


Figure 3.13: Some arcs in a Hamiltonian orientation D of W_n

does not lie on any Hamiltonian cycle in D. If D contains two arcs (v_i, v) and (v_{i+1}, v), then (v_i, v) lies on no Hamiltonian path in D and, consequently, on no Hamiltonian cycle in D either. Hence, D is not 2-path Hamiltonian. Since D is an arbitrary Hamiltonian orientation of W_n, it follows that $\mathrm{he}^+(W_n) < 2$. Therefore, $\mathrm{he}^+(W_n) = 1$ if $n \geq 3$ is odd.

Case 2. *$n \geq 4$ is even.* Let j be the minimum positive integer such that (v_j, v) is an arc of D. Thus, $2 \leq j \leq n$. Hence, $Q = (v_j, v, v_{j-1})$ is a path of order 3 in D. Since D contains the path $(v_1, v_2, \ldots, v_n)$, the path Q does not lie on any Hamiltonian cycle in D and so D is not 3-path Hamiltonian. Because D is an arbitrary Hamiltonian orientation of W_n, it follows that $\mathrm{he}^+(W_n) \leq 2$. Next, consider the Hamiltonian orientation D' of W_n containing (1) the arcs (v, v_i) if i is odd and (v_i, v) if i is even and (2) the directed cycle $(v_1, v_2, \ldots, v_n, v_1)$. Figure 3.14 shows the Hamiltonian orientation D' of W_n for $n = 8$.

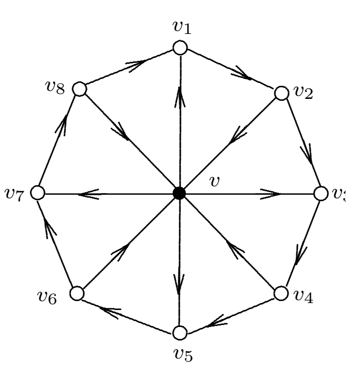


Figure 3.14: A Hamiltonian orientation D' of W_8

For each integer i with $1 \leq i \leq n$, we observe the following, where the subscript of each vertex is expressed as an integer modulo n:

⋆ for an odd integer i, the arc (v_i, v_{i+1}) lies on the Hamiltonian cycle

$(v_i,\ v_{i+1},\ v,\ v_{i+2},\ v_{i+3},\ \ldots,\ v_{i-1},\ v_i)$,

⋆ for an even integer i, the arc (v_i, v_{i+1}) lies on the Hamiltonian cycle

$(v_i,\ v_{i+1},\ v_{i+2},\ v,\ v_{i+3},\ v_{i+4},\ \ldots,\ v_{i-1},\ v_i)$,

⋆ the arc (v, v_i) lies on the Hamiltonian cycle $(v, v_i, v_{i+1}, v_{i+2}, \ldots, v_{i-1}, v)$,

⋆ the arc (v_i, v) lies on the Hamiltonian cycle $(v_i, v, v_{i+1}, v_{i+2}, \ldots, v_{i-1}, v_i)$.

Thus, $\mathrm{he}(D') \geq 2$. Since the path (v_i, v, v_{i+3}), where i is even, does not lie on any Hamiltonian cycle in D, it follows that $\mathrm{he}(D') \leq 2$ and so $\mathrm{he}(D') = 2$. Therefore, $\mathrm{he}^+(W_n) = 2$ if $n \geq 4$ is even. ■

By Proposition 3.5.2, if $n \geq 3$ is odd, then $\mathrm{he}^+(W_n) = \mathrm{he}^-(W_n)$.

Next, we consider $\mathrm{he}^+(K_{r,r})$ and $\mathrm{he}^-(K_{r,r})$ for complete r-regular bipartite graphs $K_{r,r}$ for $r \geq 2$. By Proposition 3.5.1,

$$\mathrm{he}^+(K_{2,2}) = \mathrm{he}^-(K_{2,2}) = 4$$

and $\mathrm{he}^-(K_{r,r}) = 1$ for $r \geq 3$. We begin with $r = 3$.

Proposition 3.5.3 $\mathrm{he}^+(K_{3,3}) = 3$ *and* $\mathrm{he}^-(K_{3,3}) = 1$.

Proof. As we already saw, $\mathrm{he}^-(K_{3,3}) = 1$ and so it remains to show that $\mathrm{he}^+(K_{3,3}) = 3$. There are exactly two Hamiltonian orientations D_1 and D_2 of $K_{3,3}$, both shown in Figure 3.15. Note that the only difference in these two Hamiltonian orientations of $K_{3,3}$ is the edge v_2v_5 is oriented to produce (v_2, v_5) or (v_5, v_2).

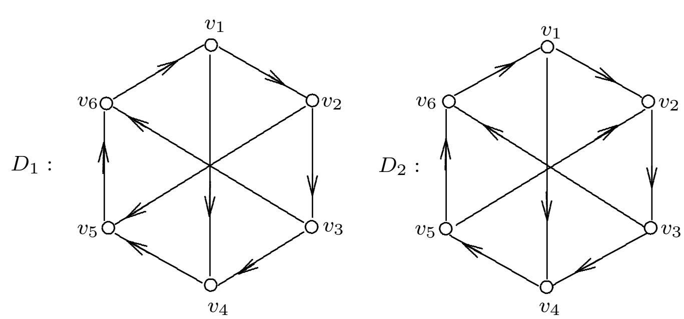


Figure 3.15: The two Hamiltonian orientations of $K_{3,3}$

Let $e_1 = (v_1, v_4)$ and $e_3 = (v_3, v_6)$. When $e_2 = (v_2, v_5)$ is an arc, observe that none of e_1, e_2, e_3 lies on a Hamiltonian cycle of D_1. Thus, D_1 is not 2-path Hamiltonian.

In D_2, the direction of the edge v_2v_5 is reversed and $e_2' = (v_5, v_2)$ is an arc. Here, every arc lies on a Hamiltonian cycle of D_2 and so D_2 is 2-path Hamiltonian. Since the path (v_1, v_2, v_3, v_6) does not lie on a Hamiltonian cycle of D_2, this digraph is not 4-path Hamiltonian. On the other hand, it is straightforward to show that every path of order 3 lies on a Hamiltonian cycle of D_2. Therefore, $\mathrm{he}^+(K_{3,3}) = 3$. ■

We now consider the values of $\mathrm{he}^+(K_{r,r})$ and $\mathrm{he}^-(K_{r,r})$ for $r \geq 4$, beginning with even integers $r \geq 4$.

Theorem 3.5.4 *For each even integer $r \geq 4$,*

$$\mathrm{he}^+(K_{r,r}) = 2r \quad \textit{and} \quad \mathrm{he}^-(K_{r,r}) = 1.$$

Proof. By Proposition 3.5.1, $\mathrm{he}^-(K_{r,r}) = 1$. Let $G = K_{r,r}$ where $r = 2s$ for some positive integer s. Let $U = U_1 \cup U_3$ and $U' = U_2 \cup U_4$ be the partite sets of G, where $|U_i| = s$ for $1 \leq i \leq 4$. Now let D be the orientation of $K_{r,r}$ obtained by directing each edge in $[U_i, U_{i+1}]$ from U_i to U_{i+1} for $1 \leq i \leq 4$ and $U_5 = U_1$ (see Figure 3.16). So $D = D_{4,s}$, which we described earlier. Since $\mathrm{he}(D_{4,s}) = 4s = 2r$ by Theorem 3.1.1, it follows that $\mathrm{he}^+(K_{r,r}) = 2r$. ■

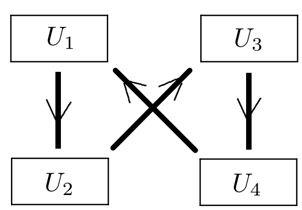


Figure 3.16: An Hamiltonian orientation of $K_{r,r}$

Next, we consider the values of $\mathrm{he}^+(K_{r,r})$ and $\mathrm{he}^-(K_{r,r})$ for odd integers $r \geq 5$.

Theorem 3.5.5 *For each odd integer $r \geq 5$,*

$$\mathrm{he}^+(K_{r,r}) \leq 2r - 5 \quad \textit{and} \quad \mathrm{he}^-(K_{r,r}) = 1.$$

Proof. By Proposition 3.5.1, $\mathrm{he}^-(K_{r,r}) = 1$. It therefore remains to show that

$$\mathrm{he}^+(K_{r,r}) \leq 2r - 5.$$

Let $G = K_{r,r}$. Assume, to the contrary, that there is a Hamiltonian orientation D of G such that D is $(2r-4)$-path Hamiltonian. Let $C = (v_1, v_2, \ldots, v_{2r}, v_{2r+1} = v_1)$ be a Hamiltonian cycle in D. Thus, $\{v_1, v_3, \ldots, v_{2r-1}\}$ and $\{v_2, v_4, \ldots, v_{2r}\}$ are the partite sets of G. We consider two cases. In each case, the subscript of a vertex is expressed as a positive integer modulo $2r$.

Case 1. *(v_i, v_{i+5}) is an arc in D for some i with $1 \le i \le 2r$.* Without loss of generality, we may assume that (v_1, v_6) is an arc in D (see Figure 3.17).

We consider the $(2r-4)$-path

$$Q_1 = (v_7, v_8, v_9, \ldots, v_{2r-1}, v_{2r}, v_1, v_6).$$

Thus, $V(Q_1) = V(G) - \{v_2, v_3, v_4, v_5\}$. Since D is $(2r-4)$-path Hamiltonian, it follows that Q_1 can be extended to a Hamiltonian cycle in D. This implies that each of the arcs (v_6, v_3), (v_5, v_2) and (v_2, v_7) must belong to D.

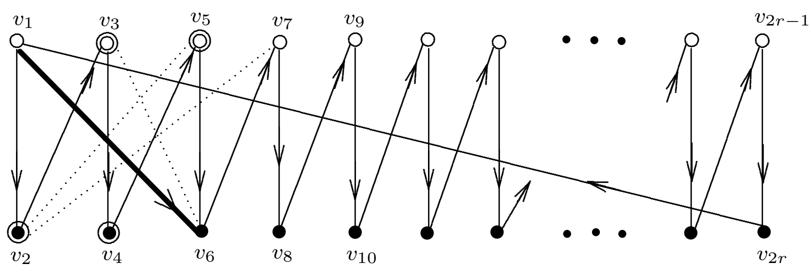


Figure 3.17: The path Q_1 lies on a Hamiltonian cycle

Next, consider the $(2r-4)$-path

$$Q_2 = (v_8, v_9, v_{10}, \ldots, v_{2r-1}, v_{2r}, v_1, v_2, v_7).$$

See Figure 3.18. Thus, $V(Q_2) = V(G) - \{v_3, v_4, v_5, v_6\}$. Since Q_2 can be extended to a Hamiltonian cycle in D, it follows that (v_7, v_4) and (v_3, v_8) are arcs in D.

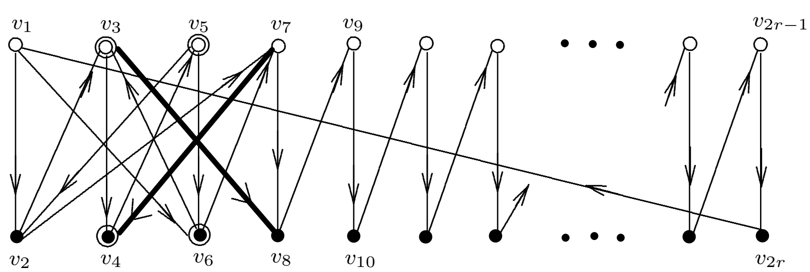


Figure 3.18: The path Q_2 lies on a Hamiltonian cycle

Assume, for each integer i with $2 \le i \le 2r-1$, that Q_{i-1} is defined and that Q_{i-1} can be extended to a Hamiltonian cycle in D. Now, consider the $(2r-4)$-path

$$Q_i = (v_{i+6}, v_{i+7}, \cdots, v_{2r-1}, v_{2r}, v_1, v_2, \cdots, v_i, v_{i+5}).$$

Then

$$V(Q_i) = V(G) - \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}.$$

Since Q_i can be extended to a Hamiltonian cycle in D, it follows that (v_{i+5}, v_{i+2}) and (v_{i+1}, v_{i+6}) are arcs in D. Hence, (v_{i+5}, v_{i+2}) and (v_{i+1}, v_{i+6}) are arcs in D for $1 \leq i \leq 2r-1$. This implies that for each integer j with $1 \leq j \leq 2r$, the arcs (v_{j+3}, v_j) and (v_j, v_{j+5}) both belong to D.

Next, consider the $(2r-4)$-path

$$\begin{aligned} P_1 \;\; = \;\; & (v_1, v_6, v_3, v_4, v_5, v_2, \; \underline{v_{2r-1}, v_{2r-4}, v_{2r-3}, v_{2r-2}}, \\ & \underline{v_{2r-5}, v_{2r-8}, v_{2r-7}, v_{2r-6}}, \ldots, \underline{v_{13}, v_{10}, v_{11}, v_{12}}). \end{aligned}$$

Thus, $V(P_1) = V(G) - \{v_7, v_8, v_9, v_{2r}\}$. Since P_1 can be extended to a Hamiltonian cycle in D and (v_7, v_8) is an arc in D, it follows that (v_9, v_{2r}), (v_{2r}, v_7) and (v_8, v_1) are arcs in D. This is illustrated in Figure 3.19 for $r = 9$.

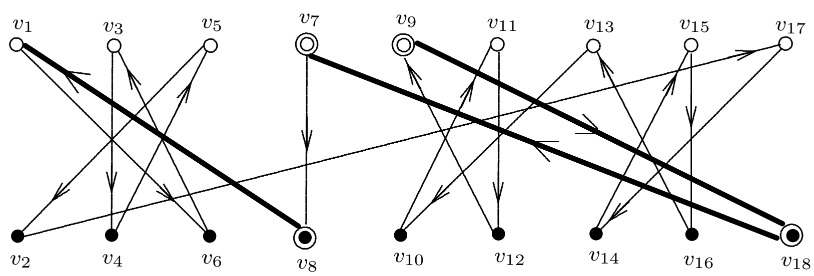


Figure 3.19: The path P_1 lies on a Hamiltonian cycle

We now consider another $(2r-4)$-path

$$\begin{aligned} P_2 \;\; = \;\; & (v_2, v_7, v_4, v_5, v_6, v_3, \; \underline{v_{2r}, v_{2r-3}, v_{2r-2}, v_{2r-1}}, \\ & \underline{v_{2r-4}, v_{2r-7}, v_{2r-6}, v_{2r-5}}, \cdots, \underline{v_{14}, v_{11}, v_{12}, v_{13}}). \end{aligned}$$

Thus, $V(P_2) = V(G) - \{v_1, v_8, v_9, v_{10}\}$ Since P_2 can be extended to a Hamiltonian cycle in D and (v_9, v_{10}) is an arc in D, it follows that (v_{10}, v_1), (v_1, v_8) and (v_9, v_2) are arcs in D. Once again, this is illustrated in Figure 3.20 for $r = 9$. However then, both (v_1, v_8) and (v_8, v_1) are arcs in D, which is a contradiction.

Case 2. *(v_{i+5}, v_i) is an arc in D for each i with $1 \leq i \leq 2r$.* We consider two subcases.

Subcase 2.1. *(v_{i+3}, v_i) is an arc in D for each $1 \leq i \leq 2r$.*

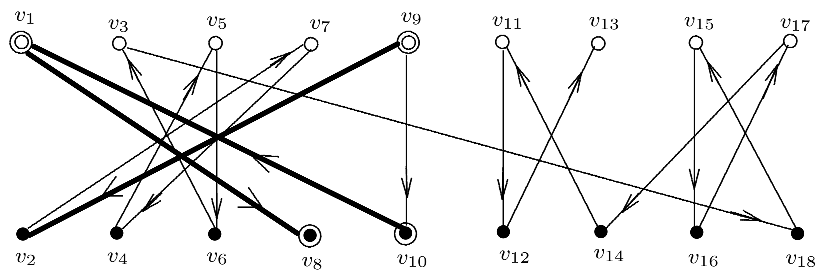


Figure 3.20: The path P_2 does not lie on a Hamiltonian cycle

Now consider the $(2r-4)$-path

$$\begin{aligned} P &= (v_4,\ \underline{v_1, v_{2r-2}, v_{2r-1}, v_{2r}},\ \underline{v_{2r-3}, v_{2r-6}, v_{2r-5}, v_{2r-4}}, \\ &\quad \underline{v_{2r-7}, v_{2r-10}, v_{2r-9}, v_{2r-8}}, \ldots, \underline{v_{11}, v_8, v_9, v_{10}}, v_7). \end{aligned}$$

Thus, $V(P) = V(G) - \{v_2, v_3, v_5, v_6\}$. Since P can be extended to a Hamiltonian cycle in D, it follows that (v_7, v_2) is an arc of D. Because (i) (v_5, v_2) and (v_6, v_3) are arcs of D and (ii) $(v_2, v_3, v_4, v_5, v_6, v_7)$ is a path in D, it follows that P cannot be extended to a Hamiltonian cycle in D, which is impossible. This is illustrated in Figure 3.21 for $r = 9$, where all bold arcs do not belong to the $v_4 - v_7$ path P in D.

Subcase 2.2. *(v_i, v_{i+3}) is an arc in D for some i with* $1 \le i \le 2r$. Without loss of generality, we may assume that (v_1, v_4) is an arc in D.

We consider the $(2r-4)$-path

$$P_1 = (v_7, v_8, v_9, \ldots, v_{2r-1}, v_{2r}, v_1, v_4).$$

Thus, $V(P_1) = V(G) - \{v_2, v_3, v_5, v_6\}$. Since D is $(2r-4)$-path Hamiltonian, it follows that P_1 can be extended to a Hamiltonian cycle in D. This implies that (v_3, v_6) and (v_5, v_2) must belong to D.

Next, consider the $(2r-4)$-path

$$P_3 = (v_9, v_{10}, \ldots, v_{2r-1}, v_{2r}, v_1, v_2, v_3, v_6).$$

Thus, $V(P_2) = V(G) - \{v_4, v_5, v_7, v_8\}$. Since Q_2 can be extended to a Hamiltonian cycle in D, it follows that (v_7, v_4) and (v_5, v_8) are arcs in D.

Assume, for each integer odd integer i with $3 \le i \le 2r-1$, that P_{i-2} is defined and that P_{i-2} can be extended to a Hamiltonian cycle in D. Now, consider the $(2r-4)$-path

$$P_i = (v_{i+6}, v_{i+7}, \cdots, v_{2r-1}, v_{2r}, v_1, v_2, \cdots, v_i, v_{i+3}).$$

Then $V(P_i) = V(G) - \{v_{i+1}, v_{i+2}, v_{i+4}, v_{i+5}\}$. Since P_i can be extended to a Hamiltonian cycle in D, it follows that (v_{i+2}, v_{i+5}) and (v_{i+4}, v_{i+1}) are arcs in D. Hence, (v_{i+2}, v_{i+5}) and (v_{i+4}, v_{i+1}) are arcs in D for $1 \leq i \leq 2r-1$. This implies that for each integer odd integer j with $1 \leq j \leq 2r-1$, the arcs (v_j, v_{j+3}) and (v_j, v_{j-3}) belong to D.

Next, consider the $(2r-4)$-path

$$Q_1 = (v_8, v_3, v_6, v_7, v_{10}, \ldots, v_{2r-1}, v_{2r}, v_1).$$

Then $V(Q_1) = V(G) - \{v_2, v_4, v_5, v_9\}$. Since Q_1 can be extended to a Hamiltonian cycle in D, it follows that (v_2, v_9) is an arc in D.

Similarly, by considering the $(2r-4)$-path

$$Q_2 = (v_7, v_4, v_5, v_2, v_9, \ldots, v_{2r-1}, v_{2r}),$$

we have that (v_1, v_8) belongs to D.

Then the $(2r-4)$-path Hamiltonian path

$$Q = (v_1, v_8, v_3, v_6, v_7, v_{10} \ldots, v_{2r-1}, v_{2r}),$$

with $V(Q) = V(G) - \{v_2, v_4, v_5, v_9\}$ cannot be extended to a Hamiltonian cycle in D, which is a contradiction. ■

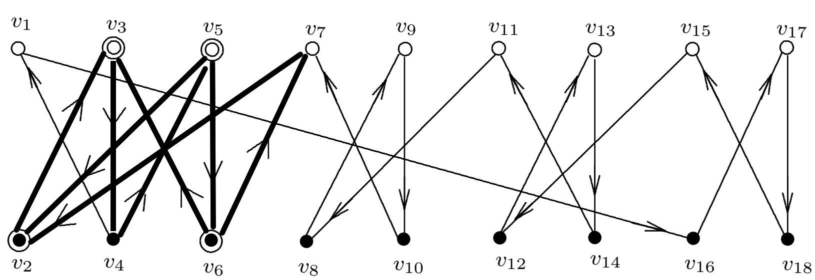


Figure 3.21: The path P does not lie on a Hamiltonian cycle

We have seen that there is only one Hamiltonian orientation of $K_{3,3}$ that is 6-path Hamiltonian. In fact, this orientation is ℓ-path Hamiltonian if and only if $\ell \in \{1, 2, 3, 5, 6\}$. By Theorem 3.5.5, $\text{he}^+(K_{5,5}) \leq 5$. In fact, $3 \leq \text{he}^+(K_{5,5}) \leq 4$, as we show next.

Proposition 3.5.6 *There exists a Hamiltonian orientation of $K_{5,5}$ having Hamiltonian extension number 3.*

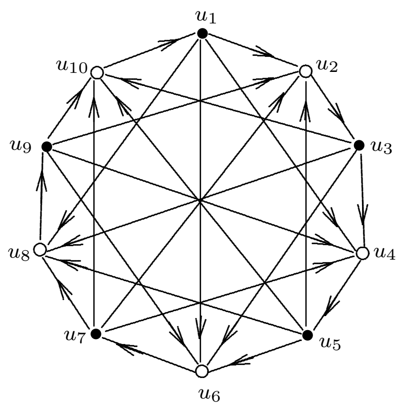


Figure 3.22: The Hamiltonian orientation D^* of $K_{5,5}$ in Proposition 3.5.6

Proof. Let $U = \{u_1, u_3, u_5, u_7, u_9\}$ and $W = \{u_2, u_4, u_6, u_8, u_{10}\}$ be the partite sets of $K_{5,5}$. Consider the orientation D^* of $K_{5,5}$ shown in Figure 3.22. Therefore, $\text{id}\, u = \text{od}\, w = 1$ for each $u \in U$ and $w \in W$.

It can be shown that D^* is ℓ-path Hamiltonian for $\ell = 1, 2, 3$ and so $\text{he}(D^*) \geq 3$. Since the path (u_5, u_2, u_3, u_4) of order 4, for example, does not lie on any Hamiltonian cycle in D^*, it follows that D^* is not 4-path Hamiltonian. Therefore, $\text{he}(D^*) \leq 3$ and so $\text{he}(D^*) = 3$. ■

Next, we show that D^* is the only Hamiltonian orientation of $K_{5,5}$ that is 5-path Hamiltonian.

