Extensions of the Sigma and Tau Functions

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EXTENSIONS OF THE SIGMA AND TAU FUNCTIONS

by

Meredith W. Potter

A Project Report
Submitted to the
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of the
Specialist in Arts Degree

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TABLE OF CONTENTS

CHAPTER  | PAGE
----------|------
INTRODUCTION | 1
I AN EXAMINATION OF ORDERED k-TUPLES WHOSE PRODUCT IS n | 2
II SPECIAL CASES AND APPLICATIONS | 12
III A GENERALIZATION OF THE SIGMA FUNCTION | 20
IV ASYMPTOTIC APPROXIMATIONS OF SUMMATORY FUNCTIONS OF T AND S | 26

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INTRODUCTION

A number-theoretic function is any function whose domain is the set of positive integers and whose range is a subset of the complex numbers. A number-theoretic function $f$ is said to be multiplicative in case $(a, b) = 1$ implies $f(ab) = f(a)f(b)$ and $f(1) = 1$. If $f$ and $g$ are multiplicative functions, the composition of $f$ and $g$, sometimes termed the Dirichlet convolution product and denoted $f \circ g$, is defined by $(f \circ g)(n) = \sum_{d | n} f(n/d)g(d)$, where the summation extends over all positive divisors $d$ of $n$.

Two of the most interesting multiplicative functions which have been studied intensively are $\tau(n)$, the number of positive divisors of $n$, and $\sigma(n)$, the sum of the positive divisors of $n$.

The purpose of this project is to examine generalizations of $\tau(n)$ and $\sigma(n)$ under the special restriction that for a given prime $p$ we shall consider only those divisors $d$ of $pn$ such that $(d, p) = 1$. Asymptotic representations of these functions and the "Dirichlet generators" will be determined and some applications given.
AN EXAMINATION OF ORDERED k-TUPLES WHOSE PRODUCT IS n

The number of distinct ordered pairs \([a, b]\) such that \(a > 0, b > 0\), and \(ab = n\) is given by \(\tau(n)\). Since \(\tau(n) = \sum_{d|n} 1\) where \(d\) ranges over all positive divisors \(d\) of \(n\), the total number of distinct ways to factor \(n\) into a product of two positive factors (where the factorization \(dd'\) is considered different from \(d'd\)) is determined by counting the number of distinct ordered pairs \([d, d']\) of positive integers \(d\) and \(d'\) such that \(dd' = n\). From this observation it is clear that in order to exhibit the total number of distinct ordered \(k\)-tuples \([a_1, \ldots, a_k]\) such that \(a_1 > 0\) \((i = 1, \ldots, k)\) and \(\prod_{i=1}^{k} a_i = n\), it is sufficient to determine the total number of distinct factorizations of \(n\) into a product of \(k\) positive factors where a change in the order of the factors will be considered to yield a different factorization.

Beumer showed in [1] that the total number \(\tau_k(n)\) of ways to factor \(n = \prod_{i=1}^{r} p_i^{a_i}\) into a product of \(k\) factors (where different orderings of factors are counted as distinct factorizations) is given by

\[
\tau_k(n) = \frac{1}{(k-1)!} \prod_{i=1}^{r} \frac{\left(\alpha_i + k - 1\right)!}{\alpha_i!}.
\]
A proof that Beumer's function is multiplicative and a new proof of his results in terms of distinct ordered k-tuples is given here.

**Theorem 1:** Let \( \tau_k(1) = 1 \) and \( \tau_k(n) = \frac{r}{i=1} \frac{(a_i+k-1)!}{a_i!(k-1)!} \), where \( n = \prod_{i=1}^{r} p_i^a_i \), \( p_i \) distinct primes for \( i = 1, \ldots, r \).

Then \( \tau_k \) is multiplicative, and \( \tau_k \) denotes the number of ordered k-tuples \([a_1, \ldots, a_k]\) of positive integers such that \( \prod_{i=1}^{k} a_i = n \).

