12-2017

Color-Connected Graphs and Information-Transfer Paths

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by

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A dissertation submitted to the Graduate College
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
Mathematics
Western Michigan University
December 2017

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The Department of Homeland Security in the United States was created in 2003 in response to weaknesses discovered in the transfer of classified information after the September 11, 2001 terrorist attacks. While information related to national security needs to be protected, there must be procedures in place that permit access between appropriate parties. This two-fold issue can be addressed by assigning information-transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage. Situations such as this can be represented by a graph whose vertices are the agencies and where two vertices are adjacent if there is direct access between them. Such graphs can then be studied by means of certain edge colorings of the graphs, where colors here refer to passwords. During the past decade, many research topics in graph theory have been introduced to deal with this type of problems. In particular, edge colorings of connected graphs have been introduced that deal with various ways every pair of vertices are connected by paths possessing some prescribed color condition.

Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P is a rainbow path in an edge-colored graph G if no two edges of P are colored the same. An edge coloring c of a connected graph G is a rainbow coloring of G if every pair of distinct vertices of G are connected by a rainbow path in G. In this case, G is rainbow-connected. The minimum number of colors needed for a rainbow coloring of G is referred to as the rainbow connection number of G and is denoted by $\mathrm{rc}(G)$. A path P is a proper path in G if no two adjacent edges of P are colored the same. An edge coloring c of a connected graph G is a proper-path coloring of G if every pair of distinct vertices of G are connected by a proper path in G. If k colors are used, then c is referred to as a proper-path k-coloring. The minimum k for which G has a proper-path k-coloring is called the proper connection number $\text{pc}(G)$ of G. In recent years, these two concepts have been studied extensively by many researchers. It has been observed that these two concepts model a communications network, where the goal is to transfer information in a secure manner between various law enforcement and intelligence agencies. Research on these two concepts has typically involved problems dealing with the minimum number of colors required for the graphical models of these communications networks to possess at least one desirable information-transfer path between each pair of agencies.

Looking at rainbow colorings and proper-path colorings in a different way brings up edge colorings that are intermediate to rainbow and proper-path colorings. Let G be a nontrivial connected edge-colored graph, where adjacent edges may be colored the same. A path P in G is a proper path if no two adjacent edges of P are colored the same and is a rainbow path if no two edges of P are colored the same. For an integer $k \geq 2$, a path P in G is a k-rainbow path if every subpath of P having length at most k is a rainbow path. An edge coloring of G is a k-rainbow coloring if every pair of distinct vertices of G are connected by a k-rainbow path in G. The minimum number of colors for which G has a k-rainbow coloring is called the k-rainbow connection number of G. Thus, if G is a nontrivial connected graph whose longest paths have length ℓ, then

$$\text{pc}(G) = \text{rc}_2(G) \leq \text{rc}_3(G) \leq \cdots \leq \text{rc}_\ell(G) = \text{rc}(G).$$

We first investigate the 3-rainbow colorings in graphs and the relationships among the 3-rainbow connection numbers and the well-studied chromatic number, chromatic index, rainbow or proper connection numbers of graphs. Since every connected graph G contains a spanning tree T and $\text{rc}_k(G) \leq \text{rc}_k(T)$, the various connection numbers of trees play an important role in the study of general graphs. It is shown that

- ⋆ for a triple (a, b, c) of integers with $2 \leq a \leq b \leq c$, there exists a tree T with $\text{pc}(T) = a$, $\text{rc}_3(T) = b$ and $\text{rc}(T) = c$ if and only if $a = b = c$ or $2 \leq a < b \leq \min\{2a - 1, c\}$ and
- ⋆ for a triple $(a, b, m)$of positive integers, there exists a tree T of size m with $\chi'(G) = a$ and $\text{rc}_3(T) = b$ if and only if $a \leq b \leq m$ such that $a = b = m$ or $2 \leq a < b \leq \min\{2a - 1, m\}$.

For each integer $k \geq 3$, the values of $\text{rc}_k(G)$ are determined for several well-known classes of graphs G. For example, if $K_{s,t}$ is the complete bipartite graph $2 \leq s \leq t$, ℓ is the length of a longest path in $K_{s,t}$ and k is an integer with $2 \le k \le \ell$, then

$$\mathrm{rc}_k(K_{s,t}) = \begin{cases} 2 & \text{if } k = 2 \\ \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 3\right\} & \text{if } k = 3 \\ \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 4\right\} & \text{if } 4 \le k \le \ell. \end{cases}$$

If T is a tree of diameter at least k for some integer $k \ge 2$, then

$$\mathrm{rc}_k(T) = \max\{m(T') :\ T' \subseteq T \text{ and } \mathrm{diam}(T') = k\}.$$

With the aid of results on trees, upper bounds for $\mathrm{rc}_k(G)$ are established in terms of the maximum degree of G. For example, if G is a connected graph of order at least $k+1 \ge 4$ and maximum degree $\Delta \ge 3$, then

$$\mathrm{rc}_k(G) \le \begin{cases} \frac{\Delta\left[(\Delta-1)^t-1\right]}{\Delta-2} & \text{if } k = 2t \ge 2 \text{ is even} \\ 1 + 2(\Delta-1)\frac{\left[(\Delta-1)^{t-1}-1\right]}{\Delta-2} & \text{if } k = 2t-1 \ge 3 \text{ is odd.} \end{cases}$$

We also establish sharp upper bounds for a connected graph in terms of its order.

- ⋆ If G is a nontrivial connected graph of order $n \ge 3$ that is not a tree such that the length of a longest path in G is ℓ, then $\mathrm{rc}_k(G) \le n-2$ for all integers k with $2 \le k \le \ell$. Furthermore, the upper bound $n-2$ is best possible.

- ⋆ Let G be a connected graph of order $n \ge 4$ and size $m \ge n+1$ and let ℓ be the length of a longest path in G. If G does not contain $K_4 - e$ as a subgraph, then $\mathrm{rc}_k(G) \le n-3$ for all integers k with $2 \le k \le \ell$. Furthermore, the upper bound $n-3$ is best possible.

If G is a nontrivial connected graph of diameter d, then $\mathrm{rc}_d(G) \ge d$. Furthermore, if G is a nontrivial tree of order n, then $\mathrm{rc}_d(G) = n-1$ and so $\mathrm{rc}_d(G) - \mathrm{diam}(G)$ can be arbitrarily large. On the other hand, if G is a connected graph of order $n \ge 3$ and diameter $d \ge 2$ that is not tree, then $d \le \mathrm{rc}_d(G) \le n-2$. It is shown that for each triple (d,k,n) of integers with $2 \le d \le k \le n-1$, there exists a connected graph G of order n that is not a tree such that $\mathrm{diam}(G) = d$ and $\mathrm{rc}_d(G) = k$ if and only if $k \ne n-1$.

One of well-known areas of research in graph theory involves the Hamiltonian properties of graphs. A Hamiltonian cycle in a graph G is a cycle containing every vertex of G and a graph having a Hamiltonian cycle is a Hamiltonian graph. A Hamiltonian path in a graph G is a path containing every vertex of G. A graph G is Hamiltonian-connected if G contains a Hamiltonian $u-v$ path for every pair u, v of distinct vertices of G. In a rainbow coloring or a proper-path coloring of a connected graph G, every two vertices u and v of G are connected by a rainbow $u-v$ path or a proper $u-v$ path and there is no condition on what the length of such a path must be. For certain graphs G, however, it is natural to ask whether there may exist an edge coloring of G using a certain number of colors such that every two vertices of G are connected by a rainbow path or proper path of a prescribed length.

For a Hamiltonian-connected graph G, an edge coloring c is called a Hamiltonian-connected rainbow coloring if every two vertices of G are connected by a rainbow Hamiltonian path in G. The minimum number of colors needed in a Hamiltonian-connected rainbow coloring of G is the rainbow Hamiltonian-connection number of G and is denoted by $\mathrm{hrc}(G)$. An edge coloring c is a proper Hamiltonian-path coloring if every two vertices of G are connected by a proper Hamiltonian path in G. The minimum number of colors needed in a proper Hamiltonian-path coloring of G is the proper Hamiltonian-connection number of G and is denoted by $\mathrm{hpc}(G)$.

Inspired by proper Hamiltonian-path colorings, k-rainbow colorings and Hamiltonian-connected rainbow colorings of Hamiltonian-connected graphs, we introduce and study the concept of k-rainbow Hamiltonian-path colorings of Hamiltonian-connected graphs. Let G be an edge-colored Hamiltonian-connected graph, where adjacent edges may be colored the same. For an integer $k \geq 2$, a Hamiltonian path P in G is a k-rainbow Hamiltonian path if every subpath of P having length at most k is a rainbow path. An edge coloring of G is a k-rainbow Hamiltonian-path coloring if every two vertices of G are connected by a k-rainbow Hamiltonian path in G. The minimum number of colors in a k-rainbow Hamiltonian-path coloring of G is the k-rainbow Hamiltonian-connection number of G. Thus, k-rainbow Hamiltonian-path colorings are intermediate to Hamiltonian-connected rainbow colorings and proper Hamiltonian-path colorings. In particular, if G is a Hamiltonian-connected graph of order $n \geq 4$ and size m, then

$$2 \leq \mathrm{hpc}(G) = \mathrm{hrc}_2(G) \leq \mathrm{hrc}_3(G) \leq \cdots \leq \mathrm{hrc}_{n-1}(G) = \mathrm{hrc}(G) \leq m.$$

We investigate the k-rainbow Hamiltonian-path colorings in two well-known classes of Hamiltonian-connected graphs, namely the join $G \vee K_1$ of a Hamiltonian graph G and the trivial graph K_1 and the prism $G \,\square\, K_2$ where G is a Hamiltonian graph of odd order. Results and open questions are also presented.

There is a rainbow concept that is somewhat reverse to rainbow connection in graphs. Let G be a nontrivial connected, edge-colored graph. An edge-cut R of G is called a

rainbow cut if no two edges in R are colored the same. An edge-coloring of G is a rainbow disconnection coloring if for every two distinct vertices u and v of G, there exists a rainbow cut in G, where u and v belong to different components of $G - R$. We introduce and study the rainbow disconnection number $\mathrm{rd}(G)$ of G, which is defined as the minimum number of colors required of a rainbow disconnection coloring of G. It is shown that the rainbow disconnection number of a nontrivial connected graph G equals the maximum rainbow disconnection number among the blocks of G. It is also shown that for a nontrivial connected graph G of order n, $\mathrm{rd}(G) = n - 1$ if and only if G contains at least two vertices of degree $n - 1$. The rainbow disconnection numbers of all grids $P_m \square P_n$ are determined. Furthermore, it is shown for integers k and n with $1 \leq k \leq n - 1$ that the minimum size of a connected graph of order n having rainbow disconnection number k is $n + k - 2$. Other results and a conjecture are also presented.

ACKNOWLEDGMENTS

I first give God all of the glory for His provision, wisdom, peace, and guidance on a wonderful journey that I would never have chosen myself.

Deepest thanks and gratitude to my advisor, Dr. Ping Zhang, who cheered for me when I felt incapable, believed I could do great things, and pushed me to accomplish more than I thought I could achieve. Her careful organization, patience, and encouragement made all of this possible.

Thank you to all of my committee members. Thanks to Dr. Chartrand for his kind words, good thoughts, and gentle encouragement. Thank you to Dr. Bennett for asking good questions that made me think about new things. Thanks to Dr. Martino for providing valuable feedback and for cheering me on. Great thanks to Dr. Hedetniemi for the plethora of comments and suggestions that made this paper the best it could be.

Thank you to my wife, Ana, who poured encouragement into me like a fountain, reminded me that my identity is not in my failures, showed me grace and love when I needed it most, and heard me talk about so much graph theory that she probably could have written these chapters herself.

Finally, thank you to all of my friends and family who supported me for the years spent on this journey. Your care for me kept me going more than you will know.

Stephen Devereaux

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Introduction

In this chapter, we provide background and motivation for research topics studied in this work and review some fundamental concepts and results on graph colorings that will be encountered as we proceed. In addition, we review some facts concerning Hamiltonian properties, vertex-cuts and connectivity concepts in graphs. We refer to the books [15, 16] for graph theoretic notation and terminology not described in this work. All graphs under consideration here are nontrivial connected graphs in general.

1.1 An Information-Transfer Problem

It was stated in [12] that the Department of Homeland Security in the United States was created in 2003 in response to weaknesses discovered in the transfer of classified information after the September 11, 2001 terrorist attacks. In [19], Ericksen made the following observation:

> *An unanticipated aftermath of those deadly attacks was the realization that law enforcement and intelligence agencies couldn't communicate with each other through their regular channels from radio systems to databases. The technologies utilized were separate entities and prohibited shared access, meaning there was no way for officers and agents to cross check information between various organizations.*

While information related to national security needs to be protected, there must be procedures in place that permit access between appropriate parties. This two-fold issue can be addressed by assigning information-transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage. An immediate question arises:

> *What is the minimum number of passwords or firewalls needed that permits the existence of one or more secure paths between every two agencies?*

To answer this question, we need an understanding of what is meant by a secure information-transfer path. There are many possible interpretations of what might be meant by such a secure path. At one extreme, a path may be considered secure only if the passwords along the path are distinct. A considerably less stringent interpretation might require only that every pair of consecutive passwords along the path be distinct. As described in [12], situations such as this can be represented by a graph whose vertices are the agencies and where two vertices are adjacent if there is direct access between them. Such graphs can then be studied by means of certain edge colorings of the graphs, where colors here refer to passwords.

1.2 Rainbow and Proper-Path Colorings

A *rainbow coloring* of a connected graph G is an edge coloring c of G with the property that for every two vertices u and v of G, there exists a $u - v$ *rainbow path* (no two edges of the path are colored the same). In this case, G is *rainbow-connected* (*with respect to* c). The minimum number of colors needed for a rainbow coloring of G is referred to as the *rainbow connection number* of G and is denoted by $\mathrm{rc}(G)$. These concepts were introduced in 2006 and studied by Chartrand, Johns, McKeon and Zhang [12]. In recent years, this topic has been studied by many and, in fact, there is a book by Li and Sun [28] on rainbow colorings, published in 2012.

While passwords have been around for ages, passwords, as we know them, are used to stop unauthorized individuals from having access to private information. In recent years, it has become increasingly clear of the necessity of using more complex passwords and of changing passwords more frequently. Indeed, some policies of agencies and institutions require their users to change passwords – every year, every 180 days or perhaps every 90 days. This, of course, makes remembering new passwords more difficult. Since protecting security is critical, some policies require any new password to be distinct from those passwords used within a certain time period. For example, one American university requires its employees to change passwords at least once a year and not use the same password that has been used during the past 20 years. This suggests another concept in graph theory, namely that of assigning colors to the edges of a connected graph G so that every two vertices of G are connected by a path P having the property that the colors on every subpath of P of length k or less are distinct (a rainbow subpath).

Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is *properly colored* or, more simply, is a *proper path* in G if no two adjacent edges of P are colored the same. An edge coloring c of a connected graph G is a *proper-path coloring* of G if every pair of distinct vertices of G are connected by a proper path in G. If k colors are used, then c is referred to as a *proper-path k-coloring*. The minimum k for which G has a proper-path k-coloring is called the *proper connection number* $\mathrm{pc}(G)$ of G. This concept was defined by Borozan et al. in [8] and studied by many (see [1, 27] for example).

While proper-path colorings were introduced to parallel concepts dealing with rainbow colorings, there is motivation for this concept corresponding to that introduced for rainbow colorings of graphs. With regard to the national security discussion mentioned above, we are then interested in the answer to the following question:

> *What is the minimum number of passwords or firewalls that allow one or more secure paths between every two agencies where, as we progress from one step to the next along such a path, we are required to change passwords*?

In [27], Li and Magnant reported that when building a communications network between wireless signal towers, one fundamental requirement is that the network be connected. If there cannot be a direct connection between two towers A and B for any variety of reasons (such as there is a mountain between the towers), then it is necessary to have a route through other towers to proceed from A to B. As a wireless transmission passes through a signal tower, it would help to avoid interference if the incoming signal and the outgoing signal do not share the same frequency. Suppose we assign a vertex to each signal tower, an edge between two vertices if the corresponding signal towers are directly connected by a signal and assign a color to each edge based on the assigned frequency used for the communication. Then the minimum number of frequencies needed to assign to the connections between towers so that there is always a path avoiding interference between each pair of towers is precisely the proper connection number of the corresponding graph.

1.3 The Knight's Tour Problem

Anther motivation has been suggested for the study of proper-path colorings of graphs. The Knight's Tour Problem is a famous problem that asks whether it's possible for a knight to tour an 8×8 chessboard where each square of the chessboard is visited exactly once (except that the final square visited is the initial square of the tour) and each step along the tour is a single legal move of a knight. It is well known that such a tour is possible. Since a single move of a knight causes the knight to move from a square of one color on the chessboard to a square of the other color, it follows that if two knights were placed on any two squares of a chessboard, then there exists a path of legal moves from one knight to the other on a chessboard, using legal knight moves, such that the squares visited along the path alternate in color. Because the Knight's Tour Problem is equivalent to that of finding a particular type of Hamiltonian cycle on the grid $P_8 \square P_8$ (the Cartesian product of P_8 and P_8), this brings up two possible graph colorings.

(1) Colors are assigned to the edges of a connected graph G so that for every two vertices u and v of G, there exists a $u - v$ path P in G such that every two adjacent edges on P have distinct colors.

(2) Colors are assigned to the vertices of a connected graph G so that for every two vertices u and v of G, there exists a $u-v$ path P in G such that every two adjacent vertices on P have distinct colors.

It is colorings of type (1), of course, that lead us once again to proper-path colorings of connected graphs.

1.4 k-Rainbow Colorings

Let us review the various edge-colorings we have discussed. First, if G is an edge-colored graph such that for every two vertices u and v, there exists a $u - v$ path P having the property that every subpath of P is a rainbow path, then this edge coloring is a rainbow coloring. On the other hand, if for every two vertices u and v, there exists a $u - v$ path Q having the property that every subpath of Q of length (at most) 2 is a rainbow path, then this edge coloring is a proper-path coloring. However, what if we require this for subpaths of length greater than 2? Looking at rainbow colorings and proper-path colorings in this way brings up, quite naturally, other edge colorings that are intermediate to rainbow and proper-path colorings.

More formally, let G be an edge-colored nontrivial connected graph, where adjacent edges may be colored the same. For an integer $k \geq 2$, a path P in G is a *k-rainbow path* (*with respect to the edge coloring*) if every subpath of P having length k or less is a rainbow path. Thus, every proper path is a 2-rainbow path and for each $k \geq 3$, a k-rainbow path is also an ℓ-rainbow path for every integer ℓ with $2 \leq \ell \leq k$. In particular, every k-rainbow path is a proper path for each integer $k \geq 2$.

For an integer $k \geq 2$, an edge coloring c is a *k-rainbow coloring* of a connected graph G if every pair of distinct vertices of G are connected by a k-rainbow path in G. In this case, the graph G is *k-rainbow connected* (*with respect to c*). If j colors are used to produce a k-rainbow coloring of G, then c is referred to as a *k-rainbow j-edge coloring* (or simply a *k-rainbow j-coloring*). The minimum j for which G has a k-rainbow j-coloring is called the *k-rainbow connection number* $\mathrm{rc}_k(G)$ of G. Hence, $\mathrm{rc}_2(G) = \mathrm{pc}(G)$ and $r_k(G) = \mathrm{rc}(G)$ if k is the length of a longest path in G. For every nontrivial connected graph G of size m and integer $k \geq 2$,

$$1 \leq \mathrm{pc}(G) \leq \mathrm{rc}_k(G) \leq \mathrm{rc}(G) \leq m. \tag{1.1}$$

To illustrate these concepts, a proper-path 2-coloring, a 3-rainbow 3-coloring and a rainbow 4-coloring are shown for the graph G in Figure 1.1. In fact, for this graph G, we have $\mathrm{pc}(G) = 2$, $\mathrm{rc}_3(G) = 3$ and $\mathrm{rc}(G) = 4$.

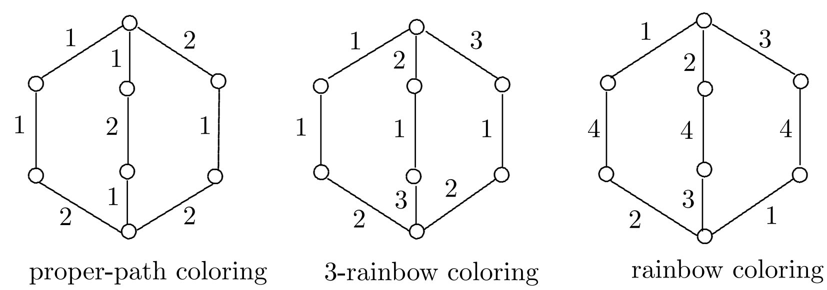


Figure 1.1: Three edge colorings of a graph G

1.5 Basic Definitions and Notation

In this section, we review some basic concepts in graph theory that we will need along with some of the theorems that have been obtained concerning them. We refer to the books [15, 16] for additional information about these concepts.

1.5.1 Proper Vertex and Edge Colorings

The vertex colorings of a graph G that have received the most attention over the years are proper colorings (see [15, 36, 37]). A *proper vertex coloring* of a graph G is a function c from $V(G)$ to some set S of objects (colors) such that $c(u) \neq c(v)$ for every pair u, v of adjacent vertices of G. In our case, $S = [k] = \{1, 2, \ldots, k\}$ for some positive integer k and so the coloring c is a *k-vertex coloring* (or, more often, simply a *k-coloring*) of G. The minimum positive integer k for which G has a k-vertex coloring is the *chromatic number* of G, denoted by $\chi(G)$. For graphs of order $n \geq 3$, it is immediate which graphs of order n have chromatic number 1, 2 or n. A graph is *empty* if it has no edges; thus, a *nonempty graph* has one or more edges.

Observation 1.5.1 *If G is a graph of order $n \geq 3$, then $\chi(G) = 1$ if and only if G is empty, $\chi(G) = n$ if and only if $G = K_n$, and $\chi(G) = 2$ if and only if G is a nonempty bipartite graph.*

By Observation 1.5.1 then, $\chi(G) \geq 3$ if and only if G contains an odd cycle. The following result, though elementary, is useful.

Proposition 1.5.2 *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

The *clique number* $\omega(G)$ of a graph G is the maximum order of a complete subgraph of G. The following result is therefore a consequence of Proposition 1.5.2.

Corollary 1.5.3 *For every graph G, $\omega(G) \leq \chi(G)$.*

There are graphs G for which $\chi(G)$ and $\omega(G)$ may differ significantly. As far as upper bounds for the chromatic number of a graph G are concerned, the following result gives such a bound in terms of the maximum degree $\Delta(G)$ of the graph G.

Theorem 1.5.4 *For every graph G, $\chi(G) \leq \Delta(G) + 1$.*

For each positive integer n, $\chi(K_n) = n = \Delta(K_n) + 1$ and for each odd integer $n \geq 3$, $\chi(C_n) = 3 = \Delta(C_n) + 1$. Brooks [6] proved that these two classes of graphs are the only connected graphs with this property.

Theorem 1.5.5 (Brooks' Theorem) *If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.*

A *proper edge coloring* c of a nonempty graph G is a function $c : E(G) \to \{1, 2, \ldots, k\}$ for some positive integer k with the property that $c(e) \neq c(f)$ for every two adjacent edges e and f of G. If the colors are chosen from a set of k colors, then c is called a *k-edge coloring* of G. The minimum positive integer k for which G has a k-edge coloring is called the *chromatic index* of G and is denoted by $\chi'(G)$. It is immediate for every nonempty graph G that $\chi'(G) \geq \Delta(G)$. The best known and most useful theorem dealing with the chromatic index is one obtained by Vizing [34].

Theorem 1.5.6 (Vizing's Theorem) *For every nonempty graph G,*

$$\chi'(G) \leq \Delta(G) + 1.$$

As a result of Vizing's theorem, the chromatic index of a nonempty graph G is one of two numbers, namely either $\Delta(G)$ or $\Delta(G)+1$. A graph G with $\chi'(G) = \Delta(G)$ is called a *class one graph* while a graph G with $\chi'(G) = \Delta(G)+1$ is called a *class two graph*. The following result is essentially due to König [26].

Theorem 1.5.7 (König's Theorem) *Every nonempty bipartite graph is of class one.*

Every complete bipartite graph is of class one, while cycles and complete graphs are of class one if their orders are even and are of class two if their orders are odd. For an r-regular graph G, either $\chi'(G) = r$ or $\chi'(G) = r+1$ by Vizing's theorem. If $\chi'(G) = r$, then there is an r-edge coloring of G, resulting in r color classes $E_1, E_2, \ldots, E_r$. Since every vertex v of G has degree r, the vertex v is incident with exactly one edge in each set E_i $(1 \le i \le r)$. Therefore, each color class E_i is a *perfect matching* (or a 1-*factor*) and G is 1-*factorable*. The following results are known (see [15]).

Theorem 1.5.8 *A regular graph G is of class one if and only if G is 1-factorable.*

Corollary 1.5.9 *Every regular graph of odd order is of class two.*

We refer to the books [15, 36, 37] for more information on graph colorings.

1.5.2 Hamiltonian Concepts

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G. A graph having a Hamiltonian cycle is a *Hamiltonian graph*. A graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u-v$ path for every pair u, v of distinct vertices of G. Among many sufficient conditions for a graph G to be Hamiltonian or Hamiltonian-connected are those concerning the minimum of the degree sums of two nonadjacent vertices in G and are those concerning the size of G. For a nontrivial graph G that is not complete, let

$$\sigma_2(G) = \min\{\deg u + \deg v : uv \notin E(G)\}.$$

For a connected graph G, let

$$\operatorname{diam}(G) = \max\{d(u,v) : u, v \in V(G)\}$$

denote the diameter of G, where $d(u,v)$ is the length of the shortest path from u to v. The following result is well known (see [16, p. 152], for example).

Theorem 1.5.10 *If G is a graph of order $n \geq 2$ such that $\sigma_2(G) \geq n-1$, then G contains a Hamiltonian path.*

The following two results are due to Ore, the first of which was obtained in 1960 [30] and the second in 1963 [31].

Theorem 1.5.11 (Ore) *If G is a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then G is Hamiltonian.*

Theorem 1.5.12 (Ore) *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n+1$, then G is Hamiltonian-connected.*

Each of Theorems 1.5.10–5.1.1 has a corollary providing a lower bound on the minimum degree $\delta(G)$ for a graph G to possess the respective property.

Corollary 1.5.13 *If G is a graph of order $n \geq 2$ with $\delta(G) \geq (n-1)/2$, then G contains a Hamiltonian path.*

The following result (which is a corollary of Theorem 1.5.11) is the first theoretical result on Hamiltonian graphs. This result occurred in 1952 and is due to Dirac [17].

Corollary 1.5.14 (Dirac) *If G is a graph of order $n \geq 3$ with $\delta(G) \geq n/2$, then G is Hamiltonian.*

The following result is a corollary of Theorem 5.1.1.

Corollary 1.5.15 *If G is a graph of order $n \geq 4$ with $\delta(G) \geq (n+1)/2$, then G is Hamiltonian-connected.*

It is well known that all bounds stated in Corollaries 1.5.13, 1.5.14 and 1.5.15 are sharp.

1.5.3 Cut-Sets and Connectivity

A *vertex-cut* of a graph G is a set S of vertices of G such that $G - S$ is disconnected. A vertex-cut of minimum cardinality in G is called a *minimum vertex-cut* of G and this cardinality is called the *vertex-connectivity* (or the *connectivity*) of G (when G is not complete) and is denoted by $\kappa(G)$. Complete graphs do not contain vertex-cuts. The connectivity of the complete graph of order n is defined as $n-1$, that is, $\kappa(K_n) = n-1$. In general, the *connectivity* $\kappa(G)$ of a graph G is the smallest number of vertices whose removal from G results in either a disconnected graph or a trivial graph. Therefore, for every graph G of order n,

$$0 \leq \kappa(G) \leq n - 1.$$

Thus, a graph G has connectivity 0 if and only if either $G = K_1$ or G is disconnected; a graph G has connectivity 1 if and only if $G = K_2$ or G is a connected graph with cut-vertices; and a graph G has connectivity 2 or more if and only if G is a nonseparable graph (connected and no cut-vertices) of order 3 or more.

A graph G is *k-connected* for some positive integer k if $\kappa(G) \geq k$. That is, G is k-connected if the removal of fewer than k vertices from G does not result in a disconnected graph. The 1-connected graphs are then the nontrivial connected graphs, while the 2-connected graphs are the nonseparable graphs of order 3 or more. Whitney [35] provided a useful characterization of k-connected graphs regarding the number of internally disjoint paths. Note that two $u - v$ paths are *internally disjoint* if they have only u and v in common.

Theorem 1.5.16 (Whitney's Theorem) *A nontrivial graph G is k-connected for some positive integer k if and only if for each pair u, v of distinct vertices of G, there are at least k internally disjoint $u - v$ paths in G.*

An *edge-cut* of a graph G is a subset X of $E(G)$ such that $G - X$ is disconnected. An edge-cut of minimum cardinality in G is a *minimum edge-cut* and this cardinality is the *edge-connectivity* of G, which is denoted by $\lambda(G)$. The trivial graph K_1 does not contain an edge-cut but its edge-connectivity is defined to be 0; that is, $\lambda(K_1) = 0$. Therefore, $\lambda(G)$ is the minimum number of edges whose removal from G results in a disconnected or trivial graph. Since the set of edges incident with any vertex of a graph G of order n is an edge-cut of G, it follows that

$$0 \leq \lambda(G) \leq \delta(G) \leq n - 1.$$

A graph G is *k-edge-connected* for some positive integer k if $\lambda(G) \geq k$; namely, G is k-edge-connected if the removal of fewer than k edges from G results in neither a disconnected graph nor a trivial graph. Thus, a 1-edge-connected graph is a nontrivial connected graph and a 2-edge-connected graph is a nontrivial connected bridgeless graph. There is a well-known edge analogue of Whitney's Theorem that appears in many sources (see [14, 102], for example).

Theorem 1.5.17 *A nontrivial graph G is k-edge-connected if and only if G contains k pairwise edge-disjoint $u - v$ paths for each pair u, v of distinct vertices of G.*

1.5.4 Distance Concepts

Let G be a nontrivial connected graph. The *distance* $d(u, v)$ between vertices u and v in G is the minimum number of edges in a $u - v$ path in G. The *eccentricity* $e(v) = \max\{d(v, w) : w \in V(G)\}$ of a vertex v of G is the distance between v and a vertex farthest from v in G. The *diameter* $\mathrm{diam}(G) = \max\{e(v) : v \in V(G)\}$ of G is the largest eccentricity among the vertices of G and the *radius* $\mathrm{rad}(G) = \min\{e(v) : v \in V(G)\}$ is the smallest eccentricity among the vertices of G. Therefore, the diameter of G is the greatest distance between any two vertices of G. A vertex v with $e(v) = \mathrm{rad}(G)$ is called a *central vertex* of G and a vertex v with $e(v) = \mathrm{diam}(G)$ is called a *peripheral vertex* of G. Two vertices u and v of G with $d(u, v) = \mathrm{diam}(G)$ are *antipodal vertices* of G. Necessarily, if u and v are antipodal vertices in G, then both u and v are peripheral vertices. The subgraph induced by the central vertices of a connected graph G is the *center* of G and the subgraph induced by the peripheral vertices of a connected graph G is the *periphery* of G. The following result is due to Hedetniemi (see [9]).

Theorem 1.5.18 [9] *Every graph is the center of some graph.*

While every graph is the center of some graph, this is not true for the periphery. The following result is due to Bielak and Sysło (see [5]).

Theorem 1.5.19 [5] *A nontrivial graph G is the periphery of some graph if and only if every vertex of G has eccentricity* 1 *or no vertex of G has eccentricity* 1.

Chapter 2

On 3-Rainbow Connected Graphs

In this chapter, we investigate k-rainbow colorings of graphs for the case where $k = 3$. Let's first review this concept. Let G be a connected graph of order at least 4. An edge coloring c of G is a 3-*rainbow coloring* if every pair of distinct vertices of G are connected by a path every subpath of which having length 3 or less is a rainbow path. The minimum integer j for which G has a 3-rainbow j-coloring is called the 3-*rainbow connection number* $\mathrm{rc}_3(G)$ of G. We investigate the relationship between the 3-rainbow connection numbers and the well-studied rainbow and proper connection numbers of graphs. The values of $\mathrm{rc}_3(G)$ are determined for several well-known classes of graphs G. With the aid of the 3-rainbow connection number of a tree, we establish sharp bounds for the 3-rainbow connection number of a graph and some realization results on trees. Furthermore, we study the relationship between the 3-rainbow connection numbers, chromatic number and chromatic index of a graph and present a realization result on the 3-rainbow connection number and chromatic index of a graph.

2.1 Preliminary Results

Let G be a nontrivial connected graph of order n and size m. In the case of the k-rainbow connection number when $k = 3$, the inequalities in

$$1 \leq \mathrm{pc}(G) \leq \mathrm{rc}_k(G) \leq \mathrm{rc}(G) \leq m \text{ for all integers } k \geq 2$$

give rise to the following inequalities:

$$1 \leq \mathrm{pc}(G) \leq \mathrm{rc}_3(G) \leq \mathrm{rc}(G) \leq m. \tag{2.1}$$

Since $\mathrm{rc}(G) = 1$ if and only if $G = K_n$, it follows that $\mathrm{pc}(G) = \mathrm{rc}_3(G) = 1$ if and only if $G = K_n$. Thus, $\mathrm{rc}_3(G) = 1$ if and only if $\mathrm{rc}(G) = 1$.

Proposition 2.1.1 *Let G be a nontrivial connected graph. Then*

$$\mathrm{rc}(G) = 2 \textit{ if and only if } \mathrm{rc}_3(G) = 2.$$

Proof. If $\mathrm{rc}(G) = 2$, then G is not complete and so $2 \leq \mathrm{rc}_3(G) \leq \mathrm{rc}(G) = 2$. Thus, $\mathrm{rc}_3(G) = 2$. For the converse, suppose that $\mathrm{rc}_3(G) = 2$. Then there exists a 3-rainbow 2-coloring c of G. Since the only 3-rainbow paths in G are proper 2-colored paths, it follows that c is also a rainbow 2-coloring of G. Hence, $\mathrm{rc}(G) = 2$. ■

The following observation is immediate.

Observation 2.1.2 *Let G be a nontrivial connected graph.*

$$\textit{If } \mathrm{rc}_3(G) = 2, \textit{ then } \mathrm{pc}(G) = 2.$$

Not surprisingly, the converse of Observation 2.1.2 is false. For example, $\mathrm{pc}(P_4) = 2$, while $\mathrm{rc}_3(P_4) = 3$. In fact, if c is a 3-rainbow coloring of a graph G and P is a 3-rainbow path of length at least 3, then the number of colors of the edges of P is at least 3. This observation gives the following result.

Proposition 2.1.3 *If G is a connected graph with $\mathrm{diam}(G) \geq 3$, then*

$$\mathrm{rc}_3(G) \geq 3.$$

It is probably not surprising that the converse of Proposition 2.1.3 is false. For example, the diameter of the Petersen graph P is 2 and it is known that $\mathrm{pc}(P) = 2$ and $\mathrm{rc}(P) = 3$. Thus, either $\mathrm{rc}_3(P) = 2$ or $\mathrm{rc}_3(P) = 3$. We show that $\mathrm{rc}_3(P) = 3$. Assume, to the contrary, that there is a 3-rainbow 2-coloring c of P using the colors 1 and 2. Since P is 3-regular, there are two adjacent edges of P that are colored the same, say $c(uv) = c(vw) = 1$. Since (u, v, w) is not a proper path and the girth of P is 5, the length of any 3-rainbow $u - v$ path of the Petersen graph P is at least 3, which is impossible. Hence, $\mathrm{rc}_3(P) = 3$. Following this example, one might hope that a small diameter would result in an upper bound on the 3-rainbow connection number. However, as we will see later, the 3-rainbow connection number of a graph can be arbitrarily large regardless of the diameter.

If H is a connected spanning subgraph of G and $c_H : E(H) \to [k]$ is a 3-rainbow k-coloring of H, then c_H can be extended to a 3-rainbow coloring c_G of G by assigning any color in $[k]$ to each edge in $E(G) - E(H)$. Since every two vertices of G are connected by a 3-rainbow path in H (and so in G), it follows that c_G is a 3-rainbow k-coloring of G. This observation gives the following result.

Proposition 2.1.4 *If H is a connected spanning subgraph of a nontrivial connected graph G, then* $\mathrm{rc}_3(G) \le \mathrm{rc}_3(H)$*. In particular, if T is a spanning tree of G, then*

$$\mathrm{rc}_3(G) \le \mathrm{rc}_3(T).$$

In the case of paths, we have the following.

Proposition 2.1.5 *For each integer $n \ge 4$,* $\mathrm{rc}_3(P_n) = 3$*.*

It is shown in [12] that $\mathrm{rc}(C_n) = \lceil n/2 \rceil$ for each integer $n \ge 4$. Thus,

$$\mathrm{rc}_3(C_4) = \mathrm{rc}(C_4) = 2 \text{ and } \mathrm{rc}_3(C_5) = \mathrm{rc}(C_5) = 3.$$

More generally, we have the following.

Proposition 2.1.6 *For each integer $n \ge 5$,* $\mathrm{rc}_3(C_n) = 3$*.*

A *Hamiltonian path* in a graph G is a path containing every vertex of G. A graph G containing a Hamiltonian path is a *traceable graph*. The following is an immediate consequence of Propositions 2.1.4 and 2.1.5.

Corollary 2.1.7 *If a graph G is traceable, then* $\mathrm{rc}_3(G) \le 3$*.*

As an illustration of Corollary 2.1.7, we determine the 3-rainbow connection number of grids. The *Cartesian product* $G \,\Box\, H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (w, x) in $G \,\Box\, H$ if and only if either $u = w$ and $vx \in E(H)$ or $uw \in E(G)$ and $v = x$. The $m \times n$ grid graph $G_{m,n} = P_m \,\Box\, P_n$ consists of m horizontal paths P_n and n vertical paths P_m.

Proposition 2.1.8 *Let $m, n \in \mathbb{N}$ with $m \ge 2$ and $n \ge 2$. Then*

$$\mathrm{rc}_3(P_m \,\Box\, P_n) = \begin{cases} 2 & m = n = 2 \\ 3 & m \ge 3 \text{ or } n \ge 3. \end{cases}$$

Proof. Let $G = P_m \,\Box\, P_n$ and suppose first that $m = n = 2$. For this proof, the vertices of G are considered as entries of a matrix, where $x_{i,j}$ denotes the vertex in row i and column j, where $1 \le i \le m$ and $1 \le j \le n$.

Since G is not complete, $\mathrm{rc}_3(G) \ge 2$. We demonstrate that G has a 3-rainbow 2-coloring. Indeed, let

- $c(x_{i_1,j_1} x_{i_2,j_2}) = 1$ if $i_1 = i_2$, and

- $c(x_{i_1,j_1}x_{i_2,j_2}) = 2$ if $i_1 \neq i_2$

In other words, all horizontal paths are colored 1 and all vertical paths are colored 2. Since every path between two vertices in G requires at most one vertical edge and at most one horizontal edge, this is a 3-rainbow 2-coloring of G.

Now suppose without loss of generality that $m \geq 3$. Since the graph G now has diameter at least 3, $\mathrm{rc}_3(G) \geq 3$ by Proposition 2.1.3. However, since G is traceable, Corollary 2.1.7 implies that $\mathrm{rc}_3(G) \leq 3$. Thus, $\mathrm{rc}_3(G) = 3$ in this case, as desired. ■

The *join* $G = G_1 \vee G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and egde set

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

One example is the wheel $W_n = C_n \vee K_1$ of order $n+1 \geq 4$, which is the join of a cycle C_n and K_1, for which we have the following result.

Proposition 2.1.9 *For the wheel W_n of order $n+1 \geq 4$,*

$$\mathrm{rc}_3(W_n) = \begin{cases} 1 & \textit{if } n = 3 \\ 2 & \textit{if } 4 \leq n \leq 6 \\ 3 & \textit{if } n \geq 7. \end{cases}$$

Proof. Since $W_3 = K_4$, it follows that $\mathrm{rc}_3(W_3) = 1$. Thus, we may assume that $n \geq 4$. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $V(K_1) = \{v\}$. First, suppose that $n = 4, 5, 6$. We define an edge coloring $c : E(W_n) \to \{1, 2\}$ as follows:

- ⋆ If $n = 4$, let c be a proper coloring of C_4 and let $c(vv_i) = 1$ for $1 \leq i \leq 4$;
- ⋆ If $n = 5$, let $c(e) = 1$ if $e \in \{vv_1, v_1v_2, v_3v_4\}$ and $c(e) = 2$ otherwise;
- ⋆ If $n = 6$, let c be a proper coloring of C_6, let $c(vv_i) = 1$ for $1 \leq i \leq 3$ and $c(vv_i) = 2$ for $4 \leq i \leq 6$.

In each case, c is a 3-rainbow 2-coloring c of W_n and so $\mathrm{rc}_3(W_n) = 2$ for $n = 4, 5, 6$.