Proposition 3.5.7 *The digraph D^* in* Figure 3.22 *is the only Hamiltonian orientation of $K_{5,5}$ that is* 5*-path Hamiltonian.*

Proof. First, using a case-by-case analysis, we see that the digraph D^* in Figure 3.22 is 5-path Hamiltonian. Now, let D be a 5-path Hamiltonian orientation of $K_{5,5}$ with a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{10}, v_{11} = v_1)$, where all subscripts of vertices are expressed modulo 10. We show that $D = D^*$. We consider two cases.

Case 1. *For each $i \in \{1, 2, 3, \ldots, 10\}$, $\vec{d}_D(v_i, v_{i+3}) > 1$.* Since $i+3$ and i are opposite parity and (v_i, v_{i+3}) is not an arc in D, it follows that (v_{i+3}, v_i) is an arc in D for each $i \in \{1, 2, 3, \ldots, 10\}$. Thus, D contains the subdigraph shown in Figure 3.24. By the symmetry of the graph, either

$$(i) \;\; (v_2, v_7), (v_6, v_1) \in E(D) \text{ or } (ii) \;\; (v_7, v_2), (v_6, v_1) \in E(D).$$

If (i) occurs, then the 5-path $(v_4, v_5, v_6, v_1, v_8)$ cannot be extended to a Hamiltonian cycle

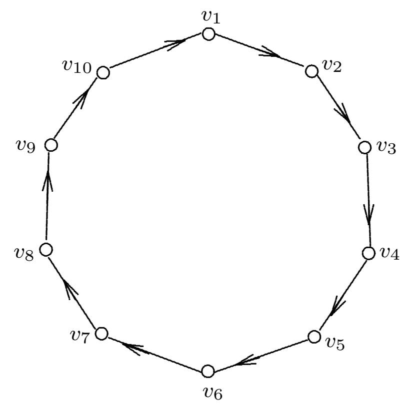


Figure 3.23: A Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{10}, v_1)$ in D

since (v_2, v_i) is an arc for $i \in \{3, 7, 9\}$. If (ii) occurs, then the 5-path $(v_4, v_5, v_2, v_9, v_{10})$ cannot be extended to a Hamiltonian cycle. Thus, this case cannot occur.

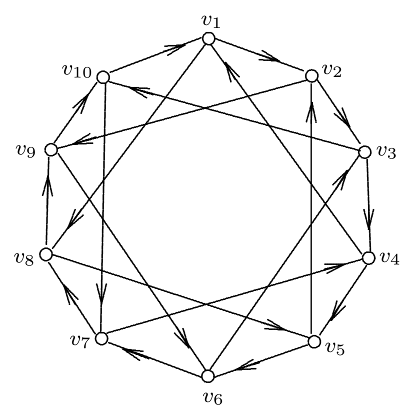


Figure 3.24: A subdigraph in D in Case 1

Case 2. *There exists* $i \in \{1, 2, 3, \ldots, 10\}$ *such that* $\vec{d}_D(v_i, v_{i+3}) = 1$ *or* $(v_i, v_{i+3}) \in E(D)$. We may assume, without loss of generality, that $(v_5, v_8) \in E(D)$. Let $\mathcal{P}$ be the set of 6-paths in D that cannot be extended to a Hamiltonian cycle. Thus, $\mathcal{P} \neq \emptyset$ by Theorem 3.5.5. We now consider the 6-path $Q = (v_1, v_2, v_3, v_4, v_5, v_8)$. Note that $(Q, v_9, v_6, v_7, v_{10}, v_1)$ is the only possible Hamiltonian cycle in D that contains Q. Hence, if $Q \in \mathcal{P}$, then

$$\text{either (i) } (v_9, v_6), (v_{10}, v_7) \in E(D) \text{ or (ii) } (v_6, v_9) \in E(D).$$

Let $Q' = Q - v_1 = (v_2, v_3, v_4, v_5, v_8)$ be the 5-path in D.

$\star$ If (i) occurs, then $(Q', v_9, v_{10}, v_1, v_6, v_7, v_2)$ is the only possible Hamiltonian cycle containing Q' and so $(v_1, v_6), (v_7, v_2) \in E(D)$. However then, no Hamiltonian cycle in D contains the 5-path $(v_{10}, v_7, v_2, v_3, v_4)$. Hence, (i) cannot occur.

- ⋆ If (ii) occurs, then every Hamiltonian cycle containing Q' must be of the form $(Q', *, *, *, *, v_7, v_2)$ and so $(v_7, v_2) \in E(D)$. Now consider the 5-path $Q'' = (v_6, v_9, v_{10}, v_1, v_2)$. Since (v_6, v_7, v_2) is in D, every Hamiltonian cycle containing Q'' must be of the form $(Q'', *, *, v_7, *, *, v_6)$. However, one can verify that this is impossible and so (ii) cannot occur either.

Consequently, $Q \notin \mathcal{P}$. In other words, if (v_5, v_8) is an arc in D, then there exists a Hamiltonian cycle in D containing the 6-path $Q = Q_1 = (v_1, v_2, v_3, v_4, v_5, v_8)$, implying that (v_7, v_{10}) and (v_9, v_6) are both arcs in D. This produces another 6-path $Q_3 = (v_3, v_4, v_5, v_6, v_7, v_{10})$ not belonging to $\mathcal{P}$ and so (v_1, v_8) and (v_9, v_2) are also arcs. Continuing in this manner, we have the following:

> Let $C = (v_1, v_2, \ldots, v_{10}, v_1)$ be a Hamiltonian cycle in a 5-path Hamiltonian orientation D of $K_{5,5}$. If $(v_\alpha, v_{\alpha+3}) \in E(D)$ for some α, then $(v_\beta, v_{\beta\pm3}) \in E(D)$ if and only if α and β are of the same parity.

Thus, we may assume that D contains the digraph shown in Figure 3.25 as a spanning subdigraph.

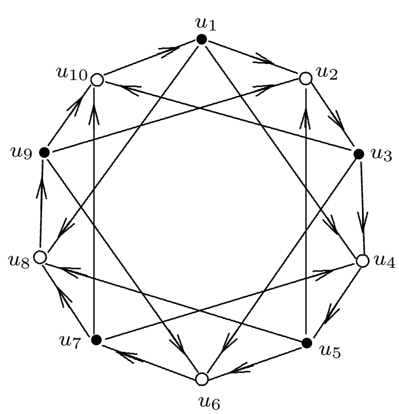


Figure 3.25: A spanning subgraph of D

We claim that $(u_\alpha, u_{\alpha+5})$ is an arc in D if and only if α is odd. If this is not the case, say (u_6, u_1) is an arc in D, then the 5-path $(u_4, u_5, u_6, u_1, u_8)$ cannot be extended to a 10-cycle since (u_i, u_{10}) is an arc for $i \in \{3, 7, 9\}$. Thus, the claim is verified and so $D = D^*$. ∎

The following is a consequence of Theorem 3.5.5 and Propositions 3.5.6 and 3.5.7.

Corollary 3.5.8 $3 \leq \mathrm{he}^+(K_{5,5}) \leq 4$.

Proof. Since $\mathrm{he}(D^*) = 3$ by Proposition 3.5.6, it follows that $\mathrm{he}^+(K_{5,5}) \geq 3$. By Proposition 3.5.7 then, D^* is the only Hamiltonian orientation of $K_{5,5}$ that is 5-path Hamiltonian. This implies that there is no Hamiltonian orientation D of $K_{5,5}$ with $\mathrm{he}(D) = 5$. It then follows by Theorem 3.5.5 that $\mathrm{he}^+(K_{5,5}) \leq 4$. ■

It is not known whether $\mathrm{he}^+(K_{5,5}) = 3$ or $\mathrm{he}^+(K_{5,5}) = 4$.

Problem 3.5.9 *Determine a lower bound for* $\mathrm{he}^+(K_{r,r})$ *for odd integers* $r \geq 7$.

Chapter 4

Rainbow Hamiltonian-Connected Graphs

4.1 Introduction

A *rainbow coloring* of a connected graph G is an edge coloring c of G (where adjacent edges may be colored the same) with the property that for every two vertices u and v of G, there exists a $u - v$ *rainbow path* (no two edges of the path are colored the same). In this case, G is said to be *rainbow-connected* (*with respect to* c). The minimum number of colors needed for a rainbow coloring of G is referred to as the *rainbow connection number* of G and is denoted by $\mathrm{rc}(G)$. These concepts were introduced and studied by Chartrand, Johns, McKeon and Zhang in 2006. The first paper [10] on this topic was published in 2008. In recent years, this topic has been studied by many and, in fact, there is a book [34] on rainbow colorings, published in 2012.

While, in a rainbow-connected graph G, every two vertices u and v of G are connected by a $u - v$ rainbow path, there is no condition on what the length of such a path must be. For certain highly Hamiltonian graphs G, however, it is natural to ask whether there exists an edge coloring of G using a certain number of colors such that every two vertices of G can be connected by a rainbow path of a prescribed length.

For a Hamiltonian-connected graph G, an edge coloring $c : E(G) \to [k] = \{1, 2, \ldots, k\}$ is called a *Hamiltonian-connected rainbow k-coloring* if every two vertices of G are connected by a rainbow Hamiltonian path in G. An edge coloring c is a *Hamiltonian-connected rainbow coloring* if c is a Hamiltonian-connected rainbow k-coloring for some positive integer k. The minimum k for which G has a Hamiltonian-connected rainbow k-coloring is the *rainbow Hamiltonian-connection number* of G, denoted by $\mathrm{hrc}(G)$.

If H is a Hamiltonian-connected spanning subgraph of a graph G and c is a Hamiltonian-connected rainbow coloring of H, then the coloring c can be extended to a Hamiltonian-connected rainbow coloring of G by assigning any color used by c to each edge in $E(G) - E(H)$. Thus, we have the following observation.

Observation 4.1.1 *If H is a Hamiltonian-connected spanning subgraph of a graph G, then*

$$\text{hrc}(G) \leq \text{hrc}(H).$$

Let G be a Hamiltonian-connected graph of order $n \geq 4$. Since (1) every Hamiltonian-connected rainbow coloring of G is a rainbow coloring, (2) there is no Hamiltonian-connected rainbow coloring of G using less than $n-1$ colors and (3) the edge coloring that assigns distinct colors to distinct edges of G is a Hamiltonian-connected rainbow coloring, we have the following observation.

Observation 4.1.2 *If G is a Hamiltonian-connected graph of order $n \geq 4$ and size m, then*

$$\max\{\text{rc}(G), n-1\} \leq \text{hrc}(G) \leq m.$$

If G is a Hamiltonian-connected graph of order $n \geq 3$ and size at most $2n-3$, then $\text{hrc}(G) \neq n-1$, for otherwise, each $(n-1)$-edge coloring of G results in some edge $e = uv$ of G having the property that e is the only edge possessing the color assigned to it. However then, there is no rainbow Hamiltonian $u - v$ path P in G. This gives rise to the following observation.

Observation 4.1.3 *If G is a Hamiltonian-connected graph of order $n \geq 3$ and size at most $2n-3$, then* $\text{hrc}(G) \geq n$.

We now present infinite classes of Hamiltonian-connected graphs G such that $\text{hrc}(G) = |V(G)|-1$. For two vertex-disjoint graphs F and H, let $F \vee H$ denote the *join* of F and H. For an integer $n \geq 3$, the wheel $W_n = C_n \vee K_1$ of order $n+1$ is Hamiltonian-connected.

Theorem 4.1.4 *For each integer $n \geq 3$,* $\text{hrc}(W_n) = n$.

Proof. Let $W_n = C_n \vee K_1$, where $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and the vertex $v \in V(K_1)$ is adjacent to each vertex of C_n. Since $\text{hrc}(W_n) \geq n$ by Observation 4.1.2, it remains to show that $\text{hrc}(W_n) \leq n$. Define an n-edge coloring $c : E(W_n) \to [n]$ by $c(v_i v_{i+1}) = c(v_i v) = i$ for $1 \leq i \leq n$, where $v_{n+1} = v_1$. We show that every two vertices x and y in W_n are connected by a rainbow Hamiltonian path in W_n. We may assume, without loss of generality, that $x = v_1$. If $y = v$, then $(v_1, v_2, \ldots, v_n, v)$ is a rainbow Hamiltonian

$v_1 - v$ path. If $y = v_n$, then $(v_1, v_2, v_3, \ldots, v_{n-1}, v, v_n)$ is a rainbow Hamiltonian $v_1 - v_n$ path. If $y = v_i$ where $2 \leq i \leq n-1$, then $(v_1, v_2, \ldots, v_{i-1}, v, v_n, v_{n-1}, \ldots, v_i)$ is a rainbow Hamiltonian $v_1 - v_i$ path. Thus, c is a Hamiltonian-connected rainbow coloring of W_n and so $\text{hrc}(W_n) \leq n$. Therefore, $\text{hrc}(W_n) = n$. ■

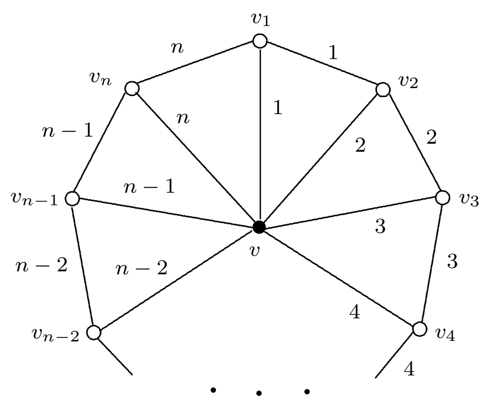


Figure 4.1: An n-edge coloring c of $W_n = C_n \vee K_1$

The following results are consequences of Observations 4.1.1 and 4.1.2 and Theorem 4.1.4.

Corollary 4.1.5 *If G is a Hamiltonian graph of order $n \geq 3$, then*

$$\text{hrc}(G \vee K_1) = n.$$

Corollary 4.1.6 *For each integer $n \geq 4$,* $\text{hrc}(K_n) = n - 1$.

If G is a graph of order $n \geq 3$ containing a Hamiltonian path, then the join $G \vee \overline{K}_2$ of G and $\overline{K}_2$ is Hamiltonian-connected. In particular, $P_n \vee \overline{K}_2$ is Hamiltonian-connected for $n \geq 3$. In this section, we determine the rainbow Hamiltonian-connection numbers graphs that are joins of graphs possessing a Hamiltonian path with $\overline{K}_2$.

Theorem 4.1.7 *For each integer $n \geq 3$,* $\text{hrc}(P_n \vee \overline{K}_2) = n + 1$.

Proof. If $n = 3$, then $P_3 \,\square\, \overline{K}_2$ is the wheel of order 5 and so $\text{hrc}(W_4) = 4$ by Theorem 4.1.4. Thus, we assume that $n \geq 4$. Let $G = P_n \vee \overline{K}_2$ where $V(\overline{K}_2) = \{u, v\}$ and $P_n = (x_1, x_2, \ldots, x_n)$. Since G has order $n+2$, it follows that $\text{hrc}(G) \geq n+1$. Thus, it remains to show that G has a Hamiltonian-connected rainbow $(n+1)$-coloring.

Define the edge coloring $c : V(G) \to [n+1]$ of G by

- $\star$ $c(ux_i) = i$ for $1 \leq i \leq n$ and $c(x_i x_{i+1}) = i + 1$ for $1 \leq i \leq n - 1$ and

$\star$ $c(vx_1) = 1$ and $c(vx_i) = i + 1$ for $2 \leq i \leq n - 2$ and $c(x_{n-1}) = c(x_n) = n + 1$.

This coloring is shown in Figure 4.2. To show that c is a Hamiltonian-connected rainbow coloring of G, we illustrate a rainbow Hamiltonian path connecting each pair of the vertices of G as listed below.

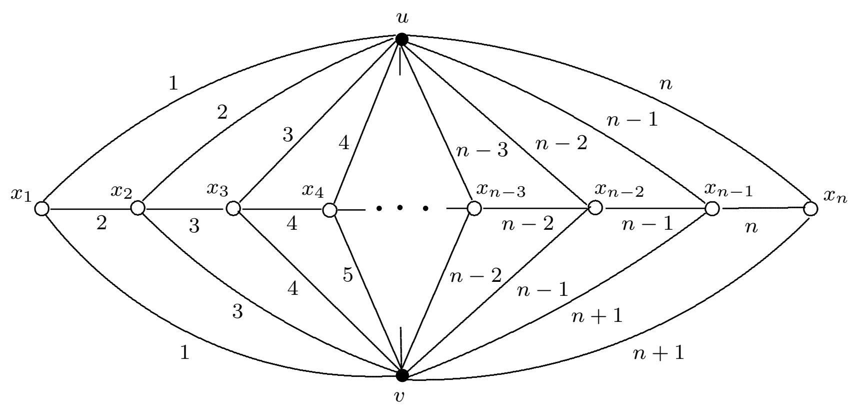


Figure 4.2: An $(n+1)$-edge coloring c of $P_n \vee \overline{K}_2$

$u - v$:	$(u, x_1, x_2, x_3, \ldots, x_n, v)$
$u - x_i$:	$(u, x_{i+1}, x_{i+2}, \ldots, x_n, v, x_1, x_2, \ldots, x_i)$ for $1 \leq i \leq n-1$
$u - x_n$:	$(u, x_1, x_2, x_3, \ldots, x_{n-2}, v, x_{n-1}, x_n)$
$v - x_i$:	$(v, x_n, x_{n-1}, \ldots, x_{i+1}, u, x_1, x_2, \ldots, x_i)$ for $1 \leq i \leq n-1$
$v - x_n$:	$(v, x_{n-1}, x_{n-2}, \cdots, x_1, u, x_n)$
$x_i - x_{i+1}$:	$(x_i, x_{i-1}, \ldots, x_1, v, x_n, x_{n-1}, \ldots, x_{i+2}, u, x_{i+1})$ for $1 \leq i \leq n-2$
$x_{n-1} - x_n$:	$(x_{n-1}, v, x_{n-2}, x_{n-3}, \ldots, x_1, u, x_n)$
$x_i - x_j$:	$(x_i, x_{i-1}, \ldots, x_1, u, x_{i+1}, \ldots, x_{j-1}, v, x_n, x_{n-1}, \ldots, x_j)$ for $1 \leq i < j \leq n-1$
$x_i - x_n$:	$(x_i, x_{i-1}, \ldots, x_1, u, x_{i+1}, \ldots, x_{n-2}, v, x_{n-1}, x_n)$ for $1 \leq i \leq n-3$
$x_{n-2} - x_n$:	$(x_{n-2}, x_{n-3}, \ldots, x_1, v, x_{n-1}, u, x_n)$.

Therefore, c is a Hamiltonian-connected rainbow coloring of G and so $\mathrm{hrc}(G) = n + 1$. ■

The following are consequences of Observations 4.1.1 and 4.1.2 and Theorem 4.1.7.

Corollary 4.1.8 *If G is a graph of order $n \geq 3$ containing a Hamiltonian path, then*

$$\mathrm{hrc}(G \vee \overline{K}_2) = n + 1.$$

We have seen in Observation 4.1.2 that if G is a Hamiltonian-connected graph of order $n \geq 4$ and size m, then

$$n - 1 \leq \mathrm{hrc}(G) \leq m. \tag{4.1}$$

In the next two sections, we will investigate the rainbow Hamiltonian-connection numbers of the *Cartesian products* $H \square K_2$ of graphs H and K_2, where H belongs to one of two infinite classes of graphs, for the purpose of verifying that the bounds in (4.1) are sharp. We will show in Section 4.2 that for graphs H in one class, $\mathrm{hrc}(H \square K_2)$ attains the lower bound in (4.1), while in Section 4.3 that for graphs H in the other class, $\mathrm{hrc}(H \square K_2)$ attains the upper bound in (4.1).

4.2 The Graphs $K_n \square K_2$

In this section, we investigate the rainbow Hamiltonian-connection numbers of the Hamiltonian-connected graphs $K_n \square K_2$ for several integers $n \geq 3$. First, we determine the rainbow Hamiltonian-connection number of $K_3 \square K_2$, the unique Hamiltonian-connected cubic graph of order 6.

Theorem 4.2.1 $\mathrm{hrc}(K_3 \square K_2) = 7$.

Proof. First, we consider the 7-edge coloring of the graph $G = K_3 \square K_2$ shown in Figure 4.3. This edge coloring is a Hamiltonian-connected rainbow 7-coloring since each of the following $\binom{6}{2} = 15$ Hamiltonian paths are rainbow paths. Therefore, $\mathrm{hrc}(G) \leq 7$.

1. rainbow Hamiltonian $u_1 - u_2$ path: $(u_1, v_1, v_2, v_3, u_3, u_2)$
2. rainbow Hamiltonian $u_1 - u_3$ path: $(u_1, u_2, v_2, v_1, v_3, u_3)$
3. rainbow Hamiltonian $u_1 - v_1$ path: $(u_1, u_2, u_3, v_3, v_2, v_1)$
4. rainbow Hamiltonian $u_1 - v_2$ path: $(u_1, v_1, v_3, u_3, u_2, v_2)$
5. rainbow Hamiltonian $u_1 - v_3$ path: $(u_1, v_1, v_2, u_2, u_3, v_3)$
6. rainbow Hamiltonian $u_2 - u_3$ path: $(u_2, u_1, v_1, v_2, v_3, u_3)$
7. rainbow Hamiltonian $u_2 - v_1$ path: $(u_2, u_1, u_3, v_3, v_2, v_1)$
8. rainbow Hamiltonian $u_2 - v_2$ path: $(u_2, u_1, u_3, v_3, v_1, v_2)$
9. rainbow Hamiltonian $u_2 - v_3$ path: $(u_2, v_2, v_1, u_1, u_3, v_3)$
10. rainbow Hamiltonian $u_3 - v_1$ path: $(u_3, v_3, v_2, u_2, u_1, v_1)$
11. rainbow Hamiltonian $u_3 - v_2$ path: $(u_3, v_3, v_1, u_1, u_2, v_2)$
12. rainbow Hamiltonian $u_3 - v_3$ path: $(u_3, u_1, u_2, v_2, v_1, v_3)$
13. rainbow Hamiltonian $v_1 - v_2$ path: $(v_1, v_3, u_3, u_1, u_2, v_2)$
14. rainbow Hamiltonian $v_1 - v_3$ path: $(v_1, v_2, u_2, u_1, u_3, v_3)$
15. rainbow Hamiltonian $v_2 - v_3$ path: $(v_2, v_1, u_1, u_2, u_3, v_3)$

It remains to show that $\mathrm{hrc}(G) \geq 7$. Of course, $\mathrm{hrc}(G) \geq n - 1 = 5$. To see that $\mathrm{hrc}(G) \geq 6$, assume, to the contrary, that $\mathrm{hrc}(G) = 5$. Then there is a Hamiltonian-connected rainbow 5-coloring of G. Since G has nine edges, some edge uv of G has

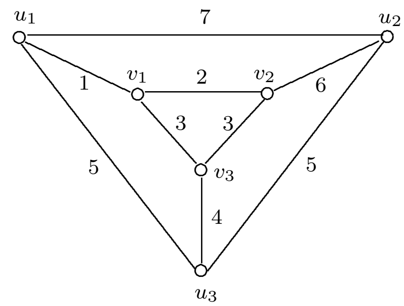


Figure 4.3: A Hamiltonian-connected rainbow coloring of $K_3 \,\square\, K_2$ using 7 colors

the property that it is the only edge possessing the color assigned to it. However, by definition, there is a rainbow Hamiltonian $u-v$ path P in G. Since the length of P is 5 and P does not contain uv, it is impossible for P to be a rainbow path. Thus, $\text{hrc}(G) \geq 6$. Hence, either $\text{hrc}(G) = 6$ or $\text{hrc}(G) = 7$. We show that $\text{hrc}(G) \neq 6$; for suppose that there is a Hamiltonian-connected rainbow 6-coloring $c : E(G) \to \{1, 2, \ldots, 6\}$ of G. For each pair x, y of vertices of G, there are exactly two Hamiltonian $x-y$ paths. At least one of these two paths is necessarily a rainbow path. These $2\binom{6}{2} = 30$ Hamiltonian paths are shown below. Also, see Figure 4.3.

