A Category of Banach Spaces

Kenneth Leroy Pothoven
Western Michigan University
A CATEGORY OF BANACH SPACES

by

Kenneth L. Pothoven

A Project Report
Submitted to the
Faculty of the School of Graduate Studies in partial fulfillment of the Specialist in Arts Degree

Western Michigan University
Kalamazoo, Michigan
July 1968
ACKNOWLEDGEMENTS

In preparing this thesis I have become above all indebted to Dr. Kung-Wei Yang for his encouragement, patient instruction, and advice. My thanks go to him, others in the Department of Mathematics for their instruction, and to Western Michigan University for financial aid enabling me to complete my work for the Specialist in Arts degree. To my wife Betty I give a special word of thanks for her patience and help in preparing the final transcript of this paper.

Kenneth L. Pothoven
MASTER'S THESIS M-1603

POTHOVEN, Kenneth Leroy
A CATEGORY OF BANACH SPACES.

Western Michigan University, Sp.A., 1968 Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
TABLE OF CONTENTS

SECTION                                      PAGE

1. PRELIMINARIES..................................  1
2. KERNELS, COKERNELS, IMAGES, COIMAGES, AND NORMALITY  6
3. PRODUCTS, COPRODUCTS, PULLBACKS, AND PUSHOUTS. 12
4. UNIONS AND INTERSECTIONS, PROJECTIVE AND INJECTIVE OBJECTS. 22
5. NORMAL MORPHISMS IN B'.......................  30
6. THE TENSOR PRODUCT OF BANACH SPACES ........  35
1. PRELIMINARIES

Definition 1.1 Let $\mathcal{A}$ be a collection of "objects", $X, Y,$ $Z,$ and so on, together with two functions:

(i) a function assigning to each ordered pair $(X, Y)$ of objects a set $\text{hom}_\mathcal{A}(X, Y)$. An element $f$ in this set is called a morphism of $\mathcal{A}$ with domain and codomain $Y$ (written $f:X \rightarrow Y$).

(ii) a function assigning to each triplet $(X, Y, Z)$ of objects of $\mathcal{A}$ a function

$$\text{hom}_\mathcal{A}(Y, Z) \times \text{hom}_\mathcal{A}(X, Y) \rightarrow \text{hom}_\mathcal{A}(X, Z).$$

For morphisms $g:Y \rightarrow Z$ and $f:X \rightarrow Y$ this function takes $(g, f) \rightarrow g \circ f$, and $g \circ f:X \rightarrow Z$ is called the composite of $g$ with $f$.

$\mathcal{A}$ is called a category when the following axioms hold.

(i) If $h:Z \rightarrow W$, $g:Y \rightarrow Z$, and $f:X \rightarrow Y$ are morphisms of $\mathcal{A}$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) For each object $Y$ of $\mathcal{A}$ there exists a morphism (necessarily unique) $l_Y: Y \rightarrow Y$ such that $f:X \rightarrow Y$ implies $l_Y \circ f = f$ and $g:Y \rightarrow Z$ implies $g \circ l_Y = g$.

Throughout this paper morphisms in a category $\mathcal{A}$ will be denoted by such letters as $f$, $g$, $h$, and $p$; while objects will be denoted by letters such as $X$, $Y$, $A$, $B$, and $C$. If $f:X \rightarrow Y$ is a morphism in a category $\mathcal{A}$, it will just be said without confusion, $f$ is in $\mathcal{A}$. 

1
Let $B^*$ be the collection of Banach spaces (scalar field is the real or complex number field and is denoted by $\mathbb{F}$) together with the following functions.

(i) For each pair of Banach spaces $(X, Y)$ assign the set $\text{hom}_B(X, Y) = B^*(X, Y)$ where $B^*(X, Y)$ is the set of all linear continuous maps $f: X \rightarrow Y$ such that $|f| = \sup_{|x| \leq 1} |f(x)| \leq 1$.

(ii) For each triplet $(X, Y, Z)$ of Banach spaces assign to each element $(g, f)$ of $B^*(Y, Z) \times B^*(X, Y)$ the function $g \circ f: X \rightarrow Z$. Since $g \circ f$ is linear and $|g \circ f| \leq |g||f| \leq 1$, $g \circ f \in B^*(X, Z)$.

It is now easy to establish the following proposition.

**Proposition 1.2** $B^*$ is a category.

In this paper the category $B^*$ will be examined. The following propositions and remarks will be essential to this study.

**Remark 1.3** It should be recalled that for a linear function $f: X \rightarrow Y$ where $X$ and $Y$ are normed linear spaces

$$|f| = \sup_{|x| \leq 1} |f(x)| = \sup_{|x| \leq 1} |f(x)|/|x|$$

$$= \inf \{ M \mid \text{for all } x \in X, |f(x)| \leq M|x| \}.$$

The terminology for the following lemma and its corollary is taken from Kelley [2]. Let $X$ be an order-complete chain with the order topology.

**Lemma 1.4** Each monotone increasing net in $X$ whose range is bounded converges to the supremum of its range.
Corollary 1.5 If \( \mathbb{R} \) is the set of real numbers with the usual order, then each monotone increasing (respectively, decreasing) net in \( \mathbb{R} \), whose range has an upper (respectively, lower) bound converges to the supremum (respectively, infimum) of its range.

Let \( B \) be a Banach space and \( I \) an arbitrary index set. Let \( \{ a_i \mid i \in I \} \) be a set of elements from \( B \) indexed by \( I \). Let \( \mathcal{F} \) be the set of all finite subsets of \( I \). \( \mathcal{F} \) becomes a directed set by the relation \( \prec \) defined by \( A \prec B \) if and only if \( A \subseteq B \). Define \( S : \mathcal{F} \to \mathbb{R} \) by \( S(A) = \sum_{i \in A} |a_i| \).

Definition 1.6 \( \sum_{i} |a_i| < \infty \) if and only if \( S \) is bounded.

The following remarks follow from the corollary above.

Remark 1.7 \( S \) converges if and only if \( \sum_{i} |a_i| < \infty \).

Remark 1.8 Absolute convergence implies convergence; that is, if \( S \) converges then \( \sum_{i} a_i \) converges to a point in \( B \). Moreover, if \( S \) converges then \( |\sum_{i} a_i| \leq \sum_{i} |a_i| \).

Remark 1.9 If \( \sum_{i} a_i \) and \( \sum_{i} b_i \) both converge, then

(i) \( \sum_{i} a_i + \sum_{i} b_i = \sum_{i} (a_i + b_i) \).

(ii) \( \sum_{i} ka_i = k \sum_{i} a_i \) for \( k \in \mathbb{F} \).

Remark 1.10 Let \( f \) be a continuous linear map from Banach space \( A \) to Banach space \( B \). If \( \sum_{i} a_i \) converges in \( A \), then \( f(\sum_{i} a_i) = \sum_{i} f(a_i) \).

Let \( X \) and \( Y \) be normed linear spaces and \( Z \) a closed subspace of \( X \). Then \( X/Z \) is a normed linear space consisting of
equivalence classes \([a.]\), formed by a relation \(R: aRb\) if \(a - b \in Z\). The norm on \(X/Z\) is given by \(|[a]| = \inf_{b \in [a]} |b|\).

Throughout this paper \([\ ]\) will denote an equivalence class in \(X/Z\).

**Lemma 1.11** Let \(f:X \rightarrow Y\) be a linear function of norm no greater than one. Then \(g:X/f^1(0) \rightarrow Y\) defined by \(g([a]) = f(a)\) is linear, one-to-one, and has norm no greater than one. \((f^1(0) = \{x \in X | f(x) = 0\}\), a subspace of \(X\))

**Proof:** That \(g\) is linear follows from the fact that \(f\) is linear. If \(f(a) = f(b)\) then \(a - b \in f^1(0)\) and \([a] = [b]\). Therefore \(g\) is one-to-one. Now by definition

\[ |g| = \sup_{[a] \neq 0} \frac{|g([a])|}{|[a]|} = \sup_{[a] \neq 0} \frac{|f(a)|}{|[a]|}. \]

However, for all \(b \in [a]\), \(f(a) = f(b)\) and hence \(\frac{|f(a)|}{|b|} \leq 1\) and \(\frac{|f(b)|}{|b|} \leq |b|\). Hence

\[ \frac{|f(a)|}{|[a]|} \leq 1 \text{ as } |[a]| = \inf_{b \in [a]} |b|. \]

Therefore \(|g| \leq 1\).

**Definition 1.12** In a category \(\mathbb{A}\), a morphism \(f:A \rightarrow B\) is a monomorphism if and only if the only pair of morphisms \(f':C \rightarrow A\) and \(f'':C \rightarrow A\) such that \(f \circ f' = f \circ f''\) is \(f' = f''\).

**Proposition 1.13** In \(\mathbb{B}\) a morphism is a monomorphism if and only if it is one-to-one.

**Proof:** Let \(f:A \rightarrow B\) be one-to-one. If \(f':C \rightarrow A\) and \(f'':C \rightarrow A\) are not equal, then \(f'(c) \neq f''(c)\) for some \(c \in C\). Then \(f(f'(c)) \neq f(f''(c))\) or \(f \circ f' \neq f \circ f''\).

Let \(f:A \rightarrow B\) be a monomorphism. If \(f\) is not one-to-one, then \(f^1(0)\), a closed subset of \(A\), is not trivial.
Consider two unequal maps $f'$ and $f''$ from $f^1(0)$ to $A$: the insertion map and the constant zero map respectively. Now $f'$ and $f''$ are in $B^*$, but $f'f' \neq f''f''$. This is a contradiction.

**Definition 1.14** A morphism $f:A \rightarrow B$ is an epimorphism in a category $\mathcal{A}$ if and only if the only pair of morphisms $f':B \rightarrow C$ and $f'':B \rightarrow C$ such that $f'f = f''f$ is $f' = f''$.

**Proposition 1.15** In $B^*$ a morphism $f:A \rightarrow B$ is an epimorphism if and only if $f[A]$ is dense in $B$. ($f[A] = \{ b \in B | b = f(a) \text{ for some } a \in A \}$)

**Proof:** Suppose $f[A]$ is dense in $B$. Let $f':B \rightarrow C$ and $f'':B \rightarrow C$ be two unequal morphisms such that $f'f = f''f$. There exists some $c \in B$ such that $f'(c) \neq f''(c)$. Let $\varepsilon > 0$ be arbitrary. Since $f[A]$ is dense in $B$, there exists $b \in B$ such that $b = f(a)$ for some $a \in A$ and $|c - b| < \varepsilon/2$. Therefore $|(f' - f'')(c)| \leq |(f' - f'')(c - b)|$ as $f'(b) = f''(b)$ and $|(f' - f'')(c - b)| \leq |f' - f''| |c - b| < (|f'| + |f''|)\varepsilon/2 < \varepsilon$. This means $f'(c) \neq f''(c)$. This is a contradiction.

Let $f:A \rightarrow B$ be an epimorphism. Suppose $f[A]$ is not dense in $B$. Then $B/f[A]$ is a non-zero Banach space. ($f[A]$ is the closure in $B$ of $f[A]$) Consider two unequal maps $f'$ and $f''$ from $B$ to $B/f[A]$: the canonical map and the constant zero map respectively. Then both are in $B^*$ but $f'f = f''f$. This is a contradiction.

**Definition 1.16** In category $\mathcal{A}$, $f:A \rightarrow B$ is an isomorphism if and only if there exists $h:B \rightarrow A$ such that $f \circ h = 1_B$.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
and $g:B \rightarrow A$ such that $g \circ f = l_A$.

Every isomorphism is both an epimorphism and monomorphism. By virtue of the preceding propositions, it is now readily seen that $f:A \rightarrow B$ is an isomorphism in $B'$ if and only if $f$ is an isometric function and $f[A] = B$.