**Proof:** We first establish that the function \( \tau_k \) is multiplicative. Let \( N = \prod_{i=1}^{s} p_i^{a_i} \), \( M = \prod_{i=s+1}^{r} p_i^{a_i} \). Then \( (N,M) = 1 \), and we can write the product \( NM = \prod_{i=1}^{r} p_i^{a_i} \).

\[
\tau_k(N)\tau_k(M) = \frac{S}{i=1} \frac{(a_i+k-1)!}{a_i!(k-1)!} \cdot \frac{1}{i=s+1} \frac{(a_i+k-1)!}{a_i!(k-1)!}
\]

\[
= \prod_{i=1}^{r} \frac{(a_i+k-1)!}{a_i!(k-1)!}
\]

\[
= \tau_k(NM).
\]

Having thus shown that \( \tau_k \) is multiplicative, in order to show that for \( n = \prod_{i=1}^{r} p_i^{a_i} \), \( \tau_k(n) \) is the number of ordered
k-tuples \([a_1, \ldots, a_k]\) such that \(a_1 > 0\) and \(\prod_{i=1}^{k} a_i = n\),

we need only prove that for every prime \(p\) and every positive integer \(\alpha\), \(\tau_k(p^\alpha)\) is the number of ordered k-tuples \([a_1, \ldots, a_k]\) such that \(a_1 > 0\) and \(\prod_{i=1}^{k} a_i = p^\alpha\),

and the assertion for \(\tau_k(n)\) will then follow by the multiplicative property of \(\tau_k\). The assertion is obvious for \(k = 1\), since \(\tau_1(p^\alpha) = \frac{\alpha!}{\alpha!0!} = 1\). For \(k = 2\), \(\tau_2(p^\alpha) = \frac{(\alpha+1)!}{\alpha!1!} = \alpha + 1 = \tau(p^\alpha)\), which we showed at the beginning to be the number of distinct ordered pairs \([a, b]\) such that \(ab = p^\alpha\). Hence the theorem is true for \(k = 1, 2\). We proceed now by induction. Assume for \(k > 1\) that the number of ordered \((k-1)\)-tuples \([a_1, \ldots, a_{k-1}]\) such that \(\prod_{i=1}^{k-1} a_i = p^\alpha\) is given by \(\tau_{k-1}(p^\alpha)\). We wish to show that the number of ordered \(k\)-tuples \([a_1, \ldots, a_k]\) such that \(\prod_{i=1}^{k} a_i = p^\alpha\) is given by \(\tau_k(p^\alpha)\). Consider an arbitrary \(k\)-tuple with the above property and let the first component of this \(k\)-tuple be fixed. Then there exist \(\tau_1(p^\beta)\) choices for this first component where \(0 \leq \beta \leq \alpha\). By our inductive assumption the number of choices for the remaining \(k - 1\) components is given by \(\tau_{k-1}(p^{\alpha-\beta})\). This implies that the number of distinct ordered \(k\)-tuples
such that \( \prod_{i=1}^{k} a_i = p^\alpha \) is given by

\[
\sum_{\beta=0}^{\alpha} \tau_1(p^\beta)\tau_{k-1}(p^{\alpha-\beta}). \quad \text{Since } \tau_1(p^\beta) = 1 \text{ for all } \beta,
\]

the summation is just

\[
\sum_{\beta=0}^{\alpha} \tau_{k-1}(p^{\alpha-\beta}) = \sum_{\beta=0}^{\alpha} \frac{(\alpha-\beta+k-2)!}{(\alpha-\beta)!(k-2)!} = \sum_{\beta=0}^{\alpha} \frac{(k-2+\alpha-\beta)}{k-2}
\]

Let \( K = k-2 \). Then by repeated application of the well-known identity:

\[
\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}
\]

we obtain

\[
\sum_{\beta=0}^{\alpha} \tau_{k-1}(p^{\alpha-\beta}) = \sum_{\beta=0}^{\alpha} \frac{(K+\beta)}{K} = \binom{K+1}{K} + \binom{K+1}{K} + \frac{\alpha}{K} \sum_{\beta=2}^{K+\beta} \binom{K+1}{K} = \binom{K+2}{K+1} + \binom{K+2}{K} + \frac{\alpha}{K} \sum_{\beta=3}^{K+\beta} \binom{K+1}{K} = \binom{K+\alpha}{K+1} + \binom{K+\alpha}{K} = \binom{K+\alpha+1}{K+1} = \binom{k-2+\alpha+1}{k-2+1} = \binom{\alpha+k-1}{k-1}
\]
\[= \frac{(a+k-1)!}{a! (k-1)!} = \tau_k(p^a)\]

and the assertion now follows by induction on \(k\).