1. $u_1 - u_2$ paths: $(u_1, v_1, v_2, v_3, u_3, u_2)$, $(u_1, u_3, v_3, v_1, v_2, u_2)$
2. $u_1 - u_3$ paths: $(u_1, u_2, v_2, v_1, v_3, u_3)$, $(u_1, v_1, v_3, v_2, u_2, u_3)$
3. $u_1 - v_1$ paths: $(u_1, u_2, u_3, v_3, v_2, v_1)$, $(u_1, u_3, u_2, v_2, v_3, v_1)$
4. $u_1 - v_2$ paths: $(u_1, v_1, v_3, u_3, u_2, v_2)$, $(u_1, u_2, u_3, v_3, v_1, v_2)$
5. $u_1 - v_3$ paths: $(u_1, v_1, v_2, u_2, u_3, v_3)$, $(u_1, u_3, u_2, v_2, v_1, v_3)$
6. $u_2 - u_3$ paths: $(u_2, u_1, v_1, v_2, v_3, u_3)$, $(u_2, v_2, v_3, v_1, u_1, u_3)$
7. $u_2 - v_1$ paths: $(u_2, u_1, u_3, v_3, v_2, v_1)$, $(u_2, v_2, v_3, u_3, u_1, v_1)$
8. $u_2 - v_2$ paths: $(u_2, u_1, u_3, v_3, v_1, v_2)$, $(u_2, u_3, u_1, v_1, v_3, v_2)$
9. $u_2 - v_3$ paths: $(u_2, v_2, v_1, u_1, u_3, v_3)$, $(u_2, u_3, u_1, v_1, v_2, v_3)$
10. $u_3 - v_1$ paths: $(u_3, v_3, v_2, u_2, u_1, v_1)$, $(u_3, u_1, u_2, v_2, v_3, v_1)$
11. $u_3 - v_2$ paths: $(u_3, v_3, v_1, u_1, u_2, v_2)$, $(u_3, u_2, u_1, v_1, v_3, v_2)$
12. $u_3 - v_3$ paths: $(u_3, u_1, u_2, v_2, v_1, v_3)$, $(u_3, u_2, u_1, v_1, v_2, v_3)$
13. $v_1 - v_2$ paths: $(v_1, u_1, u_2, u_3, v_3, v_2)$, $(v_1, v_3, u_3, u_1, u_2, v_2)$
14. $v_1 - v_3$ paths: $(v_1, v_2, u_2, u_1, u_3, v_3)$, $(v_1, u_1, u_3, u_2, v_2, v_3)$
15. $v_2 - v_3$ paths: $(v_2, u_2, u_3, u_1, v_1, v_3)$, $(v_2, v_1, u_1, u_2, u_3, v_3)$

Because of the symmetry of two Hamiltonian $u_1 - u_2$ paths, we may assume that the Hamiltonian-connected rainbow 6-coloring c is such that the first of the paths in (1) is a rainbow path. Thus, we may assume that

$$c(u_1v_1) = 1,\ c(v_1v_2) = 2,\ c(v_2v_3) = 3,\ c(v_3u_3) = 4,\ c(u_3u_2) = 5.$$

See Figure 4.4.

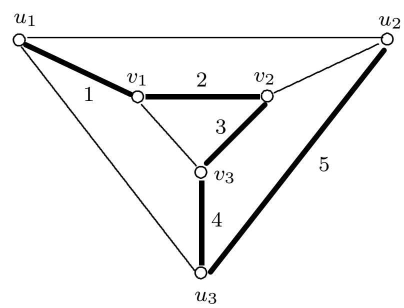


Figure 4.4: A rainbow Hamiltonian $u_1 - u_2$ path of $K_3 \,\square\, K_2$

It remains, therefore, to determine the possible colors assigned to the remaining four edges of G. For this purpose, we make a number of observations, beginning with the possible color of the edge u_1u_2. Since both Hamiltonian $u_3 - v_2$ paths in (11) contain both edges u_1v_1 and u_1u_2, the edge u_1u_2 cannot be colored 1. Similarly by (12), the edge u_1u_2 cannot be colored 2. According to (10), u_1u_2 cannot be colored 3. By (13), u_1u_2 cannot be colored 4. Consequently, this edge must be colored 5 or 6. See Figure 4.5.

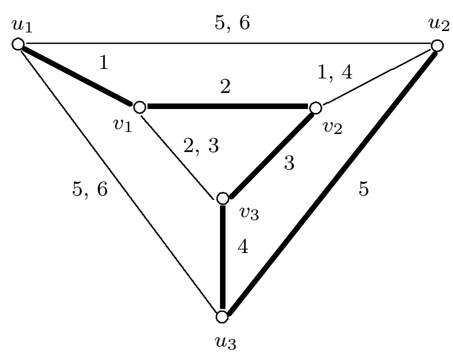


Figure 4.5: A step in the proof of Theorem 4.2.1

Next, we consider the possible color of the edge u_1u_3. Since the two Hamiltonian $u_2 - v_3$ paths in (9) contain both edges u_1v_1 and u_1u_3, it follows that $c(u_1u_3) \neq 1$. Also by (9), $c(u_1u_3) \neq 2$. By (7), $c(u_1u_3) \neq 3$ and $c(u_1u_3) \neq 4$. Therefore, either $c(u_1u_3) = 5$ or $c(u_1u_3) = 6$. Again, see Figure 4.5.

Next, we consider the possible color of the edge u_2v_2. Since the two Hamiltonian $u_1 - v_3$ paths in (5) contain both edges u_2v_2 and v_1v_2, it follows that $c(u_2v_2) \neq 2$. By (10), $c(u_2v_2) \neq 3$. By (5), $c(u_2v_2) \neq 5$. Thus, $c(u_2v_2) \in \{1, 4, 6\}$.

Next, we consider the possible color of the edge v_1v_3. The two Hamiltonian $u_3 - v_2$ paths in (11) contain both edges v_1v_3 and u_1v_1. Thus, $c(v_1v_3) \neq 1$. By (4), $c(v_1v_3) \neq 4$ and $c(v_1v_3) \neq 5$. Hence, $c(v_1v_3) \in \{2, 3, 6\}$.

If $c(u_2v_2) = 6$, then $c(u_1u_2) = 5$ by (10) and $c(u_1u_3) = 5$ by (14). Also, if $c(v_1v_3) = 6$, then $c(u_1u_2) = 5$ by (11) and $c(u_1u_3) = 5$ by (8). That is, if either $c(u_2v_2) = 6$ or $c(v_1v_3) = 6$, then $c(u_1u_2) = c(u_1u_3) = c(u_2u_3) = 5$. However, this is impossible by (3). Thus, $c(u_2v_2) \in \{1, 4\}$ and $c(v_1v_3) \in \{2, 3\}$(see Figure 4.5).

If $c(u_1u_3) = 5$, then no rainbow Hamiltonian $u_1 - v_3$ path can contain both u_1u_3 and u_2u_3. Thus, the only possible rainbow Hamiltonian $u_1 - v_3$ path contains the edges u_1v_1, u_2v_2 and u_3v_3 (see (5)). Since each of these three edges is colored 1 or 4, this is impossible. Therefore, $c(u_1u_3) = 6$. Similarly, if $c(u_1u_2) = 5$, then no rainbow Hamiltonian $u_1 - v_2$ path can contain both u_1u_2 and u_2u_3. Hence, the only possible rainbow Hamiltonian $u_1 - v_2$ path contains the edges u_1v_1, u_2v_2 and u_3v_3, again a contradiction, and so $c(u_1u_2) = 6$. Since no rainbow Hamiltonian path can contain both u_1u_2 and u_1u_3, the only possible rainbow Hamiltonian $u_2 - v_1$ path must contain the edges u_1v_1, u_2v_2 and u_3v_3, a contradiction.

Thus, c is not a Hamiltonian-connected rainbow coloring of G and so $\mathrm{hrc}(G) = 7$. ■

According to Theorem 4.2.1 then, for the graph $K_3 \,\square\, K_2$ of order $n = 6$ and size $m = 9$, we have $\mathrm{hrc}(K_3 \,\square\, K_2) = 7 = n+1$. Next, we turn our attention to the graphs $K_n \,\square\, K_2$ where $n \geq 5$. First, we determine $\mathrm{hrc}(W \,\square\, K_2)$ for all wheels W of order 5 or more.

Theorem 4.2.2 *If W is a wheel of order $n \geq 5$, then* $\mathrm{hrc}(W \,\square\, K_2) = 2n - 1$.

Proof. Let $G = W \,\square\, K_2$ be obtained from two copies F and F' of the wheel W of order $n \geq 5$, where $V(F) = \{u, u_1, u_2, \ldots, u_{n-1}\}$ and $V(F') = \{v, v_1, v_2, \ldots, v_{n-1}\}$, by adding the n edges uv and u_iv_i for $1 \leq i \leq n-1$. Furthermore, assume that $F = C_{n-1} \vee K_1$ where $C_{n-1} = (u_1, u_2, \ldots, u_{n-1}, u_n = u_1)$ and $F' = C_{n-1} \vee K_1$ where C_{n-1}=(v_1, v_2, …, v_{n-1}, $v_n = v_1$). The edge coloring $c_F : E(F) \to [n-1]$ defined by $c_F(u_iu_{i+1}) = c_F(u_iu) = i$ for $1 \leq i \leq n-1$ is a Hamiltonian-connected rainbow coloring of F and the edge coloring $c_{F'} : E(F') \to \{n, n+1, \ldots, 2n-2\}$ defined by $c_{F'}(v_iv_{i+1}) = c_{F'}(v_iv) = n - 1 + i$ for $1 \leq i \leq n-1$ is a Hamiltonian-connected rainbow coloring of F'. Define the $(2n-1)$-edge coloring $c : E(G) \to [2n-1]$ by

$$c(e) = \begin{cases} c_F(e) & \text{if } e \in E(F) \\ c_{F'}(e) & \text{if } e \in E(F') \\ 1 & \text{if } e = u_1v_1 \\ n+2 & \text{if } e = u_3v_3 \\ 2n-1 & \text{if } e = uv \text{ or } e = u_iv_i \text{ for } i = 2 \text{ or } 4 \le i \le n-1. \end{cases}$$

For $n = 6$, this coloring c of $G = W \,\Box\, K_2$ is illustrated in Figure 4.6. Since $n \ge 5$, it follows that u_1 and u_3 are nonadjacent vertices in F and v_1 and v_3 are nonadjacent vertices in F'. We show that c is a Hamiltonian-connected rainbow $(2n-1)$-coloring of G; that is, we show that every two vertices x and y of G are connected by a rainbow Hamiltonian path in G. We consider two cases, according to the locations of x and y in G.

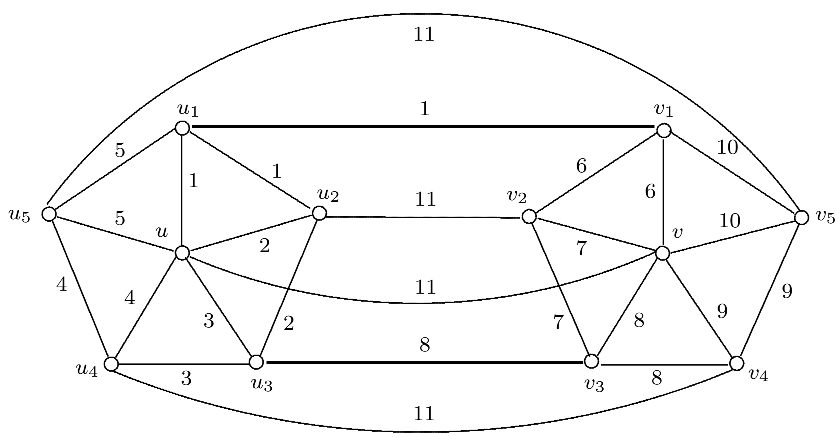


Figure 4.6: A Hamiltonian-connected rainbow 11-coloring of $W \,\Box\, K_2$ for $n = 6$

Case 1. *$x \in V(F)$ and $y \in V(F')$.* Since $n \ge 5$, there exists $z \in V(F) - \{x\}$ such that (1) the corresponding vertex z' of z in F' is not y and (2) $c(zz') = 2n-1$ (namely, zz' is not u_1v_1, u_3v_3 or the two edges between F and F' incident with x or y). Let P be a rainbow Hamiltonian $x - z$ path in F and let P' be a rainbow Hamiltonian $z' - y$ path in F'. Then the path (P, P') is a Hamiltonian $x - y$ path in G.

Case 2. *$x, y \in V(F)$ or $x, y \in V(F')$.* We may assume, without loss of generality, that $x, y \in V(F)$. Let $Q = (x = x_1, x_2, \ldots, x_n = y)$ be a rainbow Hamiltonian $x - y$ path in F. Since $\{c(x_ix_{i+1}) : 1 \le i \le n-1\} = [n-1]$, there is exactly one integer $t \in [n-2]$ such that $c(x_tx_{t+1}) = 1$. Let x'_t and x'_{t+1} be the corresponding vertices of x_t

and x_{t+1} in F', respectively. Thus, $\{c(x_t x_t'), c(x_{t+1}x_{t+1}')\} = \{1, 2n-1\}$, say $c(x_t x_t') = 1$ and $c(x_{t+1}x_{t+1}') = 2n-1$. Let Q' be a rainbow Hamiltonian $x_t' - x_{t+1}'$ path in F'. Next, let Q_1 be the $x_1 - x_t$ subpath of Q and let Q_2 be the $x_{t+1} - x_{n-1}$ subpath of Q. Then the path (Q_1, Q', Q_2) is a rainbow Hamiltonian $x - y$ in G. ■

Corollary 4.2.3 *For each integer $n \geq 5$,* $\mathrm{hrc}(K_n \,\square\, K_2) = 2n-1$.

By Corollary 4.2.3 then, for each integer $n \geq 5$, it follows that

$$\mathrm{hrc}(K_n \,\square\, K_2) - 2\,\mathrm{hrc}(K_n) = 1.$$

For every Hamiltonian-connected graph H of order $n \geq 4$, the number $\mathrm{hrc}(H \,\square\, K_2) - 2\,\mathrm{hrc}(H)$ cannot be much larger than 1.

Theorem 4.2.4 *If H is a Hamiltonian-connected graph of order $n \geq 4$, then*

$$\mathrm{hrc}(H \,\square\, K_2) \leq 2\,\mathrm{hrc}(H) + 2.$$

Proof. Suppose that $\mathrm{hrc}(H) = k$. Let $G = H \,\square\, K_2$ be obtained from two copies F and F' of the graph H of order $n \geq 4$, where $V(F) = \{u_1, u_2, \ldots, u_n\}$ and $V(F') = \{v_1, v_2, \ldots, v_n\}$, by adding the n edges $u_i v_i$ for $1 \leq i \leq n$. Since $\mathrm{hrc}(H) = k$, it follows that H has a Hamiltonian-connected rainbow k-coloring. Let $c_F : V(F) \to \{1, 2, \ldots, k\}$ and $c_{F'} : V(F') \to \{k+1, k+2, \ldots, 2k\}$ be a Hamiltonian-connected rainbow k-coloring of F and F', respectively. Define the $(2k+2)$-edge coloring $c : E(G) \to [2k+2]$ by

$$c(e) = \begin{cases} c_F(e) & \text{if } e \in E(F) \\ c_{F'}(e) & \text{if } e \in E(F') \\ 2k+1 & \text{if } e = u_i v_i \text{ and } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 2k+2 & \text{if } e = u_i v_i \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n. \end{cases}$$

We show that c is a Hamiltonian-connected rainbow coloring of G; that is, we show that every two vertices x and y of G are connected by a rainbow Hamiltonian path in G. We consider two cases, according to the locations of x and y in G.

Case 1. *$x = u_i$ and $y = v_j$ where $1 \leq i, j \leq n$.* Let $t \in [n] - \{i, j\}$. Let P be a rainbow Hamiltonian $u_i - u_t$ path in F and let P' be a rainbow Hamiltonian $v_t - v_j$ path in F'. Then the path (P, P') is a Hamiltonian $u_i - v_j$ path in G.

Case 2. *$x, y \in V(F)$ or $x, y \in V(F')$, say the former.* Suppose that $x = u_i$ and $y = u_j$ where $1 \leq i, j \leq n$ and $i \neq j$. Let Q be a rainbow Hamiltonian $u_i - u_j$ path in F, say

$Q = (u_i = x_1, x_2, \ldots, x_n = u_j)$. Thus, there is $t \in [n-1]$ such that $c(x_t x_t') \neq c(x_{t+1} x_{t+1}')$, where x_t' and x_{t+1}' are the corresponding vertices of x_t and x_{t+1} in F', respectively. Let Q_1 be the $x_1 - x_t$ subpath of Q and let Q_2 be the $x_{t+1} - x_n$ subpath of Q. Now, let Q' be a rainbow Hamiltonian $x_t' - x_{t+1}'$ path in F'. Then the path (Q_1, Q', Q_2) is a rainbow Hamiltonian $u_i - u_j$ in G.

Therefore, c is a Hamiltonian-connected rainbow coloring of G and so $\mathrm{hrc}(G) \leq 2k+2$. ■

By Theorem 4.2.1 and Corollary 4.2.3, we then have the following.

Corollary 4.2.5 $\mathrm{hrc}(K_n \square K_2) = \begin{cases} 7 & \text{if } n = 3 \\ 2n-1 & \text{if } n \geq 5. \end{cases}$

Thus, as far as the numbers $\mathrm{hrc}(K_n \square K_2)$, $n \geq 3$, are concerned, only $\mathrm{hrc}(K_4 \square K_2)$ is unknown. On the other hand, $\mathrm{hrc}(K_4 \square K_2)$ is either 7 or 8 by Theorem 4.2.4.

The upper bound in Theorem 4.2.4 can be improved for a Hamiltonian-connected graph H if H possesses a Hamiltonian-connected rainbow coloring with a particular property. Before describing this bound, we first present an additional definition. For a Hamiltonian-connected graph H, let $c : E(H) \to [k]$ be an edge coloring of H. For each integer i with $1 \leq i \leq k$, let $E_i = \{e \in E(H) : \ c(e) = i\}$ be the set of edges of H that are colored i by c and let $H[E_i]$ be the subgraph induced by E_i in H. If T_1 and T_2 are edge-disjoint stars in H such that their central vertices are not adjacent in H, then T_1 and T_2 are referred to as *nonadjacent stars* in H. The following result is a more general case of Theorem 4.2.2.

Theorem 4.2.6 *Let H be a Hamiltonian-connected graph of order $n \geq 5$ and let $c : E(H) \to [k]$ be a Hamiltonian-connected rainbow coloring of of H for some integer $k \geq n-1$. If there are two distinct colors $\alpha, \beta \in [k]$ such that $H[E_\alpha]$ and $H[E_\beta]$ are nonadjacent stars in H, then* $\mathrm{hrc}(H \square K_2) \leq 2\,\mathrm{hrc}(H) + 1$.

Proof. Let $G = H \square K_2$ be obtained from two copies F and F' of the graph H of order $n \geq 5$, where $V(F) = \{u_1, u_2, \ldots, u_n\}$ and $V(F') = \{v_1, v_2, \ldots, v_n\}$, by adding the n edges $u_i v_i$ for $1 \leq i \leq n$. Let $c_F : E(F) \to [k]$ be a Hamiltonian-connected rainbow coloring of F such that there are two distinct colors $\alpha, \beta \in [k]$ for which $F[E_\alpha]$ and $F[E_\beta]$ are nonadjacent stars in F. We may assume that the central vertex of $F[E_\alpha]$ is u_1 and the central vertex of $F[E_\beta]$ is u_2, where then u_1 and u_2 are not adjacent in F. For an edge e of F, let e' denote the corresponding edge of e in F'. The coloring

$c_{F'} : E(F') \to \{k+1, k+2, \ldots, 2k\}$ of F' defined by $c_{F'}(e') = c_F(e) + k$ for each $e' \in E(F')$ is then a Hamiltonian-connected rainbow coloring of F'.

We now define the $(2k+1)$-edge coloring $c : E(G) \to [2k+1]$ of G by

$$c(e) = \begin{cases} c_F(e) & \text{if } e \in E(F) \\ c_{F'}(e) & \text{if } e \in E(F') \\ \alpha & \text{if } e = u_1v_1 \\ k+\beta & \text{if } e = u_2v_2 \\ 2k+1 & \text{if } e = u_iv_i \text{ for } 3 \le i \le n. \end{cases}$$

We now verify that c is a Hamiltonian-connected rainbow $(2k+1)$-coloring of G by showing that every two vertices x and y of G are connected by a rainbow Hamiltonian path in G. We consider two cases, according to the location of x and y in G. In each of these cases, we denote, for a vertex w of F, the corresponding vertex of w in F' by w'.

Case 1. *$x \in V(F)$ and $y \in V(F')$.* Since $n \ge 5$, there exists $z \in V(F) - \{x\}$ such that zz' is neither u_1v_1, u_2v_2 nor either of the two edges between F and F' incident with x or y. Let P be a rainbow Hamiltonian $x - z$ path in F and let P' be a rainbow Hamiltonian $z' - y$ path in F'. Then the path (P, P') is a Hamiltonian $x - y$ path in G.