Next, suppose that $n \geq 7$. Since W_n is traceable, it follows by Corollary 2.1.7 that $\mathrm{rc}_3(W_n) \leq 3$. It remains to show that $\mathrm{rc}_3(W_n) \neq 2$. Assume, to the contrary, that there is a 3-rainbow 2-coloring c of W_n using the colors 1 and 2 for some integer $n \geq 7$. Thus, for each pair x, y of distinct vertices of W_n, there is a properly colored $x - y$ path of length 2 in W_n. First, observe that if $i, j \in \{1, 2, \ldots, n\}$ such that $|i - j| \geq 3$, then there is a unique $v_i - v_j$ path of length 2 in W_n, namely, (v_i, v, v_j). We may assume, without loss of generality, that $c(vv_1) = 1$. Hence, $c(vv_i) = 2$ for $4 \leq i \leq n-2$. Since

$c(vv_4) = c(vv_{n-2}) = 2$, it follows that $c(vv_2) = c(vv_n) = 1$. Next, since $c(vv_2) = 1$, this implies that $c(vv_{n-1}) = 2$. Similarly, since $c(vv_n) = 1$, we have $c(vv_3) = 2$. Thus, $c(vv_3) = c(vv_{n-1}) = 2$. Since the only $v_3 - v_{n-1}$ path of length 2 in W_n is (v_3, v, v_{n-1}), which is not proper, this produces a contradiction. Therefore, $\mathrm{rc}_3(W_n) = 3$ for $n \geq 7$. ■

It is shown by Chartrand et al. in [12] that if W_n is the wheel of order $n + 1 \geq 4$, then

$$\mathrm{rc}(W_n) = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } 4 \leq n \leq 6 \\ 3 & \text{if } n \geq 7. \end{cases}$$

Thus, $\mathrm{rc}_3(W_n) = \mathrm{rc}(W_n)$ for $n \geq 3$.

In Proposition 2.1.9, we saw that $\mathrm{rc}_3(W_n) = 3$ for $n \geq 7$, while $\mathrm{rc}_3(W_6) = 2$. The reason for this is that for each integer $n \geq 7$, the cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ has the property that C_n contains a vertex u_1 for which there exists a cyclic sequence

$$s : u_1, u_2, \ldots, u_k, u_{k+1} = u_1$$

of k distinct vertices of C_n for some odd integer $k \geq 3$ such that $d_{C_n}(u_i, u_{i+1}) \geq 3$ for each integer i for $1 \leq i \leq k$. In general, we will say that a graph G has Property (1) if it contains a vertex u_1 for which such a cyclic sequence exists. In fact, here every vertex of C_n has Property (1). For example, for $n = 7$, the cyclic sequence

$$s : v_1, v_4, v_7, v_3, v_6, v_2, v_5, v_1$$

has this property, while for $n = 8$, the cyclic sequence

$$s : v_1, v_4, v_7, v_2, v_5, v_1$$

has this property. Let v be the vertex of degree n in W_n. If there existed a 3-rainbow 2-coloring of W_n, then we can assume that $c(vv_1) = 1$ in each case above. For $n = 7$, we then must have $c(vv_4) = 2$, $c(vv_7) = 1$, $c(vv_3) = 2$, $c(vv_6) = 1$, $c(vv_2) = 2$, $c(vv_5) = 1$ and $c(vv_1) = 2$, which is impossible. Therefore, $\mathrm{rc}_3(W_n) \neq 2$ for $n \geq 7$. In the case of W_6, there is no vertex of C_6 for which such a sequence s exists. For example, for the vertex v_1 of C_6, the only vertex u with the property that $d_{C_6}(v_1, u) \geq 3$ is $u = v_4$ and the only vertex u with the property that $d_{C_6}(v_4, u) \geq 3$ is v_1.

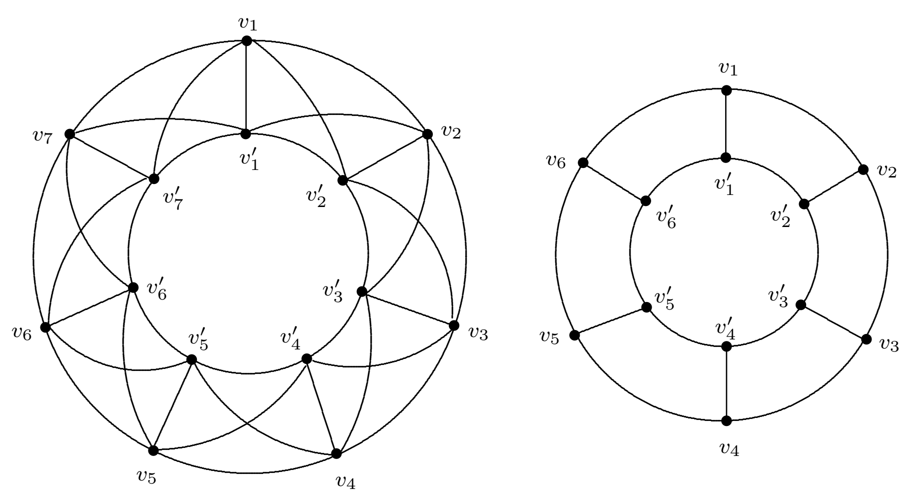
In the study of 3-rainbow colorings of graphs, it is often quite challenging to determine the exact value of the 3-rainbow connection number of a given graph or to establish a lower bound for this number. Therefore, we have a different view of the proof of

Proposition 2.1.9 and we can apply this idea to other more general graphs. As an example of other connected graphs G having radius at least 3 for which one can show that $\mathrm{rc}_3(G \vee K_1) \geq 3$, we define the *square* of a graph G to be the graph G^2 with vertex set $V(G^2) = V(G)$ and edge set $E(G^2) = \{uv \in E(G) : d_G(u, v) \leq 2\}$. Now consider the graph $G = C_{11}^2$, where $C_{11} = (v_1, v_2, \ldots, v_{11}, v_1)$. The cyclic sequence

$$s : v_1, v_6, v_{11}, v_5, v_{10}, v_4, v_9, v_3, v_8, v_2, v_7, v_1$$

has Property (1). The graph $G = C_{10}^2$ does not have this property, however. For example, the only vertex u for which $d_G(v_1, u) \geq 3$ is $u = v_6$, while the only vertex u with $d_G(v_6, u) \geq 3$ is v_1.

Two other examples of graphs with this property are shown in Figure 2.1.



G F

Figure 2.1: The graphs G and F

In the graph G of Figure 2.1, the cyclic sequence

$$s : v_1, v_4, v_7, v_3, v_6, v_2, v_5, v_1$$

has Property (1). In the graph $F = C_6 \square K_2$ of Figure 2.1, the cyclic sequence

$$s : v_1, v_4', v_1', v_5, v_3', v_1$$

has Property (1). Thus, $\mathrm{rc}_3(G \vee K_1) \geq 3$ and $\mathrm{rc}_3(F \vee K_1) \geq 3$ for the graphs G and F of Figure 2.1.

2.2 Complete Bipartite Graphs

Next, we determine the 3-rainbow connection numbers of complete bipartite graphs. Prior to doing this, we recall the rainbow connection numbers of these graphs (see [12]).

Theorem 2.2.1 *For integers s and t with $2 \le s \le t$,*

$$\mathrm{rc}(K_{s,t}) = \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 4\right\}.$$

The 3-rainbow connection numbers of complete bipartite graphs are nearly the same as their rainbow connection counterparts.

Theorem 2.2.2 *For integers s and t with $2 \le s \le t$,*

$$\mathrm{rc}_3(K_{s,t}) = \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 3\right\}.$$

Proof. Observe that $\lceil \sqrt[s]{t}\, \rceil \ge 2$ for all integers s and t with $2 \le s \le t$. First, suppose that $\lceil \sqrt[s]{t}\, \rceil = 2$. Since $2 \le \mathrm{rc}_3(K_{s,t}) \le \mathrm{rc}(K_{s,t}) = 2$ by (2.1) and Theorem 2.2.1, it follows that $\mathrm{rc}_3(G) = 2$ and so $\mathrm{rc}_3(G) = \min\{\lceil \sqrt[s]{t}\, \rceil,\ 3\}$. Thus, we may assume that $\lceil \sqrt[s]{t}\, \rceil \ge 3$ and so $t \ge 2^s + 1$. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$ be the partite sets of G.

Initially, we show that $\mathrm{rc}_3(K_{s,t}) \ge 3$. Assume, to the contrary, that there exists a 3-rainbow 2-coloring $c : E(K_{s,t}) \to \{1, 2\}$ of $K_{s,t}$. For each vertex $w \in W$, let $\mathrm{code}(w) = (a_1, a_2, \ldots, a_s)$ be the color code of w, where $a_i = c(u_i w) \in \{1, 2\}$ for $1 \le i \le s$. Since $t > 2^s$, there exist two distinct vertices w' and w'' of W such that $\mathrm{code}(w') = \mathrm{code}(w'')$. Since the edges of every $w' - w''$ path of length 2 are colored the same, there is no proper $w' - w''$ path in $K_{s,t}$, which is a contradiction. Thus, $\mathrm{rc}_3(K_{s,t}) \ge 3$.

Next, we show that $\mathrm{rc}_3(K_{s,t}) \le 3$ by defining a 3-rainbow 3-coloring of $K_{s,t}$. We consider two cases, according to whether $s = 2$ or $s \ge 3$.

Case 1. $s = 2$. Thus, $t \ge 2^2 + 1 = 5$. Let $c : E(K_{s,t}) \to \{1, 2, 3\}$ be defined by

$$c(e) = \begin{cases} 1 & \text{if } e = u_1w_1, u_2w_2 \\ 2 & \text{if } e = u_1w_2, u_2w_1 \\ 3 & \text{otherwise.} \end{cases}$$

Let $x, y \in V(K_{s,t})$ and $x \ne y$. If x and y belong to different partite sets, then (x, y) is a path. If x and y are nonadjacent vertices of $\{u_1, u_2, w_1, w_2\}$, then there is a properly colored $x - y$ path of length 2. Thus, we may assume that $x, y \in W$, where $\{x, y\} \ne \{w_1, w_2\}$, say $x = w_i$ and $y = w_j$ where $1 \le i < j$ and $j \ge 3$. If $x \in \{w_1, w_2\}$, say

$x = w_1$, then $(x = w_1, u_1, y)$ is a properly colored $x - y$ path of length 2 whose edges colored 1 and 3. If $x = w_i$ for some $i \geq 3$, then $P = (w_i, u_1, w_1, u_2, w_j)$ is a 3-rainbow path of length 4 in $K_{s,t}$, where the edges of P are colored by $3, 1, 2, 3$ in this order.

Case 2. $s \geq 3$. Let $H = K_{s,s}$ be the subgraph of $K_{s,t}$ induced by the sets U and $W' = \{w_1, w_2, \ldots, w_s\}$. First, we define three perfect matchings M_1, M_2, M_3 in H as follows: For $j = 1, 2, 3$, let $M_j = \{w_i u_{i+j-1} : \ 1 \leq i \leq s\}$ where the subscript of each vertex is expressed modulo s. The edges in M_1, M_2, M_3 are as follows:

$$\begin{aligned}
M_1 &= \{w_i u_i : \ 1 \leq i \leq s\} = \{w_1 u_1, w_2 u_2, w_3 u_2, \ldots, w_s u_s\} \\
M_2 &= \{w_i u_{i+1} : 1 \leq i \leq s\} = \{w_1 u_2, w_2 u_3, w_3 u_4, \ldots, w_{s-1} u_s, w_s u_1\} \\
M_3 &= \{w_i u_{i+2} : 1 \leq i \leq s\} = \{w_1 u_3, w_2 u_4, w_3 u_5, \ldots, w_{s-2} u_s, w_{s-1} u_1, w_s u_2\}.
\end{aligned}$$

Next, we partition the set U into two subsets U_1 and U_2 where

$$\begin{aligned}
U_1 &= \{u_i : \ \ i \text{ is odd and } 1 \leq i \leq s\} \\
U_2 &= \{u_i : \ \ i \text{ is even and } 2 \leq i \leq s\}.
\end{aligned}$$

For $i = 1, 2$, let F_i be the subgraph of $K_{s,t}$ induced by

$$U_i \cup \{w_{s+1}, w_{s+2}, \ldots, w_t\}.$$

Define the edge coloring $c : E(K_{s,t}) \to \{1, 2, 3\}$ by

$$c(e) = \begin{cases} 3 & \text{if } e \in M_3 \cup E(F_1) \\ 2 & \text{if } e \in M_2 \\ 1 & \text{otherwise.} \end{cases}$$

Let $x, y \in V(K_{s,t})$ and $x \neq y$. If x and y belong to different partite sets, then (x, y) is a path. If $x, y \in V(H)$, then there is a properly colored $x - y$ path of length 2. For example, if $x = u_1$ and $y = u_2$, then (u_1, w_1, u_2) is a properly colored path of length 2 whose edges are colored by 1 and 2; while if $x = w_1$ and $y = w_2$, then (w_1, u_2, w_2) is a properly colored path of length 2 whose edges are colored 2 and 1. Thus, we may assume that $x, y \in W$, say $x = w_i$ and $y = w_j$ where $1 \leq i < j$ and $j \geq s + 1$. There are two subcases.

Subcase 2.1. $1 \leq i \leq s$. Here, we may assume that $x = w_1$ and $y = w_{s+1}$. Then (w_1, u_1, w_{s+1}) is a properly colored path of length 2 whose edges are colored 1 and 3.

Subcase 2.2. $i \geq s + 1$. Here, we may assume that $x = w_{s+1}$ and $y = w_{s+2}$. Then $(w_{s+1}, u_1, w_2, u_3, w_{s+2})$ is a 3-rainbow path of length 4. The edges of this path are colored $3, 1, 2, 3$ in this order.

In each case, there is a 3-rainbow $x-y$ path in $K_{s,t}$. Therefore, $\mathrm{rc}_3(K_{s,t}) = 3 = \min\left\{\left\lceil \sqrt[s]{t}\,\right\rceil,\ 3\right\}$ when $t \geq 2^s+1$. ■

If $t \geq 3^s+1$ and $s \geq 2$, then $\left\lceil \sqrt[s]{t}\,\right\rceil \geq 4$ and so $\mathrm{rc}(K_{s,t}) = 4$. Since $\mathrm{pc}(K_{s,t}) = 2$ and $\mathrm{rc}_3(K_{s,t}) = 3$ in such a case, it follows that

$$\mathrm{pc}(K_{s,t}) < \mathrm{rc}_3(K_{s,t}) < \mathrm{rc}(K_{s,t})$$

when $t \geq 3^s+1$ and $s \geq 2$. We next determine the 3-rainbow connection number of complete k-partite graphs where $k \geq 3$. First, we recall the rainbow connection numbers of these graphs (see [12]).

Theorem 2.2.3 *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$\mathrm{rc}(G) = \begin{cases} 1 & \text{if } n_k = 1 \\ 2 & \text{if } n_k \geq 2 \text{ and } s \geq t \\ \min\{\left\lceil \sqrt[s]{t}\,\right\rceil,\ 3\} & \text{if } s < t. \end{cases}$$

With the aid of Theorem 2.2.3, we now show for complete k-partite graphs G with $k \geq 3$ that $\mathrm{rc}(G) = \mathrm{rc}_3(G)$.

Theorem 2.2.4 *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$\mathrm{rc}_3(G) = \begin{cases} 1 & \text{if } n_k = 1 \\ 2 & \text{if } n_k \geq 2 \text{ and } s \geq t \\ \min\{\left\lceil \sqrt[s]{t}\,\right\rceil,\ 3\} & \text{if } s < t. \end{cases}$$

Proof. If $n_k = 1$, then G is complete and $\mathrm{rc}_3(G) = 1$. Suppose next that $n_k \geq 2$ and $s \geq t$. By Theorem 2.2.3, $\mathrm{rc}(G) = 2$ and so $\mathrm{rc}_3(G) = 2$ by (2.1). Hence, we may assume that $s < t$. Let W be the partite set with $t = n_k$ vertices and $U = \{u_1, u_2, \ldots, u_s\}$ be the union of the remaining partite sets of G. Since G is not complete, $\mathrm{rc}_3(G) \geq 2$. By Theorem 2.2.3, $\mathrm{rc}(G) = \min\{\left\lceil \sqrt[s]{t}\,\right\rceil,\ 3\}$ and so $\mathrm{rc}_3(G) \leq \min\{\left\lceil \sqrt[s]{t}\,\right\rceil,\ 3\}$. Assume, to the contrary, that $\mathrm{rc}_3(G) < \min\{\left\lceil \sqrt[s]{t}\,\right\rceil,\ 3\}$. Since $2 \leq \mathrm{rc}_3(G) < \min\{\left\lceil \sqrt[s]{t}\,\right\rceil,\ 3\} \leq 3$, it follows that $\mathrm{rc}_3(G) = 2$. Thus, $\left\lceil \sqrt[s]{t}\,\right\rceil \geq 3$ and so $\sqrt[s]{t} > 2$. Hence, $t > 2^s$. Let $c : E(G) \to \{1,2\}$ be a 3-rainbow 2-coloring of G. Thus, we can associate a color code $\mathrm{code}(w) = (a_1, a_2, \ldots, a_s)$ with each vertex $w \in W$, where $a_i = c(u_i w) \in \{1,2\}$ for

$1 \leq i \leq s$. Since the maximum number of distinct color codes is 2^s and $t > 2^s$, there exist two distinct vertices $w', w'' \in W$ for which $\text{code}(w') = \text{code}(w'')$. Hence, the two edges of each $w' - w''$ path of length 2 are colored the same and so there is no 3-rainbow $w' - w''$ path in $K_{s,t}$, producing a contradiction. Thus, $\text{rc}_3(K_{s,t}) = 3 = \min\{\lceil \sqrt[s]{t}\ \rceil,\ 3\}$ in this case. ■

In [8] it was shown that $\text{rc}_2(G) \leq 3$ if G is 2-connected. This suggests the following question: if G is a 2-connected graph, is $\text{rc}_3(G) \leq 3$?

2.3 On the 3-Rainbow Connection Number of a Tree

We saw in Proposition 2.1.4 that if G is a nontrivial connected graph and T is a spanning tree of G, then $\text{rc}_3(G) \leq \text{rc}_3(T)$ and so

$$\text{rc}_3(G) \leq \min\{\text{rc}_3(T) : \quad T \text{ is a spanning subgraph of } G\}. \tag{2.2}$$

This suggests investigating $\text{rc}_3(T)$ for trees T. Since $\text{rc}_3(T) \leq m$ for a tree T of size m, we begin by characterizing those trees T of size m for which $\text{rc}_3(T) = m$. This can only occur if the diameter of T is at most 3. We will see later that this inequality can be made strict and that the gap between $\text{rc}_3(G)$ and $\text{rc}_3(T)$ can be made arbitrarily large.

Proposition 2.3.1 *Let T be a nontrivial tree of size m. Then $\text{rc}_3(T) = m$ if and only if T is a star or a double star.*

Proof. If T is a star or a double star, then $\text{diam}(T) = 2$ or $\text{diam}(T) = 3$. Let e and f be any two edges of T.

Suppose that e and f are adjacent, say $e = uv$ and $f = vw$ for $u, v, w \in V(T)$. Then (u, v, w) is the only $u - w$ path of length 2. Hence, any 3-rainbow coloring must assign different colors to $e = uv$ and $f = vw$.

If e and f are not adjacent, say $e = uv$ and $f = wx$, where then (u, v, w, x) is the only $u - x$ path of length 3. So e and f must be colored differently by any 3-rainbow coloring. In any case, e and f must be colored differently. Since e and f were chosen arbitrarily, each edge in T must have a distinct color. Thus, $\text{rc}_3(T) = m$.

For the converse, suppose T is not a star or double star. Thus, $\text{diam}(T) \geq 4$. It suffices to show that there is a 3-rainbow coloring of T in which two edges are colored the same. Let $e = uv$ and $f = wx$ be edges, where u and x are leaves and $d(u, x) \geq 4$. We define an edge coloring c on T in which we color all edges of T distinctly with the exception that $c(e) = c(f)$. The unique path from u to x is a rainbow path, since the distance between u and x is at least 4. The path between any other two vertices of T has distinctly colored edges and hence is a rainbow path. Hence c is a 3-rainbow coloring of T using $m-1$ colors. ∎

By Proposition 2.3.1, if T is a star, then $\mathrm{rc}_3(T) = \Delta(T)$; while if T is a double star with two vertices of degree $\Delta(T)$ or T is a path of order at least 4, then $\mathrm{rc}_3(T) = 2\Delta(T) - 1$. In fact, more can be said, as we show next. For integers $a, b \geq 2$, let $S_{a,b}$ denote the double star whose central vertices have degrees a and b. Thus, $S_{a,b}$ has order $a+b$ and size $a+b-1$. The following result gives the 3-rainbow connection number of every tree.

Theorem 2.3.2 *If T is a tree of diameter 3 or more, then*

$$\mathrm{rc}_3(T) = \max\{a+b-1 : \ S_{a,b} \subseteq T\}.$$

Proof. Let $k = \max\{a+b-1 : \ S_{a,b} \subseteq T\}$ and let $S_{a,b} \subseteq T$ such that $a+b-1 = k$. We may assume that $2 \leq a \leq b$. Since any 3-rainbow coloring of T must assign distinct colors to the $k = a+b-1$ edges of $S_{a,b}$, it follows that $\mathrm{rc}_3(T) \geq k$.

Next, we show that $\mathrm{rc}_3(T) \leq k$. Let u and v be the two central vertices of $S_{a,b}$ where $\deg_T u = a$ and $\deg_T(v) = b$. Root the tree T at vertex v. For each integer i with $0 \leq i \leq e(v)$, let $V_i = \{w \in V(T) : d(v,w) = i\}$. For each integer h with $1 \leq h \leq e(v)$, let $T_h = T[\cup_{i=0}^{h} V_i]$ be the subtree of T induced by the set $\cup_{i=0}^{h} V_i$ of vertices whose distance from v is at most h. We proceed by induction on h to show that every T_h, $1 \leq h \leq e(v)$, has a 3-rainbow k-coloring.

For $h = 1$, let $c_1 : E(T_1) \to [k]$ of the edge coloring of T_1 that assigns the b distinct colors $1, 2, \ldots, b$ to the b edges incident with v. Then c_1 is a 3-rainbow k-coloring of T_1. For $h = 2$, we define an edge coloring $c_2 : E(T_2) \to \{1, 2, \ldots, k\}$ such that $c_2(e) = c_1(e)$ for each $e \in E(T_1)$. It remains to define $c_2(e)$ for each uncolored edge e of T_2. Each such uncolored edge e is incident with a vertex in V_1 and a vertex in V_2. Let $v_1 \in V_1$. Since $\deg_{T_2}(v_1) + \deg_{T_2}(v) \leq a+b$, there are at least $(a+b-1) - b = a-1$ colors in $[k]$ that have not been used to color any edge of T_1 incident with v. Since there are at most $a-1$ uncolored edges incident with v_1, there are $a-1 \geq 1$ colors in $[k]$ that are available for these edges in T_2. By assigning $\deg_T(v_1) - 1$ of these $a-1$ colors to the uncolored edges of T_2 incident with v_1, a 3-rainbow k-coloring c_2 of T_2 is produced.

Assume by induction that for every integer $h2 \leq h < e(v)$, a 3-rainbow k-coloring $c_h : E(T_h) \to [k]$ of T_h has been defined. Next, we define a 3-rainbow coloring $c_{h+1} : E(T_{h+1}) \to [k]$ of T_{h+1}. First, let $c_{h+1}(e) = c_h(e)$ for each $e \in E(T_h)$. The only uncolored edges of T_{h+1} are those that join a vertex of V_h and a vertex of V_{h+1}. Let $v_{h+1} \in$

V_{h+1}. Then v_{h+1} is adjacent to a unique vertex $v_h \in V_h$ and v_h is adjacent to a unique vertex $v_{h-1} \in V_{h-1}$. Suppose that $\deg_{T_{h+1}}(v_h) = \deg_T(v_h) = a'$ and $\deg_{T_{h+1}}(v_{h-1}) = \deg_T(v_{h-1}) = b'$. Then $a' + b' \leq a + b$. The edges incident with v_{h-1} are colored with b' colors in $[k]$. Hence, there are at least $k - b' = (a + b - 1) - b' \geq (a' + b' - 1) - b' = a' - 1$ colors in $[k]$ that have not been used to color any edge of T_{h+1} incident with v_h. There are $a' - 1$ uncolored edges incident with v_h in T_{h+1} and there are $a' - 1$ colors in $[k]$ that are available for these edges in T_{h+1}. By assigning $a' - 1$ of these $k - b'$ colors to the $a' - 1$ uncolored edges of T_{h+1} incident with v_h, a 3-rainbow k-coloring of T_{h+1} is produced.

Thus, the subtree T_h of T has a 3-rainbow k-coloring for every integer h with $1 \leq h \leq e(v)$. In particular, $T_{e(v)} = T$ has a 3-rainbow k-coloring and so $\mathrm{rc}_3(T) \leq k$. Therefore, $\mathrm{rc}_3(T) = k$. ■

Notice that we may restate Theorem 2.3.2 to say that

$$\mathrm{rc}_3(T) = \max\{\deg(u) + \deg(v) - 1 : uv \in E(T)\}.$$

In [1] it is shown that for any tree T, $\mathrm{pc}(T) = \Delta(T)$, that is, the 2-rainbow connection number of T is the maximum size of a star in T. By Theorem 2.3.2, the 3-rainbow connection number of a tree T that is not a star is the maximum size of a double star in T. The following is a consequence of Theorem 2.3.2.

Corollary 2.3.3 *If T is a tree of order at least 4, then*

$$\Delta(T) \leq \mathrm{rc}_3(T) \leq 2\Delta(T) - 1.$$

Furthermore,

(i) $\mathrm{rc}_3(T) = \Delta(T)$ *if and only if T is a star and*

(ii) $\mathrm{rc}_3(T) = 2\Delta(T) - 1$ *if and only if T contains two adjacent vertices of degree $\Delta(T)$.*

By (2.1), if G is a nontrivial connected graph of order at least 3 with $\mathrm{pc}(G) = a$, $\mathrm{rc}_3(G) = b$ and $\mathrm{rc}(G) = c$, then $2 \leq a \leq b \leq c$. Next, we determine all triples (a, b, c) of integers with $2 \leq a \leq b \leq c$ that are realizable as the proper connection number, 3-rainbow connection number and rainbow connection number of some tree of order at least 3. Recall that if T is a tree of size m, then $\mathrm{rc}(T) = m$ and $\mathrm{pc}(T) = \Delta(T)$ (see [1, 12]).

Theorem 2.3.4 *Let (a, b, c) be a triple of integers with $2 \leq a \leq b \leq c$. Then there exists a tree T with $\mathrm{pc}(T) = a$, $\mathrm{rc}_3(T) = b$ and $\mathrm{rc}(T) = c$ if and only if (i) $a = b = c$ or (ii) $2 \leq a < b \leq \min\{2a - 1, c\}$.*

Proof. First, suppose that T is a tree of size m and maximum degree Δ with $\mathrm{pc}(T) = a$, $\mathrm{rc}_3(T) = b$ and $\mathrm{rc}(T) = c$. Then $\mathrm{pc}(T) = a = \Delta$ and $\mathrm{rc}(T) = c = m$. If T is a star, then $\mathrm{pc}(T) = \mathrm{rc}_3(T) = \mathrm{rc}(T) = m$ and so (i) holds. Thus, we may assume that T is not a star. Thus, $\mathrm{rc}_3(T) > \Delta = a$ by Corollary 2.3.3. Since $b \le m$ and $b \le 2\Delta - 1 = 2a - 1$, again by Corollary 2.3.3, it follows that $\Delta < b \le \min\{2\Delta - 1, m\} = \{2a - 1, c\}$.

Next, we verify the converse. For (i), the star of size a has the desired property. For (ii), let (a, b, c) be a triple of positive integers with $2 \le a < b \le \min\{2a - 1, c\}$. Let $S_{a,b-a+1}$ be the double star of size b whose central vertices u and v have degrees a and $b-a+1$, respectively. Suppose that u is adjacent to the $a-1$ end-vertices $u_1, u_2, \ldots, u_{a-1}$ and v is adjacent to the $b-a$ end-vertices $v_1, v_2, \ldots, v_{b-a}$. Since $b \le 2a-1$, it follows that $a \ge b-a+1$ and so $\Delta(S_{a,b-a+1}) = a$. If $c = b$, let $T = S_{a,b-a+1}$ and so $\mathrm{pc}(T) = \Delta(T) = a$ and $\mathrm{rc}_3(T) = \mathrm{rc}(T) = b$. If $b < c$, let T be the tree of size c obtained from the double star $S_{a,b-a+1}$ and the path $P_{c-b} = (w_1, w_2, \ldots, w_{c-b})$ by adding the edge v_1w_1. Then $\mathrm{pc}(T) = \Delta(T) = a$ and $\mathrm{rc}(T) = c$. It remains to show that $\mathrm{rc}_3(T) = b$. Since the double star $S_{a,b-a+1}$ of size b is a subtree of T, it follows that $\mathrm{rc}_3(T) \ge b$. Next, define a 3-rainbow b-coloring c of T by (1) $c(uu_i) = i$ for $1 \le i \le a - 1$, $c(uv) = a$, $c(vv_j) = a + j$ for $1 \le j \le b - a$ and (2) assigning the colors $1, 2, 3$ to the edges of the path $(v_1, w_1, w_2, \ldots w_{c-b})$ in the order $1, 2, 3, 1, 2, 3, \ldots$. Therefore, $\mathrm{rc}_3(T) \le b$ and so $\mathrm{rc}_3(T) = b$. ■

By Corollary 2.3.3, if T is a nontrivial tree of size m with $\mathrm{rc}_3(T) = k$ and $\Delta(T) = \Delta$, then $\Delta \le k \le \min\{2\Delta - 1, m\}$. Since $\mathrm{rc}_3(T) = \Delta(T)$ if and only if T is a star, it follows that there is no tree T of size m such that $\Delta(T) = \mathrm{rc}_3(T) < m$. Furthermore, we saw that if T is a tree of size m, then $\mathrm{rc}(T) = m$ and $\mathrm{pc}(T) = \Delta(T)$. Therefore, the following is a consequence of Theorem 2.3.4.

Corollary 2.3.5 *Let (Δ, k, m) be a triple of integers with $\Delta \le k \le m$. Then there exists a tree T of size m and $\Delta(T) = \Delta$ such that $\mathrm{rc}_3(T) = k$ if and only if (i) $\Delta = k = m$ or (ii) $2 \le \Delta < k \le \min\{2\Delta - 1, m\}$.*

The following is a consequence of (2.2) and Corollary 2.3.3.

Corollary 2.3.6 *If G is a connected graph of order 4 or more that is not a tree, then*

$$\mathrm{rc}_3(G) \le 2\Delta(G) - 1.$$

The upper bound for $\mathrm{rc}_3(G)$ provided in Corollary 2.3.6 is sharp for connected graphs G that are not trees. In fact, more can be said.

Proposition 2.3.7 *For each pair (Δ, k) of integers with $3 \leq k \leq 2\Delta - 1$, there is a connected graph G that is not a tree such that $\Delta(G) = \Delta$ and $\mathrm{rc}_3(G) = k$.*

Proof. We begin with the double star $H = S_{\lfloor k/2 \rfloor+1, \lceil k/2 \rceil}$ of size k and the complete graph K_Δ of order Δ. Then the graph G is obtained from H and K_Δ by identifying an end-vertex x of H and a vertex of K_Δ and labeling the identified vertex by x. Suppose that the central vertices of H are u and v where $\deg_H u = \lfloor k/2 \rfloor + 1$ and $\deg_H v = \lceil k/2 \rceil$. Since $k \leq 2\Delta - 1$, it follows that $\deg_G u = \lfloor k/2 \rfloor + 1 \leq 1 + k/2 \leq \Delta + 1/2$ and $\deg_G v = \lceil k/2 \rceil \leq 1 + k/2 \leq \Delta + 1/2$. Thus, $\deg_G u \leq \Delta$ and $\deg_G v \leq \Delta$. Furthermore, $\deg_G x = \Delta$ and so $\Delta(G) = \Delta$.

It remains to show that $\mathrm{rc}_3(G) = k$. Any 3-rainbow coloring of G must assign distinct colors to the k edges in H and so $\mathrm{rc}_3(G) \geq k$. Next, we show that there is a 3-rainbow k-coloring of G. Assume, without loss of generality that x is adjacent to u. The edge coloring that assigns the colors $1, 2, \ldots, k$ to the k edges in H such that the color 1 is assigned to a pendant edge incident with v and each edge of K_Δ is a 3-rainbow k-coloring of G. Therefore, $\mathrm{rc}_3(G) \leq k$ and $\mathrm{rc}_3(G) = k$. ■

If G is a connected graph of order at least 4 for which $\mathrm{rc}_3(G) = 2\Delta(G) - 1$, then G must contain a spanning tree containing two adjacent vertices of degree $\Delta(G)$. However, the graph G of Figure 2.2 shows that even if *every* spanning tree of a graph G contains adjacent vertices of degree $\Delta(G)$, then it need not occur that $\mathrm{rc}_3(G) = 2\Delta(G) - 1$. In this case, $\Delta(G) = 3$ and $\mathrm{rc}_3(G) = 4$. A 3-rainbow 4-coloring is shown in Figure 2.2. On the other hand, for each such spanning tree T of G, it follows that $\mathrm{rc}_3(T) = 2\Delta(G) - 1$.

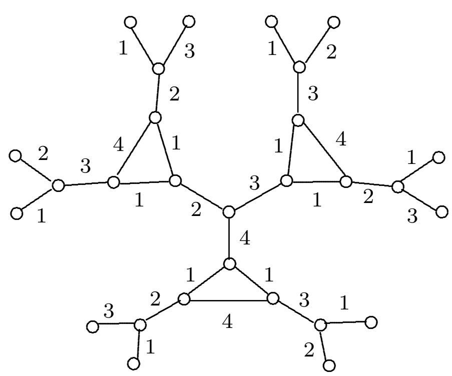


Figure 2.2: A graph G with $\mathrm{rc}_3(G) < 2\Delta(G) - 1$ for which every spanning tree contains adjacent vertices of degree $\Delta(G)$

2.4 The 3-Rainbow Connection Number and Chromatic Index

Recall that an edge coloring of a nonempty graph G is *proper* if no two adjacent edges of G are colored the same. The minimum number of colors required of a proper edge coloring is the *chromatic index* of G, denoted by $\chi'(G)$. The following result is due to König [26].

Theorem 2.4.1 (König's Theorem) *If G is a nonempty bipartite graph, then*

$$\chi'(G) = \Delta(G).$$

In particular, if T is a nontrivial tree, then $\chi'(T) = \Delta(T)$.

We discuss some facts involving the chromatic index $\chi'(G)$ and the 3-rainbow connection number $\mathrm{rc}_3(G)$ of a connected graph G. There are graphs G_1, G_2 and G_3 such that

$$\text{(i) } \chi'(G_1) = \mathrm{rc}_3(G_1), \text{ (ii) } \chi'(G_2) > \mathrm{rc}_3(G_2) \text{ and (iii) } \chi'(G_3) < \mathrm{rc}_3(G_3).$$

In fact, each of (i), (ii) and (iii) holds for infinite classes of graphs.

(i) If $G_1 = K_{1,t}$ is a star, then $\chi'(G_1) = \mathrm{rc}_3(G_1) = t$.

(ii) Let G_2 be the graph obtained from $\Delta \geq 4$ pairwise disjoint paths (u_i, v_i) of length 1 for $1 \leq i \leq \Delta$ by (1) adding two new vertices u and v and (2) joining u to each vertex u_i and v to each vertex v_i for $1 \leq i \leq \Delta$. Thus, $\Delta(G_2) = \Delta \geq 3$. Since G_2 is a bipartite graph, $\chi'(G_2) = \Delta(G_2) = \Delta$. An edge coloring $c : E(G_2) \to \{1, 2, 3\}$ of G_2 is defined by

$$c(e) = \begin{cases} 1 & \text{if } e = uu_i \text{ for } 1 \leq i \leq \Delta - 1 \text{ or } e = vv_\Delta \\ 2 & \text{if } e = u_iv_i \text{ for } 1 \leq i \leq \Delta \\ 3 & \text{if } e = vv_i \text{ for } 1 \leq i \leq \Delta - 1 \text{ or } e = uu_\Delta \end{cases}$$

is a 3-rainbow coloring, it follows that $\mathrm{rc}_3(G_2) \leq 3$. However, since every 3-rainbow coloring of G_2 must use at least three colors, it follows that $\mathrm{rc}_3(G_2) = 3$. If $\Delta \geq 4$, then $\chi'(G_2) = \Delta > 3 = \mathrm{rc}_3(G)$ and the value of $\chi'(G_2) - \mathrm{rc}_3(G_2) = \Delta - 3$ can be arbitrarily large.

(iii) If $G_3 = P_n$ for integer $n \geq 4$, then $\chi'(G_3) = 2$ and $\mathrm{rc}_3(G_3) = 3$. Therefore, $\mathrm{rc}_3(G_3) > \chi'(G_3)$.

We have seen that if T is a tree of order at least 4, then

$$\Delta(T) \leq \mathrm{rc}_3(T) \leq 2\Delta(T) - 1.$$

Thus, if T is a tree of order at least 4, then

$$\chi'(T) \leq \mathrm{rc}_3(T) \leq 2\chi'(T) - 1. \tag{2.3}$$

Hence, the example of stars and paths presented are special cases of (2.3). Therefore, in the case of trees, we ask the following question:

> *For which pairs a, b of integers with $2 \leq a \leq b$, is there a tree T such that $\chi'(T) = a$ and $\mathrm{rc}_3(T) = b$?*

Not only is there a complete answer to this question but more can be said. An immediate consequence of (2.3) and Corollary 2.3.5 is the following.

Corollary 2.4.2 *Let (a, b, m) be a triple of positive integers. There exists a tree T of size m for which $\chi'(T) = a$ and $\mathrm{rc}_3(T) = b$ if and only if $a \leq b \leq m$ such that*

$$(i)\ a = b = m \text{ or } (ii)\ 2 \leq a < b \leq \min\{2a - 1, m\}.$$

Chapter 3

On k-Rainbow Colorings of Graphs

In the preceeding chapter, we considered 3-rainbow colorings of graphs. Now, we study k-rainbow colorings and k-rainbow chromatic numbers of graphs for integers $k \geq 3$ in general and extend several results obtained in Chapter 2. In this chapter, we determine the k-rainbow connection numbers of graphs belonging to some familiar classes of graphs, including complete multipartite graphs, cycles, prisms and umbrella graphs. In addition, we present some preliminary observations and results on k-rainbow connection numbers of graphs.

3.1 Introduction

Let's first review a primary concept in this dissertation. For an integer $k \geq 2$, a path P in G is a *k-rainbow path* if every subpath of P having length k or less is a rainbow path. An edge coloring c is a *k-rainbow coloring* of a connected graph G if every pair of distinct vertices of G are connected by a k-rainbow path in G. If j colors are used to produce a k-rainbow coloring of G, then c is referred to as a k-rainbow j-edge coloring or simply a k-rainbow j-coloring. The minimum j for which G has a k-rainbow j-coloring is called the k-rainbow connection number of G, denoted by $\mathrm{rc}_k(G)$. For every nontrivial connected graph G whose longest paths have length ℓ,

$$\mathrm{pc}(G) = \mathrm{rc}_2(G) \leq \mathrm{rc}_3(G) \leq \cdots \leq \mathrm{rc}_\ell(G) = \mathrm{rc}(G). \tag{3.1}$$

First, we state some observations concerning k-rainbow connection numbers in general.

Observation 3.1.1 *If H is a connected spanning subgraph of a nontrivial connected graph G, then $\mathrm{rc}_k(G) \le \mathrm{rc}_k(H)$ for each integer $k \ge 3$. In particular, if T is a spanning tree of G, then $\mathrm{rc}_k(G) \le \mathrm{rc}_k(T)$.*

The *length* of a path is the number of edges in the path.

Lemma 3.1.2 *Let G be a connected graph of diameter $d \ge 2$ whose longest paths have length ℓ.*

(a) If $2 \le k \le d$, then $\mathrm{rc}_k(G) \ge k$.

(b) If $d+1 \le k \le \ell$, then $\mathrm{rc}_k(G) \ge d$.

Proof. Let u and v be two antipodal vertices of G such that $d(u,v) = d$. If $2 \le k \le d$, then every k-rainbow coloring of G must assign at least k distinct colors to the edges of any k-rainbow $u - v$ path in G. Hence, $\mathrm{rc}_k(G) \ge k$ and so (a) holds. If $d+1 \le k \le \ell$, then every k-rainbow coloring of G must assign at least d distinct colors to the edges of any k-rainbow $u - v$ path in G. Hence, $\mathrm{rc}_k(G) \ge d$ and so (b) holds. ■

In fact, the length of a longest path in a graph G is called the *detour number* of G and is denoted by $\tau(G)$.

