A Study of a Graphical Valued Function Associated With an Extension of Graphs

Nance

Follow this and additional works at: https://scholarworks.wmich.edu/masters_theses

Part of the Mathematics Commons

Recommended Citation
https://scholarworks.wmich.edu/masters_theses/3188

This Masters Thesis-Open Access is brought to you for free and open access by the Graduate College at ScholarWorks at WMU. It has been accepted for inclusion in Master's Theses by an authorized administrator of ScholarWorks at WMU. For more information, please contact wmu-scholarworks@wmich.edu.
A STUDY OF A GRAPHICAL VALUED
FUNCTION ASSOCIATED WITH AN
EXTENSION OF GRAPHS

by

Douglas W. Nance

A Project Report
Submitted to the
Faculty of the School of Graduate
Studies in partial fulfillment
of the
Specialist in Arts Degree

Western Michigan University
Kalamazoo, Michigan
July, 1968
ACKNOWLEDGEMENTS

The author of this thesis wishes to express his sincere gratitude and appreciation for the helpful suggestions, encouragement and seemingly infinite patience of his advisor, Dr. S. F. Kapoor. He also wishes to acknowledge Dr. Gary Chartrand for his many suggestions and constructive criticism. Finally, the author wishes to thank his wife, Helen, for her patience and encouragement.

Douglas William Nance
MASTER'S THESIS M-1629

NANCE, Douglas William
A STUDY OF A GRAPHICAL VALUED FUNCTION ASSOCIATED WITH AN EXTENSION OF GRAPHS.

Western Michigan University, Sp.A., 1968
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>ACKNOWLEDGEMENTS</th>
<th>ii</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF FIGURES</td>
<td>iv</td>
</tr>
<tr>
<td><strong>CHAPTER</strong></td>
<td></td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. BASIC DEFINITIONS</td>
<td>4</td>
</tr>
<tr>
<td>III. PARK GRAPHS</td>
<td>20</td>
</tr>
<tr>
<td>IV. HAMILTONIANISM</td>
<td>36</td>
</tr>
<tr>
<td>V. PLANARITY</td>
<td>54</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>71</td>
</tr>
<tr>
<td>INDEX OF DEFINITIONS AND SYMBOLS</td>
<td>72</td>
</tr>
<tr>
<td>Figure</td>
<td>PAGE</td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>Figure 1.1</td>
<td>1</td>
</tr>
<tr>
<td>Figure 2.1</td>
<td>5</td>
</tr>
<tr>
<td>Figure 2.2</td>
<td>7</td>
</tr>
<tr>
<td>Figure 2.3</td>
<td>9</td>
</tr>
<tr>
<td>Figure 2.4</td>
<td>11</td>
</tr>
<tr>
<td>Figure 2.5</td>
<td>11</td>
</tr>
<tr>
<td>Figure 2.6</td>
<td>13</td>
</tr>
<tr>
<td>Figure 2.7</td>
<td>14</td>
</tr>
<tr>
<td>Figure 2.8</td>
<td>16</td>
</tr>
<tr>
<td>Figure 3.1</td>
<td>21</td>
</tr>
<tr>
<td>Figure 3.2</td>
<td>23</td>
</tr>
<tr>
<td>Figure 3.3</td>
<td>24</td>
</tr>
<tr>
<td>Figure 3.4</td>
<td>25</td>
</tr>
<tr>
<td>Figure 3.5</td>
<td>26</td>
</tr>
<tr>
<td>Figure 3.6</td>
<td>28</td>
</tr>
<tr>
<td>Figure 3.7</td>
<td>29</td>
</tr>
<tr>
<td>Figure 3.8</td>
<td>30</td>
</tr>
<tr>
<td>Figure 3.9</td>
<td>35</td>
</tr>
<tr>
<td>Figure 4.1</td>
<td>36</td>
</tr>
<tr>
<td>Figure 4.2</td>
<td>49</td>
</tr>
<tr>
<td>Figure 4.3</td>
<td>51</td>
</tr>
<tr>
<td>Figure 5.1</td>
<td>54</td>
</tr>
<tr>
<td>Figure 5.2</td>
<td>55</td>
</tr>
<tr>
<td>Figure 5.3</td>
<td>56</td>
</tr>
<tr>
<td>Figure 5.4</td>
<td>57</td>
</tr>
<tr>
<td>Figure 5.5</td>
<td>57</td>
</tr>
<tr>
<td>Figure 5.6</td>
<td>60</td>
</tr>
<tr>
<td>Figure 5.7</td>
<td>61</td>
</tr>
<tr>
<td>Figure 5.8</td>
<td>62</td>
</tr>
<tr>
<td>Figure 5.9</td>
<td>63</td>
</tr>
<tr>
<td>Figure 5.10</td>
<td>64</td>
</tr>
<tr>
<td>Figure 5.11</td>
<td>65</td>
</tr>
<tr>
<td>Figure 5.12</td>
<td>67</td>
</tr>
<tr>
<td>Figure 5.13</td>
<td>68</td>
</tr>
<tr>
<td>Figure 5.14</td>
<td>68</td>
</tr>
<tr>
<td>Figure 5.15</td>
<td>69</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

If we start with a finite nonempty set $V$ and pick a set $E$ whose elements are unordered pairs of elements from $V$, we obtain an ordered pair of sets $(V, E)$ such that this yields the usual definition of an ordinary graph, i.e., finite, undirected, and without loops or multiple edges.

This raises the natural question of considering sets $\pi(k)$ whose elements are certain unordered $k$-tuples of elements from $V$ for $k \geq 2$. By imposing suitable restrictions, $\pi(2)$ turns out to be $E$. For $k = 3$ we may proceed as follows. If $p \in \pi(3)$, then $p = (uvw)$, where $u$, $v$ and $w$ are distinct elements from $V$. The following possibilities now arise for the unordered couples of elements in $p$ as shown in Figure 1.1.

![Figure 1.1](image_url)
If \( \pi(3) \) consists of elements of type (a) alone, we essentially work with triangles or 2-simplexes in graphs. Harary and Palmer have done some work in this connection in a recent paper [5]. Elements of type (b) qualify to be called "almost triangles" and this research mainly deals with sets \( \pi(3) \) consisting of all elements of type (a) or (b). Still another reason for choosing elements of these types is that the vertices in an element are "connected". We will write \( \pi(3) \) as \( P \) and say that its elements are parks in the graph. We note that if \( p = (uvw) \) is a park in a graph \( G \), then the corresponding triple in \( \overline{G} \) (the complement of \( G \)) must be of type (c) or (d). In this sense, types (a) and (b) in \( G \) are duals of types (c) and (d) in \( \overline{G} \).

In Chapter II basic definitions are presented and some preliminary results are obtained. Most of the familiar concepts known for ordinary graphs \((V,E)\) are extended to generalized graphs \((V,E,P)\).

For an arbitrary graph \( G \), a graphical valued function \( P \), called the park graph function, is defined in Chapter III. \( P(G) \) denotes the park graph of \( G \). Simple combinatorial properties are investigated. Park graphs are considered in detail for special classes of graphs including star graphs, some complete graphs and a few complete bipartite graphs. It is also proved that \( P(G) \) cannot be \( K_n \) \((n \geq 5)\) or \( K_{m,n} \) \((m \geq 1, n \geq 3)\) for any graph \( G \).

Since \( P \) is a graphical valued function, it belongs to a class which includes \( L \) (the line-graph function), and \( T \) (the
total graph function). These were studied in [2] and [1] respectively. It is known that the images of a hamiltonian graph under L and T are hamiltonian. We prove the corresponding result for P in Chapter IV.

Sedláček gave a characterization of planar line-graphs in [8]. We show in Chapter V that the park graph of a nonplanar graph is nonplanar and investigate some properties as possible candidates for obtaining a characterization similar to Sedláček.

The contents of this thesis are in the nature of a preliminary report on a new concept in graph theory. This has suggested many reasonable and interesting problems, and their investigation will serve as topics for further research.
BASIC DEFINITIONS

Following the prevailing terminology in the field of graph theory, we will mean by a graph $G$ an ordered pair of sets $(V,E)$ where $V$ is a finite nonempty set of elements called vertices (points) of $G$, and $E$ is a collection of unordered pairs of two element subsets of $V$ whose elements are called edges (lines) of $G$. To avoid ambiguity, $V$ and $E$ will sometimes be denoted by $V(G)$ and $E(G)$ respectively. In view of this, $G$ becomes an ordinary graph, i.e., finite, undirected, and without loops or multiple edges. Unless otherwise specified we will follow the standard terminology of [4].

In a graph $G = (V,E)$, consider an unordered triple $p = (v_1v_2v_3)$ with the property that at least two of the unordered couples $(v_1v_2), (v_2v_3),$ and $(v_3v_1)$ are in $E$. Call such an element $p$ a park and denote the collection of all parks of $G$ by $P$. The two element subsets of a park $p$ that are in $E$ are called the lines of the park $p$ and their number is denoted by $|p|$. Clearly $|p| = 2$ or $3$.

**Definition 2.1.** An extended graph $G$ is an ordered triple $(V,E,P)$ where $G = (V,E)$ is a graph and $P$ is the collection of all the parks of $G$.

An ordinary graph $G$ then can be thought of as being embedded in an extended graph $G$, and without any loss of generality we will...
assume that all ordinary graphs are extended graphs. For this
reason, henceforth we make no distinction between them, and we
will call them simply "graphs".

Example 2.1. For the graph \( G \) in Figure 2.1,
\[
V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}
\]
\[
E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_1, v_4), (v_4, v_5), (v_5, v_6), (v_5, v_7), (v_6, v_7)\}
\]
\[
P = \{(v_1, v_2, v_3), (v_2, v_3, v_4), (v_3, v_4, v_5), (v_3, v_4, v_6), (v_1, v_4, v_5), (v_2, v_4, v_5), (v_4, v_5, v_6), (v_4, v_5, v_7), (v_5, v_6, v_7)\}
\]

![Figure 2.1](image)

Proposition 2.1. If parks \( p_1, p_2 \) have two lines in common,
then \( p_1 = p_2 \).

Proof: Let parks \( p_1 = \text{(abc)} \) and \( p_2 = \text{(def)} \) have two common
lines. Since one line in common implies that at least two of the
vertices of \( p_1 \) and \( p_2 \) must be the same, we may assume that \( a = d \)
and \( b = e \). Hence \( p_1 = \text{(abc)} \) and \( p_2 = \text{(abf)} \). The other two possible
lines for \( p_1 \) are \( \text{(ac)} \) or \( \text{(bc)} \), and for \( p_2 \) are \( \text{(af)} \) or \( \text{(bf)} \).
Since \( a, b, \) and \( c \) are distinct, we must have \( \text{(ac)} = \text{(af)} \) or \( \text{(bf)} \),
or \( \text{(bc)} = \text{(af)} \) or \( \text{(bf)} \). But we cannot have \( \text{(ac)} = \text{(bf)} \) or
\( \text{(bc)} = \text{(af)} \). Consequently the remaining alternatives imply that
c = f. Hence $p_1 = p_2$.

**Proposition 2.2.** For any graph $G$, $|P| = \sum_{v \in V} \binom{\deg v}{2} - 2T$

where $T$ is the number of triangles of $G$.

**Proof:** Consider a vertex $v$ of $G$ with $\deg v = k \geq 2$ and let $x_1, x_2, \ldots, x_k$ denote the $k$ lines incident with $v$. For each pair of distinct integers $i, j, 1 \leq i, j \leq k$, the pair of lines $(x_i, x_j)$ form a park which includes some other line $y$ if there exists a triangle $t$ that contains $x_i, x_j,$ and $y$. Hence by considering all pairs of lines incident with $v$, we obtain $\binom{\deg v}{2}$ parks. Since all parks contain at least two lines, we need not apply the above criterion to $v \in V$ such that $\deg v < 2$. Then we get $|P| \leq \sum_{v \in V} \binom{\deg v}{2}$. Let $t$ be a triangle in $G$ with vertices $v_1, v_2, \text{ and } v_3$. Since $\deg v_i = k_i \geq 2, i = 1, 2, 3$, each $v_i$ is considered in the previous sum and the park $p = (v_1 v_2 v_3)$ gets added three times. Therefore we must subtract two for each such triangle to obtain the number of distinct parks of $G$. By Proposition 2.1 no other duplications can occur, hence

$|P| = \sum_{v \in V} \binom{\deg v}{2} - 2T.$

**Corollary 2.3.** If $G$ is bipartite, then $|P| = \sum_{v \in V} \binom{\deg v}{2}$.

**Proof:** For bipartite graphs, $T = 0$.

For a graph $G$, the definitions of the usual concepts of
degree, adjacency, incidence, sequence of lines, connectedness, paths, cycles, walks, trails, blocks, distance and planarity can be found in [4]. It is possible to define their counterparts for parks.