Case 2. *$x, y \in V(F)$ or $x, y \in V(F')$.* We may assume, without loss of generality that $x, y \in V(F)$. Let $Q = (x = x_1, x_2, \ldots, x_n = y)$ be a rainbow Hamiltonian $x - y$ path in F and let $c(Q) = \{c(x_ix_{i+1}) : 1 \le i \le n-1\}$ be the set of colors of the edges of Q. First, suppose that $\alpha \in c(Q)$. Since Q is a rainbow Hamiltonian path of F and $H[E_\alpha]$ is a star, there is exactly one integer $t \in [n-1]$ such that $c(x_tx_{t+1}) = \alpha$. Then either x_t or x_{t+1} is the central vertex of $H[E_\alpha]$. So, x'_t and x'_{t+1} are the corresponding vertices of x_t and x_{t+1} in F', respectively. Since the central vertices of $F[E_\alpha]$ and $F[E_\beta]$ are not adjacent in F, it follows that $\{c(x_tx'_t), c(x_{t+1}x'_{t+1})\} = \{\alpha, 2k+1\}$, say $c(x_tx'_t) = \alpha$ and $c(x_{t+1}x'_{t+1}) = 2k+1$. Let Q' be a rainbow Hamiltonian $x'_t - x'_{t+1}$ path in F'. Next, let Q_1 be the $x_1 - x_t$ subpath of Q and let Q_2 be the $x_{t+1} - x_n$ subpath of Q. Then the path (Q_1, Q', Q_2) is a rainbow Hamiltonian $x - y$ path in G. Next, suppose that $\alpha \notin c(Q)$. Since Q is a rainbow Hamiltonian path of F, there is exactly one integer $t \in [n-1]$ such that $x_t = u_1$ (the central vertex of $F[E_\alpha]$). Let x'_t and x'_{t+1} be the corresponding vertices of x_t and x_{t+1} in F', respectively. Let Q' be a rainbow Hamiltonian $x'_t - x'_{t+1}$ path in F'. Next, let Q_1 be the $x_1 - x_t$ subpath of Q and let Q_2 be the $x_{t+1} - x_n$ subpath of Q. Then the path (Q_1, Q', Q_2) is a rainbow Hamiltonian $x - y$ path in G. ■

A graph is a *galaxy* if each of its components is a star. If H_1 and H_2 are edge-disjoint subgraphs of a graph H where H_1 and H_2 are galaxies, such that no central vertex of any component in H_1 is adjacent to a central vertex in a component in H_2, then H_1 and H_2 are referred to as *nonadjacent galaxies* in H. The argument used in the proof of Theorem 4.2.6 yields the following result.

Corollary 4.2.7 *Let H be a Hamiltonian-connected graph of order $n \geq 5$ and let $c : E(H) \to [k]$ be a Hamiltonian-connected rainbow coloring of of H for some integer $k \geq n-1$. If there are two distinct colors $\alpha, \beta \in [k]$ such that $H[E_\alpha]$ and $H[E_\beta]$ are nonadjacent galaxies in H, then* $\mathrm{hrc}(H \,\square\, K_2) \leq 2\,\mathrm{hrc}(H) + 1$.

4.3 Hamiltonian-Connected Prisms

We have now seen (in Theorem 4.1.4, Corollary 4.1.6, Theorem 4.2.2 and Corollary 4.2.3) Hamiltonian-connected graphs G of order n for which $\mathrm{hrc}(G) = n-1$. We also saw in Theorem 4.2.1 a Hamiltonian-connected graph G of order 6 with $\mathrm{hrc}(G) = n+1$. While we have not determined the value of $\mathrm{hrc}(K_4 \,\square\, K_2)$, no graph H of order $n \geq 4$ with $\mathrm{hrc}(H) = n$ has been established. Indeed, the examples described thus far may suggest the existence of a constant c such that $\mathrm{hrc}(G) \leq n + c$ for every Hamiltonian-connected graph G of order $n \geq 4$. There is no such constant, however, as we verify in this section.

For an integer $n \geq 3$, the prism $C_n \,\square\, K_2$ is Hamiltonian-connected if and only if n is odd. Thus, if G is a Hamiltonian graph of odd order $n \geq 3$, then $G \,\square\, K_2$ is Hamiltonian-connected. For an odd integer $n \geq 3$, let $C_n \,\square\, K_2$ be obtained from two copies C and C' of the n-cycle C_n, where

$$C = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1) \text{ and } C' = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1),$$

by adding the n edges $u_i v_i$ for $1 \leq i \leq n$. We will refer to this labeling of the vertices of $C_n \,\square\, K_2$ for all results in this section. By Theorem 4.2.1, $\mathrm{hrc}(C_3 \,\square\, K_2) = 7$. Next, we show that $\mathrm{hrc}(C_n \,\square\, K_2) = 3n$ for each odd integer $n \geq 5$. In order to do this, we first present some preliminary results.

Lemma 4.3.1 *Let $n \geq 5$ be an odd integer. For each integer i with $1 \leq i \leq n$, there are exactly two $u_i - v_i$ Hamiltonian paths in $C_n \,\square\, K_2$.*

Proof. By the symmetry of the graph $G = C_n \,\square\, K_2$, it suffices to show that there are exactly two Hamiltonian $u_1 - v_1$ paths in G. First, G contains the following two Hamiltonian $u_1 - v_1$ paths:

$$P = (u_1, u_n, u_{n-1}, \ldots, u_2, v_2, v_3, v_4, \ldots, v_n, v_1) \quad (4.2)$$

$$Q = (u_1, u_2, u_3, \ldots, u_n, v_n, v_{n-1}, \ldots, v_2, v_1). \quad (4.3)$$

The Hamiltonian $u_1 - v_1$ paths P and Q are shown in Figure 4.7 for $n = 7$, where solid lines indicate edges in P or Q and dashed lines indicate edges not in P or Q.

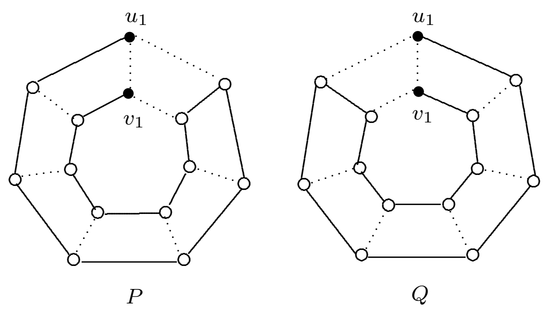


Figure 4.7: The two Hamiltonian $u_1 - v_1$ paths in $C_7 \,\square\, K_2$

Thus, it remains to show that P and Q are the only Hamiltonian $u_1 - v_1$ paths in G. Let R be a Hamiltonian $u_1 - v_1$ path in G. Since $u_1v_1 \notin E(R)$, it follows that exactly one of u_1u_n and u_1u_2 belongs to R, say $u_1u_n \in E(R)$ and $u_1u_2 \notin E(R)$. Since G is 3-regular and R is a Hamiltonian path of G, for each $x \in V(G) - \{u_1, v_1\}$, exactly one of the three edges incident with x does not belong to R. Since $u_1u_2 \notin E(R)$, it follows that (v_2, u_2, u_3) is a subpath of R and exactly one of v_1v_n and v_1v_2 belongs to R. We consider these two cases.

Case 1. *$v_1v_n \in E(R)$ and $v_1v_2 \notin E(R)$.* Since $v_1v_2 \notin E(R)$ and G is 3-regular, it follows that $v_2v_3 \in E(R)$. Hence, $u_nv_n, u_3v_3 \notin E(R)$. This implies that u_nu_{n-1}, v_nv_{n-1}, u_3u_4, v_3v_4 are edges of R. Continuing this argument, we see that $u_iv_i \notin E(R)$ for $3 \leq i \leq n$ and so R is the path P described in (4.2).

Case 2. *$v_1v_n \notin E(R)$ and $v_1v_2 \in E(R)$.* Then $v_1v_n, v_2v_3 \notin E(R)$ and so (u_n, v_n, v_{n-1}) and (u_3, v_3, v_4) are subpaths of R. Thus, $u_nu_{n-1}, u_3u_4 \notin E(R)$. If $n = 5$, then $v_{n-1} = v_4$. However then, u_4 does not belong to R, which is a contradiction. If $n \geq 7$, then (v_4, u_4, u_5) is a subpath of R and so $v_4v_5 \notin E(R)$. Continuing this argument, we conclude that

$$(v_4, u_4, u_5, v_5, v_6, u_6, \ldots, u_{n-2}, v_{n-2}, v_{n-1})$$

is a subpath of R. However then, u_{n-1} does not belong to R, which is a contradiction. Thus, Case 2 cannot occur. ■

Lemma 4.3.2 *If c is a Hamiltonian-connected rainbow coloring of $C_n \,\square\, K_2$ for some odd integer $n \geq 5$, then for each integer i with $1 \leq i \leq n$, the coloring c assigns distinct colors to the $2(n-2)$ edges in the two paths $C - u_i$ and $C' - v_i$ in $C_n \,\square\, K_2$.*

Proof. Let c be a Hamiltonian-connected rainbow coloring of $G = C_n \square K_2$ for some odd integer $n \geq 5$. By the symmetry of the graph G, it suffices to show that c must assign distinct colors to all edges in the two paths $C - u_1$ and $C' - v_1$ in G. Let $X = E(C - u_1) \cup E(C' - v_1)$. Then $|X| = 2(n-2)$. By Lemma 4.3.1, the paths P in (4.2) and Q in (4.3) are the only Hamiltonian $u_1 - v_1$ paths in G. Thus, at least one of P and Q is rainbow. Since $X \subseteq E(P) \cap E(Q)$ (see Figure 4.8 for $n = 7$), it follows that all edges in X must be assigned different colors. ∎

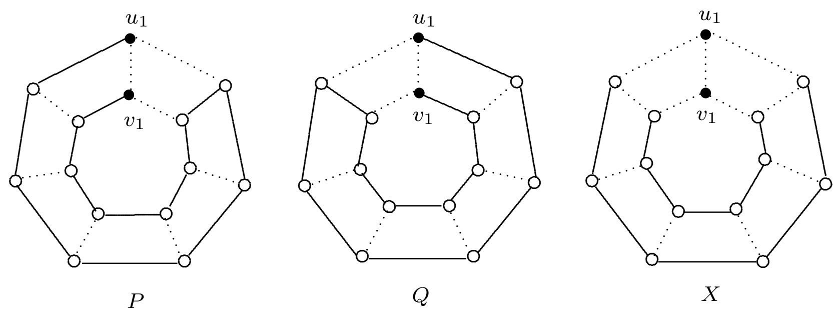


Figure 4.8: The set X of edges in $C_7 \square K_2$

Lemma 4.3.3 *Let $n \geq 5$ be an odd integer. For each integer i with $1 \leq i \leq n$, there are exactly two $u_i - v_{i+1}$ Hamiltonian paths in $C_n \square K_2$.*

Proof. By the symmetry of the graph $G = C_n \square K_2$, it suffices to show that there are exactly two Hamiltonian $u_1 - v_2$ paths in G. First, G contains the following two Hamiltonian $u_1 - v_2$ paths:

$$P = (u_1, v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, u_{n-3}, v_{n-3}, \ldots, v_3, u_3, u_2, v_2) \quad (4.4)$$
$$Q = (u_1, u_2, u_3, v_3, v_4, u_4, u_5, v_5, v_6, \ldots, v_{n-1}, u_{n-1}, u_n, v_n, v_1, v_2). \quad (4.5)$$

The Hamiltonian $u_1 - v_2$ paths P and Q are shown in Figure 4.9 for $n = 7$, where solid lines indicate edges in P or Q and dashed lines indicate edges not in P or Q.

Thus, it remains to show that P and Q are the only Hamiltonian $u_1 - v_2$ paths in G. Let R be a Hamiltonian $u_1 - v_2$ path in G. Then R contains exactly one of u_1v_1, u_1u_2 or u_1u_n. We consider these three cases.

Case 1. *$u_1v_1 \in E(R)$ and $u_1u_2, u_1u_n \notin E(R)$.* Clearly, $v_1v_2 \notin E(R)$. Hence, (v_2, u_2, u_3) and (v_1, v_n, u_n, u_{n-1}) are subpaths of R. Since v_2 is an end-vertex of R, it follows that $v_2v_3 \notin E(R)$. Now $v_2v_3, v_nv_{n-1} \notin E(R)$ and so (u_3, v_3, v_4) is a subpath of R. If $n = 5$, then

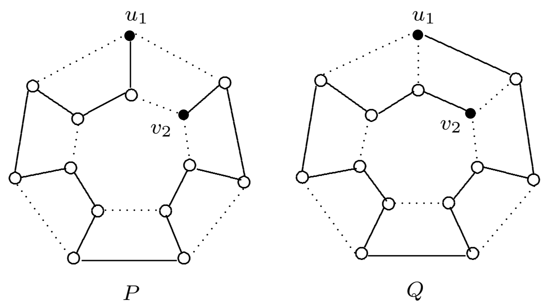


Figure 4.9: The two Hamiltonian $u_1 - v_2$ paths in $C_7 \square K_2$

$$R = (u_1, v_1, v_5, u_5, u_4, v_4, v_3, u_3, u_2, v_2),$$

which is the path P in (4.4), while if $n \geq 7$, then, by a similar argument, we conclude that R is the path P in (4.4).

Case 2. $u_1u_2 \in E(R)$ *and* $u_1v_1, u_1u_n \notin E(R)$. Then $(v_2, v_1, v_n, u_n, u_{n-1})$ is a subpath of R. Since $v_nv_{n-1} \notin E(R)$, it follows that $(u_{n-1}, v_{n-1}, v_{n-2})$ is a subpath of R. Continuing in this way, we see that R is the path Q in (4.5) in this case.

Case 3. $u_1u_n \in E(R)$ *and* $u_1v_1, u_1u_2 \notin E(R)$. Since R is a Hamiltonian path and G is 3-regular, it follows that $(v_n, v_1, v_2, u_2, u_3)$ is a subpath of R, which is impossible since v_2 is an end-vertex of R. Thus, Case 3 cannot occur. ∎

Lemma 4.3.4 *Let $n \geq 5$ be an odd integer. For each integer i with $1 \leq i \leq n$, let*

$$R_i = (u_{i+1}, u_{i+2}, v_{i+2}, v_{i+3}, u_{i+3}, u_{i+4}, v_{i+4}, v_{i+5}, \ldots, v_{i-2}, u_{i-2}, u_{i-1}, v_{i-1}, v_i) \qquad (4.6)$$

be the path in $C_n \square K_2$, where the subscript of each vertex is expressed as an integer modulo n. If c is a Hamiltonian-connected rainbow coloring of $C_n \square K_2$, then c assigns distinct colors to the edges of R_i for $1 \leq i \leq n$.

Proof. Let c be a Hamiltonian-connected rainbow coloring of $G = C_n \square K_2$ for some odd integer $n \geq 5$. It suffices to show that all edges in R_1 must be colored differently by c. Observe that

$$R_1 = (u_2, u_3, v_3, v_4, u_4, u_5, v_5, v_6, \ldots, u_{n-1}, u_n, v_n, v_{n+1} = v_1).$$

By Lemma 4.3, the paths P in (4.4) and Q in (4.5) are the only Hamiltonian $u_1 - v_2$ paths in G. Thus, at least one of P and Q is rainbow. Since $E(R_1) \subseteq E(P) \cap E(Q)$ (see Figure 4.10 for $n = 7$), it follows that all edges in $E(R_1)$ must be assigned different colors. ∎

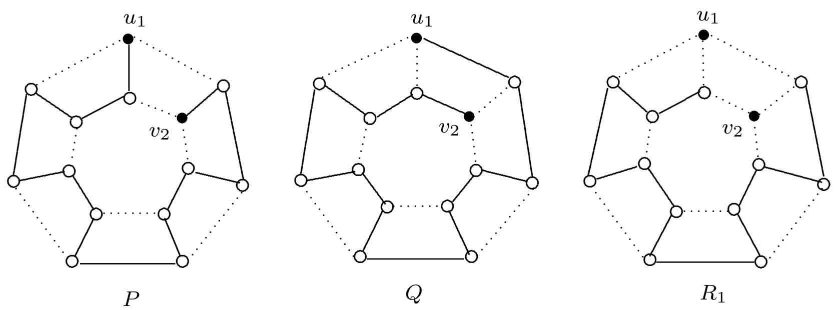


Figure 4.10: The path R_1 in $C_7 \square K_2$

We now make a useful observation. Let $E[C, C'] = \{u_i v_i : 1 \le i \le n\}$ be the set of edges of $C_n \square K_2$ that do not belong to $E(C) \cup E(C')$. For each integer i with $1 \le i \le n$, let R_i be the path of $C_n \square K_2$ defined in (4.6). Then $E[C, C'] - \{u_i v_i, u_{i+1} v_{i+1}\} \subseteq E(R_i)$ and $u_i v_i, u_{i+1} v_{i+1} \notin E(R_i)$. For example, R_1 contains all edges $u_i v_i$ for $3 \le i \le n$ and $u_1 v_1, u_2 v_2 \notin E(R_1)$.

Theorem 4.3.5 *For each odd integer $n \ge 5$, $\mathrm{hrc}(C_n \square K_2) = 3n$.*

Proof. Let $G = C_n \square K_2$ for some odd integer $n \ge 5$. Since the size of G is $3n$, it follows by Observation 4.1.2 that $\mathrm{hrc}(C_n \square K_2) \le 3n$. It remains to show that every Hamiltonian-connected rainbow coloring of G must assign distinct colors to distinct edges of G. Let c be a Hamiltonian-connected rainbow coloring and let $e, f \in E(G)$. We show that $c(e) \ne c(f)$.

First, suppose that at least one of e and f belongs to the two n-cycles C and C' in G, say $e \in E(C) \cup E(C')$. Assume, without loss of generality, that $e = u_1 u_2$. We consider two cases, according to whether $f \in E(C) \cup E(C')$ or $f \in E[C, C']$.

Case 1. $f \in E(C) \cup E(C')$. Thus, either $f = u_j u_{j+1} \in E(C)$ where $2 \le j \le n$ or $f = v_j v_{j+1} \in E(C')$ where $1 \le j \le n$. Since $n \ge 5$, it follows that there exists $i \in [n] - \{1, 2, j, j+1\}$. Thus, $e, f \in E(C - u_i) \cup E(C' - v_i)$. It then follows by Lemma 4.3.4 that $c(e) \ne c(f)$.

Case 2. $f \in E[C, C']$. Let $f = u_j v_j$ where $1 \le j \le n$. First, suppose that $f = u_1 v_1$ or $f = u_2 v_2$, say the former. Then both $e = u_1 u_2$ and $f = u_1 v_1$ lie on the path

$$R_3 = (u_4, u_5, v_5, v_6, u_6, u_7, \ldots, v_{n-1}, u_{n-1}, u_n, v_n, v_1, u_1, u_2, v_2, v_3),$$

as described in (4.6). The path R_3 is shown in Figure 4.11 for $n = 7$, where e and f are indicated by bold lines, the edges in R_3 are indicated by solid lines and all other edges (that are not on R_3) are indicated by dashed lines.

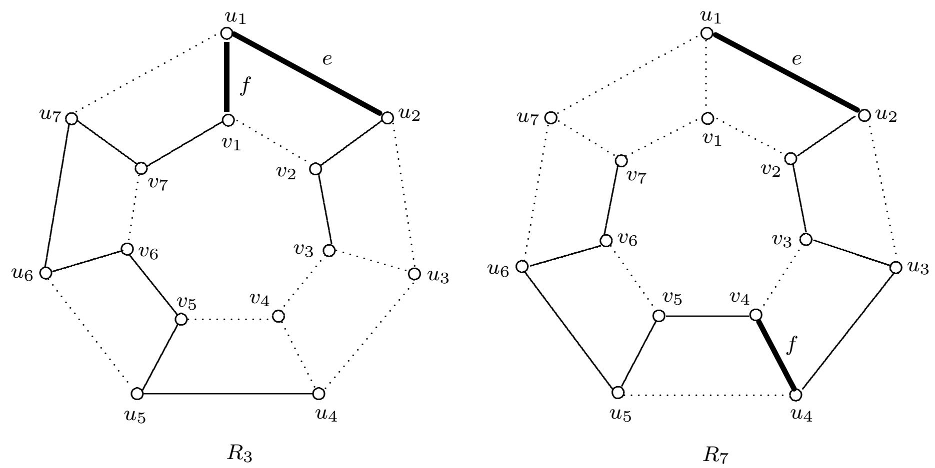


Figure 4.11: The paths R_3 and R_7 in $C_7 \,\square\, K_2$

By Lemma 4.3.4, $c(e) \neq c(f)$. Next, suppose that $f = u_i v_i$ where $3 \leq i \leq n$. By symmetry, we may assume that $3 \leq i \leq \lceil \frac{n}{2} \rceil + 1$. Then $e = u_1 u_2$ and $f = u_i v_i$ lie on the path

$$R_n = (u_1, u_2, v_2, v_3, u_3, u_4, \ldots, v_{n-2}, u_{n-2}, u_{n-1}, v_{n-1}, v_n),$$

as described in (4.6). For $f = u_4 v_4$, the path R_7 is shown in Figure 4.11 for $n = 7$ which contains both e and f. Since $u_i v_i \in E(R_n)$ for $3 \leq i \leq \lceil \frac{n}{2} \rceil + 1$, it follows that $f \in E(R_n)$. Thus, both e and f belong to R_n and so $c(e) \neq c(f)$ by Lemma 4.3.4.

Next, suppose that neither e nor f belongs to $E(C) \cup E(C')$. Assume, without loss of generality, that $e = u_1 v_1$ and $f = u_j v_j$ where $2 \leq j \leq n$. Since $n \geq 5$, there is $i \in [n] - \{1, j\}$ such that $e, f \in E(R_i)$. It then follows by Lemma 4.3.4 that $c(e) \neq c(f)$. ■

It is a consequence of Theorem 4.3.5 that there is no constant c such that $\mathrm{hrc}(G) \leq n + c$ for every Hamiltonian-connected graph G of order n. If $p \geq 5$ is an odd integer, then $C_p \,\square\, K_2$ is a Hamiltonian-connected graph $n = 2p$ and $\mathrm{hrc}(C_p \,\square K_2) = 3p = 3n/2$. This gives rise to the following question.

Problem 4.3.6 *Does there exist any Hamiltonian-connected graph G of order n for which* $\mathrm{hrc}(G) > 3n/2$*?*

4.4 The Square of Hamiltonian Graphs

During 1960-1980, there was a great deal of research activity involving Hamiltonian properties of powers of graphs. For a connected graph G and a positive integer k, the *kth power* G^k of G is the graph whose vertex set is $V(G)$ such that uv is an edge of G^k if $1 \leq d_G(u, v) \leq k$, where $d_G(u, v)$ is the distance between two vertices u and v in G (the length of a shortest $u - v$ path in G). The graph G^2 is called the *square* of G and G^3 the *cube* of G. In 1960, Sekanina [55] proved the following result.

Theorem 4.4.1 *If G is a nontrivial connected graph, then G^3 is Hamiltonian-connected.*

In the 1960s, it was conjectured independently by Nash-Williams [39] and Plummer (see [14, p. 147]) that the square of every 2-connected graph is Hamiltonian. In 1974, Fleischner [20] verified this conjecture. Also, in 1974, using Fleischner's result, Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [8] proved the following.

Theorem 4.4.2 *If G is a 2-connected graph, then G^2 is Hamiltonian-connected. In particular, the square of every Hamiltonian graph is Hamiltonian-connected.*

In 2009, Chia, Ong and Tan [53] proved the following stronger result.