3.2 k-Rainbow Colorings of Complete Multipartite Graphs

In this section, we determine the k-rainbow connection numbers of complete multipartite graphs. We saw in Chapter 2 that if s and t are integers with $2 \le s \le t$, then

$$\mathrm{rc}_3(K_{s,t}) = \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 3\right\} \text{ and } \mathrm{rc}(K_{s,t}) = \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 4\right\}.$$

Consequently, we have the following corollary by (3.1).

Corollary 3.2.1 *Let s and t be integers with $2 \le s \le t$ and let ℓ be the length of a longest path in $K_{s,t}$. If k is an integer with $4 \le k \le \ell$, then*

$$\min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 3\right\} \le \mathrm{rc}_k(K_{s,t}) \le \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 4\right\}.$$

We show, in fact, for every such integer k in Corollary 3.2.1 that $\mathrm{rc}_k(K_{s,t})$ attains the upper bound in Corollary 3.2.1.

Theorem 3.2.2 *Let s and t be integers with $2 \le s \le t$ and let ℓ be the length of a longest path in $K_{s,t}$. If k is an integer with $4 \le k \le \ell$, then*

$$\mathrm{rc}_k(K_{s,t}) = \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 4\right\}.$$

Proof. By (3.1) and Corollary 3.2.1, it suffices to show that

$$\mathrm{rc}_4(K_{s,t}) = \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 4\right\}.$$

Observe that $\left\lceil \sqrt[s]{t}\, \right\rceil \geq 2$ for all integers s and t with $2 \leq s \leq t$. Let U and W be the partite sets of $K_{s,t}$, where $|U| = s$ and $|W| = t$. First, suppose that $\left\lceil \sqrt[s]{t}\, \right\rceil = a$, where $a \in \{2, 3\}$. Since $a = \mathrm{rc}_3(K_{s,t}) \leq \mathrm{rc}_4(K_{s,t}) \leq \mathrm{rc}(K_{s,t}) = a$ by Corollary 3.2.1, it follows that $\mathrm{rc}_4(K_{s,t}) = \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 4\right\}$ if $\left\lceil \sqrt[s]{t}\, \right\rceil \in \{2, 3\}$.

We now assume that $\left\lceil \sqrt[s]{t}\, \right\rceil \geq 4$. Then $t \geq 3^s + 1$. Let $U = \{u_1, u_2, \ldots, u_s\}$. Since $\mathrm{rc}_4(K_{s,t}) \leq 4$ by Corollary 3.2.1, it remains to show that $\mathrm{rc}_4(K_{s,t}) \geq 4$. Assume, to the contrary, that there exists a 4-rainbow 3-coloring of $K_{s,t}$. Corresponding to this 4-rainbow 3-coloring of $K_{s,t}$, there is a color code, denoted by $\mathrm{code}(w)$, assigned to each vertex $w \in W$, consisting of an ordered s-tuple $(a_1, a_2, \ldots, a_s)$, where $a_i = c(u_i w) \in \{1, 2, 3\}$ for $1 \leq i \leq s$. Since $t > 3^s$, there exist two distinct vertices w' and w'' of W such that $\mathrm{code}(w') = \mathrm{code}(w'')$. Every $w' - w''$ path P in $K_{s,t}$ has even length. Since $\mathrm{code}(w') = \mathrm{code}(w'')$, the path P cannot have length 2 as the colors of the two edges of every $w' - w''$ path of length 2 are the same. However, if the path P has length 4 or more, then each subpath of length 4 in P must repeat a color as this edge coloring uses only three colors. Hence, there is no 4-rainbow $w' - w''$ path in $K_{s,t}$, a contradiction. Thus, $\mathrm{rc}_4(K_{s,t}) \geq 4$ and so $\mathrm{rc}_4(K_{s,t}) = 4$.

It then follows by (3.1) and Corollary 3.2.1 that $\mathrm{rc}_k(K_{s,t}) = \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 4\right\}$ for all integers k with $4 \leq k \leq \ell$. ■

In summary, we have the following.

Corollary 3.2.3 *Let s and t be integers with $2 \leq s \leq t$ and let ℓ be the length of a longest path in $K_{s,t}$. If k is an integer with $2 \leq k \leq \ell$, then*

$$\mathrm{rc}_k(K_{s,t}) = \begin{cases} 2 & \text{if } k = 2 \\ \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 3\right\} & \text{if } k = 3 \\ \min\left\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 4\right\} & \text{if } 4 \leq k \leq \ell. \end{cases}$$

Next, we consider complete multipartite graphs. Let $G = K_{n_1, n_2, \ldots, n_p}$ be a complete p-partite graph, where $p \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_p$ such that $s = \sum_{i=1}^{p-1} n_i$ and $t = n_p$. We have seen that

$$\mathrm{rc}_3(G) = \mathrm{rc}(G) = \begin{cases} 1 & \text{if } n_p = 1 \\ 2 & \text{if } n_p \geq 2 \text{ and } s \geq t \\ \min\{\left\lceil \sqrt[s]{t}\, \right\rceil,\ 3\} & \text{if } s < t. \end{cases} \tag{3.2}$$

Thus, the following corollary is a consequence of (3.1) and (3.2).

Corollary 3.2.4 *Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph, where $p \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_p$ such that $s = \sum_{i=1}^{p-1} n_i$ and $t = n_p$. If k is an integer with $3 \leq k \leq \ell$ where ℓ is the length of a longest path in G, then*

$$\mathrm{rc}_k(G) = \begin{cases} 1 & \text{if } n_p = 1 \\ 2 & \text{if } n_p \geq 2 \text{ and } s \geq t \\ \min\{\lceil \sqrt[s]{t}\ \rceil,\ 3\} & \text{if } s < t. \end{cases}$$

3.3 k-Rainbow Colorings of Paths, Cycles and Wheels

We now determine the k-rainbow connection numbers of graphs belonging to some other familiar classes of graphs, namely paths, cycles and wheels. We begin with an observation.

Observation 3.3.1 *For integers k and n with $2 \leq k \leq n-1$, $\mathrm{rc}_k(P_n) = k$.*

We now turn to k-rainbow connection numbers of cycles.

Theorem 3.3.2 *For integers k and n with $3 \leq k \leq n-1$ and $n \geq 5$,*

$$\mathrm{rc}_k(C_n) = \min\{\lceil n/2 \rceil,\ k\}.$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $e_i = v_i v_{i+1}$ for $1 \leq i \leq n$. The diameter of C_n is $\mathrm{diam}(C_n) = \lfloor n/2 \rfloor$. We consider two cases, according to whether $\lceil n/2 \rceil \leq k$ or $\lceil n/2 \rceil > k$.

Case 1. $\lceil n/2 \rceil \leq k$. Here, we show that $\mathrm{rc}_k(C_n) = \min\{\lceil n/2 \rceil,\ k\} = \lceil n/2 \rceil$. First, define an edge coloring c of C_n by

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \lceil n/2 \rceil \\ i - \lceil n/2 \rceil & \text{if } \lceil n/2 \rceil + 1 \leq i \leq n. \end{cases}$$

Thus, the *color sequence of the edges of* C_n with respect to c is

$$S_c = (c(e_1), c(e_2), \ldots, c(e_n)) = (1, 2, \ldots, \lceil n/2 \rceil, 1, 2, \ldots, \lfloor n/2 \rfloor).$$

Note that every subsequence of length at most $\lfloor n/2 \rfloor$ in S_c has distinct terms. For two vertices v_i and v_j of C_n where $1 \leq i < j \leq n$, there are exactly two $v_i - v_j$ paths P and Q in C_n. We may assume that $|E(P)| \leq |E(Q)|$. Then P is a rainbow path of length at most $\lfloor n/2 \rfloor$. Since $\lfloor n/2 \rfloor \leq \lceil n/2 \rceil \leq k$, it follows that P is a k-rainbow $v_i - v_j$ path. Therefore, c is a k-rainbow $\lceil n/2 \rceil$-coloring of C_n and so $\mathrm{rc}_k(C_n) \leq \lceil n/2 \rceil$.

Next, we show that $\mathrm{rc}_k(C_n) \geq \lceil n/2 \rceil$. Assume, to the contrary, that C_n has a k-rainbow coloring c^* using the colors $1, 2, \ldots, \lceil n/2 \rceil - 1$. Of the two $v_1 - v_{\lfloor n/2 \rfloor + 1}$ paths on C_n, one has length $\lceil n/2 \rceil$ and the other $\lfloor n/2 \rfloor$. If n is even, then neither path is a k-rainbow path; while if n is odd, the path of length $\lceil n/2 \rceil$ cannot be a k-rainbow path. In this case, let $n = 2t + 1$ for some integer $t \geq 2$ and consider the path $(v_1, v_2, \ldots, v_{t+1})$ of length $t = \lfloor n/2 \rfloor$, which is necessarily a k-rainbow path. Hence, we may assume that $c^*(v_i v_{i+1}) = i$ for $1 \leq i \leq t$. The path $(v_2, v_3, \ldots, v_{t+s})$ also has length t and so is a k-rainbow path, implying that $c^*(v_{t+1} v_{t+2}) = 1$. Continuing in this manner, we see that

$$c^*(v_i v_{i+1}) = \begin{cases} i & \text{if } 1 \leq i \leq t \\ t - i & \text{if } t + 1 \leq i \leq 2t \\ 1 & \text{If } i = 2t + 1. \end{cases}$$

The $v_{2t+1} - v_t$ path $P = (v_{2t+1}, v_1, v_2, \ldots, v_t)$ of length t has $c^*(v_{2t+1} v_1) = v^*(v_1 v_2) = 1$, implying that neither P nor the $v_{2t+1} - v_t$ path of length $t + 1$ is a k-rainbow path, producing a contradiction.

Case 2. $\lceil n/2 \rceil > k$. We show that $\mathrm{rc}_k(C_n) = \min\{\lceil n/2 \rceil, k\} = k$. First, by Observations 3.1.1 and 3.3.1, $\mathrm{rc}_k(C_n) \leq \mathrm{rc}_k(P_n) \leq k$. Next, we show that $\mathrm{rc}_k(C_n) \geq k$. Assume, to the contrary, that C_n has a k-rainbow coloring c^* using the $k-1$ colors $1, 2, \ldots, k-1$. There are two $v_1 - v_{\lceil n/2 \rceil + 1}$ paths in C_n, one of which has length $\lfloor n/2 \rfloor$ and the other has length $\lceil n/2 \rceil$. Since the coloring c^* only uses $k - 1$ distinct colors, neither path can be a k-rainbow $v_1 - v_{t+1}$ path in C_n, producing a contradiction. ■

Let $W_n = C_n \vee K_1$ be the wheel of order $n + 1 \geq 5$ in which the length of a longest path is n. We have seen in Chapter 2 that $\mathrm{rc}_2(W_n) = 2$ for all $n \geq 4$ and

$$\begin{aligned} \mathrm{rc}_2(W_n) &= \mathrm{pc}(W_n) = 2, \\ \mathrm{rc}(W_n) &= \mathrm{rc}_3(W_n) = \begin{cases} 2 & \text{if } 4 \leq n \leq 6 \\ 3 & \text{if } n \geq 7. \end{cases} \end{aligned}$$

By (3.1), if $n \geq 4$, then $\mathrm{rc}_i(W_n) = \mathrm{rc}_{i-1}(W_n)$ for each integer i with $2 < i < n$.

Corollary 3.3.3 *If n and k are integers with $3 \leq k \leq n-1$ and $n \geq 4$, then*

$$\mathrm{rc}_k(W_n) = \mathrm{rc}_3(W_n) = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } 4 \leq n \leq 6 \\ 3 & \text{if } n \geq 7. \end{cases}$$

3.4 k-Rainbow Colorings of Prisms

The *prism* $C_n \,\square\, K_2$ is the Cartesian product of the cycle C_n of order $n \geq 3$ and K_2. Since $C_n \,\square\, K_2$ is a Hamiltonian graph of order $2n$, the length of a longest path in G is $2n-1$.

The following is a consequence of Observation 3.1.1 and Theorem 3.3.2.

Corollary 3.4.1 *If G is a Hamiltonian graph of order $n \geq 3$, then $\mathrm{rc}_k(G) \leq k$ for each integer k with $2 \leq k \leq n-1$.*

Proof. Since G is a Hamiltonian graph of order n, the length of a longest paths in G is $n-1$. Let C be a Hamiltonian cycle of G. It then follows by Observation 3.1.1 and Theorem 3.3.2 that $\mathrm{rc}_k(G) \leq \mathrm{rc}_k(C) \leq k$ for each integer k with $2 \leq k \leq n-1$. ■

First, we present a lemma.

Lemma 3.4.2 *For each integer $n \geq 3$,* $\mathrm{diam}(C_n \,\square\, K_2) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Proof. For an integer $n \geq 3$, let $G = C_n \,\square\, K_2$ be obtained from two copies C and C' of the n-cycle C_n, where $C = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ and $C' = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$, by adding the n edges $u_i v_i$ for $1 \leq i \leq n$. Since $d(u_i, v_{\lfloor \frac{n}{2} \rfloor + i}) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ for $1 \leq i \leq n$ and $d(x, y) \leq \left\lfloor \frac{n}{2} \right\rfloor$ if $\{x, y\} \neq \{u_i, v_{\lfloor \frac{n}{2} \rfloor + i}\}$ for any i with $1 \leq i \leq n$, where the subscripts are expressed as integers modulo n, it follows that $\mathrm{diam}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. ■

We are now prepared to present the following result.

Theorem 3.4.3 *For integers k and n with $2 \leq k \leq 2n-1$ and $n \geq 3$,*

$$\mathrm{rc}_k(C_n \,\square\, K_2) = \min\left\{k,\ \left\lfloor \frac{n}{2} \right\rfloor + 1\right\}.$$

Proof. For an integer $n \geq 3$, let $G = C_n \,\square\, K_2$ be obtained from two copies C and C' of the n-cycle C_n, where

$$C = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1) \text{ and } C' = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1),$$

by adding the n edges u_iv_i for $1 \leq i \leq n$. Let $d = \lfloor \frac{n}{2} \rfloor + 1$. By Lemma 3.4.2, $\text{diam}(G) = d$. Since G is a Hamiltonian graph of order $2n$, the length of a longest path in G is $2n-1$.

First, suppose that $2 \leq k \leq d$. We show that $\text{rc}_k(G) = k$. Since $k \leq d$, it follows by Lemma 3.1.2 that $\text{rc}_k(G) \geq k$. Since G is Hamiltonian, $\text{rc}_k(G) \leq k$ by Corollary 3.4.1. Therefore, $\text{rc}_k(G) = k$.

Next, suppose that $d+1 \leq k \leq 2n-1$. We show that $\text{rc}_k(G) = d$. By Lemma 3.1.2, $\text{rc}_k(G) \geq d$. Thus, it remains to show that G has a k-rainbow d-coloring. We consider two cases, according to whether n is even or n is odd.

Case 1. *$n \geq 4$ is even.* By Theorem 3.3.2, $\text{rc}_k(C_n) = d-1$. Let $c_1 : E(C) \to [d-1]$ be a k-rainbow $(d-1)$-coloring of the n-cycle $C = (u_1, u_2, \ldots, u_n, u_1)$ and let $c_2 : E(C') \to [d-1]$ be a k-rainbow $(d-1)$-coloring of n-cycle $C' = (v_1, v_2, \ldots, v_n, v_1)$. Define an edge coloring c of G by $c(e) = c_1(e)$ if $e \in E(C)$, $c(e) = c_2(e)$ if $e \in E(C')$ and $c(u_iv_i) = d$ for $1 \leq i \leq n$. We show that c is a k-rainbow d-coloring of G. Let x and y be two nonadjacent vertices of G. First, suppose that $x, y \in V(C)$ or $x, y \in V(C')$, say the former. Since c_1 is a k-rainbow coloring of C, there is a k-rainbow $x-y$ path in C and in G as well. Next, suppose that $x = u_i$ and $y = v_j$, where $1 \leq i, j \leq n$. Let P be a k-rainbow $u_i - u_j$ path in C. Since no edge of P is colored d and $c(u_jv_j) = d$, it follows that P followed by the edge u_jv_j produces a k-rainbow $u_i - v_j$ path in G. Hence, c is a k-rainbow d-coloring of G.

Case 2. *$n \geq 3$ is odd.* In this case, $d = \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$. Define an edge coloring $c : E(G) \to [d]$ of G by

$$c(u_iu_{i+1}) = \begin{cases} i & \text{if } 1 \leq i \leq d \\ i-d & \text{if } d+1 \leq i \leq n \end{cases}$$

$$c(v_iv_{i+1}) = \begin{cases} i & \text{if } 1 \leq i \leq d-1 \\ (i+1)-d & \text{if } d \leq i \leq n \end{cases}$$

$$c(u_iv_i) = d \ \text{ if } 1 \leq i \leq n.$$

Hence, the *color sequences of the edges of C and C'* with respect to c are

$$(c(u_1u_2), c(u_2u_3), \ldots, c(u_nu_1)) = (1, 2, \ldots, d, 1, 2, \ldots, d-1) \tag{3.3}$$

$$(c(v_1v_2), c(v_2v_3), \ldots, c(v_nv_1)) = (1, 2, \ldots, d-1, 1, 2, 3, \ldots, d). \tag{3.4}$$

This coloring is illustrated in Figure 3.1 for $n = 9$. Since every subsequence of length at most $d-1$ in the sequences in (3.3) and (3.4) has distinct terms, the restriction of this coloring c to each of the n-cycles C and C' is a k-rainbow coloring. Next, we show that c is a k-rainbow d-coloring of G.

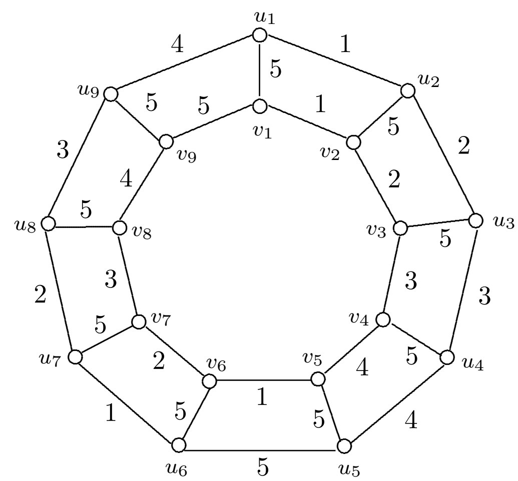


Figure 3.1: A 5-rainbow coloring c of $C_9 \,\square\, K_2$ in Case 2

Let x and y be two nonadjacent vertices of G. First, suppose that $x, y \in V(C)$ or $x, y \in V(C')$, say $x, y \in V(C)$. Since the restriction of this coloring c to the n-cycle C is a k-rainbow coloring, it follows that x and y are connected by a k-rainbow $x-y$ path in C and so in G. Hence, we may assume that $x = u_i$ and $y = v_j$, where $1 \le i, j \le n$. Note that $u_d u_{d+1}$ is the only edge colored d on C and $v_n v_1$ is the only edge colored d on C'. Let P be the k-rainbow $u_i - u_j$ path of length at most $d-1$ on C. If $u_d u_{d+1} \notin E(P)$, then no edge on P is colored by d and so P followed by the edge $u_j v_j$ produces a k-rainbow $u_i - v_j$ in G. Thus, we may assume $u_d u_{d+1} \in E(P)$. Let Q be a k-rainbow $v_i - v_j$ path of length at most $d-1$ in C'. Then Q is the path in C' that corresponds to the $u_i - u_j$ path P in C and so Q contains the edge $v_d v_{d+1}$. Since the length of Q is at most $d-1$, it follows that Q does not contain the edge $v_n v_1$ and so no edge on Q is colored d. Thus, the edge $u_i v_i$ followed by Q produces a k-rainbow $u_i - v_j$ in G. Hence, c is a k-rainbow d-coloring of G. ∎

3.5 k-Rainbow Colorings of Umbrella Graphs

An *umbrella graph* is constructed from a wheel W by attaching a path at the *central vertex* of W. More precisely, let $W_a = C_a \vee K_1$ where $C_a = (v_1, v_2, \ldots, v_a, v_1)$ and $V(K_1) = \{v\}$ and let $P_b = (u_1, u_2, \ldots, u_b)$. The vertex v is referred to as the *central* *vertex* of W_a. The umbrella graph $U(a,b)$ is constructed from W_a and P_b by adding the edge vu_1. Then the order of $U(a,b)$ is $a+b+1$, the diameter of $U(a,b)$ is $b+1$ and the length of a longest path in $U(a,b)$ is $a+b$. We now determine $\text{rc}_k(U(a,b))$ for all positive integers a, b, k with $a \geq 3$ and $2 \leq k \leq a+b$, beginning with the case when $k \leq b+1$.

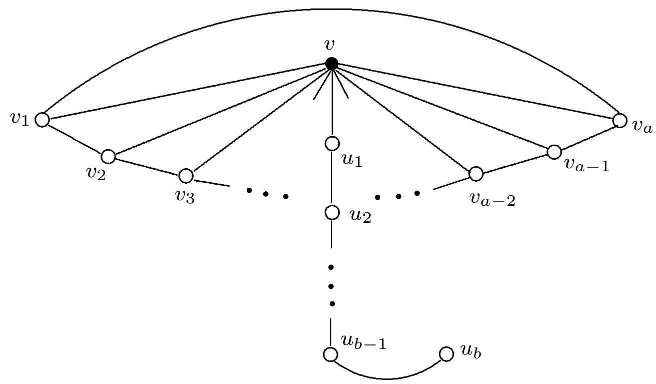


Figure 3.2: The umbrella graph $U(a,b)$

Proposition 3.5.1 *If a, b, k are integers with $a \geq 3$ and $2 \leq k \leq b+1$, then $\text{rc}_k(U(a,b)) = k$.*

Proof. In the umbrella graph $U(a,b)$, let $W_a = C_a \vee K_1$, where $C_a = (v_1, v_2, \ldots, v_a, v_1)$ and $V(K_1) = \{v\}$, and let $P_b = (u_1, u_2, \ldots, u_b)$. Then $U(a,b)$ is constructed from W_a and P_b by adding the edge vu_1. Furthermore, let $P_{b+1} = (v, u_1, u_2, \ldots, u_b)$ be the path of order $b+1$ in $U(a,b)$. First, we show that $\text{rc}_k(U(a,b)) \leq k$. The umbrella graph $U(a,b)$ has a Hamiltonian path P of order $a+b+1$ in $U(a,b)$. It then follows by Observations 3.1.1 and 3.3.1 that $\text{rc}_k(U(a,b)) \leq \text{rc}_k(P) = k$. Next, we show that $\text{rc}_k(U(a,b)) \geq k$. Since $b+1 \geq k$, it follows by Observation 3.3.1 that $\text{rc}_k(P_{b+1}) = k$. For every two vertices x and x' of P_{b+1}, there is a unique $x - x'$ path in $U(a,b)$. Therefore, every k-rainbow coloring of $U(a,b)$ must assign at least k distinct colors to the edges of P_{b+1} and so $\text{rc}_k(U(a,b)) \geq k$. Therefore, $\text{rc}_k(U(a,b)) = k$ when $b+1 \geq k$. ■

Next, we determine $\text{rc}_k(U(a,b))$ when $a \geq 3$ and $k \geq b+2 \geq 3$. First, we make a useful observation. Let $U(a,b)$ be the umbrella graph constructed from the wheel $W_a = C_a \vee K_1$, where $C_a = (v_1, v_2, \ldots, v_a, v_1)$ and $V(K_1) = \{v\}$, and the path $P_b = (u_1, u_2, \ldots, u_b)$ of order b by adding the edge vu_1. For each $i = 1, 2, \ldots, a$, let $Q_i = (v_i, v, u_1, u_2, \ldots, u_b)$ be the $v_i - u_b$ path of length $b+1$ in $U(a,b)$.

(P) Since $d(v_i, u_b) = b+1$ for $1 \leq i \leq a$, it follows that Q_i is a $v_i - u_b$ geodesic of length $b+1$ in $U(a,b)$. Hence, if $k \geq b+2$, then every k-rainbow coloring of

$U(a,b)$ must assign $b+1$ distinct colors to the $b+1$ edges of Q_i for some i and so $\text{rc}_k(U(a,b)) \geq b+1$.

We begin by determining $\text{rc}_k(U(a,b))$ for small umbrella graphs $U(a,b)$ where $a \in \{3,4,5\}$.

Theorem 3.5.2 *If a,b,k are integers with $a \in \{3,4,5\}$ and $k \geq b+2 \geq 3$, then*

$$rc_k(U(a,b)) = \begin{cases} b+1 & \text{if } (a,b,k) \neq (5,1,k) \\ b+2 & \text{if } (a,b,k) = (5,1,k). \end{cases}$$

Proof. Let $U(a,b)$ be the umbrella graph constructed from the wheel $W_a = C_a \vee K_1$, where $C_a = (v_1, v_2, \ldots, v_a, v_1)$ and $V(K_1) = \{v\}$, and the path $P_b = (u_1, u_2, \ldots, u_b)$ of order b by adding the edge vu_1. Let $P_{b+1} = (v, u_1, u_2, \ldots, u_b)$ be the $v - u_b$ path of order $b+1$ in $U(a,b)$. For each $i = 1, 2, \ldots, a$, let $Q_i = (v_i, v, u_1, u_2, \ldots, u_b)$ be the $v_i - u_b$ geodesic of length $b+1$ in $U(a,b)$.

First, suppose that $a \in \{3,4\}$ and $k \geq b+2$. We show that $\text{rc}_k(U(a,b)) = b+1$. By (P), it suffices to show that $U(a,b)$ has a k-rainbow $(b+1)$-coloring. Let $c : E(U(a,b)) \to [b+1]$ be the edge coloring of $U(a,b)$ defined by

- $\star$ $c(vu_1) = 1$ and $c(u_iu_{i+1}) = i+1$ for $2 \leq i \leq b-1$;
- $\star$ $c(vv_i) = b+1$ for $1 \leq i \leq a$;
- $\star$ $c(v_iv_{i+1}) = b$ if i is even and $c(v_iv_{i+1}) = b-1$ if i is odd for $1 \leq i \leq a$ and $v_{a+1} = v_1$.

This edge coloring c is illustrated in Figure 3.3 for $U(3,b)$ and $U(4,b)$, respectively. Next, we show that every two nonadjacent vertices of $U(a,b)$ are connected by a k-rainbow path in $U(a,b)$. For each $i = 1, 2, \ldots, a$, the path $Q_i = (v_i, v, u_1, u_2, \ldots, u_b)$ is a rainbow path of length $b+1$. Furthermore, every two nonadjacent vertices on C_a (where then $a = 4$) are connected by a rainbow path of length 2. Thus, c is a k-rainbow $(b+1)$-coloring and so $\text{rc}_k(U(a,b)) = b+1$.

Next, suppose that $a = 5$ and $k \geq b+2 \geq 3$. We show that

$$\text{rc}_k(U(5,1)) = \text{rc}_k(U(5,2)) = 3 \text{ and } \text{rc}_k(U(5,b)) = b+1 \text{ for } b \geq 3.$$

First, suppose that $b \in \{1,2\}$. To show that $\text{rc}_k(U(5,b)) \leq 3$, let the edge coloring $c : E(U(a,b)) \to [3]$ be defined by

- $\star$ $c(v_1v_2) = c(v_3v_4) = 1$, $c(v_2v_3) = c(v_4v_5) = 2$ and $c(v_5v_1) = 3$;

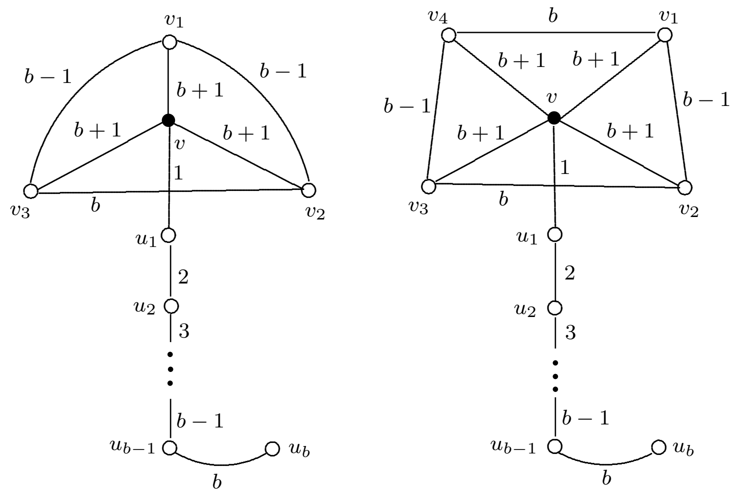


Figure 3.3: The k-rainbow $(b+1)$-colorings of $U(3,b)$ and $U(4,b)$ when $k \geq b+2$

- $\star$ $c(vv_i) = 3$ for $1 \leq i \leq 5$;
- $\star$ $c(vu_1) = 1$ and $c(u_1u_2) = 2$ if $b = 2$.

This coloring is shown in Figure 3.4 for $U(5,1)$ and $U(5,2)$. Observe that (a) for each $i = 1, 2, \ldots, 5$, the path $Q_i = (v_i, v, u_1, u_2)$ is a rainbow $v_i - u_2$ path and (b) every two nonadjacent vertices of C_5 are connected by a rainbow path of length 2. Thus, c is a k-rainbow 3-coloring of $U(5,b)$ for $b = 1, 2$ and all $k \geq 3$.

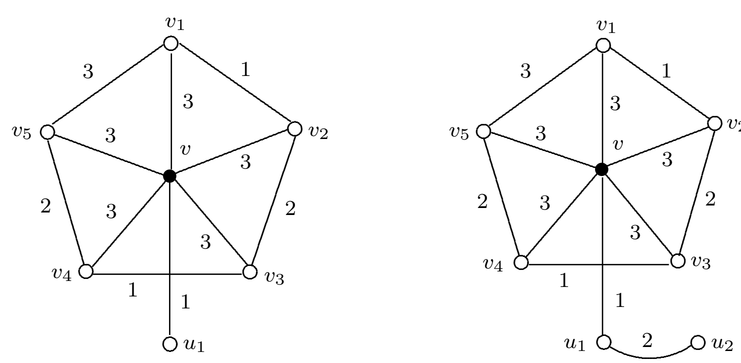


Figure 3.4: A k-rainbow 3-coloring of $U(5,b)$ for $b = 1, 2$ and $k \geq b+2$

Next, we show that $\text{rc}_k(U(5,b)) \geq 3$ for $b \in \{1,2\}$ and $k \geq 3$. If $b = 2$, then $\text{rc}_k(U(5,2)) \geq 3$ by (P). Thus, it remains to show that $\text{rc}_k(U(5,1)) \geq 3$. Assume, to the contrary, that $\text{rc}_k(U(5,1)) = 2$ for some integer $k \geq 3$. Let there be a k-rainbow 2-coloring c of $U(5,1)$. Since $\chi'(C_5) = 3$, there are two adjacent edges of C_5 that are colored the same, say $c(v_5v_1) = c(v_1v_2) = 1$. Since $k \geq 3$ and c only uses two colors, every k-rainbow $v_5 - v_2$ path must have length 2. This implies that (v_5, v, v_2) must be a rainbow path and so $\{c(v_5v), c(vv_2)\} = \{1, 2\}$, say $c(v_5v) = 1$ and $c(vv_2) = 2$. Similarly, both the $v_2 - u_1$ path (v_2, v, u_1) and $v_5 - u_1$ path (v_5, v, u_1) must be rainbow. If $c(vu_1) = 1$, then there is no k-rainbow $v_5 - u_1$ path; while if $c(vu_1) = 2$, then there is no k-rainbow $v_2 - u_1$ path, which is impossible. Thus, $\text{rc}_k(U(5, 1)) \geq 3$. Therefore, $\text{rc}_k(U(5, 1)) = \text{rc}_k(U(5, 2)) = 3$ for $k \geq 3$.

Next, suppose that $b \geq 3$ and we show that $\text{rc}_k(U(5, b) = b+1$. By (P), it suffices to show that a k-rainbow $(b + 1)$-coloring of $U(5, b)$ exists. Let $c : E(U(a, b)) \to [b + 1]$ be the edge coloring defined by

- ⋆ $c(vu_1) = 1$ and $c(u_iu_{i+1}) = i + 1$ for $2 \leq i \leq b - 1$;
- ⋆ $c(vv_i) = b + 1$ for $1 \leq i \leq 5$.
- ⋆ $c(v_1v_2) = c(v_3v_4) = b$, $c(v_2v_3) = c(v_5v_1) = b - 1$, and $c(v_4v_5) = b - 2$.

This coloring is shown in Figure 3.5. Then $Q_i = (v_i, v, u_1, \ldots, u_b)$ is a rainbow $v_i - u_b$ path for $i = 1, 2, \ldots, 5$ and every two nonadjacent vertices of C_5 are connected by a rainbow path of length 2. Hence, c is a k-rainbow $(b + 1)$-coloring and so $\text{rc}_k(U(5, b) = b + 1$.

Therefore, $\text{rc}_k(U(5, b)) = b + 2$ if $(a, b, k) = (5, 1, k)$ and $\text{rc}_k(U(5, b)) = b + 1$ otherwise. ■

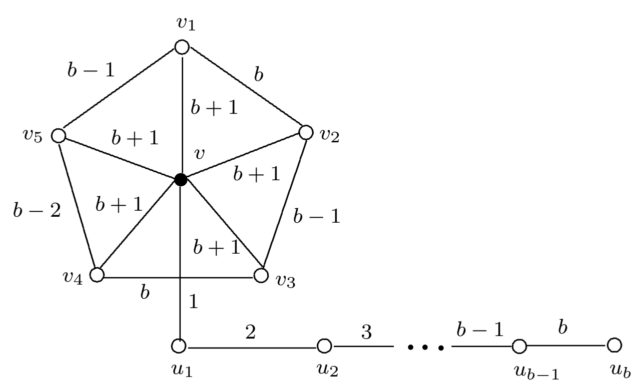


Figure 3.5: A k-rainbow 3-coloring of $U(5, b)$ for $b \geq 3$ for $k \geq b + 2$

Theorem 3.5.3 *If a, b, k are integers with $a \geq 6$ and $k \geq b + 2 \geq 3$, then*

$$rc_k(U(a, b)) = \begin{cases} b + 1 & \text{if } \lceil a/2 \rceil \leq b + 1 \\ b + 2 & \text{if } \lceil a/2 \rceil \geq b + 2. \end{cases}$$

Proof. In the umbrella graph $U(a,b)$, let $W_a = C_a \vee K_1$, where $C_a = (v_1, v_2, \ldots, v_a, v_1)$ and $V(K_1) = \{v\}$, and let $P_b = (u_1, u_2, \ldots, u_b)$. Then $U(a,b)$ is constructed from W_a and P_b by adding the edge vu_1. Furthermore, let $P_{b+1} = (v, u_1, u_2, \ldots, u_b)$ be the path of order $b+1$ in $U(a,b)$. We consider two cases, according to whether $\lceil a/2 \rceil \leq b+1$ or $\lceil a/2 \rceil \geq b+2$.

Case 1. $\lceil a/2 \rceil \leq b+1$. We show that $\mathrm{rc}_k(U(a,b)) = b+1$. Since $k \geq b+2$, it suffices to show that $U(a,b)$ has a k-rainbow $(b+1)$-coloring by (P). Since $\lceil a/2 \rceil \leq b+1 < k$, it follows by Theorem 3.3.2 that $\mathrm{rc}_k(C_a) = \min\{\lceil a/2 \rceil,\ k\} = \lceil a/2 \rceil \leq b+1$. Hence, there is a k-rainbow coloring $c_0 : E(C_a) \to [b+1]$ of C_a. Let c_1 be a *rainbow* coloring of $P_{b+1} = (v, u_1, u_2, \ldots, u_b)$ using the colors $1, 2, \ldots, b$. Now, define a k-rainbow coloring $c : E(U(a,b)) \to [b+1]$ of $U(a,b)$ by

$$c(e) = \begin{cases} c_0(e) & \text{if } e \in E(C_a) \\ c_1(e) & \text{if } e \in E(P_{b+1}) \\ b+1 & \text{if } e = vv_i \text{ for } 1 \leq i \leq a. \end{cases}$$

For $b = 3$, this k-rainbow 4-coloring is shown in Figure 3.6 for $U(6,3)$ and $U(7,3)$, respectively. Thus, $\mathrm{rc}_k(U(a,b)) \leq b+1$.

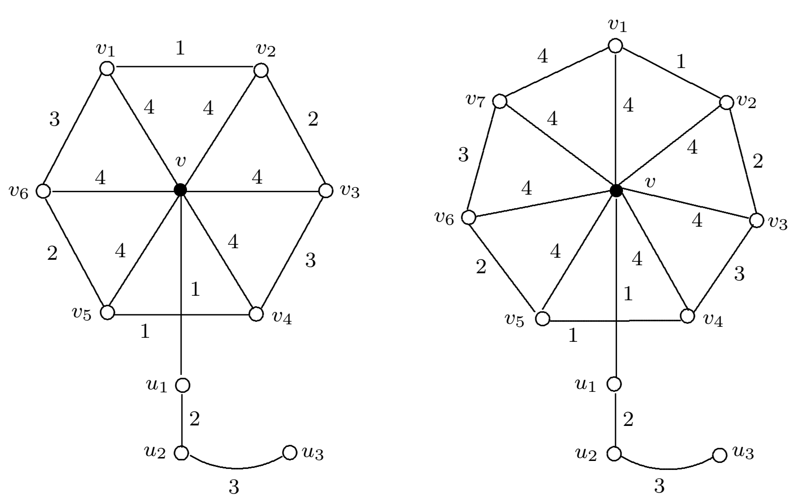


Figure 3.6: A k-rainbow 4-coloring for each of $U(6,3)$ and $U(7,3)$

Case 2.. $\lceil a/2 \rceil \geq b+2$. We show that $\mathrm{rc}_k(U(a,b)) = b+2$. First, we show that $\mathrm{rc}_k(U(a,b)) \geq b+2$. By (P), $\mathrm{rc}_k(U(a,b)) \geq b+1$. Assume, to the contrary, that $\mathrm{rc}_k(U(a,b)) = b+1$. Then there is a k-rainbow coloring $c : E(U(a,b)) \to [b+1]$ of

$U(a,b)$. Since all edges of $P_{b+1} = (v, u_1, u_2, \ldots, u_b)$ must be colored differently, we may assume that $c(vu_1) = 1$ and $c(u_iu_{i+1}) = i+1$ for $1 \le i \le b-1$. For each j with $1 \le j \le a$, there is only one $v_j - u_b$ path of length $b+1$, namely $(v_j, v, u_1, u_2, \ldots, u_b)$. Since $b+1 \le k$, it follows that $c(vv_j) = b+1$ for $1 \le j \le a$. This implies that every two vertices x and y of C_a must be connected by a k-rainbow $x-y$ path on C_a. However then, the restriction of c to the cycle C_a is a k-rainbow coloring of C_a using at most $b+1$ colors. On the other hand, since $\lceil a/2 \rceil > b+1$ and $k > b+1$, it follows that $\mathrm{rc}_k(C_a) = \min\{\lceil a/2 \rceil, k\} > b+1$, which is impossible. Hence, $\mathrm{rc}_k(U(a,b)) \ge b+2$. Next, we show that $U(a,b)$ has a k-rainbow coloring using the colors $1, 2, \ldots, b+2$. Define an edge coloring $c : E(U(a,b)) \to [b+2]$ of $U(a,b)$ by

- $\star$ $c(vu_1) = 1$ and $c(u_iu_{i+1}) = i+1$ for $1 \le i \le b-1$;
- $\star$ $c(vv_i) = b+1$ if i is odd and $c(vv_i) = b+2$ if i is even for $1 \le i \le a$;
- $\star$ $c(v_iv_{i+1}) = b+1$ if i is even and $c(v_iv_{i+1}) = 1$ if i is odd for $1 \le i \le a$ and $v_{a+1} = v_1$.

For $b = 2$, this k-rainbow 4-coloring is shown in Figure 3.7 for $U(8,2)$ and $U(9,2)$. It remains to show that c is a k-rainbow coloring of $U(a,b)$.