Since \(\tau_k\) is multiplicative, this establishes that

for \(n = \prod_{i=1}^{r} p_1^{a_i}\), \(\tau_k(n) = \prod_{i=1}^{r} \tau_k(p_1^{a_i}) = \prod_{i=1}^{r} \frac{(a_i+k-1)!}{a_i! (k-1)!}\)

is the number of ordered \(k\)-tuples \([a_1, \ldots, a_k]\) such that

\[\prod_{i=1}^{k} a_i = n.\]

Assume now that for a given prime \(p\) we desire to have one and only one component of any \(k\)-tuple congruent to zero (mod \(p\)). In order to insure that such a component exists, we consider the number of factorizations of \(pn\).

**Theorem 2:** Let \(n\) and \(k\) be positive integers and \(p\) be prime. If \(T(p,n,k)\) denotes the total number of distinct ordered \(k\)-tuples \([a_1, \ldots, a_k]\) such that \(a_i > 0\) and

\[\prod_{i=1}^{k} a_i = pn\]

and such that precisely one component \(a_i\) is congruent to zero (mod \(p\)), then

\[T(p,n,k) = k \prod_{i=1}^{s} \frac{(a_i+k-1)!}{a_i! (k-1)!},\]

where \(n = p^\beta \prod_{i=1}^{s} q_1^{a_i}\), \(\beta \geq 0\) and \(p \neq q_1\) for \(i = 1, \ldots, s\).
Proof: Let \( m = \prod_{i=1}^{S} q_i^a_i \). We first determine the number of ordered \( k \)-tuples \([a_1, \ldots, a_k]\) such that \( \prod_{i=1}^{k} a_i = m \). By Theorem 1, this is equal to \( \tau_k(m) \). Hence there exist \( \frac{s}{\prod_{i=1}^{k} a_i!(k-1)!} \) distinct ordered \( k \)-tuples with the above property and in addition no component is congruent to zero (mod \( p \)) by the assumption on \( m \). Let \( \gamma = g+1 \). For each ordered \( k \)-tuple there exist \( k \) choices for placing \( p^\gamma \) so that precisely one component will be congruent to zero (mod \( p \)) and the product of the components will be \( pn \). Hence \( T(p,n,k) = k \tau_k(m) = k \prod_{i=1}^{s} \frac{(a_i+k-1)!}{a_i!(k-1)!} \).

Example: Let \( n = 12, p = 5, k = 3 \). We wish to determine the number of ordered triples \([a_1, a_2, a_3]\) such that \( a_1 \cdot a_2 \cdot a_3 = 5 \cdot 12 = 60 \) and such that one and only one component is congruent to zero (mod \( 5 \)). We first enumerate all the ordered triples \([a_1, a_2, a_3]\) such that \( a_1 \cdot a_2 \cdot a_3 = 12 \). By Theorem 1, this is \( \tau_3(12) = 18 \), namely: 

\([1, 1, 12], [1, 2, 6], [1, 6, 2], [1, 3, 4], [1, 4, 3], [1, 12, 1], [2, 1, 6], [2, 2, 3], [2, 3, 2], [2, 6, 1], [6, 1, 2], [6, 2, 1], [3, 1, 4], [3, 2, 2], [3, 4, 1], [4, 1, 3], [4, 3, 1], \) and \([12, 1, 1]\).
Now consider the triple \([1,3,4]\). We have three choices for placing 5, i.e., \([5,3,4]\), \([1,15,4]\), and \([1,3,20]\). Each of these newly formed triples has precisely one component congruent to zero (mod 5) and the product of the components of each triple is 60. We do this for each triple. Hence there are \(3 \times 18 = 54\) distinct ordered triples with the desired properties. This is verified by Theorem 2: 
\[ T(5,12,3) = 3 \tau_3(12) = 54. \]