Theorem 4.4.3 *If G is a connected graph having only one cut-vertex, then G^2 is panconnected and, consequently, G is Hamiltonian-connected.*

By Theorem 4.4.2, the square of every Hamiltonian graph is Hamiltonian-connected. Since the square of a Hamiltonian graph G of order $n \geq 3$ contains the square C_n^2 of an n-cycle C_n as a spanning subgraph, it then follows by Observation 4.1.1 that $\text{hrc}(G^2) \leq \text{hrc}(C_n^2)$. Thus, it is of interest to investigate the rainbow Hamiltonian-connection numbers of the squares of cycles. Since $\text{hrc}(K_n) = n - 1$ for $n \geq 4$, it follows that $\text{hrc}(C_n^2) = n - 1$ for $n = 4, 5$. For $n \geq 6$, $\text{hrc}(C_n^2) \geq n - 1$ by Observation 4.1.2. The following theorem says that for every integer $n \geq 6$, either $\text{hrc}(C_n^2) = n - 1$ or $\text{hrc}(C_n^2) = n$.

Theorem 4.4.4 *For each integer $n \geq 6$, $\text{hrc}(C_n^2) \leq n$.*

Proof. Let $G = C_n^2$, where $C_n = (v_1, v_2, \ldots, v_n, v_1)$. We show that G has a Hamiltonian-connected rainbow coloring using n colors. Define the n-edge coloring $c : E(G) \to [n]$ by $c(v_i v_{i+1}) = i$ and $c(v_i v_{i+2}) = i + 1$ for $1 \leq i \leq n$, where the subscript of each vertex is expressed as a positive integer modulo n. This is illustrated in Figure 4.12.

We show that every two vertices x and y in G are connected by a rainbow Hamiltonian path in G. Since C_n is a rainbow Hamiltonian cycle in G, we may assume that x and y are not consecutive vertices of C_n. By symmetry of the graph G and the coloring c, we may further assume that $x = v_1$ and $y = v_i$ for $3 \leq i \leq n - 1$.

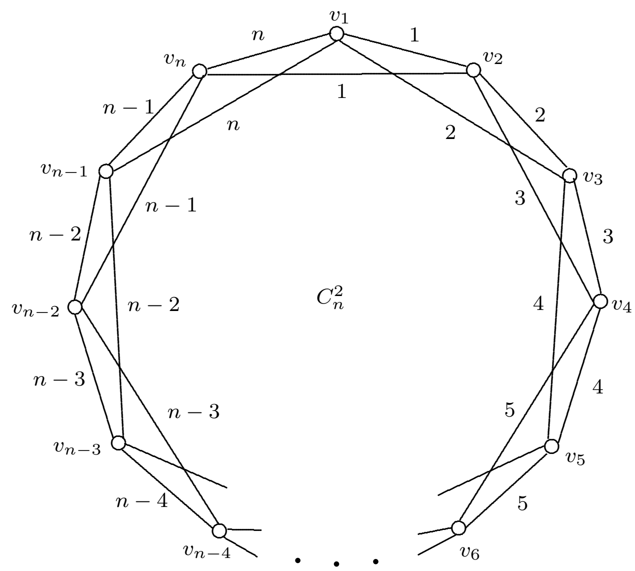


Figure 4.12: A Hamiltonian-connected rainbow coloring of C_n^2 using n colors

- ⋆ If i is odd, then $(v_1, v_n, v_{n-1}, \ldots, v_{i+1}, v_{i-1}, v_{i-3}, \ldots, v_2, v_3, v_5, v_7, \ldots, v_i)$ is a rainbow Hamiltonian $v_1 - v_i$ path.

- ⋆ If i is even, then $(v_1, v_n, v_{n-1}, \ldots, v_{i+1}, v_{i-1}, v_{i-3}, \ldots, v_3, v_2, v_4, v_6, \ldots, v_i)$ is a rainbow Hamiltonian $v_1 - v_i$ path.

For example, if $n = 10$, then $(v_1, v_{10}, v_9, v_8, v_7, v_6, v_4, v_2, v_3, v_5)$ is a rainbow Hamiltonian $v_1 - v_5$ path and $(v_1, v_{10}, v_9, v_8, v_7, v_5, v_3, v_2, v_4, v_6)$ is a rainbow Hamiltonian $v_1 - v_6$ path. Thus, c is a Hamiltonian-connected rainbow coloring of G and so $\mathrm{hrc}(G) \leq n$. ■

If G is a Hamiltonian graph of of order $n \geq 4$ having diameter at most 2, then $G^2 = K_n$ and so $\mathrm{hrc}(G^2) = n - 1$. In general, the following is an immediate consequence of Theorem 4.4.4 and Observation 4.1.1.

Corollary 4.4.5 *If G is a Hamiltonian graph of order at least $n \geq 6$, then*

$$\mathrm{hrc}(G^2) \leq n.$$

In fact, the rainbow Hamiltonian-connection number of C_6^2 is 5. To show this, consider the 5-edge coloring of the graph $G = C_6^2$ shown in Figure 4.13. This edge coloring is a Hamiltonian-connected rainbow 5-coloring since each of the following $\binom{6}{2} = 15$ Hamiltonian paths are rainbow paths. Therefore, $\mathrm{hrc}(G) = 5$ by Observation 4.1.2.

1. rainbow Hamiltonian $v_1 - v_2$ path: $(v_1, v_6, v_5, v_4, v_3, v_2)$

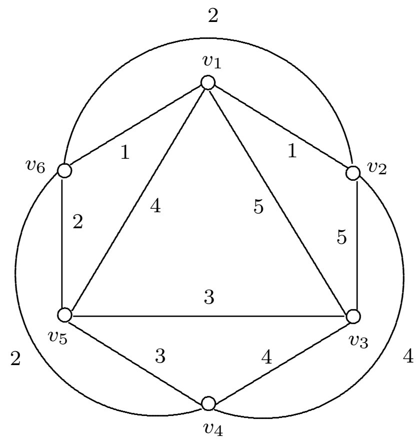


Figure 4.13: A Hamiltonian-connected rainbow coloring of C_6^2 using 5 colors

2. rainbow Hamiltonian $v_1 - v_3$ path: $(v_1, v_6, v_5, v_4, v_2, v_3)$
3. rainbow Hamiltonian $v_1 - v_4$ path: $(v_1, v_6, v_5, v_3, v_2, v_4)$
4. rainbow Hamiltonian $v_1 - v_5$ path: $(v_1, v_6, v_4, v_2, v_3, v_5)$
5. rainbow Hamiltonian $v_1 - v_6$ path: $(v_1, v_2, v_3, v_4, v_5, v_6)$
6. rainbow Hamiltonian $v_2 - v_3$ path: $(v_2, v_4, v_5, v_6, v_1, v_3)$
7. rainbow Hamiltonian $v_2 - v_4$ path: $(v_2, v_3, v_5, v_1, v_6, v_4)$
8. rainbow Hamiltonian $v_2 - v_5$ path: $(v_2, v_6, v_1, v_3, v_4, v_5)$
9. rainbow Hamiltonian $v_2 - v_6$ path: $(v_2, v_1, v_3, v_4, v_5, v_6)$
10. rainbow Hamiltonian $v_3 - v_4$ path: $(v_3, v_2, v_6, v_1, v_5, v_4)$
11. rainbow Hamiltonian $v_3 - v_5$ path: $(v_3, v_1, v_6, v_2, v_4, v_5)$
12. rainbow Hamiltonian $v_3 - v_6$ path: $(v_3, v_2, v_1, v_5, v_4, v_6)$
13. rainbow Hamiltonian $v_4 - v_5$ path: $(v_4, v_2, v_6, v_1, v_3, v_5)$
14. rainbow Hamiltonian $v_4 - v_6$ path: $(v_4, v_2, v_1, v_3, v_5, v_6)$
15. rainbow Hamiltonian $v_5 - v_6$ path: $(v_5, v_3, v_1, v_2, v_4, v_6)$

By Theorem 4.4.4, for each integer $n \geq 7$, the rainbow Hamiltonian-connection number of C_n^2 is either $n-1$ or n. By Theorem 4.4.1, if G is a nontrivial connected graph of order n, then G^3 is Hamiltonian-connected and so G^k is Hamiltonian-connected for each $k \geq 3$. Therefore, if G is a Hamiltonian graph of order $n \geq 3$, then $\text{hrc}(G^k) \in \{n-1, n\}$ for each integer $k \geq 3$.

4.5 Minimum Hamiltonian-Connected Graphs

If G is a Hamiltonian-connected graph that is not complete and u and v are nonadjacent vertices of G, then $G + uv$ is also Hamiltonian-connected and $\text{hrc}(G + uv) \leq \text{hrc}(G)$ by

Observation 4.1.1. This suggests that Hamiltonian-connected graphs having the greatest rainbow Hamiltonian-connection numbers are minimal Hamiltonian-connected graphs. This leads us to consider Hamiltonian-connected graphs of order n and minimum size. Every Hamiltonian-connected graph of order at least 4 is 3-connected. Therefore, if G is a Hamiltonian-connected graph of $n \geq 4$, then $\delta(G) \geq 3$, which implies that the minimum size of a Hamiltonian-connected graph of order n is $\lfloor \frac{3n+1}{2} \rfloor$. The following result is due to Moon [36].

Theorem 4.5.1 *For each integer $n \geq 4$, there exists Hamiltonian-connected graph of order n and size $\lfloor \frac{3n+1}{2} \rfloor$.*

For each integer $k \geq 2$, let $P_k \,\square\, K_2$ be the grid of order $2k$ in which two paths of order k are $P_k = (x_1, x_2, \ldots, x_k)$ and $P'_k = (y_1, y_2, \ldots, y_k)$ such that $x_i y_i \in E(P_k \,\square\, K_2)$ for $1 \leq i \leq k$. Now, let H_k be the cubic graph of order $2k+2$ obtained by adding two adjacent vertices u and v to the grid $P_k \,\square\, K_2$ and joining (1) the vertex u to x_1 and y_1 and (2) the vertex v to x_k and y_k in $P_k \,\square\, K_2$ (see Figure 4.14).

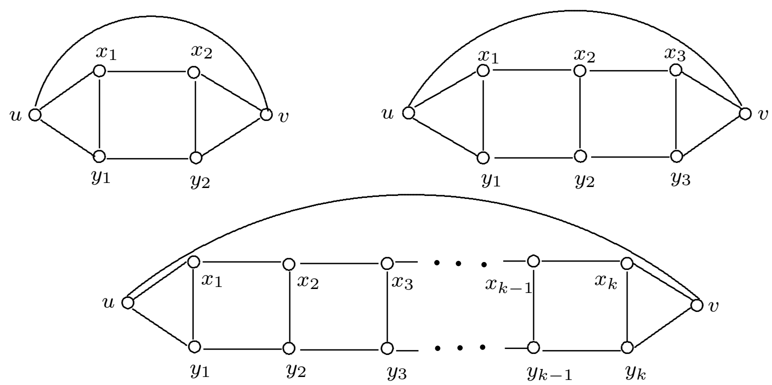


Figure 4.14: Graphs H_2, H_3 and H_k

The graph H_3 has order 8 and rainbow Hamiltonian-connection number 10, as we show next.

Theorem 4.5.2 $\text{hrc}(H_3) = 10$.

Proof. Let $G = H_3$. First, we make an observation. For each pair w, z of distinct vertices of G, there are exactly two Hamiltonian $w-z$ paths except for $\{w, z\} = \{x_1, y_3\}$ and $\{w, z\} = \{x_3, y_1\}$, in which case, there are exactly four Hamiltonian $w-z$ paths. Since there are $\binom{8}{2} = 28$ pairs of vertices of G, there are $2\binom{8}{2} + 4 = 60$ Hamiltonian paths in G. For each pair w, z of distinct vertices of G, we list the Hamiltonian $w-z$ paths in G as well as the edges commonly belonging to each of these $w-z$ paths. We label the vertices and the edges of G as indicated in Figure 4.15.

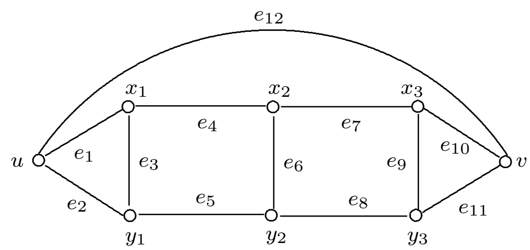


Figure 4.15: The graph H_3

1. $u - v$: $(u, x_1, y_1, y_2, x_2, x_3, y_3, v)$, $(u, y_1, x_1, x_2, y_2, y_3, x_3, v)$, e_3, e_6, e_9
2. $u - x_1$: $(u, v, y_3, x_3, x_2, y_2, y_1, x_1)$, $(u, y_1, y_2, y_3, v, x_3, x_2, x_1)$, e_5, e_7, e_{11}
3. $u - y_1$: $(u, v, x_3, y_3, y_2, x_2, x_1, y_1)$, $(u, x_1, x_2, x_3, v, y_3, y_2, y_1)$, e_4, e_8, e_{10}
4. $u - x_2$: $(u, v, x_3, y_3, y_2, y_1, x_1, x_2)$, $(u, x_1, y_1, y_2, y_3, v, x_3, x_2)$, e_3, e_5, e_8, e_{10}
5. $u - y_2$: $(u, v, y_3, x_3, x_2, x_1, y_1, y_2)$, $(u, y_1, x_1, x_2, x_3, v, y_3, y_2)$, e_3, e_4, e_7, e_{11}
6. $u - x_3$: $(u, v, y_3, y_2, y_1, x_1, x_2, x_3)$, $(u, y_1, x_1, x_2, y_2, y_3, v, x_3)$, e_3, e_4, e_8, e_{11}
7. $u - y_3$: $(u, v, x_3, x_2, x_1, y_1, y_2, y_3)$, $(u, x_1, y_1, y_2, x_2, x_3, v, y_3)$, e_3, e_5, e_7, e_{10}
8. $x_1 - v$: $(x_1, u, y_1, y_2, x_2, x_3, y_3, v)$, $(x_1, x_2, x_3, y_3, y_2, y_1, u, v)$, e_2, e_5, e_7, e_9
9. $x_1 - y_1$: $(x_1, u, v, y_3, x_3, x_2, y_2, y_1)$, $(x_1, x_2, y_2, y_3, x_3, v, u, y_1)$, e_6, e_9, e_{12}
10. $x_1 - x_2$: $(x_1, u, y_1, y_2, y_3, v, x_3, x_2)$, $(x_1, y_1, u, v, x_3, y_3, y_2, x_2)$, e_2, e_8, e_{10}
11. $x_1 - y_2$: $(x_1, y_1, u, v, y_3, x_3, x_2, y_2)$, $(x_1, x_2, x_3, y_3, v, u, y_1, y_2)$, $e_2, e_7, e_9, e_{11}, e_{12}$
12. $x_1 - x_3$: $(x_1, x_2, y_2, y_1, u, v, y_3, x_3)$, $(x_1, y_1, u, v, y_3, y_2, x_2, x_3)$, e_2, e_6, e_{11}, e_{12}
13. $x_1 - y_3$: $(x_1, x_2, y_2, y_1, u, v, x_3, y_3)$, $(x_1, u, y_1, y_2, x_2, x_3, v, y_3)$,
    $(x_1, x_2, x_3, v, u, y_1, y_2, y_3)$, $(x_1, y_1, u, v, x_3, x_2, y_2, y_3)$, e_2, e_{10}
14. $y_1 - v$: $(y_1, u, x_1, x_2, y_2, y_3, x_3, v)$, $(y_1, y_2, y_3, x_3, x_2, x_1, u, v)$, e_1, e_4, e_8, e_9
15. $y_1 - x_2$: $(y_1, x_1, u, v, x_3, y_3, y_2, x_2)$, $(y_1, y_2, y_3, x_3, v, u, x_1, x_2)$, $e_1, e_8, e_9, e_{10}, e_{12}$
16. $y_1 - y_2$: $(y_1, x_1, u, v, y_3, x_3, x_2, y_2)$, $(y_1, u, x_1, x_2, x_3, v, y_3, y_2)$, e_1, e_7, e_{11}
17. $y_1 - x_3$: $(y_1, y_2, x_2, x_1, u, v, y_3, x_3)$, $(y_1, y_2, y_3, v, u, x_1, x_2, x_3)$
    $(y_1, x_1, u, v, y_3, y_2, x_2, x_3)$, $(y_1, u, x_1, x_2, y_2, y_3, v, x_3)$, e_1, e_{11}
18. $y_1 - y_3$: $(y_1, y_2, x_2, x_1, u, v, x_3, y_3)$, $(y_1, x_1, u, v, x_3, x_2, y_2, y_3)$, e_1, e_6, e_{10}, e_{12}
19. $x_2 - v$: $(x_2, x_3, y_3, y_2, y_1, x_1, u, v)$, $(x_2, x_1, u, y_1, y_2, y_3, x_3, v)$, e_1, e_5, e_8, e_9
20. $x_2 - y_2$: $(x_2, x_1, y_1, u, v, x_3, y_3, y_2)$, $(x_2, x_3, y_3, v, u, x_1, y_1, y_2)$, e_3, e_9, e_{12}
21. $x_2 - x_3$: $(x_2, y_2, y_1, x_1, u, v, y_3, x_3)$, $(x_2, x_1, u, y_1, y_2, y_3, v, x_3)$, e_1, e_5, e_{11}
22. $x_2 - y_3$: $(x_2, y_2, y_1, x_1, u, v, x_3, y_3)$, $(x_2, x_3, v, u, x_1, y_1, y_2, y_3)$, $e_1, e_3, e_5, e_{10}, e_{12}$
23. $y_2 - v$: $(y_2, y_3, x_3, x_2, x_1, y_1, u, v)$, $(y_2, y_1, u, x_1, x_2, x_3, y_3, v)$, e_2, e_4, e_7, e_9
24. $y_2 - x_3$: $(y_2, x_2, x_1, y_1, u, v, y_3, x_3)$, $(y_2, y_3, v, u, y_1, x_1, x_2, x_3)$, $e_2, e_3, e_4, e_{11}, e_{12}$
25. $y_2 - y_3$: $(y_2, y_1, u, x_1, x_2, x_3, v, y_3)$, $(y_2, x_2, x_1, y_1, u, v, x_3, y_3)$, e_2, e_4, e_{10}
26. $x_3 - v$: $(x_3, x_2, x_1, u, y_1, y_2, y_3, v)$, $(x_3, y_3, y_2, x_2, x_1, y_1, u, v)$, e_2, e_4, e_8
27. $x_3 - y_3$: $(x_3, v, u, y_1, x_1, x_2, y_2, y_3)$, $(x_3, x_2, y_2, y_1, x_1, u, v, y_3)$, e_3, e_6, e_{12}
28. $y_3 - v$: $(y_3, y_2, y_1, u, x_1, x_2, x_3, v)$, $(y_3, x_3, x_2, y_2, y_1, x_1, u, v)$, e_1, e_5, e_7.

For each Hamiltonian-connected rainbow coloring of G and each pair w, z of distinct vertices of G, at least one of these $w - z$ paths is necessarily a rainbow path. First, we consider the 10-edge coloring of the graph G shown in Figure 4.16. A Hamiltonian path P is a rainbow path if P contains at most one edge in $\{ux_1, uy_1\}$ and at most one edge from $\{vx_3, vy_3\}$. For each pair w, z of distinct vertices of G, at least one of these $w - z$ paths contains at most one edge in $\{ux_1, uy_1\}$ and at most one edge from

$\{vx_3, vy_3\}$. Thus, this edge coloring is a Hamiltonian-connected rainbow 10-coloring and so $\mathrm{hrc}(G) \leq 10$.

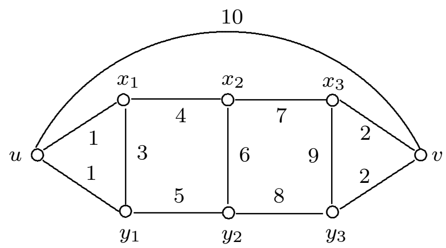


Figure 4.16: A rainbow Hamiltonian coloring of G

It remains to show that $\mathrm{hrc}(G) \geq 10$. Of course, $\mathrm{hrc}(G) \geq n - 1 = 7$. To see that $\mathrm{hrc}(G) \geq 8$, assume, to the contrary, that $\mathrm{hrc}(G) = 7$. Then there is a Hamiltonian-connected rainbow 7-coloring of G. Since G has 12 edges, some edge $e = wz$ of G has the property that it is the only edge possessing the color assigned to it. However, by definition, there is a rainbow Hamiltonian $w - z$ path P in G. Since the length of P is 7 and P does not contain wz, it is impossible for P to be a rainbow path. Thus, $\mathrm{hrc}(G) \geq 8$. In fact, every Hamiltonian-connected rainbow coloring of G must use at least 10 colors, as we show next.

Let c be a Hamiltonian-connected rainbow coloring of G. If e and f are two distinct edges of G such that e and f belong to *every* Hamiltonian $w - z$ path of G for some pair w, z of distinct vertices of G, then e and f cannot be assigned the same color by c. For example, since e_3 and e_9 belong to both two Hamiltonian $u - v$ paths of G, it follows that $c(e_3) \neq c(e_9)$. We now construct a graph G^* with $V(G^*) = E(G)$ such that two vertices x and y of G^*, that is, two edges x and y of G, are adjacent in G^* if the edges x and y of G belong to *every* Hamiltonian $w - z$ path of G for some pair w, z of distinct vertices of G. The graph G^* is shown in Figure 4.17. The degrees of the vertices in G^* are

$$\deg_{G^*} e_i = \begin{cases} 7 & i = 6 \\ 9 & i = 4, 5, 7, 8 \\ 10 & i = 1, 2, 10, 11 \\ 11 & i = 3, 9, 12. \end{cases}$$

The minimum number of independent sets into which $V(G^*)$ can be partitioned is therefore 7. Since $\deg_{G^*} e_i = 11$ for $i = 3, 9, 12$, it follows that $\{e_3\}$, $\{e_9\}$, $\{e_{12}\}$ are maximal independent sets. Since $\deg_{G^*} e_i = 10$ for $i = 1, 2, 10, 11$ and $\{e_1, e_2\}$ and $\{e_{10}, e_{11}\}$ are independent sets, they are both maximal independent sets. This leaves the five vertices e_4, e_5, e_6, e_7, e_8. The two sets $\{e_4, e_5\}$ and $\{e_7, e_8\}$ are independent, while

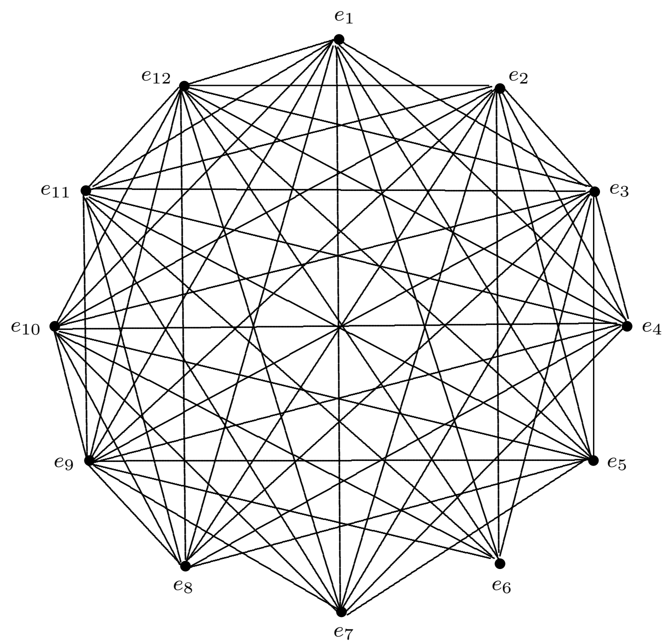


Figure 4.17: The graph G^*

each vertex of $\{e_4, e_5\}$ is adjacent to each vertex of $\{e_7, e_8\}$. The vertex e_6 of G^* is adjacent to none of e_4, e_5, e_7, e_8. Hence, a Hamiltonian-connected rainbow coloring c of G must assign distinct colors to the vertices of G^* in the seven sets $\{e_3\}$, $\{e_9\}$, $\{e_{12}\}$, $\{e_1, e_2\}$, $\{e_{10}, e_{11}\}$, $\{e_4, e_5\}$, $\{e_7, e_8\}$, while e_6 may be assigned the same color that is assigned to e_4 and e_5 or to e_7 and e_8. Next, let $A = \{\{e_1, e_2\}, \{e_{10}, e_{11}\}, \{e_4, e_5\}, \{e_7, e_8\}\}$. The digraph D of Figure 4.18 has vertex set A. For $S, T \in A$, (S, T) is an arc of D if the vertices in S are colored the same by c, then the vertices of T must be colored differently. For example, let $S = \{e_4, e_5\}$ and $T = \{e_7, e_8\}$. Since (i) $Q_1 = (u, v, y_3, x_3, x_2, x_1, y_1, y_2)$ and $Q_2 = (u, y_1, x_1, x_2, x_3, v, y_3, y_2)$ are the only Hamiltonian $u - y_2$ paths in H_3 and (ii) $S \subseteq E(Q_1)$ and $T \subseteq E(Q_2)$, it follows that if $c(e_4) = c(e_5)$, then $c(e_7) \neq c(e_8)$. Thus, (S, T) is an arc of D. Similarly, (T, S) is also an arc of D. While with the aid of the Hamiltonian $u - y_3$ paths and Hamiltonian $x_1 - v$ paths, the remaining arcs of D in Figure 4.18 are determined. Consequently, the minimum number of colors can be obtained when $\{e_1, e_2\}$ and $\{e_{10}, e_{11}\}$ are color classes.

