A Category of Banach Spaces

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A CATEGORY OF BANACH SPACES

by

Kenneth L. Pothoven

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MASTER'S THESIS

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1. PRELIMINARIES

Definition 1.1 Let \( \mathcal{A} \) be a collection of "objects", \( X, Y, Z \), and so on, together with two functions:

(i) a function assigning to each ordered pair \( (X,Y) \) of objects a set \( \text{hom}_\mathcal{A}(X,Y) \). An element \( f \) in this set is called a morphism of \( \mathcal{A} \) with domain and codomain \( Y \) (written \( f:X \rightarrow Y \)).

(ii) a function assigning to each triplet \( (X,Y,Z) \) of objects of \( \mathcal{A} \) a function \( \text{hom}_\mathcal{A}(Y,Z) \times \text{hom}_\mathcal{A}(X,Y) \rightarrow \text{hom}_\mathcal{A}(X,Z) \).

For morphisms \( g:Y \rightarrow Z \) and \( f:X \rightarrow Y \) this function takes \( (g,f) \rightarrow g \circ f \), and \( g \circ f: X \rightarrow Z \) is called the composite of \( g \) with \( f \).

\( \mathcal{A} \) is called a category when the following axioms hold.

(i) If \( h:Z \rightarrow W \), \( g:Y \rightarrow Z \), and \( f:X \rightarrow Y \) are morphisms of \( \mathcal{A} \), then \( h \circ (g \circ f) = (h \circ g) \circ f \).

(ii) For each object \( Y \) of \( \mathcal{A} \) there exists a morphism (necessarily unique) \( l_Y:Y \rightarrow Y \) such that \( f:X \rightarrow Y \) implies \( l_Y \circ f = f \) and \( g:Y \rightarrow Z \) implies \( g \circ l_Y = g \).

Throughout this paper morphisms in a category \( \mathcal{A} \) will be denoted by such letters as \( f, g, h, \) and \( p \); while objects will be denoted by letters such as \( X, Y, A, B, \) and \( C \). If \( f:X \rightarrow Y \) is a morphism in a category \( \mathcal{A} \), it will just be said without confusion, \( f \) is in \( \mathcal{A} \).
Let $\mathcal{B}^*$ be the collection of Banach spaces (scalar field is the real or complex number field and is denoted by $\mathbb{F}$) together with the following functions.

(i) For each pair of Banach spaces $(X,Y)$ assign the set $\text{hom}_B(X,Y) = \mathcal{B}^*(X,Y)$ where $\mathcal{B}^*(X,Y)$ is the set of all linear continuous maps $f:X \rightarrow Y$ such that $|f| = \sup_{|x| \leq 1} |f(x)| \leq 1$.

(ii) For each triplet $(X,Y,Z)$ of Banach spaces assign to each element $(g,f)$ of $\mathcal{B}^*(Y,Z) \times \mathcal{B}^*(X,Y)$ the function $g \circ f:X \rightarrow Z$. Since $g \circ f$ is linear and $|g \circ f| \leq |g||f| \leq 1$, $g \circ f \in \mathcal{B}^*(X,Z)$.

It is now easy to establish the following proposition.

**Proposition 1.2** $\mathcal{B}^*$ is a category.

In this paper the category $\mathcal{B}^*$ will be examined. The following propositions and remarks will be essential to this study.

**Remark 1.3** It should be recalled that for a linear function $f:X \rightarrow Y$ where $X$ and $Y$ are normed linear spaces

$$|f| = \sup_{|x| \leq 1} |f(x)| = \sup_{|x| \leq 1} |f(x)|/|x| \quad \text{for } x \neq 0$$

$$= \inf \left\{ M \mid \text{for all } x \in X, |f(x)| \leq M|x| \right\}.$$  

The terminology for the following lemma and its corollary is taken from Kelley [21]. Let $X$ be an order-complete chain with the order topology.

**Lemma 1.4** Each monotone increasing net in $X$ whose range is bounded converges to the supremum of its range.
**Corollary 1.5** If \( R \) is the set of real numbers with the usual order, then each monotone increasing (respectively, decreasing) net in \( R \), whose range has an upper (respectively, lower) bound converges to the supremum (respectively, infimum) of its range.

Let \( B \) be a Banach space and \( I \) an arbitrary index set. Let \( \{a_i|i \in I\} \) be a set of elements from \( B \) indexed by \( I \). Let \( \mathcal{F} \) be the set of all finite subsets of \( I \). \( \mathcal{F} \) becomes a directed set by the relation \(<\) defined by \( A < B \) if and only if \( A \subseteq B \). Define \( S: \mathcal{F} \to \mathbb{R} \) by \( S(A) = \sum_{i \in A} |a_i| \).

**Definition 1.6** \( \sum_I |a_i| < \infty \) if and only if \( S \) is bounded.

The following remarks follow from the corollary above.

**Remark 1.7** \( S \) converges if and only if \( \sum_I |a_i| < \infty \).

**Remark 1.8** Absolute convergence implies convergence; that is, if \( S \) converges then \( \sum_I a_i \) converges to a point in \( B \). Moreover, if \( S \) converges then \( |\sum_I a_i| \leq \sum_I |a_i| \).

**Remark 1.9** If \( \sum_I a_i \) and \( \sum_I b_i \) both converge, then

1. \( \sum_I a_i + \sum_I b_i = \sum_I (a_i + b_i) \).
2. \( \sum_I ka_i = k \sum_I a_i \) for \( k \in \mathbb{R}^+ \).

**Remark 1.10** Let \( f \) be a continuous linear map from Banach space \( A \) to Banach space \( B \). If \( \sum_I a_i \) converges in \( A \), then \( f(\sum_I a_i) = \sum_I f(a_i) \).

Let \( X \) and \( Y \) be normed linear spaces and \( Z \) a closed subspace of \( X \). Then \( X/Z \) is a normed linear space consisting of
equivalence classes \([a.\)], formed by a relation \(R: aRb\) if \(a - b \in \mathbb{Z}\). The norm on \(X/\mathbb{Z}\) is given by \(|[a]| = \inf_{b \in [a]} |b|\).

Throughout this paper \(\left[\right]\) will denote an equivalence class in \(X/\mathbb{Z}\).

**Lemma 1.11** Let \(f: X \rightarrow Y\) be a linear function of norm no greater than one. Then \(g: X/f^1(0) \rightarrow Y\) defined by \(g([a]) = f(a)\) is linear, one-to-one, and has norm no greater than one. \((f^1(0) = \{x \in X \mid f(x) = 0\}\), a subspace of \(X)\)

**Proof:** That \(g\) is linear follows from the fact that \(f\) is linear. If \(f(a) = f(b)\) then \(a - b \in f^1(0)\) and \(\left[a\right] \neq \left[b\right]\). Therefore \(g\) is one-to-one. Now by definition

\[|g| = \sup_{[a] \neq [0]} |g([a])|/|[a]| = \sup_{[a] \neq [0]} |f(a)|/|[a]|.\]

However, for all \(b \in [a]\), \(f(a) = f(b)\) and hence \(|f(a)|/|b| = |f(b)|/|b| \leq 1\) and \(|f(b)| \leq |b|\). Hence

\[|f(a)|/|[a]| \leq 1\]  \(\text{as } |[a]| = \inf_{b \in [a]} |b|\).

Therefore \(|g| \leq 1\).

**Definition 1.12** In a category \(A\), a morphism \(f: A \rightarrow B\) is a monomorphism if and only if the only pair of morphisms \(f': C \rightarrow A\) and \(f'': C \rightarrow A\) such that \(f \circ f' = f \circ f''\) is \(f' = f''\).

**Proposition 1.13** In \(B\) a morphism is a monomorphism if and only if it is one-to-one.

**Proof:** Let \(f: A \rightarrow B\) be one-to-one. If \(f': C \rightarrow A\) and \(f'': C \rightarrow A\) are not equal, then \(f'(c) \neq f''(c)\) for some \(c \in C\). Then \(f(f'(c)) \neq f(f''(c))\) or \(f \circ f' \neq f \circ f''\).

Let \(f: A \rightarrow B\) be a monomorphism. If \(f\) is not one-to-one, then \(f^1(0)\), a closed subset of \(A\), is not trivial.
Consider two unequal maps $f'$ and $f''$ from $f^1(0)$ to $A$: the insertion map and the constant zero map respectively. Now $f'$ and $f''$ are in $B'$, but $f \circ f' \neq f \circ f''$. This is a contradiction.

**Definition 1.14** A morphism $f:A \to B$ is an epimorphism in a category $\mathcal{A}$ if and only if the only pair of morphisms $f':B \to C$ and $f'':B \to C$ such that $f' \circ f = f'' \circ f$ is $f' = f''$.

**Proposition 1.15** In $B^*$ a morphism $f:A \to B$ is an epimorphism if and only if $f[A]$ is dense in $B$. ($f[A] = \{ b \in B | b = f(a) \text{ for some } a \in A \}$)

**Proof:** Suppose $f[A]$ is dense in $B$. Let $f':B \to C$ and $f'':B \to C$ be two unequal morphisms such that $f' \circ f = f'' \circ f$. There exists some $c \in B$ such that $f'(c) \neq f''(c)$. Let $\varepsilon > 0$ be arbitrary. Since $f[A]$ is dense in $B$, there exists $b \in B$ such that $b = f(a)$ for some $a \in A$ and $|c - b| < \varepsilon/2$. Therefore $|(f' - f'')(c)| = |(f' - f'')(c - b)|$ as $f'(b) = f''(b)$ and $|(f' - f'')(c - b)| \leq |f' - f''| |c - b| < (|f'| + |f''|) \varepsilon/2 < \varepsilon$. This means $f'(c) \neq f''(c)$. This is a contradiction.

Let $f:A \to B$ be an epimorphism. Suppose $f[A]$ is not dense in $B$. Then $B/f[A]$ is a non-zero Banach space. ($\overline{f[A]}$ is the closure in $B$ of $f[A]$) Consider two unequal maps $f'$ and $f''$ from $B$ to $B/\overline{f[A]}$: the canonical map and the constant zero map respectively. Then both are in $B'$ but $f' \circ f = f'' \circ f$. This is a contradiction.

**Definition 1.16** In category $\mathcal{A}$, $f:A \to B$ is an isomorphism if and only if there exists $h:B \to A$ such that $f \circ h = 1_B$.
and \( g: B \to A \) such that \( g \circ f = l_A \).

Every isomorphism is both an epimorphism and monomorphism. By virtue of the preceding propositions, it is now readily seen that \( f: A \to B \) is an isomorphism in \( B' \) if and only if \( f \) is an isometric function and \( f[A] = B \).

2. KERNELS, COKERNELS, IMAGES, COIMAGES, AND NORMALITY.

Definition 2.1 An object 0 in a category \( \mathcal{A} \) is an initial object if \( \text{hom}_{\mathcal{A}}(0, A) \) has exactly one element for each object \( A \) in \( \mathcal{A} \). It is a terminal object if \( \text{hom}_{\mathcal{A}}(A, 0) \) has exactly one element for each object \( A \) in \( \mathcal{A} \). It is a zero object if it is both an initial and terminal object.

Definition 2.2 Let \( \mathcal{A} \) be a category with a zero object. The kernel of a morphism \( f: A \to B \) is a morphism \( k: K \to A \) such that \( f \circ k = 0 \), and if \( k': K' \to A \) is any morphism such that \( f \circ k' = 0 \) then there is a unique map \( u: K' \to K \) such that \( k \circ u = k' \).