Since \(\tau_k\) is multiplicative, \(\tau_k(p^n) = \tau_k(p)^n\)

\[ = \tau_k(p^n) = \tau_k(m) \text{ where } n = p^g \prod_{i=1}^{s} \frac{a_i}{q_i}, m = \frac{S}{\prod_{i=1}^{s} q_i} \text{ and } \gamma = g+1. \]

So we can also write \(T(p,n,k) = k^{\gamma!(k-1)!/(\gamma+k-1)!} \tau_k(p^n)\)

\[ = \frac{k! \cdot \gamma!}{(\gamma+k-1)!} \tau_k(p^n). \]

Note that in case \(g = 0\),
\[ T(p,n,k) = \frac{k! \cdot (k-1)!}{(1+k-1)!} \tau_k(p^n) = \tau_k(p^n). \]

With this motivation, we extend our definition of \(T(p,n,k)\) to include the case \(p = 1\) and define \(T(1,n,k) = \tau_k(n)\).

In order to obtain a multiplicative function we note that since \(\tau_k\) is multiplicative by Theorem 1, we need only divide \(T(p,n,k)\) by \(k\) to obtain a function which is multiplicative in \(n\). This proves

**Theorem 2:** Let \(T(p,n,k)\) be as defined in Theorem 2.

Then \(\frac{T(p,n,k)}{k}\) is multiplicative in the argument \(n\).
We pause briefly before proceeding with our development to recall a well-known computational method. Let
\[ G(n) = \sum_{d|n} f(n/d)g(d). \]
This is called the Dirichlet convolution of \( f \) and \( g \), which employing the notation of Cashwell and Everett in [2] can be expressed equivalently by:
\[ G = f \circ g. \]
Since the set of all multiplicative functions forms a commutative group under convolution, inverses of such functions exist, and if \( f^{-1} \) denotes the inverse of \( f \), it follows that \( f^{-1} \circ G = g \) or \( g(n) = \sum_{d|n} f^{-1}(n/d)G(d). \)

Recall that the Möbius function \( \mu \) is defined as follows: \( \mu(1) = 1; \mu(n) = 0 \) if \( n \) is divisible by a square larger than 1; \( \mu(n) = (-1)^t \) if \( n = p_1 \cdots p_t \), where the \( p_i \) are distinct primes. It is clear from the definition that \( \mu \) is multiplicative, and it is also shown in [4, p. 87] that \( \mu^{-1}(n) = 1 \) for all \( n \). Hence if \( G(n) = \sum_{d|n} g(n/d), \)
we can write \( G = \mu^{-1} \circ g \) which implies that \( g = \mu \circ G \). This is nothing more than the Möbius inversion formula, namely:
\[ G(n) = \sum_{d|n} g(n/d) \text{ if and only if } g(n) = \sum_{d|n} \mu(n/d)G(d). \]
We can now make the following definition.

**Definition:** Let \( f(n) = \sum_{d|n} \mu(n/d)F(d) \). Then \( f \) is called the Dirichlet generator of \( F \). Since the summation extends over the divisors \( d \) of \( n \), \( t(n) \) will be called the Dirichlet
generator of $T(p,n,k)$ if $t(n) = \sum_{d|n} \mu(n/d) \frac{T(p,d,k)}{k}$.

We will now determine $t(n)$ explicitly.

**Theorem 4:** Let $t(n)$ be the Dirichlet generator of $T(p,n,k)$. Let $n = p^\beta \prod_{i=1}^{s} q_i^{a_i}$, $q_i \neq p$ for $i = 1, \ldots, s$. Then

$$t(n) = \prod_{i=1}^{s} \tau_{k-1}(q_i^{a_i}) \text{ if } \beta = 0, \text{ and } t(n) = 0 \text{ otherwise.}$$

**Proof:** Let $p,k$ be arbitrary and fixed and let $T(q^\alpha) = T(p,q^\alpha,k)$ for brevity. We wish to determine the values of $t = \mu \circ T$. It is well-known and the proof can be found in [4, p.88] that if $T$ is multiplicative, $t$ will be also. So we need only evaluate $t$ at the prime powers. For any prime $q$ different from the given prime $p$ and for $\alpha \geq 1$:

$$t(q^\alpha) = \mu(1)T(q^\alpha) + \mu(q)T(q^{\alpha-1}).$$

Note that all other terms vanish since $\mu(q^\alpha) = 0$ for all $\alpha > 1$.