Next, we show that the remaining eight vertices of G^* are in individual color classes. Recall that a Hamiltonian-connected rainbow coloring c of G must assign distinct colors to the vertices of G^* in the seven sets $\{e_3\}$, $\{e_9\}$, $\{e_{12}\}$, $\{e_1, e_2\}$, $\{e_{10}, e_{11}\}$, $\{e_4, e_5\}$, $\{e_7, e_8\}$, while e_6 may be assigned the same color that is assigned to e_4 and e_5 or to e_7 and e_8. Thus, it suffices to show that e_6 must be assigned a different color from the colors as-

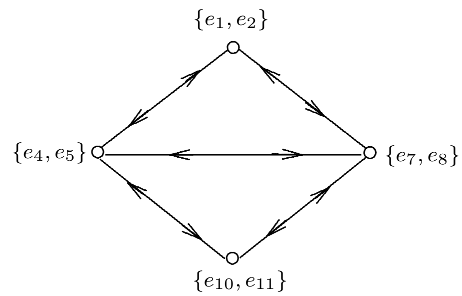


Figure 4.18: The digraph D in the proof of Theorem 4.5.2

signed to any of e_4, e_5, e_7, e_8; that is, $\{e_6\}$ must be a color class. Consider the two Hamiltonian $y_2 - y_3$ paths $R_1 = (y_2, y_1, u, x_1, x_2, x_3, v, y_3)$ and $R_2 = (y_2, x_2, x_1, y_1, u, v, x_3, y_3)$ in H_3. Since $c(e_1) = c(e_2)$ in R_2, it follows that R_1 is a rainbow Hamiltonian $y_2 - y_3$ path. Thus, $c(e_6) \neq c(e_4)$. Similarly, by considering the Hamiltonian $x_2 - x_3$ paths, the Hamiltonian $y_1 - y_2$ paths and the Hamiltonian $x_1 - x_2$ paths, we obtain $c(e_6) \notin \{c(e_5), c(e_7), c(e_8)\}$ and so $\{e_6\}$ is a color class. Therefore, $\text{hrc}(G) = 10$. ■

We saw that there are infinitely many Hamiltonian-connected graphs G of order n for which $\text{hrc}(G) = n - 1$. If $G = K_3 \,\square\, K_2$, then $\text{hrc}(G) = 7$, while if $G = H_3$, then $\text{hrc}(G) = 10$. Thus, there exist Hamiltonian-connected graphs G of order n for which $\text{hrc}(G) \in \{n + 1, n + 2\}$. However, it is not known whether there exists a Hamiltonian-connected graphs F of order n for which $\text{hrc}(F) = n$. A more challenging question is to determine an infinite class of Hamiltonian-connected graphs G of order n such that $\text{hrc}(G) = k$ for each integer $k \in \{n, n + 1, n + 2\}$.

Chapter 5

Rainbow Hamiltonian-Connected Digraphs

5.1 Hamiltonian-Connected Digraphs

We now turn our attention from rainbow Hamiltonian-connected graphs to rainbow Hamiltonian-connected digraphs. A nontrivial digraph D is *Hamiltonian-connected* if for every pair u, v of distinct vertices of D, there exists both a Hamiltonian $u - v$ path and a Hamiltonian $v - u$ path. This has been referred to as a *strongly Hamiltonian-connected digraph* by some. A nontrivial digraph D is sometimes called *weakly* or *unilaterally Hamiltonian-connected* if for every pair u, v of distinct vertices of D, there is either a Hamiltonian $u - v$ path or a Hamiltonian $v - u$ path.

Not a great deal of research seems to have been done on Hamiltonian-connected digraphs. There is a discussion of Hamiltonian-connected tournaments and related topics by Thomassen [45]. Roberts [41] obtained the following result.

Theorem 5.1.1 (Roberts) *If D is a digraph of order $n \geq 2$ such that $\operatorname{od} u + \operatorname{id} v \geq n + 1$ for every pair u, v of distinct vertices of D for which $(u, v) \notin E(D)$, then D is Hamiltonian-connected.*

A digraph D is *symmetric* if whenever (u, v) is an arc of D, then (v, u) is an arc of D as well. For a Hamiltonian-connected graph G, let G^* denote the symmetric (Hamiltonian-connected) digraph whose underlying graph is G. For the symmetric digraph $K^*_{r,r}$ of order $n = 2r$, $\operatorname{od} u + \operatorname{id} v = n$ for every pair u, v of distinct vertices of $K^*_{r,r}$ for which $(u, v) \notin E(K^*_{r,r})$; yet, $K^*_{r,r}$ is not Hamiltonian-connected. That is, the result stated in Theorem 5.1.1 is best possible. On the other hand, the condition stated in Theorem 5.1.1 for a digraph D to be Hamiltonian-connected is only sufficient. For $n \geq 5$, the symmetric digraph W^*_n obtained from the wheel W_n of order $n^* = n + 1$ has the property that

$\operatorname{od} u + \operatorname{id} v = 6 \leq n^*$ for every pair u, v of distinct vertices of W_n^* for which (u, v) is not an arc in W_n^*; yet, W_n^* is Hamiltonian-connected. For example, let the wheel W_6 of order 7 be obtained from the 6-cycle $C = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$ by adding the vertex v, which is adjacent to each vertex of C. Then $(v_1, v_2, v_3, v, v_6, v_5, v_4)$ and $(v_4, v_5, v_6, v, v_3, v_2, v_1)$ are Hamiltonian $v_1 - v_4$ and $v_4 - v_1$ paths in W_6^*, respectively.

5.2 Hamiltonian-Connected Rainbow Colorings

For a nontrivial Hamiltonian-connected digraph D, an arc coloring

$$c : E(D) \to [k] = \{1, 2, \ldots, k\}$$

of D is called a *Hamiltonian-connected rainbow k-coloring* if for every pair u, v of distinct vertices of D, there is both a rainbow Hamiltonian $u-v$ path and a rainbow Hamiltonian $v-u$ path. The minimum k for which D has a Hamiltonian-connected rainbow k-coloring is the *rainbow Hamiltonian-connection number* of D, denoted by $\operatorname{hrc}(D)$. Clearly, if D is a Hamiltonian-connected digraph of order $n \geq 2$ and size m, then

$$n - 1 \leq \operatorname{hrc}(D) \leq m. \tag{5.1}$$

To illustrate the concepts just described, we determine $\operatorname{hrc}(D)$ for the Hamiltonian-connected digraph D of order 4 of Figure 5.1(a).

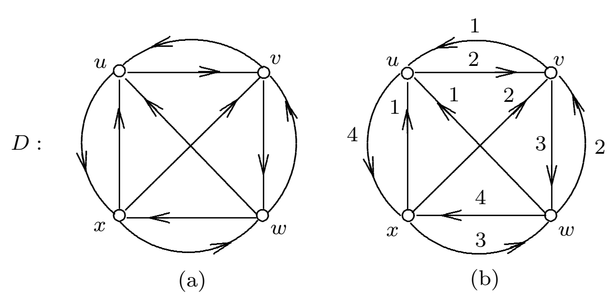


Figure 5.1: A digraph D of order 4 and a 4-arc coloring of D

Proposition 5.2.1 *If D is the Hamiltonian-connected digraph of order* 4 *shown in* Figure 5.1(a), *then* $\operatorname{hrc}(D) = 4$.

Proof. The 4-arc coloring shown in Figure 5.1(b) has the property that for every two vertices s and t, there is both a rainbow Hamiltonian $s - t$ path and a rainbow Hamiltonian $t - s$ path. Thus, this coloring is a Hamiltonian-connected rainbow coloring and so $\operatorname{hrc}(D) \leq 4$.

To verify that $\text{hrc}(D) \geq 4$, we show that no 3-arc coloring of D is a Hamiltonian-connected rainbow coloring. Assume, to the contrary, that there is a Hamiltonian-connected rainbow 3-arc coloring $c : E(D) \to \{1,2,3\}$ of D. Since there is only one Hamiltonian $w - x$ path $P_1 = (w, v, u, x)$ in D, it follows that P_1 is rainbow. Thus, we may assume that $c(wv) = 1$, $c(vu) = 2$ and $c(ux) = 3$. Because there is only one Hamiltonian $v - w$ path $P_2 = (v, u, x, w)$ in D, it follows that P_2 is rainbow and so $c(xw) = 1$. Next, consider the unique Hamiltonian $u - v$ path $P_3 = (u, x, w, v)$ in D. Since $c(xw) = c(wv) = 1$, it follows that P_3 is not rainbow and so there is no rainbow Hamiltonian $u - v$ path, which is impossible. Consequently, for this digraph D of order $n = 4$, we have $\text{hrc}(D) = n$. ■

5.3 Rainbow Hamiltonian-Connection Numbers

We now show that the possible values of the rainbow Hamiltonian-connection numbers $\text{hrc}(D)$ of Hamiltonian-connected digraphs D of order n are quite limited, namely $\text{hrc}(D)$ is either n or $n - 1$.

Theorem 5.3.1 *If D is a nontrivial Hamiltonian-connected digraph of order n, then $\text{hrc}(D) = n - 1$ or $\text{hrc}(D) = n$.*

Proof. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$. For $n = 1, 2, \ldots, n$, assign the color i to each arc directed into v_i. Now, let u and v be two distinct vertices of D. Since D is Hamiltonian-connected, there exists a Hamiltonian $u - v$ path

$$P = (u = v_{i_1}, v_{i_2}, \ldots, v_{i_{n-1}}, v_{i_n} = v).$$

For $k = 1, 2, \ldots, n-1$, the color of the arc $(v_{i_k}, v_{i_{k+1}})$ on P is i_{k+1}. Hence, the set of colors of the arcs in P is $\{i_2, i_3, \ldots, i_n\} = [n] - \{i_1\}$, which consists of $n - 1$ distinct colors. Thus, P is a rainbow Hamiltonian $u - v$ path and therefore this arc coloring is a Hamiltonian-connected rainbow n-coloring. Hence, $\text{hrc}(D) \leq n$. It then follows by (5.1) that $\text{hrc}(D) = n - 1$ or $\text{hrc}(D) = n$. ■

Proposition 5.3.2 *If G is a Hamiltonian-connected graph, then $\text{hrc}(G^*) \leq \text{hrc}(G)$. Consequently, if G has order $n \geq 4$ and $\text{hrc}(G) = n - 1$, then $\text{hrc}(G^*) = n - 1$.*

Proof. Let $\text{hrc}(G) = k$ and let $c : E(G) \to [k]$ be a Hamiltonian-connected rainbow coloring of G. For each edge uv of G, assign the color $c(uv)$ to the two arcs (u, v) and (v, u) of G^*, which produces an edge coloring $c^* : E(G^*) \to [k]$ of G^*. Since for every two vertices u and v of G, there is a rainbow Hamiltonian path $(u = v_1, v_2, \ldots, v_n = v)$, it follows that $(u = v_1, v_2, \ldots, v_n = v)$ a rainbow Hamiltonian (directed) $u - v$ path in G^* and $(v = v_n, v_{n-1}, \ldots, v_1 = u)$ is a rainbow Hamiltonian (directed) $v - u$ path in G^*. Thus, c^* is a Hamiltonian-connected rainbow k-coloring of G^* and so $\text{hrc}(G^*) \leq k = \text{hrc}(G)$. Furthermore, if G has order $n \geq 4$ and $\text{hrc}(G) = n - 1$, then $\text{hrc}(G^*) = n - 1$ by (5.1). ■

For each integer $n \geq 2$, the *complete symmetric digraph* K_n^* of order n has both arcs (u, v) and (v, u) for every two distinct vertices u and v. Since $\text{hrc}(K_n) = n-1$, it follows that $\text{hrc}(K_n^*) = n - 1$ for each integer $n \geq 3$. Therefore, there is an infinite class of Hamiltonian-connected digraphs D of order n for which $\text{hrc}(D) = n - 1$. The digraph of Figure 5.1 is Hamiltonian-connected by by Theorem 5.1.1. This digraph shows that Hamiltonian-connected digraphs D of order n exist for which $\text{hrc}(D) = n$. This gives rise to the following question:

> *Is there an infinite class of Hamiltonian-connected digraphs of order n having rainbow Hamiltonian connection number n?*

If T is an r-regular tournament of order $2r + 1 \geq 3$, then T is strong and is therefore Hamiltonian by Theorem 1.4.1. Let C be a Hamiltonian cycle in T. If for each arc (u, v) on C, we add the arc (v, u) to T, then the resulting digraph D is Hamiltonian-connected by Theorem 5.1.1. Such a Hamiltonian-connected digraph D is shown in Figure 5.2. This is another Hamiltonian-connected digraph D of order n with $\text{hrc}(D) = n$.

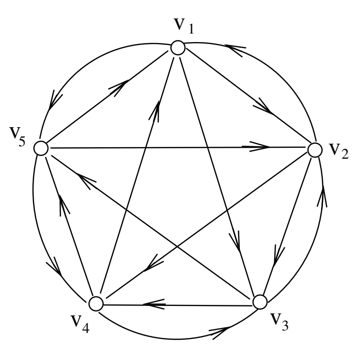


Figure 5.2: A Hamiltonian-connected digraph D of order 5

Proposition 5.3.3 *If D is the Hamiltonian-connected digraph of order* 5 *shown in* Figure 5.2*, then* $\text{hrc}(D) = 5$.

Proof. By Theorem 5.3.1, either $\text{hrc}(D) = 4$ or $\text{hrc}(D) = 5$. Assume, to the contrary, that $\mathrm{hrc}(D) = 4$. Then there is a Hamiltonian-connected rainbow 4-coloring c of D. First, we make the following observation.

Observation For $i = 1, 2, 3, 4, 5$, there is a unique Hamiltonian $v_i - v_{i+2}$ path in D, namely $Q_i = (v_i, v_{i-1}, v_{i+1}, v_{i+3}, v_{i+2})$, where each subscript is expressed as an integer modulo 5.

Next, we verify the following claim.

Claim. The restriction of the coloring c to the 5-cycle C'=(v_1, v_5, v_4, v_3, v_2, v_1) is a proper arc coloring.

Assume, to the contrary, that the Claim is false. Then we may assume that (i) $c(v_2, v_1) = c(v_1, v_5) = 1$ and (ii) the colors of the arcs in the unique Hamiltonian $v_1 - v_3$ path $Q_1 = (v_1, v_5, v_2, v_4, v_3)$ are

$$c(v_1, v_5) = 1, c(v_5, v_2) = 2, c(v_2, v_4) = 3 \text{ and } c(v_4, v_3) = 4. \tag{5.2}$$

Since $Q_4 = (v_4, v_3, v_5, v_2, v_1)$ is the the unique Hamiltonian $v_4 - v_1$ path where $c(v_4, v_3) = 4$, $c(v_5, v_2) = 2$ and $c(v_2, v_1) = 1$, it follows that $c(v_3, v_5) = 3$. This is illustrated in Figure 5.3.

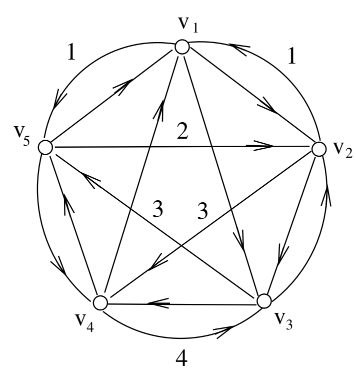


Figure 5.3: A step in the proof of Proposition 5.3.3

Next, we show that $c(v_3, v_2) \notin \{1, 2, 3, 4\}$, which is a contradiction.

- ⋆ First, consider the unique Hamiltonian $v_3 - v_5$ path $Q_3 = (v_3, v_2, v_4, v_1, v_5)$. Since $c(v_2, v_4) = 3$ and $c(v_1, v_5) = 1$, it follows that $c(v_3, v_2) \notin \{1, 3\}$.
- ⋆ Next, consider the unique Hamiltonian $v_2 - v_4$ path $Q_2 = (v_2, v_1, v_3, v_5, v_4)$. Since $c(v_2, v_1) = 1$ and $c(v_3, v_5) = 3$, it follows that $\{c(v_1, v_3), c(v_5, v_4)\} = \{2, 4\}$.

- $\star$ Finally, consider the unique Hamiltonian $v_5 - v_2$ path $Q_5 = (v_5, v_4, v_1, v_3, v_2)$. Since $\{c(v_5, v_4), c(v_1, v_3)\} = \{2, 4\}$, it follows that $c(v_3, v_2) \notin \{2, 4\}$.

Hence, $c(v_3, v_2) \notin \{1, 2, 3, 4\}$, a contradiction. Thus, the Claim holds.

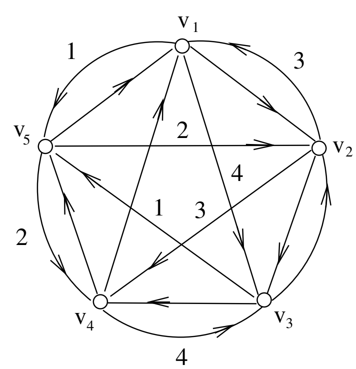
Suppose that the colors of the arcs in $Q_1 = (v_1, v_5, v_2, v_4, v_3)$ are as described in (5.2). By the Claim, $c(v_2, v_1) \neq c(v_1, v_5) = 1$ and $1 = c(v_1, v_5) \neq c(v_5, v_4) \neq c(v_4, v_3) = 4$. Hence,

$$c(v_2, v_1) \neq 1 \text{ and } c(v_5, v_4) \notin \{1, 4\}. \tag{5.3}$$

Once again, we show that $c(v_3, v_2) \notin \{1, 2, 3, 4\}$, which implies that there is no Hamiltonian-connected rainbow 4-coloring of D.

- $\star$ First, consider the the unique Hamiltonian $v_4 - v_1$ path $Q_4 = (v_4, v_3, v_5, v_2, v_1)$. Since $c(v_2, v_1) \neq 1$ by (5.3), it follows that $c(v_2, v_1) = 3$ and so $c(v_3, v_5) = 1$.

- $\star$ Next, consider the unique Hamiltonian $v_2 - v_4$ path $Q_2 = (v_2, v_1, v_3, v_5, v_4)$. Since $c(v_5, v_4) \notin \{1, 4\}$ by (5.3) and $c(v_2, v_1) = 3$, it follows that $c(v_5, v_4) = 2$ and so $c(v_1, v_3) = 4$.



- $\star$ By the Claim, $c(v_3, v_2) \notin \{1, 2, 3, 4\}$, a contradiction. ■

Chapter 6

Proper Hamiltonian-Connected Graphs

6.1 Introduction

A *proper edge coloring* c of a nonempty graph G is a function c on $E(G)$ with the property that $c(e) \neq c(f)$ for every two adjacent edges e and f of G. If the colors are chosen from a set of k colors, then c is called a *proper k-edge coloring* of G. The minimum positive integer k for which G has a k-edge coloring is called the *chromatic index* of G and is denoted by $\chi'(G)$. It is immediate for every nonempty graph G that $\chi'(G) \geq \Delta(G)$. The most important theorem dealing with chromatic index is one obtained by Vizing [46].

Theorem 6.1.1 (Vizing's Theorem) *For every nonempty graph G, $\chi'(G) \leq \Delta(G) + 1$.*

As a result of Vizing's theorem, the chromatic index of every nonempty graph G is one of two numbers, namely $\Delta(G)$ or $\Delta(G)+1$. A graph G with $\chi'(G) = \Delta(G)$ is called a *class one graph* while a graph G with $\chi'(G) = \Delta(G) + 1$ is called a *class two graph.*

Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is *properly colored* or, more simply, P is a *proper path* in G if no two adjacent edges of P are colored the same. An edge coloring c is a *proper-path coloring* of a connected graph G if every pair u, v of distinct vertices of G are connected by a proper $u - v$ path in G. If k colors are used, then c is referred to as a *proper-path k-coloring.* The minimum k for which G has a proper-path k-coloring is called the *proper connection number* $\mathrm{pc}(G)$ of G. Recently, this topic has been studied by many (see [2, 4] for example). In fact, there is a dynamic survey of this topic due to Li and Magnant [33].

6.2 Proper Hamiltonian-Path Colorings

If G is a Hamiltonian-connected graph with a proper edge coloring, then for every two vertices u and v of G, there is a proper Hamiltonian $u - v$ path in G. Indeed, every Hamiltonian path in G is a proper Hamiltonian path. However, if our primary interest concerns edge colorings of graphs G with the property that for every two vertices u and v of G, there exists a proper Hamiltonian $u - v$ path in G, then this may very well be possible using fewer than $\chi'(G)$ colors. Of course, graphs possessing such edge colorings are necessarily Hamiltonian-connected. For a Hamiltonian-connected graph G, an edge coloring $c : E(G) \to [k]$ is a *proper Hamiltonian-path k-coloring* if every two vertices of G are connected by a proper Hamiltonian path in G. An edge coloring c is a *proper Hamiltonian-path coloring* if c is a proper Hamiltonian-path k-coloring for some positive integer k. The minimum number of colors required of a proper Hamiltonian-path coloring of G is the *proper Hamiltonian-connection number* of G, denoted by $\text{hpc}(G)$. Since every proper edge coloring of a Hamiltonian-connected graph G is a proper Hamiltonian-path coloring of G and there is no proper Hamiltonian-path 1-coloring of G, it follows that

$$2 \leq \text{hpc}(G) \leq \chi'(G). \tag{6.1}$$

To illustrate these concepts, consider the graph $G = C_6^2$. Since $\Delta(G) = 4$ and the edge coloring of G in Figure 6.1(a) is a proper 4-edge coloring, it follows that $\chi'(G) = \Delta(G) = 4$. Next, consider the 2-edge coloring c of G shown in Figure 6.1(b).