**Definition 2.2.** In a graph G, two distinct parks will be said to be **adjacent** if they have a line in common.

**Example 2.2.** Let \( p_1 = (v_1v_2v_3) \), \( p_2 = (v_2v_3v_4) \) and \( p_3 = (v_3v_4v_5) \) be the parks (shown by broken arcs) of the graph G in Figure 2.2. Then \( p_1 \) is adjacent to \( p_2 \) but not \( p_3 \).

![Figure 2.2](image)

According to the definition then, it is possible for parks to have a common vertex but not be adjacent. This provides the motivation for the following definition.

**Definition 2.3.** In a graph G, two distinct parks are said to be **weakly adjacent** if they have at least one vertex in common.

In Example 2.2, \( p_1 \) is weakly adjacent to \( p_3 \), but \( p_1 \) is not adjacent to \( p_3 \). Hence "adjacent" implies "weakly adjacent" but the converse does not necessarily hold.

**Definition 2.4.** The **degree** of a park \( p \) in a graph G is the number of distinct parks adjacent to \( p \) and will be written as \( \text{DEG}(p) \).

It is convenient to introduce the following notation. For a line \( x \) in a graph G, let \( T(x) \) and \( P(x) \) denote respectively the
number of distinct triangles and parks which contain $x$.

**Proposition 2.4.** If $x$ is any line in a graph $G$, then

$$P(x) + T(x) = \deg x.$$  

**Proof:** Let $\deg x = q$ and $x_1, x_2, \ldots, x_q$ be the $q$ lines incident with $x$. Then any park containing $x$ must contain at least one of the lines $x_i, 1 \leq i \leq q$. If $x$ is not contained in any triangle, then the pairs of lines $x$ and $x_i, i = 1, 2, \ldots, q$, form $q$ distinct parks containing $x$, and $P(x) = \deg x$. On the other hand, suppose $x$ is in a triangle $H$. Then there exist distinct integers $i, j, 1 \leq i, j \leq q$ such that $x_i$ and $x_j$ are incident with $x$. $H$ then contains the lines $x, x_i$ and $x_j$. Since $x_i$ and $x_j$ contribute two to the degree of $x$ but form only one park containing $x$, $\deg x$ must be decreased by one for every triangle that contains $x$ to obtain $P(x)$. Therefore

$$P(x) = \deg x - T(x).$$

**Proposition 2.5.** Let $p$ be any park in a graph $G$. Then

$$\text{DEG}(p) = \sum_{x \in p} (P(x) - 1).$$

**Proof:** Let $x$ be a line of $p$ and suppose that $P(x) = k$. Then there exist $k$ distinct parks containing the line $x$. Since $p$ is one of these $k$ parks, there are $k - 1$ parks different from $p$ that contain $x$, hence there are $k - 1$ parks adjacent to $p$ by virtue of having the line $x$ in common. Since $x \in p$ was arbitrary,

$$\text{DEG}(p) = \sum_{x \in p} (P(x) - 1).$$
Definition 2.5. A p-walk in G is an alternating sequence of parks and lines, beginning and ending with a park, such that each line in the sequence is contained in the park immediately preceding and following it in the sequence.

Example 2.3. The sequence $W = p_1, x_1, p_2, x_2, p_2, x_2, p_3,$ $x_3, p_5$ of parks and lines of the graph G in Figure 2.3 is a p-walk.

![Figure 2.3](image-url)

Figure 2.3

Note that in a p-walk both lines and parks may be repeated. If a p-walk begins with $p_1$ and ends with $p_n$, we speak of the p-walk joining $p_1$ and $p_n$. A p-walk joining $p_1$ and $p_n$ is called closed if $p_1 = p_n$, and open otherwise.

Lemma 2.6. Let $G$ be a path of length $r \geq 2$. Then the parks of $G$ form a p-walk.

Proof: Let $V = \{v_0, v_1, \ldots, v_r\}$ be the vertex set of $G$
with \( r \geq 2 \). Then \( E = \{ x_i = (v_{i-1}v_i) | i = 1,2,\ldots,r \} \) is the edge set of \( G \) and \( P = \{ p_i = (v_{i-1}v_i v_{i+1}) | i = 1,2,\ldots,r-1 \} \) is the park set. Clearly, two distinct parks \( p_i \) and \( p_j \) are adjacent if and only if \( |i - j| = 1 \). Thus \( W = p_1, x_2, p_2, \ldots, x_{r-1}, p_{r-1} \) is the desired p-walk. In fact, \( W \) turns out to be a p-path (see Definition 2.6) of length \( r - 2 \).

**Proposition 2.7.** In a graph \( G \), every walk with two or more distinct lines yields a p-walk, and conversely.

**Proof:** Let \( W = v_1, x_1, v_2, x_2, \ldots, v_n, x_n, v_{n+1} \) be a walk in \( G \) possessing at least two distinct lines. Let \( G' \) denote the subgraph of \( G \) formed by the points and lines in \( W \). Then \( G' \) is connected. If \( U \) is a path of maximum length \( s \) in \( G' \), then \( s \geq 2 \). By lemma 2.6, the parks of \( U \) form a p-walk.

To establish the converse, let \( W = p_1, x_1, p_2, x_2, \ldots, p_n \) be a p-walk in \( G \). Then \( G' \), the subgraph of \( G \) formed by the points and lines in \( W \), is connected. Since \( W \) has at least one distinct park, \( G' \) has a path of length two or more. This completes the proof.

**Definition 2.6.** In a graph \( G \), a p-trail is a p-walk in which no line is repeated. A p-walk in which neither lines nor parks repeat is called a p-path. The number of lines in a p-trail (p-path) will be called the length of the p-trail (p-path).

**Example 2.4.** In the graph \( G \) of Figure 2.4,
\[ \mathcal{J} = p_1, x_4, p_2, x_2, p_2, x_3, p_3 \] is a p-trail and
\[ \mathcal{F} = p_1, x_4, p_2, x_3, p_3 \] is a p-path. (Recall that by Proposition
2.1, two lines of a park completely determine that park. Thus $p_2$ is the triangle containing lines $x_2, x_3,$ and $x_4$.)

By a construction very similar to that employed in the proof of Proposition 2.7, it is readily seen that a trail (path) with two or more lines gives rise to a $p$-trail (p-path) and conversely.

**Definition 2.7.** A graph $G$ is **p-connected** if between every two parks $p, p'$ in $G$ there is a $p$-path beginning at $p$ and ending at $p'$.

In view of this definition, the graphs $G_1$ and $G_2$ of Figure 2.5 are seen to be p-connected but not connected.
These graphs show that connectedness is not equivalent to p-connectedness. However, the following result is possible.

**Proposition 2.8.** (a) If $G$ is a connected graph, then $G$ is p-connected. (b) If $G$ is a p-connected graph which contains no isolated points and each of whose components has at least two lines, then $G$ is connected.

**Proof:** (a) If $G$ has only one park, the result follows vacuously. Otherwise let $p'$ and $p''$ be any two distinct parks of $G$. If $p'$ and $p''$ have a common line $x$, then $p'$, $x$, $p''$ is the desired p-path. If $p'$ and $p''$ have no line in common, there exists a path $P$ from a point $v_1$ of $p'$ to a point $v_{n+1}$ of $p''$ such that $P$ contains no other vertices of $p'$ or $p''$. Then $P$ is of the form $v_1$, $x_1$, $v_2$, $x_2$, ..., $v_n$, $x_n$, $v_{n+1}$. Let $x_0$ be a line in $p'$ with endpoints $v_1$ and $v_0$, and $x_{n+1}$ be a line in $p''$ joining $v_{n+1}$ and $v_{n+2}$. Now consider the path $P' = v_0$, $x_0$, $v_1$, $x_1$, ..., $v_n$, $x_n$, $v_{n+1}$, $x_{n+1}$, $v_{n+2}$ whose first line is in $p'$ and last line is in $p''$. Then the path $P'$ gives rise to a p-path joining $p'$ and $p''$.

(b) Let $u$, $v$ be vertices in $G$. Since $G$ contains no isolated vertices and each component of $G$ has at least two lines, there exist parks $p_u$, $p_v$ such that $u$ is in $p_u$ and $v$ is in $p_v$. If $p_u = p_v$, there is nothing to prove. Assume that $p_u 
eq p_v$ and the p-path from $p_u$ to $p_v$ is of length one (i.e. $P = p_u$, $x_1$, $p_v$). Since $x_1$ is in $p_u$ and $p_v$, there is a path from $u$ to $v$ in $G$. Now assume that for a p-path of length $k \leq n$ from $p_u$ to $p_v$ there is
a path from $u$ to $v$, and let $P$ be a $p$-path of length $n + 1$ from $p_u$ to $p_v$. Then $P$ is of the form $p_u, x_1, p_1, x_2, \ldots, p_{n-1}, x_n, p_n, x_{n+1}, p_v$. By assumption there is a path from $u$ to some vertex $u'$ in $p_n$. Since $p_n$ and $p_v$ have $x_{n+1}$ in common, there is a path from $u'$ to $v$. Since the relation "is joined by a path to" is an equivalence relation on the vertex set of $G$, there is a path from $u$ to $v$ in $G$.

**Definition 2.8.** A $p$-walk $p_1, x_1, p_2, \ldots, p_n, x_n, p_{n+1}$ ($n \geq 3$) in which no park repeats, and where $p_1 = p_{n+1}$, is called a $p$-cycle and will be denoted $p_1, p_2, \ldots, p_n, p_1$.

**Example 2.5.** In the graph $G$ of Figure 2.6, $p_1, p_2, p_3, p_4, p_1$ is a $p$-cycle.

![Figure 2.6](image)

**Proposition 2.9.** If a graph $G$ has a cycle of length four or more, then $G$ has a $p$-cycle.

**Proof:** Let a graph $G$ have a cycle $C = v_1, v_2, \ldots, v_n, v_1$, $n \geq 4$. Let $p_i = (v_{i}, v_{i+1}, v_{i+2})$, $i = 1, 2, \ldots, n - 2$,
$p_{n-1} = (v_{n-1}, v_n, v_1)$ and $p_n = (v_n, v_1, v_2)$. Then the sequence $W = p_1, p_2, \ldots, p_n, p_1$ is a $p$-cycle in $G$.

The graph $G$ in Figure 2.7 shows that the converse of the preceding result fails, for $W = p_1, p_2, p_3, p_1$ is a $p$-cycle but $G$ does not contain a cycle.

**Figure 2.7**

**Definition 2.9.** A graph $G$ is called $p$-hamiltonian if it has a $p$-cycle which contains every park of $G$. Such a cycle is called a $p$-hamiltonian cycle.

**Example 2.6.** In the graph $G$ of Figure 2.6, $W = p_1, p_2, p_3, p_5, p_6, p_4, p_1$ is a $p$-hamiltonian cycle.

It is proved in Proposition 4.3 that if a graph $G$ is hamiltonian, then it is necessarily $p$-hamiltonian. That the converse of this result does not necessarily hold follows from the consideration of the graph in Figure 2.7, since $p_1, p_2, p_3$ is a $p$-hamiltonian cycle.

**Definition 2.10.** A graph $G$ is called a $p$-tree if it contains no $p$-cycles.
Proposition 2.10. A connected graph $G$ with $n \geq 4$ vertices is a $p$-tree if and only if $G$ is a path.

Proof: Let a graph $G$ be a $p$-tree and assume $G$ is not a path. Since $G$ is connected and not a path, there exists a vertex of degree three, or $G$ is a cycle of length $k \geq 4$. In either case $G$ has a $p$-cycle, which is a contradiction. On the other hand, if $G$ is a path on four or more points, the technique used in Lemma 2.6 can be employed to prove the converse.

Definition 2.11. A $p$-bridge is a park $p$ in a graph $G$ such that the removal of the lines of $p$ will disconnect $G$.

Example 2.7. In the graph $G$ of Figure 2.4, the park $p_2$ is a $p$-bridge.