Notation: The object \( K \) is generally denoted by \( \text{Ker } f \), while the morphism \( k \) is sometimes denoted by \( \text{ker } f \).

Remark 2.3 If \( k: K \to A \) and \( k': K' \to A \) are both kernels of \( f: A \to B \) then \( K \) and \( K' \) are isomorphic.

Proposition 2.4 In \( B' \), a kernel of \( f: A \to B \) is \( k: K \to A \).
where \( K = \{ a \in A | f(a) = 0 \} \) and \( k \) is the insertion map, 
\( k(b) = b \) for all \( b \in K \).

**Proof:** As a closed subset of \( A \), \( K \) is a Banach space and clearly \( k \in B^* \). Let \( k' : K' \rightarrow A \) be a morphism such that \( f \circ k' = 0 \). Define \( u : K' \rightarrow K \) by \( u(x) = k'(x) \). Since \( k'(x) \in K \) the map is well-defined and clearly \( k \circ u = k' \).

**Definition 2.5** Let \( A \) be a category with zero object, and let \( f : A \rightarrow B \) be a morphism. A morphism \( g : B \rightarrow F \) is called a cokernel of \( f \) if \( g \circ f = 0 \), and if for every morphism \( g' : B \rightarrow F' \) such that \( g' \circ f = 0 \) there is a unique morphism \( u : F \rightarrow F' \) such that \( u \circ g = g' \).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g'} & & \downarrow{u} \\
F' & & F
\end{array}
\]

**Notation:** The object \( F \) is generally denoted by \( \text{Coker } f \), while the morphism \( g \) is sometimes denoted by \( \text{coker } f \).

**Remark 2.6** If \( g : B \rightarrow F \) and \( g' : B \rightarrow F' \) are both cokernels of \( f : A \rightarrow B \), then \( F \) and \( F' \) are isomorphic.

**Proposition 2.7** In \( B^* \), a cokernel of \( f : A \rightarrow B \) is \( g : B \rightarrow B/\overline{f[A]} \) where \( g \) is the canonical map.

**Proof:** Clearly \( g \circ f = 0 \) as \( g(f(x)) = [f(x)] = [0] \) where \([ \_ \]\) denotes an equivalence class in \( B/\overline{f[A]} \). Let \( g' : B \rightarrow F' \) be a morphism such that \( g' \circ f = 0 \). Define \( u : B/\overline{f[A]} \rightarrow F' \) by \( u[y] = g'(y) \). Let \([y] = [x] \); then \( x - y \in \overline{f[A]} \subseteq \ker g' \).

This says \( g'(x) = g'(y) \) and \( u \) is well defined. If \([ \_ \]_g
denotes an equivalence class in $B/K \ker g'$, then by lemma 1.11 and the fact that $\overline{f[A]} \subseteq \ker g'$,
\[
|u| = \sup_{[y] \neq 0} |u[y]| / |[y]| = \sup_{[y] \neq 0} |g'(y)| / |[y]| \leq \sup_{[y] \neq 0} |g'(y)| / |[y]| g'[y] \leq 1.
\]
Clearly $uog = g'$. If $u':B/\overline{f[A]}\to F'$ is any morphism such that $u'og = g'$, then $u'[y] = u'(g(y)) = g'(y) = u(g(y)) = u[y]$, and $u$ is unique.

Definition 2.8 If $f:A\to A$ is a monomorphism, $A'$ is called a subobject of $A$ with respect to $f$.

The statement - let $f:A\to B$ be a subobject of $B$ - will be used to mean $A$ is a subobject of $B$ with respect to $f$.

Terminology: If $f':A\to A$ and $f'':A''\to A$ are subobjects of $A$, $A'$ is smaller than $A''$ if there is a morphism $u:A'\to A''$ such that $f''\circ u = f'$.

Definition 2.9 The image of a morphism $f:A\to B$ is the smallest subobject $f':I\to B$ through which $f$ factors, that is, $f = f'f_1$ for some $f_1:A\to I$.

Notation: The image of $f:A\to B$ will be denoted by $\text{im } f$.

Proposition 2.10 In $\mathcal{B}$ the image of $f:A\to B$ is $g:A/\ker f \to B$ where $g$ is defined by $g([a]) = f(a)$.

Proof: Using lemma 1.11, $g$ is a monomorphism in $\mathcal{B}$, and therefore $g:A/\ker f \to B$ is a subobject of $B$. Also if $c:A\to A/\ker f$ is the canonical mapping; then $g\circ c = f$, that is, $f$ factors through $A/\ker f$. Let $g':I'\to B$ be
any other subobject of B through which f factors, that is, 
\[ f = g' \circ c' \]  for \( c' : A \to I' \).

\[ \text{A/Ker f} \]  
\[ \ker f \]  
\[ \text{Ker f} \]  
\[ \text{f} \]  
\[ \text{c'} \]  
\[ \text{u} \]  
\[ \text{g} \]  
\[ \text{g'} \]  
\[ \text{B} \]  
\[ \text{A} \]  
\[ \text{f} \]  
\[ \text{I} \]  

Since \( g' \) is a monomorphism, \( c' \circ \ker f = 0 \); since \( c : A \to A/\ker f \) is a cokernel of \( \ker f : \ker f \to A \), there exists a unique \( u : A/\ker f \to I \) such that \( c' = u \circ c \). Therefore \( g \circ c = g' \circ u \circ c \) and since \( c \) is an epimorphism, \( g = g' \circ u \).

**Definition 2.11** If \( f : A \to A' \) is an epimorphism, \( A' \) is called a quotient object of \( A \) with respect to \( f \).

The statement - let \( f : A \to A' \) be a quotient object of \( A \) - will be used to mean \( A' \) is a subobject of \( A \) with respect to the epimorphism \( f \).

**Terminology:** If \( f_1 : A \to A_1 \) and \( f_2 : A \to A_2 \) are quotient objects of \( A \), \( A_1 \) is smaller than \( A_2 \) if there is a morphism \( u : A_2 \to A_1 \) such that \( u \circ f_2 = f_1 \).

**Definition 2.12** The coimage of a morphism \( f : A \to B \) is defined as the smallest quotient object of \( A \), \( f_1 : A \to A_1 \), through which \( f \) factors, that is, \( f = f' \circ f_1 \) for some \( f' : A_1 \to B \).

**Notation:** The coimage of \( f : A \to B \) will be denoted by \( \text{coim } f \).

**Proposition 2.13** In \( B' \), the coimage of \( f : A \to B \) is
f_1 : A \to \overline{f[A]} \text{ where } f_1 \text{ is defined by } f_1(a) = f(a).

Proof: The morphism f_1 : A \to \overline{f[A]} is an epimorphism as 
\overline{f_1[A]} = \overline{f[A]} \text{ so that } \overline{f[A]} \text{ is a quotient object of } A.

The morphism f : A \to B \text{ factors through } \overline{f[A]} \text{ with } f = f' \circ f_1 \text{ where } f' : \overline{f[A]} \to B \text{ is defined by } f'(b) = b \text{ for } b \in \overline{f[A]}.

Let f_2 : A \to A_2 \text{ be any other quotient object of } A \text{ through which } f \text{ factors, that is } f = f'' \circ f_2 \text{ for } f'' : A_2 \to B.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & \overline{f[A]} & \xrightarrow{f'} & B \\
\downarrow{f_1} & & \downarrow{\overline{f[A]}} & & \downarrow{coker \ f} & \xrightarrow{\overline{f[A]}} & B/\overline{f[A]} \\
\downarrow{f_2} & & \downarrow{f''} & & \downarrow{coker \ f} & & \downarrow{coker \ f'} \\
A_2 & & & & & & \end{array}
\]

Since f_2 \text{ is an epimorphism and } (coker \ f) \circ f'' \circ f_2 = 0, 
(coker \ f) \circ f' = 0; \text{ and since } f' : \overline{f[A]} \to B \text{ is a kernel of } coker \ f : B \to B/\overline{f[A]}; \text{ there exists a unique morphism } u : A_2 \to \overline{f[A]} \text{ such that } f'' = f' \circ u. \text{ Therefore } f' \circ f_1 = f' \circ (u \circ f_2) \text{ and since } f' \text{ is a monomorphism, } f_1 = u \circ f_2.

Definition 2.14 A category is balanced if every morphism which is both a monomorphism and an epimorphism is also an isomorphism.

Definition 2.15 If A' \to A \text{ is the kernel of some morphism, then } A' \text{ is called a normal subobject of } A. \text{ If every monomorphism in a category is normal, then the category is called normal.}

The following example (taken from [6]) will show B' is not balanced (in fact, a morphism with norm one
which is both an epimorphism and monomorphism is not an isomorphism) and not normal. Let $c_0$ be the Banach space of all sequences $b = (b_n)_{n \in \mathbb{N}}$, $b_n \in \mathbb{F}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$, converging to zero with norm defined by $|b| = \sup_{n} |b_n|$.

Define $g: c_0 \rightarrow c_0$ by $g((a_n)_{n \in \mathbb{N}}) = (a_n/n)_{n \in \mathbb{N}}$. Then $g$ is in $B^*$ and $|g| = 1 \left( |g((1,0,0,\ldots))| = |(1,0,0,\ldots)| = 1 \right)$. Clearly $g$ is one-to-one and since $g(a) \neq (1/n)_{n \in \mathbb{N}}$ for any $a \in c_0$, $g$ is not an onto map. However, $g$ is an epimorphism, for its range contains the dense subspace of $c_0$, $c_0^\infty$, the space of all sequences with, at most, a finite number of non-zero terms. Therefore, $g$ is both a monomorphism and an epimorphism with $|g| = 1$. However $g$ is not an isomorphism, as it has no inverse in $B^*$. To see this, let $b_0 = (0,0,0,2,0,0,\ldots)$. Then $g(b_0) = (0,0,0,\frac{1}{2},0,0,\ldots)$. If $g^{-1}$ existed in $B^*$, then

$$|g^{-1}| = \sup_{|a| = 1} |g^{-1}(a)| \geq |g^{-1}(g(b_0))| = |b_0| = 2.$$ 

This is a contradiction.

The following proposition is proved in Mitchell [3, p 17].

**Proposition 2.16** Let $f:A \rightarrow B$ be a monomorphism with co-kernel 0 in a normal category. Then $f$ is an isomorphism. Hence, a normal category is balanced.

This proposition shows that since $B^*$ is not balanced, it is not normal.
Definition 3.1 In $\mathbb{B}'$, let $\{B_i\}_I$ be a family of objects indexed by a set $I$. Then $\bigotimes_{i} - \text{join}(\{B_i\}_I)$ is the subset of the Cartesian product of $\{B_i\}_I$ consisting of all elements $f$ such that $|f|_1 = \sum_{i} |f_i| < \infty$.

Proposition 3.2 With $| \cdot |_1$ as norm $\bigotimes_{i} - \text{join}(\{B_i\}_I)$ is a Banach space with addition and scalar multiplication defined component-wise, that is, $(f + g)_i = f_i + g_i$ and $(kf)_i = kf_i$ for $k \in \mathbb{K}$.