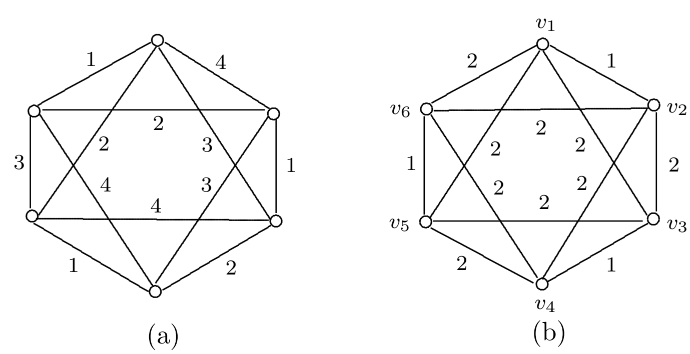


Figure 6.1: A proper 4-edge coloring and a proper Hamiltonian-path 2-coloring of C_6^2

We show that c is a proper Hamiltonian-path coloring of G; that is, every two vertices u and v of G are connected by a proper Hamiltonian $u-v$ path P in G. If $\{u, v\} = \{v_1, v_2\}$ or $\{u, v\} = \{v_1, v_6\}$, say the former, let $P = (v_1, v_6, v_5, v_4, v_3, v_2)$; if $\{u, v\} = \{v_1, v_3\}$ or $\{u, v\} = \{v_1, v_5\}$, say the former, let $P = (v_1, v_2, v_6, v_5, v_4, v_3)$; while if $\{u, v\} = \{v_1, v_4\}$, let $P = (v_1, v_2, v_6, v_5, v_3, v_4)$. By the symmetry of this edge coloring, c is a proper

Hamiltonian-path 2-coloring and so $\text{hpc}(G) = 2$. Therefore, $\text{hpc}(G) < \chi'(G)$.

Next, we give an example of a graph G with $\text{hpc}(G) = \chi'(G)$. Let $G = K_3 \square K_2$, where the two triangles K_3 in G are (u, x, w, u) and (v, y, z, v) and $uv, xy, wz \in E(G)$. Since there is a proper 3-edge coloring of G shown in Figure 6.2 and $\Delta(G) = 3$, it follows that $\chi'(G) = 3$. Hence, $\text{hpc}(G) \leq 3$.

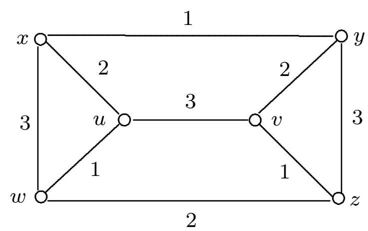


Figure 6.2: A proper 3-edge coloring of $K_3 \square K_2$

We now show that $\text{hpc}(G) \geq 3$. Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring c of G using the colors red (color 1) and blue (color 2). There are only two Hamiltonian $u-v$ paths, namely (u, w, x, y, z, v) and (u, x, w, z, y, v). Because of the symmetry of these paths, we may assume that the first path is a proper Hamiltonian $u - v$ path and its edges are colored as $c(uw) = c(xy) = c(zv) = 1$ and $c(wx) = c(yz) = 2$. Next, we consider a proper Hamiltonian $x - z$ path. There are only two Hamiltonian $x - z$ paths in G, namely, $Q_1 = (x, w, u, v, y, z)$ and $Q_2 = (x, y, v, u, w, z)$. Since the path $Q = (w, u, v, y)$ lies on both Q_1 and Q_2, it follows that Q must be proper. This implies that $c(uv) = 2$ and $c(vy) = 1$. Similarly, there are only two Hamiltonian $w - y$ paths in G, each of which contains the path (x, u, v, z), and so this path must be proper. This implies that $c(ux) = 1$. We now consider a proper Hamiltonian $x - v$ path. There are only two Hamiltonian $x - v$ paths in G, namely, $R_1 = (x, u, w, z, y, v)$ and $R_2 = (x, y, z, w, u, v)$. Since the path $R = (y, z, w, u)$ lies on both R_1 and R_2, it follows that R must be properly colored by the colors 1 and 2. Since $c(yz) = 2$ and $c(wu) = 1$, this is impossible. Thus, there is no proper Hamiltonian $x - v$ path in G, which is a contradiction. Therefore, $\text{hpc}(G) \geq 3$ and so $\text{hpc}(G) = 3$.

We now consider some well-known Hamiltonian-connected graphs, beginning with complete graphs, which are supergraphs of all Hamiltonian-connected graphs. It is easy to see that $\text{hpc}(K_3) = 3$. When $n \geq 4$, $\text{hpc}(K_n) = 2$, however, which we verify next.

Theorem 6.2.1 *For every integer $n \geq 4$,* $\text{hpc}(K_n) = 2$.

Proof. We consider two cases, according to whether n is even or n is odd.

Case 1. *n is even.* The complete graph $G = K_n$ contains a 1-factor F. Define an edge coloring c of G by assigning the color red to each edge of F and the color blue to the remaining edges of G. We show that c is a proper Hamiltonian-path 2-coloring of G; that is, for every two vertices u and v of G, there is a proper Hamiltonian $u-v$ path in G. Let $n = 2k$ and let $V(G) = \{v_1, v_2, \ldots, v_{2k}\}$. Suppose that $E(F) = \{v_{2i-1}v_{2i} : 1 \le i \le k\}$. There are two possibilities, depending on whether uv is a blue edge or uv is a red edge. Thus, we may assume that either (1) $u = v_1$ and $v = v_{2k}$ or (2) $u = v_2$ and $v = v_1$. Consider the properly colored Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{2k}, v_1)$ of G. If (1) occurs, then $(u = v_1, v_2, \ldots, v_{2k} = v)$ is a proper Hamiltonian $u - v$ path in G; while if (2) occurs, then $(u = v_2, v_3, \ldots, v_{2k}, v_1 = v)$ is a proper Hamiltonian $u - v$ path in G. Therefore, $\mathrm{hpc}(K_n) = 2$.

Case 2. *$n \ge 5$ is odd.* Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamiltonian cycle in $G = K_n$. Define a coloring c of G by assigning the color red to each edge of C and the color blue to the remaining edges of G. We show that c is a proper Hamiltonian-path 2-coloring of G; that is, for every two vertices u and v of G, there is a proper Hamiltonian $u - v$ path in G. We may assume that $v = v_n$ and $u = v_i$ for some integer i with $1 \le i \le (n-1)/2$.

First, suppose that $u = v_1$. If $n \equiv 1 \pmod 4$, then

$$(u = v_1, v_2, v_4, v_3, v_5, v_6, v_8, v_7, v_9, \ldots, v_{n-3}, v_{n-1}, v_{n-2}, v_n = v)$$

is a proper Hamiltonian $u - v$ path in G; while if $n \equiv 3 \pmod 4$, then

$$(u = v_1, v_2, v_4, v_3, v_5, v_6, v_8, v_7, v_9, \ldots, v_{n-5}, v_{n-3}, v_{n-4}, v_{n-1}, v_{n-2}, v_n = v)$$

is a proper Hamiltonian $u - v$ path in G.

Next, suppose that $u = v_j$ where $2 \le j \le (n-1)/2$. If $n = 5$, then $u = v_2$ and $(v_5, v_3, v_4, v_1, v_2)$ is a proper Hamiltonian $u - v$ path in G. Thus, we may assume that $n \ge 7$ is odd. Let $A = \{v_1, v_2, \ldots, v_{j-1}\}$ and $B = \{v_{j+1}, v_{j+2}, \ldots, v_{n-1}\}$. Let $|A| = a$ and $|B| = b$. Since $n \ge 7$ is odd, it follows that (1) $b \ge 3$ and (2) $a + b = n - 2$ is odd and so a and b are of opposite parity. We consider two subcases, according to whether a is even or a is odd.

Subcase 2.1. *a is even.* Then

$$Q = (u = v_j, v_{j-2}, v_{j-1}, v_{j-4}, u_{j-3}, v_{j-6}, v_{j-5}, \ldots, v_1, v_2, v_{j+2})$$

is a proper $u - v_{j+2}$ path in G with $V(Q) = \{v_1, v_2, \ldots, v_j\} \cup \{v_{j+2}\}$ and

$$Q' = (v_{j+2}, v_{j+1}, v_{j+4}, v_{j+3}, u_{j+6}, v_{j+5}, v_{j+8}, u_{j+7}, \ldots, v_{n-2}, v_{n-3}, v_{n-1}, v_n = v)$$

is a proper $v_{j+2} - v$ path in G with $V(Q') = \{v_{j+1}, v_{j+2}, \ldots, v_n\}$. Thus, $V(Q) \cup V(Q') = V(G)$, $V(Q) \cap V(Q') = \{v_{j+1}\}$ and v_2v_{j+2} and $v_{j+1}v_{j+2}$ have distinct colors (namely,

v_2v_{j+2} is blue and $v_{j+1}v_{j+2}$ is red). Therefore, the path Q followed by Q' produces a proper Hamiltonian $u-v$ path in G.

Subcase 2.2. *a is odd.* If $a \equiv 3 \pmod 4$, then

$$Q = (u = v_j, v_{j-1}, v_{j-3}, v_{j-2}, u_{j-4}, v_{j-5}, v_{j-7}, u_{j-6}, \ldots, v_1, v_2, v_{j+1})$$

is a proper $u - v_{j+1}$ path in G; while if $a \equiv 1 \pmod 4$, then

$$Q = (u = v_j, v_{j-1}, v_{j-3}, v_{j-2}, u_{j-4}, v_{j-5}, v_{j-7}, u_{j-6}, \ldots, v_3, v_4, v_1, v_2, v_{j+1})$$

is a proper $u - v_{j+1}$ path in G. We now show that Q can be extended to a proper Hamiltonian $u - v$ path in G. If $b \equiv 0 \pmod 4$, then

$$Q' = (v_{j+1}, v_{j+2}, v_{j+4}, v_{j+3}, u_{j+5}, v_{j+6}, v_{j+8}, u_{j+7}, \ldots, v_{n-3}, v_{n-1}, v_{n-2}, v_n = v)$$

is a proper $v_{j+1} - v$ path in G; while if $b \equiv 2 \pmod 4$, then $b \geq 6$ (since $b \geq 3$) and

$$Q' = (v_{j+1}, v_{j+2}, v_{j+4}, v_{j+3}, u_{j+5}, v_{j+6}, v_{j+8}, u_{j+7}, \ldots, v_{n-4}, v_{n-1}, v_{n-2}, v_n = v)$$

is a proper $v_{j+1} - v$ path in G. Thus, as in Case 1, the path Q followed by Q' produces a proper Hamiltonian $u - v$ path in G. ■

We saw that if G is a Hamiltonian-connected graph of order at least 4, then $\delta(G) \geq 3$. There are infinitely many Hamiltonian-connected cubic graphs. For each odd integer $n \geq 3$, the prism $C_n \,\square\, K_2$ is cubic and Hamiltonian-connected (see [31]). We saw that $\mathrm{hpc}(C_3 \,\square\, K_2) = 3$. In fact, $\mathrm{hpc}(C_n \,\square\, K_2) = 3$ for all odd integers $n \geq 3$, which we now verify.

Theorem 6.2.2 *For each odd integer $n \geq 3$,* $\mathrm{hpc}(C_n \,\square\, K_2) = 3$.

Proof. For an odd integer $n \geq 3$, let $G = C_n \,\square\, K_2$, which is constructed from the two n-cycles $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ by adding the n edges u_iv_i for $1 \leq i \leq n$. Since $\chi'(G) = 3$, it follows by (6.1) that $\mathrm{hpc}(G) \leq 3$. It remains to show that $\mathrm{hpc}(G) \geq 3$. Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring c of G using the colors 1 and 2.

First, consider a proper Hamiltonian $u_1 - u_3$ path P in G. Observe that either P begins with u_1, u_2 or P ends with u_2, u_3. Suppose that P begins with u_1, u_2. Hence, P must begin with u_1, u_2, v_2 and so $u_1u_n, u_1v_1 \notin E(P)$. Since each vertex in $V(G) - \{u_1, u_3\}$ has degree 2 in P, it follows that $v_1v_n, v_1v_2 \in E(P)$ and so P begins with the subpath $(u_1, u_2, v_2, v_1, v_n)$. Since $u_nu_1 \notin E(P)$ and u_n has degree 2 in P, it follows that $u_nv_n, u_nu_{n-1} \in E(P)$ and so P contains the subpath $(u_1, u_2, v_2, v_1, v_n, u_n, u_{n-1})$.

Similarly, $v_nv_{n-1} \notin E(P)$ and $u_{n-1}v_{n-1}, v_{n-1}v_{n-2} \in E(P)$. Continuing in this way, we see that P is the following path

$$P_1 = (u_1, u_2, v_2, v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_4, v_4, v_3, u_3). \tag{6.2}$$

Next, suppose that P ends with u_2, u_3. This implies that $u_1u_2, u_3v_3, u_3u_4 \notin E(P)$ and so $u_2v_2, v_2v_3, v_3v_4 \in E(P)$. Hence, P ends with the subpath $(v_4, v_3, v_2, u_2, u_3)$. An argument similar to the one above shows that P is the following path

$$P_2 = (u_1, v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_4, v_4, v_3, v_2, u_2, u_3).$$

In either case, P must contain the subpath

$$P' = (v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_4, v_4, v_3).$$

The paths P_1 and P_2 are illustrated in Figure 6.3 for $C_9 \,\square\, K_2$.

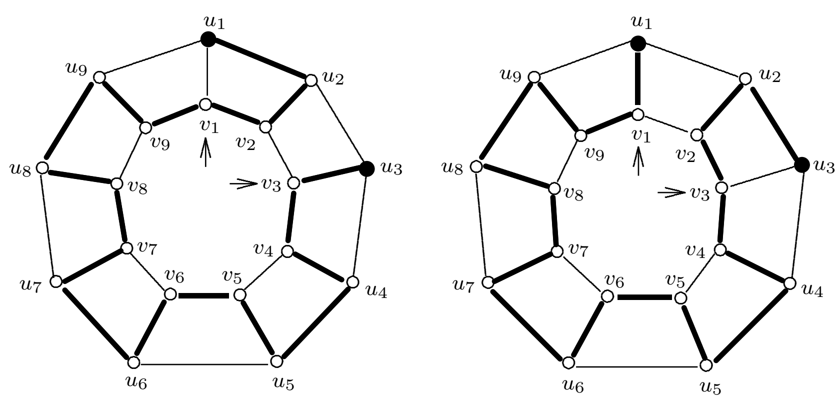


Figure 6.3: Two Hamiltonian $u_1 - u_3$ paths in $C_9 \,\square\, K_2$

By the symmetry of the graph G, we may assume, without loss of generality, that $P = P_1$, described in (6.2). Since c is a proper Hamiltonian-path 2-coloring of G using the colors 1 and 2, we may assume, without loss of generality, that $c(u_1u_2) = 1$. Since P_1 is a proper path and $c(u_1u_2) = 1$, it follows that $c(u_2v_2) = 2$ and $c(v_1v_2) = 1$. For the remaining edges e of P_1, it follows that $c(e) = 1$ if $e = u_iv_i$ and $c(e) = 2$ if e belongs to one of the two n-cycles. In particular, $c(v_1v_n) = 2$. Next, consider a proper Hamiltonian $u_3 - u_5$ path Q in G. The argument above shows that there are two possibilities for Q. This is illustrated in Figure 6.4 for $C_9 \,\square\, K_2$. Furthermore, Q must contain the subpath $Q' = (v_3, v_2, u_2, u_1, v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_6, v_6, v_5)$. Since Q' is proper and $c(u_2v_2) = 2$, it follows that $c(v_3v_2) = 1$ and so the colors of Q' are alternately colored by 1 and 2, beginning with 1. In particular, $c(v_1v_n) = 1$, which contradicts the fact that $c(v_1v_n) = 2$. ■

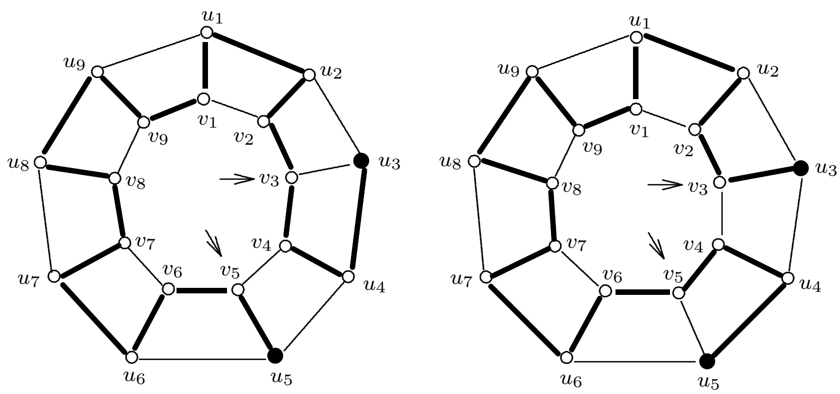


Figure 6.4: Two Hamiltonian $u_3 - u_5$ paths in $C_9 \square K_2$

6.3 Two Minimum Hamiltonian-Connected Graphs

We start this section with a useful observation.

Observation 6.3.1 *If H is a Hamiltonian-connected spanning subgraph of a graph G, then* $\mathrm{hpc}(G) \leq \mathrm{hpc}(H)$.

If G is a Hamiltonian-connected graph that is not complete and u and v are nonadjacent vertices of G, then $G + uv$ is also Hamiltonian-connected and $\mathrm{hpc}(G + uv) \leq \mathrm{hpc}(G)$ by Observation 6.3.1. This suggests that Hamiltonian-connected graphs having the greatest proper Hamiltonian connection numbers are minimal Hamiltonian-connected graphs. This leads us to consider Hamiltonian-connected graphs of order n and minimum possible size. Every Hamiltonian-connected graph of order at least 4 is 3-connected. Therefore, if G is a Hamiltonian-connected graph of $n \geq 4$, then $\delta(G) \geq 3$, which implies that the minimum size of a Hamiltonian-connected graph of order n is $\lfloor \frac{3n+1}{2} \rfloor$. The following result is due to Moon [36].

Theorem 6.3.2 *For each integer $n \geq 4$, there exists a Hamiltonian-connected graph of order n and size $\lfloor \frac{3n+1}{2} \rfloor$.*

We now determine the proper Hamiltonian connection numbers of graphs belonging to two classes of Hamiltonian-connected graphs of order n and size $\lfloor \frac{3n+1}{2} \rfloor$, one class for n even and the other class for n odd, beginning with the case when n is even.

For each integer $k \geq 2$, let $P_k \square K_2$ be the grid of order $2k$ in which two paths of order k are $P_k = (x_1, x_2, \ldots, x_k)$ and $P'_k = (y_1, y_2, \ldots, y_k)$ such that $x_i y_i \in E(P_k \square K_2)$ for $1 \leq i \leq k$. Now, let H_k be the cubic graph of order $2k + 2$ obtained by adding two adjacent vertices u and v to the grid $P_k \square K_2$ and joining (1) the vertex u to x_1 and y_1

and (2) the vertex v to x_k and y_k in $P_k \,\square\, K_2$. (see Figure 6.5). Each graph H_k has the property that it is Hamiltonian-connected (see [36]) and $\mathrm{hpc}(H_k) = \chi'(H_k) = \Delta(H_k) = 3$. We verify this now.

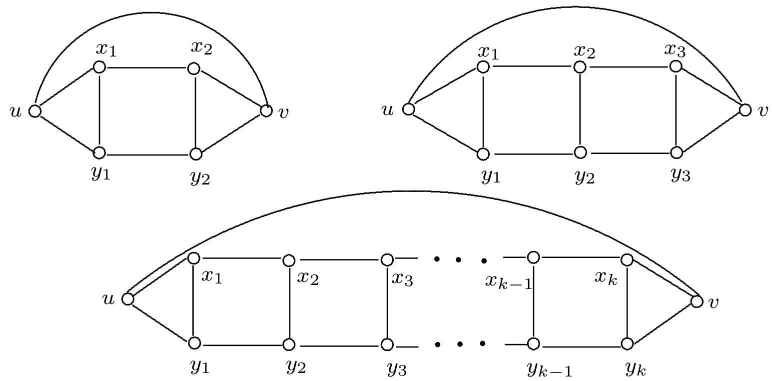


Figure 6.5: Graphs H_2, H_3 and H_k

Theorem 6.3.3 *For each integer $k \geq 2$,* $\mathrm{hpc}(H_k) = 3$.

Proof. Let $C = (u, x_1, x_2, \ldots, x_k, v, y_k, y_{k-1}, \ldots, y_3, y_2, y_1, u)$ be a Hamiltonian cycle of H_k. Define a proper 3-edge coloring of H_k by alternately assigning the colors 1 and 3 to the edges of C and assigning the color 2 to the remaining edges of H_k. Thus, $\mathrm{hpc}(H_k) \leq \chi'(H_k) = 3$. Figure 6.6(a) shows this edge coloring for the case when k is odd and Figure 6.6(b) shows this edge coloring for the case when k is even.

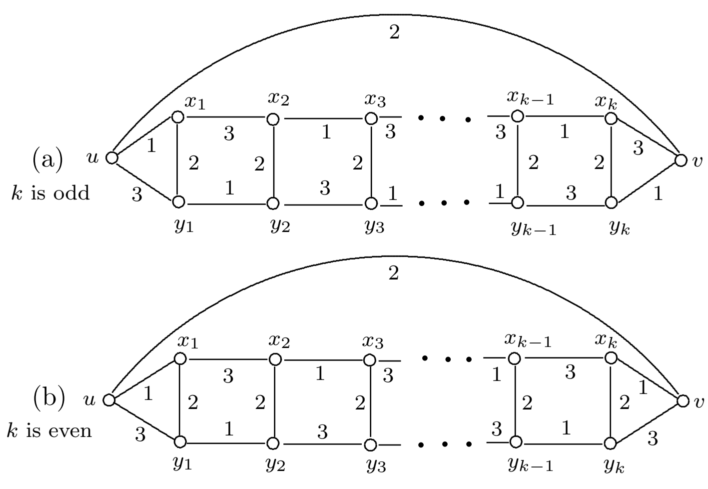


Figure 6.6: Edge colorings of H_k

It therefore remains to show that $\mathrm{hpc}(H_k) \geq 3$. Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring c of H_k using the colors 1 and 2. First, consider a proper Hamiltonian $u-v$ path. There are only two Hamiltonian $u-v$ paths in G. Because of the symmetry of these paths, we consider the path $(u, x_1, y_1, y_2, x_2, x_3, y_3, \ldots, x_k, y_k, v)$ if k is odd and $(u, x_1, y_1, y_2, x_2, x_3, y_3, \ldots, y_k, x_k, v)$ if k is even. Choosing $c(ux_1) = 1$, the colors of the remaining edges on the path are determined as shown in Figure 6.7 when k is odd.