Proposition 2.11. If $p$ is a $p$-bridge in a graph $G$ such that $\deg v = k \geq 3$ for some $v$ in $p$, then $p$ contains a cut point of $G$.

Proof: Let $p = (v_1v_2v_3)$ be a $p$-bridge in $G$ and let $G'$ be the graph obtained by removing the lines of $p$ from $G$. Then there exist distinct components $C_1, C_2$ of $G'$ such that exactly one vertex of $p$, say $v_1$, is in $C_1$ and at least one other vertex of $p$, say $v_2$, is in $C_2$. Without loss of generality, assume that $\deg v_1 = k \geq 3$. Since only the lines of $p$ have been removed from $G$, there exists a vertex $u_1$ in $C_1$ such that $u_1$ is adjacent to $v_1$. But $u_1 \in C_1$, $v_2 \in C_2$ implies that there does not exist a path in $G'$ from $u_1$ to $v_2$. Hence every path in $G$ from $u_1$ to $v_2$ contains $v_1$, which shows that $v_1$ is a cut point of $G$. 
Definition 2.12. The park connectivity $\mu(G)$ of a graph $G$ is the minimum number of parks such that the removal of the lines of these parks will disconnect $G$. If $G$ has no parks, we say that $\mu(G) = 0$.

Example 2.8. In the graph $G$ of Figure 2.8, the removal of lines from parks $p_1$ and $p_2$ will disconnect $G$, but the removal of the lines of any one park will not disconnect $G$. Hence $\mu(G) = 2$.

![Figure 2.8](image)

The above definition of park connectivity is analogous to the definition of connectivity (line connectivity), denoted by $\kappa(G)$ (or $\lambda(G)$), which is the minimum number of vertices (lines) whose removal will disconnect $G$ or reduce it to a single point.

The following proposition establishes a relationship between $\mu(G)$ and $\lambda(G)$. In the proposition, $\min \deg G = \min \{\deg v\}$. \forall v \in V(G)

**Proposition 2.12.** For any graph $G$,

(a) $\mu(G) \leq \lambda(G)$

(b) $\mu(G) \leq \left\lfloor \frac{\min \deg G}{2} \right\rfloor$

**Proof:** Without loss of generality we may assume that $G$ is connected, $|P| > 0$ and $\lambda(G) = m \geq 1$. For if $G$ is discon-
nected, or \( |P| = 0 \), we get \( \mathcal{P}(G) = 0 \). Let \( E_1 = \{ x_i \mid i = 1, 2, \ldots, m \} \) be the set of \( m \) lines whose removal will disconnect \( G \). Since every line in a connected graph must belong to a park, let \( p_i \) be a park containing \( x_i \), \( i = 1, 2, \ldots, m \). By removing the lines of the parks \( p_i \), \( i = 1, 2, \ldots, m-1 \), at least \( m-1 \) of the lines in \( E_1 \) will be removed. If this disconnects \( G \), \( \mu(G) < \lambda(G) \). If not, the removal of the lines of \( p_m \) will disconnect \( G \), in which case \( \mu(G) = \lambda(G) \). This proves part (a).

To prove (b), let \( v \) be a point of minimum degree in \( G \) such that \( \text{deg } v = m \). Label the lines incident with \( v \) as \( x_1, x_2, \ldots, x_m \). If \( m \) is even, let \( p_i \) be the park containing lines \( x_{2i-1} \) and \( x_{2i} \), \( i = 1, 2, \ldots, \frac{m}{2} \). If \( m \) is odd, let \( p_i \) be the park containing lines \( x_{2i-1} \) and \( x_{2i} \), \( i = 1, 2, \ldots, \frac{m-1}{2} \), and let \( p_{m+1} \) be the park containing lines \( x_m \) and \( x_1 \). In any case, the removal of the lines of the parks \( p_i \) thus formed for \( i = 1, 2, \ldots, \left\lfloor \frac{m}{2} \right\rfloor \) will make \( v \) an isolated point, and hence \( \mu(G) \leq \left\lfloor \frac{m}{2} \right\rfloor \).

For the graph \( G \) of Figure 2.8, \( K(G) = 1 \) and \( \mu(G) = 2 \). For a complete graph \( K_n \), \( n \geq 3 \), \( \mu(K_n) = \left\lfloor \frac{n-1}{2} \right\rfloor \) and \( K(K_n) = n - 1 \). Thus we see that \( \mu(G) \) and \( K(G) \) are not comparable.

Let \( d \) be the usual distance function on the point set \( V \) of a connected graph \( G \). It is known that \( d \) is a metric for \( V \). We now define a "distance" function on the park set of a connected graph as follows.
**Definition 2.13.** Let $G$ be a connected graph.

Let $D: P \times P \rightarrow \mathbb{R}$ be defined by

\[
D(p_1, p_2) = \begin{cases} 
0, & \text{if } p_1 = p_2 \\
1, & \text{if } p_1 \text{ is adjacent to } p_2 \\
2 + \min \{d(u, v)\}, & \text{otherwise}
\end{cases}
\]

We conclude this chapter by proving that $D$ is a metric on the set of parks of a connected graph having at least one park.

To show that $D$ is a metric on $P$, we must prove the following:

(a) $D(p_1, p_2) \geq 0 \ \forall \ p_1, p_2 \in P$

(b) $D(p_1, p_2) = 0 \iff p_1 = p_2$

(c) $D(p_1, p_3) \leq D(p_1, p_2) + D(p_2, p_3) \ \forall \ p_1, p_2, p_3 \in P$

Parts (a) and (b) follow immediately from the definition and the fact that $d$ is a metric for $V$. To prove (c) we may assume $D(p_1, p_3) \geq 2$ and $p_2$ is distinct from $p_1$ and $p_3$, since otherwise the inequality is obviously satisfied. Now consider the following cases:

(i) $p_2$ is adjacent to $p_1$ and $p_3$,

(ii) $p_2$ is adjacent to $p_1$ but not $p_3$, and

(iii) $p_2$ is adjacent to neither $p_1$ nor $p_3$.

In case (i), $p_1$ and $p_3$ have a common vertex, hence

$D(p_1, p_3) = 2$. Since $p_2$ is adjacent to $p_1$ and $p_3$, $D(p_1, p_2) = D(p_2, p_3) = 1$ and the inequality is satisfied.
For cases (ii) and (iii), let \( \min \{d(u,v)\} = k \), \( u \in p_1 \), \( v \in p_3 \)

\[ \min \{d(u,v)\} = m \text{ and } \min \{d(u,v)\} = n. \]

In case (ii) \( m = 1 \), \( u \in p_1 \), \( v \in p_2 \), so assume \( k > 1 + n \). Let \( u_2 \in p_2 \), \( u_3 \in p_3 \) be such that

\[ d(u_2, u_3) = n. \]

Since \( p_2 \) and \( p_1 \) have two common vertices, there exists a vertex \( u \in p_1 \) such that \( d(u, u_2) \leq 1 \). Thus

\[ d(u, u_3) \leq d(u, u_2) + d(u_2, u_3) \leq 1 + n < k, \]

which is a contradiction. Hence \( k \leq 1 + n \), and this implies

\[ D(p_1, p_3) \leq D(p_1, p_2) + D(p_2, p_3). \]

In case (iii), assume \( k > m + n + 2 \) and let \( u_1 \in p_1 \), \( u_2 \in p_2 \), \( u'_2 \in p_2 \) and \( u_3 \in p_3 \) be such that \( d(u_1, u_2) = m \) and \( d(u'_2, u_3) = n \). Since \( d(u_2, u'_2) \leq 2 \), we have

\[ d(u_1, u_3) \leq d(u_1, u'_2) + d(u'_2, u_3) \leq d(u_1, u_2) + d(u_2, u_2') + d(u'_2, u_3) \leq m + 2 + n < k \]

which is a contradiction. Therefore \( k \leq m + n + 2 \), and consequently \( D(p_1, p_3) \leq D(p_1, p_2) + D(p_2, p_3) \).

This completes the proof of part (c).

We now state these results in the form of a proposition.

**Proposition 2.13.** The distance function \( D \) is a metric on the set of all parks of a connected graph.
CHAPTER III

PARK GRAPHS

In the literature of graph theory, we come across many graphical valued functions defined on the set of all graphs. Among these functions are the line graph, total graph, block graph, block-cut point graph and clique graph functions. The concept of a park in a graph was introduced in the last chapter, and we now define a graphical valued function $P$ (the park graph function) on the set of all graphs.

Definition 3.1. The park graph of a graph $G$, denoted $P(G)$, is a graph whose vertex set can be put in 1-1 correspondence with the set of parks of $G$ such that two vertices in $P(G)$ are adjacent if and only if the corresponding parks are adjacent.

It is readily seen that the degree of a vertex in $P(G)$ is the same as the degree of the corresponding park in $G$.

Example 3.1. In Figure 3.1 we illustrate a graph $G$ and its park graph $P(G)$, and note that $p_i \leftrightarrow u_i$ is a 1-1 correspondence which has the property that for $i \neq j$, $p_i$ and $p_j$ are adjacent if and only if $u_i$ and $u_j$ are adjacent.
Recall that the number of triangles in a graph $G$ was denoted by $T$. We now have the following proposition.

**Proposition 3.1.** Let $G$ be a graph with park graph $P(G)$. Then $|V(P(G))| = \sum_{v \in V(G)} \binom{\deg v}{2} - 2T$.

**Proof:** Clearly $P(G)$ has as many vertices as the number of parks in $G$; and by Proposition 2.2, this number is

\[
\sum_{v \in V(G)} \binom{\deg v}{2} - 2T.
\]

The next proposition determines the number of lines in the park graph of a graph $G$. Recall that for any line $x$ of $G$, $\mathcal{P}(x)$ denotes the number of distinct parks which contain $x$.

**Proposition 3.2.** If $G$ is a graph with park graph $P(G)$, then $|E(P(G))| = \sum_{x \in E(G)} \binom{\mathcal{P}(x)}{2}$.

**Proof:** Let $\mathcal{P}$ denote the set of parks in a graph $G$ and let
Let $H_x$ be the subgraph of $P(G)$ induced by the vertices in $S_x$. Since every vertex in $H_x$ corresponds to a park in $G$ which contains $x$, $H_x$ is a complete subgraph of $P(G)$ on $p(x)$ vertices. By Proposition 2.1, distinct parks can have at most one common line, hence for $x \neq x'$ in $G$, $H_x$ and $H_{x'}$ have no common lines. Also, a complete graph on $m \geq 2$ vertices has $\binom{m}{2}$ lines, hence $H_x$ has $\binom{p(x)}{2}$ lines and the total number of lines in $P(G)$ is given by $\sum_{x \in S(G)} \binom{p(x)}{2}$.

It follows from Definition 2.7 that a graph $G$ is $p$-connected if and only if $P(G)$ is connected. We also see by considering the park graphs of the graphs $G_1$ and $G_2$ of Figure 2.5 that it is possible for $P(G)$ to be connected while $G$ is not connected. In view of these remarks, the following analogue to Proposition 2.8 is stated without proof.

**Proposition 3.3.** (a) If $G$ is a connected graph, then $P(G)$ is connected. (b) If each component of a graph $G$ has at least two lines and if $P(G)$ is connected, then $G$ is connected.

We now describe some of the properties of park graphs of some special classes of graphs. The first to be considered are paths.

**Example 3.2.** The graph $P(G)$ in Figure 3.2 is the park graph of the path $G$ in Figure 3.2.
Proposition 3.2. Let $G$ be a connected graph on $k \geq 4$ vertices. Then $G$ is a path of length $k - 1$ if and only if $P(G)$ is a path of length $k - 3$.

Proof: If $G$ is a path of length $k - 1$, then clearly $P(G)$ is a path of length $k - 3$. On the other hand, let $P(G)$ be a path on $u_1, u_2, \ldots, u_{k-2}$ and of length $k - 3$. Since any vertex of degree exceeding 2 in $G$ gives rise to a triangle in $P(G)$, $\max \text{deg } G \leq 2$. Furthermore, since $G$ is connected and has $k \geq 4$ vertices, it follows that $G$ is a path or a cycle of length four or more. By Proposition 3.1 and 3.2, and the fact that $P(G)$ is a path, we have

$$\sum_{x \in E(G)} \binom{P(x)}{2} = \sum_{v \in V(G)} \binom{\text{deg } v}{2} - 2T - 1.$$  

If $G$ has no triangles, this reduces to

$$\sum_{x \in E(G)} \binom{\text{deg } x}{2} = \sum_{v \in V(G)} \binom{\text{deg } v}{2} - 1.$$  

If $G$ is a cycle, $\sum_{x \in E(G)} \binom{\text{deg } x}{2} = \sum_{v \in V(G)} \binom{\text{deg } v}{2}$, which is a contradiction. Thus $G$ is a path, and in order for $G$ to have $k - 2$ parks, $G$ must be a path of length $k - 1$. 