2. KERNELS, COKERNELS, IMAGES, COIMAGES, AND NORMALITY.

**Definition 2.1** An object $0$ in a category $\mathbb{C}$ is an initial object if $\text{hom}_\mathbb{C}(0,A)$ has exactly one element for each object $A$ in $\mathbb{C}$. It is a terminal object if $\text{hom}_\mathbb{C}(A,0)$ has exactly one element for each object $A$ in $\mathbb{C}$. It is a zero object if it is both an initial and terminal object.

**Definition 2.2** Let $\mathbb{C}$ be a category with a zero object. The kernel of a morphism $f:A \rightarrow B$ is a morphism $k:K \rightarrow A$ such that $f \circ k = 0$, and if $k':K' \rightarrow A$ is any morphism such that $f \circ k' = 0$ then there is a unique map $u:K' \rightarrow K$ such that $k \circ u = k'$.

![Diagram](https://via.placeholder.com/150)

**Notation:** The object $K$ is generally denoted by $\text{Ker } f$, while the morphism $k$ is sometimes denoted by $\text{ker } f$.

**Remark 2.3** If $k:K \rightarrow A$ and $k':K' \rightarrow A$ are both kernels of $f:A \rightarrow B$ then $K$ and $K'$ are isomorphic.

**Proposition 2.4** In $B'$, a kernel of $f:A \rightarrow B$ is $k:K \rightarrow A$
where \( K = \{ a \in A | f(a) = 0 \} \) and \( k \) is the insertion map, \( k(b) = b \) for all \( b \in K \).

**Proof:** As a closed subset of \( A \), \( K \) is a Banach space and clearly \( k \in B^* \). Let \( k':K' \rightarrow A \) be a morphism such that \( f \circ k' = 0 \). Define \( u:K' \rightarrow K \) by \( u(x) = k'(x) \). Since \( k'(x) \in K \) the map is well-defined and clearly \( k \circ u = k' \).

**Definition 2.5** Let \( \mathcal{A} \) be a category with zero object, and let \( f:A \rightarrow B \) be a morphism. A morphism \( g:B \rightarrow F \) is called a cokernel of \( f \) if \( g \circ f = 0 \), and if for every morphism \( g':B \rightarrow F' \) such that \( g' \circ f = 0 \) there is a unique morphism \( u:F \rightarrow F' \) such that \( u \circ g = g' \).

\[
\begin{array}{c}
A \\
\downarrow f \\
B \\
\downarrow g \\
F \\
\downarrow u \\
F'
\end{array}
\]

**Notation:** The object \( F \) is generally denoted by \( \text{Coker } f \), while the morphism \( g \) is sometimes denoted by \( \text{coker } f \).

**Remark 2.6** If \( g:B \rightarrow F \) and \( g':B \rightarrow F' \) are both cokernels of \( f:A \rightarrow B \), then \( F \) and \( F' \) are isomorphic.

**Proposition 2.7** In \( B^* \) a cokernel of \( f:A \rightarrow B \) is \( g:B \rightarrow B/F[A] \) where \( g \) is the canonical map.

**Proof:** Clearly \( g \circ f = 0 \) as \( g(f(x)) = [f(x)] = [0] \) where \([ \_ ]\) denotes an equivalence class in \( B/F[A] \). Let \( g':B \rightarrow F' \) be a morphism such that \( g' \circ f = 0 \). Define \( u:B/F[A] \rightarrow F' \) by \( u[y] = g'(y) \). Let \([y] = [x] \); then \( x - y \in F[A] \leq \ker g' \). This says \( g'(x) = g'(y) \) and \( u \) is well defined. If \([ \_ ]_g \)
denotes an equivalence class in $B/\text{Ker } g'$, then by lemma 1.11 and the fact that $\mathcal{F}[\mathcal{A}] \subseteq \text{Ker } g'$,
\[
|u| = \sup_{[y] \neq 0} \frac{|u[y]|}{|[y]|} = \sup_{[y] \neq 0} \frac{|g'(y)|}{|[y]|} = \sup_{[y] \neq 0} \frac{|g'(y)|}{|[y]|} |g'| \leq 1.
\]
Clearly $u \circ g = g'$. If $u':B/\mathcal{F}[\mathcal{A}] \to F'$ is any morphism such that $u' \circ g = g'$, then $u'[y] = u'(g(y)) = g'(y) = u(g(y)) = u[y]$, and $u$ is unique.

**Definition 2.8** If $f:A' \to A$ is a monomorphism, $A'$ is called a **subobject** of $A$ with respect to $f$.

The statement - let $f:A \to B$ be a subobject of $B$ - will be used to mean $A$ is a subobject of $B$ with respect to $f$.

**Terminology:** If $f':A' \to A$ and $f'':A'' \to A$ are subobjects of $A$, $A'$ is smaller than $A''$ if there is a morphism $u:A' \to A''$ such that $f'' \circ u = f'$.

**Definition 2.9** The **image** of a morphism $f:A \to B$ is the smallest subobject $f':I \to B$ through which $f$ factors, that is, $f = f' \circ f_1$ for some $f_1:A \to I$.

**Notation:** The image of $f:A \to B$ will be denoted by $\text{im } f$.

**Proposition 2.10** In $\mathcal{B}'$ the image of $f:A \to B$ is $g:A/\text{Ker } f \to B$ where $g$ is defined by $g([a]) = f(a)$.

**Proof:** Using lemma 1.11, $g$ is a monomorphism in $\mathcal{B}'$, and therefore $g:A/\text{Ker } f \to B$ is a subobject of $B$. Also if $c:A \to A/\text{Ker } f$ is the canonical mapping; then $g \circ c = f$, that is, $f$ factors through $A/\text{Ker } f$. Let $g':I' \to B$ be
any other subobject of B through which \( f \) factors, that is, \( f = g' \circ c' \) for \( c':A \rightarrow I' \).

\[
\begin{array}{ccc}
A/Ker f & \overset{g}{\rightarrow} & B \\
\downarrow{c} & \nearrow{u} & \downarrow{g'} \\
\ker f & \rightarrow & A \\
\end{array}
\]

Since \( g' \) is a monomorphism, \( c' \circ \ker f = 0 \); since \( c:A \rightarrow A/Ker f \) is a cokernel of \( \ker f:Ker f \rightarrow A \), there exists a unique \( u:A/Ker f \rightarrow I \) such that \( c' = u \circ c \). Therefore \( g \circ c = g' \circ u \circ c \) and since \( c \) is an epimorphism, \( g = g' \circ u \).

**Definition 2.11** If \( f:A \rightarrow A' \) is an epimorphism, \( A' \) is called a **quotient object** of \( A \) with respect to \( f \).

The statement - let \( f:A \rightarrow A' \) be a quotient object of \( A \) - will be used to mean \( A' \) is a subobject of \( A \) with respect to the epimorphism \( f \).

**Terminology:** If \( f_1:A \rightarrow A_1 \) and \( f_2:A \rightarrow A_2 \) are quotient objects of \( A \), \( A_1 \) is smaller than \( A_2 \) if there is a morphism \( u:A_2 \rightarrow A_1 \) such that \( u \circ f_2 = f_1 \).

**Definition 2.12** The **coimage** of a morphism \( f:A \rightarrow B \) is defined as the smallest quotient object of \( A \), \( f_1:A \rightarrow A_1 \), through which \( f \) factors, that is, \( f = f' \circ f_1 \) for some \( f':A_1 \rightarrow B \).

**Notation:** The coimage of \( f:A \rightarrow B \) will be denoted by \( \text{coim } f \).

**Proposition 2.13** In \( B' \), the coimage of \( f:A \rightarrow B \) is
$f_1: A \rightarrow \overline{f[A]}$ where $f_1$ is defined by $f_1(a) = f(a)$.

**Proof:** The morphism $f_1: A \rightarrow \overline{f[A]}$ is an epimorphism as $\overline{f_1[A]} = \overline{f[A]}$ so that $\overline{f[A]}$ is a quotient object of $A$.

The morphism $f: A \rightarrow B$ factors through $\overline{f[A]}$ with $f = f' \circ f_1$ where $f': \overline{f[A]} \rightarrow B$ is defined by $f'(b) = b$ for $b \in \overline{f[A]}$.

Let $f_2: A \rightarrow A_2$ be any other quotient object of $A$ through which $f$ factors, that is $f = f'' \circ f_2$ for $f'': A_2 \rightarrow B$.

Since $f_2$ is an epimorphism and $(\text{coker } f) \circ f'' \circ f_2 = 0$, $(\text{coker } f) \circ f'' = 0$; and since $f': \overline{f[A]} \rightarrow B$ is a kernel of $\text{coker } f: B \rightarrow B/\overline{f[A]}$, there exists a unique morphism $u: A_2 \rightarrow \overline{f[A]}$ such that $f'' = f' \circ u$. Therefore $f' \circ f_1 = f' \circ (u \circ f_2)$ and since $f'$ is a monomorphism, $f_1 = u \circ f_2$.

**Definition 2.14** A category is balanced if every morphism which is both a monomorphism and an epimorphism is also an isomorphism.

**Definition 2.15** If $A' \rightarrow A$ is the kernel of some morphism, then $A'$ is called a normal subobject of $A$. If every monomorphism in a category is normal, then the category is called normal.

The following example (taken from [6]) will show $B'$ is not balanced (in fact, a morphism with norm one
which is both an epimorphism and monomorphism is not an isomorphism) and not normal. Let $c_0$ be the Banach space of all sequences $b = (b_n)_{n \in \mathbb{N}}$, $b_n \in \mathbb{F}$ and $N = \{1, 2, 3, \ldots\}$, converging to zero with norm defined by $|b| = \sup_{n} |b_n|$. Define $g: c_0 \to c_0$ by $g((a_n)_{n \in \mathbb{N}}) = (a_n/n)_{n \in \mathbb{N}}$. Then $g$ is in $B^*$ and $|g| = 1$ (if $g((1, 0, 0, \ldots)) = (1, 0, 0, \ldots)$). Clearly $g$ is one-to-one and since $g(a) \neq (1/\sqrt{n})_{n \in \mathbb{N}}$ for any $a \in c_0$, $g$ is not an onto map. However, $g$ is an epimorphism, for its range contains the dense subspace of $c_0$, $\ell^\infty$, the space of all sequences with, at most, a finite number of non-zero terms. Therefore, $g$ is both a monomorphism and an epimorphism with $|g| \leq 1$. However $g$ is not an isomorphism, as it has no inverse in $B^*$. To see this, let $b_0 = (0, 0, 0, 2, 0, 0, \ldots)$. Then $g(b_0) = (0, 0, 0, 1/2, 0, 0, \ldots)$. If $g^{-1}$ existed in $B^*$, then

$$|g^{-1}| = \sup_{|a| \leq 1} |g^{-1}(a)| \geq |g^{-1}(g(b_0))| = |b_0| = 2.$$  

This is a contradiction.

The following proposition is proved in Mitchell [3, p 17].

**Proposition 2.16** Let $f:A \to B$ be a monomorphism with co-kernel 0 in a normal category. Then $f$ is an isomorphism. Hence, a normal category is balanced.

This proposition shows that since $B^*$ is not balanced, it is not normal.
Definition 3.1 In $\mathcal{B}$, let $\{B_i\}_I$ be a family of objects indexed by a set $I$. Then $\bigoplus \limits_I \text{join}(\{B_i\}_I)$ is the subset of the Cartesian product of $\{B_i\}_I$ consisting of all elements $f$ such that $|f|_1 = \sum_1^I |f_i| < \infty$.

Proposition 3.2 With $| \cdot |_1$ as norm $\bigoplus \limits_I \text{join}(\{B_i\}_I)$ is a Banach space with addition and scalar multiplication defined component-wise, that is, $(f + g)_1 = f_1 + g_1$ and $(kf)_1 = kf_1$ for $k \in \mathbb{F}$.