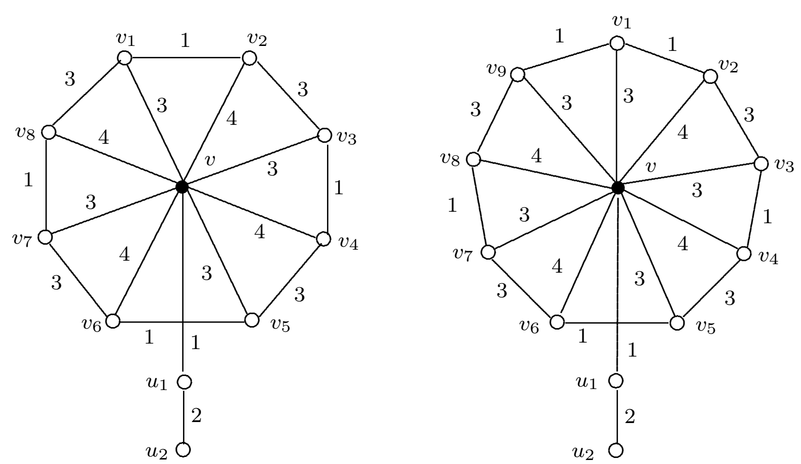


Figure 3.7: A k-rainbow 4-coloring for each of $U(8,2)$ and $U(9,2)$

Let x and y be two nonadjacent vertices of $U(a,b)$. Since, for each integer i with $1 \le i \le a$, the path $(v_i, v, u_1, u_2, \ldots, u_b)$ is a rainbow path in $U(a,b)$, it follows that if at least one of x and y does not belong to the cycle C_a, then x and y are connected by a rainbow $x-y$ path. Thus, we may assume that $x, y \in V(C_a)$. Then $x = v_i$ and $y = v_j$ for some $i, j \in \{1, 2, \ldots, a\}$ such that $|i-j| \ge 2$. If i and j are of opposite parity, where $\{i, j\} \neq \{1, a\}$, then (v_i, v, v_j) is a rainbow path. Thus, we may assume that i and j are of the same parity. Hence, either $c(v_i v_{i-1}) = 1$ or $c(v_i v_{i+1}) = 1$, where the subscript of a vertex is expressed as an integer modulo a. Suppose, without loss of generality, that $c(v_i v_{i-1}) = 1$. This implies that $i - 1$ and j are of opposite parity and so $c(vv_{i-1}) \neq c(vv_j)$. Thus, the path (v_i, v_{i-1}, v, v_j) is a k-rainbow path. Therefore, c is a k-rainbow coloring of $U(a, b)$ and so $\mathrm{rc}_k(U(a, b)) = b + 2$. ■

Proposition 3.5.1 and Theorems 3.5.2 and 3.5.3 give rise to the following result, which provides the exact value of the k-rainbow connection number of each umbrella graph.

Corollary 3.5.4 *Let a, b, k be positive integers with $a \geq 3$ and $2 \leq k \leq a + b$.*

(i) If $k \leq b + 1$, then $rc_k(U(a, b)) = k$.

(ii) If $k \geq b + 2$, then $rc_k(U(a, b)) \in \{b + 1, b + 2\}$. Furthermore, $rc_k(U(a, b)) = b + 1$ if and only if $(a, b) = (5, 1)$ or $3 \leq \lceil a/2 \rceil \leq b + 1$.

Chapter 4

Bounds for k-Rainbow Connection Numbers

In this chapter, we establish several sharp upper bounds for k-rainbow connection numbers of graphs in terms of their maximum degree and order. In order to do this, we first study k-rainbow colorings of trees, unicyclic graphs and connected graphs of cycle rank 2. Several realization results involving k-rainbow connection numbers and other graphical parameters are also presented.

4.1 k-Rainbow Colorings of Trees

Recall for integers $a, b \geq 2$ that $S_{a,b}$ denotes the double star whose central vertices have degrees a and b. Thus, $S_{a,b}$ has order $a+b$ and size $a+b-1$. It was shown in Chapter 2 that if T is a tree of diameter 3 or more, then

$$\mathrm{rc}_3(T) = \max\{a+b-1 : \ S_{a,b} \subseteq T\}.$$

We now extend this to k-rainbow colorings of trees for other values of k. First, we present a lower bound for the k-rainbow connection number of a tree. The size of a tree T' is denoted by $m(T')$.

Proposition 4.1.1 *If T is a tree of diameter at least $k \geq 2$ for some integer k, then*

$$\mathrm{rc}_k(T) \geq \max\{m(T') : \ T' \textit{ is a subtree of } T \textit{ with } \mathrm{diam}(T') = k\}.$$

Proof. Suppose that T is a tree of diameter at least $k \geq 2$. Let

$$m_k = \max\{m(T') : \ T' \text{ is a subtree of } T \text{ with } \mathrm{diam}(T') = k\}.$$

Let there be given a k-rainbow coloring c of T and let T^* be a subtree of T with $\text{diam}(T^*) = k$ such that $m(T^*) = m_k$. For every two edges $e, f \in E(T^*)$, there is an $x-y$ path P in T^* such that $e, f \in E(P)$. The path P is the only $x-y$ path in T. Since $\text{diam}(T^*) = k$, the length of P is at most k. Thus, $c(e) \neq c(f)$ and so $\text{rc}_k(T) \geq m_k$. ■

The following is an immediate consequence of Proposition 4.1.1.

Corollary 4.1.2 *If T is a tree of order n and diameter $d \geq 2$, then* $\text{rc}_d(T) = n - 1$.

Proof. By Proposition 4.1.1,

$$\begin{aligned} \text{rc}_d(T) &\geq \max\{m(T') :\ T' \text{ is a subtree of } T \text{ with } \text{diam}(T') = d\} \\ &= m(T) = n - 1. \end{aligned}$$

Since $\text{rc}_d(T) \leq n - 1$, it follows that $\text{rc}_d(T) = n - 1$. ■

Next, we show that the lower bound for the k-rainbow connection number of a tree T in Proposition 4.1.1 is, in fact, the value of $\text{rc}_k(T)$ for $2 \leq k \leq 5$. First, we introduce an additional notation. For two disjoint sets U and W of $V(G)$, let $[U, W]$ denote the set of edges joining a vertex of U and a vertex of W.

Theorem 4.1.3 *If T is a tree of diameter at least k for some integer k with $2 \leq k \leq 5$, then*

$$\text{rc}_k(T) = \max\{m(T') :\ T' \textit{ is a subtree of } T \textit{ with } \text{diam}(T') = k\}.$$

Proof. We have seen that the result is true for $k = 2, 3$. Thus, we may assume that $k \geq 4$. Let

$$m_k = \max\{m(T') :\ T' \text{ is a subtree of } T \text{ with } \text{diam}(T') = k\}.$$

By Proposition 4.1.1, it suffices to show that $\text{rc}_k(T) \leq m_k$; that is, we show that T has a k-rainbow coloring using m_k colors. Let v be a peripheral vertex of T. Then v is an end-vertex of T and $e(v) = \text{diam}(T)$. Express the tree T as a rooted tree whose root is v. For each integer i with $0 \leq i \leq \text{diam}(T)$, let

$$V_i = \{w \in V(T) : d(v, w) = i\} = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\},$$

where then $|V_i| = n_i$. For each integer h with $0 \leq h \leq \text{diam}(T)$, let $T_h = T[\cup_{i=0}^{h} V_i]$ be the subtree of T induced by the set $\cup_{i=0}^{h} V_i$ of vertices whose distance from v is at most h.

We proceed by induction to show that every subtree T_h, $0 \leq h \leq \text{diam}(T)$, has a k-rainbow coloring using at most m_k colors. This is true vacuously for T_0. Let h_0 be the largest integer h with $1 \leq h \leq \text{diam}(T)$ such that $\text{diam}(T_{h_0}) \leq k$. [In fact, either $\text{diam}(T_{h_0}) = k - 1$ or $\text{diam}(T_{h_0}) = k$; for otherwise, assume that $\text{diam}(T_{h_0}) \leq k - 2$. Since v is a peripheral vertex of T and $\text{diam}(T) = k$, it follows that

$$\text{diam}(T_{h_0}) + 1 \leq \text{diam}(T_{h_0+1}) \leq \text{diam}(T_{h_0}) + 2 \leq k,$$

which contradicts the defining property of h_0.] Since $\text{diam}(T_{h_0}) \leq k$, it follows that $m(T_{h_0}) \leq m_k$. Thus, for $h = 0, 1, 2, \ldots, h_0$, there is a k-rainbow coloring $c_h : E(T_h) \to [m_k]$ of T_h that assigns distinct colors to distinct edges of T_h. Assume, for an integer h with $h_0 \leq h < \text{diam}(T)$, that there is a k-rainbow coloring $c_h : E(T_h) \to [m_k]$ of T_h. Next, we define a k-rainbow coloring $c_{h+1} : E(T_{h+1}) \to [m_k]$ of T_{h+1}. First, define $c_{h+1}(e) = c_h(e)$ for each $e \in E(T_h)$. At this point, the only uncolored edges of T_{h+1} are those that join a vertex of V_h and a vertex of V_{h+1}, namely the edges in $[V_h, V_{h+1}]$.

Divide the set V_h into n_{h-1} subsets $V_{h,1}, V_{h,2}, \ldots, V_{h,n_{h-1}}$ such that $V_{h,j}$ $(1 \leq j \leq n_{h-1})$ is the set of all children of the vertex $v_{h-1,j} \in V_{h-1}$ in the rooted tree T. Consequently, $V_{h,j} = N(v_{h-1,j}) \cap V_h$ for $1 \leq j \leq n_{h-1}$. Next, divide the set V_{h+1} into n_{h-1} subsets $V_{h+1,1}$, $V_{h+1,2}$, $\ldots$, $V_{h+1,n_{h-1}}$ such that each set $V_{h+1,j}$ $(1 \leq j \leq n_{h-1})$ consists of all children of vertices in $V_{h,j}$ that belong to V_{h+1} (namely, the grandchildren of the vertex $v_{h-1,j}$). Thus, $V_{h+1,j} = N(V_{h,j}) \cap V_{h+1}$. Then

$$[V_h, V_{h+1}] = \bigcup_{j=1}^{n_{h-1}} [V_{h,j}, V_{h+1,j}].$$

For each integer j with $1 \leq j \leq n_{h-1}$ for which $[V_{h,j}, V_{h+1,j}] \neq \emptyset$, we now define a coloring of the edges in $[V_{h,j}, V_{h+1,j}]$ such that the resulting coloring is a k-rainbow m_k-coloring of T. In what follows, we assume that each set $[V_{h,j}, V_{h+1,j}]$ $(1 \leq j \leq n_{h-1})$ under consideration is not empty.

First, suppose that $\text{diam}(T_h) < k$. Since $h \geq h_0$ and h_0 is the largest integer h with $1 \leq h \leq \text{diam}(T)$ such that $\text{diam}(T_h) \leq k$, it follows that $h = h_0$ and so $\text{diam}(T_h) = k-1$. Since $\text{diam}(T_h) = k - 1$, the subtree $T(j)$ of T_{h+1}, where $1 \leq j \leq n_{h-1}$, induced by the set $V(T_h) \cup V_{h+1,j}$ has diameter at most k. We may assume, without loss of generality, that

$$\gamma = |[V_{h,1}, V_{h+1,1}]| \geq |[V_{h,j}, V_{h+1,j}]|$$

for $2 \leq j \leq n_{h-1}$. Thus, the size of $T(1)$ is at most m_k. Let S be the set of the $m(T_h)$ colors used to color the edges of T_h and let $\overline{S} = [m_k] - S$. Then $|\overline{S}| \geq \gamma \geq |[V_{h,j}, V_{h+1,j}]|$ for $1 \leq j \leq n_{h-1}$. If $e_i \in [V_{h,i}, V_{h+1,i}]$ and $e_j \in [V_{h,j}, V_{h+1,j}]$, where $1 \leq i, j \leq n_{h-1}$

and $i \neq j$, then e_i and e_j do not lie on a path of length k or less in T and so e_i and e_j can be colored the same. Thus, assign the distinct colors in $\overline{S}$ to the edges in each set $[V_{h,j}, V_{h+1,j}]$ for $1 \leq j \leq n_{h-1}$, producing a k-rainbow m_k-coloring of T_{h+1}.

Next, suppose that $\text{diam}(T_h) \geq k$. By the induction hypothesis, we have a k-rainbow m_k-coloring of T_h. Suppose that $[V_{h,1}, V_{h+1,1}] \neq \emptyset$. We wish to define a coloring of the edges in $[V_{h,1}, V_{h+1,1}]$ such that the resulting coloring of the subtree induced by the set $V(T_h) \cup V_{h+1,1}$ is a k-rainbow m_k-coloring of this subtree. Let $v_{h-1,1} \in V_{h-1}$ and let $V_{h+1,1}$ denote the set of children of the vertices of $V_{h,1}$. Next, let $E = [\{v_{h-1,1}\}, V_{h,1}] \cup [V_{h,1}, V_{h+1,1}]$. Then the diameter of the subtree $T[E]$ of T is at most 4. Let T' be the subtree of T_{h+1} having maximum size such that $T[E] \subseteq T'$ and $\text{diam}(T') = k$. Next, let T'' be the subtree of T_h having maximum size such that $T' - V_{h+1} \subseteq T''$ and $\text{diam}(T'') = k$. Let $E^* = E(T'') - E(T')$. By construction, if $e \in [V_{h,1}, V_{h+1,1}]$ and $e^* \in E^*$, then e and e^* do not lie on a path of length k or less in T_{h+1} (and therefore in T as well). This is illustrated in Figure 4.1, where the subtree $T[E]$ is indicated by bold lines, the subtree T' by solid thin lines and the subtree T'' by dashed lines.

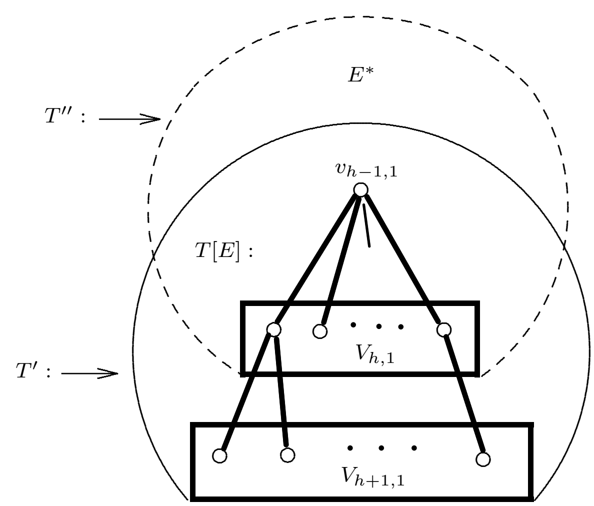


Figure 4.1: The subtrees $T[E]$, T' and T'' of T

By the induction hypothesis, there is a k-rainbow m_k-coloring of the tree T''. Let

$$|E(T'') - E^*| = \alpha,\ |E^*| = \beta \text{ and, as before, } |[V_{h,1}, V_{h+1,1}]| = \gamma.$$

Therefore, $m(T'') = \alpha + \beta$ and $m(T') = \alpha + \gamma$, where then $\alpha + \beta \leq m_k$ and $\alpha + \gamma \leq m_k$. Let S be the set of α colors used to color the edges of $E(T'') - E^*$ and let $\overline{S} = [m_k] - S$. Thus, the color of each edge in E^* belongs to $\overline{S}$. Since $\gamma \leq m_k - \alpha = |\overline{S}|$, there are γ distinct colors in $\overline{S}$ that can be assigned to the γ edges in $[V_{h,1}, V_{h+1,1}]$. As we mentioned earlier, if $e \in [V_{h,1}, V_{h+1,1}]$ and $e^* \in E^*$, then e and e^* do not lie on a path of length k or less in T and so e and e^* can be colored the same. This produces a coloring of the edges of $[V_{h,1}, V_{h+1,1}]$ such that every path P of length at most k such that $E(P) \subseteq E(T_h) \cup [V_{h,1}, V_{h+1,1}]$ is a rainbow path. Repeating this method, we color the edges in $[V_{h,j}, V_{h+1,j}]$ for each j with $2 \leq j \leq n_{h-1}$ such that if Q is a path of length at most k such that $E(Q) \subseteq E(T_h) \cup [V_{h,j}, V_{h+1,j}]$, then Q is a rainbow path. Since $k \leq 5$, it follows that if $e_i \in [V_{h,i}, V_{h+1,i}]$ and $e_j \in [V_{h,j}, V_{h+1,j}]$ where $1 \leq i, j \leq n_{h-1}$ $i \neq j$, then e_i and e_j do not lie on a path of length k or less in T and so e_i and e_j can be colored the same. Thus, every path of length at most k in T_{h+1} is a rainbow path and so this coloring is a k-rainbow m_k-coloring of T_{h+1}.

By the Principle of Mathematical Induction, the subtree T_h of T has a k-rainbow m_k-coloring for every integer h with $0 \leq h \leq \text{diam}(T)$. In particular, $T_{\text{diam}(T)} = T$ has a k-rainbow m_k-coloring and so $\text{rc}_k(T) \leq m_k$. Therefore, $\text{rc}_k(T) = m_k$. ■

As an illustration of Theorem 4.1.4 for $k = 4$ and $k = 5$, consider the tree T of Figure 4.2 whose diameter is 6. The solid vertex v (or root) of T has eccentricity 6.

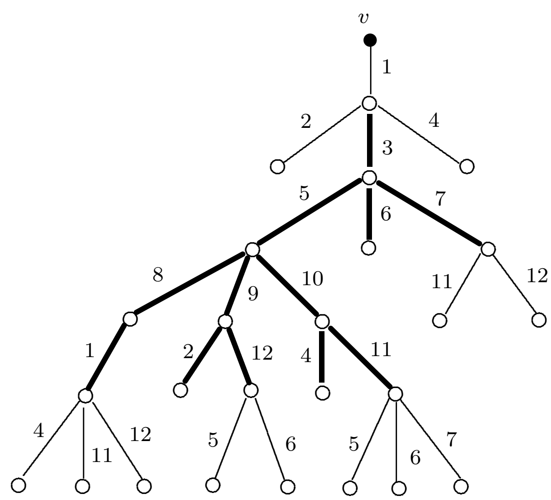


Figure 4.2: A 4-rainbow 12-coloring of T

- ⋆ For $k = 4$, the maximum size of a subtree of diameter 4 in T is $m_4 = 12$. The subtree of T, whose edges are indicated by bold lines, has size 12 and diameter 4. A 4-rainbow 12-coloring of T is shown in Figure 4.2.

- ⋆ For $k = 5$, the maximum size of a subtree of diameter 5 in T is $m_5 = 17$. The subtree of T, whose edges are indicated by bold lines, has size 17 and diameter 5. A 5-rainbow 17-coloring of T is shown in Figure 4.3.

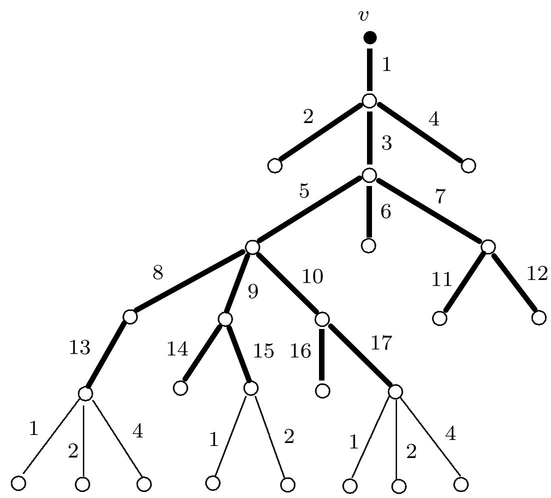


Figure 4.3: A 5-rainbow 17-coloring of T

By Theorem 4.1.4, $\mathrm{rc}_4(T) = 12$ and $\mathrm{rc}_5(T) = 17$.

In fact, we have a more general result. First, we introduce some additional definitions and notation. For two edges e and f of a connected graph, an $e - f$ path has e as its initial edge and f as its terminal edge. Let T' be a proper subtree of a tree T and $e = uv$ an edge of T not in T' such that e is incident with the vertex u of T'. Then $T' + e$ denotes that subtree of T obtained by adding the vertex v and the edge e to T'.

Theorem 4.1.4 *If T is a tree of diameter at least $k \geq 2$ for some integer k, then*

$$\mathrm{rc}_k(T) = \max\{m(T') : \ T' \textit{ is a subtree of } T \textit{ with } \mathrm{diam}(T') = k\}.$$

Proof. We have seen that the result is true for every integer k with $2 \leq k \leq 5$. Thus, we may assume that $k \geq 6$. Let

$$m_k = \max\{m(T') : \ T' \text{ is a subtree of } T \text{ with } \mathrm{diam}(T') = k\}.$$

By Proposition 4.1.1, it suffices to show that T has a k-rainbow coloring using m_k colors.

Let T_0 be a subtree of maximum size m_k in T such that $\mathrm{diam}(T_0) = k$. Color the m_k edges of T_0 with distinct colors from the set $[m_k]$. We consider two cases, according to whether k is even or k is odd.

Case 1. *k is even.* Thus, $k = 2r$ for some integer $r \geq 3$. Therefore, T_0 has a unique central vertex v_0. Let $e_T(v_0) = t$, where then $t \geq r$. We express the tree T as a rooted tree whose root is v_0. For each integer i with $0 \leq i \leq t$, let $V_i = \{v \in V(T) : d(v_0, v) = i\}$. Hence, $V_0 = \{v_0\}$ and for each vertex $v \in V_i$, where $1 \leq i \leq t$, there is a unique $v' \in V_{i-1}$ such that $vv' \in E(T)$. Furthermore, the subtree $T[\cup_{i=0}^{r} V_i]$ of T induced by $\cup_{i=0}^{r} V_i$ is T_0.

If $T = T_0$, then the proof is complete. Thus, we may assume that $E(T) - E(T_0) \neq \emptyset$ and so $\text{diam}(T) \geq k + 1$. Thus, $e_T(v_0) = t > r$.

Let $e_1 \in [V_r, V_{r+1}]$ and let $T_1 = T_0 + e_1$. Hence, $\text{diam}(T_1) = k + 1$. Let T_1' be the subtree of T_1 consisting of e_1 and all those edges of T_1 lying on a path of length k or less having initial edge e_1. We claim that $\text{diam}(T_1') = k$. Suppose that this is not the case. Then there is an $e - f$ path in T_1' of length $k + 1$ for some $e, f \in E(T_1')$. Since $\text{diam}(T_0) = k$, one of e and f must be e_1, say $e = e_1$. However, from the definition of T_1', there is no $e_1 - f$ path of length $k + 1$ in T_1', a contradiction. Thus, as claimed, $\text{diam}(T_1') = k$ and so the size of T_1' is at most m_k. Hence, at most $m_k - 1$ edges of T_1' have been assigned colors from $[m_k]$ and therefore there is at least one color in $[m_k]$ that has not been assigned to any edge of T_1'. Assigning such a color to the edge e_1 results in a k-rainbow coloring of T_1' using the colors of $[m_k]$. If $T = T_1$, then the proof is complete. Hence, we may assume that $E(T) - E(T_1) \neq \emptyset$.

If there is an edge $e' \in (E(T) - E(T_1)) \cap [V_r, V_{r+1}]$, where then e' is incident with a vertex of T_1, and $T_1 + e'$ has diameter $k + 1$, then we denote this edge e' by e_2 and let $T_2 = T_1 + e_2$. Next, if there is an edge $e' \in (E(T) - E(T_1)) \cap [V_r, V_{r+1}]$, again incident with a vertex of T_2, such that $T_2 + e'$ has diameter $k + 1$, then we denote this edge e' by e_3 and let $T_3 = T_2 + e_3$. We continue this procedure until no such edges e' exist, obtaining a sequence $e_1, e_2, \ldots, e_p$ $(p \geq 1)$ of edges and a sequence $T_1, T_2, \ldots, T_p$ of subtrees of diameter $k+1$. Next, if there exists an edge $e' \in (E(T) - E(T_p)) \cap [V_r, V_{r+1}]$ such that e' is incident with a vertex of T_p, then denote this edge e' by e_{p+1} and let $T_{p+1} = T_p + e_{p+1}$. Then $\text{diam}(T_p + e') = k + 2$. We continue this procedure until no edges in $[V_r, V_{r+1}]$ remain, say arriving at the tree $T_{p'}$. If $E(T) - E(T_{p'}) \neq \emptyset$, then let $e' \in (E(T) - E(T_{p'})) \cap [V_{r+1}, V_{r+2}]$. We continue this procedure until no edges of $[V_{r+1}, V_{r+2}]$ remain. We then continue this, obtaining a sequence $e_1, e_2, \ldots, e_q$ of all edges of $E(T) - E(T_0)$ and a sequence $T_1, T_2, \ldots, T_q$ of subtrees, where $q = m(T) - m(T_0)$ and $T_q = T$. In summary, after selecting the edge e_1, we select other edges in $[V_r, V_{r+1}]$, one edge at a time, such that the addition of each such edge to the preceding subtree obtained results in a subtree of diameter $k + 1$. When no such edges remain, we then select other edges in $[V_r, V_{r+1}]$, one edge at a time, such that the addition of each such edge to the preceding subtree obtained results in a subtree of diameter $k + 2$. Once no such edges remain in $[V_r, V_{r+1}]$, we turn to edges in $[V_{r+1}, V_{r+2}]$, the addition of which to the preceding subtree obtained results in a subtree of diameter $k + 3$, and so on. Hence, $T_i \subseteq T_{i+1}$ and $\text{diam}(T_i) \leq \text{diam}(T_{i+1}) \leq \text{diam}(T_i) + 1$ for $0 \leq i \leq q - 1$.

We claim, for each integer i with $0 \leq i \leq q$, that a k-rainbow coloring of T_i exists using the colors of $[m_k]$. We proceed by induction. We have already seen that such a coloring exists for both T_0 and T_1. Assume that such a coloring exists for T_j for some integer j with $1 \leq j < q$. We show that such a coloring exists for $T_{j+1} = T_j + e_{j+1}$. Let B be the branch of T at v_0 containing e_{j+1}. Suppose that $e_{j+1} \in [V_s, V_{s+1}]$, where then $s+1 \leq t$. Let u be the vertex incident with e_{j+1} in V_{s+1}. Consequently, there is no edge e' in T_{j+1} that belongs to $[V_{s+1}, V_{s+2}]$.

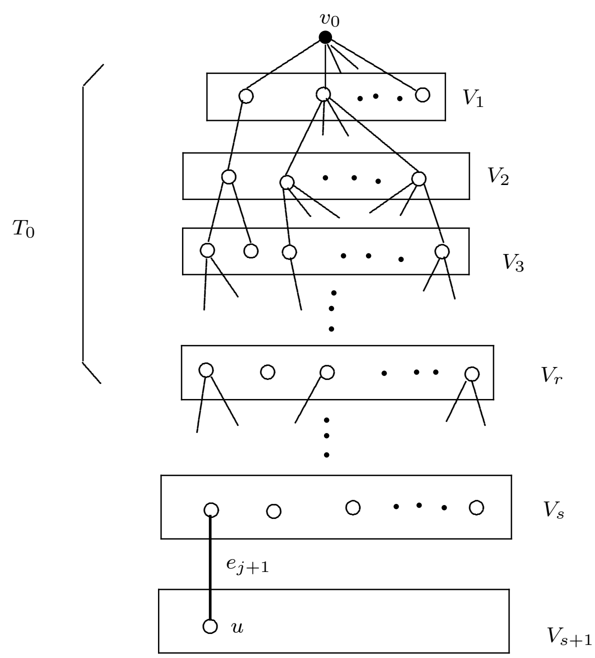


Figure 4.4: A step in the proof of Case 1

Let T'_{j+1} be the subtree of T_{j+1} consisting of e_{j+1} and all those edges of T_{j+1} lying on a path of length k or less having initial edge e_{j+1}. We claim that $\text{diam}(T'_{j+1}) = k$. Clearly, $\text{diam}(T'_{j+1}) \geq k$. Let e and f be two distinct edges of T'_{j+1}. We claim that e and f lie on a path of length k or less in T'_{j+1}. Clearly, this is the case if either e or f is e_{j+1}. Thus, we may assume that neither e nor f is e_{j+1}. Let P be the unique $e_{j+1} - e$ path in T'_{j+1} and Q the unique $e_{j+1} - f$ path in T'_{j+1}. Suppose that P is a $u - x$ path and Q is a $u - y$ path. Let v be the last vertex that P and Q have in common. Let P' be the $v - x$ subpath of P and Q' the $v - y$ subpath of Q. Thus, either $d(x, v_0) \geq d(v, v_0)$ or $d(y, v_0) \geq d(v, v_0)$. Let $d_T(u, v) = a$, $d_T(v, x) = b$ and $d_T(v, y) = c$. Thus, $a + b \leq k$ and $a + c \leq k$. There are four possibilities for the locations of e and f.

1. Both edges e and f belong to the branch B.
2. Exactly one of e and f belongs to a branch of T at v_0 distinct from the branch B.
3. The edges e and f lie on the same branch of T at v_0 but this branch is distinct

from the branch B.

4. The edges e and f lie on distinct branches of T at v_0, neither of which is the branch B.

Regardless of which situation (1)–(4) occurs, either $b \leq a$ or $c \leq a$ (or both). Since $b \leq k-a$ and $c \leq k-a$, it follows that $b+c \leq k$. Thus, e and f lie on a path of length at most k in T'_{j+1} and so $\text{diam}(T'_{j+1}) = k$. Consequently, $m(T'_{j+1}) \leq m_k$. Hence, at most $m_k - 1$ edges of T'_{j+1} have been assigned colors from $[m_k]$, resulting in at least one color in $[m_k]$ that has not been assigned to any edge of T'_{j+1}. Hence, there is at least one color in $[m_k]$ available for e_{j+1}. Assigning e_{j+1} such a color results in a k-rainbow coloring of T_{j+1} using the colors of $[m_k]$. Therefore, there exists a k-rainbow coloring of T_i with the colors of $[m_k]$ for each integer i with $1 \leq i \leq q$. In particular, there exists a k-rainbow coloring of $T_q = T$ using the colors from the set $[m_k]$. Therefore, $\text{rc}_k(T) = m_k$ for each even integer $k \geq 6$.

Case 2. *k is odd.* Thus, $k = 2r+1$ for some integer $r \geq 3$. Therefore, T_0 has two central vertices u_0 and w_0 and a central edge $e_0 = u_0w_0$. Let t_1 be the length of the longest branch of T at u_0 that does not contain w_0 and let t_2 be the length of the longest branch of T at w_0 that does not contain u_0. Hence, $t_1 \geq r$ and $t_2 \geq r$. For $i = 0, 1, 2, \ldots, t_1$, let U_i be the set of all vertices of T at distance i from u_0 that lie on a branch of T at u_0 not containing w_0. For $i = 0, 1, 2, \ldots, t_2$, let W_i be the set of all vertices of T at distance i from w_0 that lie on a branch of T at w_0 not containing u_0. In particular, $U_0 = \{u_0\}$ and $W_0 = \{w_0\}$. Therefore, $T[\cup_{i=0}^{r}(U_i \cup W_i)] = T_0$. If $T = T_0$, then the proof is complete. Thus, we may assume that $E(T) - E(T_0) \neq \emptyset$ and so $\text{diam}(T) \geq k+1$.

Let $e_1 \in [U_r, U_{r+1}] \cup [W_r, W_{r+1}]$ and let $T_1 = T_0 + e_1$. Hence, $\text{diam}(T_1) = k+1$. Let T'_1 be the subtree of T_1 consisting of e_1 and all those edges of T_1 lying on a path of length k or less having initial edge e_1. We claim that $\text{diam}(T'_1) = k$. Suppose that this is not the case. Then there is an $e-f$ path in T'_1 of length $k+1$ for some $e, f \in E(T'_1)$. Since $\text{diam}(T_0) = k$, one of e and f must be e_1, say $e = e_1$. However, from the definition of T'_1, there is no $e_1 - f$ path of length $k+1$ in T'_1, a contradiction. Thus, as claimed, $\text{diam}(T'_1) = k$ and so the size of T'_1 is at most m_k. Hence, at most $m_k - 1$ edges of T'_1 have been assigned colors from $[m_k]$ and therefore there is at least one color in $[m_k]$ that has not been assigned to any edge of T'_1. Assigning such a color to the edge e_1 results in a k-rainbow coloring of T'_1 using the colors of $[m_k]$. If $T = T_1$, then the proof is complete. Hence, we may assume that $E(T) - E(T_1) \neq \emptyset$.

If there is an edge $e' \in (E(T) - E(T_1)) \cap ([U_r, U_{r+1}] \cup [W_r, W_{r+1}])$, where then e'

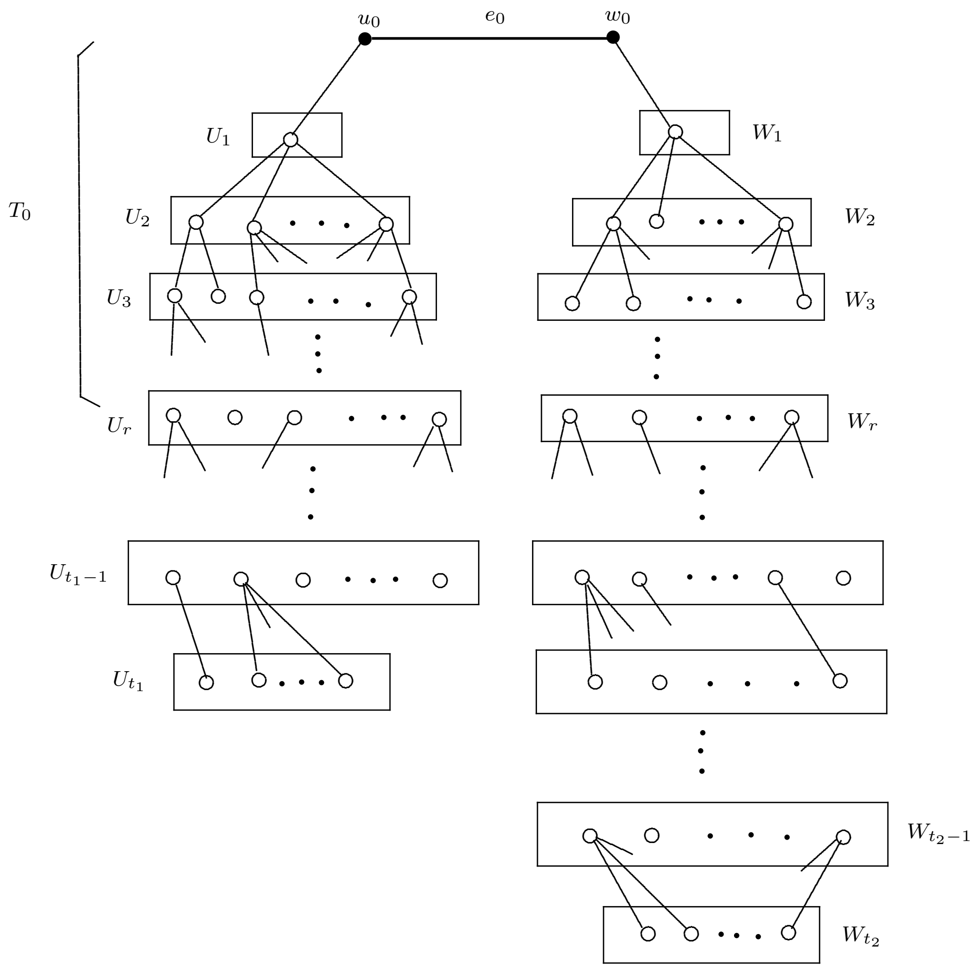


Figure 4.5: The sets U_i and W_i of the tree T in Case 2

is incident with a vertex of T_1 and $T_1 + e'$ has diameter $k + 1$, then we denote this edge e' by e_2 and let $T_2 = T_1 + e_2$. Next, if there is an edge $e' \in (E(T) - E(T_2)) \cap ([U_r, U_{r+1}] \cup [W_r, W_{r+1}])$ such that $T_2 + e'$ has diameter $k + 1$, then we denote this edge e' by e_3 and let $T_3 = T_2 + e_3$. We continue this procedure until no such edges e' exist, obtaining a sequence $e_1, e_2, \ldots, e_p$ ($p \geq 1$) of edges and a sequence $T_1, T_2, \ldots, T_p$ of subtrees of T having diameter $k + 1$. Next, if there exists an edge $e' \in (E(T) - E(T_p)) \cap ([U_r, U_{r+1}] \cup [W_r, W_{r+1}])$, then we denote this edge e' by e_{p+1} and let $T_{p+1} = T_p + e_{p+1}$. Then $\mathrm{diam}(T_p + e') = k + 2$. We continue this procedure until no edges in $[U_r, U_{r+1}] \cup [W_r, W_{r+1}]$ remain, say arriving at the tree $T_{p'}$. If $E(T) - E(T_{p'}) \neq \emptyset$, then let $e' \in (E(T) - E(T_{p'})) \cap ([U_{r+1}, U_{r+2}] \cup [W_{r+1}, W_{r+2}])$. We continue this procedure until no edges of $[U_{r+1}, U_{r+2}] \cup [W_{r+1}, W_{r+2}]$ remain. We then continue this, obtaining a sequence $e_1, e_2, \ldots, e_q$ of all edges of $E(T) - E(T_0)$ and a sequence $T_1, T_2, \ldots, T_q$ of subtrees, where

$q = m(T) - m(T_0)$ and $T_q = T$. In summary, after selecting the edge e_1, we select other edges in $[U_r, U_{r+1}] \cup [W_r, W_{r+1}]$, one edge at a time, such that the addition of each such edge to the preceding subtree obtained results in a subtree of diameter $k+1$. When no such edges remain, we then select other edges in $[U_r, U_{r+1}] \cup [W_r, W_{r+1}]$, one edge at a time, such that the addition of each such edge to the preceding subtree obtained results in a subtree of diameter $k+2$. Once no such edges remain in $[U_r, U_{r+1}] \cup [W_r, W_{r+1}]$, we turn to edges in $[U_{r+1}, U_{r+2}] \cup [W_{r+1}, W_{r+2}]$, the addition of which to the preceding subtree obtained results in a subtree of diameter $k+3$, and so on. Hence, $T_i \subseteq T_{i+1}$ and $\operatorname{diam}(T_i) \le \operatorname{diam}(T_{i+1}) \le \operatorname{diam}(T_i) + 1$ for $0 \le i \le q-1$.

We claim, for each integer i with $0 \le i \le q$, that a k-rainbow coloring of T_i exists using the colors of $[m_k]$. We proceed by induction. We have already seen that such a coloring exists for both T_0 and T_1. Assume that such a coloring exists for T_j for some integer j with $1 \le j < q$. We show that such a coloring exists for $T_{j+1} = T_j + e_{j+1}$. We may assume that e_{j+1} belongs to a branch B at u_0 that does not contain w_0 and that $e_{j+1} \in [U_s, U_{s+1}]$, where then $s+1 \le t_1$. Let u be the vertex incident with e_{j+1} in U_{s+1}. Consequently, there is no edge in T_{j+1} that belongs to $[U_{s+1}, U_{s+2}]$.

Let T'_{j+1} be the subtree of T_{j+1} consisting of e_{j+1} and all those edges of T_{j+1} lying on a path of length k or less having initial edge e_{j+1}. We claim that $\operatorname{diam}(T'_{j+1}) = k$. Clearly, $\operatorname{diam}(T'_{j+1}) \ge k$. Let e and f be two distinct edges of T'_{j+1}. We claim that e and f lie on a path of length k or less in T'_{j+1}. Clearly, this is the case if either e or f is e_{j+1}. Thus, we may assume that neither e nor f is e_{j+1}. Let P be the unique $e_{j+1} - e$ path in T'_{j+1} and let Q be the unique $e_{j+1} - f$ path in T'_{j+1}. Suppose that P is a $u - x$ path and Q is a $u - y$ path. Let v be the last vertex that P and Q have in common. Let P' be the $v - x$ subpath of P and Q' the $v - y$ subpath of Q. Thus, either $d(x, v_0) \ge d(v, v_0)$ or $d(y, v_0) \ge d(v, v_0)$. Let $d_T(u, v) = a$, $d_T(v, x) = b$ and $d_T(v, y) = c$. Thus, $a + b \le k$ and $a + c \le k$, and so $c \le k - a$.

Recall that e_{j+1} belongs to the branch B of T at u_0 that does not contain w_0. To simplify terminology, when we refer to a branch at u_0, we mean a branch of T at u_0 that does not contain w_0. Similarly, a branch at w_0 is a branch of T at w_0 that does not contain u_0. For the locations of e and f, the following situations are possible:

- ⋆ one of e and f is e_0 and the other lies on (i) the branch B, (ii) a branch $B' \ne B$ at u_0 or (iii) a branch at w_0,
- ⋆ e and f lie on the same branch at u_0 or at w_0; so this branch is (i) the branch B, (ii) a branch $B' \ne B$ at u_0 or (iii) a branch at w_0,
- ⋆ e and f lie on two different branches at u_0, one of which is B or neither of which

is B,

- ⋆ e and f lie on two different branches at w_0,
- ⋆ e and f lie on two different branches, one of which is a branch at w_0 and the other is either (i) the branch B or (ii) a branch $B' \neq B$ at u_0.

More precisely, there are eleven possibilities for the locations of e and f:

1. One of e and f is the edge e_0 and the other belongs to B.
2. One of e and f is the edge e_0 and the other belongs to a branch B' at u_0 distinct from B.
3. One of e and f is the edge e_0 and the other belongs to a branch B' at w_0.
4. Both e and f belong to B.
5. Both e and f belong to the same branch B' at u_0 distinct from B.
6. Both e and f belong to the same branch B' at w_0.
7. One of e and f belongs to B and the other belongs to a branch B' at u_0 distinct from B.
8. One of e and f belongs to B and the other belongs to a branch B' at w_0.
9. The edges e and f belong to distinct branches at u_0, neither of which is B.
10. One of e and f belongs to a branch B' at u_0 distinct from B and the other belongs to a branch at w_0.
11. The edges e and f belong to two distinct branches at w_0.