Hence

$$t(q^\alpha) = \frac{(\alpha+k-1)!}{\alpha!(k-1)!} + \frac{(-1)(\alpha-1+k-1)!}{(\alpha-1)!(k-1)!}$$

$$= \frac{(\alpha+k-1)}{\alpha} - \frac{(\alpha+k-2)}{\alpha-1} = \frac{(\alpha+k-2)}{\alpha} \text{ by identity (1).}$$

Therefore, $t(q^\alpha) = \frac{(\alpha+k-2)!}{\alpha!(k-2)!} = \tau_{k-1}(q^\alpha)$. 

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For the given prime $p$ and for $\beta \geq 1$:

$$t(p^\beta) = \mu(1)T(p^\beta) + \mu(p)T(p^{\beta-1})$$

$$= \frac{(\beta+1)'(k-1)!}{(\beta+k)!} \cdot \frac{(\beta+k)!}{(\beta+1)!(k-1)!} + (-1)^{\beta'(k-1)'} \frac{(\beta+k-1)!}{(\beta+k)!} \beta'(k-1)!$$

$$= 1 \cdot 1 - 1 \cdot 1 = 0.$$

We have thus shown that the number of distinct ordered $k$-tuples $[a_1, \ldots, a_k]$ such that $\prod_{i=1}^{k} a_i = pn$ and which have precisely one component congruent to zero (mod $p$) is given by $T(p,n,k) = \prod_{i=1}^{k} \frac{a_i + k-1}{a_i} \cdot \frac{1}{(k-1)!}$ where $n = p^\beta \prod_{i=1}^{S} q_i^a$.

In addition the function $\frac{T(p,n,k)}{k}$ is multiplicative in $n$ and its Dirichlet generator $t(n)$ was determined. We now turn our attention to special cases of this function and some applications.
SPECIAL CASES AND APPLICATIONS

Let us restrict our attention to the case \( k = 2 \).

Then for \( n = p^s \prod_{1 \leq i \leq s} q_i^{a_i} \) the equation reduces to the more simple form:

\[
T(p, n, 2) = \frac{s}{2} \prod_{i=1}^{s} \frac{(a_i+1)!}{a_i!} = \frac{s}{2} \prod_{i=1}^{s} (a_i+1) = \frac{\tau(n)}{\beta+1}.
\]

Then by our development of \( T(p, n, k) \) and the definition of \( T(1, n, k) \), it is clear that \( T(1, n, 2) = \tau(n) \). A more interesting example is \( T(2, n, 2) \), which gives the total number of distinct ordered pairs having the property that the two components of each pair are of opposite parity. Observe that the Dirichlet generator in this case is the characteristic function of the set of odd integers: \( \chi_T(n) = 1 \) if \( n \equiv 1 \pmod{2} \) and \( \chi_T(n) = 0 \) otherwise, and thus \( T(2, n, 2) \) is the number of odd divisors of \( n \).

In this particular case the function has an interesting application if we consider an arithmetic progression of the form: \( a + (a+1) + \ldots + (a+k) = \frac{(2a+k)(k+1)}{2} \) where \( k \) is a non-negative integer and \( a \) is a positive integer. If we now let \( a \) and \( n \) be positive integers, \( k \) a non-negative integer such that \( \sum_{j=0}^{k} (a+j) = n \), then \( 2n = \frac{(2a+k)(k+1)}{2} \). Notice that these two factors are of opposite parity.
For if \( k \equiv 0 \pmod{2} \), then \( k+1 \equiv 1 \pmod{2} \) and
\( 2a+k \equiv 0 \pmod{2} \) for any positive integer \( a \). On the other hand, if \( k \equiv 1 \pmod{2} \) then \( 2a+k \equiv 1 \pmod{2} \) and \( k+1 \equiv 0 \pmod{2} \). Hence \( T\left(\frac{2}{2},n,2\right) \) is the number of distinct arithmetic progressions with common difference one whose sum is \( n \). An example illustrates how \( a \) and \( k \)
are uniquely determined from each ordered pair \([c,d]\) such that \( cd = 2n \) and \( c \not\equiv d \pmod{2} \).