Proof: If $f, g \in \bigoplus \limits_I \text{join}(\{B_i\}_I)$, then $f + g \in \bigoplus \limits_I \text{join}(\{B_i\}_I)$ as:

$$|f + g|_1 = \sum_1^I |(f + g)_i| \leq \sum_1^I |f_i| + |g_i| = \sum_1^I |f_1| + \sum_1^I |g_1| < \infty.$$  

Also for $\alpha \in \mathbb{F}$, $\alpha f \in \bigoplus \limits_I \text{join}(\{B_i\}_I)$ as:

$$|\alpha f|_1 = \sum_1^I |\alpha f_i| = |\alpha| \sum_1^I |f_i| < \infty.$$  

Thus it can be immediately seen that for $f, g \in \bigoplus \limits_I \text{join}(\{B_i\}_I)$ the following are true:

(i) $|f|_1 \geq 0$; $|f|_1 = 0$ if and only if $f_i = 0$ for all $i \in I$

(ii) $|\alpha f|_1 = |\alpha||f|_1$

(iii) $|f + g|_1 \leq |f|_1 + |g|_1$.

Thus $\bigoplus \limits_I \text{join}(\{B_i\}_I)$ forms a normed linear space.

This normed linear space is also complete in its norm topology. Let $(f^n)_N$ be a Cauchy sequence in $\bigoplus \limits_I \text{join}(\{B_i\}_I)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
For each $\varepsilon > 0$, there exists $N_0 > 0$ such that for $m, n > N_0$,
\[ |f^n - f^m|_1 = \sum_{i} |f^n_i - f^m_i| < \varepsilon. \]

Therefore for each $i \in I$, $|f^n_i - f^m_i| < \varepsilon$ and $(f^n_i)_N$ is a Cauchy sequence in $B_i$. Since $B_i$ is complete, $f^n_i \rightarrow g_i$, $g_i \in B_i$. It shall be shown that $g : I \rightarrow \bigcup_{i} B_i$ defined by $g(i) = g_i$ is in $\prod_{i} \operatorname{Join}(B_i)$ and $f^n \rightarrow g$. Since for $n, m > N_0$ and any finite subset $\cap_{i} I$,
\[ \sum_{i \in \cap_{i} I} |f^n_i - g_i| = \lim_{m \rightarrow \infty} \sum_{i \in \cap_{i} I} |f^n_i - f^m_i| < \varepsilon, \]
then $\sum_{i \in I} |f^n_i - g_i| < \varepsilon$ or $f^n \rightarrow g$. Also as
\[ |g|_1 = \sum_{i} |g_i| < \sum_{i} |g_i - f^n_i| + \sum_{i} |f^n_i| < \varepsilon + \sum_{i} |f^n_i| < \infty \text{ for some } n > N_0, \]
$g \in \prod_{i} \operatorname{Join}(\{B_i\}_I)$. This completes the proof.

**Definition 3.3** Let $\{A_i\}_I$ be a family of objects in a category $A$ indexed by a set $I$. The **coproduct** (sum) of $\{A_i\}_I$ is an object $\sum_{i} A_i$ in $A$ together with a family of morphisms $\{u_i : A_i \rightarrow \sum_{i} A_i\}_I$ such that for any family of morphisms $\{x_i : A_i \rightarrow X\}_{i \in I'}$, $X$ an object in $A$, there exists a unique morphism $u : \sum_{i} A_i \rightarrow X$ such that $u \circ u_i = x_i$ for all $i \in I$.

\[ \begin{tikzpicture}
    \node (X) at (0,0) {$X$};
    \node (A1) at (3,0) {$A_1$};
    \node (A) at (1.5,0) {$\sum_{i} A_i$};
    \node (x1) at (2,1) {$x_1$};
    \draw[->] (X) -- (A1);
    \draw[->] (X) -- (A);
    \draw[->,dashed] (A) -- (x1);
    \draw[->] (A1) -- (x1);
\end{tikzpicture} \]

**Remark 3.4** The objects of any two coproducts of the

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
family, \( \{A_i\}_I \), are isomorphic.

**Proposition 3.5** In \( B' \) the coproduct of the family of objects \( \{B_i\}_{i \in I} \) is the space \( \bigoplus_i \text{join}(\{B_i\}_I) \) together with the family of morphisms \( \{u_i : B_i \to \bigoplus_i \text{join}(\{B_i\}_I)\}_I \) where for \( t \in I \), \( u_t(b_t) = f \) in the Cartesian product of \( \{B_i\}_I \) defined by \( f_i = 0_i \) if \( i \neq t \), \( f_t = b_t \).

**Proof:** For all \( i \in I \), \( u_i \in B' \) as each is linear, and

\[
|u_i| = \sup_{b_i \leq 1} |u_i(b_i)| = \sup_{b_i \leq 1} \sum_{i \in I} |f_i| = \sup_{b_i \leq 1} \sum_{i \in I} |f_i| \leq 1.
\]

Now let \( X \) be any Banach space and \( \{x_i : B_i \to X\}_I \) a family of morphisms in \( B' \). A unique morphism \( u : \bigoplus_i \text{join}(\{B_i\}_I) \to X \) must be found so that for all \( i \in I \) a diagram like that above commutes.

Define \( u : \bigoplus_i \text{join}(\{B_i\}_I) \to X \) by \( u(f) = \sum_I x_i(f_i) \).

Since \( \sum_I |x_i(f_i)| \leq \sum_I |f_i| < \infty \), \( \sum_I x_i(f_i) \) converges and the function is well-defined. Also \( u \) is linear since for \( f, g \in \bigoplus_i \text{join}(\{B_i\}_I) \) and \( \alpha \in \mathbb{F} \),

\[
u(f + g) = \sum_I x_i(f_i + g_i) = \sum_I x_i(f_i) + x_i(g_i)
= \sum_I x_i(f_i) + \sum_I x_i(g_i) = u(f) + u(g),
\]

and

\[
u(\alpha f) = \sum_I x_i(\alpha f_i) = \sum_I \alpha x_i(f_i) = \alpha \sum_I x_i(f_i) = \alpha u(f).
\]

The norm of \( u \) satisfies \( |u| \leq 1 \) as

\[
|u| = \sup_{|f_i| \leq 1} |u(f)| = \sup_{|f_i| \leq 1} |\sum_I x_i(f_i)| = \sup_{|f_i| \leq 1} (\sum_I |f_i|) \leq 1.
\]

Also for all \( i \in I \), \( u_i u_i = x_i \). Let \( t \in I \). Then for \( b_t \in B_t \),

\[(u \circ u_t)(b_t) = u(f) = \sum_I x_i(f_i) = x_t(b_t).
\]

The map \( u \) is unique. Let \( f \notin \bigoplus_i \text{join} \); then \( f \) can
be written as $\sum_I u_i(f_1)$. If $u': l_1^* \to \text{Join} \to X$ is any map in $B^*$ such that $u'(u_1) = x_1$ for all $i \in I$, then as $u'$ is continuous (see Remark 1.10),

$$u'(f) = u'(\sum_I u_i(f_1)) = \sum_I x_i(f_1) = u(f).$$

**Definition 3.6** In $\mathcal{B}^*$, let $\{B_i\}_I$ be a family of Banach spaces indexed by $I$. Then $\bigcup_{\infty} \text{Join}(\{B_i\}_I)$ is the subset of the Cartesian product of $\{B_i\}_I$ consisting of all elements $f$ such that $|f|_\infty = \sup_{i \in I} |f_i| < \infty$.

**Proposition 3.7** $\bigcup_{\infty} \text{Join}(\{B_i\}_I)$ is a Banach space with addition and scalar multiplication defined component-wise and $| |_\infty$ as norm.

**Proof:** If $g, f \in \bigcup_{\infty} \text{Join}(\{B_i\}_I)$, then $f + g \in \bigcup_{\infty} \text{Join}(\{B_i\}_I)$ as

$$|f + g|_\infty = \sup_I |f_i + g_i| \leq \sup_I (|f_i| + |g_i|)$$

$$\leq \sup_I |f_i| + \sup_I |g_i| < \infty.$$  

Also for $\alpha \in \mathcal{F}$, $\alpha f \in \bigcup_{\infty} \text{Join}(\{B_i\}_I)$ as

$$|\alpha f|_\infty = \sup_I |\alpha f_i| = |\alpha| \sup_I |f_i| = |\alpha| |f|_\infty < \infty.$$  

From this it can be seen that

(i) $|f|_\infty \geq 0$; $|f|_\infty = 0$ if and only if $f_i = 0_1$ for all $i \in I$.

(ii) $|\alpha f|_\infty = |\alpha| |f|_\infty$ for all $\alpha \in \mathcal{F}$.

(iii) $|f + g|_\infty \leq |f|_\infty + |g|_\infty$

Thus $\bigcup_{\infty} \text{Join}(\{B_i\}_I)$ forms a normed linear space.

This normed linear space is also complete in its norm topology. Let $(f_n)_N$ be a Cauchy sequence in
$L_\infty$-join($\{B_i\}_I$). For each $\epsilon > 0$, there exists $N_0 > 0$ such that for $n,m > N_0$,

$$|f_n^i - f_m^i|_\infty = \sup_I |f_n^i - f_m^i| < \epsilon.$$ 

Therefore for each $i \in I$, $|f_n^i - f_m^i| < \epsilon$ and $(f_n^i)_N$ is a Cauchy sequence in $B_i$. Since $B_i$ is complete, $f_n^i \rightarrow g_i$ for $g_i \in B_i$. It will be shown that $g : I \rightarrow \bigcup B_i$, defined by $g(i) = g_i$, is in $L_\infty$-join($\{B_i\}_I$); and $f_n \rightarrow g$. Since $|f_n|_\infty < \infty$, for each $n$, there exists $M_n$ such that $|f_n^i| < M_n - \epsilon$ for all $i \in I$. Pick $m,n > N_0$. Then

$$|g_i| = |g_i - f_n^i + f_m^i - f_n^i + f_m^i|$$

$$\leq |g_i - f_n^i| + |f_n^i - f_m^i| + |f_m^i|$$

$$\leq |g_i - f_n^i| + \epsilon + M_m - \epsilon.$$ 

Letting $m \rightarrow \infty$, since the left side is independent of $n$ and $|g_i - f_n^i| \rightarrow 0$, $|g_i| \leq M_m$ or $g \in L_\infty$-join($\{B_i\}_I$).

Now $f_n \rightarrow g$ if and only if for all $\epsilon > 0$, there exists $N$ such that for $n > N$, $|f_n^i - g_i| < \epsilon$ for all $i \in I$. Pick $n,m > N_0$. Then for all $i \in I$

$$|f_n^i - g_i| \leq |f_n^i - f_m^i| + |f_m^i - g_i| < \epsilon + |f_m^i - g_i|.$$ 

Letting $m \rightarrow \infty$, $|f_n^i - g_i| < \epsilon$ and the result follows.

**Definition 3.8** Let $\{A_i\}_I$ be a family of objects in a category $\mathcal{A}$. The **product** of $\{A_i\}_I$ is an object, $\prod A_i^I$, in $\mathcal{A}$ together with a family of morphisms, $\{p_i : \prod A_i^I \rightarrow A_i\}_I$, such that for any family of morphisms $\{x_i : X \rightarrow A_i\}_I$, $X$ an object in $\mathcal{A}$, there is a unique morphism $u : X \rightarrow \prod A_i^I$ such that $p_i^u = x_i$ for all $i \in I$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Remark 3.9 The objects of any two products of the family \(\{A_i\}_I\) are isomorphic.

**Proposition 3.10** In \(\mathcal{B}'\), \(\bigwedge \cap \) join(\(\{B_i\}_I\)) together with the family, \(\{p_i: \bigwedge \cap \) join(\(\{B_i\}_I\)) \(\longrightarrow B_i\}_I\) where for \(t \in I\), \(p_t(f) = f_t\) is the product of the family \(\{B_i\}_I\).