Regardless of which situation (1)–(11) occurs, either $b \leq a$ or $c \leq a$ (or both). Since $b \leq k-a$ and $c \leq k-a$, it follows that $b+c \leq k$. Thus, e and f lie on a path of length at most k in T'_{j+1} and so $\text{diam}(T'_{j+1}) = k$. Consequently, $m(T'_{j+1}) \leq m_k$. Hence, at most $m_k - 1$ edges of T'_{j+1} have been assigned colors from $[m_k]$, resulting in at least one color in $[m_k]$ that has not been assigned to any edge of T'_{j+1}. Hence, there is at least one color in $[m_k]$ available for e_{j+1}. Assigning e_{j+1} such a color results in a k-rainbow coloring of T_{j+1} using the colors of $[m_k]$. Therefore, there exists a k-rainbow coloring of T_i with the colors of $[m_k]$ for each integer i with $1 \leq i \leq q$. In particular, there exists a k-rainbow coloring of $T_q = T$ using the colors from the set $[m_k]$. Therefore, $\text{rc}_k(T) = m_k$ for each odd integer $k \geq 7$. ■

It was observed in Chapter 2 that if T is a nontrivial tree of size m, then $\mathrm{rc}_3(T) = m$ if and only if T is a star or a double star. This observation can be extended to the following, which is a consequence of Theorem 4.1.4.

Corollary 4.1.5 *Let T be a nontrivial tree of size m and let $k \geq 2$ be an integer. Then $\mathrm{rc}_k(T) = m$ if and only if $\mathrm{diam}(T) \leq k$.*

Proof. First, suppose that $\mathrm{diam}(T) \leq k$. Let e and f be any two edges of T. Then there is an $x - y$ path P in T such that $e, f \in E(P)$. The path P is the only $x - y$ path in T. Since $\mathrm{diam}(T) \leq k$, the length of P is at most k. Thus, $c(e) \neq c(f)$ and so $\mathrm{rc}_k(T) = m$.

For the converse, suppose that T is a tree with $\mathrm{diam}(T) > k$. Let T^* be a subtree of T having maximum size m_k in T such that $\mathrm{diam}(T^*) = k$. Since $\mathrm{diam}(T) > k$, it follows that T^* is a proper subtree of T and so $m_k < m$. It then follows by Theorem 4.1.4 that $\mathrm{rc}_3(T) = m_k < m$. ∎

Theorem 4.1.4 provides a formula for $\mathrm{rc}_k(T)$ of a tree T for each integer $k \geq 2$, namely $\mathrm{rc}_k(T)$ is the maximum size m_k of a subtree of T having diameter k. However, when k is large, the value m_k may not be easy to compute without the aid of technology (however, the value m_k may be computed in polynomial time when technology is implemented). Therefore, this suggests the problem of obtaining bounds for $\mathrm{rc}_k(T)$ in terms of more easily computable expressions. For integers $\Delta \geq 3$ and $d \geq 2$, let $\mathcal{T}_{\Delta,d}$ denote the set of trees having maximum degree Δ and diameter d and denote the minimum and maximum sizes of trees in $\mathcal{T}_{\Delta,d}$ by

$$\begin{aligned} m(\Delta, d) &= \min \{m(T) : T \in \mathcal{T}_{\Delta,d}\} \\ M(\Delta, d) &= \max \{m(T) : T \in \mathcal{T}_{\Delta,d}\}. \end{aligned}$$

We will soon derive expressions for these numbers.

A tree is *central* if its center is K_1 and *bicentral* if its center is K_2. It is known that a tree is central if and only if it has even diameter. Furthermore, if T is central, then

$$\mathrm{diam}(T) = 2\,\mathrm{rad}(T);$$

while if T is bicentral, then

$$\mathrm{diam}(T) = 2\,\mathrm{rad}(T) - 1$$

(see [14, pp. 88]).

Proposition 4.1.6 *For integers $\Delta \geq 3$ and $d \geq 2$,*

$$m(\Delta, d) \;=\; \Delta + d - 2,$$

$$M(\Delta, d) \;=\; \begin{cases} \frac{\Delta\left[(\Delta-1)^t-1\right]}{\Delta-2} & \text{if } d = 2t \geq 2 \text{ is even} \\ 1 + 2(\Delta-1)\frac{\left[(\Delta-1)^{t-1}-1\right]}{\Delta-2} & \text{if } d = 2t-1 \geq 3 \text{ is odd.} \end{cases}$$

Proof. Since the result is true for $d = 2, 3$, we may assume that $d \geq 4$. First, we show that $m(\Delta, d) = \Delta + d - 2$. Let $T \in T_{\Delta,d}$ such that T has size m and let v be a vertex of T with $\deg v = \Delta$. Then $T - v$ is a forest with Δ components, say $T_1, T_2, \ldots, T_\Delta$. Let m_i denote the size of T_i for $i = 1, 2, \ldots, \Delta$. Therefore,

$$m = \Delta + \sum_{i=1}^{\Delta} m_i.$$

Let P be a path of size d in T. Suppose first that v lies on P. Then exactly two of the trees $T_1, T_2, \ldots, T_\Delta$, say T_1 and T_2, contain vertices of P. Hence, $m_1 + m_2 \geq d - 2$ and so

$$m = \Delta + \sum_{i=1}^{\Delta} m_i \geq \Delta + (d-2) + \sum_{i=3}^{\Delta} m_i \geq \Delta + d - 2.$$

Next, suppose that v is not a vertex of P. Hence, P lies entirely in one of the trees $T_1, T_2, \ldots, T_\Delta$, say T_1 and so $m_1 \geq d$. Thus,

$$m = \Delta + \sum_{i=1}^{\Delta} m_i \geq \Delta + d + \sum_{i=2}^{\Delta} m_i \geq \Delta + d.$$

Therefore, $m(\Delta, d) \geq \Delta + d - 2$. Now, let T' be the tree obtained from the path $(v_0, v_1, \ldots, v_d)$ where $d \geq 2$ by adding $\Delta - 2$ vertices and joining all these vertices to v_1. Then $T' \in T_{\Delta,d}$. Since the size of T' is $\Delta + d - 2$, it follows that $m(\Delta, d) \leq \Delta + d - 2$ and so $m(\Delta, d) = \Delta + d - 2$.

It remains to determine $M(\Delta, d)$. We consider two cases, according to whether d is even or d is odd.

Case 1. *$d \geq 4$ is even.* Then $d = 2t$ for some integer $t \geq 2$. Let

$$K(\Delta, d) = \frac{\Delta\left[(\Delta-1)^t-1\right]}{\Delta-2}.$$

We show that $M(\Delta, d) = K(\Delta, d)$. Now, let $T \in \mathcal{T}_{\Delta,d}$ be an arbitrary tree. Since $\mathrm{diam}(T) = d$ is even, T has a unique central vertex v. Then $e(v) = t$. We may assume that T is a rooted tree with root v. For each integer i with $0 \le i \le t$, let

$$V_i = \{w \in V(T) : d(v, w) = i\}.$$

Since $V_0 = \{v\}$ and $\deg v \le \Delta$, it follows that $|[V_0, V_1]| \le \Delta$. Furthermore, $|[V_1, V_2]| \le \Delta(\Delta - 1)$ and $|[V_2, V_3]| \le \Delta(\Delta - 1)^2$. In general,

$$|[V_i, V_{i+1}]| \le \Delta(\Delta - 1)^i \text{ for } 0 \le i \le t - 1.$$

Thus,

$$\begin{aligned} m(T) &= \sum_{i=0}^{t-1} |[V_i, V_{i+1}]| \le \Delta \sum_{i=0}^{t-1} (\Delta - 1)^i \\ &= \frac{\Delta\left[(\Delta - 1)^t - 1\right]}{\Delta - 2} = K(\Delta, d). \end{aligned}$$

Since T is an arbitrary tree in $\mathcal{T}_{\Delta,d}$, it follows that $M(\Delta, d) \le K(\Delta, d)$. On the other hand, let $T_M \in \mathcal{T}_{\Delta,d}$ be a tree with central vertex u such that for each integer i with $0 \le i \le t - 1$, every vertex in $\{w \in V(T) : d(u, w) = i\}$ has degree Δ and every vertex in $\{w \in V(T) : d(u, w) = t\}$ is an end-vertex of T_M. Hence, $|[V_i, V_{i+1}]| = \Delta(\Delta - 1)^i$ for $0 \le i \le t - 1$ and so $m(T_M) = K(\Delta, d)$. Hence, $M(\Delta, d) \ge K(\Delta, d)$ and so $M(\Delta, d) = K(\Delta, d)$ when $d \ge 4$ is even.

Case 2. *$d \ge 5$ is odd.* Then $d = 2t - 1$ for some integer $t \ge 3$. Let

$$K(\Delta, d) = 1 + 2(\Delta - 1)\frac{\left[(\Delta-1)^{t-1}-1\right]}{\Delta-2}.$$

We show that $M(\Delta, d) = K(\Delta, d)$. Next, let $T \in \mathcal{T}_{\Delta,d}$. Since $\mathrm{diam}(T) = d$ is odd, the center of T is K_2 and so T has exactly two central vertices u and v. Thus, u and v are adjacent vertices with $e(u) = e(v) = t$. Express the tree T as "a double rooted tree" whose roots are u and v. For each integer i with $0 \le i \le t - 1$, let

$$\begin{aligned} W_{i,u} &= \{w \in V(T) : \ d(u, w) = i \text{ and } d(v, w) = i + 1\}, \\ W_{i,v} &= \{w \in V(T) : \ d(v, w) = i \text{ and } d(u, w) = i + 1\}, \end{aligned}$$

and $W_i = W_{i,u} \cup W_{i,v}$ for $0 \le i \le t - 1$. Thus, $W_0 = \{u, v\}$ and W_1 consits of those vertices in $V(T) - \{u, v\}$ that are adjacent to either u or v. Then $|[W_0, W_1]| \le 2(\Delta - 1)$. Furthermore, $|[W_1, W_2]| \le 2(\Delta - 1)^2$ and $|[W_2, W_3]| \le 2(\Delta - 1)^3$. More generally,

$$|[W_i, W_{i+1}]| \le 2(\Delta-1)^i \text{ for } 0 \le i \le t-2.$$

Hence,

$$\begin{aligned}
m(T) &= 1 + \sum_{i=0}^{t-2} |[W_i, W_{i+1}]| \le 1 + \sum_{i=0}^{t-2} 2(\Delta-1)^{i+1} \\
&\le 1 + 2(\Delta-1)\sum_{i=0}^{t-2}(\Delta-1)^i \\
&= 1 + 2(\Delta-1)\frac{\left[(\Delta-1)^{t-1}-1\right]}{\Delta-2} = K(\Delta, d).
\end{aligned}$$

Since T is an arbitrary tree in $\mathcal{T}_{\Delta,d}$, it follows that $M(\Delta, d) \le K(\Delta, d)$. On the other hand, let $T_M \in \mathcal{T}_{\Delta,d}$ be a tree with the central vertices x and y such that for each integer i with $0 \le i \le t-2$, every vertex in $W_i = W_{i,x} \cup W_{i,y}$ has degree Δ and every vertex in $W_{t-1} = W_{t-1,x} \cup W_{t-1,y}$ is an end-vertex of T_M. Then $|[W_i, W_{i+1}]| = 2(\Delta-1)^{i+1}$ for $0 \le i \le t-2$ and so $m(T_M) = K(\Delta, d)$. Hence, $M(\Delta, d) = K(\Delta, d)$. ■

The following corollary is a consequence of Theorem 4.1.4 and Proposition 4.1.6

Corollary 4.1.7 *If T is a tree having maximum degree $\Delta \ge 3$ and diameter at least k, where $k \ge 2$, then*

$$\mathrm{rc}_k(T) \ge \Delta + k - 2$$

$$\mathrm{rc}_k(T) \le \begin{cases} \frac{\Delta\left[(\Delta-1)^t-1\right]}{\Delta-2} & \text{if } k = 2t \ge 2 \text{ is even} \\ 1 + 2(\Delta-1)\frac{\left[(\Delta-1)^{t-1}-1\right]}{\Delta-2} & \text{if } k = 2t-1 \ge 3 \text{ is odd.} \end{cases}$$

The following is an immediate consequence of Observation 3.1.1 and Corollary 4.1.7.

Corollary 4.1.8 *If G is a connected graph of order at least $k+1 \ge 4$ and maximum degree $\Delta \ge 3$, then*

$$\mathrm{rc}_k(G) \le \begin{cases} \frac{\Delta\left[(\Delta-1)^t-1\right]}{\Delta-2} & \text{if } k = 2t \ge 2 \text{ is even} \\ 1 + 2(\Delta-1)\frac{\left[(\Delta-1)^{t-1}-1\right]}{\Delta-2} & \text{if } k = 2t-1 \ge 3 \text{ is odd.} \end{cases}$$

4.2 Unicyclic Graphs

In [32, 33], upper bounds for the rainbow connection numbers have been established for connected graphs G having minimum degree $\delta(G) \geq 2$.

Theorem 4.2.1 [33] *Let G be a connected graph of order $n \geq 4$ with minimum degree $\delta(G) = 2$. If $G \notin \{C_4, K_4 - e, C_5\}$, then $\mathrm{rc}(G) \leq n - 3$.*

Theorem 4.2.2 [32] *If G is a connected graph of order $n \geq 4$ with minimum degree $\delta(G) \geq 3$, then $\mathrm{rc}(G) \leq \frac{3n-1}{4}$.*

The following is a consequence of (3.1) and Theorems 4.2.1 and 4.2.2.

Corollary 4.2.3 *Let G be a connected graph of order $n \geq 6$ whose longest paths have length ℓ and let k be an integer with $2 \leq k \leq \ell$.*

(1) *If $\delta(G) = 2$, then $\mathrm{rc}_k(G) \leq n - 3$.*

(2) *If $\delta(G) \geq 3$, then $\mathrm{rc}_k(G) \leq \frac{3n-1}{4}$.*

Thus, we study connected graphs containing end-vertices, namely those graphs with minimum degree 1. Every nontrivial tree contains end-vertices. If T is a tree of order $n \geq 3$, then $\mathrm{diam}(T) = d$ is the length of a longest path in T and $\mathrm{rc}_d(T) = n - 1$ by Corollary 4.1.2. Thus, $\mathrm{rc}_k(T) \leq n - 1$ for all integers k with $2 \leq k \leq d$ by (3.1). It can be shown that if G is a connected graph of order $n \geq 3$ and ℓ is the length of a longest path in G, then G contains a spanning tree T such that $\mathrm{diam}(T) = \ell$. It then follows by Observation 3.1.1 and Corollary 4.1.2 that $\mathrm{rc}_k(G) \leq n - 1$ for all integers k with $2 \leq k \leq \ell$. In this section, we study the k-rainbow colorings of a well-known class of graphs, namely unicyclic graphs. As a consequence, it is shown that if G is a connected graph of order $n \geq 3$ that is not a tree and the length of a longest path in G is ℓ, then $\mathrm{rc}_k(G) \leq n - 2$ for all integers k with $2 \leq k \leq \ell$.

A *unicyclic graph* is a connected graph containing exactly one cycle. Thus, if G is a unicyclic graph of order $n \geq 3$, then the size of G is also n. In particular, each cycle is a unicyclic graph. Since $\mathrm{rc}(K_n) = 1$ for all integer $n \geq 3$ (and so $\mathrm{rc}(C_3) = 1$), it then follows by (3.1) and Theorem 3.3.2 that $\mathrm{rc}_k(C_n) \leq n - 2$ for integers k and n with $2 \leq k \leq n - 1$. Next, we show that $\mathrm{rc}_k(G) \leq n - 2$ for all unicyclic graphs G of order $n \geq 3$ in general. First, we introduce additional notation. If P and Q are two paths in a graph G such that P and Q have exactly one vertex in common and this vertex is an end-vertex of P and Q, say P is $u - v$ path and Q is a $v - w$ path where

$V(P) \cap V(Q) = \{v\}$, then let (P, Q) denote the $u - w$ path in G constructed from P and Q, namely P followed by Q.

Proposition 4.2.4 *If G is a unicyclic graph of order $n \geq 3$ whose longest paths have length ℓ, then $\mathrm{rc}_k(G) \leq n - 2$ for all integers k with $2 \leq k \leq \ell$.*

Proof. Let G be a unicyclic graph of order $n \geq 3$. Thus, G has n edges. Since K_3 is the only unicyclic graph of order 3 and $\mathrm{rc}(K_3) = 1$, we may assume that $n \geq 4$ and so $\mathrm{diam}(G) \geq 2$. Let $C = (u_1, u_2, \ldots, u_p, u_{p+1} = u_1)$ be the unique cycle in G. By Theorem 3.3.2, we may assume that $G \neq C$ and so $3 \leq p \leq n - 1$. We show that G has a rainbow coloring using $n - 2$ colors. Define the edge coloring $c : E(G) \to [n - 2]$ by assigning the color 1 to the three edges u_1u_2, u_2u_3, u_3u_4 on C (where $u_3u_4 = u_3u_1$ if $p = 3$) and assigning the $n - 3$ distinct colors $2, 3, \ldots, n - 2$ to the remaining $n - 3$ edges of G. It remains to show that c is a rainbow coloring of G.

Let x and y be two nonadjacent vertices of G. We show that there is a rainbow $x - y$ path in G. First, suppose that $x, y \in V(C)$. Then there are two $x - y$ paths Q and Q' on C. Since only three edges on C are colored 1, at least one of Q and Q' contains at most one edge colored 1, say Q contains at most one edge colored 1. Thus, Q is a rainbow $x - y$ path. Next, suppose that at least one of x and y does not belong to C, say $y \notin V(C)$. Let $u_i \in V(C)$ where $1 \leq i \leq n$ such that $d(y, u_i) = \min\{d(y, u) : u \in V(C)\}$ and let P be the $u_i - y$ geodesic in G. Then P is a rainbow $u_i - y$ path in G. If $x \in V(C)$, then let Q be the rainbow $x - u_i$ path on C (where Q is a trivial path if $x = u_i$). Hence, (Q, P) is a rainbow $x - y$ path in G. If $x \notin V(C)$, then let $u_j \in V(C)$ such that $d(x, u_j) = \min\{d(x, u) : u \in V(C)\}$ where it is possible that $u_i = u_j$. Let P' be the $x - u_j$ geodesic in G and so P' is a rainbow $x - u_j$ path. Now let Q be the $u_i - u_j$ rainbow path on C (where Q is a trivial path if $u_i = u_j$). Then (P', Q, P) is a rainbow $x - y$ path in G. Therefore, c is a rainbow coloring of G and so $\mathrm{rc}(G) \leq n - 2$. It then follows by (3.1) that $\mathrm{rc}_k(G) \leq \mathrm{rc}(G) \leq n - 2$ for all integers k with $2 \leq k \leq \ell$. ■

The bound $n - 2$ established for $\mathrm{rc}_k(G)$ in Proposition 4.2.4 is best possible as we will see soon. In order to extend Proposition 4.2.4 to all connected graphs that are not trees, we first present a lemma.

Lemma 4.2.5 *If G is a nontrivial connected graph that is not a tree such that the length of a longest path in G is ℓ, then G contains a unicyclic spanning subgraph whose longest paths have length ℓ.*

Proof. Since the result is certainly true if G is a unicyclic graph, we may assume that G is not unicyclic. Let L be a path of length ℓ in G. Since G is not a tree, there is an edge $e_1 \in E(G) - E(L)$ that lies on a cycle of G. Then $G_1 = G - e_1$ is a connected spanning subgraph of G. Since G_1 contains L, the length of a longest path in G_1 is ℓ. If G_1 is a unicyclic graph, then G_1 has the desired property. Otherwise, there is $e_2 \in E(G_1) - E(L)$ that lies on a cycle of G_1. Then $G_2 = G - e_2$ is a connected spanning subgraph of G_1 and G. Since G_2 contains L, the length of a longest path in G_2 is ℓ. If G_2 is a unicyclic graph, then G_2 has the desired property; if not, we continue this procedure until eventually arriving at a unicyclic spanning subgraph of G whose longest paths have length ℓ. ■

Theorem 4.2.6 *If G is a nontrivial connected graph of order $n \geq 3$ that is not a tree such that the length of a longest path in G is ℓ, then $\mathrm{rc}_k(G) \leq n-2$ for all integers k with $2 \leq k \leq \ell$.*

Proof. Let G be a connected graph of order n that is not a tree such that the length of a longest path in G is ℓ. By Lemma 4.2.5, G contains a uncyclic spanning subgraph H such that the length of a longest path in H is ℓ. By Proposition 4.2.4, $\mathrm{rc}_k(H) \leq n-2$ for all integers k with $2 \leq k \leq \ell$. It then follows by Observation 3.1.1 that $\mathrm{rc}_k(G) \leq \mathrm{rc}_k(H) \leq n-2$ for all integers k with $2 \leq k \leq \ell$. ■

The bound $n-2$ established for $\mathrm{rc}_k(G)$ in Theorem 4.2.6 is best possible. In fact, more can be said. A tree T has the property that there exists a 3-rainbow coloring of T such that every two vertices u and v are connected by a unique 3-rainbow $u-v$ path in T. This gives rise to the following question: Is there a connected graph that is not a tree with this property? We provide an affirmative answer to this question. First, we give an example of a class of graphs of diameter 3. The *corona* $\mathrm{cor}(H)$ of a graph H is the graph obtained from H by attaching a pendant edge to each vertex of H. Thus, if H has order n, then the corona $\mathrm{cor}(H)$ has order $2n$ and has precisely n leaves. Let $G = \mathrm{cor}(K_n)$ for some integer $n \geq 3$, where $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(G) - V(K_n) = \{v_1, v_2, \ldots, v_n\}$ such that u_iv_i is the pendant edge at u_i for $1 \leq i \leq n$. Define the edge coloring $c : E(G) \to [n+1]$ by $c(u_iv_i) = i$ for $1 \leq i \leq n$ and $c(e) = n+1$ for each $e \in E(K_n)$. Then c is a 3-rainbow $(n+1)$-coloring of G. Furthermore, every two vertices are connected by a unique 3-rainbow path (or a rainbow path) in G.

Theorem 4.2.7 *For two integers d and k with $d \geq k \geq 3$, there exists a connected graph G of diameter d that is not a tree with the property that $\mathrm{rc}_k(G) = k$ and G has a k-rainbow coloring such that every two vertices of G are connected by a unique k-rainbow path.*

Proof. For an integer $k \geq 3$, let G_k be the graph constructed from the graph $K_2 = (u, v)$ and the path $P_k = (w_1, w_2, \ldots, w_k)$ of order k by joining u and v to w_1. (see Figure 4.6). Since $\text{diam}(G_k) = k$, it follows that $\text{rc}_k(G_k) \geq k$. Next, define the edge coloring c of G_k defined by $c(e) = 1$ if $e \in \{uv, uw_1, vw_1\}$ and $c(w_i w_{i+1}) = i + 1$ for $1 \leq i \leq k - 1$. Since c is a k-rainbow k-coloring of G_k, it follows that $\text{rc}_k(G) = k$.

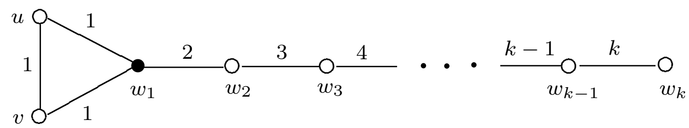


Figure 4.6: The graph G_k

Next, we show that every two vertices of G are connected by a unique k-rainbow path. Let x and y be any two vertices of G_k. If x and y belong to P_k, then the $x - y$ subpath of P_k is the unique k-rainbow $x - y$ path in G_k. If x and y belong to $\{u, v, w_1\}$, then (x, y) is the unique k-rainbow $x - y$ path in G_k. If x belongs to $\{u, v, w_1\}$ and y belongs to P_k, say $y = w_i$ for some i with $2 \leq i \leq k$, then the path $(x, w_1, w_2, \ldots, w_i)$ is the unique k-rainbow $x - y$ path in G_k. Thus, the graph G_k (together with the k-rainbow coloring c) has the desired property.

A graph G_d of diameter $d > k$ with the desired property can be constructed from the graph G_k and the path $(w_{k+1}, w_{k+2}, \ldots, w_d)$ of order $d - k$ by adding the edge $w_k w_{k+1}$. We can then extend the k-rainbow coloring c of G_k to a k-rainbow coloring of G_d by assigning the colors $1, 2, \ldots, k$ *in this order* to the edges of the path $Q = (w_k, w_{k+1}, w_{k+2}, \ldots, w_d)$ beginning with $w_k w_{k+1}$ (namely, assign the color $i + 1$ to the edge $w_{k+i} w_{k+i+1}$ for $0 \leq i \leq k - 1$ if $d - k \geq k$) and repeat this procedure if $d - k > k$. ■

By Theorem 4.2.7, for each integer $n \geq 4$, there is a unicyclic graph G of order n such that $\text{rc}_{n-2}(G) = n - 2$. Therefore, as we mentioned, the bound $n - 2$ established for $\text{rc}_k(G)$ where $2 \leq k \leq n - 1$ in Theorem 4.2.6 is the best possible

4.3 Graphs of Cycle Rank 2

In this section, we show that if G is a connected graph of order $n \geq 3$ and size at least $n+1$ that is $(K_4 - e)$-free and the length of a longest path is ℓ, then $\text{rc}_k(G) \leq n - 3$ for all integers k with $2 \leq k \leq \ell$. In order to show this, we first introduce a another class of graphs. Let G be a connected graph of order n and size m. The number of edges that must be deleted from G to obtain a spanning tree of G is $m - n + 1$. The number $m - n + 1$ is called the *cycle rank* (or *Betti number*) of G. Thus, the cycle rank of a tree is 0 and the cycle rank of a unicyclic graph (a connected graph with exactly one cycle) is 1. The cycle rank of a connected graph of order n and size $m = n + 1$ is therefore 2. A graph H is called a *subdivision* of a graph G if H is obtained from G by inserting vertices of degree 2 into one or more edges of G. For this purpose, we also say that a graph is vacuously a subdivision of itself. If H is a subdivision of a graph G, then H and G have the same cycle rank.

As is the case with trees, there is a formula that gives the number of end-vertices in a connected graph G having cycle rank ψ in terms of ψ and the number of vertices of G having degree 3 or more. Although the following result is known, we present a proof for completion.

Proposition 4.3.1 *Let G be a nontrivial connected graph having maximum degree Δ and cycle rank ψ. If n_i is the number of vertices of degree i in G, where $1 \leq i \leq \Delta$, then*

$$n_1 = (2 - 2\psi) + n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta. \tag{4.1}$$

Proof. Suppose that G has order n and size m. Then $m = (n-1) + \psi$,

$$n = \textstyle\sum_{i=1}^{\Delta} n_i, \text{ and } 2m = \sum_{i=1}^{\Delta} i n_i.$$

Therefore, $2m = 2(n - 1 + \psi) = 2n - 2 + 2\psi$ and so

$$\sum_{i=1}^{\Delta} i n_i = 2\sum_{i=1}^{\Delta} n_i - 2 + 2\psi. \tag{4.2}$$

Solving for n_1 in (4.2), we obtain (4.1). ∎

For each integer $n \geq 4$, let $\mathcal{G}_{2,n}$ denote the set of all connected graphs of order n with cycle rank 2. If $G \in \mathcal{G}_{2,n}$, then the size of G is $n+1$. If $G \in \mathcal{G}_{2,n}$, then G contains at least two cycles and so G has a subgraph F that is isomorphic to one of three types of graphs in Figure 4.7.

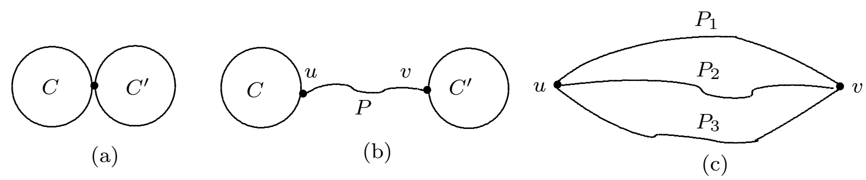


Figure 4.7: Three possible types of subgraphs

(1) a graph obtained from two cycles C and C', by identifying a vertex in C and a vertex in C', as shown in Figure 4.7(a),

(2) a graph obtained from two disjoint cycles C and C' and a path P of length 1 or more by identifying an end-vertex u of P with a vertex of C and identifying the other end-vertex v of P with a vertex of C', as shown in Figure 4.7(b),

(3) a subdivision of $K_4 - e$, that is, a graph consisting of three internally disjoint $u - v$ paths P_i $(1 \leq i \leq 3)$, as shown in Figure 4.7(c), where at least two paths P_i $(1 \leq i \leq 3)$ have length 2 or more.

The graph $K_4 - e$ is the only connected graph of order 4 and cycle rank 2 and so $\mathcal{G}_{2,4} = \{K_4 - e\}$. Figure 4.8 shows all graphs in in $\mathcal{G}_{2,5}$ and $\mathcal{G}_{2,6}$ together with a rainbow coloring for each of these graphs. If G is a connected graph of order $n \geq 4$ and cycle rank 2, then $\mathrm{diam}(G) \leq n-2$. If $G = K_4 - e$, then $\mathrm{rc}(G) = \mathrm{diam}(G) = 2 = n-2$. There are three graphs $G \in \mathcal{G}_{2,5} \cup \mathcal{G}_{2,6}$ such that $\mathrm{rc}(G) = \mathrm{diam}(G) = n-2$, each of which is placed inside a box shown in Figure 4.8. Notice that these are the only graphs in $\mathcal{G}_{2,5}$ and $\mathcal{G}_{2,6}$ containing $K_4 - e$ as a subgraph and having maximum degree 3. The following result can be verified.

Proposition 4.3.2 *Let G be a connected graph of order $n \in \{4, 5, 6\}$ and cycle rank 2 whose longest paths have length ℓ. Then $\mathrm{rc}(G) = n-2$ if and only if $\mathrm{diam}(G) = n-2$ and $\mathrm{rc}(G) \leq n-3$ otherwise. Consequently, if $\mathrm{diam}(G) \neq n-2$, then $\mathrm{rc}_k(G) \leq n-3$ for each integer k with $2 \leq k \leq \ell$.*

We now turn our attention to connected graphs of order $n \geq 7$ having cycle rank 2. A graph G having cycle rank 2 is of *type I* if the two cycles in G are edge-disjoint. Thus,

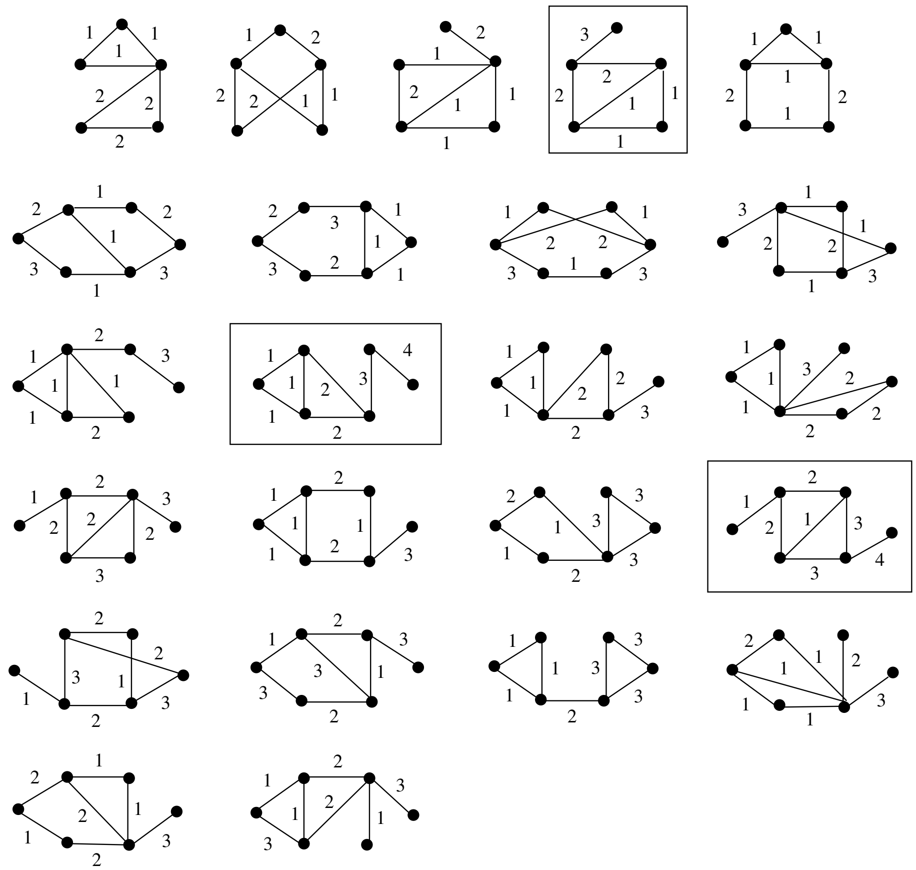


Figure 4.8: Graphs in $\mathcal{G}_{2,5}$ and $\mathcal{G}_{2,6}$

G contains a subgraph that is isomorphic either to the graph in Figure 4.7(a) or to the graph in Figure 4.7(b).

Proposition 4.3.3 *Let G be a connected graph of order $n \geq 7$ having cycle rank 2 and let ℓ be the length of a longest path in G. If G is of type I, then $\mathrm{rc}_k(G) \leq n-3$ for all integers k with $2 \leq k \leq \ell$.*

Proof. Let $C_p = (u_1, u_2, \ldots, u_p, u_1)$ and $C_q = (v_1, v_2, \ldots, v_q, v_1)$ be the two edge-disjoint cycles in G. We may assume that $q \geq p \geq 3$. By (3.1), it suffices to show that $\mathrm{rc}(G) \leq n-3$; that is, there is a rainbow coloring of G using at most $n-3$ colors. First, observe that if $p = 3$, then $\mathrm{rc}(C_p) = 1 = p-2$; while if $p \geq 4$, then $\mathrm{rc}(C_p) \leq \lfloor p/2 \rfloor \leq p-2$. Thus, $\mathrm{rc}(C_p) \leq p-2$ for each integer $p \geq 3$. Similarly, $\mathrm{rc}(C_q) \leq q-2$ for each integer $q \geq 3$. Suppose that $\mathrm{rc}(C_p) = a$ and $\mathrm{rc}(C_q) = b$. Then $a + b \leq p + q - 4$. Let $c_p : E(C_p) \to [a]$ be a rainbow coloring of C_p and let $c_q : E(C_q) \to \{a+1, a+2, \ldots, a+b\}$ be a rainbow coloring of C_q. Let $X = E(G) - (E(C_p) \cup E(C_q))$. Since the size of G is

$n+1$, it follows that $|X| = n+1-p-q \geq 0$.

- ⋆ If $X = \emptyset$, then $n = p+q-1$. Define the edge coloring $c : E(G) \to [a+b]$ by $c(e) = c_p(e)$ if $e \in E(C_p)$ and $c(e) = c_q(e)$ if $e \in E(C_q)$. Then c is a rainbow coloring of G and so $\text{rc}(G) \leq a+b \leq p+q-4 = n-3$.

- ⋆ If $X \neq \emptyset$, then let $X = \{e_1, e_2, \ldots, e_\gamma\}$, where $\gamma = n+1-p-q \geq 1$. Define the edge coloring $c : E(G) \to [a+b+\gamma]$ by $c(e) = c_p(e)$ if $e \in E(C_p)$ and $c(e) = c_q(e)$ if $e \in E(C_q)$ and $c(e_i) = a+b+i$ for $1 \leq i \leq \gamma$. Then c is a rainbow coloring of G and so $\text{rc}(G) \leq a+b+\gamma \leq (p+q-4)+(n+1-p-q) = n-3$.

It then follows by (3.1) that $\text{rc}_k(G) \leq n-3$ for all integers k with $2 \leq k \leq \ell$. ■

A connected graph G of order $n \geq 4$ having cycle rank 2 is of *type II* if the two cycles in G have at least one edge in common. Thus, G contains a subdivision of $K_4 - e$. Therefore, G contains a subgraph that is isomorphic to the graph in Figure 4.7(c).

Proposition 4.3.4 *Let G be a connected graph of order $n \geq 7$ having cycle rank 2 and let ℓ be the length of a longest path in G. If G is of type II but does not contain $K_4 - e$ as a subgraph, then $\text{rc}_k(G) \leq n-3$ for all integers k with $2 \leq k \leq \ell$.*

Proof. By (3.1), it suffices to show that $\text{rc}(G) \leq n-3$, that is, there is a rainbow coloring of G using at most $n-3$ colors. Let H be the subgraph of order p that is isomorphic to a subdivision of $K_4 - e$ in G. Since $H \neq K_4 - e$, it follows that $p \geq 5$. Since H is a connected graph of cycle rank 2 itself, the size of H is $p+1$. If $G = H$, then G is a connected graph of order $n \geq 7$ and $\delta(G) = 2$. It then follows by Theorem 4.2.1 that $\text{rc}(G) \leq n-3$. Thus, we may assume that $G \neq H$. Then

$$|E(G) - E(H)| = (n+1) - (p+1) = n-p \geq 1.$$

Let $E(G) - E(H) = \{e_1, e_2, \ldots, e_{n-p}\}$. Since $H \neq C_5$ and $\delta(H) = 2$, it follows by Theorem 4.2.1 that $\text{rc}(H) = a \leq p-3$. Let $c_H : E(H) \to [a]$ be a rainbow coloring of H. Define the edge coloring $c : E(G) \to [a+(n-p)]$ by $c(e) = c_H(e)$ if $e \in E(H)$ and $c(e_i) = a+i$ for $1 \leq i \leq n-p$. It can be shown that c is a rainbow coloring of G. Thus,

$$\text{rc}(G) \leq a+(n-p) \leq (p-3)+(n-p) = n-3.$$

Therefore, $\text{rc}_k(G) \leq n-3$ for all integers k with $2 \leq k \leq \ell$ by (3.1). ■

The following is a consequence of the three preceding propositions.

Theorem 4.3.5 *Let G be a connected graph of order $n \geq 4$ having cycle rank 2 and let ℓ be the length of a longest path in G. If G does not contain $K_4 - e$ as a subgraph, then $\mathrm{rc}_k(G) \leq n - 3$ for all integers k with $2 \leq k \leq \ell$.*

By an argument similar to that used in the proof of Lemma 4.2.5, we obtain the following lemma.

Lemma 4.3.6 *If G is a nontrivial connected graph of order $n \geq 4$ and size $m \geq n+1$, the length of whose longest path is ℓ, then G contains a connected spanning subgraph of size $n+1$ whose longest paths have length ℓ.*

The following is a consequence of Observation 3.1.1, Lemma 4.3.6 and Theorem 4.3.5.

Theorem 4.3.7 *Let G be a connected graph of order $n \geq 4$ and size $m \geq n+1$ and let ℓ be the length of a longest path in G. If G does not contain $K_4 - e$ as a subgraph, then $\mathrm{rc}_k(G) \leq n - 3$ for all integers k with $2 \leq k \leq \ell$.*

The results stated in Theorems 4.3.5 and 4.3.7 are best possible. In fact, for each integer $n \geq 4$, there are connected graphs G_n of order $n \geq 4$ and size $n+1$ that contain $K_4 - e$ as a subgraph such that $\mathrm{rc}_k(G_n) = n - 2$ for some integer k where $2 \leq k \leq \ell$ and ℓ is the length of a longest path in G_n. To see this, let $G_4 = K_4 - e$ and for $n \geq 5$, let G_n be constructed from the graph $K_4 - e$ and $P_{n-4} = (v_1, v_2, \ldots, v_{n-4})$ by joining v_1 to a vertex of degree 2 in $K_4 - e$. Since $\mathrm{diam}(G_n) = n - 2$, it follows by Lemma 3.1.2 and Theorem 4.2.6 that $\mathrm{rc}_k(G_n) = n - 2$ for $k = n - 2, n - 1$. Furthermore, for each integer $n \geq 5$, there are connected graphs F_n of order $n \geq 4$ and size $n + 1$ that do not contain $K_4 - e$ as a subgraph such that $\mathrm{rc}_k(F_n) = n - 3$ for some integer k where $2 \leq k \leq \ell$ and ℓ is the length of a longest path in F_n. To see this, let $F_5 = C_5 + e$ and for $n \geq 6$, let F_n be constructed from the graph $C_5 + e$ and $P_{n-5} = (v_1, v_2, \ldots, v_{n-5})$ by joining v_1 to a vertex of degree 2 in $C_5 + e$. Since $\mathrm{diam}(F_n) = n - 3$, it follows by Lemma 3.1.2 and Theorem 4.3.5 that $\mathrm{rc}_k(F_n) = n - 3$ for $k \in \{n - 3, n - 2, n - 1\}$.

4.4 Diametric-Rainbow Colorings in Graphs

We have seen in Lemma 3.1.2 that if G is a nontrivial connected graph of diameter d, then $\mathrm{rc}_d(G) \geq d$. By Corollary 4.1.2, if G is a nontrivial tree of order n, then $\mathrm{rc}_d(G) = n - 1$ and so $\mathrm{rc}_d(G) - \mathrm{diam}(G)$ can be arbitrarily large. This gives rise to the following question:

If G is not a tree, how large can $\mathrm{rc}_d(G) - \mathrm{diam}(G)$ be?