Proof: If $f \in \bigotimes_{i} - \text{join}(\{B_i\}_I)$ and $g \in \bigotimes_{i} - \text{join}(\{B_i\}_I)$, then $f + g \in \bigotimes_{i} - \text{join}(\{B_i\}_I)$ as:

$$|f + g|_1 = \sum_{i} |f_i + g_i| \leq \sum_{i} |f_i| + |g_i|$$

$$= \sum_{i} |f_i| + \sum_{i} |g_i| < \infty.$$ 

Also for $\alpha \in \mathbb{K}$, $\alpha f \in \bigotimes_{i} - \text{join}(\{B_i\}_I)$ as

$$|\alpha f|_1 = \sum_{i} |\alpha f_i| = |\alpha| \sum_{i} |f_i| < \infty.$$ 

Thus it can be immediately seen that for $g, f \in \bigotimes_{i} - \text{join}(\{B_i\}_I)$ the following are true:

(i) $|f|_1 \geq 0$; $|f|_1 = 0$ if and only if $f_i = 0$ for all $i \in I$ 

(ii) $|\alpha f|_1 = |\alpha| |f|_1$

(iii) $|f + g|_1 \leq |f|_1 + |g|_1$.

Thus $\bigotimes_{i} - \text{join}(\{B_i\}_I)$ forms a normed linear space.

This normed linear space is also complete in its norm topology. Let $(f^n)_N$ be a Cauchy sequence in $\bigotimes_{i} - \text{join}(\{B_i\}_I)$.
For each $\varepsilon > 0$, there exists $N_0 > 0$ such that for $m,n > N_0$,
\[ |f_n - f_m|_1 = \sum_{i \in I} |f^n_i - f^m_i| < \varepsilon. \]

Therefore for each $i \in I$, $|f^n_i - f^m_i| < \varepsilon$ and $(f^n_i)$ is a Cauchy sequence in $B_i$. Since $B_i$ is complete, $f^n_i \rightarrow g_i$, $g_i \in B_i$. It shall be shown that $g : I \rightarrow \bigcup_{i \in I} B_i$ defined by $g(i) = g_i$ is in $\bigcup_{i \in I} \text{Join}(\{B_i\}_i)$ and $f^n \rightarrow g$. Since for $n,m > N_0$ and any finite subset $\mathcal{F} \subseteq I$,
\[ \sum_{i \in \mathcal{F}} |f^n_i - g_i| = \lim_{m \to \infty} \sum_{i \in \mathcal{F}} |f^n_i - f^m_i| < \varepsilon, \]
then $\sum_{i \in \mathcal{F}} |f^n_i - g_i| < \varepsilon$ or $f^n \rightarrow g$. Also as
\[ |g|_1 = \sum_{i \in I} |g_i| \leq \sum_{i \in I} |g_i - f^n_i| + \sum_{i \in I} |f^n_i| < \varepsilon + \sum_{i \in I} |f^n_i| < \infty \text{ for some } n > N_0, \]
$g \in \bigcup_{i \in I} \text{Join}(\{B_i\}_i)$. This completes the proof.

**Definition 3.3** Let $\{A_i\}_i$ be a family of objects in a category $A$ indexed by a set $I$. The **coproduct** (sum) of $\{A_i\}_i$ is an object $\sum_i A_i$ in $A$ together with a family of morphisms $\{u_i : A_i \rightarrow \sum_i A_i\}_i$ such that for any family of morphisms $\{x_i : A_i \rightarrow X\}_{i \in I}$, $X$ an object in $A$, there exists a unique morphism $u : \sum_i A_i \rightarrow X$ such that $u \circ u_i = x_i$, for all $i \in I$.

![Diagram](image)

**Remark 3.4** The objects of any two coproducts of the
family, \( \{ A_i \}_{i \in I} \), are isomorphic.

**Proposition 3.5** In \( \mathcal{B} \) the coproduct of the family of objects \( \{ B_i \}_{i \in I} \) is the space \( \bigoplus_{i \in I} - \text{join}(\{ B_i \}_{i \in I}) \) together with the family of morphisms \( \{ u_i : B_i \rightarrow \bigoplus_{i \in I} - \text{join}(\{ B_i \}_{i \in I}) \}_{i \in I} \) where for \( t \in I \), \( u_t(b_t) = f \) in the Cartesian product of \( \{ B_i \}_{i \in I} \) defined by \( f_i = 0 \) if \( i \neq t \), \( f_t = b_t \).

**Proof:** For all \( i \in I \), \( u_i \in \mathcal{B} \) as each is linear, and

\[
|u_i| = \sup_{|b_i| \leq 1} |u_i(b_i)| = \sup_{|b_i| \leq 1} \sum_{i \in I} |f_i| = \sup_{|b_i| \leq 1} |b_i| \leq 1.
\]

Now let \( X \) be any Banach space and \( \{ x_i : B_i \rightarrow X \}_{i \in I} \) a family of morphisms in \( \mathcal{B} \). A unique morphism \( u : \bigoplus_{i \in I} - \text{join}(\{ B_i \}_{i \in I}) \rightarrow X \) must be found so that for all \( i \in I \) a diagram like that above commutes.

Define \( u : \bigoplus_{i \in I} - \text{join}(\{ B_i \}_{i \in I}) \rightarrow X \) by \( u(f) = \sum_{i \in I} x_i(f_i) \).

Since \( \sum_{i \in I} |x_i(f_i)| \leq \sum_{i \in I} |f_i| < \infty \), \( \sum_{i \in I} x_i(f_i) \) converges and the function is well-defined. Also \( u \) is linear since for \( f, g \in \bigoplus_{i \in I} - \text{join}(\{ B_i \}_{i \in I}) \) and \( \alpha \in \mathbb{F} \),

\[
u(f + g) = \sum_{i \in I} x_i(f_i + g_i) = \sum_{i \in I} x_i(f_i) + x_i(g_i),
\]

and

\[
u(\alpha f) = \sum_{i \in I} x_i(\alpha f_i) = \sum_{i \in I} \alpha x_i(f_i) = \alpha \sum_{i \in I} x_i(f_i) = \alpha u(f).
\]

The norm of \( u \) satisfies \( |u| \leq 1 \) as

\[
|u| = \sup_{|f_i| \leq 1} |u(f)| = \sup_{|f_i| \leq 1} |\sum_{i \in I} x_i(f_i)| = \sup_{|f_i| \leq 1} (\sum_{i \in I} |f_i|) \leq 1.
\]

Also for all \( i \in I \), \( u \circ u_i = x_i \). Let \( t \in I \). Then for \( b_t \in B_t \),

\[
(u \circ u_t)(b_t) = u(f) = \sum_{i \in I} x_i(f_i) = x_t(b_t).
\]

The map \( u \) is unique. Let \( f \in \bigoplus_{i \in I} - \text{join} \); then \( f \) can
be written as \( \sum_I u_\ell(f_\ell) \). If \( u': l_1^\infty - \text{join} \to X \) is any map in \( B^* \) such that \( u' \circ u^1 = x^1 \) for all \( i \in I \), then as \( u' \) is continuous (see Remark 1.10),

\[
u'(f) = u'(\sum_I u_\ell(f_\ell)) = \sum_I x_\ell(f_\ell) = u(f).
\]

**Definition 3.6** In \( B^* \), let \( \{B_i\}_I \) be a family of Banach spaces indexed by \( I \). Then \( l_\infty^\infty - \text{join}(\{B_i\}_I) \) is the subset of the Cartesian product of \( \{B_i\}_I \) consisting of all elements \( f \) such that \( |f|_\infty = \sup_{i \in I} |f_\ell| < \infty \).

**Proposition 3.7** \( l_\infty^\infty - \text{join}(\{B_i\}_I) \) is a Banach space with addition and scalar multiplication defined component-wise and \( |\cdot|_\infty \) as norm.

**Proof:** If \( g, f \in l_\infty^\infty - \text{join}(\{B_i\}_I) \), then \( f + g \in l_\infty^\infty - \text{join}(\{B_i\}_I) \) as

\[
|f + g|_\infty = \sup_I |f_\ell + g_\ell| \leq \sup_I (|f_\ell| + |g_\ell|) \\
\leq \sup_I |f_\ell| + \sup_I |g_\ell| < \infty.
\]

Also for \( \alpha \in \mathbb{F}, \alpha f \in l_\infty^\infty - \text{join}(\{B_i\}_I) \) as

\[
|\alpha f|_\infty = \sup_I |\alpha f_\ell| = |\alpha| \sup_I |f_\ell| = |\alpha||f|_\infty < \infty.
\]

From this it can be seen that

(i) \( |f|_\infty \geq 0; \ |f|_\infty = 0 \) if and only if \( f_\ell = 0 \) for all \( i \in I \).

(ii) \( |\alpha f|_\infty = |\alpha||f|_\infty \) for all \( \alpha \in \mathbb{F} \).

(iii) \( |f + g|_\infty \leq |f|_\infty + |g|_\infty \)

Thus \( l_\infty^\infty - \text{join}(\{B_i\}_I) \) forms a normed linear space.

This normed linear space is also complete in its norm topology. Let \( (f^n)_N \) be a Cauchy sequence in
\[ l_\infty - \text{join}(\{B_i\}_i). \] For each \( \varepsilon > 0 \), there exists \( N_0 > 0 \) such that for \( n,m > N_0 \),
\[ |f^n_m - f^n_m|_\infty = \sup_I |f^n_i - f^n_i| < \varepsilon. \]

Therefore for each \( i \in I \), \( |f^n_i - f^n_i| < \varepsilon \) and \( (f^n_i)_N \) is a Cauchy sequence in \( B_i \). Since \( B_i \) is complete, \( f^n_i \rightarrow g_i \) for \( g_i \in B_i \). It will be shown that \( g: I \rightarrow \bigcup B_i \), defined by \( g(i) = g_i \), is in \( l_\infty - \text{join}(\{B_i\}_i) \); and \( f^n \rightarrow g \). Since \( |f^n|_\infty < \infty \), for each \( n \), there exists \( M_n \) such that \( |f^n_i| < M_n - \varepsilon \) for all \( i \in I \). Pick \( m,n > N_0 \). Then
\[ |g_i| = |g_i - f^n_i| + |f^n_i - f^m_i| + |f^m_i| \]
\[ \leq |g_i - f^n_i| + |f^n_i - f^m_i| + |f^m_i| \]
\[ \leq |g_i - f^n_i| + \varepsilon + M_m - \varepsilon. \]

Letting \( n \rightarrow \infty \), since the left side is independent of \( n \) and \( |g_i - f^n_i| \rightarrow 0 \), \( g_i \leq M \) or \( g \in l_\infty - \text{join}(\{B_i\}_i) \).

Now \( f^n \rightarrow g \) if and only if for all \( \varepsilon > 0 \), there exists \( N \) such that for \( n > N \), \( |f^n_i - g_i| < \varepsilon \) for all \( i \in I \). Pick \( n,m > N \). Then for all \( i \in I \)
\[ |f^n_i - g_i| \leq |f^n_i - f^n_i| + |f^n_i - g_i| < \varepsilon + |f^n_i - g_i|. \]

Letting \( m \rightarrow \infty \), \( |f^n_i - g_i| < \varepsilon \) and the result follows.

\textbf{Definition 3.8} Let \( \{A_i\}_I \) be a family of objects in a category \( \mathcal{A} \). The \textit{product} of \( \{A_i\}_I \) is an object, \( \prod A_i \), in \( \mathcal{A} \) together with a family of morphisms, \( \{p_i:\prod A_i \rightarrow A_i\}_I \), such that for any family of morphisms \( \{x_i:X \rightarrow A_i\}_I \), \( X \) an object in \( \mathcal{A} \), there is a unique morphism \( u:X \rightarrow \prod A_i \) such that \( p_i u = x_i \) for all \( i \in I \).
Remark 3.9 The objects of any two products of the family \( \{A_i\}_I \) are isomorphic.