Figure 3.2

Proposition 3.4. Let $G$ be a connected graph on $k \geq 4$ vertices. Then $G$ is a path of length $k - 1$ if and only if $P(G)$ is a path of length $k - 3$.

Proof: If $G$ is a path of length $k - 1$, then clearly $P(G)$ is a path of length $k - 3$. On the other hand, let $P(G)$ be a path on $u_1, u_2, \ldots, u_{k-2}$ and of length $k - 3$. Since any vertex of degree exceeding 2 in $G$ gives rise to a triangle in $P(G)$, $\max \text{deg } G \leq 2$. Furthermore, since $G$ is connected and has $k \geq 4$ vertices, it follows that $G$ is a path or a cycle of length four or more. By Proposition 3.1 and 3.2, and the fact that $P(G)$ is a path, we have

$$\sum_{x \in E(G)} \binom{P(x)}{2} = \sum_{v \in V(G)} \binom{\text{deg } v}{2} - 2T - 1.$$  

If $G$ has no triangles, this reduces to

$$\sum_{x \in E(G)} \binom{\text{deg } x}{2} = \sum_{v \in V(G)} \binom{\text{deg } v}{2} - 1.$$  

If $G$ is a cycle, $\sum_{x \in E(G)} \binom{\text{deg } x}{2} = \sum_{v \in V(G)} \binom{\text{deg } v}{2}$, which is a contradiction. Thus $G$ is a path, and in order for $G$ to have $k - 2$ parks, $G$ must be a path of length $k - 1$. 

Figure 3.2
Corollary 3.5. If $G$ is a path on $k \geq 4$ vertices, then $|V(P(G))| = k - 2$ and $|E(P(G))| = k - 3$.

Example 3.3. A cycle $C$ on 6 vertices has park graph $P(C)$ as illustrated in Figure 3.3. The 1-1 mapping $f$ between the parks of $G$ and $V(P(G))$ is given by $f(u_i) = p_i$.

![Diagram](image)

Figure 3.3

Proposition 3.6. A graph $G$ is a cycle of length $k \geq 4$ if and only if $P(G)$ is a cycle of length $k$.

Proof: Let $G$ be a cycle of $k \geq 4$ vertices $v_1, v_2, \ldots, v_k$ arranged cyclically. Let $p_i = (v_{i-1}v_iv_{i+1})$, $i = 2, 3, \ldots, k-1$, $p_1 = (v_kv_1v_2)$, and $p_k = (v_{k-1}v_kv_1)$. This lists all the parks in $G$, and the mapping $f^{-1}(p_i) = u_i$ establishes that $P(G)$ is a cycle of length $k$. On the other hand, let $P(G)$ be a cycle whose vertices are $u_1, u_2, \ldots, u_k$, $k \geq 4$, with the mapping $f(u_i) = p_i$. Suppose that for some $i$, $1 \leq i \leq k$, $|p_i| = 3$ (i.e. the number of lines in the park $p_i$ is 3) where $p_i = (v_1v_2v_3)$. Since $p_i$ is adjacent to some park $p_j$, $i \neq j$, there exists a vertex $v_4$ in $V(G)$.
such that \( v_4 \) is adjacent to say \( v_1 \). Then \( p' = (v_1v_2v_4) \), \( p'' = (v_1v_3v_4) \), and \( p_1 = (v_1v_2v_3) \) are mutually adjacent, hence \( f^{-1}(p'), f^{-1}(p''), \text{and } f^{-1}(p_1) \) form a triangle in \( P(G) \) which is a contradiction. Therefore \( |p_i| = 2 \) for \( i = 1, 2, \ldots, k \). Since \( P(G) \) has no triangles, \( \max \deg G \leq 2 \). But \( \text{DEG}(p) = 2 \) for all \( p \) in \( P \), hence \( G \) has no vertex of degree 1. Consequently, \( G \) is a cycle. It is readily seen by the first part of the proof that \( G \) must be a cycle of length \( k \).

Proposition 3.6 gives a class of graphs that are isomorphic to their park graphs. It is then natural to ask if there are other graphs that have this property. The answer to the question is yes, and is illustrated in the graphs of Figure 3.4.

\[ \text{Figure 3.4} \]
Definition 3.2. An \( n \)-star is a connected graph \( G \) on \( n \) vertices such that one vertex has degree \( n - 1 \) and the remaining \( n - 1 \) vertices have degree one each.

Example 3.4. The graph \( G \) of Figure 3.5 is a 5-star and has park graph \( P(G) \) as indicated.

![Figure 3.5](image)

**Proposition 3.7.** If \( G \) is an \( n \)-star, \( n \geq 4 \), then

(i) \( |V(P(G))| = \binom{n-1}{2} \)

(ii) \( P(G) \) is regular of degree \( 2(n-3) \)

(iii) \( K_{n-2} \) occurs as a subgraph of \( P(G) \)

**Proof:**

(i) The only parks in an \( n \)-star are formed by pairs of lines incident at the vertex of degree \( (n-1) \). Since there are \( \binom{n-1}{2} \) such distinct pairs, \( \binom{n-1}{2} = |P| = |V(P(G))| \).

(ii) Let \( v_0 \) in \( V(G) \) be such that \( \deg v_0 = n - 1 \), and let \( p \) be an arbitrary park in \( G \). Then \( p \) is of the form \( (v_i v_0 v_j) \), \( i \neq j \) and \( i, j \neq 0 \). For \( v' \neq v_i \) or \( v_j \), \( (v_i v_0 v') \) and
(v'v_o v_j) form distinct parks in G that are adjacent to p. Since there are n - 3 such vertices v', DEG(p) = 2(n-3). But p was arbitrary, hence DEG(p) = 2(n-3) for all p in P. Therefore P(G) is regular of degree 2(n-3).

(iii) Let deg v_o = n - 1, deg v_i = 1, i = 1, 2, ..., n-1, and let P' = { p ∈ P | p = (v_i v_o v_j), j ≠ l, j ≠ 0, v_j ∈ V(G) }. Then |P'| = n - 2 and the parks in P are mutually adjacent. The corresponding vertices in P(G) then induce a complete subgraph of P(G) on n - 2 vertices.

We now turn to the study of park graphs of complete, and complete bipartite graphs. The park graphs of K_3, K_4, K_5, K_2,2 and K_3,3 are illustrated in Figures 3.6, 3.7 and 3.8.
Figure 3.6

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 3.7

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ p_1 = (v_4 v_1 v_5) \quad p_{10} = (v_3 v_4 v_2) \]
\[ p_2 = (v_5 v_1 v_6) \quad p_{11} = (v_2 v_4 v_1) \]
\[ p_3 = (v_4 v_1 v_6) \quad p_{12} = (v_3 v_4 v_1) \]
\[ p_4 = (v_4 v_2 v_5) \quad p_{13} = (v_3 v_5 v_2) \]
\[ p_5 = (v_5 v_2 v_6) \quad p_{14} = (v_2 v_5 v_1) \]
\[ p_6 = (v_4 v_2 v_6) \quad p_{15} = (v_3 v_5 v_1) \]
\[ p_7 = (v_4 v_3 v_5) \quad p_{16} = (v_3 v_6 v_2) \]
\[ p_8 = (v_5 v_3 v_6) \quad p_{17} = (v_2 v_6 v_1) \]
\[ p_9 = (v_4 v_3 v_5) \quad p_{18} = (v_3 v_6 v_1) \]

Figure 3.8
It is interesting to note that the Petersen graph occurs as a subgraph of $P(K_5)$. According to a well known theorem due to Kuratowski $[7]$, a graph $G$ is nonplanar iff it has a subgraph homeomorphic to $K_5$ or $K_{3,3}$, and it is shown in $[6]$ that the Petersen graph contains a subgraph which is homeomorphic to $K_{3,3}$. Hence $P(K_5)$ is nonplanar. $P(K_{3,3})$ is also seen to be nonplanar since $\min \deg P(K_{3,3}) = 6$ (see $[3]$).

**Proposition 3.8.** Let $G$ be a complete graph $K_n$, $n \geq 3$. Then

$$|V(P(G))| = \binom{n}{3}.$$

**Proof:** The vertices of a complete graph are mutually adjacent, hence every combination of three vertices out of $n$ forms a park.

As a trivial consequence of Proposition 3.7, we note that complete graphs of many orders may occur as subgraphs of $P(G)$. Furthermore, Figure 3.6 shows that $P(K_4) \cong K_4$ and $P(K_{2,2}) \cong K_{2,2}$. Thus $P(G)$ can be $K_n$ for $n \leq 4$ and $K_{m,n}$ for $m, n \leq 2$. It is natural to ask whether there are graphs $G, G'$ such that $P(G) \cong K_n$ for $n \geq 5$, and $P(G') \cong K_{m,n}$ for $m \geq 1$ and $n \geq 3$. This is answered in the negative in the following propositions.

**Proposition 3.9.** For any graph $G$, $P(G)$ cannot be $K_n$ for $n \geq 5$.

**Proof:** If $P(G)$ is the complete graph on $m$ points, there are precisely $m$ distinct parks in $G$ that are mutually adjacent.
Let \( p_1 \) be a park in \( G \) determined by a triple \((abc)\) and let \( d \) be another point of \( G \). This can give rise to at most four parks \( p_1, p_2, p_3, \) and \( p_4 \), determined by the triples \((abc), (abd), (adc)\) and \((dbc)\) respectively, such that they are mutually adjacent. Since \( n \geq 5 \) there must be at least one point \( e \) of \( G \) that is distinct from \( a, b, c \) and \( d \) which has the property that it gives rise to a park \( p_5 \) that is adjacent to \( p_1, p_2, p_3 \) and \( p_4 \). Without any loss of generality, let \( p_5 \) be determined by the triple \((abe)\). This is impossible since \( p_3 \) and \( p_5 \) cannot be adjacent, and consequently \( P(G) \) cannot be the complete graph on five or more points.

**Proposition 3.10.** For any connected graph \( G \), \( P(G) \) cannot be \( K_{m,n} \) for \( m \geq 1 \) and \( n \geq 3 \).

**Proof:** Assume \( P(G) = K_{m,n} \) for some \( m \geq 1 \) and \( n \geq 3 \). Clearly \( G \) cannot have a vertex of degree exceeding 2 since \( P(G) \) would then contain a triangle. Thus \( \max \deg G \leq 2 \), and the connected component of \( G \) which gives rise to \( K_{m,n} \) must be a path or a cycle. Since \( P(G) \) is \( K_{m,n} \), \( m \geq 1 \) and \( n \geq 3 \), there exists a vertex \( u \in V(P(G)) \) such that \( \deg u = q \geq 3 \). But \( G \) is a path or a cycle implies that \( \max \deg P(G) = 2 \). This is a contradiction.

For a connected graph \( G \) with \( \min \deg G = 2 \), the presence of triangles has a big influence on the structure of \( P(G) \). However, it is possible to destroy a triangle by inserting a vertex in any one of its lines. It this is done for every
triangle, the resulting graph, denoted $G^\Delta$, is without triangles, and is actually homeomorphic from $G$ (see [4]). The next proposition shows that $P(G)$ has a contraction $H$ (see [3]) such that $H \cong G$.

**Proposition 3.11.** If $G$ is a connected graph with $n \geq 4$ vertices and $\text{min deg } G \geq 2$, then there exists a graph $G^\Delta$ such that $G^\Delta$ is homeomorphic from $G$ and $P(G^\Delta)$ has a contraction to a graph isomorphic to $G$.