For a nontrivial connected graph G of diameter d, a d-rainbow coloring of G is also referred to as a *diametric-rainbow coloring* of G. Thus, a *diametric-rainbow coloring* of a connected graph G with $\operatorname{diam}(G) = d$ is an edge coloring of G such that every pair of distinct vertices of G are connected by a d-rainbow path in G. By Theorem 4.2.6, if G is a connected graph of order $n \geq 3$ and diameter $d \geq 2$ that is not tree, then

$$d \leq \operatorname{rc}_d(G) \leq n - 2.$$

In fact, if d, γ, n are integers with $2 \leq d \leq \gamma \leq n-2$, then there exists a connected graph G of order n that is not a tree such that $\operatorname{diam}(G) = d$ and $\operatorname{rc}_d(G) = \gamma$. We begin with case where $2 \leq d \leq k = n - 2$.

Theorem 4.4.1 *For every triple (d, k, n) of integers with $2 \leq d \leq k$ and $n = k + 2$, there exists a connected graph G of order n that is not a tree such that*

$$\operatorname{diam}(G) = d \text{ and } \operatorname{rc}_d(G) = k.$$

Proof. First, suppose that $k = d$. Let G_d be the graph of order $d+2$ constructed from the graph $K_2 = (u, v)$ and the path $P_d = (w_1, w_2, \ldots, w_d)$ of order d by joining u and v to w_1 (see Figure 4.6). We saw, in the proof of Proposition 4.2.7, that $\operatorname{diam}(G_d) = d$ and $\operatorname{rc}_d(G_d) = d$. Next, suppose that $k > d$. We consider two cases, according to whether $k = d + 1$ or $k \geq d + 2$.

Case 1. $k = d + 1$. Let G be the graph of order $k + 2 = d + 3$ obtained from G_d by adding a new vertex x and joining x to w_{d-1}. Then $\operatorname{diam}(G) = d$. We claim that $\operatorname{rc}_d(G) = d + 1$. First, we show that $\operatorname{rc}_d(G) \leq d + 1$. Define the edge coloring $c : E(G) \to [d + 1]$ of G defined by

$$c(e) = \begin{cases} 1 & \text{if } e \in \{uv, uw_1, vw_1\} \\ i+1 & \text{if } e = w_i w_{i+1} \text{ for } 1 \leq i \leq d-1 \\ d+1 & \text{if } e = w_{d+1}x. \end{cases}$$

This coloring is shown in Figure 4.9. Since every two vertices are connected by a rainbow path, it follows that c is a d-rainbow $(d + 1)$-coloring of G. Hence, $\operatorname{rc}_d(G) \leq d + 1$.

Next, we show that $\operatorname{rc}_d(G) \geq d + 1$. Assume, to the contrary, that $\operatorname{rc}_d(G) \leq d$. It then follows by Lemma 3.1.2 that $\operatorname{rc}_d(G) = d$. Let there be given a d-rainbow d-coloring $c : E(G) \to [d]$ of G. We may assume, without loss of generality, that $c(uw_1) = 1$ and the color 1 is also assigned to another edge e of H. First, suppose that $c(e) = 1$ for some

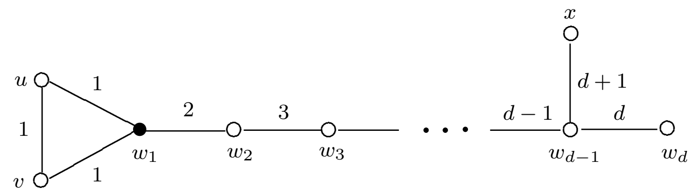


Figure 4.9: A graph G with $\mathrm{diam}(G) = d$ in Case 1

$e \in E(H) - \{xw_{d-1}, w_{d-1}w_d\}$. Since $k \ge d$ and only d colors are used by c, it follows that a k-rainbow $u - w_{d-1}$ path must be a rainbow path. There are exactly two $u - w_{d-1}$ paths in G, namely

$$Q_1 = (u, w_1, w_2, \ldots, w_{d-1}) \text{ and } Q_2 = (u, v, w_1, w_2, \ldots, w_{d-1}).$$

Since Q_1 is not a rainbow path, it follows that Q_2 must be a rainbow path. Thus, $\{c(uv), c(vw_1)\}= \{2, 3\}$, say $c(uv) = 2$ and $c(vw_1) = 3$. This implies that $c(f) \notin \{2, 3\}$ for each $f \in E(H) - \{xw_{d-1}, w_{d-1}w_d\}$ and so $\{c(xw_{d-1}), c(w_{d-1}w_d)\} = \{2, 3\}$, say $c(xw_{d-1}) = 2$ and $c(w_{d-1}w_d) = 3$. However then, there is no rainbow $v - w_d$ path in G, which is a contradiction. Therefore, $\mathrm{rc}_d(G) \ge d + 1$ and so $\mathrm{rc}_d(G) = d + 1$.

Case 2. $k \ge d + 2$. Let G be the graph of order $k + 2$ obtained from G_d by adding $k - d + 1$ new vertices $x_1, x_2, \ldots, x_{k-d+1}$ and joining x_i to w_{d-1} for $1 \le i \le k - d + 1$. Then $\mathrm{diam}(G) = d$. It remains to show that $\mathrm{rc}_d(G) = k$. The subgraph $G - \{u, v\}$ is a tree of size k. For every pair x, y of distinct vertices of H, there is a unique $x - y$ path in G. Hence, every d-rainbow coloring must assign k distinct colors to the k edges of H, implying that $\mathrm{rc}_d(G) \ge k$. Next, define the edge coloring $c : E(G) \to [k]$ by

- ⋆ assigning the colors $1, 2, 3$ to the three edges uv, uw_1, vw_1,
  namely $\{c(uv), c(uw_1), c(vw_1)\} = \{1, 2, 3\}$,
- ⋆ assigning the colors $1, 2, 3$ to the three edges $w_{d-1}x_i$ for $i = 1, 2, 3$,
  namely $\{c(w_{d-1}x_1, c(w_{d-1}x_2, c(w_{d-1}x_3\} = \{1, 2, 3\}$,
- ⋆ assigning the $k - 3$ colors in $\{4, 5, \ldots, k\}$ to the remaining $k - 3$ edges of G.

We may assume that the coloring c is the one as shown in Figure 4.10.

Since H is a rainbow tree in G, every two vertices of H are connected by a rainbow path. In fact, every two vertices of G are connected by a rainbow path. For example, the vertices u and x_2 are connected by the rainbow $u - x_2$ path $(u, v, w_1, \ldots, w_{d-1}, x_2)$ in G. Thus, c is a rainbow coloring of G and so c is a d-rainbow k-coloring of G. Therefore, $\mathrm{rc}_d(G) \le k$ and so $\mathrm{rc}_d(G) = k$. ■

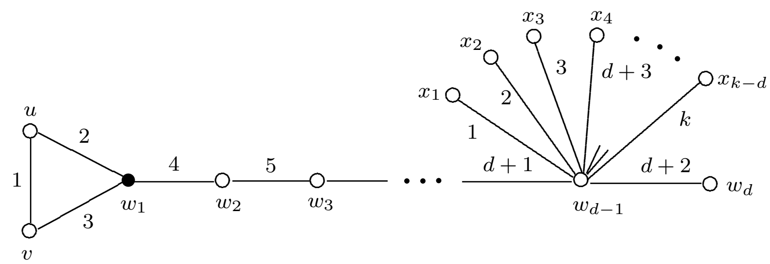


Figure 4.10: A graph G with $\mathrm{diam}(G) = d$ in Case 2

Theorem 4.4.2 *For every triple (d, k, n) of integers with $2 \le d \le k \le n-3$, there exists a connected graph G of order n that is not a tree such that*

$$\mathrm{diam}(G) = d \textit{ and } \mathrm{rc}_d(G) = k.$$

Proof. First, suppose that $2 \le d = k \le n-3$. Let G be the graph obtained from the path $P_d = (u_1, u_2, \ldots, u_d)$ of order d and the complete graph K_{n-d} by joining u_d to each vertex of K_{n-d}. Then the order of G is n and diameter of G is d. We show that $\mathrm{rc}_d(G) = d$. Since $\mathrm{rc}_d(G) \ge d$, it remains to show that G has a d-rainbow d-coloring. Define the edge coloring $c : E(G) \to [d]$ by

$$c(e) = \begin{cases} 1 & \text{if } e \in E(K_{n-d}) \\ i & \text{if } e = u_i u_{i+1} \text{ for } 1 \le i \le d-1 \\ d & \text{if } e = u_d x \text{ for each } x \in V(K_{n-d}). \end{cases}$$

Then c is a d-rainbow d-coloring and so $\mathrm{rc}_d(G) = d$.

Next, suppose that $2 \le d < k \le n-3$. Then $n \ge k+3 \ge 6$ and $k-d+1 \ge 2$. We begin with the tree T of order $k+1$ and diameter $d \ge 2$ obtained from the path $P_d = (u_1, u_2, \ldots, u_d)$ of order d and the $k-d+1 \ge 2$ vertices $v_1, v_2, \ldots, v_{k-d+1}$ by joining the end-vertex u_d of P_d to each vertex v_i for $1 \le i \le k-d+1$. Let $c_T : E(T) \to [k]$ be a k-rainbow coloring of T defined by

$$c_T(e) = \begin{cases} i & \text{if } e = u_i u_{i+1} \text{ for } 1 \le i \le d-1 \\ d+i-1 & \text{if } e = u_d v_i \text{ for } 1 \le i \le k-d+1. \end{cases} \tag{4.3}$$

The tree T and the coloring c_T of T are shown in Figure 4.11. Next, we construct a graph G of order n that is not a tree such that $\mathrm{diam}(G) = d$ and $\mathrm{rc}_d(G) = k$. There are two cases, according to whether n and k are of opposite parity or of the same parity.

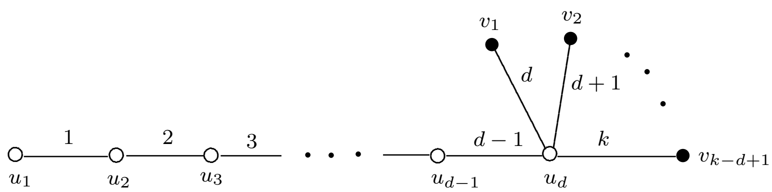


Figure 4.11: A k-rainbow coloring c_T of the tree T

Case 1. *n and k are of opposite parity.* Write $n - k = 2p + 1 \geq 3$ for some positive integer p. Let $F = pK_2$, where

$$V(F) = \{w_1, w'_1, w_2, w'_2, \ldots, w_p, w'_p\}$$

such that $w_i w'_i \in E(F)$ for $1 \leq i \leq p$. Now, let G be the graph obtained from the tree T and the graph F by joining the vertex u_d of T to each vertex of F. Then the order of G is $k + 1 + (n - k - 1) = n$ and $\text{diam}(G) = d$. Since every d-rainbow coloring of G must assign k distinct colors to the k edges of T, it follows that $\text{rc}_d(G) \geq k$. Next, we show that G has a d-rainbow k-coloring. Let c_T be the coloring of T described in (4.3). Define the edge coloring $c : E(G) \to [k]$ of G by

$$c(e) = \begin{cases} c_T(e) & \text{if } e \in E(T) \\ 1 & \text{if } e = w_i w'_i \text{ for } 1 \leq i \leq p \\ d & \text{if } e = u_d w_i \text{ for } 1 \leq i \leq p \\ d+1 & \text{if } e = u_d w'_i \text{ for } 1 \leq i \leq p. \end{cases}$$

The graph G and the coloring c of G are shown in Figure 4.12. We show that c is a d-rainbow coloring of G.

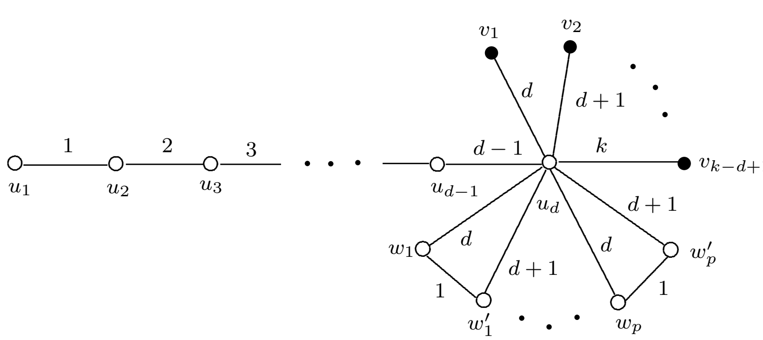


Figure 4.12: A k-rainbow coloring c of the graph G in Case 1

Let x and y be two nonadjacent vertices of G. Since T is a rainbow subgraph of G, every two vertices of T are connected by a rainbow path in T and so in G. Thus, we may assume that one of x and y is not a vertex of T, say $y \notin V(T)$. By symmetry, we may assume that $y = w_1$. Thus, $x \notin N(w_1) = \{u_d, w_1'\}$.

- ⋆ If $x = u_i$ where $1 \le i \le d-1$, then $(u_i, u_{i+1}, \ldots, u_d, w_1)$ is a rainbow $x-y$ path.
- ⋆ If $x = v_1$, then (v_1, u_d, w_1', w_1) is a rainbow $x-y$ path.
- ⋆ If $x = v_i$ where $2 \le i \le k-d+1$, then (v_i, u_d, w_1) is a rainbow $x-y$ path.
- ⋆ If $x = w_i$ or $x = w_i'$ where $2 \le i \le p$, then, since (w_i, w_i', u_d, w_1) is a rainbow path, there is a rainbow $x-y$ path.

Thus, c is a d-rainbow coloring of G and so $\mathrm{rc}_d(G) \le k$. Hence, $\mathrm{rc}_d(G) = k$.

Case 2. *n and k are of the same parity.* Write $n-k = 2p \ge 4$ for some integer $p \ge 2$. For $p = 2$, let $F = P_3 = (w_1, w, w_1')$ and for $p \ge 3$, let $F = P_3 + (p-2)K_2$, where $P_3 = (w_1, w, w_1')$ and $w_{i+1}w_{i+1}' \in E(F)$ for $1 \le i \le p-2$. The graph G is obtained from the tree T and the graph F by joining the vertex u_d of T to each vertex of F. The order of G is $k+1+(n-k-1) = n$ and $\mathrm{diam}(G) = d$. Since every d-rainbow coloring of G must assign k distinct colors to the k edges of T, it follows that $\mathrm{rc}_d(G) \ge k$. Next, we show that G has a d-rainbow k-coloring. Let c_T be the coloring of T described in (4.3). Define the edge coloring $c : E(G) \to [k]$ of G by

$$c(e) = \begin{cases} c_T(e) & \text{if } e \in E(T) \\ 1 & \text{if } e = w_1w \text{ or } e = w_iw_i' \text{ for } 2 \le i \le p-2 \text{ if } p \ge 3 \\ 2 & \text{if } e = ww_1' \\ d-1 & \text{if } e = u_dw \\ d & \text{if } e = u_dw_i \text{ for } 1 \le i \le p-2 \text{ if } p \ge 3 \\ d+1 & \text{if } e = u_dw_i' \text{ for } 1 \le i \le p-2 \text{ if } p \ge 3. \end{cases}$$

The graph G and the coloring c of G are shown in Figure 4.13. We show that c is a d-rainbow coloring of G.

Let x and y be two nonadjacent vertices of G. Since T is a rainbow subgraph of G, every two vertices of T are connected by a rainbow path in T and so in G. Thus, we may assume that one of x and y is not a vertex of T, say $y \notin V(T)$. By symmetry, it suffices to consider $y = w_1$ or $y = w$. First, suppose that $y = w_1$. Since (w_1, w, w_1') is a rainbow path, we may assume that $x \notin N(w_1) \cup \{w_1'\}$.

- ⋆ If $x = u_i$ where $1 \le i \le d-1$, then $(u_i, u_{i+1}, \ldots, u_d, w_1)$ is a rainbow $x-y$ path.

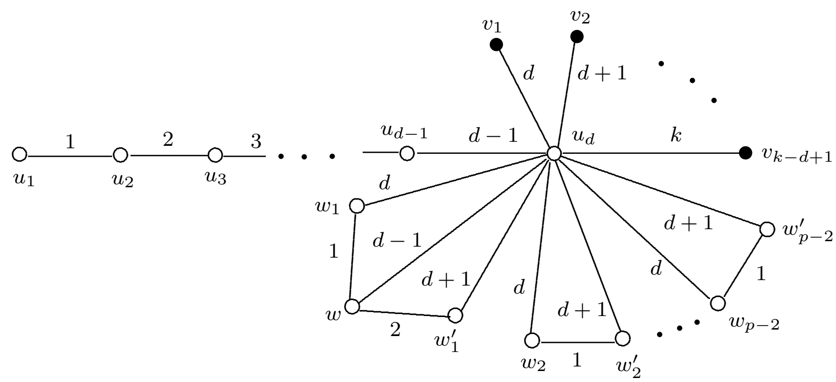


Figure 4.13: A k-rainbow coloring c of the graph G in Case 2

- $\star$ If $x = v_1$, then (v_1, u_d, w'_1, w, w_1) is a rainbow $x - y$ path.
- $\star$ If $x = v_i$ where $2 \le i \le k - d + 1$, then (v_i, u_d, w_1) is a rainbow $x - y$ path.
- $\star$ If $x = w_i$ or $x = w'_i$ where $2 \le i \le p - 2$, then, since (w_i, w'_i, u_d, w_1) is a rainbow path, there is a rainbow $x - y$ path.

Next, suppose that $y = w$. Thus, $x \notin N(w)$.

- $\star$ If $x = u_i$ where $1 \le i \le d - 1$, then $(u_i, u_{i+1}, \ldots, u_d, w_1, w)$ is a d-rainbow $x - y$ path.
- $\star$ If $x = v_i$ where $1 \le i \le k - d + 1$, then (v_i, u_d, w) is a rainbow $x - y$ path.
- $\star$ If $x = w_i$ or $x = w'_i$ where $2 \le i \le p - 2$, then, since (w_i, w'_i, u_d, w) is a rainbow path, there is a rainbow $x - y$ path.

Thus, c is a d-rainbow coloring of G and so $\mathrm{rc}_d(G) \le k$. Hence, $\mathrm{rc}_d(G) = k$. ■

Combining Theorems 4.4.1 and 4.4.2, we have the following characterization of all triples (d, k, n) of integers with $2 \le d \le k \le n-1$ that can be realizable as the diameter, d-rainbow connection number and order, respectively, of some connected graph.

Theorem 4.4.3 *Let (d, k, n) be a triple of integers with $2 \le d \le k \le n - 1$. Then there exists a connected graph G of order n that is not a tree such that* $\mathrm{diam}(G) = d$ *and* $\mathrm{rc}_d(G) = k$ *if and only if* $k \ne n - 1$.

Chapter 5

k-Rainbow Hamiltonian-Connected Graphs

In this dissertation, we introduce the concept of k-rainbow Hamiltonian paths in Hamiltonian-connected graphs and investigate k-rainbow Hamiltonian-path colorings in two well-known classes of Hamiltonian-connected graphs, namely the join $G \vee K_1$ of a Hamiltonian graph G and the trivial graph K_1 and the prism $G \,\square\, K_2$ where G is a Hamiltonian graph of odd order. Other results and open questions are also presented.

5.1 Introduction

First, we review some definitions and results on Hamiltonian graphs and Hamiltonian-connected graphs. A *Hamiltonian cycle* in a graph G is a cycle containing every vertex of G and a graph having a Hamiltonian cycle is a *Hamiltonian graph.* A *Hamiltonian path* in a graph G is a path containing every vertex of G. A graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u-v$ path for every pair u, v of distinct vertices of G. Observe that every Hamiltonian-connected graph is Hamiltonian. However, the converse is not true. For example, for $n \geq 4$, C_n is Hamiltonian but not Hamiltonian-connected.

For a nontrivial graph G, recall that $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G, respectively, and

$$\sigma_2(G) = \min\{\deg u + \deg v : uv \notin E(G)\}$$

where $\deg w$ is the degree of a vertex w in G. Ore [31] proved the following results in 1963.

Theorem 5.1.1 *If G is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n + 1$, then G is Hamiltonian-connected.*

Corollary 5.1.2 *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq (n+1)/2$, then G is Hamiltonian-connected.*

In both a rainbow coloring and a proper-path coloring of a connected graph G, every two vertices u and v of G are connected by a $u - v$ path, the colors of whose edges satisfy some specified required property; however, there is no condition on the length of such a path. For certain graphs G, though, it is natural to require the existence of such an edge coloring of G for which every two vertices of G are connected by a path of some prescribed length. For a Hamiltonian-connected graph G, an edge coloring c is called a *Hamiltonian-connected rainbow coloring* if every two vertices of G are connected by a rainbow Hamiltonian path in G. The minimum number of colors needed in a Hamiltonian-connected rainbow coloring of G is the *rainbow Hamiltonian-connection number* of G and is denoted by $\text{hrc}(G)$. If G is a Hamiltonian-connected graph of order $n \geq 4$ and size m, then

$$\max\{\text{rc}(G), n-1\} \leq \text{hrc}(G) \leq m. \tag{5.1}$$

For a Hamiltonian-connected graph G, an edge coloring c is a *proper Hamiltonian-path coloring* if every two vertices of G are connected by a proper Hamiltonian path in G. The minimum number of colors needed in a proper Hamiltonian-path coloring of G is the *proper Hamiltonian-connection number* of G and is denoted by $\text{hpc}(G)$. Since every proper edge coloring of a Hamiltonian-connected graph G of order at least 3 is a proper Hamiltonian-path coloring of G and there is no proper Hamiltonian-path 1-coloring of G, it follows that

$$2 \leq \text{hpc}(G) \leq \chi'(G). \tag{5.2}$$

These concepts were introduced and studied by Bi et al. in [2, 3, 4]. Inspired by proper Hamiltonian-path colorings, k-rainbow colorings and Hamiltonian-connected rainbow colorings, we study the concept of k-rainbow Hamiltonian-path colorings of Hamiltonian-connected graphs.

Let G be an edge-colored Hamiltonian-connected graph, where adjacent edges may be colored the same. For an integer $k \geq 2$, a Hamiltonian path P in G is a *k-rainbow Hamiltonian path* if every subpath of P having length at most k is a rainbow path. An edge coloring c of G is a *k-rainbow Hamiltonian-path coloring* if every two vertices of G are connected by a k-rainbow Hamiltonian path in G. If j colors are used in a k-rainbow Hamiltonian-path coloring c of G, then c is referred to as a *k-rainbow Hamiltonian-path j-edge coloring* (or simply a *k-rainbow Hamiltonian-path j-coloring*). The minimum number of colors required of a k-rainbow Hamiltonian-path coloring of G is the *k-rainbow*

Hamiltonian-connection number of G and is denoted by $\mathrm{hrc}_k(G)$. As expected, k-rainbow Hamiltonian-path colorings are intermediate to Hamiltonian-connected rainbow colorings and proper Hamiltonian-path colorings. In particular, if G is a Hamiltonian-connected graph of order $n \geq 4$ and size m, then

$$2 \leq \mathrm{hpc}(G) = \mathrm{hrc}_2(G) \leq \mathrm{hrc}_3(G) \leq \cdots \leq \mathrm{hrc}_{n-1}(G) = \mathrm{hrc}(G) \leq m. \tag{5.3}$$

If H is a Hamiltonian-connected spanning subgraph of a graph G and c is a k-rainbow Hamiltonian-path coloring of H for some integer $k \geq 2$, then the coloring c can be extended to a k-rainbow Hamiltonian-path coloring of G by assigning any color used by c to each edge in $E(G) - E(H)$. Thus, we have the following observation.

Observation 5.1.3 *If H is a Hamiltonian-connected spanning subgraph of a graph G, then $k \leq \mathrm{hrc}_k(G) \leq \mathrm{hrc}_k(H)$ for every integer k with $2 \leq k \leq n-1$.*

5.2 k-Rainbow Hamiltonian-Path Colorings of Wheels

For two vertex-disjoint graphs F and H, let $F \vee H$ denote the *join* of F and H. In particular, the join $G \vee K_1$ of a graph G and the trivial graph K_1 is obtained by joining the vertex of K_1 to each vertex of G. If G is a Hamiltonian graph of order $n \geq 3$, then $G \vee K_1$ is a Hamiltonian-connected graph of order $n+1$ and so the length of a longest path in $G \vee K_1$ is n. Since the wheel $W_n = C_n \vee K_1$ is a spanning Hamiltonian-connected subgraph of $G \vee K_1$, it follows by Observation 5.1.3 that $\mathrm{hrc}_k(G \vee K_1) \leq \mathrm{hrc}_k(W_n)$. This suggests investigating k-rainbow Hamiltonian-path colorings of the wheels W_n since $\mathrm{hrc}_k(W_n)$ is an upper bound for $\mathrm{hrc}_k(G \vee K_1)$ for every Hamiltonian graph G of order n. We have seen in Chapters 2 and 3 that $\mathrm{rc}_2(W_3) = \mathrm{rc}(W_3) = 1$ and for integers k and n with $3 \leq k \leq n$ and $n \geq 4$,

$$\begin{aligned} \mathrm{rc}_2(W_n) &= \mathrm{pc}(W_n) = 2, \\ \mathrm{rc}_k(W_n) &= \begin{cases} 2 & \text{if } 4 \leq n \leq 6 \\ 3 & \text{if } n \geq 7. \end{cases} \end{aligned}$$

The rainbow Hamiltonian-connection number $\mathrm{hrc}(W_n)$ has been determined for each integer $n \geq 3$.

Theorem 5.2.1 [2] *For each integer $n \geq 3$, $\mathrm{hrc}(W_n) = n$.*

By Theorem 5.2.1 then, $\mathrm{hrc}_n(W_n) = n$. We now investigate the following problem:

What can be said about the value of $\mathrm{hrc}_k(W_n)$ *for integers* k *and* n *with* $2 \leq k \leq n-1$*?*

It is known (see [2, 3, 4]) that

(i) $\mathrm{hpc}(K_3) = 3$ and $\mathrm{hpc}(K_n) = 2$ for $n \geq 4$ and

(ii) $\mathrm{hrc}(K_n) = n - 1$ for $n \geq 4$.

Since $W_3 = K_4$, it follows that $\mathrm{hpc}(K_4) = \mathrm{hrc}_2(K_4) = 2$ and $\mathrm{hrc}(K_4) = \mathrm{hrc}_3(W_3) = 3$. Thus, we now assume that $n \geq 4$. In order to present a lower bound for $\mathrm{hrc}_k(W_n)$, we first present a lemma.

Lemma 5.2.2 *Let* k *and* n *be integers with* $2 \leq k \leq n-2$*. If* c *is a* k*-rainbow Hamiltonian-path coloring of* $W_n = C_n \vee K_1$*, then every path of length* k *in* C_n *is a rainbow path. In particular, the restriction of* c *to* C_n *is a proper edge coloring of* C_n*.*

Proof. For an integer $n \geq 3$, let $W_n = C_n \vee K_1$ where C_n=(u_1, u_2, …, u_n, $u_{n+1} = u_1$) and $V(K_1) = \{u_0\}$. For each integer i with $1 \leq i \leq n$, there are exactly two $u_i - u_0$ Hamiltonian paths in W_n, namely

$$\begin{aligned} P_{i,0} &= (u_i, u_{i+1}, \ldots, u_n, u_1, u_2, \ldots, u_{i-1}, u_0) \\ Q_{i,0} &= (u_i, u_{i-1}, \ldots, u_n, u_{n-1}, u_{n-2}, \ldots, u_{i+1}, u_0) \end{aligned}$$

where all subscripts are expressed as positive integers modulo n. Assume, to the contrary, there is a path of length k in C_n that is not a rainbow path, say $P = (u_1, u_2, \ldots, u_{k+1})$ is not a rainbow path. Since P is a subpath of the two $u_{k+2} - u_0$ Hamiltonian paths $P_{k+2,0}$ and $Q_{k+2,0}$ in W_n, there is no k-rainbow Hamiltonian $u_{k+2} - u_0$ path in W_n, which is a contradiction. Consequently, the restriction of c to C_n is a proper edge coloring of C_n. ■

Theorem 5.2.3 *If* k *and* n *are integers with* $2 \leq k \leq n-1$ *and* $n \geq 4$*, then*

$$\mathrm{hrc}_k(W_n) \geq k + 1.$$

In particular, $\mathrm{hpc}(W_n) \geq 3$*.*

Proof. For an integer $n \geq 4$, let $W_n = C_n \vee K_1$ where C_n=(u_1, u_2, …, u_n, $u_{n+1} = u_1$) and $V(K_1) = \{u_0\}$. By Lemma 5.2.2, it follows that $\mathrm{hrc}_k(W_n) \geq k$. We now show that $\mathrm{hrc}_k(W_n) \neq k$. Assume, to the contrary, that there is a k-rainbow Hamiltonian-path

k-coloring $c : E(W_n) \to [k]$ of W_n. As we saw in the proof of Lemma 5.2.2, for each integer i with $1 \le i \le n$, there are exactly two $u_i - u_0$ Hamiltonian paths in W_n, namely

$$\begin{aligned} P_{i,0} &= (u_i, u_{i+1}, \ldots, u_n, u_1, u_2, \ldots, u_{i-1}, u_0) \\ Q_{i,0} &= (u_i, u_{i-1}, \ldots, u_n, u_{n-1}, u_{n-2}, \ldots, u_{i+1}, u_0) \end{aligned}$$

where all subscripts are expressed as the integers $1, 2, \ldots n$ modulo n. Since c is a k-rainbow Hamiltonian-path coloring of W_n, at least one of $P_{1,0}$ and $Q_{1,0}$ is a k-rainbow Hamiltonian path. By the symmetry of W_n, we may assume that $P_{1,0} = (u_1, u_2, \ldots, u_n, u_0)$ is a k-rainbow Hamiltonian path and $c(u_n u_0) = k$. Let $n = kq + r$ for some integers q and r where $q \ge 1$ and $0 \le r \le k-1$ and let $C_{P_{1,0}} = (c(u_1u_2), c(u_2u_3), \ldots, c(u_nu_0))$. By Lemma 5.2.2, we may assume, without loss of generality, that

$$C_{P_{1,0}} = \begin{cases} (1, 2, \ldots, k, ***, 1, 2, \ldots, k) & \text{if } r = 0 \\ (1, 2, \ldots, k, ***, 1, 2, \ldots, k, 1, 2, \ldots, r) & \text{if } 1 \le r \le k-1. \end{cases} \tag{5.4}$$

where $***$ represents repeating the next k-tuple as the preceding k-tuple. We consider two cases, according to whether $r = 0$ or $1 \le r \le k-1$.

Case 1. $r = 0$. There are exactly two Hamiltonian $u_1 - u_{n-1}$ paths in W_n, namely,

$$\begin{aligned} P_{1,n-1} &= (u_1, u_n, u_0, u_2, u_3, \ldots, u_{n-1}) \\ Q_{1,n-1} &= (u_1, u_2, \ldots, u_{n-2}, u_0, u_n, u_{n-1}). \end{aligned}$$

Since $c(u_1u_n) = c(u_nu_0) = k$, it follows that $P_{1,n-1}$ is not a k-rainbow Hamiltonian path. Because $c(u_{n-k}u_{n-k+1}) = c(u_0u_n) = k$, the path

$$(u_{n-k}, u_{n-k+1}, \ldots, u_{n-2}, u_0, u_n)$$

of length k in $Q_{1,n-1}$ is not a rainbow path and so $Q_{1,n-1}$ is not a k-rainbow Hamiltonian path. Thus, u_1 and u_{n-1} are not connected by any k-rainbow Hamiltonian path in W_n, a contradiction.

Case 2. $1 \le r \le k-1$. Consider the path $P = (u_n, u_1, u_2, \ldots, u_k)$ of length k on the cycle C_n. By (5.4) then, $c(P - u_n) = \{1, 2, \ldots, k-1\}$ and $c(u_nu_1) = r$. Since $1 \le r \le k-1$, it follows that u_nu_1 has the same color as one of the edges on $P - u_n$ and so P is not a rainbow path, which contradicts Lemma 5.2.2. ■

Next, we present an upper bound for $\text{hrc}_k(W_n)$ where $2 \le k \le n-1$ and $n \ge 4$.

Theorem 5.2.4 *For integers k and n with $2 \leq k \leq n-1$ and $n \geq 4$, let $n = kq + r$ where $0 \leq r < k$. If n is even and $n > 2k + 2r$ or n is odd and $n > 2k + 2r + 1$, then*

$$\operatorname{hrc}_k(W_n) \leq k + r + \lceil \tfrac{n}{2} \rceil.$$

Proof. Let $W_n = C_n \vee K_1$ where $C_n=(u_1,\ u_2,\ \ldots,\ u_n,\ u_{n+1} = u_1)$ and $V(K_1) = \{u_0\}$. It suffices to show that there is a k-rainbow Hamiltonian-path coloring c of W_n using $k + r + \lceil \frac{n}{2} \rceil$ colors. First, we assign the $k + r$ colors $1, 2, \ldots, k + r$ to the edges of C_n, where

$$c(C_n) = (1, 2, \ldots, k, *\,*\,*, 1, 2, \ldots, k, k+1, k+2, \ldots, k+r),$$

where $*\,*\,*$ represents repeating the next k-tuple as the preceding k-tuple. Next, we assign $\lceil \frac{n}{2} \rceil$ new colors to the spokes of W_n. We consider two cases, according to whether n is even or n is odd.

Case 1. *n is even and $n > 2k + 2r$.* Then we define

$$c(u_0u_{2t-1}) = c(u_0u_{2t}) = k + r + t \text{ for } 1 \leq t \leq \tfrac{n}{2}.$$

Thus, we obtain an edge coloring of W_n with $k + r + \frac{n}{2}$ colors. Since $n > 2k + 2r$, it follows that $k + r + \frac{n}{2} < n$.

Case 2. *n is odd and $n > 2k + 2r + 1$.* Then we define

$$c(u_0u_{2t-1}) = c(u_0u_{2t}) = k + r + t \text{ for } 1 \leq t \leq \tfrac{n-1}{2}$$

$$c(u_0u_n) = k + r + \frac{n+1}{2}.$$

Thus, we obtain an edge coloring of W_n with $k + r + \frac{n+1}{2}$ colors. Since $n > 2k + 2r + 1$, it follows that $k + r + \frac{n+1}{2} < n$.

It remains to show that c is a k-rainbow Hamiltonian-path coloring of W_n. In what follows, all subscripts are expressed as the integers $1, 2, \ldots, n$ modulo n. For every two distinct vertices u_i and u_j of W_n, we show that there is a k-rainbow Hamiltonian $u_i - u_j$ path $P_{i,j}$ in W_n as follows:

- ⋆ For $1 \leq i \leq n$, $P_{i,0} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_n, u_1, u_2, \ldots, u_{i-1}, u_0)$.
- ⋆ For $i \neq n$, $P_{i,i+1} = (u_i, u_{i-1}, u_{i-2}, \ldots, u_1, u_0, u_n, u_{n-1}, u_{n-2} \ldots, u_{i+1})$.
- ⋆ For $i = n$, $P_{n,1} = (u_n, u_{n-1}, u_0, u_{n-2}, u_{n-3}, \ldots, u_2, u_1)$ if n is even and $P_{n,1} = (u_n, u_0, u_{n-1}, u_{n-2}, u_{n-3}, \ldots, u_2, u_1)$ if n is odd.

⋆ For $1 \leq i, j \leq n$ and $j \geq i+2$,

$$P_{i,j} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_{j-1}, u_0, u_{i-1}, u_{i-2}, \ldots, u_n, u_{n-1}, \ldots, u_j).$$

Therefore, c is a k-rainbow Hamiltonian-path $(k + r + \lceil \frac{n}{2} \rceil)$-coloring of W_n and so $\text{hrc}_k(W_n) \leq k + r + \lceil \frac{n}{2} \rceil$. ■

The upper bound for $\text{hrc}_k(W_n)$ in Theorem 5.2.4 can be improved. As an example, we consider the case when $k = 2$. By Theorem 5.2.4, $\text{hpc}(W_n) \leq 2 + \lceil \frac{n}{2} \rceil$ if $n \geq 4$ is even and $\text{hpc}(W_n) \leq 3 + \lceil \frac{n}{2} \rceil$ if $n \geq 3$ is odd. For each integer $n \in \{4, 5, 6\}$, a proper Hamiltonian-path coloring of W_n using the colors $1, 2, 3$ is shown in Figure 5.1 and so $\text{hpc}(W_n) = 3$ by Theorem 5.2.3. Therefore, $\text{hpc}(W_n) = \lceil \frac{n}{2} \rceil$ for $n = 5, 6$. In general, $\lceil \frac{n}{2} \rceil$ is an upper bound for $\text{hpc}(W_n)$ for all $n \geq 5$, as we show next.

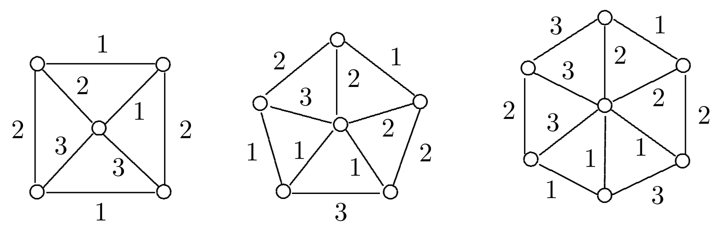


Figure 5.1: Proper Hamiltonian-path colorings of W_n for $n = 4, 5, 6$

Theorem 5.2.5 *For each integer $n \geq 5$,* $\text{hpc}(W_n) \leq \lceil \frac{n}{2} \rceil$.

Proof. For an integer $n \geq 5$, let $W_n = C_n \vee K_1$ where $C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ and $V(K_1) = \{u_0\}$. We consider two cases, according to whether n is even or n is odd.

Case 1. *$n \geq 6$ is even.* Let $n = 2t$ for some integer $t \geq 2$. Define an edge coloring $c : E(W_n) \to [t]$ of W_n by

$$c(u_i u_{i+1}) = \begin{cases} i & \text{if } 1 \leq i \leq t \\ i - t & \text{if } t+1 \leq i \leq n \end{cases}$$

$$c(u_i u_0) = \begin{cases} 2 & \text{if } i = 1, 2 \\ 1 & \text{if } i = 3, 4 \\ p & \text{if } i = 2p-1, 2p \text{ and } 3 \leq p \leq t. \end{cases}$$

It remains to show that c is a proper Hamiltonian-path coloring of W_n. In what follows, all subscripts are expressed as the integers $1, 2, \ldots, n$ modulo n. For every two distinct vertices u_i and u_j of W_n, we show that there is a proper Hamiltonian $u_i - u_j$ path $P_{i,j}$ in W_n as follows:

⋆ For $1 \le i \le n$, $P_{i,0} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_n, u_1, u_2, \ldots, u_{i-1}, u_0)$.

⋆ For $i \ne n$, $P_{i,i+1} = (u_i, u_{i-1}, u_{i-2}, \ldots, u_1, u_0, u_n, u_{n-1}, u_{n-2} \ldots, u_{i+1})$.

⋆ For $i = n$, $P_{n,1} = (u_n, u_{n-1}, u_0, u_{n-2}, u_{n-3}, \ldots, u_2, u_1)$.

⋆ For $1 \le i, j \le n$ and $j \ge i + 2$,

$P_{i,j} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_{j-1}, u_0, u_{i-1}, u_{i-2}, \ldots, u_n, u_{n-1}, \ldots, u_j)$.

Therefore, c is a proper Hamiltonian t-coloring and so $\mathrm{hpc}(W_n) \le t = \frac{n}{2}$.

Case 2. *$n \ge 5$ is odd.* Let $n = 2t + 1$ for some integer $t \ge 2$. Define an edge coloring $c : E(W_n) \to [t + 1]$ of W_n by

$$c(u_i u_{i+1}) = \begin{cases} i & \text{if } 1 \le i \le t+1 \\ i - (t+1) & \text{if } t+2 \le i \le n \end{cases}$$

$$c(u_i u_0) = \begin{cases} 2 & \text{if } i = 1, 2 \\ 1 & \text{if } i = 3, 4 \\ p & \text{if } i = 2p-1, 2p \text{ and } 3 \le p \le t \\ t+1 & \text{if } i = n. \end{cases}$$

It remains to show that c is a proper Hamiltonian-path coloring of W_n. As before, all subscripts are expressed as the integers $1, 2, \ldots, n$ modulo n. For every two distinct vertices u_i and u_j of W_n, we show that there is a proper Hamiltonian $u_i - u_j$ path $P_{i,j}$ in W_n as follows:

⋆ For $1 \le i \le n$, $P_{i,0} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_n, u_1, u_2, \ldots, u_{i-1}, u_0)$.

⋆ For $i \ne n$, $P_{i,i+1} = (u_i, u_{i-1}, u_{i-2}, \ldots, u_1, u_0, u_n, u_{n-1}, u_{n-2} \ldots, u_{i+1})$.

⋆ For $i = n$, $P_{n,1} = (u_n, u_0, u_{n-1}, u_{n-2}, u_{n-3}, \ldots, u_2, u_1)$.