Proposition 3.10 In \( B^* \), \( \bigwedge_{\infty} \text{join}(\{B_i\}_I) \) together with the family, \( \{p_i: \bigwedge_{\infty} \text{join}(\{B_i\}_I) \rightarrow B_i \}_{i \in I} \), where for \( i \in I \), \( p_i(f) = f_i \) is the product of the family \( \{B_i\}_I \).

Proof: For all \( i \in I \), \( p_i \in B^* \) as

\[
|p_i| = \sup_{|f| \leq 1} |p_i(f)| = \sup_{|f| \leq 1} |f_i| = \sup_{|f| \leq 1} |f| \leq 1.
\]

Let \( X \) be any object in \( B^* \) and \( \{x_i : X \rightarrow B_i \}_{i \in I} \) be a family of morphisms in \( B^* \). Define \( u : X \rightarrow \bigwedge_{\infty} \text{join}(\{B_i\}_I) \) by \( u(x) = f \) where \( f_i = x_i(x) \). Then \( u \in B^* \) as

\[
|u| = \sup_{|x| \leq 1} |u(x)| = \sup_{|x| \leq 1} |f|_{\infty} = \sup \sup_{|x| \leq 1} |x_i(x)|
\]

\[
= \sup \sup_{|x| \leq 1} |x_i(x)| \leq 1.
\]

Also it is clear that \( p_i \circ u = x_i \) for each \( i \in I \). If \( u' \) is any map \( X \rightarrow \bigwedge_{\infty} \text{join}(\{B_i\}_I) \) such that \( p_i \circ u' = x_i \) for all \( i \in I \), then \( u' = u \); since for all \( i \in I \), \( (u'(x))_i = (p_i \circ u')(x) = (p_i \circ u)x = (u(x))_i \), that is, each component of \( u'(x) \) and \( u(x) \) are equal; and hence \( u'(x) = u(x) \).

It should be noted that the fact that the product and coproduct in \( B^* \) are \( \bigwedge_{\infty} \text{join} \) and \( \bigwedge_1 \text{join} \) respectively is given by Semadeni [5, p. 559].

Definition 3.11 Let \( f_1 : A_1 \rightarrow A \) and \( f_2 : A_2 \rightarrow A \) be morphisms
in a category \( \mathcal{A} \). A commutative diagram,

\[
\begin{array}{ccc}
P & \xrightarrow{g_2} & A_2 \\
\downarrow{g_1} & & \downarrow{f_2} \\
A_1 & \xrightarrow{f_1} & A
\end{array}
\]

with \( g_2 \in \text{hom}_\mathcal{A}(P_1, A_2) \) and \( g_1 \in \text{hom}_\mathcal{A}(P, A_1) \), is called a pullback for \( f_1 \) and \( f_2 \) if for every pair of morphisms \( g_1: P' \to A_1 \) and \( g_2: P' \to A_2 \) in \( \mathcal{A} \) such that \( f_1 \circ g_1 = f_2 \circ g_2 \), there exists a unique morphism, \( u: P' \to P \), such that \( g_1 = g_1 \circ u \) and \( g_2 = g_2 \circ u \).

**Remark 3.12** If

\[
\begin{array}{ccc}
P' & \xrightarrow{g_1'} & A_1 \\
\downarrow{g_2'} & & \downarrow{f_2} \\
A_1 & \xrightarrow{f_1} & A
\end{array}
\]

is also a pullback for \( f_1 \) and \( f_2 \), then \( P' \) and \( P \) are isomorphic.

**Proposition 3.13** In \( \mathcal{B} \), the diagram,

\[
\begin{array}{ccc}
P & \xrightarrow{g_2} & B_2 \\
\downarrow{g_1} & & \downarrow{f_2} \\
B_1 & \xrightarrow{f_1} & B
\end{array}
\]

where \( P = \{ b = (b_1, b_2) \mid \prod_{i=1}^{2} B_i | f_1(b_1) = f_2(b_2) \} \) and \( g_1 \) and \( g_2 \) are the restrictions of the projections from \( \prod_{i=1}^{2} B_i \) to \( B_1 \) and \( B_2 \) respectively, is a pullback of \( f_1 \) and \( f_2 \).

**Proof:** As a closed subspace of \( \prod_{i=1}^{2} B_i \), \( P \) is itself a Banach space. To see that \( P \) is closed, let \( b \) be in the closure of \( P \) and \( b^n \) a sequence in \( P \) converging to \( b \). Then the
coordinates $b^n_1$ and $b^n_2$ of $b^n$ converge to the coordinates of $b$, $b_1$, and $b_2$ respectively. However, $b \in P$ as

$$f_1(b_1) = f_1(\lim_n b^n_1) = \lim_n f_1(b^n_1)$$

$$= \lim_n f_2(b^n_2) = f_2(b_2).$$

Since $g^1$ and $g^2$ are restrictions to $P$ of the respective projections, each is in $B^*$. Now let $g^1_1 : P' \rightarrow B_1$ and $g^1_2 : P' \rightarrow B_2$ be two morphisms in $B^*$ such that $f_1 \circ g^1_1 = f_2 \circ g^1_2$. Define $u : P' \rightarrow P$ by $u(p') = (g^1_1(p'), g^1_2(p'))$.

Then $u$ is well-defined as $f_1(g^1_1(p')) = f_2(g^1_2(p'))$ because $f_1 \circ g^1_1 = f_2 \circ g^1_2$. Also $u \in B^*$ since

$$|u| = \sup |u(p)| = \sup \sup |g^1_i(p)|$$

$$\sup \sup \sup \sup |g^1_i(p)| \leq 1$$

Also $g^1 \circ u = g^1_i$, $i = 1, 2$, as

$$(g^1 \circ u)p = g^1_i(g^1_1(p), g^1_2(p)) = g^1_i(p).$$

If $u' : P' \rightarrow P$, $u' \neq u$, also satisfies the conditions $g^1 \circ u' = g^1_i$, $i = 1, 2$; then for some $p' \in P'$, $u(p') \neq u'(p)$.

This means that for $i = 1$ or $2$, $g^1_i(p') = g^1_i(u'(p')) \neq g^1_i(u(p')) = g^1_i(p')$. This is a contradiction.

**Definition 3.14** Let $f_1 : A \rightarrow A_1$ and $f_2 : A \rightarrow A_2$ be morphisms in a category $\mathcal{A}$. A commutative diagram,

$$\begin{array}{ccc}
A & \xrightarrow{f_2} & A_2 \\
\downarrow{f_1} & \downarrow{g_2} & \downarrow{g_1} \\
A_1 & \xrightarrow{g_1} & P
\end{array}$$

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with $g_1 \in \text{hom}_A(A_1, P)$ and $g_2 \in \text{hom}_A(A_2, P)$ is called a \textbf{pushout} for $f_1$ and $f_2$ if for every pair of morphisms $g_1': A_1 \rightarrow P'$ and $g_2': A_2 \rightarrow P'$ in $\mathcal{A}$ such that $g_1' \circ f_1 = g_2' \circ f_2$, there exists a unique morphism $u: P \rightarrow P'$, such that $u \circ g_1 = g_1'$ and $u \circ g_2 = g_2'$.

\textbf{Remark 3.15} If

is also a pushout for $f_1$ and $f_2$, then $P'$ and $P$ are isomorphic.

\textbf{Proposition 3.16} In $\mathcal{B}'$, the diagram,

\begin{equation}
\begin{array}{ccc}
A & \xrightarrow{f_2} & A_2 \\
\downarrow f_1 & & \downarrow g_2' \\
A_1 & \xrightarrow{g_1'} & P'
\end{array}
\end{equation}

where $P = \sum_{i=1,2} A_i / I$, $I = \{(x, y) \in \sum_{i=1,2} A_i \mid \text{there exists } a \in A \text{ such that } f_1(a) = -x \text{ and } f_2(a) = y\}$, and $g_1: A_1 \rightarrow P$ is defined by $g_1(a_1) = [(a_1, 0_2)]$ while $g_2: A_2 \rightarrow P$ is defined by $g_2(a_2) = [(0_1, a_2)]$, is the pushout of $f_1$ and $f_2$.

\textbf{Proof:} Let $(x_1, x_2) \in I$ and $(y_1, y_2) \in I$ with $f_1(a) = -x_1$, $f_2(a) = x_2$, $f_1(b) = -y$, and $f_2(b) = y_2$. Then $(x_1, x_2) + (y_1, y_2) \in I$ as $f_1(a + b) = -(x_1 + y_1)$ and $f_2(a + b) = (x_2 + y_2)$. Similarly for $\alpha \in \mathcal{F}$, $\alpha(x_1, x_2) \in \mathcal{F}$. Thus $I$ is a subspace of $\sum_{i=1,2} A_i$. Therefore $\sum_{i=1,2} A_i / I$ is a Banach space with norm $\|([x, y])\| = \inf \{\|x'\| \mid ([x', y']) \in ([x, y]) \times \{y'\}\}$. 

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The morphisms \( g_1 \) and \( g_2 \) are in \( \mathcal{B}^* \) since
\[
|g_1| = \sup |g_1(a_1)| = \sup |(a_1, a_2)| = \sup \inf(|x| + |y|) \leq \sup |a_1| \leq 1.
\]
A similar inequality involving \( g_2 \) can be formed. Also
\[
g_2 \circ f_2 = g_1 \circ f_1 \text{ as } g_1(f_1(a)) = [(f_1(a), 0_2)] = g_2(f_2(a)) \text{ since } (f_1(a), 0_2) = (0_1, f_2(a_2)) \in \mathcal{I}, \text{ for all } a \in A.
\]
Now let \( g'_1 : A_1 \rightarrow P' \) and \( g'_2 : A_2 \rightarrow P' \) be morphisms in \( \mathcal{B}^* \) such that \( g'_1 \circ f_1 = g'_2 \circ f_2 \). Define \( u : P \rightarrow P' \) by \( u([a_1, a_2]) = g'_1(a_1) + g'_2(a_2) \). Note that \( u \) is well-defined for if
\[
[(a_1, a_2)] = [(b_1, b_2)], \text{ then there exists } a \in A \text{ such that } f_1(a) = a_1 - b_1 \text{ and } f_2(a) = b_2 - a_2.
\]
Therefore
\[
g'_1(a_1) - g'_1(b_1) = g'_1(a_1 - b_1) = g'_1(f_1(a)) = g'_2(f_2(a)) = g'_2(b_2) - g'_2(a_2)
\]
which implies \( g'_1(a_1) + g'_2(a_2) = g'_1(b_1) + g'_2(b_2) \). Also
\[
|u| \leq 1. \text{ To understand this it must be noted that for all } (a_1, a_2) \in [(b_1, b_2)] \triangleq [(0_1, 0_2)],
\]
\[
\frac{|u([b_1, b_2])|}{|a_1, a_2|} = \frac{|u([a_1, a_2])|}{|a_1, a_2|} = \frac{|g'_1(a_1) + g'_2(a_2)|}{|a_1| + |a_2|} \leq \frac{|a_1| + |a_2|}{|a_1| + |a_2|} \leq 1.
\]
Hence
\[
\frac{|u([b_1, b_2])|}{|a_1, a_2|} \leq 1.
\]
Therefore
\[
\frac{|u([b_1, b_2])|}{|a_1, a_2|} \leq 1.
\]
It is clear that \( u \) satisfies the conditions \( u \circ g_1 = g'_1 \), \( i = 1, 2 \). If \( u' : P \rightarrow P' \) is any map satisfying these conditions,
then \((u' \circ g_1)(a_1) = u'(g_1(a_1)) = u'[(a_1, 0_2)] = g_1'(a_1)\)
and \((u' \circ g_2)(a_2) = u'(g_2(a_2)) = u'[(0_1, a_2)] = g_2'(a_2)\).
Hence \(u'[[a_1, 0_2]] + [0_1, a_2]] = u'[(a_1, a_2)] = g_1'(a_1) + g_2'(a_2)\),
and \(u' = u\).