**Proof:** For all \(i \in I\), \(p_i \in \mathcal{B}'\) as
\[
|p_i| = \sup_{|f| \leq 1} |p_i(f)| = \sup_{|f| \leq 1} |f|_I = \sup_{|f| \leq 1} |f|_I.
\]
Let \(X\) be any object in \(\mathcal{B}'\) and \(\{x_i:X \longrightarrow B_i\}_I\) be a family of morphisms in \(\mathcal{B}'\). Define \(u:X \longrightarrow \bigwedge \cap \) join(\(\{B_i\}_I\)) by
\[
u(x) = f \text{ where } f_i = x_i(x).\]
Then \(u \in \mathcal{B}'\) as
\[
|u| = \sup_{|x| \leq 1} |u(x)| = \sup_{|x| \leq 1} |f|_I = \sup \sup_{|x| \leq 1} |x_i(x)|
\]
\[
= \sup \sup_{|x| \leq 1} |x_i(x)| \leq 1.
\]
Also it is clear that \(p_i \circ u = x_i\) for each \(i \in I\). If \(u'\) is any map \(X \longrightarrow \bigwedge \cap \) join(\(\{B_i\}_I\)) such that \(p_i \circ u' = x_i\) for all \(i \in I\), then \(u' = u\); since for all \(i \in I\), \((u'(x))_1 \equiv (p_i \circ u')(x) = (p_i \circ u)x = (u(x))_1\), that is, each component of \(u'(x)\) and \(u(x)\) are equal; and hence \(u'(x) = u(x)\).

It should be noted that the fact that the product and coproduct in \(\mathcal{B}'\) are \(\bigwedge \cap \) join and \(\bigvee \cap \) join respectively is given by Semadeni [5, p. 359].

**Definition 3.11** Let \(f_1:A_1 \longrightarrow A\) and \(f_2:A_2 \longrightarrow A\) be morphisms...
in a category $\mathcal{A}$. A commutative diagram,

\[
\begin{array}{ccc}
P & \xrightarrow{g_2} & A_2 \\
| & \downarrow{g_1} & \\
A_1 & \xrightarrow{f_1} & A
\end{array}
\]

with $g_2 \in \text{hom}_\mathcal{A}(P_1, A_2)$ and $g_1 \in \text{hom}_\mathcal{A}(P, A_1)$, is called a pullback for $f_1$ and $f_2$ if for every pair of morphisms $g_1': P' \rightarrow A_1$ and $g_2': P' \rightarrow A_2$ in $\mathcal{A}$ such that $f_1 \circ g_1' = f_2 \circ g_2'$, there exists a unique morphism, $u: P' \rightarrow P$, such that $g_1' = g_1 \circ u$ and $g_2' = g_2 \circ u$.

**Remark 3.12** If

\[
\begin{array}{ccc}
P' & \xrightarrow{g_2'} & A_2 \\
| & \downarrow{g_1'} & \\
A_1 & \xrightarrow{f_1} & A
\end{array}
\]

is also a pullback for $f_1$ and $f_2$, then $P'$ and $P$ are isomorphic.

**Proposition 3.13** In $\mathcal{B}^*$ the diagram,

\[
\begin{array}{ccc}
P & \xrightarrow{g_2} & B_2 \\
| & \downarrow{g_1} & \\
B_1 & \xrightarrow{f_1} & B
\end{array}
\]

where $P = \{ b = (b_1, b_2) | (b_1, b_2) \in \bigsqcup_{i=1,2} B_i | f_1(b_1) = f_2(b_2) \}$ and $g_1$ and $g_2$ are the restrictions of the projections from $\bigsqcup_{i=1,2} B_i$ to $B_1$ and $B_2$ respectively, is a pullback of $f_1$ and $f_2$.

**Proof:** As a closed subspace of $\bigsqcup_{i=1,2} B_i$, $P$ is itself a Banach space. To see that $P$ is closed, let $b$ be in the closure of $P$ and $b^n$ a sequence in $P$ converging to $b$. Then the
coordinates $b_1^B$ and $b_2^B$ of $b^B$ converge to the coordinates of $b$, $b_1$ and $b_2$ respectively. However, $b \in P$ as
\[
    f_1(b_1) = f_1(\lim_{n} b_1^n) = \lim_{n} f_1(b_1^n) = \lim_{n} f_2(b_2^n) = f_2(b_2).
\]

Since $g_1$ and $g_2$ are restrictions to $P$ of the respective projections, each is in $B^*$. Now let $g_1^p : P^p \rightarrow B_1$ and $g_2^p : P^p \rightarrow B_2$ be two morphisms in $B^*$ such that $f_1 \circ g_1^p = f_2 \circ g_2^p$. Define $u : P^p \rightarrow P$ by $u(p') = (g_1^p(p'), g_2^p(p'))$.

Then $u$ is well-defined as $f_1(g_1^p(p')) = f_2(g_2^p(p'))$ because $f_1 \circ g_1^p = f_2 \circ g_2^p$. Also $u \in B^*$ since
\[
    |u| = \sup_{p} |u(p)| = \sup_{i=1}^{\infty} \sup_{p} \sup_{i=1}^{\infty} |g_1^p(p)| \leq 1
\]
and
\[
    |g_1^p(p)| \leq 1.
\]

Also $g_1 \circ u = g_1^p$, $i = 1, 2$, as
\[
    (g_1 \circ u)(p) = (g_1(g_1^p(p)), g_2(g_1^p(p))) = g_1^p(p).
\]

If $u' : P^p \rightarrow P$, $u' \neq u$, also satisfies the conditions $g_1 \circ u' = g_1^p$, $i = 1, 2$; then for some $p' \in P^p$, $u(p') \neq u'(p)$. This means that for $i = 1$ or $2$, $g_1^p(p') = g_1(u'(p')) \neq g_1(u(p')) = g_1^p(p')$. This is a contradiction.

**Definition 3.14** Let $f_1 : A \rightarrow A_1$ and $f_2 : A \rightarrow A_2$ be morphisms in a category $\mathcal{A}$. A commutative diagram,

\[
\begin{array}{ccc}
A & \xrightarrow{f_2} & A_2 \\
\downarrow{f_1} & & \downarrow{g_2} \\
A_1 & \rightarrow & P \\
\end{array}
\]
with \( g_1 \in \text{hom}_A(A_1, P) \) and \( g_2 \in \text{hom}_A(A_2, P) \) is called a pushout for \( f_1 \) and \( f_2 \) if for every pair of morphisms \( g_1': A_1 \rightarrow P' \) and \( g_2': A_2 \rightarrow P' \) in \( \mathcal{A} \) such that \( g_1' \circ f_1 = g_2' \circ f_2 \), there exists a unique morphism \( u: P \rightarrow P' \), such that 
\[ u \circ g_1 = g_1' \text{ and } u \circ g_2 = g_2'. \]

**Remark 3.15** If 
\[ \xymatrix{ A \ar[r]^{f_2} \ar[d]_{f_1} & A_2 \ar[d]^{g_2'} \ar[dl]_{g_1'} \ar[l]_{g_1} & \} \]
is also a pushout for \( f_1 \) and \( f_2 \), then \( P' \) and \( P \) are isomorphic.

**Proposition 3.16** In \( \mathcal{B} \), the diagram,
\[ \xymatrix{ A \ar[r]^{f_2} \ar[d]_{f_1} & A_2 \ar[d]^{g_2} \ar[dl]_{g_1} \ar[l]_{g_1} \ar[dr]_{g_2} & \} \]
where \( P = \sum_{i=1}^{2} A_i / I, I = \{(x, y) \in \sum_{i=1}^{2} A_i \mid \text{there exists } a \in A \text{ such that } f_1(a) = -x \text{ and } f_2(a) = y\} \), and \( g_1: A_1 \rightarrow P \) is defined by \( g_1(a_1) = [(a_1, 0)] \) while \( g_2: A_2 \rightarrow P \) is defined by \( g_2(a_2) = [(0, a_2)] \), is the pushout of \( f_1 \) and \( f_2 \).

**Proof**: Let \( (x_1, x_2) \in I \) and \( (y_1, y_2) \in I \) with \( f_1(a) = -x_1 \), \( f_2(a) = x_2 \), \( f_1(b) = -y \), and \( f_2(b) = y_2 \). Then \( (x_1, x_2) + (y_1, y_2) \in I \) as \( f_1(a + b) = -(x_1 + y_1) \) and \( f_2(a + b) = (x_2 + y_2) \). Similarly for \( \alpha \in \Phi \), \( \alpha(x_1, x_2) \in I \). Thus \( I \) is a subspace of \( \sum_{i=1}^{2} A_i / I \). Therefore \( \sum_{i=1}^{2} A_i / I \) is a Banach space with norm \( \|(x, y)\| = \inf \{(x, y') \mid (x, y') \in [(x, y)] \} \).
The morphisms $g_1$ and $g_2$ are in $\mathcal{B}^*$ since

$$|g_1| = \sup_{|a_1|, |a_1|} |g_1(a_1)| = \sup_{|a_1|} |[a_1, O_2]|$$

$$= \sup_{|a_1|} \inf_{(x, y) \in [a_1, O_2]} (|x| + |y|) \leq \sup_{|a_1|} |a_1| \leq 1.$$ 

A similar inequality involving $g_2$ can be formed. Also

$$g_2 \circ f_2 = g_1 \circ f_1$$

as $g_1(f_1(a)) = [(f_1(a), O_2)] = [(0_1, f_2(a))] = g_2(f_2(a))$ since $(f_1(a), O_2) - (0_1, f_2(a_2)) \notin I$, for all $a \in A$.

Now let $g_i : A_i \to P'$ and $g_i : A_2 \to P'$ be morphisms in $\mathcal{B}$ such that $g_i \circ f_1 = g_i \circ f_2$. Define $u : P \to P'$ by $u([(a_1, a_2)]) = g_1(a_1) + g_2(a_2)$. Note that $u$ is well-defined for if $[(a_1, a_2)] = [(b_1, b_2)]$, then there exists $a \in A$ such that

$$f_1(a) = a_1 - b_1 \text{ and } f_2(a) = b_2 - a_2.$$ 

Therefore

$$g_1(a_1) - g_1(b_1) = g_2(a_2) - g_2(b_2) = g_1(f_1(a)) = g_2(f_2(a))$$

which implies $g_1(a_1) + g_2(a_2) = g_1(b_1) + g_2(b_2)$. Also

$$|u| \leq 1.$$ 

To understand this it must be noted that for all $(a_1, a_2) \in [(b_1, b_2)] \neq [(0_1, O_2)]$,

$$|u([(b_1, b_2)])| = \frac{|u([(a_1, a_2)])|}{|(a_1, a_2)|} = \frac{|g_1(a_1) + g_2(a_2)|}{|a_1| + |a_2|} \leq \frac{|a_1| + |a_2|}{|a_1| + |a_2|} \leq 1.$$ 

Hence

$$\frac{|u([(b_1, b_2)])|}{|(a_1, a_2)|} \leq 1.$$ 

Therefore

$$|u| \leq \sup_{[(b_1, b_2)] \neq [(0_1, O_2)]} \frac{|u([(b_1, b_2)])|}{|[(b_1, b_2)]|} \leq 1.$$ 

It is clear that $u$ satisfies the conditions $u \circ g_i = g_i'$, $i = 1, 2$. If $u' : P \to P'$ is any map satisfying these conditions,
\[
\begin{align*}
(u' \circ g_1)(a_1) &= u'(g_1(a_1)) = u'[([a_1, 0_2])] = g_1'(a_1) \\
(u' \circ g_2)(a_2) &= u'(g_2(a_2)) = u'([0_1, a_2]) = g_2'(a_2) \\
\text{Hence} \\
\quad&u'(\llbracket a_1, 0_2 \rrbracket + \llbracket 0_1, a_2 \rrbracket) = u'([a_1, a_2]) = g_1'(a_1) + g_2'(a_2),
\end{align*}
\]

and \(u' = u\).

\[\text{4. UNIONS AND INTERSECTIONS} \]
\[\text{PROJECTIVE AND INJECTIVE OBJECTS}\]

**Definition 4.1** Let \(f: A' \to A\) and \(g: B' \to B\) be monomorphisms and \(h: A \to B\) any morphism (refer to diagram below). Subobject \(A'\) is said to be \textit{carried into} the subobject \(B'\) by \(h\) if there exists a morphism \(u: A' \to B'\) such that \(g \circ u = h \circ f\).

\[
\begin{tikzcd}
A' \arrow{r}{u} \arrow{d}{f} & B' \arrow{d}{g} \\
A \arrow{r}{h} & B
\end{tikzcd}
\]

**Definition 4.2** Let \(\{A_i\}_{i \in I}\) be a family of subobjects of \(A\) corresponding to monomorphisms \(\{f_i\}_{i \in I}, f_i: A_i \to A\). The \textit{union} of \(\{A_i\}_{i \in I}\) is a subobject \(A'\) of \(A\) corresponding to some monomorphism \(g: A' \to A\) such that

1. for each \(A_i\) there exists \(g_i: A_i \to A'\) such that \(g \circ g_i = f_i\).
2. if \(f: A \to B\) is any morphism to an object \(B\) and each \(A_i\) is carried into some \(B'\) of \(B\) by \(f\), then \(A'\) is also carried into \(B'\) by \(f\).