Example: Let \( n = 15 \). Then the ordered pairs \([c,d]\) are:
and \([6,5] \). Since \( T\left(\frac{2}{2},15,2\right) = 4 \), we expect these pairs to yield precisely four distinct progressions with common difference one whose sum is \( n \). Clearly, we need only consider the first four ordered pairs if we let each factor be set equal to either component. In addition a direct computation shows that a progression arises only if \( k+1 \) is set equal to the smaller component. Hence from the eight ordered pairs we can obtain only four distinct progressions. Consider the ordered pair \([1,30] \). We set
\( k+1 = 1 \) and so \( 2a+k = 30 \). Solving simultaneously, \( k = 0 \)
and \( a = 15 \). Examining now each ordered pair \([c,d]\) where \( c \leq d \), we obtain:
<table>
<thead>
<tr>
<th>Number of Terms (k+1)</th>
<th>First Term (a)</th>
<th>Progression Enumerated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>7, 8</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4, 5, 6</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1, 2, 3, 4, 5</td>
</tr>
</tbody>
</table>

Hence our function tells us exactly how many distinct progressions there are, and then an examination of each ordered pair determines the progression explicitly.

The example can readily be extended for a progression with common difference $b > 1$. If $a$, $b$, and $n$ are positive integers, $k$ a non-negative integer such that

$$\sum_{j=0}^{k} (a+bj) = n$$

we see that the two factors $(2a+bk)(k+1)$

$$= 2 \sum_{j=0}^{k} (a+bj)$$

will be of opposite parity whenever $k \equiv 1 \pmod{2}$ and $b$ is odd. Hence $\frac{T(2,n,2)}{2}$ is the exact number of progressions of the form $\sum_{j=0}^{k} (a+bj)$ for fixed $a$, $b$, and $k$ which have an even number of terms and whose sum is $n$.

A quite different application of $T(2,n,2)$ arises from a consideration of the Diophantine equation $x^2 + y^2 = z^2$. It is proved in [3, p.190] that the most general solution of $x^2 + y^2 = z^2$ is of the form: $x = 2ab$, $y = a^2 - b^2$, $z = a^2 + b^2$, where $a$ and $b$ are positive integers of opposite parity such that $(a,b) = 1$ and $a > b > 0$. We know $T(2,ab,2)$ is the total number of ways
2ab = x can be factored into two factors of opposite parity, but our function in no way rules out the possibility that \( a \equiv b \equiv 1 \pmod{2} \) nor does it impose the restrictions that \( a \) and \( b \) be relatively prime or that \( a > b > 0 \). Hence before we can determine whether or not \( T(2,ab,2) \) counts the number of Pythagorean triples (i.e., solutions of \( x^2 + y^2 = z^2 \)) for fixed \( x \), we must first investigate what effect removal of coprime and parity requirements for \( a \) and \( b \) will have on solutions of the form: \( x = 2ab, y = |a^2 - b^2| \) and \( z = a^2 + b^2 \).

In order to facilitate our discussion we establish the following two lemmas.

**Lemma 1:** Assume solutions of \( x^2 + y^2 = z^2 \) are of the form \( x = 2ab, y = |a^2 - b^2| \) and \( z = a^2 + b^2 \), where \( a \) and \( b \) are positive integers such that \( a = dA, b = dB, \) and \( (A,B) = 1 \). Then \( (a,b) = d \) implies \( (x,y) = kd^2 \) where \( k = 2 \) if \( A \equiv B \pmod{2} \) and \( k = 1 \) otherwise.