4. UNIONS AND INTERSECTIONS
PROJECTIVE AND INJECTIVE OBJECTS

**Definition 4.1** Let \(f:A' \to A\) and \(g:B' \to B\) be monomorphisms and \(h:A \to B\) any morphism (refer to diagram below). Subobject \(A'\) is said to be **carried into** the subobject \(B'\) by \(h\) if there exists a morphism \(u:A' \to B'\) such that \(g \circ u = h \circ f\).

\[
\begin{array}{ccc}
A' & \to & B' \\
\downarrow f \downarrow & & \downarrow g \\
A & \to & B \\
\downarrow h \\
B
\end{array}
\]

**Definition 4.2** Let \(\{A_i\}_{i \in I}\) be a family of subobjects of \(A\) corresponding to monomorphisms \(\{f_i\}_{i \in I}, f_i:A_i \to A\). The union of \(\{A_i\}_{i \in I}\) is a subobject \(A'\) of \(A\) corresponding to some monomorphism \(g:A' \to A\) such that

(i) for each \(A_i\) there exists \(g_i:A_i \to A'\) such that \(g \circ g_i = f_i\).

(ii) if \(f:A \to B\) is any morphism to an object \(B\) and each \(A_i\) is carried into some \(B'\) of \(B\) by \(f\), then \(A'\) is also carried into \(B'\) by \(f\).

All these morphisms are shown in the following diagram.
**Remark 4.3** Any other subobject of $A$ which behaves as a union of the family $\{A_i\}_I$ must be isomorphic to $A'$.

**Proposition 4.4** $B'$ has unions; that is, a union exists for every family of subobjects of any object $A$ in $B'$.

**Proof:** Let $\{A_i\}_I$ be a family of subobjects of a Banach space $A$ with corresponding monomorphisms $\{f_i:A_i \rightarrow A\}_I$.

Define $g': \sum A_i \rightarrow A$ by $g'(a) = \sum f_i(a_i)$.

Since $\sum f_i(a_i)$ is absolutely convergent, it is convergent; and $|g'| \leq 1$ as $|\sum f_i(a_i)| \leq \sum |f_i(a_i)| \leq \sum |a_i|$. Using remark 1.9 above, $g'$ is also linear. Therefore $g'$ is in $B'$. Now define $g: \sum A_i/\text{Ker } g' \rightarrow A$ by $g([a]) = g'(a)$. By lemma 1.11 above $g$ is in $B'$. Also $g$ is a monomorphism, for if $g([a]) = g([b])$ then $g'(a) = g'(b)$ or $[a] = [b]$. Therefore $A' = \sum A_i/\text{Ker } g'$ is a subobject of $A$.

Now for each $k \in I$ and for all $x_k \in A_k$ define $g_k:A_k \rightarrow A'$ by $g_k(x_k) = [a]$ where $a_1 = 0$ if $1 \nmid k$ and $a_k = x_k$. Since $|g_k| = \sup |g_k(x_k)| = \sup |[a]| = \sup \inf |b| \leq \sup |x_k| = 1$, $g$ is in $B'$. Clearly for all $i \in I$, $g_ig_i = f_i$.

Now let $f:A \rightarrow B$ be a morphism in $B'$ and $B'$ a subobject of $B$ with respect to a monomorphism $h:B' \rightarrow B$ such
that each $A_i$ is carried by $f$ to $B'$, that is, for all $i \in I$ there exists $h_i:A_i \rightarrow B'$ such that $f \circ h_i = h \circ h_i$. Define $u:A' \rightarrow B'$ by $u([a]) = \sum h_i(a_i)$. This is a well-defined function, as $\sum h_i(a_i)$ is absolutely convergent and hence convergent. Also if $[a] = [b]$, then $\sum f_i(a_i) - f_i(b_i) = g'(a) - g'(b) = \sum f_i(b_i)$. Hence $f(\sum f_i(a_i) - f_i(b_i)) = 0$, which means $\sum f_i(a_i) = \sum f_i(b_i)$, for

$$h(\sum h_i(a_i - b_i)) = \sum h_i(a_i - b_i) = 0,$$

since $h$ is a monomorphism. In this same way $h \circ u = f \circ g$, for

$$(f \circ g)([a]) = f(g'(a)) = f(\sum f_i(a_i)) = \sum f_i(a_i) = \sum h_i(a_i)) = (h \circ u)([a]).$$

Thus $A'$ is also carried by $f$ into $B'$.

**Definition 4.5** Let $\{f_i:A_i \rightarrow A\}_{i \in I}$ be a family of subobjects of $A$ in $\mathbf{A}$. A morphism $f:A' \rightarrow A$ is called the **intersection** of the family if

1. for each $i \in I$, $f = f_i \circ h_i$ for some morphism $g_i:A' \rightarrow A_i$ (necessarily unique), and
2. for every morphism $h:A'' \rightarrow A$ such that for all $i$, $h = f_i \circ h_i$ for some $h_i:A'' \rightarrow A_i$ there exists a unique morphism $g:A'' \rightarrow A'$ such that $h = f \circ g$. 

\[\begin{array}{ccc}
A' & \xrightarrow{f} & A \\
\downarrow{g} & & \downarrow{h} \\
A'' & \xrightarrow{h} & A_i & \xrightarrow{f_i} & A \\
\end{array}\]
Remark 4.6 From the uniqueness of \( g \) it can be shown that if \( h: A' \to A \) is also an intersection for the family 
\[ \{ f_i: A_i \to A \}_{i \in I}, \]
then \( A' \) and \( A'' \) are isomorphic.

Proposition 4.7 \( B^* \) has intersections; that is, the intersection exists for every family of subobjects of any object in \( B^* \).

Proof: Let \( \{ f_i: A_i \to A \}_{i \in I} \) be a family of subobjects of \( A \).
Let \( A' = \{ b \in \prod_i A_i \mid \text{for all } i, j \in I, f_i(b_i) = f_j(b_j) \} \).
As a closed subspace of \( \prod_i A_i \), \( A' \) is also a Banach space.
To see that \( A' \) is closed, let \( A'_{k_j} = \{ b \in \prod_i A_i \mid f_k(b_k) = f_j(b_j) \} \). Each \( A'_{k_j} \) is closed (see proposition 3.13) and \( A' = \bigcap_{k_j} A'_{k_j} \). Now let \( p \) be a fixed element of \( I \) and define \( f: A' \to A \) by \( f(b) = f_p(b_p) \). The function \( f \) is a morphism in \( B^* \) since \( f_p \) linear makes \( f \) linear, and
\[
|f| = \sup_{|b| \leq 1} |f(b)| = \sup_{|b| \leq 1} |f_p(b_p)| \leq \sup_{|b_p| \leq 1} |f_p(b_p)| \leq 1.
\]
If for each \( i \in I \), \( g_i: A' \to A_i \) is the restriction of the projection from \( \prod_i A_i \to A_i \) then \( f = (f_1 \circ g_1) \), then \( f = (f_1 \circ g_1) \) since \( f_i(b_i) = f_p(b_p) \) for all \( i \in I \).

Now let \( h: A' \to A \) be a morphism such that for all \( i \in I \), \( h = f_1 \circ h_i \) for some \( h_i: A'' \to A_i \). Then for all \( a \in A'' \), \( \bar{a} \in A' \) where \( \bar{a}_i = h_i(a) \) since for all \( i, j \in I \),
\[
f_i(\bar{a}_i) = f_i(h_i(a)) = h(a) = f_j(h_j(a)) = f_j(\bar{a}_j).
\]
Define \( g: A'' \to A' \) by \( g(a) = \bar{a} \). Because of the inequality
\[
|g| = \sup_{|a| \leq 1} |\bar{a}| = \sup_{|a| \leq 1} \sup_{i \in I} |h_i(a)| = \sup_{|a| \leq 1} \sup_{i \in I} |h_i(a)| \leq 1,
\]
g is a morphism in \( B^* \). The equality
\[(f \circ g)a = f(a) = f_p(h_p(a)) = h(a)\]

shows \(f \circ g = h\).

Now if \(f(b) = f(c)\), that is, \(f_p(b_p) = f_p(c_p)\); then for all \(i \in I\), \(f_i(b_i) = f_i(c_i)\). Since each \(f_i\) is a monomorphism, \(c_i = b_i\) for all \(i \in I\). Therefore \(b = c\) and \(f\) is a monomorphism. This fact shows that if \(g':A' \to A\) is any morphism such that \(f \circ g' = h\), then \(g = g'\) as \(f \circ g = f \circ g'\).

**Definition 4.8** A category \(A\) with coproducts is called a \(C_1\) category if for every family of monomorphisms \(\{u_i:A_i \to B_i\}_I\) the unique morphism \(u: \sum A_i \to \sum B_i\) such that the following diagram commutes is a monomorphism.

Since the unique map \(u: \sum A_i \to \sum B_i\) is defined by \(u(a) = b\) where \(b_i = u_i(a_i)\) and \(u\) is a monomorphism, \(B\) is a \(C_1\) category.

**Definition 4.9** A category \(A\) is called a \(C_2\) category if it has products, coproducts, and a zero object and if the morphism \(f: \prod A_i \to A\) so that the diagram

\[
\begin{array}{ccc}
\sum A_i & \xrightarrow{f} & \prod A_i \\
\downarrow p' & & \downarrow p_j \\
A_j & & \\
\end{array}
\]
commutes for all \( j \in I \) is a monomorphism (\( p_j \) and \( p'_j \) are the projections from \( \prod_i A_i \) and \( \sum_i A_i \) respectively).

\( \mathcal{B} \) is a \( \mathcal{C}_2 \) category with the unique map \( f \) given by
\[ f(b) = b \] which is clearly a monomorphism.

**Definition 4.10** An object \( P \) in a category \( \mathcal{A} \) is **projective** if for every epimorphism \( f: A \rightarrow B \) and morphism \( g: P \rightarrow B \), there exists a morphism \( h: P \rightarrow A \) such that \( f \circ h = g \).

\[ \begin{array}{ccc}
A & \rightarrow & P \\
\downarrow h & & \downarrow g \\
B & \rightarrow & B
\end{array} \]

**Definition 4.11** An object \( Q \) in a category \( \mathcal{A} \) is **injective** if for every monomorphism \( f: A \rightarrow B \) and morphism \( g: A \rightarrow Q \), there exists a morphism \( h: B \rightarrow Q \) such that \( h \circ f = g \).

\[ \begin{array}{ccc}
A & \rightarrow & Q \\
\downarrow h & & \downarrow f \\
B & \rightarrow & B
\end{array} \]

**Definition 4.12** A morphism \( f: A \rightarrow B \) is called a **coretraction** if there is a morphism \( g: B \rightarrow A \) such that \( g \circ f = 1_A \). Then \( A \) is called a **retract** of \( B \). The morphism \( f \) is called a **retraction** if there is a morphism \( g': B \rightarrow A \) such that \( f \circ g' = 1_B \).

The following two lemmas are established in categorical algebra [3, p.70].

**Lemma 4.13** If for objects \( A_1 \) and \( A_2 \) in a category \( \mathcal{A} \), \( A_1 \) is a retract of \( A_2 \) and \( A_2 \) is injective (projective),
then $A_1$ is injective (projective).