All these morphisms are shown in the following diagram.
Remark 4.3 Any other subobject of $A$ which behaves as a union of the family $\{A_i\}_{i \in I}$ must be isomorphic to $A'$.

Proposition 4.4 $B'$ has unions; that is, a union exists for every family of subobjects of any object $A$ in $B'$.

Proof: Let $\{A_i\}_{i \in I}$ be a family of subobjects of a Banach space $A$ with corresponding monomorphisms $\{f_i: A_i \to A\}_{i \in I}$.

Define $g': \sum_{i \in I} A_i \to A$ by $g'(a) = \sum_{i \in I} f_i(a_i)$.

Since $\sum_{i \in I} f_i(a_i)$ is absolutely convergent, it is convergent; and $|g'| \leq 1$ as $|\sum_{i \in I} f_i(a_i)| \leq \sum_{i \in I} |f_i(a_i)| \leq \sum_{i \in I} |a_i|$. Using remark 1.9 above, $g'$ is also linear. Therefore $g'$ is in $B'$. Now define $g: \sum_{i \in I} A_i / \ker g' \to A$ by $g([a]) = g'(a)$. By lemma 1.11 above $g$ is in $B'$. Also $g$ is a monomorphism, for if $g([a]) = g([b])$ then $g'(a) = g'(b)$ or $[a] = [b]$. Therefore $A' = \sum_{i \in I} A_i / \ker g'$ is a subobject of $A$.

Now for each $k \in I$ and for all $x_k \in A_k$ define $g_k: A_k \to A'$ by $g_k(x_k) = [a]$ where $a_i = 0$ if $i \neq k$ and $a_k = x_k$. Since $|g_k| = \sup_{|x_k| \leq 1} |g_k(x_k)| = \sup_{|x_k| \leq 1} |[a]| = \sup_{|x_k| \leq 1} \inf_{b \in [a]} |b| \leq \sup_{|x_k| \leq 1} |x_k| = 1$, $g$ is in $B'$. Clearly for all $i \in I$, $g_fg_i = f_i$.

Now let $f: A \to B$ be a morphism in $B'$ and $B'$ a subobject of $B$ with respect to a monomorphism $h: B' \to B$ such
that each $A_i$ is carried by $f$ to $B'$, that is, for all $i \in I$ there exists $h_i:A_i \rightarrow B'$ such that $f \circ f_i = h \circ h_i$. Define $u:A' \rightarrow B'$ by $u([a]) = \sum h_i(a_i)$. This is a well-defined function, as $\sum h_i(a_i)$ is absolutely convergent and hence convergent. Also if $[a] = [b]$, then $\sum f_i(a_i) = g'(a) = g'(b) = \sum f_i(b_i)$. Hence $\sum (f(a_i) - f(b_i)) = 0$, which means $\sum h_i(a_i) = \sum h_i(b_i)$, for

$$h_i(a_i) = \sum h_i(a_i) = \sum h_i(b_i).$$

since $h$ is a monomorphism. In this same way $h \circ u = f \circ g$, for

$$(f \circ g)([a]) = f(g'(a)) = f(\sum f_i(a_i)) = \sum f(f_i(a_i)) = \sum h(h_i(a_i)) = h(\sum h_i(a_i)) = (h \circ u)([a]).$$

Thus $A'$ is also carried by $f$ into $B'$.

**Definition 4.5** Let $\{f_i:A_i \rightarrow A\}_{i \in I}$ be a family of subobjects of $A$ in $\mathbf{A}$. A morphism $f:A' \rightarrow A$ is called the **intersection** of the family if

1. for each $i \in I$, $f = f_i \circ g_i$ for some morphism $g_i:A' \rightarrow A_i$ (necessarily unique), and

2. for every morphism $h:A' \rightarrow A$ such that for all $i$, $h = f_i \circ h_i$ for some $h_i:A'' \rightarrow A_i$ there exists a unique morphism $g:A'' \rightarrow A'$ such that $h = f \circ g$. 

\[ \text{Diagram:} \]

\begin{tikzcd}
A' \arrow{dr}{g} \arrow[swap]{d}{h_i} & \arrow{r}{f} & A' \\
A'' \arrow{ur}{g_i} & \arrow{r}{f_i} & A
\end{tikzcd}
Remark 4.6 From the uniqueness of \( g \) it can be shown that if \( h:A'' \longrightarrow A \) is also an intersection for the family \( \{ f_i:A_i \longrightarrow A \} \), then \( A' \) and \( A'' \) are isomorphic.

Proposition 4.7 \( B^* \) has intersections; that is, the intersection exists for every family of subobjects of any object in \( B^* \).

Proof: Let \( \{ f_i:A_i \longrightarrow A \} \) be a family of subobjects of \( A \).

Let \( A' = \{ b \in \bigcap A_i \mid \text{for all } i, j \in I, f_i(b_i) = f_j(b_j) \} \).

As a closed subspace of \( \bigcap A_i \), \( A' \) is also a Banach space.

To see that \( A' \) is closed, let \( A'_{k_j} = \{ b \in \bigcap A_i \mid f_k(b_k) = f_j(b_j) \} \). Each \( A'_{k_j} \) is closed (see proposition 3.13) and \( A' = \bigcap_{k_j} A'_{k_j} \). Now let \( p \) be a fixed element of \( I \) and define \( f:A'' \longrightarrow A \) by \( f(b) = f_p(b_p) \). The function \( f \) is a morphism in \( B^* \) since \( f_p \) linear makes \( f \) linear, and

\[
|f| = \sup_{|b| \leq 1} |f(b)| = \sup_{|b| \leq 1} |f_p(b_p)| \leq \sup_{|b_p| \leq 1} |f_p(b_p)| \leq 1.
\]

If for each \( i \in I \), \( g_i:A_i \longrightarrow A_i \) is the restriction of the projection from \( \bigcap A_i \longrightarrow A_i \) then \( f = (f_i \circ g_i) \), then \( f = (f_i \circ g_i) \) since \( f_i(b_i) = f_p(b_p) \) for all \( i \in I \).

Now let \( h:A'' \longrightarrow A \) be a morphism such that for all \( i \in I \), \( h = f_i \circ h_i \) for some \( h_i:A_i \longrightarrow A_i \). Then for all \( a \in A'' \), \( \bar{a} \in A' \) where \( \bar{a} = h_1(a) \) since for all \( i, j \in I \), \( f_i(\bar{a}_i) = f_i(h_1(a)) = h(a) = f_j(h_j(a)) = f_j(\bar{a}_j) \).

Define \( g:A'' \longrightarrow A' \) by \( g(a) = \bar{a} \). Because of the inequality

\[
|g| = \sup_{|a| \leq 1} |\bar{a}| = \sup_{|a| \leq 1} \sup_{I} |h_i(a)| = \sup_{|a| \leq 1} \sup_{I} |h_i(a)| \leq 1,
\]

\( g \) is a morphism in \( B^* \). The equality

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[(f \circ g)a = f(a) = f_p(h_p(a)) = h(a)\]

shows \(f \circ g = h\).

Now if \(f(b) = f(c)\), that is, \(f_p(b_p) = f_p(c_p)\); then for all \(i \in I\), \(f^i_1(b_i) = f^i_1(c_i)\). Since each \(f^i_1\) is a monomorphism, \(c_i = b_i\) for all \(i \in I\). Therefore \(b = c\) and \(f\) is a monomorphism. This fact shows that if \(g' : A' \rightarrow A\) is any morphism such that \(f \circ g' = h\), then \(g = g'\) as \(f \circ g = f \circ g'\).

**Definition 4.8** A category \(\mathbf{A}\) with coproducts is called a \(\mathbf{C}_1\) category if for every family of monomorphisms \(\{u_i : A_i \rightarrow B_i\}_I\) the unique morphism \(u : \sum_i A_i \rightarrow \sum_i B_i\) such that the following diagram commutes is a monomorphism.

\[
\begin{array}{ccc}
A_i & \rightarrow & \sum_i A_i \\
\downarrow u_i & & \downarrow u \\
B_i & \rightarrow & \sum_i B_i \\
\end{array}
\]

Since the unique map \(u : \sum_i A_i \rightarrow \sum_i B_i\) is defined by \(u(a) = b\) where \(b_i = u_1(a_1)\) and \(u\) is a monomorphism, \(\mathbf{B}\) is a \(\mathbf{C}_1\) category.

**Definition 4.9** A category \(\mathbf{A}\) is called a \(\mathbf{C}_2\) category if it has products, coproducts, and a zero object and if the morphism \(f : \prod_i A_i \rightarrow \prod_i A_i\) so that the diagram

\[
\begin{array}{ccc}
\sum_i A_i & \hbox{ } \xrightarrow{f} \hbox{ } & \prod_i A_i \\
\downarrow p^i & & \downarrow p_j \\
A_i & \hbox{ } \xrightarrow{p^j} \hbox{ } & A_j \\
\end{array}
\]
commutes for all \( j \in I \) is a monomorphism (\( p_j \) and \( p'_j \) are the projections from \( \prod A_i \) and \( \sum A_i \) respectively).

\( B \) is a \( C_2 \) category with the unique map \( f \) given by \( f(b) = b \) which is clearly a monomorphism.

**Definition 4.10** An object \( P \) in a category \( A \) is projective if for every epimorphism \( f : A \rightarrow B \) and morphism \( g : P \rightarrow B \) there exists a morphism \( h : P \rightarrow A \) such that \( f \circ h = g \).

\[ \begin{array}{ccc}
A & \rightarrow & B \\
\downarrow h \quad & & \downarrow g \\
\uparrow f 
\end{array} \]

**Definition 4.11** An object \( Q \) in a category \( A \) is injective if for every monomorphism \( f : A \rightarrow B \) and morphism \( g : A \rightarrow Q \) there exists a morphism \( h : B \rightarrow Q \) such that \( h \circ f = g \).

\[ \begin{array}{ccc}
A & \rightarrow & Q \\
\downarrow f \\
\uparrow h 
\end{array} \]

**Definition 4.12** A morphism \( f : A \rightarrow B \) is called a coretraction if there is a morphism \( g : B \rightarrow A \) such that \( g \circ f = 1_A \).

Then \( A \) is called a retract of \( B \). The morphism \( f \) is called a retraction if there is a morphism \( g' : B \rightarrow A \) such that \( f \circ g' = 1_B \).