**Proof:** Partition the vertex set $V(G)$ of $G$ into $n$ singletons, $C_1, C_2, \ldots, C_n$ where $C_i = \{v_i\}$. For each triangle $t$ in $G$, subdivide exactly one of the lines of $t$ and include the new vertex $w$ in one of the $C_i$ that contains a vertex of the triangle $t$. Subdividing one line of each triangle in this manner and associating the new vertices as described above, a graph $G^\Delta$ is obtained that is homeomorphic from $G$, and the vertices of $G^\Delta$ are partitioned into $n$ subsets. If $G$ has lines common to more than one triangle, it is sufficient to subdivide $G$ in the prescribed manner until the subdivided graph contains no triangles. Since $G^\Delta$ contains no triangles, all parks of $G^\Delta$ will consist of two lines. Let $C_1, C_2, \ldots, C_n$ denote the vertex sets in the partition of $V(G^\Delta)$. Note that by construction each $C_i$ contains $v_i$ from $V(G)$ with the possible addition of some vertices obtained from the subdivision.

For each $i, i = 1, 2, \ldots, n$, consider all parks formed by
pairs of lines incident with vertices in $C_i$. Let $C_i^P$ denote the corresponding set of vertices in $P(G^A)$. $C_i^P$ is nonempty for each $i$ because $\text{min deg } G \geq 2$, and each vertex in $V(P(G^A))$ is in exactly one of the $C_i^P$. Hence $V(P(G^A))$ is partitioned into $n$ subsets.

Let $\langle C_i^P \rangle$ denote the subgraph of $P(G^A)$ induced by the vertices in $C_i^P$ for $i = 1, 2, \ldots, n$. Clearly then $\langle C_i^P \rangle$ is connected for each $i$. If $v_i$ is adjacent to $v_k$ ($i \neq k$) in $G$, there exists a line $x_{i,k}$ from $C_i$ to $C_j$ in $G^A$. Since $\text{min deg } G \geq 2$, there exist lines $x_1, x_2$, distinct from $x_{i,k}$ incident with $v_i$ and $v_k$ respectively. Then there exist parks $(x_{i,k} x_1)$ and $(x_{i,k} x_2)$ such that the vertices of $P(G^A)$ that correspond to these parks are in $C_i^P$ and $C_k^P$ respectively. Since both parks have the line $x_{i,k}$ in common, there exists a line in $P(G^A)$ from $C_i^P$ to $C_k^P$. If $v_i$ is not adjacent to $v_k$ in $G$, there does not exist a line from $C_i$ to $C_j$ in $G^A$. In this case, any park formed by a pair of lines incident with a vertex in $C_i$ will not have a line in common with a park formed by a pair of lines incident with a vertex in $C_j$. Hence there does not exist a line from $C_i^P$ to $C_k^P$ in $P(G^A)$. $P(G^A)$ now has the property that if each $\langle C_i^P \rangle$ is contracted to a single point, the resulting graph will
be isomorphic to $G$.

**Figure 3.9**

**Corollary 3.12.** If $G$ has no triangles and $\min \text{deg } G \geq 2$, then $P(G)$ has a contraction to a graph $H$ such that $H \cong G$.

**Proof:** Let $G = G^\Delta$ in the above proposition.
It was shown in [2] and [1] that if $G$ is a hamiltonian graph, then the graphical valued functions $L$ and $T$ (line-graph and total graph functions respectively) are hamiltonian functions, i.e., $L(G)$ and $T(G)$ are hamiltonian graphs. Since we see in Figure 4.1 that the park graph of an arbitrary graph is not necessarily hamiltonian, it is natural to look for conditions on a graph $G$ which will insure that $P(G)$ be hamiltonian. First we prove the following result.

**Proposition 4.1.** For a graph $G$, $P(G)$ is hamiltonian if and only if the parks of $G$ can be ordered as $p_1, p_2, \ldots, p_k$ such that $p_i$ is adjacent to $p_{i+1}$, $i = 1, 2, \ldots, k-1$, and $p_k$ is adjacent to $p_1$.

**Proof:** Let $P(G)$ be the park graph of $G$ with the mapping $f(u_i) = p_i$. If $P(G)$ is hamiltonian, the vertices of $P(G)$ can be ordered in a hamiltonian cycle $C = \langle u_1, u_2, \ldots, u_i, \ldots, u_k \rangle$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
such that \( u_i \) is adjacent to \( u_{i+1} \); \( i = 1, 2, ..., k-1 \), and \( u_k \) is adjacent to \( u_1 \). Obviously then \( W = p_1, p_2, ..., p_k \) is a sequence of parks of \( G \) which has the desired property. On the other hand, assume the parks of \( G \) can be ordered as \( p_1, p_2, ..., p_k \) as stated in the hypothesis. Then \( C = \langle f^{-1}(p_1), f^{-1}(p_2), ..., f^{-1}(p_k) \rangle \)
\( \langle u_1, u_2, ..., u_k \rangle \) is a hamiltonian cycle in \( P(G) \).

**Corollary 4.2.** If \( G \) is an \( n \)-star, \( n \geq 4 \), then \( P(G) \) is hamiltonian.

**Proof:** Denote the vertices of \( G \) as \( v_0, v_1, ..., v_{n-1} \) where \( \deg v_0 = n - 1 \) and \( \deg v_i = 1, i = 1, 2, ..., n-1 \). Without any loss of generality we may assume that the vertices are labeled in a counter-clockwise manner. Since each triple \( (v_i v_0 v_j), i \neq j \), forms a park and all parks in \( G \) are of this form, the following sequence satisfies the hypothesis of Proposition 4.1.
\( W = (v_1 v_0 v_{n-1}), (v_1 v_0 v_{n-2}), ..., (v_1 v_0 v_2), (v_2 v_0 v_{n-1}), (v_2 v_0 v_{n-2}), ..., (v_2 v_0 v_3), (v_3 v_0 v_{n-1}), ..., (v_{n-2} v_0 v_{n-1}) \).

In view of Proposition 2.1 we note that parks may be denoted be an unordered pair of lines even if they contain three lines.

**Proposition 4.3.** If a graph \( G \) is hamiltonian, then \( P(G) \) is hamiltonian.

**Proof:** To show that \( P(G) \) is hamiltonian, we will construct a sequence of parks of \( G \) that satisfies the conditions of Proposition 4.1. Since \( G \) is hamiltonian, let its vertices be arranged
cyclically and be labeled \(v_1, v_2, \ldots, v_m\) in a counterclockwise manner. Label the lines of this cycle \(y_1, y_2, \ldots, y_i, \ldots, y_m\) such that \(y_i\) joins \(v_i\) to \(v_{i+1}\), \(i = 1, 2, \ldots, m-1\), and \(y_m\) joins \(v_m\) to \(v_1\). We will now define the desired sequence in steps.

**Step I.** The first park in the sequence is \((y_m, y_1)\). If \(\deg v_1 \leq 3\), the second park in the sequence will be \((y_1, y_2)\), and we proceed to step II. Otherwise, label the diagonals incident with \(v_1\) as \(x_1^1, x_2^1, \ldots, x_{k_1}^1\) in a clockwise manner where \(\deg v_1 = k_1 + 2\), and consider the following. If \(x_1^1\) does not join \(v_1\) to \(v_{m-1}\), let \(H_{x_1^1} (y_1, x_2^1)\) be the set of all parks containing \(x_1^1\) and ordered such that \((x_1^1, y_1)\) occurs first and \((x_1^1, x_2^1)\) occurs last. Since \(\deg v_1 \geq 4\), we know that \((x_1^1, y_1)\) and \((x_1^1, x_2^1)\) both exist. If \(x_1^1\) joins \(v_1\) to \(v_{m-1}\), let \(H_{x_1^1} (y_1, x_2^1)\) be all parks containing \(x_1^1\) except the park \((y_{m-1}, y_m)\), and ordered such that \((x_1^1, y_1)\) occurs first and \((x_1^1, x_2^1)\) occurs last. Adjoin the collection \(H_{x_1^1} (y_1, x_2^1)\) to \((y_m, y_1)\). Note that since the park \((y_m, y_1)\) is not in \(H_{x_1^1} (y_1, x_2^1)\), all parks ordered so far are distinct.

Now define \(H_{x_2^1} (x_3^1)\) to be the set of all parks containing \(x_2^1\) that have not already been ordered, and order them such that \((x_2^1, x_3^1)\) occurs last. If \(k_1 = 2\), we will let \(x_3^1 = y_1\). Since the

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
park \((x_2^1 x_3^1)\) has not been previously ordered, \(H_{x_2^1} (x_3^1)\) is non-empty. The sequence of parks ordered thus far is \((y_m y_1)\), \(H_{x_1^1} (y_1, x_2^1), H_{x_2^1} (x_3^1)\). Clearly all parks in this sequence are distinct. We continue this process by defining \(H_{x_1^i} \left(x_{i+1}^1\right)\), 

\[i = 3, 4, \ldots, k_1,\]

to be the set of all parks containing \(x_1^1\) that have not already been ordered, and order them such that \((x_i^1 x_{i+1}^1)\) occurs last. For \(i = k_1\) we let \(x_{k+1}^1 = y_1\), and \(H_{x_{k_1}^1} \left(x_{k_1+1}^1\right) = H_{x_{k_1}^1} (y_1)\). In general, \(H_{x_{k_1}^1} \left(x_{i+1}^1\right)\) will be nonempty since the park \((x_i^1 x_{i+1}^1)\), \(i = 3, 4, \ldots, k_1\), will not have been previously ordered. Then the partial sequence of parks so constructed becomes \((y_m y_1), H_{x_1^1} (y_1, x_2^1), H_{x_2^1} (x_3^1), \ldots, H_{x_{k_1-1}^1} (x_{k_1}^1)\), 

\(H_{x_{k_1}^1} (y_1)\), and no park occurs more than once in this sequence.

If \(x_{k_1}^1\) does not join \(v_1\) to \(v_3\), the next park in the sequence will be \((y_1 y_2)\). If \(x_{k_1}^1\) joins \(v_1\) to \(v_3\), \((y_1 y_2)\) is the last park in \(H_{x_{k_1}^1} (y_1)\). In any case, \((y_1 y_2)\) is the last park of the sequence as constructed thus far. Hence without loss of generality we can write \((y_1 y_2)\) in the sequence, realizing that it might be the last park in \(H_{x_{k_1}^1} (y_1)\).
Step II. If deg $v_2 \leq 3$, let $(y_2 y_3)$ be the park following $(y_1 y_2)$ and proceed to step III. Otherwise, label the diagonals incident with $v_2$ as $x_1^2$, $x_2^2$, ..., $x_k^2$ in a clockwise manner and continue the sequence as follows.

Define $H_{x_1^2} (y_2, x_2^2)$ to be the set of all parks containing $x_1^2$ that have not already been ordered, and order these parks such that $(x_1^2 y_2)$ occurs first and $(x_1^2 x_2^2)$ occurs last.

$H_{x_1^2} (y_2, x_2^2)$ exists because the parks $(x_1^2 y_2)$ and $(x_1^2 x_2^2)$ have not been previously ordered. Continue by letting $H_{x_2^2} (x_3^2)$ be the set of all parks containing $x_2^2$ that have not already been ordered, together with an ordering in which the park $(x_2^2 x_3^2)$ occurs last.

In general, let $H_{x_i^2} (x_{i+1}^2)$, $i = 3, 4, \ldots, k_2$, be the set of all parks containing $x_i^2$ that have not already been ordered, together with an ordering of these in which the park $(x_i^2 x_{i+1}^2)$ occurs last.

For $i = k_2$, let $x_{i+1}^2 = y_2$ and $H_{x_k^2} (x_{k+1}^2) = H_{x_k^2} (y_2)$. We again point out that each $H_{x_i^2} (x_{i+1}^2)$ is nonempty since the park $(x_i^2 x_{i+1}^2)$ will not have been previously ordered. This follows because $(x_i^2 x_{i+1}^2)$ contains two diagonals incident with $v_2$, hence it could not be in the parks ordered in step I. Furthermore, it could not
be ordered prior to the insertion of $H_{x_{i+1}^2}$ in this step since
this would require a park to contain three lines incident with $v_2$.

The sequence under construction can now be extended to the
following:

$$(y \_1 \_1, H \_x \_1 \_1 \_1 (x \_1 \_1 \_1), H \_x \_2 \_1 \_1 \_1 (x \_2 \_1 \_1 \_1), \ldots, H \_x \_k \_1 \_1 \_1 (y \_1 \_1)\_1,\_1,\_1) \_2 \_1 \_1 \_1, H \_x \_2 \_2 \_2 \_2 (y \_2 \_2 \_2 \_2), \ldots, H \_x \_k \_2 \_2 \_2 \_2 (y \_2 \_2 \_2 \_2).$$

If $x_{k+1}^2$ joins $v_2$ to $v_4$, the last park in this sequence is
$(y_2^2, y_3^2)$. If not, let the next park be $(y_2^2, y_3^2)$. Notice that in this
ordering there are consecutive parks containing the line $y_1$ and
consecutive parks containing $y_2$.