**Lemma 4.14** If $A = \prod_{i} A_i$, and each $A_i$ is injective (projective), then $A$ is injective (projective). Conversely, in a category with zero if $A$ is injective (projective), then each $A_i$ is injective (projective).

**Lemma 4.15** The scalar field $\mathcal{F}$ is a retract of every non-zero Banach space.

**Proof:** Let $B$ be a nonzero Banach space. Fix $b_0 \in B$,

$b_0 \neq 0$. Let $b^* : B \to \mathcal{F}$ be a morphism with $|b^*_0| = 1$ and $b^*_0(b_0) = |b_0|$. Then $b^*_0(b_0/|b_0|) = 1$. Define $f : \mathcal{F} \to B$ by $f(m) = mb_0/|b_0|$. Then $f$ is a morphism in $B^*$ and $b^*_0f = 1$ (the identity morphism of $\mathcal{F}$). Therefore $\mathcal{F}$ is a retract of $B$.

**Lemma 4.16** The scalar field $\mathcal{F}$ is not injective in $B^*$.

**Proof:** Let $\ell_1$ be the Banach space consisting of sequences $a = (a_i)_{i \in \mathbb{N}}$, $a_i \in \mathcal{F}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$, such that the norm $\|a\| = \sum_{i \in \mathbb{N}} |a_i| < \infty$. Define $g : \ell_1 \to \ell_1$ by $g((a_i)_{i \in \mathbb{N}}) = (a_i/1 + 1)_{i \in \mathbb{N}}$. The map $g$ is a monomorphism in $B^*$. Let $b = (1/i!)_{i \in \mathbb{N}}$. Since $\|b\| = \sum_{i \in \mathbb{N}} 1/i! = e - 1$, $b \in \ell_1$. Now let $b^* : \ell_1 \to \mathcal{F}$ be a linear functional with $|b^*| = 1$ and $b^*(b) = |b|$. There is no morphism $u : \ell_1 \to \mathcal{F}$ in $B^*$ such that $u \circ g = b^*$, that is, so that the following diagram commutes.

![Diagram](image)
If such a morphism $u$ existed in $B^*$, then $(u \circ g)(b)$ must be $|b|$. However $|g(b)| = e - 2 < 1$ but $|u(g(b))| = |b| > 1$. This is a contradiction.

**Lemma 4.17** The scalar field $\mathbb{F}$ is not projective in $B^*$.

**Proof:** Let $c_0$ be the Banach space consisting of sequences $a = (a_i)_{i \in \mathbb{N}}$, $a_i \in \mathbb{F}$, and $N = \{1, 2, \ldots\}$ converging to zero with norm $|a| = \sup_{i \in \mathbb{N}} |a_i|$. Define $g: c_0 \to c_0$ by $g((a_i)_{i \in \mathbb{N}}) = (a_i/1)_{i \in \mathbb{N}}$. The map $g$ is an epimorphism and a monomorphism in $B^*$. Let $b$ be the sequence $(0, 2, 0, 0, \ldots)$ in $c_0$. Then $g(b) = (0, 1, 0, \ldots)$. Define $f: \mathbb{F} \to c_0$ by $f(n) = (0, n, 0, 0, \ldots)$. There does not exist any morphism $u: \mathbb{F} \to c_0$ in $B^*$ such that $g \circ u = f$, that is, so the following diagram commutes.

```
\begin{diagram}
\node{c_0} \arrow{e}{u} \arrow{s}{g} \node{c_0} \\
\node{c_0} \arrow{s}{f} \node{c_0}
\end{diagram}
```

If such a morphism $u$ existed in $B^*$, then $u(1)$ must be $b$ as $g$ is one-to-one. However then $|u| = \sup_{|a| \neq 0} |u(a)| \geq |u(1)| = |b| = 2 > 1$. This is a contradiction.

**Proposition 4.18** In $B^*$ if $B$ is injective, then $B = 0$.

**Proof:** The result follows from lemmas 4.13, 4.15, and 4.16.

**Proposition 4.19** In $B^*$ if $B$ is projective, then $B = 0$.

**Proof:** The result follows from lemmas 4.13, 4.15, and 4.17.
5. NORMAL MORPHISMS IN $B^*$

**Definition 5.1** In $B^*$ a morphism $f: A \rightarrow B$ is a normal morphism if the map $f': A/\text{Ker } f \rightarrow f[A]$ defined by $f'([a]) = f(a)$ is an isometric map.

**Remark 5.2** A morphism $f: A \rightarrow B$ in $B^*$ is a normal monomorphism if and only if it is an isometric map. If $f: A \rightarrow B$ is a normal epimorphism, then $f[A] = B$ (norm of $f[A]$ is the same as that of $A/\text{Ker } f$). If $f: A \rightarrow B$ is a normal monomorphism and an epimorphism, then it is an isometric isomorphism. If $f: A \rightarrow B$ is a normal morphism in $B^*$, then $\text{Cok Ker } f = \text{im } f = \text{coim } f = \text{Ker Cok } f$.

In $B^*$ the following definition is equivalent to the preceding definition of a normal subobject (Definition 2.15).

**Definition 5.3** If $f: A \rightarrow B$ is a subobject of $B$ where $f$ is a normal monomorphism in $B^*$, then $A$ is called a normal subobject of $B$.

The union of a family of subobjects $\{f_i: A_i \rightarrow A\}_I$ has been shown to be the subobject $\sum_I A_i/\text{Ker } g'$ with respect to $g: \sum_I A_i/\text{Ker } g' \rightarrow A$ where $g': \sum_I A_i \rightarrow A$ is the map $g'(a) = \sum_I f_i(a_i)$ ($a_i$ is the $i$th coordinate of $a$) and $g([a]) = g'(a)$. The morphism $u: \sum_I A_i/\text{Ker } g' \rightarrow B$ in the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{h_1} & B' \\
\downarrow{g_1} & & \downarrow{h} \\
\sum_I A_i/\text{Ker } g' & \xrightarrow{f_1} & A \\
\downarrow{g} & & \downarrow{u} \\
A & \xrightarrow{f} & B
\end{array}
\]
where for all $i \in I$, $g \circ g_i = f_i$, was given explicitly so that whenever $f$ carried each $A_i$ into $B'$, it also carried $\sum_{i \in I} A_i / \ker g'$ into $B'$. Now if $B'$ is a normal subobject of $B$, then since $h : B' \to B$ is the kernel of $\coker h : B \to B/h[B']$ and since $(\coker h) \circ (f \circ g) = 0$, the existence of a unique $u : \sum_{i \in I} A_i / \ker g' \to B'$ such that $h \circ u = f \circ g$ is guaranteed.

If $\left\langle \bigcup_{i \in I} [A_i] \right\rangle$ denotes the Banach space $\bigcap B$ (the intersection of all closed subspaces $B$ of $A$ containing $\bigcup_{i \in I} [A_i]$), a subspace of $A$ which contains $\left\{ \sum_{i \in I} f_i(a_i) \mid \text{each sum finite} \right\}$, then $\left\langle \bigcup_{i \in I} [A_i] \right\rangle$ is a subobject of $A$ with respect to the inclusion map $q : \left\langle \bigcup_{i \in I} [A_i] \right\rangle \to A$. In the diagram

\[ \begin{array}{ccc}
\left\langle \bigcup_{i \in I} [A_i] \right\rangle & \xrightarrow{g_1} & A_i \\
\downarrow q & & \downarrow h \\
A & \xrightarrow{f} & A
\end{array} \]

$g_1 : A_i \to \left\langle \bigcup_{i \in I} [A_i] \right\rangle$ is defined by $g_1(a_i) = f_i(a_i)$, then for all $i \in I$, $q \circ g_i = f_i$. If $B'$ is a normal subobject of $B$ and each $A_i$ is carried by $f$ into $B'$, then since $h : B' \to B$ is the kernel of $\coker h : B \to B/h[B']$ and $(\coker h) \circ (f \circ g) = 0$, the existence of a unique $u : \left\langle \bigcup_{i \in I} [A_i] \right\rangle \to B'$ is guaranteed such that $\left\langle \bigcup_{i \in I} [A_i] \right\rangle$ is carried by $f$ into $B'$. Thus $\left\langle \bigcup_{i \in I} [A_i] \right\rangle$ acts as the union of the $A_i$ in the special case when the $h : B' \to B$ is restricted to a normal monomorphism and as such might be called a "normal union". However, $\left\langle \bigcup_{i \in I} [A_i] \right\rangle$ is not nec-
essarily isomorphic to \( \sum_{i} A_i / \ker g' \) in diagram (*) as 
\( g: \sum_{i} A_i / \ker g' \rightarrow A \) is not generally a normal monomorphism.

It has been shown that the intersection of a family
of subobjects \( \{f_i:A_i \rightarrow A\}_{i} \) of A is \( f:A' \rightarrow A \) where
\( A' = \{ b \in \bigcap_i A_i \mid \text{for all } i,j \in I, f_i(b_i) = f_j(b_j) \} \) and \( f \) is
defined by \( f(b) = f_p(b_p) \) for a fixed element \( p \) of I. If
each \( A_i \) is a normal subobject of A, then each \( f_i[A_i] \) is
a subspace of A and \( \bigcap f_i[A_i] \) (set intersection) is a Banach
space. If in the diagram below \( j: \bigcap f_i[A_i] \rightarrow A \) is the
insertion map, and \( p_i: \bigcap f_i[A_i] \rightarrow A_i \) is the map defined
by \( p_i(f_i(a_i)) = a_i \); then for each \( i \in I \), \( f_i p_i = j \).

\[ \bigcap f_i[A_i] \quad \xrightarrow{p_i} \quad A_i \quad \xrightarrow{f_i} \quad A \]

If also for each \( i \in I \), \( h = f_i \circ h_i \), then by defining
\( u:A'' \rightarrow \bigcap f_i[A_i] \) by \( u(a'\prime) = f_p(h_p(a'')) \) for \( p \) fixed in I,
\( j \circ u = h \) as \( (j \circ u)(a''') = j(f_p(h_p(a''')) = f_p(h_p(a''')) = h(a''')) \)
for all \( a'' \in A'' \). The morphism \( u \) is unique since \( j \) is
a monomorphism. Hence if \( f_i:A_i \rightarrow A \) are normal, \( \bigcap f_i[A_i] \)
(set intersection) is the intersection of the subobjects
\( \{A_i\}_I \) of A and in this case is isomorphic to \( A' = \{ b \in \bigcap_i A_i \mid \text{for all } i,j \in I, f_i(b_i) = f_j(b_j) \} \).

**Definition 5.4** An object \( Q \) in \( B^* \) is called normal injective
if for any normal monomorphism \( f:A \rightarrow B \) and any morphism

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There is a morphism \( h: B \to Q \) making the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{h} \\
Q & & 
\end{array}
\]

commutative.

The following two lemmas are proved in the same way as lemmas 4.13 and 4.14.

**Lemma 5.5** If \( M \) is normal injective and \( r: M \to M_1 \) is a retraction, then \( M_1 \) is normal injective.

**Lemma 5.6** If \( M \) is the product of \( \{ M_i \}_I \) and all \( M_i \) are normal injective, then \( M \) is normal injective.

**Lemma 5.7** If \( M \) is normal injective and \( f: M \to A \) is any normal monomorphism, then \( f \) admits a retraction, that is, there exists \( g: A \to M \) such that \( g \circ f = 1_M \).

**Proof:** Let \( 1_M: M \to M \) be the identity morphism. Since \( f: M \to A \) is a normal monomorphism and \( M \) is normal injective, there exists \( g: A \to M \) such that \( 1_M = g \circ f \).