The following two lemmas are established in categorical algebra [3, p.70].

**Lemma 4.13** If for objects \( A_1 \) and \( A_2 \) in a category \( A \), \( A_1 \) is a retract of \( A_2 \) and \( A_2 \) is injective (projective),
Lemma 4.14 If $A = \prod_{\mathcal{I}} A_\mathcal{I}$ (sum $A_\mathcal{I}$ for each $A_\mathcal{I}$ is injective (projective)), then $A$ is injective (projective). Conversely, in a category with zero if $A$ is injective (projective), then each $A_\mathcal{I}$ is injective (projective).

Lemma 4.15 The scalar field $\mathbb{F}$ is a retract of every non-zero Banach space.

Proof: Let $B$ be a nonzero Banach space. Fix $b_0 \in B$, $b_0 \neq 0$. Let $b_0^*: B \to \mathbb{F}$ be a morphism with $|b_0^*| = 1$ and $b_0^*(b_0) = |b_0| [1, p.65]$. Then $b_0^*(b_0/|b_0|) = 1$. Define $f: \mathbb{F} \to B$ by $f(m) = mb_0/|b_0|$. Then $f$ is a morphism in $B^*$ and $b_0^*f = 1$ (the identity morphism of $\mathbb{F}$). Therefore $\mathbb{F}$ is a retract of $B$.

Lemma 4.16 The scalar field $\mathbb{F}$ is not injective in $B^*$.

Proof: Let $\ell_1$ be the Banach space consisting of sequences $a \in (a_i)_{i \in \mathbb{N}}, a_i \in \mathbb{F}$ and $\mathbb{N} = \{1,2,3,\ldots\}$, such that the norm $|a| = \sum_{i \in \mathbb{N}} |a_i| < \infty$. Define $g: \ell_1 \to \ell_1$ by $g((a_i)_{i \in \mathbb{N}}) = (a_i/1 + 1)_{i \in \mathbb{N}}$. The map $g$ is a monomorphism in $B^*$. Let $b = (1/i!)_{i \in \mathbb{N}}$. Since $|b| = \sum_{i \in \mathbb{N}} 1/i! = e - 1$, $b \in \ell_1$. Now let $b^*: \ell_1 \to \mathbb{F}$ be a linear functional with $|b^*| = 1$ and $b^*(b) = |b|$. There is no morphism $u: \ell_1 \to \mathbb{F}$ in $B^*$ such that $u \circ g = b^*$, that is, so that the following diagram commutes.

```
\[ \begin{array}{ccc}
\ell_1 & \xrightarrow{g} & \ell_1 \\
\downarrow{b} & & \downarrow{b^*} \\
\mathbb{F} & \xrightarrow{u} & \mathbb{F}
\end{array} \]
```
If such a morphism \( u \) existed in \( B^* \), then \( (u \circ g)(b) \) must be \( |b| \). However \( |g(b)| = e - 2 < 1 \) but \( |u(g(b))| = |b| > 1 \). This is a contradiction.

**Lemma 4.17** The scalar field \( \mathbb{F} \) is not projective in \( B^* \).

**Proof:** Let \( c_0 \) be the Banach space consisting of sequences \( a = (a_i)_{i \in \mathbb{N}, a_i \in \mathbb{F}, and N = \{1,2,\ldots\} \} \) converging to zero with norm \( |a| = \sup_{i \in \mathbb{N}} |a_i| \). Define \( g: c_0 \rightarrow c_0 \) by \( g((a_i)_{i \in \mathbb{N}}) = (a_i/i)_{i \in \mathbb{N}} \). The map \( g \) is an epimorphism and a monomorphism in \( B^* \). Let \( b \) be the sequence \((0,2,0,0,\ldots)\) in \( c_0 \). Then \( g(b) = (0,1,0,\ldots) \). Define \( f: \mathbb{F} \rightarrow c_0 \) by \( f(n) = (0,n,0,0,\ldots) \). There does not exist any morphism \( u: \mathbb{F} \rightarrow c_0 \) in \( B^* \) such that \( g \circ u = f \), that is, so the following diagram commutes.

\[
\begin{array}{ccc}
\mathbb{F} & \xrightarrow{u} & c_0 \\
\downarrow{g} & & \downarrow{f} \\
c_0 & & c_0
\end{array}
\]

If such a morphism \( u \) existed in \( B^* \), then \( u(1) \) must be \( b \) as \( g \) is one-to-one. However then \( |u| = \sup_{i \in \mathbb{N}} |u(a)| \geq |a|/i \) \( |u(1)| = |b| = 2 > 1 \). This is a contradiction.

**Proposition 4.18** In \( B^* \) if \( B \) is injective, then \( B = 0 \).

**Proof:** The result follows from lemmas 4.13, 4.15, and 4.16.

**Proposition 4.19** In \( B^* \) if \( B \) is projective, then \( B = 0 \).

**Proof:** The result follows from lemmas 4.13, 4.15, and 4.17.
5. NORMAL MORPHISMS IN $\mathbb{B}^*$

**Definition 5.1** In $\mathbb{B}^*$ a morphism $f:A \to B$ is a normal morphism if the map $f':A/\text{Ker } f \to f[A]$ defined by $f'([a]) = f(a)$ is an isometric map.

**Remark 5.2** A morphism $f:A \to B$ in $\mathbb{B}^*$ is a normal monomorphism if and only if it is an isometric map. If $f:A \to B$ is a normal epimorphism, then $f[A] = B$ (norm of $f[A]$ is the same as that of $A/\text{Ker } f$). If $f:A \to B$ is a normal monomorphism and an epimorphism, then it is an isometric isomorphism. If $f:A \to B$ is a normal morphism in $\mathbb{B}^*$, then $\text{Cok } \text{Ker } f = \text{im } f \cong \text{coim } f = \text{Ker } \text{cok } f$.

In $\mathbb{B}^*$ the following definition is equivalent to the preceding definition of a normal subobject (Definition 2.15).

**Definition 5.3** If $f:A \to B$ is a subobject of $B$ where $f$ is a normal monomorphism in $\mathbb{B}^*$, then $A$ is called a normal subobject of $B$.

The union of a family of subobjects $\{f_i:A_i \to A\}_I$ has been shown to be the subobject $\sum_I A_i/\text{Ker } g'$ with respect to $g: \sum_I A_i/\text{Ker } g' \to A$ where $g': \sum_I A_i \to A$ is the map $g'(a) = \sum_i f_i(a_i)$ ($a_i$ is the $i^{th}$ coordinate of $a$) and $g([a]) = g'(a)$. The morphism $\sum_I A_i/\text{Ker } g' \to B$ in the diagram

\[
\begin{array}{c}
\sum_I A_i/\text{Ker } g' \\
\downarrow g \\
A \\
\downarrow f \\
B
\end{array}
\]

\[
\begin{array}{c}
A_i \\
\downarrow f_i \\
\text{B'}
\end{array}
\]

\[
\begin{array}{c}
\text{g'} \\
\downarrow h \\
\text{A'}
\end{array}
\]

\[
\begin{array}{c}
\text{u} \\
\downarrow h
\end{array}
\]

\[\star\]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
where for all $i \in I$, $g \circ g_1 = f_1$, was given explicitly so that whenever $f$ carried each $A_i$ into $B'$, it also carried $\sum A_i/\ker g'$ into $B'$. Now if $B'$ is a normal subobject of $B$, then since $h:B' \to B$ is the kernel of $\text{coker } h:B \to B/h[B']$ and since $(\text{coker } h) \circ f \circ g = 0$, the existence of a unique $u: \sum A_i/\ker g' \to B'$ such that $h \circ u = f \circ g$ is guaranteed.

If $\left( \bigcup_{i \in I} f_1[A_i] \right)$ denotes the Banach space $\bigcap B$ (the intersection of all closed subspaces $B$ of $A$ containing $\bigcup_{i \in I} f_1[A_i]$), a subspace of $A$ which contains $\left\{ \sum f_1(a_i) \mid \text{each sum finite} \right\}$, then $\left( \bigcup_{i \in I} f_1[A_i] \right)$ is a subobject of $A$ with respect to the inclusion map $q: \left( \bigcup_{i \in I} f_1[A_i] \right) \to A$. In the diagram

\[ \begin{array}{ccc}
A_1 & \xrightarrow{h_1} & B' \\
\downarrow{g_1} & & \downarrow{u} \\
\left( \bigcup_{i \in I} f_1[A_i] \right) & \xrightarrow{f} & A \\
\end{array} \]

$g_1:A_1 \to \left( \bigcup_{i \in I} f_1[A_i] \right)$ is defined by $g_1(a_1) = f_1(a_1)$, then for all $i \in I$, $q \circ g_1 = f_1$. If $B'$ is a normal subobject of $B$ and each $A_i$ is carried by $f$ into $B'$, then since $h:B' \to B$ is the kernel of $\text{coker } h:B \to B/h[B']$ and $(\text{coker } h) \circ (f \circ g) = 0$, the existence of a unique $u: \left( \bigcup_{i \in I} f_1[A_i] \right) \to B'$ is guaranteed such that $\left( \bigcup_{i \in I} f_1[A_i] \right)$ is carried by $f$ into $B'$. Thus $\left( \bigcup_{i \in I} f_1[A_i] \right)$ acts as the union of the $A_i$ in the special case when the $h:B' \to B$ is restricted to a normal monomorphism and as such might be called a "normal union". However, $\left( \bigcup_{i \in I} f_1[A_i] \right)$ is not nec-
essarily isomorphic to $\sum_{I} A_i / \ker g'$ in diagram (**) as $g: \sum_{I} A_i / \ker g' \longrightarrow A$ is not generally a normal monomorphism.

It has been shown that the intersection of a family of subobjects $\{f_i: A_i \longrightarrow A\}_{i \in I}$ of $A$ is $f:A' \longrightarrow A$ where $A' = \{ b \in \bigcap_{I} A_i \}$ for all $i, j \in I$, $f_i(b_i) = f_j(b_j)$ and $f$ is defined by $f(b) = f_p(b_p)$ for a fixed element $p$ of $I$. If each $A_i$ is a normal subobject of $A$, then each $f_i[A_i]$ is a subspace of $A$ and $\bigcap_{I} f_i[A_i]$ (set intersection) is a Banach space. If in the diagram below $j: \bigcap_{I} f_i[A_i] \longrightarrow A$ is the insertion map, and $p_i: \bigcap_{I} f_i[A_i] \longrightarrow A_i$ is the map defined by $p_i(f_i(a_i)) = a_i$; then for each $i \in I$, $f_i \circ p_i = j$.

If also for each $i \in I$, $h = f_i \circ h_i$, then by defining $u:A'' \longrightarrow \bigcap_{I} f_i[A_i]$ by $u(a'') = f_p(h_p(a''))$ for $p$ fixed in $I$, $j \circ u = h$ as $(j \circ u)(a'') = j(f_p(h_p(a''))) = f_p(h_p(a'')) = h(a'')$ for all $a'' \in A''$. The morphism $u$ is unique since $j$ is a monomorphism. Hence if $f_i:A_i \longrightarrow A$ are normal, $\bigcap_{I} f_i[A_i]$ (set intersection) is the intersection of the subobjects $\{A_i\}_{I}$ of $A$ and in this case is isomorphic to $A' = \{ b \in \bigcap_{I} A_i \}$ for all $i, j \in I$, $f_i(b_i) = f_j(b_j)$.

Definition 5.4 An object $Q$ in $B^*$ is called normal injective if for any normal monomorphism $f:A \longrightarrow B$ and any morphism
g:A → Q there is a morphism h:B → Q making the diagram commutative.

The following two lemmas are proved in the same way as lemmas 4.13 and 4.14.

Lemma 5.5 If M is normal injective and r:M → M¹ is a retraction, then M¹ is normal injective.