⋆ For $1 \le i, j \le n$ and $j \ge i + 2$,

$P_{i,j} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_{j-1}, u_0, u_{i-1}, u_{i-2}, \ldots, u_n, u_{n-1}, \ldots, u_j)$.

Therefore, this coloring c is a proper Hamiltonian $(t + 1)$-coloring and so $\mathrm{hpc}(W_n) \le t + 1 = \lceil \frac{n}{2} \rceil$. ■

The following is a consequence of Theorems 5.2.3 and 5.2.5

Corollary 5.2.6 *If $n \ge 5$, then* $3 \le \mathrm{hpc}(W_n) \le \lceil \frac{n}{2} \rceil$.

By Corollary 5.2.6 then, $\mathrm{hpc}(W_7) = 3$ or $\mathrm{hpc}(W_7) = 4$. In fact, our conjecture is that $\mathrm{hpc}(W_7) = 4$.

5.3 k-Rainbow Hamiltonian-Path Colorings in Prisms

For a graph G, let $G \,\square\, K_2$ denote the *Cartesian product* of G and K_2. It is known that if G is a Hamiltonian graph of odd order $n \geq 3$, then $G \,\square\, K_2$ is a Hamiltonian-connected graph of order $2n$ and so the length of a longest path in $G \,\square\, K_2$ is $2n-1$. Since the prism $C_n \,\square\, K_2$ is a spanning Hamiltonian-connected subgraph of $G \,\square\, K_2$, it follows by Observation 5.1.3 that $\text{hrc}_k(G \,\square\, K_2) \leq \text{hrc}_k(C_n \,\square\, K_2)$ for each integer k with $2 \leq k \leq 2n-1$. This leads us to a study of the k-rainbow Hamiltonian-path colorings of the prisms $C_n \,\square\, K_2$ for odd integers $n \geq 3$. As we saw in Theorem 3.4.3, the k-rainbow connection numbers $\text{rc}_k(C_n \,\square\, K_2)$ have been determined for all integers k and n with $2 \leq k \leq 2n-1$, namely

$$\text{rc}_k(C_n \,\square\, K_2) = \min\left\{k,\ \left\lfloor \frac{n}{2} \right\rfloor + 1\right\}.$$

Recall that $C_n \,\square\, K_2$ is Hamiltonian-connected if and only if $n \geq 3$ is odd. The proper and rainbow Hamiltonian-connection numbers $\text{hpc}(C_n \,\square\, K_2)$ and $\text{hrc}(C_n \,\square\, K_2)$ have been determined for each integer $n \geq 3$.

Theorem 5.3.1 [3] *For each odd integer $n \geq 3$, $\text{hpc}(C_n \,\square\, K_2) = 3$.*

Theorem 5.3.2 [2] *For each odd integer $n \geq 3$, $\text{hrc}(C_3 \,\square\, K_2) = 7$ and $\text{hrc}(C_n \,\square\, K_2) = 3n$ for $n \geq 5$.*

This leads us to the following problem:

> *Investigate $\text{hrc}_k(C_n \,\square\, K_2)$ for integers k and n where $3 \leq k \leq 2n-2$ and $n \geq 3$ is odd.*

It is often quite challenging to determine the exact value of $\text{hrc}_k(G)$ for a given graph G even when the order of G is relatively small. Next, we determine the value of $\text{hrc}_k(C_3 \,\square\, K_2)$ for $2 \leq k \leq 5$.

Theorem 5.3.3 *If $G = C_3 \,\square\, K_2$, then $\text{hrc}_2(G) = 3$ and $\text{hrc}_k(G) = 7$ for $k = 3, 4, 5$.*

Proof. By Theorems 5.3.1 and 5.3.2, $\text{hrc}_2(C_3 \,\square\, K_2) = 3$ and $\text{hrc}_5(C_3 \,\square\, K_2) = 7$. It remains therefore to determine $\text{hrc}_k(C_3 \,\square\, K_2)$ for $k = 3, 4$. Let $G = C_3 \,\square\, K_2$ where the vertices and the edges of G are labeled as shown in Figure 5.2. Since $\text{hrc}(G) = 7$, it follows that $\text{hrc}_k(G) \leq 7$ for $k = 3, 4$. By (5.3), it suffices to show that there is no 3-rainbow Hamiltonian-path coloring of $C_3 \,\square\, K_2$ using at most six colors.

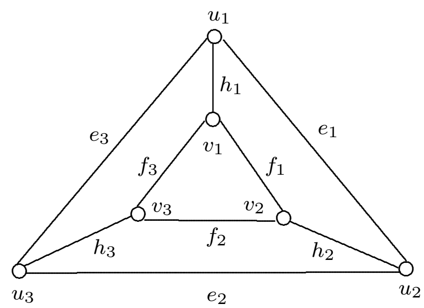


Figure 5.2: The graph $C_3 \,\square\, K_2$

Assume, to the contrary, that there is a 3-rainbow Hamiltonian-path coloring c of $C_3 \,\square\, K_2$ using at most six colors. For each of the $\binom{6}{2} = 15$ pairs x, y of vertices of G, there are exactly two Hamiltonian $x - y$ paths. At least one of these two paths is necessarily a 3-rainbow Hamiltonian path. These $2\binom{6}{2} = 30$ Hamiltonian paths are shown below.

1. $u_1 - u_2$ paths: $(h_1, f_1, f_2, h_3, e_2)$, $(e_3, h_3, f_3, f_1, h_2)$
2. $u_1 - u_3$ paths: $(e_1, h_2, f_1, f_3, h_3)$, $(h_1, f_3, f_2, h_2, e_2)$
3. $u_1 - v_1$ paths: $(e_1, e_2, h_3, f_2, f_1)$, $(e_3, e_2, h_2, f_2, f_3)$
4. $u_1 - v_2$ paths: $(h_1, f_3, h_3, e_2, h_2)$, $(e_1, e_2, h_3, f_3, f_1)$
5. $u_1 - v_3$ paths: $(h_1, f_1, h_2, e_2, h_3)$, $(e_3, e_2, h_2, f_1, f_3)$
6. $u_2 - u_3$ paths: $(e_1, h_1, f_1, f_2, h_3)$, $(h_2, f_2, f_3, h_1, e_3)$
7. $u_2 - v_1$ paths: $(e_1, e_3, h_3, f_2, f_1)$, $(h_2, f_2, h_3, e_3, h_1)$
8. $u_2 - v_2$ paths: $(e_1, e_3, h_3, f_3, f_1)$, $(e_2, e_3, h_1, f_3, f_2)$
9. $u_2 - v_3$ paths: $(h_2, f_1, h_1, e_3, h_3)$, $(e_2, e_3, h_1, f_1, f_2)$
10. $u_3 - v_1$ paths: $(h_3, f_2, h_2, e_1, h_1)$, $(e_3, e_1, h_2, f_2, f_3)$
11. $u_3 - v_2$ paths: $(h_3, f_3, h_1, e_1, h_2)$, $(e_2, e_1, h_1, f_3, f_2)$
12. $u_3 - v_3$ paths: $(e_3, e_1, h_2, f_1, f_3)$, $(e_2, e_1, h_1, f_1, f_2)$
13. $v_1 - v_2$ paths: $(h_1, e_1, e_2, h_3, f_2)$, $(f_3, h_3, e_3, e_1, h_2)$
14. $v_1 - v_3$ paths: $(f_1, h_2, e_1, e_3, h_3)$, $(h_1, e_3, e_2, h_2, f_2)$
15. $v_2 - v_3$ paths: $(h_2, e_2, e_3, h_1, f_3)$, $(f_1, h_1, e_1, e_2, h_3)$

If e and f are two distinct edges of G belonging to a 4-subpath (a subpath of order 4) in *both* Hamiltonian $w - z$ paths of G for some pair w, z of distinct vertices of G, then e and f cannot be assigned the same color by c. For example, since f_1 and h_3 belong to a 4-subpath in each of the two Hamiltonian $u_1 - u_2$ paths of G, it follows that $c(f_1) \neq c(h_3)$. As another example, since (e_3, h_3, f_2) is a subpath in each of the two Hamiltonian $u_2 - v_1$ paths of G, it follows that $|\{c(e_3), c(h_3), c(f_2)\}| = 3$.

We now construct a graph G^* with $V(G^*) = E(G)$ such that two vertices x and y of G^*, that is, two edges x and y of G, are adjacent in G^* if $c(x) \neq c(y)$. Thus, if the edges x and y of G belong to a 4-subpath for *every* Hamiltonian $w - z$ path of G for some pair w, z of distinct vertices of G, then $c(x) \neq c(y)$ and so $xy \in E(G^*)$. For example, $f_1h_3, e_3h_3, h_3f_2, e_3f_2 \in E(G^*)$. It can be shown that the graph G^* contains the complete tripartite graph $K_{3,3,3}$ as a subgraph whose partite sets are $V_1 = \{e_1, e_2, e_3\}$, $V_2 = \{f_1, f_2, f_3\}$ and $V_3 = \{h_1, h_2, h_3\}$. This verifies the following claim.

> **Claim 1.** *If $x \in V_i$ and $y \in V_j$, where $1 \leq i, j \leq 3$ and $i \neq j$, then $c(x) \neq c(y)$.*

For $i = 1, 2, 3$, let $F_i^* = G^*[V_i]$ be the subgraph of G^* induced by the set V_i and let $c(V_i) = \{c(x) : x \in V_i\}$.

> **Claim 2.** *For $i = 1, 2, 3$, $F_i^* = G^*[V_i]$ is not empty and so $|c(V_i)| \geq 2$.*

- ⋆ For $F_1^* = G^*[\{e_1, e_2, e_3\}]$, since (e_1, e_2) is a 3-subpath in one of the two Hamiltonian $u_1 - v_1$ paths and (e_3, e_2) is a 3-subpath in the other Hamiltonian $u_1 - v_1$ path, it follows that either $c(e_1) \neq c(e_2)$ or $c(e_3) \neq c(e_2)$. Thus, either $e_1e_2 \in E(F_1^*)$ or $e_2e_3 \in E(F_1^*)$.

- ⋆ For $F_2^* = G^*[\{f_1, f_2, f_3\}]$, since (f_1, f_2) is a 3-subpath in one of the two Hamiltonian $u_1 - u_2$ paths and (f_3, f_1) is a 3-subpath in the other Hamiltonian $u_1 - u_2$ path, it follows that either $c(f_1) \neq c(f_2)$ or $c(f_3) \neq c(f_1)$. Thus, either $f_1f_2 \in E(F_2^*)$ or $f_1f_3 \in E(F_2^*)$.

- ⋆ For $F_3^* = G^*[\{h_1, h_2, h_3\}]$, assume, to the contrary, that F_3^* is empty or $c(h_1) = c(h_2) = c(h_3)$. Then $(e_1, e_2, h_3, f_3, f_1)$ (in 4.), $(e_3, e_2, h_2, f_1, f_3)$ (in 5.), $(e_1, e_3, h_3, f_2, f_1)$ (in 7.) and $(e_3, e_1, h_2, f_2, f_3)$ (in 10.) are 3-rainbow paths. This implies that

$$|\{c(e_1), c(e_2), c(e_3)\}| = 3 \text{ and } |\{c(f_1), c(f_2), c(f_3)\}| = 3.$$

  However then, c uses at least 7 colors, a contradiction. Thus, F_3^* is nonempty.

Thus, Claim 2 holds. By Claims 1 and 2 then, c uses exactly six colors and $|c(V_i)| = 2$ for $i = 1, 2, 3$.

- ⋆ If $c(h_1) = c(h_2)$, then $(e_2, e_3, h_1, f_1, f_2)$, $(e_3, e_1, h_2, f_2, f_3)$ and $(e_2,\ e_1,\ h_1,\ f_3,\ f_2)$ (in 9., 10., 11.) are 3-rainbow paths and so $|c(V_1)| = 3$, which implies that c uses at least 7 colors, a contradiction.

- ⋆ If $c(h_1) = c(h_3)$, then $(e_1, e_2, h_3, f_3, f_1)$ (in 4.), $(e_1, e_3, h_3, f_2, f_1)$ (in 7.) and $(e_2, e_3, h_1, f_1, f_2)$ (in 9.) are 3-rainbow paths and so $|c(V_1)| = 3$, which implies that c uses at least 7 colors, a contradiction.
- ⋆ If $c(h_2) = c(h_3)$, then $(e_1, e_2, h_3, f_3, f_1)$ (in 4.), $(e_3, e_2, h_2, f_1, f_3)$ (in 5.) and $(e_1, e_3, h_3, f_2, f_1)$ (in 7.) are 3-rainbow paths and so $|c(V_1)| = 3$, which implies that c uses at least 7 colors, a contradiction. ■

Next, we establish an upper bound for $\mathrm{hrc}_k(C_n \,\square\, K_2)$ in terms of k and the remainder when n is divided by k. In order to do this, we introduce an additional definition. For two sets S and T in a connected graph G, the *distance between S and T* is defined as $d(S,T) = \min\{d(u,v) :\ u \in S, v \in T\}$. For two edges $e = uv$ and $f = xy$ in a connected graph G, the *distance* $d(e,f)$ is defined as the distance between the sets $\{u,v\}$ and $\{x,y\}$.

Theorem 5.3.4 *Let k and n be integers where $2 \le k \le n$ and $n \ge 5$ is odd. If $n = qk + r$ for some integers q and r with $q \ge 1$ and $0 \le r \le k-1$, then*

$$\mathrm{hrc}_k(C_n \,\square\, K_2) \le 3(k+r).$$

Proof. For an integer $n \ge 3$, let $G = C_n \,\square\, K_2$ be obtained from two copies C and C' of the n-cycle C_n, where $C = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ and $C' = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$, by adding the n edges u_iv_i for $1 \le i \le n$. We define a k-rainbow Hamiltonian-path coloring $c : E(G) \to [3(k+r)]$ as follows. Let S_u, S_v and S_{uv} denote the color sequences of the edges of C, C' and the edges between C and C'; that is,

$$\begin{aligned} S_u &= (c(u_1u_2), c(u_2u_3), \ldots, c(u_nu_1)) \\ S_v &= (c(v_1v_2), c(v_2v_3), \ldots, c(v_nv_1)) \\ S_{uv} &= (c(u_1v_1), c(u_2v_2), \ldots, c(u_nv_n)) \end{aligned}$$

- ⋆ If $r = 0$, then

$$\begin{aligned} S_u &= (1, 2, \ldots, k, ***, 1, 2, \ldots, k) \\ S_v &= (k+1, k+2, \ldots, 2k, ***, k+1, k+2, \ldots, 2k) \\ S_{uv} &= (2k+1, 2k+2, \ldots, 3k, ***, 2k+1, 2k+2, \ldots, 3k), \end{aligned}$$

  where $***$ represents repeating the next k-tuple as the preceding k-tuple.

$\star$ If $1 \leq r < k$, then

$$\begin{aligned}
S_u &= (1, 2, \ldots, k, *\!*\!*, 1, 2, \ldots, k, \quad k+1, \ldots, k+r) \\
S_v &= (k+r+1, k+r+2, \ldots, 2k+r, *\!*\!*, \\
&\quad\; k+r+1, k+r+2, \ldots, 2k+r, \\
&\quad\; 2k+r+1, \ldots, 2k+2r) \\
S_{uv} &= (2k+2r+1, 2k+2r+2, \ldots, 3k+2r, *\!*\!*, \\
&\quad\; 2k+2r+1, 2k+2r+2, \ldots, 3k+r, \\
&\quad\; 3k+2r+1, \ldots, 3k+3r),
\end{aligned}$$

again, where $*\!*\!*$ represents repeating the next k-tuple as the preceding k-tuple.

Such a 4-rainbow Hamiltonian-path coloring of $C_9 \,\square\, K_2$ is illustrated in Figure 5.3.

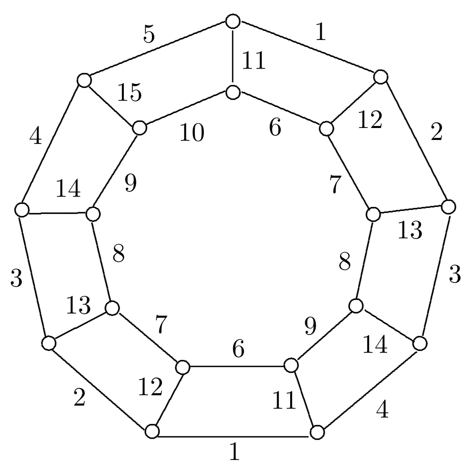


Figure 5.3: A 4-rainbow Hamiltonian-path coloring of $C_9 \,\square\, K_2$ for $k = 4$

It remains to show that c is a k-rainbow Hamiltonian-path coloring. First, we show that if e and f are two edges of G such that $c(e) = c(f)$, then $d(e, f) \geq k - 1$. Since $c(e) = c(f)$, it follows that

$$\text{(i) } e, f \in E(C) \text{ or } e, f \in E(C') \text{ or (ii) } e, f \in \{u_i v_i : 1 \leq i \leq n\}.$$

If (i) occurs, say $e, f \in E(C)$, then $d(e, f) = k - 1$; while if (ii) occurs, then $d(e, f) = k$. This implies that for every two vertices x and y in G, any $x - y$ path in G is a k-rainbow path in G. Since G is Hamiltonian-connected, it follows that every two vertices are connected by a k-rainbow Hamiltonian path in G. Thus, c is a k-rainbow Hamiltonian-path coloring of G and so $\text{hrc}_k(C_n \,\square\, K_2) \leq 3(k + r)$. ■

For $k = 3$, it follows by Theorem 5.3.4 that

$$\mathrm{hrc}_3(C_n \,\square\, K_2) \le \begin{cases} 9 & \text{if } n \equiv 0 \pmod 3 \\ 12 & \text{if } n \equiv 1 \pmod 3 \\ 15 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

In particular, $\mathrm{hrc}_3(C_5 \,\square\, K_2) \le 15$, $\mathrm{hrc}_3(C_7 \,\square\, K_2) \le 12$ and $\mathrm{hrc}_3(C_9 \,\square\, K_2) \le 9$. If G is a Hamiltonian graph of odd order $n \ge 3$, then $G \,\square\, K_2$ is Hamiltonian-connected. Thus, the following is a consequence of Observation 5.1.3 and Theorem 5.3.4.

Corollary 5.3.5 *Let G be a Hamiltonian graph of odd order $n \ge 3$ and k an integer with $2 \le k \le n$. If $n = qk + r$ for some integers q and r with $q \ge 1$ and $0 \le r \le k-1$, then*

$$\mathrm{hrc}_k(G \,\square\, K_2) \le 3(k+r).$$

In [2], it is shown that if H is a Hamiltonian-connected graph H of order $n \ge 4$, then the number $\mathrm{hrc}(H \,\square\, K_2) - 2\,\mathrm{hrc}(H)$ cannot be much larger than 1.

Theorem 5.3.6 [2] *If H is a Hamiltonian-connected graph of order $n \ge 4$, then*

$$\mathrm{hrc}(H \,\square\, K_2) \le 2\,\mathrm{hrc}(H) + 2.$$

Theorem 5.3.6 can be extended to the k-rainbow Hamiltonian-connection numbers.

Theorem 5.3.7 *If H is a Hamiltonian-connected graph of order $n \ge 4$ and k is an integer with $2 \le k \le 2n-1$, then*

$$\mathrm{hrc}_k(H \,\square\, K_2) \le 2\,\mathrm{hrc}_k(H) + 2.$$

Proof. Suppose that $\mathrm{hrc}_k(H) = s$. Let $G = H \,\square\, K_2$ be obtained from two copies F and F' of the graph H of order $n \ge 4$, where $V(F) = \{u_1, u_2, \ldots, u_n\}$ and $V(F') = \{v_1, v_2, \ldots, v_n\}$, by adding the n edges u_iv_i for $1 \le i \le n$. Since $\mathrm{hrc}_k(H) = s$, it follows that H has a k-rainbow Hamiltonian-path s-coloring. Let

$$c_F : V(F) \to \{1, 2, \ldots, s\} \text{ and } c_{F'} : V(F') \to \{s+1, s+2, \ldots, 2s\}$$

be a k-rainbow Hamiltonian-path s-coloring of F and F', respectively. Define the $(2s+2)$-edge coloring $c : E(G) \to [2s+2]$ by

$$c(e) = \begin{cases} c_F(e) & \text{if } e \in E(F) \\ c_{F'}(e) & \text{if } e \in E(F') \\ 2s+1 & \text{if } e = u_iv_i \text{ and } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2s+2 & \text{if } e = u_iv_i \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n. \end{cases}$$

We show that c is a k-rainbow Hamiltonian-path coloring of G; that is, we show that every two vertices x and y of G are connected by a k-rainbow Hamiltonian path in G. First, suppose that $x = u_i$ and $y = v_j$ where $1 \leq i, j \leq n$. Let $t \in [n] - \{i, j\}$. Let P be a k-rainbow Hamiltonian $u_i - u_t$ path in F and let P' be a k-rainbow Hamiltonian $v_t - v_j$ path in F'. Then the path (P, P') is a k-rainbow Hamiltonian $u_i - v_j$ path in G. Next, suppose that $x, y \in V(F)$ or $x, y \in V(F')$, say the former. Suppose that $x = u_i$ and $y = u_j$ where $1 \leq i, j \leq n$ and $i \neq j$. Let Q be a k-rainbow Hamiltonian $u_i - u_j$ path in F, say $Q = (u_i = x_1, x_2, \ldots, x_n = u_j)$. Thus, there is $t \in [n-1]$ such that $c(x_t x'_t) \neq c(x_{t+1} x'_{t+1})$, where x'_t and x'_{t+1} are the corresponding vertices of x_t and x_{t+1} in F', respectively. Let Q_1 be the $x_1 - x_t$ subpath of Q and let Q_2 be the $x_{t+1} - x_n$ subpath of Q. Now, let Q' be a k-rainbow Hamiltonian $x'_t - x'_{t+1}$ path in F'. Then the path (Q_1, Q', Q_2) is a k-rainbow Hamiltonian $u_i - u_j$ in G. Therefore, c is a k-rainbow Hamiltonian-path coloring of G and so $\text{hrc}_k(G) \leq 2s + 2$. ■

Chapter 6

Rainbow Disconnection in Graphs

The object of this chapter is to introduce the rainbow disconnection number $\mathrm{rd}(G)$ of a graph G, which is somewhat reverse to rainbow connection and to present some results dealing with this new concept. While rainbow connection concerns connecting each pair of vertices by a rainbow set of edges, the concept we describe here concerns disconnecting each pair of vertices by a rainbow set of edges.

6.1 Introduction

An *edge-cut* of a nontrivial connected graph G is a set R of edges of G such that $G-R$ is disconnected. The minimum number of edges in an edge-cut of G is its *edge-connectivity* $\lambda(G)$. We then have the well-known inequality $\lambda(G) \leq \delta(G)$. For two distinct vertices u and v of G, let $\lambda(u,v)$ denote the minimum number of edges in an edge-cut R of G such that u and v lie in different components of $G-R$. Thus,

$$\lambda(G) = \min\{\lambda(u,v) : \ u, v \in V(G)\}.$$

The following result of Elias, Feinstein and Shannon [18] and Ford and Fulkerson [21] presents an alternate interpretation of $\lambda(u,v)$.

Theorem 6.1.1 *For every two vertices u and v in a graph G, $\lambda(u,v)$ is the maximum number of pairwise edge-disjoint $u-v$ paths in G.*

The *upper edge-connectivity* $\lambda^+(G)$ is defined by

$$\lambda^+(G) = \max\{\lambda(u,v) : \ u,v \in V(G)\}.$$

Consider, for example, the graph $K_n + v$ obtained from the complete graph K_n, one vertex of which is attached to a single leaf v. For this graph, $\lambda(K_n + v) = 1$ while $\lambda^+(K_n + v) = n - 1$. Thus, $\lambda(G)$ denotes the global minimum edge-connectivity of a graph, while $\lambda^+(G)$ denotes the local maximum edge-connectivity of a graph.

A set R of edges in a connected edge-colored graph G is a *rainbow set* if no two edges in R are colored the same. A set R of edges in a nontrivial connected, edge-colored graph G is a *rainbow cut* of G if R is both a rainbow set and an edge-cut. A rainbow cut R is said to *separate* two vertices u and v of G if u and v belong to different components of $G - R$. Any such rainbow cut in G is called a $u - v$ *rainbow cut* in G. An edge-coloring of G is a *rainbow disconnection coloring* if for every two distinct vertices u and v of G, there exists a $u - v$ rainbow cut in G. The *rainbow disconnection number* $\mathrm{rd}(G)$ of G is the minimum number of colors required of a rainbow disconnection coloring of G. A rainbow disconnection coloring with $\mathrm{rd}(G)$ colors is called an rd-*coloring* of G. We now present bounds for the rainbow disconnection number of a graph.

Proposition 6.1.2 *If G is a nontrivial connected graph, then*

$$\lambda(G) \le \lambda^+(G) \le \mathrm{rd}(G) \le \chi'(G) \le \Delta(G) + 1.$$

Proof. First, by Vizing's theorem (Theorem 1.5.6), $\chi'(G) \le \Delta(G) + 1$. Now, let there be given a proper edge-coloring of G using $\chi'(G)$ colors. Then, for each vertex x of G, the set E_x of edges incident with x is a rainbow set and $|E_x| = \deg x \le \Delta(G) \le \chi'(G)$. Furthermore, E_x is a rainbow cut in G and so $\mathrm{rd}(G) \le \chi'(G)$.

Next, let there be given an rd-coloring of G. Let u and v be two vertices of G such that $\lambda^+(G) = \lambda(u,v)$ and let R be a $u - v$ rainbow cut with $|R| = \lambda(u,v)$. Then $|R| \le \mathrm{rd}(G)$. Thus, $\lambda(G) \le \lambda^+(G) = |R| \le \mathrm{rd}(G)$. ■

We now present examples of two classes of connected graphs G for which $\lambda(G) = \mathrm{rd}(G)$, namely cycles and wheels.

Proposition 6.1.3 *If C_n is a cycle of order $n \ge 3$, then* $\mathrm{rd}(C_n) = 2$.

Proof. Since $\lambda(C_n) = 2$, it follows by Proposition 6.1.2 that $\mathrm{rd}(C_n) \ge 2$. To show that $\mathrm{rd}(C_n) \le 2$, let c be an edge-coloring of C_n that assigns the color 1 to exactly $n-1$ edges of C_n and the color 2 to the remaining edge e of C_n. Let u and v be two vertices of C_n. There are two $u - v$ paths P and Q in C_n, exactly one of which contains the edge e, say

$e \in E(P)$. Then any set $\{e, f\}$, where $f \in E(Q)$, is a $u - v$ rainbow cut. Thus, c is a rainbow disconnection coloring of C_n using two colors. Hence, $\mathrm{rd}(C_n) = 2$. ■

Proposition 6.1.4 *If $W_n = C_n \vee K_1$ is the wheel of order $n+1 \geq 4$, then* $\mathrm{rd}(W_n) = 3$.

Proof. Since $\lambda(W_n) = 3$, it follows by Proposition 6.1.2 that $\mathrm{rd}(W_n) \geq 3$. It remains to show that there is a rainbow disconnection coloring of W_n using only the colors $1, 2, 3$. Suppose that $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and that v is the center of W_n. Define an edge-coloring $c : E(W_n) \to \{1, 2, 3\}$ of W_n as follows. First, let c be a proper edge-coloring of C_n using the colors $1, 2$ when n is even and the colors $1, 2, 3$ when n is odd. For each integer i with $1 \leq i \leq n$, let

$$a_i \in \{1, 2, 3\} - \{c(v_{i-1}v_i), c(v_iv_{i+1})\}$$

where each subscript is expressed as an integer $1, 2, \ldots, n$ modulo n, and let $c(vv_i) = a_i$. This coloring is illustrated for each of W_6 and W_7 in Figure 6.1.

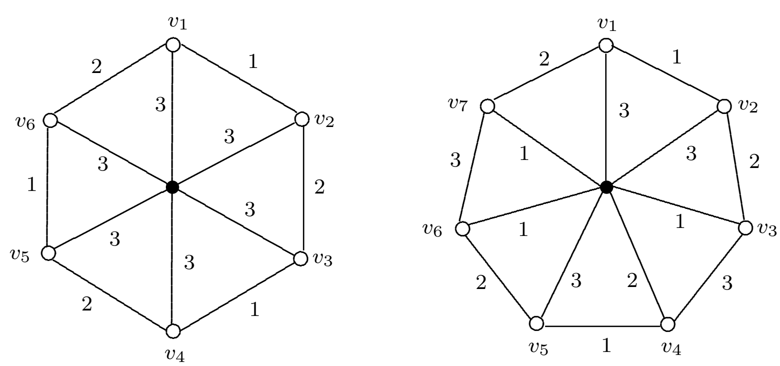


Figure 6.1: Rainbow disconnection colorings of W_6 and W_7

Thus, the set E_{v_i} of the three edges incident with v_i is a rainbow set for $1 \leq i \leq n$. Let x and y be two distinct vertices of W_n. Then at least one of x and y belongs to C_n, say $x \in V(C_n)$. Since E_x separates x and y, it follows that c is a rainbow disconnection coloring of W_n using three colors. Hence, $\mathrm{rd}(W_n) = 3$. ■

Since $\chi'(C_n) = 3$ when $n \geq 3$ is odd and $\chi'(W_n) = n$ for each integer $n \geq 3$, it follows that $\mathrm{rd}(G) < \chi'(G)$ if G is an odd cycle or if G is a wheel of order at least 4. Wheels therefore illustrate that there are graphs G for which $\chi'(G) - \mathrm{rd}(G)$ can be arbitrarily large. We now give an example of a graph G for which $\lambda^+(G) < \mathrm{rd}(G) = \chi'(G)$.

Proposition 6.1.5 *The rainbow disconnection number of the Petersen graph is* 4.

Proof. Let P denote the Petersen graph where $V(P) = \{v_1, v_2, \ldots, v_{10}\}$. Since $\lambda(P) = 3$ and $\chi'(P) = 4$, it follows by Proposition 6.1.2 that $\mathrm{rd}(P) = 3$ or $\mathrm{rd}(P) = 4$. Assume, to the contrary, that $\mathrm{rd}(P) = 3$ and let there be given a rainbow disconnection 3-coloring of P. Now, let u and v be two vertices of P and let R be a $u-v$ rainbow cut. Hence, $|R| \leq 3$ and $P - R$ is disconnected, where u and v belong to different components of $P - R$. Let U be the vertex set of the component of $P - R$ containing u, where $|U| = k$. We may assume that $1 \leq k \leq 5$. First, suppose that $1 \leq k \leq 4$. Since the girth of P is 5, the subgraph $P[U]$ induced by U contains $k-1$ edges. Therefore, $|R| = 3k-(2k-2) = k+2$, where then $3 \leq k+2 \leq 6$. If $k = 5$, then $P[U]$ contains at most five edges and so $|R| \geq 5$, which is impossible. Since $\mathrm{rd}(P) = 3$, it follows that $|R| \leq 3$ and so $k = 1$. Hence, the only possible $u - v$ rainbow cut is the set consisting of the three edges incident with u (or with v).

Let the colors assigned to the edges of P be red, blue and green. Since $\chi'(P) = 4$, there is at least one vertex of P that is incident with two edges of the same color. We claim, in fact, that there are at least two such vertices. Let E_R, E_B and E_G denote the sets of edges of P colored red, blue and green, respectively, and let P_R, P_B and P_G be the spanning subgraphs of P with edge sets E_R, E_B and E_G. We may assume that $|E_R| \geq |E_B| \geq |E_G|$ and so $|E_R| \geq 5$. If $|E_R| \geq 7$, then $\sum_{i=1}^{10} \deg_{P_R} v_i \geq 14$. Since $\deg_{P_R} v_i \leq 3$ for each i with $1 \leq i \leq 10$, at least two vertices are incident with two red edges, verifying the claim. If $|E_R| = 6$, then $\sum_{i=1}^{10} \deg_{P_R} v_i = 12$. Then either (i) at least two vertices are incident with two red edges or (ii) there is a vertex, say v_{10}, incident with three red edges and each of $v_1, v_2, \ldots, v_9$ is incident with exactly one red edge. If (ii) occurs, then either $|E_B| = 6$ or $|E_B| = 5$ and so $\sum_{i=1}^{9} \deg_{P_B} v_i \geq 10$, which implies that at least one of the vertices $v_1, v_2, \ldots, v_9$ is incident with two blue edges, again verifying the claim.

The only remaining possibility is therefore $|E_R| = |E_B| = |E_G| = 5$. If E_R is an independent set of five edges, then $P - E_R$ is a 2-regular graph. Since the girth of P is 5 and P is not Hamiltonian, it follows that $P - E_R$ consists of two vertex-disjoint 5-cycles. Thus, there is a vertex of P in each cycle incident with two blue edges or with two green edges, verifying the claim. Hence, none of E_R, E_B or E_G is an independent set. This implies that for each of these colors, there is a vertex of P incident with two edges of this color, verifying the claim in general.

Thus, P contains two vertices u and v, each of which is incident with two edges of the same color. Since the only $u - v$ rainbow cut is the set of edges incident with u or v, this is a contradiction. ■

The following two results are useful.

Proposition 6.1.6 *If H is a connected subgraph of a graph G, then* $\mathrm{rd}(H) \leq \mathrm{rd}(G)$.

Proof. Let c be an rd-coloring of G and let u and v are two vertices of G. Suppose that R is a $u-v$ rainbow cut. Then $R \cap E(H)$ is a $u-v$ rainbow cut in H. Hence, c restricted to H is a rainbow disconnection coloring of H. Thus, $\mathrm{rd}(H) \leq \mathrm{rd}(G)$. ■

A *block* of a graph is a maximal connected graph of G containing no cut-vertices. The *block decomposition* of G is the set of blocks of G.

Proposition 6.1.7 *Let G be a nontrivial connected graph, and let B be a block of G such that $rd(B)$ is maximum among all blocks of G. Then* $\mathrm{rd}(G) = \mathrm{rd}(B)$.

Proof. Let G be a nontrivial connected graph. Let $\{B_1, B_2, ..., B_t\}$ be a block decomposition of G, and let $k = \max\{\mathrm{rd}(B_i) \,|1 \leq i \leq t\}$. If G has no cut-vertex, then $G = B_1$ and the result follows. Hence, we may assume that G has at least one cut-vertex. By Proposition 6.1.6, $k \leq \mathrm{rd}(G)$.

Let c_i be an rd-coloring of B_i. We define the edge-coloring $c : E(G) \to [k]$ of G by $c(e) = c_i(e)$ if $e \in E(B_i)$.

Let $x, y \in V(G)$. If there exists a block, say B_i, that contains both x and y, then any $x-y$ rainbow cut in B_i is an $x-y$ rainbow cut in G. Hence, we can assume that no block of G contains both x and y, and that $x \in B_i$ and $y \in B_j$, where $i \neq j$. Now every $x-y$ path contains a cut-vertex, say v, of G in B_i and a cut-vertex, say w, of G in B_j. Note that v could equal w. If $x \neq v$, then any $x-v$ rainbow cut of B_i is an $x-y$ rainbow cut in G. Similarly, if $y \neq w$, then any $y-w$ rainbow cut of B_j is an $x-y$ rainbow cut in G. Thus, we may assume that $x = v$ and $y = w$. It follows that $v \neq w$. Consider the $x-y$ path $P = (x = v_1, v_2, ..., v_p = y)$. Since x and y are cut-vertices in different blocks and no block contains both x and y, P contains a cut-vertex z of G in B_i, that is, $z = v_k$ for some k $(2 \leq k \leq p-1)$. Then any $x-z$ rainbow cut of B_i is an $x-y$ rainbow cut of G. Hence, $\mathrm{rd}(G) \leq k$, and so $\mathrm{rd}(G) = k$. ■

As a consequence of Proposition 6.1.7, the study of rainbow disconnection numbers can be restricted to 2-connected graphs. We now present several corollaries of Proposition 6.1.7.

Corollary 6.1.8 *Let G and H be any two nontrivial connected graphs, and let GvH be a graph formed by identifying a vertex in G with a vertex in H. Then*

$$\mathrm{rd}(GvH) = \max\{\mathrm{rd}(G), \mathrm{rd}(H)\}.$$

Corollary 6.1.9 *Let G and H be any two nontrivial connected graphs, and let $GuvH$ be a graph formed by adding an edge between any vertex u in G and any vertex v in H. Then*

$$\mathrm{rd}(GuvH) = \max\{\mathrm{rd}(G), \mathrm{rd}(H)\}.$$

Corollary 6.1.10 *Let G be a nontrivial connected graph and G' the graph obtained by attaching a pendant edge uv to some vertex u of G. Then* $\mathrm{rd}(G') = \mathrm{rd}(G)$.

The *corona* $G \circ K_1$ is the graph obtained from G by attaching a leaf to each vertex of G. Thus, if G has order n, then the corona $G \circ K_1$ has order $2n$ and has precisely n leaves.

Corollary 6.1.11 *If G is a nontrivial connected graph, then* $\mathrm{rd}(G \circ K_1) = \mathrm{rd}(G)$.

Corollary 6.1.12 *Let G be a nontrivial connected graph, let T be a nontrivial tree and let u and v be vertices of G and T, respectively. If H is the graph obtained from G and T by identifying u and v, then* $\mathrm{rd}(H) = \mathrm{rd}(G)$.

Corollary 6.1.13 *If G is a unicyclic graph G, then* $\mathrm{rd}(G) = 2$.

6.2 Graphs with Prescribed Rainbow Disconnection Number

In this section, we characterize all those nontrivial connected graphs of order n with rainbow disconnection number k for each $k \in \{1, 2, n-1\}$. The result for graphs having rainbow disconnection number 1 follows directly from Propositions 6.1.6 and 6.1.7.

Proposition 6.2.1 *Let G be a nontrivial connected graph. Then* $\mathrm{rd}(G) = 1$ *if and only if G is a tree.*

Next, we characterize all nontrivial connected graphs of order n having rainbow disconnection number 2. By Proposition 6.2.1, such a graph must contain a cycle. An *ear* of a graph G is a maximal path whose internal vertices have degree 2 in G. An *ear decomposition* of a graph is a decomposition $H_0, H_1, \ldots, H_k$ such that H_0 is a cycle in G and H_i is an ear of the subgraph of G with edge set $E(H_0) \cup E(H_1) \cup \ldots \cup E(H_i)$ for each integer i with $1 \leq i \leq k$. Whitney [35] proved the following result in 1932.

Theorem 6.2.2 *A graph G is 2-connected if and only if G has an ear decomposition. Furthermore, every cycle is the initial cycle in some ear decomposition of G.*

The following is a consequence of Theorem 6.2.2.

Lemma 6.2.3 *A 2-connected graph G is a cycle if and only if for every two vertices u and v of G, there are exactly two internally disjoint $u - v$ paths in G.*

Also, by Theorem 6.2.2, if G is a 2-connected, non-Hamiltonian graph, then G contains a theta subgraph (a subgraph consisting of two vertices connected by three internally disjoint paths of length 2 or more).

Theorem 6.2.4 *Let G be a nontrivial connected graph. Then $\mathrm{rd}(G) = 2$ if and only if each block of G is either K_2 or a cycle and at least one block of G is a cycle.*

Proof. If G a nontrivial connected graph, each block of which is either K_2 or a cycle and at least one block of G is a cycle, then Propositions 6.1.3 and 6.1.7 imply that $\mathrm{rd}(G) = 2$.

We now verify the converse. Assume, to the contrary, that there is a connected graph G with $\mathrm{rd}(G) = 2$ that does not have the property that each block of G is either K_2 or a cycle and at least one block of G is a cycle. First, not all blocks can be K_2, for otherwise, G is a tree and so $\mathrm{rd}(G) = 1$ by Proposition 6.2.1. Hence, G contains a block that is neither K_2 nor a cycle. By Lemma 6.2.3, there exist two distinct vertices u and v of G for which G contains at least three internally disjoint $u - v$ paths P_1, P_2 and P_3. Thus, any $u - v$ rainbow cut R must contain at least one edge from each of P_1, P_2 and P_3 and so $|R| \geq 3$, which is impossible. ■

We now consider those graphs that are, in a sense, opposite to trees.

Proposition 6.2.5 *For each integer $n \geq 4$, $\mathrm{rd}(K_n) = n - 1$.*

Proof. Suppose first that $n \geq 4$ is even. Then $\lambda(K_n) = \chi'(K_n) = n-1$. It then follows by Proposition 6.1.2 that $\mathrm{rd}(K_n) = n - 1$. Next, suppose that $n \geq 5$ is odd. Then

$$n - 1 = \lambda(K_n) \leq \mathrm{rd}(K_n) \leq \chi'(K_n) = n$$

by Proposition 6.1.2. To show that $\mathrm{rd}(K_n) = n - 1$, it remains to show that there is a rainbow disconnection coloring of K_n using $n - 1$ colors. Let $x \in V(K_n)$. Then $K_n - x = K_{n-1}$. Since $n - 1$ is even, it follows that $\chi'(K_{n-1}) = n - 2$. Thus, there is a proper edge-coloring c_0 of K_{n-1} using the colors $1, 2, \ldots, n - 2$. We now extend c_0 to an edge-coloring c of K_n by assigning the color $n - 1$ to each edge of K_n that is incident with x. We show that c is a rainbow disconnection coloring of K_n. Let u and v be two vertices of K_n, where say $u \neq x$. Then the set E_u of edges incident with u is a $u - v$

rainbow cut. Thus, c is a rainbow disconnection coloring of K_n and so $\mathrm{rd}(K_n) \leq n-1$ and so $\mathrm{rd}(K_n) = n-1$. ■

By Propositions 6.1.2, 6.1.6 and 6.2.5, if G is a nontrivial connected graph of order n, then

$$1 \leq \mathrm{rd}(G) \leq n-1. \tag{6.1}$$

Furthermore, $\mathrm{rd}(G) = 1$ if and only if G is a tree by Proposition 6.2.1. We have seen that the complete graphs K_n of order $n \geq 2$ have rainbow disconnection number $n-1$. We now characterize all nontrivial connected graphs of order n having rainbow disconnection number $n-1$.