**Proposition 5.8** For every object \( A \) in \( B' \), there exists a normal monomorphism \( f \) from \( A \) into \( \prod_T^i \) where \( T \) is a suitably chosen index set.

**Proof:** Let \( n: A \to A^{**} \) be the natural embedding of \( A \) into the second conjugate space. \( A \) and \( n[A] \) are isometrically isomorphic. Let \( B^* = \{ b^* \in A^* \mid |b^*| \leq 1 \} \) where \( A^* \) is the first conjugate space. Consider \( \prod_{B^*}^i \), the set of all functions \( f: B^* \to \bar{T} \) such that \( \sum_{b^* \in B^*} |f(b^*)| \) is finite.
Define $g:A^{**} \rightarrow \bigcap_{B^*} \Phi$ by $g(a^{**}) = a^{**} |_{B^*} : B^* \rightarrow \Phi$, the restriction of $a^{**}$ to $B^*$. Then $g$ is an isometric map for $|g(a^{**})| = |a^{**} |_{B^*} | = \sup_{b^* \in B^*} |a^{**}(b^*)| = |a^{**}|$. The required normal monomorphism $f$ is given $A \rightarrow \bigcap_{B^*} \Phi$.

**Proposition 5.9** In $\mathcal{B}^*$, $\Phi$ is normal injective.

**Proof**: Let $f:A \rightarrow B$ be any normal monomorphism in $\mathcal{B}$ and $g:A \rightarrow \Phi$ any morphism. Define $h':f[A] \rightarrow \Phi$ by $h'(f(a)) = g(a)$. The map $h'$ is in $\mathcal{B}$ as $\sup |h'(f(a))| = |f(a)| \leq \sup |g(a)|$. Using the Hahn-Banach theorem [1, p.63], there is a linear map $h:B \rightarrow \Phi$ such that $|h| = |h'|$ and $h(f(a)) = h'(f(a))$ for all $f(a) \in f[A]$. Thus $h \in \mathcal{B}$ and $h \circ f = g$.

A proposition similar to the following is given in Semadeni [5, p. 363].

**Proposition 5.10** In $\mathcal{B}$ the following are equivalent.

(i) $M$ is normal injective.

(ii) $M$ is a retract of $\bigcap_{T} \Phi$ for some index set $T$.

(iii) $M$ is a normal retract, that is, if there exists a normal monomorphism from $M$ into an object $A$, then $M$ is a retract of $A$.

**Proof**:

(1) $\Rightarrow$ (iii) by lemma 5.7.

(iii) $\Rightarrow$ (ii) by proposition 5.8.

(ii) $\Rightarrow$ (1) by proposition 5.9 and lemmas 5.6 and 5.5.
6. THE TENSOR PRODUCT OF BANACH SPACES

In the following development A and B will always denote two Banach spaces. Let \( \mathcal{F}^{A \times B} \) be the vector space over the field \( \mathcal{F} \) consisting of all functions \( f: A \times B \to \mathcal{F} \). Addition and scalar multiplication in \( \mathcal{F}^{A \times B} \) are defined pointwise, that is,

\[
(f + g)(a, b) = f(a, b) + g(a, b) \\
(\theta f)(a, b) = \theta f(a, b) \quad \theta \in \mathcal{F}.
\]

For each \((a, b)\) in \(A \times B\) let \(a \ast b\) be the element in \(\mathcal{F}^{A \times B}\) defined by

\[
a \ast b(p, q) = 1 \quad \text{if} \quad (p, q) = (a, b) \\
a \ast b(p, q) = 0 \quad \text{if} \quad (p, q) \neq (a, b).
\]

Let \( \mathcal{F}(A \times B) \) be the subspace of \( \mathcal{F}^{A \times B} \) spanned by the elements of the type \(a \ast b\). Thus \( \mathcal{F}(A \times B) \) is the space consisting of all functions \( f: A \times B \to \mathcal{F} \) given by

\[
f = \sum_{i=1}^{n} \theta_i a_i \ast b_i, \quad n \text{ some positive integer}
\]

and \( f(a_1, b_1) = \theta_1 \).

Let \( K: A \times B \to \mathcal{F}(A \times B) \) be a function defined by \( K(a, b) = a \ast b \). It may happen that \( a_1 \ast b + a_2 \ast b \neq (a_1 + a_2) \ast b \) and hence \( K \) is not bilinear. Let \( S \) be the subspace of \( \mathcal{F}(A \times B) \) spanned by all elements of the type

\[
(\theta_1 a_1 + \theta_2 a_2) \ast b_1 \ast (a_1 \ast b_1) - \theta_2 (a_2 \ast b_1)
\]

and

\[
a_1 \ast (\theta_1 b_1 + \theta_2 b_2) \ast (a_1 \ast b_1) - \theta_1 (a_1 \ast b_1) - \theta_2 (a_1 \ast b_2)
\]

for all \( \theta_1, \theta_2 \in \mathcal{F} ; a_1, a_2 \in A ; b_1, b_2 \in B \). Let \( A \otimes B \) denote...
\( (A \times B)/S \). The equivalence class of the element
\[ f = \sum_{i=1}^{n} \theta_i(a_i \circ b_i) \] will be denoted by \( \sum_{i=1}^{n} \theta_i a_i \circ b_i \). It is now easy to see that
\[ (\theta_1 a_1 + \theta_2 a_2) \circ b = \theta_1(a_1 \circ b) + \theta_2(a_2 \circ b) \]
and
\[ a \circ (\theta_1 b_1 + \theta_2 b_2) = \theta_1(a \circ b_1) + \theta_2(a \circ b_2), \]
and that the function \( \circ : A \times B \rightarrow A \circ B \), defined as the composition of \( K : A \times B \rightarrow (A \times B) \) and the canonical map \( (A \times B) \rightarrow A \circ B \), is bilinear.

From a different point of view one may define a relation \( \sim \) on \( \tilde{\phi}(A \times B) \) subject to the following rules:

1. \( (a_1 + a_1') \circ b_1 + a_2 \circ b_2 + \ldots + a_n \circ b_n \sim a_1 \circ b_1 + a_1' \circ b_1 + a_2 \circ b_2 + \ldots + a_n \circ b_n \)
2. \( a_1 \circ (b_1 + b_1') + a_2 \circ b_2 + \ldots + a_n \circ b_n \sim a_1 \circ b_1 + a_1 \circ b_1' + a_2 \circ b_2 + \ldots + a_n \circ b_n \)
3. \( \theta_1(a_1 \circ b_1) + \theta_2(a_2 \circ b_2) + \ldots + \theta_n(a_n \circ b_n) \sim (\theta_1 a_1) \circ b_1 + \ldots + (\theta_n a_n) \circ b_n \)
4. \( (\theta_1 a_1) \circ b_1 + (\theta_2 a_2) \circ b_2 + \ldots + (\theta_n a_n) \circ b_n \sim a_1 \circ (\theta_1 b_1) + \ldots + a_n \circ (\theta_n b_n) \)

Now define the equivalence relation \( \sim \) on \( \tilde{\phi}(A \times B) \) by
\[ \sum_{i=1}^{n} \theta_i(a_i \circ b_1) \sim \sum_{i=1}^{m} \lambda_i(c_i \circ d_i) \] if \( \sum_{i=1}^{n} \theta_i(a_i \circ b_1) \) can be transformed into \( \sum_{i=1}^{m} \lambda_i(c_i \circ d_i) \) by a finite number of applications of rules (1) - (4). It is apparent that the two quotient spaces \( \tilde{\phi}(A \times B)/\sim \) and \( \tilde{\phi}(A \times B)/S \) are identical, that is, \( S \) is the zero class of the relation \( \sim \).

The objective is to make \( A \circ B \) into a normed linear
space and then let the Banach space $A \otimes B$ be its completion.

To this end, define a norm of $A \otimes B$ by

$$
|u| = \inf \left\{ \sum_{i=1}^{n} |\theta_i| |a_i| |b_i| \mid u = \sum_{i=1}^{n} \theta_i a_i \otimes b_i \right\},
$$

$u \in A \otimes B$. It will be shown that this is a crossnorm on $A \otimes B$, that is, a norm with the additional property that $|a \otimes b| = |a||b|$, $a \in A$, $b \in B$. The following development to show this is a crossnorm is based on work by Robert Schatten [4]. He proves the following lemma [4, p.201].

**Lemma 6.2** If $F$ is a linear functional on $A$ and

$$
\sum_{i=1}^{n} a_i \otimes b_i = \sum_{j=1}^{m} a_j' \otimes b_j',
$$

then

$$
\sum_{i=1}^{n} F(a_i)b_i = \sum_{j=1}^{m} F(a_j')b_j'.
$$

The following definitions are given for clarity.

**Definition 6.3** A norm on $A \otimes B$ is a non-negative function $N: A \otimes B \to \mathbb{R}$ satisfying the following conditions:

(i) $N(\sum_{i=1}^{n} \theta_i a_i \otimes b_i) = 0$ if and only if $\sum_{i=1}^{n} \theta_i a_i \otimes b_i = 0 \otimes 0$.

(ii) $N(\sum_{i=1}^{n} \theta_i a_i \otimes b_i) = |\theta|N(\sum_{i=1}^{n} a_i \otimes b_i)$ for all $\theta \in \mathbb{F}$.

(iii) $N(\sum_{i=1}^{n} a_i \otimes b_i + \sum_{i=1}^{n} c_i \otimes d_i) \leq N(\sum_{i=1}^{n} a_i \otimes b_i) + N(\sum_{i=1}^{n} c_i \otimes d_i)$.

**Definition 6.4** A norm $N$ on $A \otimes B$ is continuous at $\sum_{i=1}^{n} a_i \otimes b_i$ if and only if
(iv) given $\varepsilon > 0$, there exists a $\delta(a_1, \ldots, a_n, b_1, \ldots, b_n) > 0$ such that for $|a_i - a_j| < \delta$ and $|b_i - b_j| < \delta$, $i = 1, \ldots, n$, $N(\sum_{i=1}^{n} a_i \otimes b_i - \sum_{i=1}^{n} a_i' \otimes b_i') < \varepsilon$.

**Definition 6.5** A norm $N$ on $A \otimes B$ is a crossnorm if it satisfies the property

$$N(a \otimes b) = |a||b| \quad a \in A, b \in B.$$  

**Lemma 6.6** A crossnorm satisfies condition (iv).

**Proof:** $N(\sum_{i=1}^{n} a_i \otimes b_i - \sum_{i=1}^{n} a_i' \otimes b_i') \leq N(\sum_{i=1}^{n} (a_i - a_i') \otimes b_i) + N(\sum_{i=1}^{n} a_i \otimes (b_i - b_i')) + N(\sum_{i=1}^{n} a_i' \otimes (b_i' - b_i)) \leq \sum_{i=1}^{n} |a_i - a_i'| |b_i| + \sum_{i=1}^{n} |a_i| |b_i - b_i'| + \sum_{i=1}^{n} |a_i - a_i'| |b_i - b_i'|$

Let $\varepsilon > 0$ be given. Let $\delta = \min(\varepsilon/3n|b_1|, \varepsilon/3n|a_1|, \sqrt{\varepsilon}/3n)$. The result follows using this $\delta$.