Before proceeding further, it is necessary to introduce
some terminology. Let all the diagonals labeled thus far be
called **sequential** diagonals. As we continue constructing the
sequence, we will label certain diagonals in the same manner as
in steps I and II. Since the diagonals will have been labeled
with respect to a vertex (the superscript refers to the vertex),
we will call the labeled diagonal $x_{\alpha}^\beta$ sequential with respect to
the vertex $v_{\alpha}$. All other unlabeled diagonals will be called
**nonsequential** with respect to $v_{\alpha}$.

**Step III.** If there are less than two nonsequential diagonals
incident with $v_3$, let the next park in the sequence be $(y_3^2, y_4)$ and
proceed to step IV. Otherwise, label the nonsequential diagonals
as $x_1^3, x_2^3, \ldots, x_{k+1}^3$ in a clockwise manner (note that the only
possible sequential diagonal incident with \( v_3 \) must also be incident with \( v_1 \). We continue the construction of the sequence by defining \( H_{x_3} (y_3, x_2^3) \) to be the set of all parks containing \( x_1^3 \) that have not previously been ordered, together with an ordering such that \( (x_1^3 y_3) \) occurs first and \( (x_1^3 x_2^3) \) occurs last. To verify that this ordering is well defined, we must show that \( (x_1^3 y_3) \) and \( (x_1^3 x_2^3) \) have not been previously ordered. Since \( (x_1^3 y_3) \) does not contain two lines of the cycle, the only way it could have been previously ordered would be to have been in \( H_{x_3} (x_3^\alpha, x_2^\beta) \) (or \( H_{x_3} (y_3, x_2^\alpha) \)) for some \( \alpha \) and \( \beta \). But \( x_1^3 \) is non-sequential, hence there must exist a sequential diagonal \( x \) joining \( v_4 \) and the other endpoint of \( x_1^3 \). However, the only sequential diagonal incident with \( v_4 \) that has the same endpoint as \( x_1^3 \) must necessarily be incident with \( v_1 \). This would imply that \( x_1^3 \) is sequential which is a contradiction. Similarly, if \( (x_1^3 x_2^3) \) is in \( H_{x_3} (x_3^\alpha, x_2^\beta) \) (or \( H_{x_3} (y_3, x_2^\alpha) \)) for some \( \alpha \) and \( \beta \), then there exists a sequential diagonal \( x \) joining \( x_1^3 \) to \( x_2^3 \) (i.e., \( x \in p = (x_1^3 x_2^3) \)). But this can only happen when \( x \) and \( x_1^3 \) are incident with \( v_1 \), which contradicts the fact that \( x_1^3 \) is non-sequential. Thus \( H_{x_1} (y_3, x_2^3) \) with the prescribed ordering is well defined.
Next let $H_{x_i^3} (x_i^3)$ be the set of all parks containing $x_i^3$ that have not previously been ordered, together with an ordering such that $(x_i^3 x_i^3)$ occurs last. Again we need to verify that $(x_2^3 x_3^3)$ has not been previously ordered. This is easily shown by considering the cases where either $x_3^3 = y^3$ or $x_3^3 \neq y_3$ and repeating the above argument.

In general, define $H_{x_i^3} (x_{i+1}^3)$ to be the set of all parks containing $x_i^3$ that have not already been ordered, together with an ordering such that the park $(x_i^3 x_{i+1}^3)$ occurs last. For $i = k_3$, let $x_{i+1}^3 = y_{-3}$ and $H_{x_3^3} (x_{k_3+1}^3) = H_{x_k^3} (y^3)$. The ordering of parks in this step will be well defined if we can show that $(x_i^3 x_{i+1}^3)$, $i = 3, 4, \ldots, k_3$, has not been used prior to $H_{x_i^3} (x_{i+1}^3)$. If $i \neq k_3$, the only way for $(x_i^3 x_{i+1}^3)$ to have been previously used would be if a sequential diagonal $x$ joined $x_i^3$ to $x_{i+1}^3$. This would imply that $x$ and $x_i^3$ are incident with $v_1$. Hence $x_i^3$ would be sequential which is a contradiction. If $i = k_3$, the park $(x_{k_3}^3 y_3)$ does not contain $v_1$ or $y_2$, hence has not been previously ordered unless there exists a diagonal $x$ such that $x$ joins $v_4$ to $v_1$ and $x$ is adjacent to $x_{k_3}^3$. This would imply that $x_{k_3}^3$ is
sequential which is a contradiction. Hence the ordering of parks in step III is well defined.

If $x_{k_3}^3$ does not join $v_3$ to $v_5$, the next park in the sequence will be $(y_3v_4)$. If $x_{k_3}^3$ does join $v_3$ to $v_5$, $(y_3v_4)$ is already the last park in $H_{x_{k_3}^3} (y_3)$. In either case, $(y_3v_4)$ is the last park ordered thus far.

The sequence constructed so far is:

$$(y_1v_1), H_{x_{1}^1} (y_1, x_2^1), H_{x_2^1} (x_3^1), ..., H_{x_{k_1}^1} (y_1),$$

$$(y_1v_2), H_{x_{2}^2} (y_2, x_2^2), H_{x_2^2} (x_3^2), ..., H_{x_{k_2}^2} (y_2),$$

$$(y_2v_3), H_{x_3^3} (y_3, x_2^3), H_{x_3^3} (x_3^3), ..., H_{x_{k_3}^3} (y_3),$$

$$(y_3v_4).$$

Notice that there exist consecutive parks in this sequence that contain the line $y_3$. Also, no park occurs twice in this ordering except perhaps $(y_i v_{i+1})$, $i = 1, 2, 3$, in which case $(y_i v_{i+1})$ is the last park in $H_{x_{k_i}^i} (y_i)$. With this convention, we may write $(y_i v_{i+1})$ in the sequence.

Step IV. We now continue our construction in an inductive manner as follows. Suppose the sequence of parks has been defined such that the last park in the sequence is $(y_{q-1} v_q)$, $q < m$, and consider the diagonals incident with $v_q$. If there are less than
two nonsequential diagonals incident with \( v_q \), let the next park in the sequence be \((y_q, y_{q+1})\). If there are two or more nonsequential diagonals incident with \( v_q \), label them \( x_1^q, x_2^q, \ldots, x_{k_q}^q \).

Then define \( H_{x_1^q}(y_q, x_2^q), H_{x_2^q}(x_3^q), \ldots, H_{x_{k_q}^q}(x_{k_q+1}^q) \), \((i = 3, 4, \ldots, k_q - 1)\), and \( H_{x_{k_q}^q}(y_q) \) in a manner analogous to that in steps I, II, and III. If \( x_{k_q}^q \) joins \( v_q \) to \( v_{q+2} \) (or perhaps \( v_{m-1} \) to \( v_1 \)), then the last park is \((y_q, y_{q+1})\).

By construction, only distinct parks have been ordered in the sequence. Furthermore, for \( i < q \), there exist consecutive parks \( p', p'' \) in the sequence such that \( y_i \in p' \) and \( y_1 \in p'' \). It must be shown that the above ordering is well defined. In order to do this we must show that \((x_1^q, y_q), (x_2^q, x_{i+1}^q), (x_3^q, x_{i+1}^q), \ldots, (x_{k_q}^q, y_q)\) have not been ordered prior to where they appear in this step.

Suppose \((y_q, x_1^q)\) has been previously ordered. Since this park contains only one line of the cycle, there must exist \( \alpha, \beta \) such that \((y_q, x_1^q) \in H_{x_2^\alpha}(x_{k_q+1}^\beta) \). Since \( x_1^\alpha \) is adjacent to \( y_q \) and \( x_1^q \), \( x_2^\alpha \) must join \( v_{q+1} \) to vertex \( v_r \), \( r < q - 1 \). But for \( q < m \), the fact that \( x_1^\alpha \) and \( x_1^q \) are both incident with \( v_r \) (actually \( r = \alpha \) in this case) and that \( x_1^\alpha \) is sequential imply \( x_1^q \) is sequential which
is a contradiction.

Next suppose that for some \( i, i = 1, 2, \ldots, k_{q-1} \), the park \((x^q_i, x^q_{i+1})\) has been ordered prior to the insertion of \(H_{x^q_i} (x^q_{i+1})\) in the sequence. This means that there exist \( \alpha, \beta \) such that a sequential diagonal \( x^{\alpha}_{\beta} \) is adjacent to \( x^q_i \) and \( x^q_{i+1} \). But this implies that either \( x^q_i \) or \( x^q_{i+1} \) is sequential which is a contradiction. Finally, if \((x^q_{k_q}, y_q)\) has been ordered prior to the insertion of \(H_{x^q_{k_q}} (y_q)\), we have that there exists a sequential diagonal \( x \) joining \( v_{q+1} \) to the other endpoint of \( x^q_{k_q} \). But \( q + 1 \leq m \) implies \( x \) is sequential with respect to the other endpoint of \( x^q_{k_q} \) which implies \( x^q_{k_q} \) must be sequential. This is a contradiction. Hence the sequence is well defined.

Now suppose that \( q = m \). The last park in the sequence described earlier in this step is \((y_m, y_m)\). If there are less than two nonsequential diagonals incident with \( v_m \), proceed to step V. Otherwise label the nonsequential diagonals incident with \( v_m \) as \( x^m_1, x^m_2, \ldots, x^m_{k_m} \) and consider the following ordering.

Define \( H_{x^m_1} (y_m) \), \( H_{x^m_2} (x^m_2) \), \( H_{x^m_i} (x^m_{i+1}) \), \((i = 1, 2, \ldots, k_m - 1)\), and \( H_{x^m_{k_m}} (y_m) \) as before. Note that the last park contains \( y_m \) as
does the first park in step I. Actually this last park could be \((y_m, v_1)\) in the event that \(x_{k_m}^m\) joins \(v_m\) to \(v_2\). In any case, the parks ordered in steps I, II, III and IV will form a \(p\)-cycle. It must now be shown that the ordering for \(q = m\) in this step is well defined.

Assume \((x_1^m, y_m)\) has been previously ordered. Since this park contains only one line of the cycle, there must exist a sequential diagonal joining \(v_1\) to the other endpoint of \(x_1^m\). Let this diagonal be \(x_{\beta}^1\) (i.e. \((x_1^m, y_m) \in H_{x_{\beta}^1} (x_{\beta+1}^1)\)). In this situation, we may delete the park \((x_1^m, y_m)\) from the sequence of parks in step I and still have the ordering in step I well defined. This is because the parks necessary for the well defined ordering in step I contain either two diagonals incident with \(v_1\) or one such diagonal and the line \(y_1\). Hence without loss of generality we may assume \((x_1^m, y_m)\) has not been previously ordered.

By an argument similar to that used previously, it is easily shown that \((x_1^m, x_{i+1}^m)\) could not have been ordered prior to the insertion of \(H_{x_{i}^m} (x_{i+1}^m)\) in the sequence. Next suppose that \((x_{k_m}^m, y_m)\) has been ordered prior to the insertion of \(H_{x_{k_m}^m} (y_m)\) in the sequence. If \(x_{k_m}^m\) joins \(v_m\) to \(v_2\), \((x_{k_m}^m, y_m)\) is the first park ordered in step I and we have established a
p-cycle. In this case proceed to step V. If \( x_k^m \) does not join
\( v_m \) to \( v_2 \), there must exist a sequential diagonal \( x_1^\beta \) joining \( v_1 \)
to the other endpoint of \( x_k^m \). But as previously stated, a park
of this type is not essential for the ordering of parks in step I
to be well defined. Hence we may assume that \((x_k^m, y_m)\) has not
been ordered before \( H_{x_k^m} (y_m) \) appeared in the sequence. Thus for
q = m, the ordering is well defined. Furthermore, the last park
ordered in step IV is adjacent to the first park ordered in
step I. We now have the following sequence:

\[
(y_m, y_1), H_{x_1^1} (y_1, x_2^1), H_{x_2^1} (x_3^1), \ldots, H_{x_k^1} (y_1),
\]

\[
\vdots
\]

\[
(y_{i-1}, y_i), H_{x_1^i} (y_i, x_2^i), H_{x_2^i} (x_3^i), \ldots, H_{x_k^i} (y_i),
\]

\[
\vdots
\]

\[
(y_{m-1}, y_m), H_{x_1^m} (y_m, x_2^m), H_{x_2^m} (x_3^m), \ldots, H_{x_k^m} (y_m).
\]

**Step V.** If every diagonal of the graph G is now a sequential
diagonal, the sequence of parks ordered in steps I, II, III and IV
contains every park of G and \( P(G) \) is hamiltonian by Proposition
4.1. This is easily seen by considering the types of parks in the
graph G. If a park contains two lines of the cycle it has defi-
nitely been ordered. Since any other park must contain at least
one diagonal, and every diagonal is sequential, all parks of this
type have been ordered in the sequence.
On the other hand, suppose there are some nonsequential diagonals after the ordering in step IV has been completed. Label them $x_{i,j}, 1 \leq i < j - 1$, where $x_{i,j}$ is the diagonal joining $v_i$ to $v_j$. Clearly there can be at most one nonsequential diagonal incident with a vertex $v_k$. For an arbitrary nonsequential diagonal $x_{i,j}$, it is possible that the parks $(y_{i-1}x_{i,j}), (y_ix_{i,j}), (y_{j-1}x_{i,j})$, and $(y_jx_{i,j})$ have not been ordered in the sequence of parks described in steps I, II, III and IV.