Lemma 5.6 If M is the product of \{M_i\}_i and all M_i are normal injective, then M is normal injective.

Lemma 5.7 If M is normal injective and f:M → A is any normal monomorphism, then f admits a retraction, that is, there exists g:A → M such that gof = 1_M.

Proof: Let 1_M:M → M be the identity morphism. Since f:M → A is a normal monomorphism and M is normal injective, there exists g:A → M such that 1_M = gof.

Proposition 5.8 Per every object A in B*, there exists a normal monomorphism f from A into \( \prod_T f \) where T is a suitably chosen index set.

Proof: Let n:A → A'' be the natural embedding of A into the second conjugate space. A and n[A] are isometrically isomorphic. Let B'' = \{ b* ∈ A* | |b*| ≤ 1 \} where A* is the first conjugate space. Consider \( \prod_{B''} f \), the set of all functions f:B'' → R such that \( \sup_{b^* ∈ B''} |f(b^*)| \) is finite.
Define \( g: A^{**} \to B^{*} \) by \( g(a^{**}) = a^{**} \mid_{B^{*}} \), the restriction of \( a^{**} \) to \( B^{*} \). Then \( g \) is an isometric map for
\[
|g(a^{**})| = |a^{**} \mid_{B^{*}}| = \sup_{b^{*} \in B^{*}}|a^{**}(b^{*})| = |a^{**}|.
\]
The required normal monomorphism \( f \) is given:
\[
A \to \prod_{B^{*}} B^{*}.
\]

**Proposition 5.9** In \( B^{*} \), \( \prod_{B^{*}} B^{*} \) is normal injective.

**Proof:** Let \( f: A \to B \) be any normal monomorphism in \( B^{*} \) and \( g: A \to \prod_{B^{*}} B^{*} \) any morphism. Define \( h': f[A] \to \prod_{B^{*}} B^{*} \) by \( h'(f(a)) = g(a) \). The map \( h' \) is in \( B^{*} \) as
\[
\sup |h'(f(a))| = \sup |f(a)| \leq 1.
\]
Using the Hahn-Banach theorem \([1, p.63]\), there is a linear map \( h: B \to \prod_{B^{*}} B^{*} \) such that \( |h| = |h'| \) and \( h(f(a)) = h'(f(a)) \) for all \( f(a) \in f[A] \). Thus \( h \in B^{*} \) and \( h \circ f = g \).

A proposition similar to the following is given in Semadeni \([5, p.363]\).

**Proposition 5.10** In \( B^{*} \) the following are equivalent.

(i) \( M \) is normal injective.

(ii) \( M \) is a retract of \( \prod_{T} \prod_{B^{*}} B^{*} \) for some index set \( T \).

(iii) \( M \) is a normal retract, that is, if there exists a normal monomorphism from \( M \) into an object \( A \), then \( M \) is a retract of \( A \).

**Proof:**

(1) \( \implies \) (iii) by lemma 5.7.

(iii) \( \implies \) (ii) by proposition 5.8.

(ii) \( \implies \) (i) by proposition 5.9 and lemmas 5.6 and 5.5.
6. THE TENSOR PRODUCT OF BANACH SPACES

In the following development A and B will always denote two Banach spaces. Let $\mathcal{A}^A \otimes B$ be the vector space over the field $\mathcal{A}$ consisting of all functions $f: A \times B \rightarrow \mathcal{A}$. Addition and scalar multiplication in $\mathcal{A}^A \otimes B$ are defined pointwise, that is,

$$(f + g)(a,b) = f(a,b) + g(a,b)$$

$$(\theta f)(a,b) = \theta f(a,b) \quad \theta \in \mathcal{A}.$$

For each $(a,b)$ in $A \times B$ let $a^*b$ be the element in $\mathcal{A}^A \otimes B$ defined by

$$a^*b(p,q) = 1 \text{ if } (p,q) = (a,b)$$

$$a^*b(p,q) = 0 \text{ if } (p,q) \neq (a,b).$$

Let $\mathcal{A}(A \times B)$ be the subspace of $\mathcal{A}^A \otimes B$ spanned by the elements of the type $a^*b$. Thus $\mathcal{A}(A \times B)$ is the space consisting of all functions $f: A \times B \rightarrow \mathcal{A}$ given by

$$f = \sum_{i=1}^{n} \theta_i a_i^* b_i, \text{ } n \text{ some positive integer}$$

and $f(a_i, b_i) = \theta_i$.

Let $K: A \times B \rightarrow \mathcal{A}(A \times B)$ be a function defined by $K(a,b) = a^*b$. It may happen that $a_1^*b + a_2^*b \neq (a_1 + a_2)^*b$ and hence $K$ is not bilinear. Let $S$ be the subspace of $\mathcal{A}(A \times B)$ spanned by all elements of the type

$$(\theta_1 a_1 + \theta_2 a_2)^* b_1 - \theta_1 (a_1^* b_1) - \theta_2 (a_2^* b_1)$$

and

$$a_1^*(\theta_1 b_1 + \theta_2 b_2) - \theta_1 (a_1^* b_1) - \theta_2 (a_1^* b_2)$$

for all $\theta_1, \theta_2 \in \mathcal{A}$; $a_1, a_2 \in A$; $b_1, b_2 \in B$. Let $A \otimes B$ denote
The equivalence class of the element 
\[ f = \sum_{i=1}^{n} \theta_i (a_i \otimes b_i) \] will be denoted by \( \sum_{i=1}^{n} \theta_i a_i \otimes b_i \). It is now easy to see that

\[
(\theta_1 a_1 + \theta_2 a_2) \otimes b = \theta_1 (a_1 \otimes b) + \theta_2 (a_2 \otimes b)
\]

and

\[
a \otimes (\theta_1 b_1 + \theta_2 b_2) = \theta_1 (a \otimes b_1) + \theta_2 (a \otimes b_2),
\]

and that the function \( \otimes : \text{AXB} \rightarrow A \otimes B \), defined as the composition of \( K: \text{AXB} \rightarrow \overline{\phi} (\text{AXB}) \) and the canonical map \( \overline{\phi} (\text{AXB}) \rightarrow A \otimes B \), is bilinear.

From a different point of view one may define a relation \( \sim \) on \( \overline{\phi} (\text{AXB}) \) subject to the following rules:

1. \( (a_1 + a_1') \otimes b_1 + a_2 \otimes b_2 + \ldots + a_n \otimes b_n \sim a_1 \otimes b_1 + a_1' \otimes b_1 + a_2 \otimes b_2 + \ldots + a_n \otimes b_n \)

2. \( a_1 \otimes (b_1 + b_1') + a_2 \otimes b_2 + \ldots + a_n \otimes b_n \sim a_1 \otimes b_1 + a_1' \otimes b_1' + a_2 \otimes b_2 + \ldots + a_n \otimes b_n \)

3. \( \theta_1 (a_1 \otimes b_1) + \theta_2 (a_2 \otimes b_2) + \ldots + \theta_n (a_n \otimes b_n) \sim (\theta_1 a_1) \otimes b_1 + \ldots + (\theta_n a_n) \otimes b_n \)

4. \( (\theta_1 a_1) \otimes b_1 + (\theta_2 a_2) \otimes b_2 + \ldots + (\theta_n a_n) \otimes b_n \sim a_1 \otimes (\theta_1 b_1) + \ldots + a_n \otimes (\theta_n b_n) \)

Now define the equivalence relation \( \sim \) on \( \overline{\phi} (\text{AXB}) \) by

\[
\sum_{i=1}^{n} \theta_i (a_i \otimes b_i) \sim \sum_{i=1}^{m} \lambda_i (c_i \otimes d_i)
\]

if \( \sum_{i=1}^{n} \theta_i (a_i \otimes b_i) \) can be transformed into \( \sum_{i=1}^{m} \lambda_i (c_i \otimes d_i) \) by a finite number of applications of rules (1) - (4). It is apparent that the two quotient spaces \( \overline{\phi}(\text{AXB})/\sim \) and \( \overline{\phi}(\text{AXB})/S \) are identical, that is, \( S \) is the zero class of the relation \( \sim \).

The objective is to make \( A \otimes B \) into a normed linear
space and then let the Banach space $A \hat{\otimes} B$ be its completion. To this end, define a norm of $A \hat{\otimes} B$ by

$$(6.1) \quad |u| = \inf \left\{ \sum_{i=1}^{n} |\theta_i| |a_i| |b_i| \left| u = \sum_{i=1}^{n} \theta_i a_i \hat{\otimes} b_i \right| \right\},$$

$u \in A \hat{\otimes} B$. It will be shown that this is a crossnorm on $A \hat{\otimes} B$, that is, a norm with the additional property that $|a \hat{\otimes} b| = |a||b|, a \in A, b \in B$. The following development to show this is a crossnorm is based on work by Robert Schatten [4]. He proves the following lemma [4, p.201].

**Lemma 6.2** If $F$ is a linear functional on $A$ and

$$\sum_{i=1}^{n} a_i \hat{\otimes} b_i = \sum_{j=1}^{m} a_j \hat{\otimes} b_j,$$

then $\sum_{i=1}^{n} F(a_i)b_i = \sum_{j=1}^{m} F(a_j)b_j$.

The following definitions are given for clarity.

**Definition 6.3** A norm on $A \hat{\otimes} B$ is a non-negative function $N: A \hat{\otimes} B \to \mathbb{R}$ satisfying the following conditions:

1. $N(\sum_i \theta_i a_i \hat{\otimes} b_i) = 0$ if and only if $\sum_i \theta_i a_i \hat{\otimes} b_i = 0 \otimes 0$.
2. $N(\sum_i \theta_i a_i \hat{\otimes} b_i) = |\theta| N(\sum_i a_i \hat{\otimes} b_i)$ for all $\theta \in \mathbb{C}$.
3. $N(\sum_i a_i \hat{\otimes} b_i + \sum_i c_i \hat{\otimes} d_i) \leq N(\sum_i a_i \hat{\otimes} b_i) + N(\sum_i c_i \hat{\otimes} d_i)$.

**Definition 6.4** A norm $N$ on $A \hat{\otimes} B$ is continuous at $\sum_i a_i \hat{\otimes} b_i$ if and only if
(iv) given $\varepsilon > 0$, there exists a $\delta(a_1, \ldots, a_n, b_1, \ldots, b_n) > 0$ such that for $|a_i' - a_i| < \delta$
and $|b_i' - b_i| < \delta$, $i = 1, \ldots, n$, $N(\sum_{i=1}^{n} a_i \otimes b_i - \sum_{i=1}^{n} a_i' \otimes b_i') < \varepsilon$.

**Definition 6.5** A norm $N$ on $A \otimes B$ is a crossnorm if it satisfies the property

$$(v) \quad N(a \otimes b) = |a||b| \quad a \in A, b \in B.$$  

**Lemma 6.6** A crossnorm satisfies condition (iv).

**Proof:**  

$$N(\sum_{i=1}^{n} a_i \otimes b_i - \sum_{i=1}^{n} a_i' \otimes b_i') \leq N(\sum_{i=1}^{n} (a_i - a_i') \otimes b_i) +$$

$$N(\sum_{i=1}^{n} a_i \otimes (b_i - b_i')) + N(\sum_{i=1}^{n} (a_i - a_i') \otimes (b_i - b_i')) \leq$$

$$\sum_{i=1}^{n} |a_i - a_i'| |b_i| + \sum_{i=1}^{n} |a_i| |b_i - b_i'| + \sum_{i=1}^{n} |a_i - a_i'| |b_i - b_i'|$$

Let $\varepsilon > 0$ be given. Let $\delta = \min(\varepsilon/3n|b_1|, \varepsilon/3n|a_1|, \sqrt{\varepsilon/3n})$.

The result follows using this $\delta$.