**Proof:** Case 1. Let \( A \equiv B \pmod{2} \). Then \( x = 2ab = 2d^2AB, y = |a^2 - b^2| = d^2|A - B| = 2d^2M \) for some non-negative integer \( M \). Now suppose \( g = (x,y) \). Then \( g|x \) and \( g|y \) implies \( g|z \). Hence \( g|2d^2AB \) and \( g|((a^2 - b^2) + (a^2 + b^2)) \) by standard congruence arguments. Thus \( g|2a^2 = 2d^2A^2 \) and \( g|2b^2 = 2d^2B^2 \). Since \( (A,B) = 1, g|2d^2 \). Conversely, since \( d|a \) and \( d|b, \)
2d^2|2ab = x and also 2d^2|2d^2M = y. Therefore, 2d^2|(x,y) = g. We conclude that g = (x,y) = 2d^2.

**Case 2.** Let A \not\equiv B \pmod{2}, and again let g = (x,y). Then g|x = (2AB)d^2 and g|y = d^2|A - B| implies g|d^2 since for (A,B) = 1, (A,A - B) = 1 and (B,A - B) = 1 and hence (AB,A - B) = 1 by well-known congruence arguments. However, since by assumption A \not\equiv B \pmod{2}, (2,A - B) = 1 and therefore (2AB,A - B) = 1. On the other hand, d|a and d|b implies d^2|2ab = x and d^2| |a^2 - b^2| = y. Hence d^2|(x,y) = g. Therefore, g = (x,y) = d^2.

□

**Lemma 2:** Let \(x^2 + y^2 = z^2\). If (x,y) = k, then (y,z) = k.

**Proof:** Let \(x = kX, y = kY\) where (X,Y) = 1. Then \(x^2 + y^2 = z^2\) can be written \(k^2X^2 + k^2Y^2 = z^2\). Hence \(k^2|z^2\) and so \(k|z\). But we know \(k|y\) and thus \(k|(y,z)\). On the other hand, suppose \((y,z) = g\). Then \(g|y\) and \(g|z\) implies \(g|x\). Therefore, \(g|(x,y) = k\) and we conclude \(g = (y,z) = k\).

□

We are now ready to examine solutions of the Diophantine equation \(x^2 + y^2 = z^2\). If we assume solutions are of the form \(x = 2ab, y = |a^2 - b^2|\) and
z = a^2 + b^2, where a = dA, b = dB, and \((A,B) = 1\), then direct computation shows that

\[(2d^2AB)^2 = ((dA)^2 + (dB)^2)^2 - ((dA)^2 - (dB)^2)^2.\]

Hence the above expressions for \(x, y,\) and \(z\) do in fact yield solutions of the Diophantine equation \(x^2 + y^2 = z^2\) for \((a,b) = d, d \geq 1.\)

Letting \((a,b) = d,\) then for \(A \not\equiv B \pmod{2}\) by Lemma 1, \((x,y) = d^2.\) So we can write \(x = d^2X\) and \(y = d^2Y\) where \((X,Y) = 1.\) Then by Lemma 2, \(z = d^2Z.\) Therefore,

\[x^2 + y^2 = (d^2X)^2 + (d^2Y)^2\]

or

\[d^4X^2 + d^4Y^2 = d^4Z^2\]

the solution of which we recognize as a multiple of a primitive triple (i.e., a solution for which \((x,y,z) = 1).\)

The argument is clearly analogous for the case in which \(A \equiv B \pmod{2}\) and consequently \((x,y) = 2d^2.\)

Therefore, we have shown that dropping the coprime and parity requirements on \(a\) and \(b\) still gives us a solution of the general form, but for \(d > 1\) we obtain a multiple of a primitive triple.

In addition we can assert that all triples will be of this form. For let \(X^2 + Y^2 = Z^2,\) then \((\frac{X}{2})^2 = (\frac{Z+Y}{2})(\frac{Z-Y}{2}).\)

Since \((Z,Y) = 1\) both factors on the right are coprime and hence \(\frac{Z+Y}{2}\) and \(\frac{Z-Y}{2}\) must each be a perfect square by unique

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factorization. We write $Z + Y = 2m^2$ for some positive integer $m$ and $Z - Y = 2n^2$ for some $n$. Hence $X^2 = Z^2 - Y^2 = (Z + Y)(Z - Y) = 2m^2 \cdot 2n^2 = (2mn)^2$.