![Figure 4.2](image)

However, this sequence was constructed such that for each $y_k, k = 1, 2, \ldots, m$, there exist consecutive parks $p', p''$ in the sequence that contain $y_k$. Since each of these parks under consideration contains a line of the cycle, it is then possible to insert each such park between the consecutive parks in the sequence that contain the same line of the cycle. Note that if two such parks contain the same line $y_k$ of the cycle, they may both be inserted between the consecutive parks in the sequence that contain $y_k$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Now consider the various types of parks in $G$. If a park contains two lines of the cycle, it was ordered in one of the first four steps. If a park $p$ contains only one diagonal $x$, it must contain at least one line of the cycle. In this case, if $x$ is sequential, $p$ was ordered in one of the first four steps. If $x$ is nonsequential, $p$ is ordered in step $V$. Finally, if a park $p$ contains two or three diagonals, at least one of the diagonals is sequential (otherwise two nonsequential diagonals would be incident with the same vertex) and $p$ will have been ordered in one of the first four steps. Thus all the parks of $G$ have been ordered in steps $I$, $II$, $III$, $IV$, or $V$ so that the conditions of Proposition 4.1 have been satisfied and $P(G)$ is hamiltonian.

This completes the proof.

A few remarks are in order. For a cubic hamiltonian graph with sufficiently many vertices, we note that every diagonal is nonsequential and the parks of such a graph are ordered only in step $V$. On the other hand, for a complete graph $K_m$ ($m \geq 5$) with vertices $v_1$, $v_2$, ..., $v_m$, the number of sequential diagonals increases from 1 at $v_3$ to $(m-3)$ at $v_{m-1}$; also, $v_m$ has no nonsequential diagonals.

Figure 4.3 illustrates the various steps described in the last proposition for the construction of a hamiltonian cycle in the park graph of a hamiltonian graph. The ordering induced by step $V$ is shown by "parking" a park $(x,y)$ in the main sequence.
by inserting it at the appropriate place and writing it as \( \langle (x \ y) \rangle \).

This ordering also permits an easy verification of Proposition 3.1.

The relevant sets as used in the proof of Proposition 4.3 are shown on page 52 and an ordering of the parks as required in Proposition 4.1 is exhibited on page 53.
H_{x_1}^{12} (y_2, x_2^{12}) = \{(y_2^{12}, x_2^{12}), (x_2^{12}y_6), (x_2^{12}y_{11}), (x_2^{12}y_{12})\}

H_{x_2}^{12} (y_2^{12}) = \{(x_2^{12}y_4), (x_2^{12}y_{11}), (x_2^{12}y_{12})\}
\( (y_1 y_{12}) \), \( H_{x_1^1} (y_1, x_2^1) \), \( H_{x_2^2} (x_3^1) \), \( H_{x_3^3} (y_1) \)

\( (y_1 y_2) \)

\( (y_2 y_3) \)

\( (y_3 y_4) \)

\( (y_4 y_5) \)

\( (y_5 y_6) \)

\( (y_6 y_7) \)

\( (y_7 y_8) \), \( H_{x_1^8} (y_8, x_2^8) \), \( H_{x_2^2} (x_3^8) \), \( H_{x_3^3} (y_8) \)

\( (y_8 y_9) \)

\( (y_9 y_{10}) \)

\( (y_{10} y_{11}) \)

\( (y_{11} y_{12}) \), \( H_{x_1^1} (y_{12}, x_2^{12}) \), \( H_{x_2^2} (y_{12}) \)

\[ \sum_{v \in \mathcal{V}(G)} (\deg_v v) - 2 \tau = 3 \binom{5}{2} + 3 \binom{4}{2} + 3 \binom{3}{2} + 3 \binom{2}{2} = 54 \]
CHAPTER V

PLANARITY

It was shown in the last chapter that the park graph function preserves certain properties of a graph $G$ while destroying others. In this chapter, we will examine what happens to planar and non-planar graphs when operated on by the park graph function.

**Proposition 5.1.** If a graph $G$ is non-planar, then $P(G)$ is non-planar.

**Proof:** According to a theorem of Kuratowski [7], if a graph $G$ is non-planar, then $G$ has a subgraph homeomorphic from $K_5$ or $K_{3,3}$. We now consider two cases.

**Case I.** Assume $G$ has a subgraph $G'$ homeomorphic from $K_5$. Let $v_1, v_2, v_3, v_4,$ and $v_5$ in $V(G')$ denote the vertices of $K_5$ from which $G'$ is homeomorphic (Figure 5.1).

![Figure 5.1](image_url)
Since $G'$ is homeomorphic from $K_5$, there exists a path $W_{i,j}$ from $v_i$ to $v_j$ that does not contain any other $v_k$, $k \neq i,j$, and for $i,j = 1,2,3,4,5$. Let the length of such a path be denoted by $m_{i,j}$. Clearly $m_{i,j} \geq 1$.

Form $P(G')$, a subgraph of $P(G)$, in the following manner. Let $p_1$ denote the park formed by lines from $W_{1,5}$ and $W_{1,2}$ that are incident with $v_1$ (i.e. the first line of the paths $W_{1,5}$ and $W_{1,2}$ starting from $v_1$). Let $p_2$ be the park formed by the lines from $W_{2,1}$ and $W_{2,3}$ that are incident with $v_2$. Continue in this manner to obtain $p_3$, $p_4$ and $p_5$. Let $u_1$ denote the vertex in $P(G')$ that corresponds to $p_1$ in $G'$, $i = 1,2,3,4,5$. Since there is a path of length $m_{1,2}$ from $v_1$ to $v_2$ in $G'$ that contains no other $v_1$, $i = 3,4,5$, there is a $p$-path (see Definition 2.6) from $p_1$ to $p_2$ in $G'$, hence a path of length $m_{1,2}$ from $u_1$ to $u_2$ in $P(G')$. Continue this process to obtain paths in $P(G')$ from $u_2$ to $u_3$, $u_3$ to $u_4$, $u_4$ to $u_5$, and $u_5$ to $u_1$ with respective lengths $m_{2,3}$, $m_{3,4}$, $m_{4,5}$ and $m_{5,1}$. Note that each of these paths in $P(G')$ contains only two of the $u_i$, $i = 1,2,3,4,5$, because each corresponding $p$-path in $G'$ contains only two of the $p_i$, $i = 1,2,3,4,5$. We now have a subgraph of $P(G')$ as illustrated in Figure 5.2.

![Figure 5.2](image-url)
Now consider the park formed by the two lines incident with \(v_1\) that are in \(W_{1,5}\) and \(W_{1,3}\). Call this park \(p_6\) and let \(u_6\) be the corresponding vertex in \(P(G')\). Since \(p_1\) and \(p_6\) are adjacent, \(u_1\) and \(u_6\) are adjacent in \(P(G')\). Let \(p_7\) be the park formed by the two lines incident with \(v_2\) from \(W_{2,1}\) and \(W_{2,4}\); \(p_8\) be the park formed by the two lines incident with \(v_3\) from \(W_{3,2}\) and \(W_{3,5}\); \(p_9\) be the park formed by the two lines incident with \(v_4\) from \(W_{4,3}\) and \(W_{4,1}\); and \(p_{10}\) be the park formed by the two lines incident with \(v_5\) from \(W_{5,4}\) and \(W_{5,2}\). If \(u_7\), \(u_8\), \(u_9\) and \(u_{10}\) denote the corresponding vertices in \(P(G')\), we have that \(u_i\) is adjacent to \(u_{i-5}\) for \(i = 6, 7, 8, 9, 10\) (see Figure 5.3).

![Figure 5.3](image-url)

Now consider a path from \(v_1\) to \(v_3\) via \(v_5\) (i.e. \(W_{1,5}\) followed by \(W_{5,3}\)) of length \(m_{1,5} + m_{5,3}\). If it happens that \(v_1\), \(v_3\), \(v_5\) form a triangle, then \(u_6\) is adjacent to \(u_8\) in \(P(G')\). Otherwise there exists a p-path from \(p_6\) to \(p_8\) in \(G'\). Hence there is a path of length \(m_{1,5} + m_{5,3}\) from \(u_6\) to \(u_8\) in \(P(G')\) that consists of vertices distinct from those already considered. Repeat the argument for a path of length \(m_{1,3} + m_{1,4}\) from
$v_1$ to $v_4$ via $v_3$ ($W_{1,3}$ and $W_{3,4}$) to get a path from $u_6$ to $u_9$ in $P(G')$. Continuing this process we have distinct paths from $u_7$ to $u_{10}$ and $u_9$, and from $u_8$ to $u_{10}$. This now establishes the existence of a subgraph $H$ of $P(G')$ as shown in Figure 5.4.

This subgraph is homeomorphic from the Petersen graph which in turn is homeomorphic from $K_{3,3}$ [4]. Since "homeomorphic from" is transitive, $P(G)$ has a subgraph homeomorphic from $K_{3,3}$ whenever $G$ has a subgraph homeomorphic from $K_5$. Again, by Kuratowski's theorem, $P(G)$ is non-planar.

**Case II.** Assume $G$ has a subgraph $H$ homeomorphic from $K_{3,3}$ with vertices as in Figure 5.5.
Let $t_i$ denote the triangle in $P(H)$, a subgraph of $P(G)$, formed by the three lines incident with $v_i$, $i = 1, 2, \ldots, 6$. In $H$, either $v_i$, $i = 1, 2, 3$ is adjacent to $v_j$, $j = 4, 5, 6$, or there exists a $p$-path from $v_i$ to $v_j$. If $v_i$ is adjacent to $v_j$, there exists a line from $t_i$ to $t_j$ in $P(H)$. If there exists a $p$-path from $v_i$ to $v_j$ in $H$, then there exists a path from $t_i$ to $t_j$ in $P(H)$. In either case, there exist paths from $t_i$, $i = 1, 2, 3$, to $t_j$, $j = 4, 5, 6$. Form a contraction $P(H)'$ of $P(H)$ by associating each $t_i$, $i = 1, 2, \ldots, 6$, with a vertex $u_i$, and associating every other vertex with itself. This contraction is homeomorphic from $K_{3,3}$. According to a theorem by Harary and Tutte [6], $P(G)$ is then nonplanar. This completes the proof of the proposition.

Since the subgraph $H$ of $G$ which is homeomorphic from $K_{3,3}$ in Case II of the preceding proposition contains no triangles, we may apply Corollary 3.12 to show that $P(H)$ has a contraction which is isomorphic to $H$. Hence $P(H)$ cannot be planar.

It is easily seen that the converse to Proposition 5.1 is not true. For consider a graph $G$ which is an $n$-star for $n \geq 6$. Then clearly $G$ is planar. But by Proposition 3.7 (ii), $P(G)$ is regular of degree $k = 2(n-3) \geq 6$, and hence $P(G)$ is nonplanar (see [3]).