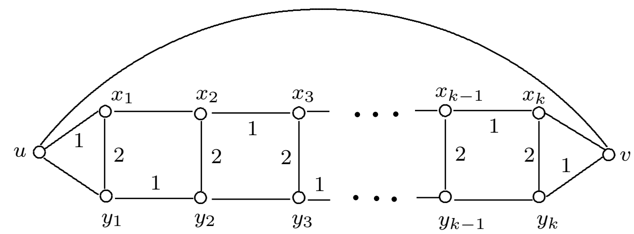


Figure 6.7: A step in the proof of Proposition 6.3.3 when k is odd

Next, consider a proper Hamiltonian $u - x_2$ path P in H_k. If P begins with u, y_1, then P cannot contain x_1, which is impossible. Suppose that P begins with u, v. Then P must end as $x_3, y_3, y_2, y_1, x_1, x_2$. Since $c(x_3y_3) = 2$, it follows that $c(y_2y_3) = 1$, which is impossible as $c(y_1y_2) = 1$. Hence, P must begin with u, x_1 and so $P = (u, x_1, y_1, y_2, \ldots, y_k, v, x_k, x_{k-1}, \ldots, x_2)$.
Furthermore,

$$\text{the edges of } P \text{ are alternately colored 1 and 2.} \tag{6.3}$$

We now consider the Hamiltonian $x_1 - x_2$ paths in G. There are only two Hamiltonian $x_1 - x_2$ paths Q and Q' in G, where

$$Q = (x_1, u, y_1, y_2, \ldots, y_k, v, x_k, x_{k-1}, \ldots, x_2)$$

and

$$Q' = \begin{cases} (x_1, y_1, u, v, y_k, x_k, x_{k-1}, y_{k-1}, \ldots, y_2, x_2) & \text{if } k \text{ is even} \\ (x_1, y_1, u, v, x_k, y_k, y_{k-1}, x_{k-1}, \ldots, y_2, x_2) & \text{if } k \text{ is odd.} \end{cases}$$

If $c(uy_1) = 1$, then Q is not proper and so Q' must be proper. However then, $c(x_ix_{i+1}) = 1$ for each integer i with $1 \le i \le k-1$, which contradicts (6.3). Hence, the edges of the Hamiltonian $x_1 - x_2$ path Q are alternately colored 1 and 2, beginning and ending with 1. Now, consider a Hamiltonian $u - y_2$ path Q. Proceeding as above with the path P, we see that Q must contain x_1y_1, x_1x_2, x_2x_3 as consecutive edges on Q. Since $c(x_1y_1) = 2$, it follows that $c(x_1x_2) = 1$. However, $c(x_2x_3) = 1$, which is impossible. Thus, no such proper Hamiltonian $u - y_2$ path exists. Therefore, $\text{hpc}(H_k) \ge 3$ and so $\text{hpc}(H_k) = 3$. ■

For each integer $k \ge 3$, recall that $P_k \square K_2$ is the grid of order $2k$ in which two paths of order k are $P_k = (x_1, x_2, \ldots, x_k)$ and $P'_k = (y_1, y_2, \ldots, y_k)$ such that $x_iy_i \in E(P_k \square K_2)$

for $1 \leq i \leq k$. The graph F_k of order $2k+1$ is constructed from $P_k \square K_2$ by adding a new vertex u and joining u to each vertex in $\{x_1, x_k, y_1, y_k\}$ (as shown in Figure 6.8). Thus, F_k has $2k$ vertices of degree 3 and one vertex of degree 4. It is known [36] that F_k is a Hamiltonian-connected graph of odd order and has the minimum size of a Hamiltonian-connected graph of order $2k+1$ for each integer $k \geq 3$. Furthermore, $\chi'(F_k) = \Delta(F_k) = 4$. We show that $\mathrm{hpc}(F_k) = 3$.

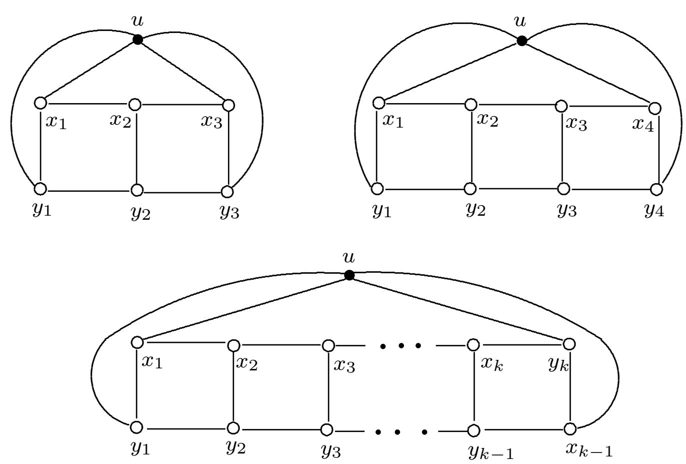


Figure 6.8: Graphs F_3, F_4 and F_k

Theorem 6.3.4 *For each integer $k \geq 3$,* $\mathrm{hpc}(F_k) = 3$.

Proof. For each integer $k \geq 3$, let $P_k \square K_2$ be the grid of order $2k$ in which two paths of order k are $P_k = (x_1, x_2, \ldots, x_k)$ and $P'_k = (y_1, y_2, \ldots, y_k)$ such that $x_i y_i \in E(P_k \square K_2)$ for $1 \leq i \leq k$. The graph F_k of order $2k+1$ is constructed from $P_k \square K_2$ by adding a new vertex u and joining u to each vertex in $\{x_1, x_k, y_1, y_k\}$. Define an edge coloring $c : E(F_k) \to \{1, 2, 3\}$ of F_k by alternately assigning the colors 1 and 3 to the edges of P_k and P'_k beginning with 1 and assigning the color 2 to the remaining edges of $P_k \square K_2$. Furthermore, if $k \geq 3$ is odd, then let $c(ux_1) = c(uy_1) = 3$ and $c(ux_k) = c(uy_k) = 1$ and if $k \geq 4$ is even, then let $c(ux_1) = c(uy_1) = 3$ and $c(ux_k) = c(uy_k) = 2$. Figure 6.9(a) shows this edge coloring for the case when k is odd and Figure 6.9(b) shows this edge coloring for the case when k is even.

Next, we show that the 3-edge coloring of F_k described in Figure 6.9 is a proper Hamiltonian-path 3-coloring of F_k; that is, we show that F_k contains a proper Hamiltonian $w - z$ path for each pair w, z of distinct vertices of F_k. First, observe that every Hamiltonian path P of F_k is proper unless P contains both ux_1 and uy_1 or contains both ux_k and uy_k. Hence, if either w or z is u, then F_k contains a proper Hamiltonian

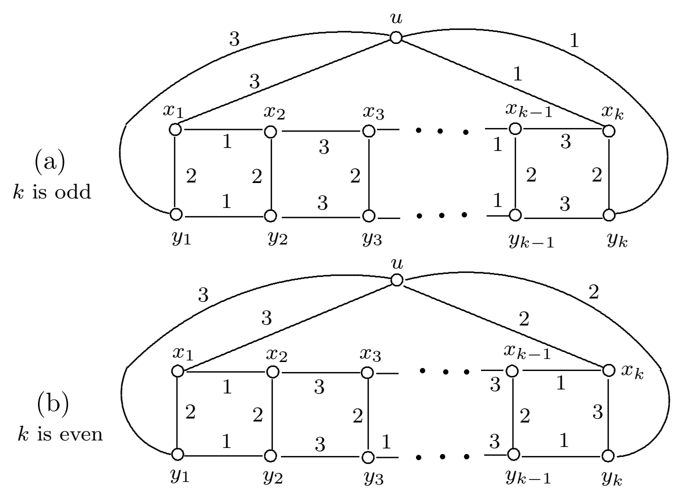


Figure 6.9: Edge colorings of F_k

$w - z$ path with initial vertex u. Therefore, we may assume that neither w nor z is u. We consider the following cases.

Case 1. $\{w, z\} = \{x_i, x_j\}$ *or* $\{w, z\} = \{y_i, y_j\}$, *where* $i < j$, *say the former.* If i is even, then consider the $x_i - u$ path

$$P' = (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, y_1, x_1, u);$$

while if i is odd, then consider the $x_i - u$ path

$$P' = (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, x_1, y_1, u).$$

Next, if $k - j$ is even, then consider the $u - x_j$ path

$$P'' = (u, y_k, x_k, x_{k-1}, y_{k-1}, y_{k-2}, \ldots, y_j, x_j);$$

while if $k - j$ is odd, then consider the $u - x_j$ path

$$P'' = (u, x_k, y_k, y_{k-1}, x_{k-1}, x_{k-2}, y_{k-2}, \ldots, y_j, x_j).$$

Then, P' followed by P'' is a proper Hamiltonian $x_i - x_j$ path.

Case 2. $\{w, z\} = \{x_i, y_j\}$. We may assume that $i \leq j$. There are two subcases.

Subcase 2.1. $i = j$. If i is even, then consider the $x_i - u$ path

$$P' = (x_i, x_{i-1}, y_{i-1}, y_{i-2}, x_{i-2}, x_{i-3}, \ldots, x_1, y_1, u);$$

while if i is odd, then consider the $x_i - u$ path

$$P' = (x_i, x_{i-1}, y_{i-1}, y_{i-2}, x_{i-2}, x_{i-3}, \ldots, y_1, x_1, u).$$

Next, if $k - i$ is even, then consider the $u - y_i$ path

$$P'' = (u, y_k, x_k, x_{k-1}, y_{k-1}, y_{k-2}, \ldots, x_{i+1}, y_{i+1}, y_i);$$

while if $k - i$ is odd, then consider the $u - y_i$ path

$$P'' = (u, x_k, y_k, y_{k-1}, x_{k-1}, x_{k-2}, \ldots, x_{i+1}, y_{i+1}, y_i).$$

Then, P' followed by P'' is a proper Hamiltonian $x_i - y_i$ path.

Subcase 2.2. $i < j$. If i is even, then consider the $x_i - u$ path

$$P' = (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, y_1, x_1, u);$$

while if i is odd, then consider the $x_i - u$ path

$$P' = (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, x_1, y_1, u).$$

If $k - j$ is even, then consider the $u - y_j$ path

$$P'' = (u, x_k, y_k, y_{k-1}, x_{k-1}, x_{k-2}, y_{k-2}, \ldots, x_j, y_j);$$

while if $k - j$ is odd, then consider the $u - y_j$ path

$$P'' = (u, y_k, x_k, x_{k-1}, y_{k-1}, y_{k-2}, x_{k-2}, \ldots, x_j, y_j).$$

Then, P' followed by P'' is a proper Hamiltonian $x_i - y_j$ path.

It therefore remains to show that $\text{hpc}(F_k) \geq 3$. Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring c of F_k using the colors 1 and 2. First, consider a proper Hamiltonian $u - v$ path. We consider two cases, according to whether k is odd or k is even.

Case 1. *$k \geq 3$ is odd.* Let $k = 2t + 1$ for some positive integer t. First, consider the vertices x_{t+1} and u. Let P be a proper Hamiltonian $x_{t+1} - u$ path in F_k. First, observe that P cannot start with x_{t+1}, y_{t+1}. Thus, either P starts with x_{t+1}, x_t or starts with x_{t+1}, x_{t+2}. Suppose, without loss of generality, that P starts with x_{t+1}, x_t. Since $x_{t+1}x_{t+2}, x_{t+1}y_{t+1} \notin E(P)$ and y_{t+1} and x_{t+2} have degree 2 on P, it follows that

$$(y_t, y_{t+1}, y_{t+2}, x_{t+2}, x_{t+3}) \text{ is a subpath of } P. \tag{6.4}$$

If $t \geq 2$, then $x_t y_t \notin E(P)$ (for otherwise, y_{t-1} cannot belong to P). Similarly, $x_i y_i \notin E(P)$ for $2 \leq i \leq t$. Hence, P contains the subpath $(x_{t+1}, x_t, \ldots, x_1, y_1, y_2, \ldots, y_{t+1}, y_{t+2})$. By (6.4), if t is odd, then

$$P = (x_{t+1}, x_t, \ldots, x_1, y_1, y_2, \ldots, y_{t+1}, y_{t+2}, x_{t+2}, x_{t+3}, y_{t+3}, \ldots, y_k, x_k, u);$$

while if t is even, then

$$P = (x_{t+1}, x_t, \ldots, x_1, y_1, y_2, \ldots, y_{t+1}, y_{t+2}, x_{t+2}, x_{t+3}, y_{t+3}, \ldots, x_k, y_k, u).$$

Since c is a proper Hamiltonian-path 2-coloring of F_k using the colors 1 and 2, we may assume that P is alternately colored 1 and 2, beginning with 1 and ending with 2. Thus, the colors of some edges of $P_k \,\square\, K_2$ are determined. This is shown for $k \in \{5, 7\}$ in Figure 6.10 where each bold edge belongs to the path P. In particular, $\{c(y_1y_2), c(x_2x_3)\} = \{c(x_{t+1}x_t), c(x_{t+2}x_{t+3})\} = \{1, 2\}$.

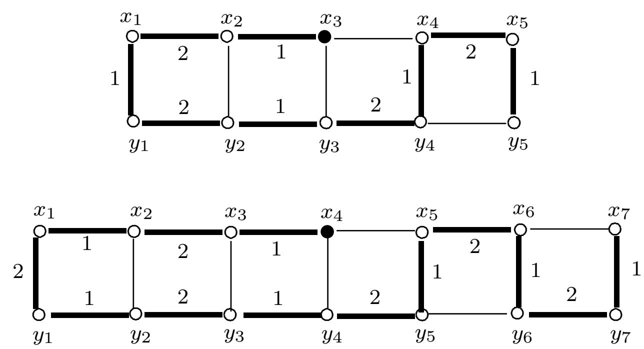


Figure 6.10: The colors of some edges of $P_k \,\square\, K_2$ in Case 1 for $k \in \{5, 7\}$

Next, consider the vertices x_1 and u. Let Q be a proper Hamiltonian $x_1 - u$ path in F_k. Since Q cannot begin with x_1, u, exactly one of x_1x_2 and x_1y_1 is an edge of Q. We consider these two subcases.

Subcase 1.1. $x_1x_2 \in E(Q)$ *and* $x_1y_1 \notin E(Q)$. Then

$$Q = (x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_1, u).$$

Since $c(x_tx_{t+1}) = 1$, it follows that $c(x_{t+1}x_{t+2}) = 2$ and $c(x_{t+2}x_{t+3}) = 1$, which is a contradiction.

Subcase 1.2. $x_1x_2 \notin E(Q)$ *and* $x_1y_1 \in E(Q)$. Here,

$$Q = (x_1, y_1, y_2, x_2, x_3, \ldots, y_{k-2}, y_{k-1}, x_{k-1}, x_k, y_k, u).$$

Since $\{c(y_1y_2), c(x_2x_3)\} = \{1, 2\}$, there is no color for y_2x_2 and so Q is not proper.

Case 2. $k \geq 4$ *is even.* Let $k = 2t$ for some integer $t \geq 2$. First, consider the vertices x_t and u. Let P be a proper Hamiltonian $x_t - u$ path in F_k. As in Case 1, the path P cannot start with x_t, y_t. Thus, either P starts with x_t, x_{t-1} or x_t, x_{t+1}. We consider these two subcases.

Subcase 2.1. *P starts with x_t, x_{t-1}.* Since y_t and x_{t+1} have degree 2 in P, it follows that

$$(y_{t-1}, y_t, y_{t+1}, x_{t+1}, x_{t+2}) \text{ is a subpath of } P. \tag{6.5}$$

If $t \geq 3$, then $x_{t-1}y_{t-1} \notin E(P)$ (for otherwise, y_{t-2} cannot belong to P). Hence, P begins with the subpath $(x_t, x_{t-1}, \ldots, x_1, y_1, y_2, \ldots, y_t)$. Because of (6.5), if $t \geq 3$ is odd, then

$$P = (x_t, x_{t-1}, \ldots, x_1, y_1, y_2, \ldots, y_t, y_{t+1}, x_{t+1}, x_{t+2}, \ldots, y_{k-1}, y_k, x_k, u);$$

while if $t \geq 2$ is even, then

$$P = (x_t, x_{t-1}, \ldots, x_1, y_1, y_2, \ldots, y_t, y_{t+1}, x_{t+1}, x_{t+2}, \ldots, x_{k-1}, x_k, y_k, u).$$

Since c is a proper Hamiltonian-path 2-coloring of F_k using the colors 1 and 2, we may assume that P is alternately colored 1 and 2, beginning with 1 which is shown in Figure 6.11. In particular, $c(x_{t-1}x_t) = 1$ and $c(x_{t+1}x_{t+2}) = 2$ whether t is odd or even.

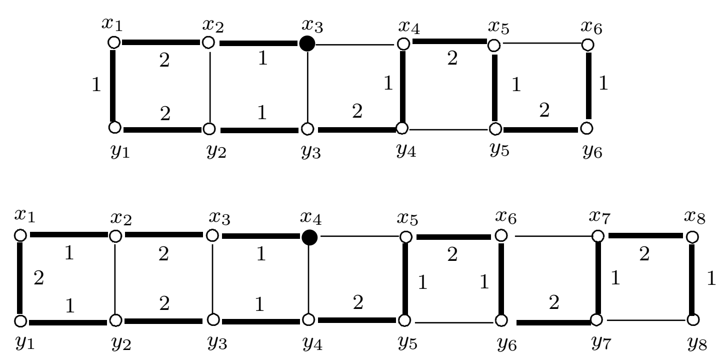


Figure 6.11: The colors of some edges of $P_k \,\Box\, K_2$ in Subcase 2.1 for $k \in \{6, 8\}$

Next, consider the vertices x_1 and u. Let Q be a proper Hamiltonian $x_1 - u$ path in F_k. Since Q cannot begin with x_1, u, exactly one of x_1x_2 and x_1y_1 is an edge of Q.

- ⋆ First, suppose that x_1x_2 is an edge of Q and x_1y_1 is not an edge of Q. Since each of x_2 and y_1 has degree 2 in Q, it follows that Q starts with (x_1, x_2, x_3) and ends at (y_2, y_1, u). This forces Q to be the following path

  $$Q = (x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_2, y_1, u).$$

  Since $c(x_{t-1}x_t) = 1$ and $c(x_{t+1}x_{t+2}) = 2$, regardless of the color of x_tx_{t+1}, it follows that Q is not proper.

- ⋆ Next, suppose that x_1y_1 is an edge of Q and x_1x_2 is not an edge of Q. Since each of x_2 and y_1 has degree 2 in Q, it follows that Q must start with $(x_1, y_1, y_2, x_2, x_3)$. This forces Q to be the following path

$$Q = (x_1, y_1, y_2, x_2, x_3, y_3, y_4, \ldots, x_{k-2}, x_{k-1}, y_{k-1}, y_k, x_k, u).$$

  Since $\{c(y_1y_2), c(x_2x_3)\} = \{1, 2\}$ (see Figure 6.11), regardless of the color of x_2y_2, it follows that Q is not proper.

Subcase 2.2. P starts with x_t, x_{t+1}. Since $x_tx_{t-1}, x_ty_t \notin E(P)$, it follows that $(x_{t-2}, x_{t-1}, y_{t-1}, y_t, y_{t+1})$ is a subpath of P. Thus, if $t \geq 3$ is odd, then

$$P = (x_t, x_{t+1}, \ldots, x_k, y_k, y_{k-1}, \ldots, y_t, y_{t-1}, x_{t-1}, x_{t-2}, \ldots, x_2, x_1, y_1, u);$$

while if $t \geq 2$ is even, then

$$P = (x_t, x_{t+1}, \ldots, x_k, y_k, y_{k-1}, \ldots, y_t, y_{t-1}, x_{t-1}, x_{t-2}, \ldots, y_2, y_1, x_1, u).$$

Since c is a proper Hamiltonian-path 2-coloring of F_k using the colors 1 and 2, we may assume that P is alternately colored 1 and 2, beginning with 1 which is shown in Figure 6.12.

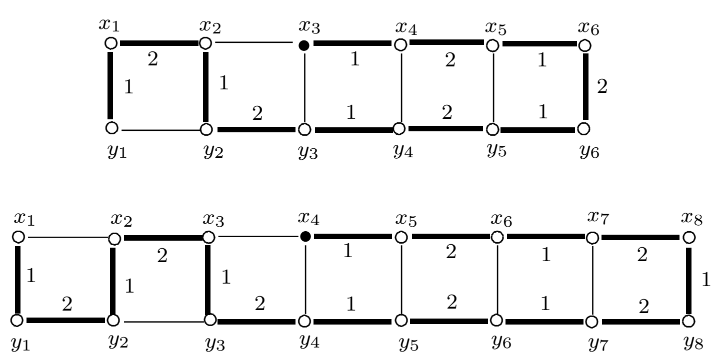


Figure 6.12: The colors of some edges of $P_k \,\square\, K_2$ in Subcase 2.2

Next, consider the vertices x_1 and u. Let Q be a proper Hamiltonian $x_1 - u$ path in F_k. Since Q cannot begin with x_1, u, exactly one of x_1x_2 and x_1y_1 is an edge of Q.

- ⋆ First, suppose that x_1x_2 is an edge of Q and x_1y_1 is not an edge of Q. Since y_1 has degree 2 in Q, it follows that Q ends with (y_2, y_1, u). Furthermore, $x_2y_2 \notin E(Q)$ and so $x_2x_3, y_2y_3 \in E(Q)$. This forces Q to be the following path

$$Q = (x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_2, y_1, u).$$

Since $\{c(x_t x_{t+1}), c(x_{t-2}x_{t-1})\} = \{1, 2\}$, there is no color for $x_{t-1}x_t$ and so Q is not proper.

⋆ Next, suppose that x_1y_1 is an edge of Q and x_1x_2 is not an edge of Q. Since each of x_2 and y_1 has degree 2 in Q and $y_1u \notin E(Q)$, it follows that

$$Q = (x_1, y_1, y_2, x_2, x_3, y_3, \ldots, y_{t-1}, y_t, x_t, x_{t+1}, \ldots, y_k, x_k, u).$$

Since $\{c(y_{t-1}y_t), c(x_t x_{t+1})\} = \{1, 2\}$, there there is no color for $c(x_t y_t)$ and so Q is not proper. ∎

It was shown in [4] that if G is a 2-connected graph, then the proper connection number of G is at most 3. Since every Hamiltonian-connected graph G of order at least 4 is 2-connected (in fact, 3-connected), $\mathrm{pc}(G) \le 3$. Since we have seen no Hamiltonian-connected graph G for which $\mathrm{hpc}(G) > 3$, we are led to the following conjecture.

Conjecture 6.3.5 *If G is a Hamiltonian-connected graph, then* $\mathrm{hpc}(G) \le 3$.

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