Let $F$ be a linear functional on $A$. Sometimes a norm on $A \otimes B$ will satisfy the condition

$$(vi) \quad |(\sum_{i=1}^{n} F(a_i)b_i)| \leq |F|N(\sum_{i=1}^{n} a_i \otimes b_i).$$

**Definition 6.7** Let $T_u$ be a transformation from the vector space of linear functionals on $A$ to the vector space $B$ defined by

$$T_u F = \sum_{i=1}^{n} F(a_i) b_i \text{ where } u = \sum_{i=1}^{n} a_i \otimes b_i.$$ 

This definition describes a well-defined function since by lemma 6.2, $\sum_{i=1}^{n} F(a_i)b_i = \sum_{i=1}^{m} F(c_i)d_i$ whenever

$$\sum_{i=1}^{n} a_i \otimes b_i = \sum_{i=1}^{m} c_i \otimes d_i.$$
Lemma 6.8 If $T_u = 0$, then $u = 0 \otimes 0$.

**Proof:** Suppose $u = \sum a_i \otimes b_i \neq 0 \otimes 0$. It can be assumed that both the sets $\{a_i \mid i = 1, \ldots, n\}$ and $\{b_i \mid i = 1, \ldots, n\}$ are linearly independent. Thus $a_i \neq 0$. A linear functional $F$ can be found for which $F(a_i) \neq 0$. Hence $\sum F(a_i)b_i \neq 0$. Thus $T_uF \neq 0$.

**Lemma 6.9** Conditions (ii) and (vi) for a norm $N$ on $A \otimes B$ imply (i).

**Proof:** From condition (ii), if $\sum a_i \otimes b_i = 0 \otimes 0$, then $N(\sum a_i \otimes b_i) = N(0 \otimes 0) = 0$. Now suppose $N(\sum a_i \otimes b_i) = 0$. By (vi), $|\sum F(a_i)b_i| = 0$ or $\sum F(a_i)b_i = 0$. Hence $T_uF = 0$ where $u = \sum a_i \otimes b_i$. By the preceding lemma $u = 0 \otimes 0$.

Now let $N:A \otimes B \to \mathbb{R}$ be the non-negative function given by equation (6.1), that is,

$$N(u) = \|u\| = \inf \left\{ \sum |\theta_i||a_i||b_i| \mid u = \sum \theta_i a_i \otimes b_i, \right\},$$

$u \in A \otimes B$.

**Proposition 6.9** $N$ is a norm for $A \otimes B$ satisfying conditions (iv), (v), and (vi).

**Proof:** It must only be shown that $N$ satisfies conditions (ii), (iii), (v), and (vi); since (v) implies (iv) while (ii) and (vi) imply (i). Condition (ii) is clear. To prove (iii) let $\sum a_i \otimes b_i$ and $\sum c_i \otimes d_i$ be two elements of $A \otimes B$. Let $\varepsilon > 0$ be given. Let $\sum a_i \otimes b_i = \sum a_i \otimes b_i$ such that

$$\sum |a_i||b_i| \leq N(\sum a_i \otimes b_i) + \varepsilon/2.$$
Similarly let \( \sum_1 c'_1 \otimes d'_1 = \sum_1 c_1 \otimes d_1 \) such that
\[
\sum_1 |c'_1| |d'_1| \leq N(\sum_1 c_1 \otimes d_1) + \varepsilon/2.
\]

Then
\[
N(\sum_1 a_1 \otimes b_1 + \sum_1 c_1 \otimes d_1) = N(\sum_1 a'_1 \otimes b'_1 + \sum_1 c'_1 \otimes d'_1)
\]
\[
\leq \sum_1 |a'_1| |b'_1| + \sum_1 |c'_1| |d'_1|
\]
\[
\leq N(\sum_1 a_1 \otimes b_1) + N(\sum_1 c_1 \otimes d_1) + \varepsilon.
\]

This proves (iii).

To prove (vi) let \( u = \sum_1 a_1 \otimes b_1 \) and \( T_u F = \sum_1 F(a_1)b_1 \).

Then \( |T_u F| = |\sum_1 F(a_1)b_1| \leq \sum_1 |F(a_1)| |b_1| \leq |F| \sum_1 |a_1| |b_1| \).

This is true for all \( \sum_1 c_1 \otimes d_1 = u \). Hence
\[
|T_u F| = |\sum_1 F(a_1)b_1| \leq |F| N(u).
\]

To prove that \( N \) is a crossnorm, let \( a \otimes b \) be in \( A \otimes B \), \( a \neq 0 \) and \( b \neq 0 \). Suppose \( \sum_1 c_1 \otimes d_1 = a \otimes b \). Let \( T_{a \otimes b} F = F(a)b \) which equals \( \sum_1 F(c_1)d_1 \). Let \( F \) be the particular linear functional given by \( F(a) = |a|, |F| = 1 \). Then
\[
|T_{a \otimes b} F| = |F(a)b| = |a| |b| = |a| |b| = \sum_1 |F(c_1)d_1| \leq |F| N(\sum_1 c_1 \otimes d_1) = N(\sum_1 c_1 \otimes d_1). \text{ Hence } |a| |b| = N(a \otimes b).
\]

\( A \otimes B \) is thus a normed linear space with the norm
\[
|u| = \inf \left\{ \sum_1 \theta_1 |a_1| |b_1| \left| \theta : \sum_1 \theta a_1 \otimes b_1 \right| \right\}.
\]

The completion of \( A \otimes B \), denoted by \( \hat{A \otimes B} \), is the Banach space known as the tensor product of Banach spaces \( A \) and \( B \).
Let $\mathcal{B}(A,B)$ denote the Banach space of continuous linear functions from Banach space $A$ into Banach space $B$.

**Proposition 6.10** If $A$, $B$, and $C$ are Banach spaces, then $\mathcal{B}(A \otimes B, C)$ and $\mathcal{B}(B, \mathcal{B}(A, C))$ are isometrically isomorphic.

**Proof:** Define $\mathcal{I} : \mathcal{B}(A \otimes B, C) \to \mathcal{B}(B, \mathcal{B}(A, C))$ by $(\mathcal{I}f)(b)(a) = f(a \otimes b)$ for $f \in \mathcal{B}(A \otimes B, C)$, and note that $\mathcal{I}f$ is easily seen to be linear over $B$, while $(\mathcal{I}f)b$ is linear over $A$ for all $b$ in $B$. Also $\mathcal{I}$ can easily be seen to be a linear map. By definition

$$|\mathcal{I}f| = \inf \{ M | \text{for all } b \in B, |(\mathcal{I}f)b| \leq M|b| \}.$$  

and

$$|(\mathcal{I}f)b| = \inf \{ N_b | \text{for all } a \in A, |(\mathcal{I}f)b| \leq N_b|a| \}.$$  

Define $|\mathcal{I}f|'$ by:

$$|\mathcal{I}f|' = \inf \{ M | \text{for all } a \in A, b \in B, |(\mathcal{I}f)b| \leq M|a| |b| \}.$$  

Since $|((\mathcal{I}f)b)a| \leq |\mathcal{I}f||b||a|$, then $|\mathcal{I}f|' \leq |\mathcal{I}f|$. Also since $|((\mathcal{I}f)b)a| \leq |\mathcal{I}f|'|a||b|$, then $|(\mathcal{I}f)b| \leq |\mathcal{I}f|'|b|$, which means $|\mathcal{I}f| \leq |\mathcal{I}f|'$.

Hence $|\mathcal{I}f| = |\mathcal{I}f|'$.

Therefore $|\mathcal{I}f| = |\mathcal{I}f|' = \inf \{ M | |(\mathcal{I}f)b| \leq M|a| |b| \} = \inf \{ M | |f(a \otimes b)| \leq M|a| |b| \} = \inf \{ M | |f(a \otimes b)| \leq M|a \otimes b| \} \leq |f|$.

It will now be shown that $\mathcal{I}$ has an inverse $\mu : \mathcal{B}(B, \mathcal{B}(A, C)) \to \mathcal{B}(A \otimes B, C)$ such that $|\mu| \leq 1$. Define $\mu : \mathcal{B}(B, \mathcal{B}(A, C)) \to \mathcal{B}(A \otimes B, C)$ by $\mu(g)(a \otimes b) = (g(b))a$ for $g \in \mathcal{B}(B, \mathcal{B}(A, C))$, and extend $\mu(g)$ linearly to $A \otimes B$. Since
\[ \hat{\mu}(g)(0 \otimes 0) = (g(0))0 = 0 \] and \( g \) is linear over \( B \); while for each \( b \), \( g(b) \) is linear over \( A \), the map \( \hat{\mu}(g) \) is well-defined. Also \( |\hat{\mu}(g)| \leq |g| \) for
\[
|\hat{\mu}(g)(\sum_{i} a_i \otimes b_i)| \leq \sum_{i} |\hat{\mu}(g)(a_i \otimes b_i)| \\
\leq \sum_{i} |\hat{\mu}(g)(a_i \otimes b_i)| = \sum_{i} |(g(b_i))a_i| \\
\leq |g| \sum_{i} |b_i| |a_i|.
\]
Hence
\[
\frac{|\hat{\mu}(g)(\sum_{i} a_i \otimes b_i)|}{\sum_{i} |b_i| |a_i|} \leq |g| \text{ if } \sum_{i} a_i \otimes b_i \neq 0 \otimes 0.
\]
Hence
\[
\frac{|\hat{\mu}(g)(\sum_{i} a_i \otimes b_i)|}{\sum_{i} |a_i| |b_i|} \leq |g| \text{ using the same reasoning as in lemma 1.11. Therefore } |\hat{\mu}(g)| \leq |g|, \text{ which also means that for each } g \in B(B,B(A,C)), \hat{\mu}(g) : A \otimes B \to C \text{ is uniformly continuous.}
\]
By the Principle of extension by continuity [1, p.23], \( \hat{\mu}(g) : A \otimes B \to C \) has a unique uniformly continuous extension \( \mu(g) : A \hat{\otimes} B \to C \). Let \( \mu : B(B,B(A,C)) \to B(A \hat{\otimes} B,C) \) map \( g \) to the unique extension of \( \hat{\mu}(g) \). Let \( x \in A \hat{\otimes} B \). Then there exists a sequence \((x_n)_{n \in \mathbb{N}}\) such that \( x_n \in A \otimes B \) and \( x_n \to x \).
\[
|\mu(g)x| = |\mu(g)\lim x_n| \\
= \lim_{n} |\hat{\mu}(g)x_n| \leq \lim_{n} |g||x_n| \\
= |g||\lim x_n| = |g||x|.
\]
Hence \( |\mu(g)| \leq |g| \) or \( |\mu| \leq 1 \).
It is also easily seen that $\hat{\mu}$ is a linear function. Hence $\mu$ is also a linear function, for if $\theta \in \mathcal{T}$ and $x \in A \hat{\otimes} B$ with $(x_n)_N$ a sequence in $A \hat{\otimes} B$ such that $x_n \to x$; then

$$\mu(\theta f_1 + f_2)x = \mu(\theta f_1 + f_2)\lim_{n} x_n$$

$$= \lim_{n} (\mu(\theta f_1 + f_2))x_n$$

$$= \lim_{n} (\hat{\mu}(\theta f_1 + f_2))x_n$$

$$= \lim_{n} \theta(\hat{\mu}(f_1))x_n + \mu(f_2)x_n$$

$$= \lim_{n} \theta(\mu(f_1))x_n + \mu(f_2)x_n$$

$$= \theta(\mu(f_1))x + \mu(f_2)x$$

$$= (\theta \mu(f_1) + \mu(f_2))(x).$$

This means $\mu(\theta f_1 + f_2) = \theta \mu(f_1) + \mu(f_2)$.

Because $\mu$ is an inverse for $\mathcal{T}$, the proposition is proved.

Corollary 6.11 $B^*(A \hat{\otimes} B, C)$ is isometrically isomorphic to $B^*(B, B(A, C))$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
BIBLIOGRAPHY