Let $F$ be a linear functional on $A$. Sometimes a norm on $A \otimes B$ will satisfy the condition

$$(vi) \quad |(\sum_{i=1}^{n} F(a_i)b_i)| \leq |F||N(\sum_{i=1}^{n} a_i \otimes b_i)|.$$

**Definition 6.7** Let $T_u$ be a transformation from the vector space of linear functionals on $A$ to the vector space $B$ defined by

$$T_uF = \sum_{i=1}^{n} F(a_i)b_i \quad \text{where} \quad u = \sum_{i=1}^{n} a_i \otimes b_i.$$

This definition describes a well-defined function since by lemma 6.2, $\sum_{i=1}^{n} F(a_i)b_i = \sum_{i=1}^{m} F(c_i)d_i$ whenever

$$\sum_{i=1}^{n} a_i \otimes b_i = \sum_{i=1}^{m} c_i \otimes d_i.$$
Lemma 6.8 If $T_u = 0$, then $u = 0 \otimes 0$.

Proof: Suppose $u = \sum_i a_i \otimes b_i \neq 0 \otimes 0$. It can be assumed that both the sets $\{a_i \mid i = 1, \ldots, n\}$ and $\{b_i \mid i = 1, \ldots, n\}$ are linearly independent. Thus $a_i \neq 0$. A linear functional $F$ can be found for which $F(a_i) \neq 0$. Hence $\sum_i F(a_i)b_i \neq 0$.

Thus $T_uF \neq 0$.

Lemma 6.9 Conditions (ii) and (vi) for a norm $N$ on $A \otimes B$ imply (i).

Proof: From condition (ii), if $\sum_i a_i \otimes b_i = 0 \otimes 0$, then $N(\sum_i a_i \otimes b_i) = N(0 \otimes 0) = 0$. Now suppose $N(\sum_i a_i \otimes b_i) = 0$.

By (vi), $|\sum_i F(a_i)b_i| = 0$ or $\sum_i F(a_i)b_i = 0$. Hence $T_uF = 0$ where $u = \sum_i a_i \otimes b_i$. By the preceding lemma $u = 0 \otimes 0$.

Now let $N: A \otimes B \to \mathbb{R}$ be the non-negative function given by equation (6.1), that is,

$$N(u) = |u| = \inf \left\{ \sum_i |\theta_i| |a_i| |b_i| \mid u = \sum_i \theta_i a_i \otimes b_i \right\},$$

$u \in A \otimes B$.

Proposition 6.9 $N$ is a norm for $A \otimes B$ satisfying conditions (iv), (v), and (vi).

Proof: It must only be shown that $N$ satisfies conditions (ii), (iii), (v), and (vi); since (v) implies (iv) while (ii) and (vi) imply (i). Condition (ii) is clear. To prove (iii) let $\sum_i a_i \otimes b_i$ and $\sum_i c_i \otimes d_i$ be two elements of $A \otimes B$. Let $\epsilon > 0$ be given. Let $\sum_i a_i \otimes b_i = \sum_i a_i \otimes b_i$ such that

$$\sum_i |a_i'| |b_i'| < N(\sum_i a_i \otimes b_i) + \epsilon/2.$$
Similarly let \( \sum_1 c_i \otimes d_i = \sum_1 c_i \otimes d_i \) such that

\[
\sum_1 |c_i| |d_i| \leq N(\sum_1 c_i \otimes d_i) + \varepsilon/2.
\]

Then

\[
N(\sum_1 a_i \otimes b_i + \sum_1 c_i \otimes d_i) = N(\sum_1 a_i \otimes b_i + \sum_1 c_i \otimes d_i)
\]

\[
\leq \sum_1 |a_i| |b_i| + \sum_1 |c_i| |d_i|
\]

\[
\leq N(\sum_1 a_i \otimes b_i) + N(\sum_1 c_i \otimes d_i) + \varepsilon.
\]

This proves (iii).

To prove (vi) let \( u = \sum_1 a_i \otimes b_i \) and \( T_u F = \sum_1 F(a_i)b_i \).

Then \( |T_u F| = |\sum_1 F(a_i)b_i| \leq \sum_1 |F(a_i)||b_i| \leq |F| \sum_1 |a_i||b_i| \).

This is true for all \( \sum_1 c_i \otimes d_i = u \). Hence

\[
|T_u F| = |\sum_1 F(a_i)b_i| \leq |F| N(u).
\]

To prove that \( N \) is a crossnorm, let \( a \otimes b \) be in \( A \otimes B \), \( a \neq 0 \) and \( b \neq 0 \). Suppose \( \sum_1 c_i \otimes d_i = a \otimes b \). Let \( T_{a \otimes b} F = F(a)b \) which equals \( \sum_1 F(c_i)d_i \). Let \( F \) be the particular linear functional given by \( F(a) = |a|, \ |F| = 1 \).

Then

\[
|T_{a \otimes b} F| = |F(a)b| = |a| |b| = |a| |b| = |\sum_1 F(c_i)d_i| \leq
\]

\[
|F| N(\sum_1 c_i \otimes d_i) = N(\sum_1 c_i \otimes d_i) . \text{ Hence } |a||b| = N(a \otimes b).
\]

\( A \otimes B \) is thus a normed linear space with the norm

\[
|u| = \inf \left\{ \sum_1 \theta_1 |a_i| |b_i| \mid u = \sum_1 \theta_1 a_i \otimes b_i \right\}.
\]

The completion of \( A \otimes B \), denoted by \( A^\Delta \otimes B \), is the Banach space known as the **tensor product** of Banach spaces \( A \) and \( B \).
Let $\mathbb{B}(A,B)$ denote the Banach space of continuous linear functions from Banach space $A$ into Banach space $B$.

**Proposition 6.10** If $A$, $B$, and $C$ are Banach spaces, then $\mathbb{B}(A \otimes B, C)$ and $\mathbb{B}(B, \mathbb{B}(A, C))$ are isometrically isomorphic.

**Proof:** Define $\mathcal{T}: \mathbb{B}(A \otimes B, C) \rightarrow \mathbb{B}(B, \mathbb{B}(A, C))$ by $[(Tf)b](a) = f(a \otimes b)$ for $f \in \mathbb{B}(A \otimes B, C)$, and note that $Tf$ is easily seen to be linear over $B$, while $(Tf)b$ is linear over $A$ for all $b$ in $B$. Also $T$ can easily be seen to be a linear map. By definition

$$|Tf| = \inf \{ M | \text{for all } b \in B, |(Tf)b| \leq M|b| \}.$$

and

$$|(Tf)b| = \inf \{ N_b | \text{for all } a \in A, |(Tf)b| \leq N_b|a| \}.$$

Define $|Tf|'$ by:

$$|Tf|' = \inf \{ M | \text{for all } a \in A, b \in B, |(Tf)b| \leq M|a| |b| \}.$$

Since $|(Tf)b| \leq |Tf| |b| |a|$, then $|Tf|' \leq |Tf|$. Also since $|(Tf)b| \leq |Tf|' |a| |b|$, then $|(Tf)(b)| \leq |Tf|' |b|$, which means $|Tf| \leq |Tf|'$. Hence $|Tf| = |Tf|'$. Therefore

$$|Tf| = |Tf|' = \inf \{ M | |(Tf)b| \leq M|a| |b| \} = \inf \{ M | |f(a \otimes b)| \leq M|a| |b| \} = \inf \{ M | |f(a \otimes b)| \leq M|a \otimes b| \} \leq |f|.$$

It will now be shown that $\mathcal{T}$ has as inverse $\mu: \mathbb{B}(B, \mathbb{B}(A, C)) \rightarrow \mathbb{B}(A \otimes B, C)$ such that $|\mu| \leq 1$. Define $\hat{\mu}: \mathbb{B}(B, \mathbb{B}(A, C)) \rightarrow \mathbb{B}(A \otimes B, C)$ by $\hat{\mu}(g)(a \otimes b) = (g(b))a$ for $g \in \mathbb{B}(B, \mathbb{B}(A, C))$, and extend $\hat{\mu}(g)$ linearly to $A \otimes B$. Since
\( \hat{\mu}(g)(0 \otimes 0) = (g(0))0 = 0 \) and \( g \) is linear over \( \mathbb{B} \); while for each \( b \), \( g(b) \) is linear over \( \mathbb{A} \), the map \( \hat{\mu}(g) \) is well-defined. Also \( |\hat{\mu}(g)| \leq |g| \) for

\[
|\hat{\mu}(g)(\sum_{i=1}^n a_i \otimes b_i)| \leq |\sum_{i=1}^n \hat{\mu}(g)(a_i \otimes b_i)| \\
\leq \sum_{i=1}^n |\hat{\mu}(g)(a_i \otimes b_i)| \\
\leq |\sum_{i=1}^n g(b_i) a_i| \\
\leq |g| \sum_{i=1}^n |b_i| |a_i|.
\]

Hence \( |\hat{\mu}(g)(\sum_{i=1}^n a_i \otimes b_i)| \leq |g| \) if \( \sum_{i=1}^n a_i \otimes b_i \neq 0 \otimes 0 \).

Hence \( |\hat{\mu}(g)(\sum_{i=1}^n a_i \otimes b_i)| \leq |g| \) using the same reasoning as in lemma 1.11. Therefore \( |\hat{\mu}(g)| \leq |g| \), which also means that for each \( g \in \mathbb{B}^{(B,B(A,C))} \), \( \hat{\mu}(g) : \mathbb{A} \otimes \mathbb{B} \to \mathbb{C} \) is uniformly continuous.

By the Principle of extension by continuity [1, p.23], \( \hat{\mu}(g) : \mathbb{A} \otimes \mathbb{B} \to \mathbb{C} \) has a unique uniformly continuous extension \( \mu(g) : \mathbb{A} \hat{\otimes} \mathbb{B} \to \mathbb{C} \). Let \( \mu : \mathbb{B}(B,B(A,C)) \to \mathbb{B}(A \hat{\otimes} B,C) \) map \( g \) to the unique extension of \( \hat{\mu}(g) \). Let \( x \in \mathbb{A} \hat{\otimes} \mathbb{B} \). Then there exists a sequence \( (x_n)_N \) such that \( x_n \in \mathbb{A} \otimes \mathbb{B} \) and \( x_n \to x \).

\[
|\mu(g)x| = |\mu(g)\lim_n x_n| \\
\leq \lim_n |\hat{\mu}(g)x_n| \\
\leq \lim_n |g||x_n| \\
= |g|\lim_n |x_n| = |g||x|.
\]

Hence \( |\mu(g)| \leq |g| \) or \( |\mu| \leq 1 \).
It is also easily seen that $\hat{\mu}$ is a linear function. Hence $\mu$ is also a linear function, for if $\theta \in \mathcal{F}$ and $x \in A \hat{\otimes} B$ with $(x_n)_N$ a sequence in $A \otimes B$ such that $x_n \rightarrow x$, then 

\begin{align*}
(\mu(\theta f_1 + f_2))x &= (\mu(\theta f_1 + f_2)) \lim_{n} x_n \\
&= \lim_{n} (\mu(\theta f_1 + f_2))x_n \\
&= \lim_{n} (\mu(\theta f_1 + f_2))x_n \\
&= \lim_{n} \theta(\hat{\mu}(f_1))x_n + (\hat{\mu}(f_2))x_n \\
&= \lim_{n} \theta(\mu(f_1))x_n + (\mu(f_2))x_n \\
&= \theta(\mu(f_1))x + (\mu(f_2))x \\
&= (\theta \mu(f_1) + \mu(f_2))(x).
\end{align*}

This means $\mu(\theta f_1 + f_2) = \theta \mu(f_1) + \mu(f_2)$.

Because $\mu$ is an inverse for $\mathcal{L}$, the proposition is proved.

**Corollary 6.11** $B^*(A \hat{\otimes} B, C)$ is isometrically isomorphic to $B^*(B, B(A, C))$. 

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