Since the proposition "$G$ planar implies $P(G)$ is planar" is false, it is natural to look for restrictions on the graph $G$ that would ensure that $P(G)$ be a planar graph. For the graphical valued function $L$ (the line-graph function), Sédlaček proved
in [8] that for a graph G, L(G) is planar if and only if G is planar and (a) max deg G ≤ 4, and (b) if deg ν = 4 then ν is a cut-point of G. In the case of the function P then, ideally we would like to find some property Π of the graph G such that P(G) is planar if and only if G is planar and G has property Π. Although a characterization similar to that of Sedláček has not been obtained, the following propositions give certain properties of the graph G that are sufficient for P(G) to be planar. We first prove a lemma that is needed for later work.

**Lemma 5.2.** Let G be a graph such that P(x) ≤ k for all x ∈ E(G). Then max deg G ≤ k + 1.

**Proof:** Assume there exists a vertex v in V(G) such that deg v = m ≥ k + 2. Let \(x_1, x_2, \ldots, x_m\) denote the lines incident with v. The pairs of lines \((x_i, x_j)\), \(i = 2, 3, \ldots, m\), then form \(m - 1 ≥ k + l ≥ k\) distinct parks that contain \(x_1\). Let u be the other endpoint of \(x_1\). For each triangle t containing \(x_1\), there exists a distinct line \(x_t\) incident with u. Using the notation of Proposition 2.4, if \(T(x_1) = q\), then \(deg x_1 =\)

\[deg u + deg v - 2 ≥ (q+1) + m - 2 = m - 1 + q.\]

By Proposition 2.4, \(P(x_1) = deg x_1 - T(x_1)\); thus \(P(x_1) ≥ (m-1+q) - q = m - l ≥ k + l ≥ k\), which is a contradiction.

**Proposition 5.3.** If G is a planar graph and max \(P(x) ≤ 2,\) \(x ∈ E(G)\), then \(P(G)\) is planar.

**Proof:** If \(P(x) ≤ 2\) for all \(x ∈ E(G)\), then max deg \(P(G) ≤ 3\). Hence \(P(G)\) cannot have a subgraph homeomorphic from \(K_5\). Assume
P(G) has a subgraph H homeomorphic from $K_{3,3}$ (see Figure 5.5). Let $p_1$ be the park in G to which corresponds the point $u_1$ of P(G). Assume that $p_1$ consists of only two lines, say $(x_1, x_2)$. Since $\text{max deg } P(G) \leq 3$ and $\text{deg } u_1 = 3$ in H, it must be that $\text{deg } u_1 = 3$ in P(G) also. So there exist three parks $p_1', p_1'', p_1'''$ in G such that each park is distinct from $p_1$ but adjacent to $p_1$. This implies that either $x_1$ or $x_2$ has park degree exceeding 2, and is a contradiction since $P(x) \leq 2$ for all $x \in E(G)$. Hence $p_1$ is a triangle with lines $x_1$, $x_2$, and $x_3$, and vertices $v_1$, $v_2$ and $v_3$. Let $x_4$ be incident with $v_1$ and let $v_4$ be the other endpoint of $x_4$ (see Figure 5.6). Then $P(x_1) = P(x_3) = P(x_4) = 2$.

![Figure 5.6](image)

If $x_5$ is another line incident with $v_1$, $i = 1, 2, 3, 4$, then $x_5$ must be incident with two of the $v_i$, $i = 1, 2, 3, 4$, for otherwise $P(x) > 2$ for some $x$ in $E(G)$. Since the maximum number of parks in a graph G constructed in this manner is 4 (i.e. when $G \equiv K_4$), no such G can exist so that $P(G)$ contains a subgraph homeomorphic from $K_{3,3}$. This completes the proof.

**Proposition 5.4.** If a graph G is planar, connected and $\text{max } \text{deg } x \leq 3$, then $P(G)$ is planar.
Proof: Clearly $G$ cannot have any vertex of degree $k \geq 5$.

We now consider two cases.

**Case I.** Assume $G$ has a vertex $v$ of degree 4. Let the adjacent vertices be labeled $v_1$, $v_2$, $v_3$ and $v_4$. Then $\deg x_i \geq 3$, $i = 1,2,3,4$, where $x_i$ denotes the line joining $v_i$ to $v$. By hypothesis $\max_{x \in E(G)} \deg x \leq 3$, hence $\deg x_i = 3$, and $x_i$, $i = 1,2,3,4$, cannot be incident with any other vertices, and $G$ must be a 5-star. Since the park graph of a 5-star is planar (see Figure 4.12), we are done.

**Case II.** Assume $\max \deg G \leq 3$. If $\max \deg G \leq 2$, $P(G)$ is obviously planar. Now suppose $G$ has vertices of degree 3 and consider the two cases: (a), $G$ has triangles: (b), $G$ has no triangles.

(a) Let $t$ denote a triangle with vertices $v_1$, $v_2$, $v_3$ and lines $x_{i,j}$, $i,j = 1,2,3$, $i \neq j$. Let $x$ be a distinct line incident with $v_1$. Then $\deg x_{1,2} = \deg x_{1,3} = 3$, which implies no other line can be incident with $v_1$, $i = 1,2,3$. Furthermore, at most one other line $x'$ can be incident with $x$ (see Figure 5.7).

![Figure 5.7](image)

Since the park graph of this graph is planar, we need only
consider graphs without triangles.

(b) Assume \( G \) has no triangles. Let \( v_1, v_2 \) be vertices in \( V(G) \) such that \( \deg v_1 = \deg v_2 = 3 \). Then \( d(v_1, v_2) \geq 2 \) for all such vertices (for otherwise the line joining \( v_1 \) and \( v_2 \) would have degree \( 4 \)). Let \( v \) in \( V(G) \) be such that \( \deg v = 3 \) where \( v_1, v_2 \) and \( v_3 \) denote the vertices adjacent to \( v \) and \( x_1, x_2, x_3 \) denote the lines incident with \( v \). By considering all parks containing \( v \) with two lines incident with \( v \), \( K_3 \) is obtained as a subgraph of \( P(G) \). Let \( u_1, u_2, u_3 \) be the vertices of such a subgraph. If \( \deg v_j = 2 \), then two of the \( u_i, i = 1,2,3, i \neq j \), are adjacent to a vertex \( u_4 \) in \( V(P(G)) \) where \( u_4 \) corresponds to the park formed by the two lines incident with \( v_j \). If all three vertices adjacent to \( v \) have degree 2, the subgraph \( H \) of \( P(G) \) in Figure 5.8 is obtained.

![Figure 5.8](image)

Any other vertex of \( P(G) \) adjacent to a vertex in \( H \) must be adjacent to \( u_4, u_5 \) or \( u_6 \). Since at most two other lines may be incident with say \( u_4 \), \( P(G) \) must be planar by an inductive argument. This completes the proof.

Since Propositions 5.3 and 5.4 have somewhat similar hypoth-
eses, it might be thought that one implies the other. That this is not the case is seen be the graphs in Figure 5.9.

\[ G_1: \]

\[ G_1 \text{ is planar,} \]
\[ P(x) = 2 \forall x \in E(G_1), \]
\[ \text{and } \deg x_1 = 4 > 3 \]

\[ G_2: \]

\[ G_2 \text{ is planar,} \]
\[ \max \deg x = 3, \]
\[ x \in E(G_2) \]
\[ \text{and } P(x) = 3 \forall x \in E(G_2) \]

Figure 5.9

Summarizing the results of Lemma 5.2, Proposition 5.3 and Proposition 5.4, we have that for a planar graph \( G \), each of the following conditions are sufficient for \( P(G) \) to be planar:

(a) \[ \max \deg G \leq 2 \]

(b) \[ \max \deg x \leq 3, \]
\[ x \in E(G) \]

(c) \[ \max P(x) \leq 2 \]
\[ x \in E(G) \]

The immediate question then arises as to whether any of these conditions can be relaxed and still imply that \( P(G) \) is planar. That this is not the case is seen in the next proposition.

**Proposition 5.5.** Let \( G \) be a planar graph. Then none of the following conditions are sufficient to imply that \( P(G) \) is planar:
(a) \( \max \deg G \leq 3 \),
(b) \( \max \deg x \leq 4 \),
\( x \in E(G) \)
(c) \( \max P(x) \leq 3 \).
\( x \in E(G) \)

Proof: Counterexamples are illustrated in Figures 5.10 (for (a) and (b)) and 5.11 (for (c)). We note that the graph \( H \) as shown is a subgraph of \( P(G) \).

Figure 5.10
If it were possible to find a property $\mathcal{P}$ for a planar graph $G$ which characterizes $P(G)$ being planar, it must be that $G$ and $P(G)$ being planar implies that $G$ has property $\mathcal{P}$. Some natural candidates for $\mathcal{P}$ are considered in the next result.

**Proposition 5.6.** Let $G$ be a graph with park graph $P(G)$. If $P(G)$ is planar, then:
\( (a) \) \( \max \deg G \leq 4 \)

\( (b) \) \( \max_{x \in E(G)} \deg x \leq 6 \)

\( (c) \) \( \max_{x \in E(G)} P(x) \leq 4 \)

**Proof:** If \( G \) has a line \( x \) such that \( \deg x > 6 \), then one of the endpoints of \( x \) must have degree \( k > 4 \). Thus \( (a) \) implies \( (b) \), and we need only prove parts \( (a) \) and \( (c) \).

To prove \( (a) \), suppose \( G \) has a vertex \( v \) such that \( \deg v = q \geq 5 \). Then the parks formed by pairs of lines incident with \( v \) yield a subgraph \( H \) of \( P(G) \) that is isomorphic to the park graph of a \((q+1)\)-star. Since \( q > 5 \), \( H \) is regular of degree \( 2(q-2) \geq 6 \) and is nonplanar [3]. This is a contradiction, so \( \max \deg G \leq 4 \).

To prove \( (c) \), suppose there exists a line \( x \) in \( E(G) \) such that \( P(x) = k \geq 5 \). Then there exists a subgraph \( H \) of \( P(G) \) such that \( H \cong K_5 \). Hence \( P(G) \) is nonplanar [7]. Again this is a contradiction, and \( \max_{x \in E(G)} P(x) \leq 4 \).

It is now natural to ask if for a graph \( G \) with planar park graph \( P(G) \) whether or not it might be possible to impose some conditions on the graph \( G \) that are more restrictive than those in Proposition 5.6. This is answered in the negative as follows.

**Proposition 5.7.** Let \( G \) be a graph with park graph \( P(G) \). Then \( P(G) \) being planar does not imply any of the following:

\( (a) \) \( \max \deg G < 4 \),

\( (b) \) \( \max_{x \in E(G)} \deg x < 6 \),
(c) $\max_{x \in E(G)} P(x) < 4$.

**Proof:** We again consider counterexamples for each case. These are illustrated in Figures 5.12, 5.13 and 5.14 for (a), (b) and (c) respectively.

![Diagram of graph G and P(G)](image)
\[ p_1 = (v_1, v_2, v_5) \quad p_6 = (v_1, v_1, v_3) \]
\[ p_2 = (v_1, v_2, v_4) \quad p_7 = (v_5, v_2, v_4) \]
\[ p_3 = (v_1, v_2, v_3) \quad p_8 = (v_5, v_2, v_3) \]
\[ p_4 = (v_4, v_1, v_5) \quad p_9 = (v_4, v_2, v_3) \]

Figure 5.13

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Another possibility for the desired property \( P \) might be to find a suitable bound for the number of parks containing a vertex. For a vertex \( v \) of a graph \( G \), let \( P(v) \) denote this number.

If \( v \) is a vertex in \( K_6 \), \( P(v) = 6 \). Furthermore, we see in Figure 3.7 that \( P(K_6) \) is nonplanar. Hence if a bound \( k \) for \( \max_{v \in V(G)} P(v) \) exists which is implied by \( P(G) \) being planar, it must be that \( k \leq 5 \).

On the other hand, the graph \( G \) with park graph \( P(G) \) as illustrated in Figure 5.15 has a vertex \( v \) such that \( P(v) = 9 \) and \( P(G) \) is planar. Obviously then there does not exist a number \( k \) such that \( P(G) \) is planar if and only if \( G \) is planar and \( \max_{v \in V(G)} P(v) \leq k \).

![Figure 5.15](image-url)
As remarked earlier, we have investigated some possibilities of obtaining a characterization for the planarity of \( P(G) \) as Sedlacek did for \( L(G) \) in [8]. The results of this chapter show that restrictions on various types of degrees of vertices and edges of \( G \) are not very helpful and we may have to search elsewhere to obtain a suitable property \( \mathcal{P} \) which might work.
REFERENCES