Theorem 6.2.6 *Let G be a nontrivial connected graph of order n. Then* $\mathrm{rd}(G) = n-1$ *if and only if G contains at least two vertices of degree $n-1$.*

Proof. First, suppose that G is a nontrivial connected graph of order n containing at least two vertices of degree $n-1$. Since $\mathrm{rd}(G) \leq n-1$ by (6.1), it remains to show that $\mathrm{rd}(G) \geq n-1$. Let $u, v \in V(G)$ such that $\deg u = \deg v = n-1$. Among all sets of edges that separate u and v in G, let S be one of minimum size. We show that $|S| \geq n-1$. Let U be a component of $G - S$ that contains u and let $W = V(G) - U$. Thus, $v \in W$ and $S = [U, W]$ consists of those edges in $G - S$ joining a vertex of U and a vertex of W. Suppose that $|U| = k$ for some integer k with $1 \leq k \leq n-1$ and then $|W| = n-k$. The vertex u is adjacent to each of the $n-k$ vertices of W and each of the remaining $k-1$ vertices in U is adjacent to at least one vertex in W. Hence,

$$|S| \geq n-k+(k-1) = n-1.$$

This implies that every $u-v$ rainbow cut contains at least $n-1$ edges of G and so $\mathrm{rd}(G) \geq n-1$.

For the converse, suppose that G is a nontrivial connected graph of order n having at most one vertex of degree $n-1$. We show that $\mathrm{rd}(G) \leq n-2$. We consider two cases.

Case 1. *Exactly one vertex v of G has degree $n-1$.* Let $H = G - v$. Thus, $\Delta(H) \leq n-3$. Since

$$\chi'(H) \leq \Delta(H) + 1 = n-2,$$

there is a proper edge-coloring of H using $n-2$ colors. We now define an edge-coloring $c : E(G) \to [n-2]$ of G. First, let c be a proper $(n-2)$-edge-coloring of H. For each vertex $x \in V(H)$, since $\deg_H x \leq n-3$, there is $a_x \in [n-2]$ such that a_x is not assigned to any edge incident with x. Define $c(vx) = a_x$. Thus, the set E_x of edges incident with

x is a rainbow set for each $x \in V(H)$. Let u and w be two distinct vertices of G. Then at least one of u and w belongs to H, say $u \in V(H)$. Since E_u separates u and w, it follows that c is a rainbow disconnection coloring of G using $n-2$ colors. Hence, $\mathrm{rd}(G) \leq n-2$.

Case 2. *No vertex of G has degree $n-1$.* Therefore $\Delta(G) \leq n-2$. If $\Delta(G) \leq n-3$, then $\mathrm{rd}(G) \leq \chi'(G) \leq n-2$ by Proposition 6.1.2. Thus, we may assume that $\Delta(G) = n-2$. Suppose first that G is not $(n-2)$-regular. We claim that G is a connected spanning subgraph of some graph G^* of order n having exactly one vertex of degree $n-1$. Let u be a vertex of degree $k \leq n-3$ in G. Let $N(u)$ be the neighborhood of u and $W = V(G) - N[u]$, where $N[u] = N(u) \cup \{u\}$ is the closed neighborhood of u. Then $|N(u)| = k$ and $|W| = n-k-1 \geq 2$. If W contains a vertex v of degree $n-2$ in G, then v is the only vertex of degree $n-1$ in $G^* = G + uv$. If no vertex in W has degree $n-2$ in G, then let G^* be the graph obtained from G by joining u to each vertex in W. In this case, u is the only vertex of degree $n-1$ in G^*. It then follows by Case 1 that $\mathrm{rd}(G^*) \leq n-2$. Since G is a connected spanning subgraph of G^*, it follows by Proposition 6.1.6 that

$$\mathrm{rd}(G) \leq \mathrm{rd}(G^*) \leq n-2.$$

Finally, suppose that G is $(n-2)$-regular. Thus, G is 1-factorable and so $\chi'(G) = \Delta(G) = n-2$. Therefore, $\mathrm{rd}(G) \leq \chi'(G) = n-2$ by Proposition 6.1.2. ■

6.3 Rainbow Disconnection in Grids and Prisms

We now determine the rainbow disconnection numbers of graphs belonging to one of two well-known classes formed by Cartesian products.

The *Cartesian product* $G \,\square\, H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V(G) \times V(H)$, where (u,v) is adjacent to (w,x) in $G \,\square\, H$ if and only if either $u = w$ and $vx \in E(H)$ or $uw \in E(G)$ and $v = x$. We consider the $m \times n$ grid graph $G_{m,n} = P_m \,\square\, P_n$, which consists of m horizontal paths P_n and n vertical paths P_m.

Theorem 6.3.1 *The rainbow disconnection numbers of the grid graphs $G_{m,n}$ are as follows:*

(*i*) *for all $n \geq 2$,* $\mathrm{rd}(G_{1,n}) = \mathrm{rd}(P_n) = 1$.

(*ii*) *for all $n \geq 3$,* $\mathrm{rd}(G_{2,n}) = 3$.

(*iii*) *for all $n \geq 4$,* $\mathrm{rd}(G_{3,n}) = 3$.

(iv) for all $4 \leq m \leq n$, $\mathrm{rd}(G_{m,n}) = 4$.

Proof. (i) That $\mathrm{rd}(G_{1,n}) = \mathrm{rd}(P_n) = 1$ for $n \geq 2$ is a consequence of Proposition 6.2.1.

For the remainder of the proof, the vertices of $G_{m,n}$ are considered as the entries of a matrix, where $x_{i,j}$ denotes the vertex in row i and column j, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

(ii) For the graph $G_{2,n}$, $n \geq 3$, $\Delta(G_{2,n}) = 3$. First, we define an edge-coloring c of $G_{2,n}$. It is convenient to use the elements of the set $\mathbb{Z}_3$ of integer modulo 3 as colors here. Define the edge-coloring $c : E(G_{2,n}) \to \mathbb{Z}_3$ by

- $\star$ $c(x_{i,j}x_{i,j+1}) = i + j + 1$ for $1 \leq i \leq 2$ and $1 \leq j \leq n - 1$ (where addition takes place in $\mathbb{Z}_3$);
- $\star$ $c(x_{1,j}x_{2,j}) = j$ for $1 \leq j \leq n$.

This is illustrated in Figure 6.3 for $G_{2,7}$.

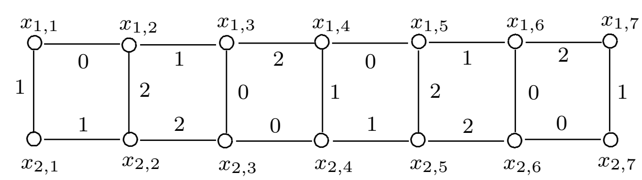


Figure 6.2: A rainbow disconnection coloring of $G_{2,7}$

Next, we show that c is a rainbow disconnection coloring of $G_{2,n}$. Let u and v be any two vertices of $G_{2,n}$. If u and v belong to two different columns, then there exist two parallel edges joining vertices in the same two columns whose removal separates u and v. Each such set of two edges is a $u - v$ rainbow cut. Next, suppose that u and v belong to the same column. We may assume that u belongs to the first row and v belongs to the second row. Then either u and v both have degree 2 or both have degree 3. Therefore, the edges incident with u form a rainbow cut, and so, $\mathrm{rd}(G_{2,n}) \leq 3$.

Observe that $\lambda(u, v) = 2$ if u and v are two vertices of $G_{2,n}$ (1) belonging to the same row, (2) belonging to different rows and columns or (3) of degree 2 belonging to the same column; while $\lambda(u, v) = 3$ if u and v are (adjacent) vertices of degree 3 belonging to the same column. It then follows by Proposition 6.1.2 that $3 = \lambda^+(G_{2,n}) \leq \mathrm{rd}(G_{2,n})$, and so $\mathrm{rd}(G_{2,n}) = 3$.

(iii) As with $G_{2,n}$, we first define an edge-coloring c of $G_{3,n}$. Once again, we use the elements of the set $\mathbb{Z}_3$ of integers modulo 3 as colors. Define the edge-coloring $c : E(G_{3,n}) \to \mathbb{Z}_3$ by

★ $c(x_{i,j}x_{i,j+1}) = i + j + 1$ for $1 \leq i \leq 3$ and $1 \leq j \leq n - 1$;

★ $c(x_{1,j}x_{2,j}) = j$ for $1 \leq j \leq n$;

★ $c(x_{2,j}x_{3,j}) = j + 2$ for $1 \leq j \leq n$.

This is illustrated in Figure 6.3 for $G_{3,7}$.

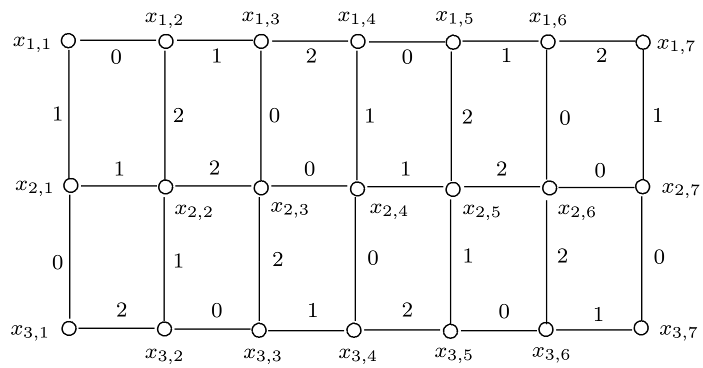


Figure 6.3: A rainbow disconnection coloring of $G_{3,7}$

Next, we show that c is a rainbow disconnection coloring of $G_{3,n}$. Let u and v be any two vertices of $G_{3,n}$. If u and v belong to two different columns, then there exist three parallel edges joining vertices in the same two columns whose removal separates u and v. Each such set of three edges is a $u - v$ rainbow cut. Next, suppose that u and v belong to the same column. Then at least one of u and v belongs to the top or bottom row, say u is such a vertex, which has degree 2 or 3. Then the edges incident with u is a $u - v$ rainbow cut. Thus, $\text{rd}(G_{3,n}) \leq 3$.

It remains to show that $\text{rd}(G_{3,n}) \geq 3$. Let u and v be two adjacent vertices of degree 4 in $G_{3,n}$ (necessarily in the middle row). Then $\lambda(u, v) = \lambda^+(G_{3,n}) = 3$. By Proposition 6.1.2, $3 \leq \lambda^+(G_{3,n}) \leq \text{rd}(G_{3,n}) \leq 3$ and so $\text{rd}(G_{3,n}) = 3$.

(iv) Finally, we consider $G_{m,n}$ for $4 \leq m \leq n$. For every two vertices u and v of degree 4, there are four pairwise edge-disjoint $u - v$ paths in $G_{m,n}$. By Theorem 6.1.1, $\lambda(u, v) = 4$. For any other pair u, v of vertices of $G_{m,n}$, $\lambda(u, v) \leq 3$. By Proposition 6.1.2 then, $4 = \lambda^+(G_{m,n}) \leq \text{rd}(G_{m,n})$. On the other hand, since $G_{m,n}$ is bipartite, $\chi'(G_{m,n}) = \Delta(G_{m,n}) = 4$. Again, by Proposition 6.1.2, $\text{rd}(G_{m,n}) \leq 4$ and so $\text{rd}(G_{4,n}) = 4$. ■

Next we determine the rainbow disconnection number of prisms, namely those graphs of the type $G \,\square\, K_2$ for some graph G.

Proposition 6.3.2 *If G is a nontrivial connected graph, then the rainbow disconnection number of the prism $G \,\square\, K_2$ is*

$$\mathrm{rd}(G \,\Box\, K_2) = \Delta(G) + 1.$$

Proof. Let G and G' be the two copies of G in the prism $G \,\Box\, K_2$, and for each $v \in V(G)$, let v' be its corresponding vertex in G'. We first show that $G \,\Box\, K_2$ has a proper edge coloring using $\Delta(G \,\Box\, K_2) = \Delta(G)+1$ colors, that is, $\chi'(G \,\Box\, K_2) = \Delta(G)+1$. Let C be a proper edge coloring of G using $\chi'(G)$ colors. Color the edges of G and G' using C, that is, G and G' have an identical edge coloring C. By Vizing's Theorem, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. First assume that $\chi'(G) = \Delta(G)$. Then assigning the color $\Delta(G) + 1$ to each edge vv' for every $v \in V(G)$ gives a proper edge-coloring of $G \,\Box\, K_2$ with $\Delta(G) + 1$ colors. Next assume that $\chi'(G) = \Delta(G) + 1$. Then for each $v \in V(G)$, at least one of the $\Delta(G) + 1$ colors is missing from the colors of the edges incident to v. Let c_v be one such missing color. Note that c_v is also missing from the colors of the edges incident to v' in G' because G and G' have the identical colorings. Hence, assigning c_v to vv' for each $v \in V(G)$ yields a proper edge-coloring of $G \,\Box\, K_2$ having $\Delta(G) + 1$ colors. By Proposition 6.1.2, it follows that $\mathrm{rd}(G \,\Box\, K_2) \leq \Delta(G) + 1$.

To establish the lower bound, let u be a vertex of G with $\deg u = \Delta(G) = \Delta$. In $G \,\Box\, K_2$, there exist $\Delta + 1$ edge-disjoint $u - u'$ paths, one of which is the edge uu' and the remaining Δ of which have the form (u, w, w', u'), where $w \in N_G(u)$ and w' is the corresponding vertex of w in G'. Thus, $\lambda(u, u') = \Delta + 1$. It again follows by Proposition 6.1.2 that $\mathrm{rd}(G \,\Box\, K_2) \geq \lambda^+(G \,\Box\, K_2) \geq \Delta(G) + 1$. ■

We now determine the rainbow disconnection numbers of the cylinder graphs $P_m \,\Box\, C_n$ for all integers $m \geq 2$ and $n \geq 3$.

Proposition 6.3.3 *For integers m and n with $m \geq 2$ and $n \geq 3$,*

$$\mathrm{rd}(P_m \,\Box\, C_n) = \begin{cases} 3 & \text{if } m = 2 \\ 4 & \text{if } m \geq 3. \end{cases}$$

Proof. If $m = 2$, then $P_2 \,\Box\, C_n = K_2 \,\Box\, C_n$ is a prism. Since $\mathrm{rd}(P_2 \,\Box\, C_n) = \Delta(C_n)+1 = 3$ by Proposition 6.3.2, we may assume that $m \geq 3$. Let $G = P_m \,\Box\, C_n$. For each integer i with $1 \leq i \leq m$, let $F_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n}, v_{i,n+1} = v_{i,1})$ be a copy of the cycle C_n of order n in G. For each integer j with $1 \leq j \leq n$, let $H_j = (v_{1,j}, v_{2,j}, \ldots, v_{m,j})$ is a copy of the path P_m of order m in G. First, we show that $\lambda^+(G) \geq 4$. Consider the vertices

$v_{2,2}$ and $v_{2,3}$ of G. Since the four $v_{2,2} - v_{2,3}$ paths

$$\begin{aligned} Q_1 &= (v_{2,2}, v_{2,3}) \\ Q_2 &= (v_{2,2}, v_{3,2}, v_{3,3}, v_{2,3}) \\ Q_3 &= (v_{2,2}, v_{1,2}, v_{1,3}, v_{2,3}) \\ Q_4 &= (v_{2,2}, v_{2,1}, v_{2,n}, v_{2,n-1}, \ldots, v_{2,4}, v_{2,3}) \end{aligned}$$

are pairwise edge-disjoint in G, it follows by Theorem 6.1.1 that $\lambda(v_{2,2}, v_{2,3}) \geq 4$. Hence, $\lambda^+(G) \geq 4$. By Proposition 6.1.2 then, $\mathrm{rd}(G) \geq 4$.

Next, we show that $\mathrm{rd}(G) \leq 4$. Again, by Proposition 6.1.2, it suffices to show that $\chi'(G) = 4$. If $n \geq 4$ is even, then G is a bipartite graph and so $\chi'(G) = \Delta(G) = 4$. Thus, we may assume that $n \geq 3$ is odd. Define the edge coloring $c : E(G) \to \{1, 2, 3, 4\}$ as follows.

- ⋆ For each integer i with $1 \leq i \leq m$, color the edges of the n-cycle F_i by

$$c(v_{i,t}v_{i,t+1}) = \begin{cases} 1 & \text{if } t \text{ is odd and } 1 \leq t \leq n-2 \\ 2 & \text{if } t \text{ is even and } 2 \leq t \leq n-1 \\ 3 & \text{if } t = n. \end{cases}$$

- ⋆ Color the edges of the m-path H_1 by

$$c(v_{s,1}v_{s+1,1}) = \begin{cases} 2 & \text{if } s \text{ is odd and } 1 \leq s \leq m-1 \\ 4 & \text{if } s \text{ is even and } 2 \leq s \leq m-1. \end{cases}$$

- ⋆ For each integer j with $2 \leq j \leq n-1$, color the edges of the m-path H_j by

$$c(v_{s,j}v_{s+1,j}) = \begin{cases} 3 & \text{if } s \text{ is odd and } 1 \leq s \leq m-1 \\ 4 & \text{if } s \text{ is even and } 2 \leq s \leq m-1. \end{cases}$$

- ⋆ Color the edges of the m-path H_n by

$$c(v_{s,n}v_{s+1,n}) = \begin{cases} 1 & \text{if } s \text{ is odd and } 1 \leq s \leq m-1 \\ 4 & \text{if } s \text{ is even and } 2 \leq s \leq m-1. \end{cases}$$

This 4-edge coloring is shown in Figure 6.4 for the graph $P_6 \square C_7$. Since c is a proper edge coloring of G using four colors, it follows that $\chi'(G) \leq 4$. Therefore, $\mathrm{rd}(G) = 4$ by Proposition 6.1.2. ■

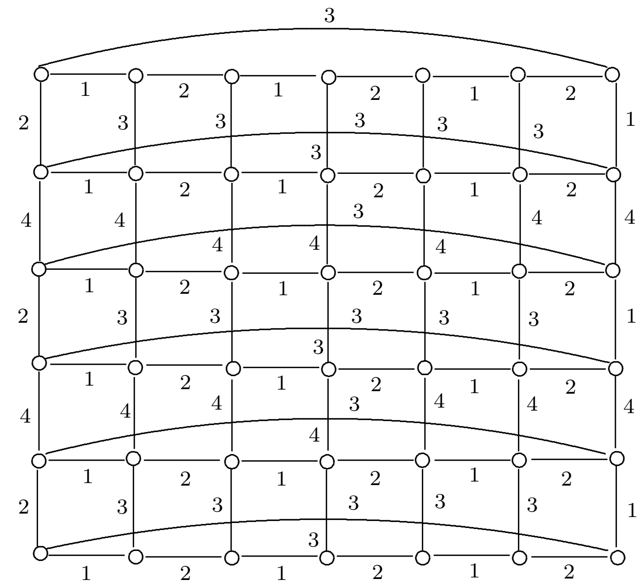


Figure 6.4: A proper 4-edge coloring of $P_6 \,\square\, C_7$

Complementary products were introduced in [22] as a generalization of Cartesian products. We consider a subfamily of complementary products, namely, complementary prisms. For a graph $G = (V, E)$, the *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex in $\overline{G}$ corresponding to v. Formally, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$. We note that complementary prisms are a generalization of the Petersen graph. In particular, the Petersen graph is the complementary prism $C_5\overline{C}_5$. For another example of a complementary prism, the corona $K_n \circ K_1$ is the complementary prism $K_n\overline{K}_n$.

We refer to the complementary prism $G\overline{G}$ as a copy of G and a copy of $\overline{G}$ with a perfect matching between corresponding vertices. For a set $S \subseteq V(G)$, let $\overline{S}$ denote the corresponding set of vertices in $V(\overline{G})$. We note that $G\overline{G}$ is isomorphic to $\overline{G}G$.

Since $\Delta(G\overline{G}) = \max\{\Delta(G), \Delta(\overline{G})\} + 1$, Proposition 6.1.2 implies that $\mathrm{rd}(G\overline{G}) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2$. This bound is sharp for the Petersen graph $P = C_5\overline{C}_5$ since by Proposition 6.1.5, $\mathrm{rd}(P) = 4 = \Delta(C_5) + 2$. On the other hand, for the complementary prisms $K_n\overline{K}_n$, Corollary 6.1.11 and Proposition 6.2.5 imply that $\mathrm{rd}(K_n\overline{K}_n) = \mathrm{rd}(K_n) = n-1 = \Delta(K_n) < \max\{\Delta(K_n), \Delta(\overline{K}_n\} + 2 = n+1$. Our next result shows that for graphs G with sufficiently large girth, $\mathrm{rd}(G\overline{G})$ is strictly greater than the maximum degree of G.

Proposition 6.3.4 *If G is a graph of order n, maximum degree $\Delta(G) < n - 1$, and* *girth at least* 5*, then*

$$\Delta(G) + 1 \leq \mathrm{rd}(\mathrm{G}\overline{\mathrm{G}}).$$

Proof. Consider a vertex u in G such that $\deg_G u = \Delta(G)$. Let $A = N_G(u)$ and $B = V - N_G[u]$. Thus, in $G\overline{G}$, $N(\overline{u}) = \overline{B} \cup \{u\}$. Note that since $n-1 > \Delta(G)$, it follows that $\overline{B} \neq \emptyset$.

We claim there are $\Delta(G)+1$ edge-disjoint u-$\overline{b}$ paths, where $\overline{b} \in \overline{B}$. To see this note that one such path is $(u, \overline{u}, \overline{b})$. Next consider the u-$\overline{b}$ paths containing a vertex $a \in A$. If a is not adjacent to b in G, then $\overline{a}$ is adjacent to $\overline{b}$ in $\overline{G}$ and $(u, a, \overline{a}, \overline{b})$ is a u-$\overline{b}$ path. If $ab \in E(G)$, then $(u, a, b, \overline{b})$ is a u-$\overline{b}$ path. Moreover, since $g(G) \geq 5$, at most one vertex in A is adjacent to b, else a 4-cycle is formed. In any case, the collection of these $|A| + 1 = \Delta(G) + 1$ paths are edge-disjoint. Hence, by Proposition 6.1.2, it follows that $\mathrm{rd}(G\overline{G}) \geq \lambda^+(G\overline{G}) \geq \Delta(G) + 1$. ■

For an example of a complementary prism attaining the lower bound, let G be the graph formed from a 5-cycle by attaching a leaf x to a vertex v of the cycle. Then, $\Delta(G) = 3$. We show that $\mathrm{rd}(G\overline{G}) = 4$. First note that the Petersen graph P is a proper subgraph of $G\overline{G}$, and by Propositions 6.1.5 and 6.1.6, $\mathrm{rd}(G\overline{G}) \geq \mathrm{rd}(P) = 4$. Furthermore, there is a proper edge-coloring c of P using four colors such that three colors are used to color C_5 and $\overline{C}_5$ and the fourth color is used on the matching edges. Thus, we may assume, without loss of generality, that v is incident to the edges colored 1 and 2 in G and that $v\overline{v}$ is assigned color 4. We extend c to a rainbow disconnection coloring of $G\overline{G}$ as follows: let $c(vx) = 3$, $c(x\overline{x}) = 4$, and $c(\overline{xu})$ be the color missing from the edges incident to $\overline{u}$ for each $\overline{u}$ adjacent to $\overline{x}$ in $\overline{G}$. Consider two arbitrary vertices of $G\overline{G}$. At least one of the vertices, say u, is not $\overline{x}$. Thus, the edges incident with u are a rainbow cut separating the two vertices. Since every such vertex u has degree at most 4, it follows that $\mathrm{rd}(G\overline{G}) \leq 4$ and so $\mathrm{rd}(G\overline{G}) = 4$.

6.4 Extremal Problems

In this section, we investigate the following problem:

> For a given pair k, n of positive integers with $k \leq n-1$, what are the minimum possible size and maximum possible size of a connected graph G of order n such that the rainbow disconnection number of G is k?

We have seen in Proposition 6.2.1 that the only connected graphs of order n having rainbow disconnection number 1 are the trees of order n. That is, the connected graphs of order n having rainbow disconnection number 1 have size $n-1$. We have also seen in Theorem 6.2.4 that the minimum size of a connected graph of order $n \geq 3$ having rainbow disconnection number 2 is n. Furthermore, we have seen in Theorem 6.2.6 that the minimum size of a connected graph of order $n \geq 2$ having rainbow disconnection number $n-1$ is $2n-3$. In fact, these are special cases of a more general result. In order to show this, we first present a lemma.

Lemma 6.4.1 *Let H be a connected graph of order n that is not complete and let x and y be two nonadjacent vertices of H. Then* $\mathrm{rd}(H+xy) \leq \mathrm{rd}(H)+1$.

Proof. Suppose that $\mathrm{rd}(H)=k$ for some positive integer k and let c_0 be a rainbow disconnection coloring of H using the colors $1,2,\ldots,k$. Extend the coloring c_0 to the edge-coloring c of $H+xy$ by assigning the color $k+1$ to the edge xy. Let u and v be two vertices of H and let R be a $u-v$ rainbow cut in H. Then $R \cup \{xy\}$ is a $u-v$ rainbow cut in $H+xy$. Hence, c is a rainbow disconnection $(k+1)$-coloring of $H+xy$. Therefore, $\mathrm{rd}(H+xy) \leq k+1 = \mathrm{rd}(H)+1$. ∎

Theorem 6.4.2 *For integers k and n with $1 \leq k \leq n-1$, the minimum size of a connected graph of order n having rainbow disconnection number k is $n+k-2$.*

Proof. By Theorem 6.2.6, the result is true for $k=n-1$. Hence, we may assume that $1 \leq k \leq n-2$. First, we show that if the size of a connected graph G of order n is $n+k-2$, then $\mathrm{rd}(G) \leq k$. We proceed by induction on k. We have seen that the result is true for $k=1,2$ by Proposition 6.2.1 and Theorem 6.2.4. Suppose that if the size of a connected graph H of order n is $n+k-2$ for some integer k with $2 \leq k \leq n-3$, then $\mathrm{rd}(H) \leq k$. Let G be a connected graph of order n and size $n+(k+1)-2 = n+k-1$. We show that $\mathrm{rd}(G) \leq k+1$. Since G is not a tree, there is an edge e such that $H = G-e$ is a connected spanning subgraph of G. Since the size of H is $n+k-2$, it follows by induction hypothesis that $\mathrm{rd}(H) \leq k$. Hence, $\mathrm{rd}(G) = \mathrm{rd}(H+e) \leq k+1$ by Lemma 6.4.1. Therefore, the minimum possible size for a connected graph G of order n to have $\mathrm{rd}(G)=k$ is $n+k-2$. [Note that if F is a connected graph of order n and size $m < n+k-2$, then $m = n+k-a$ for some integer $a \geq 3$. Since $m = n+(k-a+2)-2$, it follows that $\mathrm{rd}(F) \leq k-a+2 \leq k-1$.]

It remains to show that for each pair k,n of integers with $1 \leq k \leq n-1$ there is a connected graph G of order n and size $n+k-2$ such that $\mathrm{rd}(G)=k$. Since this is true for $k=1,2,n-1$, we now assume that $3 \leq k \leq n-2$. Let $H = K_{2,k}$ with partite set $U=\{u_1,u_2\}$ and $W=\{w_1,w_2,\ldots,w_k\}$. Now, let G be the graph of order n

and size $n+k-2$ obtained from H by subdividing the edge u_1w_1 a total of $n-k-2$ times, producing the path $P = (u_1, v_1, v_2, \ldots, v_{n-k-2}, w_1)$ in G. Since $\chi'(H) = k$, there is a proper edge-coloring c_H of H using the colors $1, 2, \ldots, k$. We may assume that $c(u_1w_1) = 1$ and $c(u_2w_1) = 2$. Next, we extend the coloring c_H to a proper edge-coloring c_G of G using the colors $1, 2, \ldots, k$ by defining $c_G(u_1v_1) = 1$ and alternating the colors of the edges of P with 3 and 1 thereafter. Hence, $\chi'(G) = k$ and so $\mathrm{rd}(G) \le \chi'(G) = k$ by Proposition 6.1.2. Furthermore, since $\lambda(u_1, u_2) = k$ and $\lambda(x, y) = 2$ for all other pairs x, y of vertices of G, it follows that $\lambda^+(G) = k$. Again, by Proposition 6.1.2, $\mathrm{rd}(G) \ge \lambda^+(G) = k$ and so $\mathrm{rd}(G) = k$. ■

For given integers k and n with $1 \le k \le n-1$, we've determined the minimum size of a connected graph G of order n with $\mathrm{rd}(G) = k$. So, this brings up the question of determining the maximum size of a connected graph G of order n with $\mathrm{rd}(G) = k$. Of course, we know this size when $k = 1$; it's $n-1$. Also, we know this size when $k = n-1$; it's $\binom{n}{2}$. For odd integers n, we have the following conjecture.

Conjecture 6.4.3 *Let k and n be integers with $1 \le k \le n-1$ and $n \ge 5$ is odd. Then the maximum size of a connected graph G of order n with $\mathrm{rd}(G) = k$ is $\frac{(k+1)(n-1)}{2}$.*

Notice that when $k = 1$, then $\frac{(k+1)(n-1)}{2} = n-1$ and when $k = n-1$, then $\frac{(k+1)(n-1)}{2} = \binom{n}{2}$. Also, when $k = 2$, then $\frac{(k+1)(n-1)}{2} = \frac{3n-3}{2}$. This is the size of the so-called *friendship graph* $\left(\frac{k-1}{2}\right) K_2 \vee K_1$ of order n (every two vertices has a unique friend). Since each block of a friendship graph is a triangle, it follows by Theorem 6.2.4 that each such graph has rainbow disconnection number 2.

For given integers k and n with $1 \le k \le n-1$ and $n \ge 5$ is odd, let H_k be a $(k-1)$-regular graph of order $n-1$. Since $n-1$ is even, such graphs H_k exist. Now, let $G_k = H_k \vee K_1$ be the join of H_k and K_1. Thus, G_k is a connected graph of order n having one vertex of degree $n-1$ and $n-1$ vertices of degree k. The size m of G_k satisfies the equation:

$$2m = (n-1) + (n-1)k = (k+1)(n-1)$$

and so $m = \frac{(k+1)(n-1)}{2}$. The graph H_k can be selected so that it is 1-factorable and so $\chi'(H_k) = k-1$. If a proper $(k-1)$-edge-coloring of H_k is given using the colors $1, 2, \ldots k-1$, and every edge incident with the vertex of G_k of degree $n-1$ is assigned the color k, then the edges incident with each vertex of degree k are properly colored with k colors. For any two vertices u and v of G_k, at least one of u and v has degree k in G_k, say $\deg_{G_k} u = k$. Then the set of edges incident with u is a $u-v$ rainbow cut in H.

Since this is a rainbow disconnection k-coloring of G, it follows that $\mathrm{rd}(G_k) \leq k$. It is reasonable to conjecture that $\mathrm{rd}(G_k) = k$.

We would still be left with the question of whether every graph H of order n and size $\frac{(k+1)(n-1)}{2} + 1$ must have $\mathrm{rd}(H) > k$. Certainly, every such graph H must contain at least two vertices whose degrees exceed k.

Chapter 7

Topics for Further Study

In this chapter, we describe some open questions and concepts resulting from this work. We also discuss some related topics for further study.

7.1 Problems on 3-Rainbow Connection

1. **2-Connected Graphs**

   In [8], it was shown that $\mathrm{rc}_2(G) \leq 3$ if G is 2-connected and there is a 2-connected graph H with $\mathrm{rc}_2(H) = 3$. This graph H is shown in Figure 7.1. In fact, $\mathrm{rc}_3(H) = 3$ as well and a 3-rainbow coloring using three colors is shown shown in Figure 7.1.

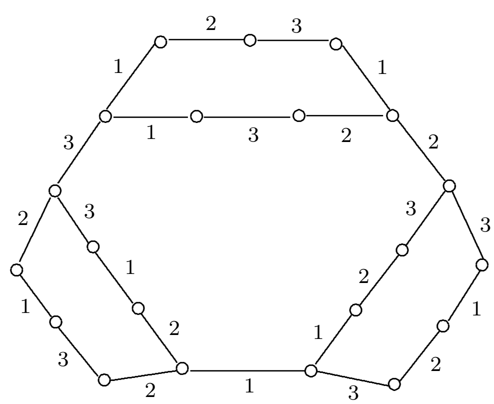


Figure 7.1: A 2-connected graph H with $\mathrm{rc}_2(H) = \mathrm{rc}_3(H) = 3$

There is reason to believe that for a 2-connected graph, the following is also true.

Conjecture 7.1.1 *If G is a 2-connected graph, then $\mathrm{rc}_3(G) \leq 3$.*

For connected graphs G with cut-vertices, there is a lower bound for the 3-rainbow connection number $\mathrm{rc}_3(G)$ in terms of the number of components in the subgraph of G that results when a cut-vertex is removed. For each cut-vertex v of G, let τ_v be the number of components in $G - v$ and let $\tau(G) = \max\{\tau_v : v \text{ is a cut-vertex of } G\}$.

Proposition 7.1.2 *If G is connected graph with cut-vertices, then* $\mathrm{rc}_3(G) \geq \tau(G)$.

Proof. Let $v \in V(G)$ such that $\tau_v = \tau(G) = \tau$ and let $G_1, G_2, \ldots, G_\tau$ be the components of $G - v$. For each i with $1 \leq i \leq \tau$, let $v_i \in V(G_i)$. Since every 3-rainbow coloring of G must assign distinct colors to the τ edges $vv_1, vv_2, \ldots, vv_\tau$ of G, it follows that $\mathrm{rc}_3(G) \geq \tau(G)$. ■

If $G = K_{1,\tau}$ is a star of order $\tau + 1 \geq 3$, then $\tau(G) = \tau$ and $\mathrm{rc}_3(G) = \tau(G)$. Next, let $G = 2P \vee K_1$, where P is the Petersen graph. Then G has a unique cut-vertex and $\tau(G) = 2$. Since $\mathrm{rc}_3(G) = 3$, it follows that $\mathrm{rc}_3(G) > \tau(G)$. Thus, it is possible that $\mathrm{rc}_3(G) = \tau(G)$ or $\mathrm{rc}_3(G) < \tau(G)$. These two examples also show that the 3-rainbow connection number of a connected graph of diameter 2 can be arbitrarily large. However, in each case, the graph under study is not 2-connected.

2. **Realizable Triples**

   If G is a nontrivial connected graph with $\mathrm{pc}(G) = a$, $\mathrm{rc}_3(G) = b$ and $\mathrm{rc}(G) = c$, then $1 \leq a \leq b \leq c$. This suggests the following question.

   Problem 7.1.3 *For which triples a, b, c, does there exist a nontrivial connected graph such that* $\mathrm{pc}(G) = a$, $\mathrm{rc}_3(G) = b$ *and* $\mathrm{rc}(G) = c$*?*

3. **Unique 3-Rainbow Paths**

   A tree T has the property that there exists a 3-rainbow coloring of T such that every two vertices u and v of T are connected by a unique 3-rainbow $u - v$ path in T.

   Conjecture 7.1.4 *If G is a connected graph with a 3-rainbow coloring such that every two vertices u and v of G are connected by a unique 3-rainbow $u - v$ path in G, then each block of G is a complete graph.*

4. **Subgraphs of Diameter 3**

   For a connected graph G of diameter 3 or more, let

   $$\rho_3 = \max\{\mathrm{rc}_3(H) : \ H \subseteq G \text{ and } \mathrm{diam}(H) = 3\}.$$

   What is the relationship between ρ_3 and $\mathrm{rc}_3(G)$?

7.2 Rainbow Sequences

If G is a nontrivial connected graph and ℓ is the length of a longest path in G, then $\mathrm{pc}(G) = \mathrm{rc}_2(G) \leq \mathrm{rc}_3(G) \leq \cdots \leq \mathrm{rc}_\ell(G) = \mathrm{rc}(G)$ and that the sequence

$$\mathrm{pc}(G), \mathrm{rc}_3(G), \mathrm{rc}_4(G), \cdots, \mathrm{rc}(G)$$

is referred to as the *rainbow sequence of* G, denoted by $\mathcal{S}_r(G)$.

A nondecreasing sequence of positive integers is a *rainbow sequence* if it is the rainbow sequence of some connected graph G.

Problem 7.2.1 *Which sequences of positive integers are rainbow sequences?*

1. There exist connected graphs G whose longest path has length ℓ for which

$$\mathrm{rc}_i(G) = \mathrm{rc}_{i-1}(G) \text{ for each integer } i \text{ with } 2 < i < \ell. \tag{7.1}$$

In particular, this is true for all connected graphs G with $\mathrm{rc}_3(G) = \mathrm{rc}(G)$. For example, let $W_n = C_n \vee K_1$ be the wheel of order $n + 1 \geq 5$, where n is the length of a longest path. If n and k are integers with $2 < k < n$ and $n \geq 4$, then

$$\mathrm{rc}_k(W_n) = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } 4 \leq n \leq 6 \\ 3 & \text{if } n \geq 7. \end{cases}$$

Which connected graphs G with $\mathrm{rc}_3(G) < \mathrm{rc}(G)$ *have the property described in* (7.1)?

2. There exist connected graphs G whose longest path has length ℓ for which

$$\mathrm{rc}_i(G) - \mathrm{rc}_{i-1}(G) = 1 \text{ for each integer } i \text{ with } 2 < i < \ell. \tag{7.2}$$

For integers k and ℓ with $2 \leq k \leq \ell - 1$, $\mathrm{rc}_k(P_\ell) = k$. Thus, if $G = P_\ell$ is the path of order $\ell \geq 3$, it follows that $\mathrm{rc}_i(G) - \mathrm{rc}_{i-1}(G) = 1$ for each integer i with $2 < i < \ell$.

Which other graphs have the property described in (7.2)?

3. *If* $\mathrm{rc}_i(G) = \mathrm{rc}_{i+1}(G)$ *for some integer i, does $rc_j(G) = \mathrm{rc}_i(G)$ for all $j > i$?*

7.3 k-Rainbow Connectivity

Let G be a graph with connectivity $\ell \geq 1$ and let $k \geq 3$. The following concepts can be studied.

1. The k-*rainbow connectivity* of G is the minimum number of colors needed in an edge-coloring of G such that every two distinct vertices u and v of G are connected by ℓ internally vertex-disjoint k-rainbow $u-v$ paths.

2. The k-*rainbow edge connectivity* of G is the minimum number of colors needed in an edge-coloring of G such that every two distinct vertices u and v of G are connected by ℓ edge-disjoint k-rainbow $u-v$ paths.

3. The 2-*mixed connectivity* of G is the minimum number of colors needed in an edge-coloring of G such that every two distinct vertices u and v of G are connected by both a rainbow $u-v$ path and a non-rainbow proper $u-v$ path or by both a rainbow $u-v$ path and a non-rainbow k-rainbow $u-v$ path.

7.4 Color Disconnection in Graphs

There are several open questions dealing with rainbow disconnection in edge-colored graphs. We list some of them below.

1. Are there classes of graphs G for which $\text{rd}(G) = \Delta(G) + 1$?

2. Does there exist a prism which does not have a proper rd-coloring?

3. Do 3-by-n grid graphs, for $n \geq 3$, have proper rd-colorings?

4. Does there exist a graph G for which $\lambda^+(G) < \text{rd}(G)$? Can this gap be arbitrarily large?

5. Does there exist a graph G for which $\lambda^+(G) < \text{rd}(G) < \chi(G)$?

There is a related disconnection concept that is closely related to rainbow disconnection in edge-colored graphs. A set R of edges in an edge-colored graph G is called a *proper* set if no two *adjacent* edges in R are colored the same. A set R of edges is a *proper cut* of G if R is both a proper set and an edge-cut. A proper cut R is said to *separate* two vertices u and v of G if u and v belong to different components of $G - R$. Any such proper cut in G is called a $u-v$ *proper cut* in G. An edge coloring of G is a *proper disconnection coloring* if for every two distinct vertices u and v of G, there exists a $u-v$ proper cut in G. The *proper disconnection number* $\text{pd}(G)$ of G is the minimum number of colors required of a proper disconnection coloring of G. A proper disconnection coloring with $\text{pd}(G)$ colors is called an pd-*coloring* of G. Since every rainbow-cut is a proper cut, it follows that $\text{pd}(G) \leq \text{rd}(G)$ for every graph G. We plan to study this concept.

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