If we can establish that for a given factorization of $x = 2ab$, $y$ and $z$ are uniquely determined, then each distinct factorization will give rise to a different triple. Let $x = 2ab = 2cd$ be two different factorizations of a given $x$, and assume for a moment that $y = c^2 - d^2 = a^2 - b^2$. Then

$$x^2 + y^2 = 4a^2b^2 + (a^2 - b^2)^2 = z^2 = (a^2 + b^2)^2.$$

Also

$$x^2 + y^2 = 4c^2d^2 + (c^2 - d^2)^2 = z^2 = (c^2 + d^2)^2.$$

Adding the identity $a^2 - b^2 = c^2 - d^2$ to $a^2 + b^2 = c^2 + d^2$ we obtain $a = c$ and so then $b = d$. Thus if $2ab = 2cd$ represent two different factorizations of $x$ such that $a$ is different from both $c$ and $d$, then $c^2 - d^2 \neq a^2 - b^2$ and hence we conclude that each distinct factorization of $x$ gives rise to a different triple. Thus by considering all factorizations of $x$ where $x \equiv 0 \pmod{2}$ we obtain all Pythagorean triples associated with this given $x$.


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and [60,1]. Of these ordered pairs, eight have components of opposite parity. Since our triples are symmetric in a and b we let 2a equal the even component, b equal the odd component without restricting the generality of our argument.

<table>
<thead>
<tr>
<th>Ordered pairs of opposite parity</th>
<th>a</th>
<th>b</th>
<th>Triple</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1,60], [60,1]</td>
<td>30</td>
<td>1</td>
<td>60, 899, 901</td>
</tr>
<tr>
<td>[4,15], [15,4]</td>
<td>2</td>
<td>15</td>
<td>60, 221, 229</td>
</tr>
<tr>
<td>[5,12], [12,5]</td>
<td>6</td>
<td>5</td>
<td>60, 11, 61</td>
</tr>
<tr>
<td>[3,20], [20,3]</td>
<td>10</td>
<td>3</td>
<td>60, 91, 109</td>
</tr>
</tbody>
</table>

These are the only triples associated with \( x = 60 \), for if we examine the remaining four ordered pairs: [2,30], [30,2], [6,10], and [10,6] we note that a = 1, 15, 5 and 3 respectively. However, these new choices for a are the b's of (2), and hence again by symmetry we have formed no new triples.

It is now apparent that \( T\left(\frac{2}{2},ab,2\right) \) is the total number of distinct Pythagorean triples associated with a given \( x = 2ab \).
A GENERALIZATION OF THE SIGMA FUNCTION

We continue to restrict our attention to the case $k = 2$ and recall that $T(p,n,2)$ is the number of distinct ordered pairs $[a,b]$ such that $ab = pn$ and either $a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$, but not both. This implies that $\frac{T(p,n,2)}{2}$ is the number of divisors $d$ of $n$ such that $(d,p) = 1$. This extension of the $\tau$ function suggests that we can develop an extended $\sigma$ function in an analogous manner.

Theorem 6: If $\frac{S(1,p,n)}{2}$ is the sum of divisors $d$ of $pn$ such that $(d,p) = 1$, then $\frac{S(1,p,n)}{2}$ is multiplicative in $n$ and $\frac{S(1,p,n)}{2} = \frac{(p-1)\sigma(n)}{(p^s+1-1)}$, where $n = p^s \prod_{i=1}^{s} q_i^a_i$, and $\sigma(n)$ is the sum of the divisors of $n$.

Proof: If $pn = p^s + 1 \prod_{i=1}^{s} q_i^a_i$ then clearly the divisors incongruent to zero $\pmod{p}$ are those $d$ such that $d|n = \prod_{i=1}^{s} q_i^a_i$. The sum of these divisors is given by $\sigma(m)$. Therefore, $\frac{S(1,p,n)}{2} = \sigma(m)$. Observe that since $\sigma$ is multiplicative, we can thus assert that $\frac{S(1,p,n)}{2}$ is
multiplicative in the argument \( n \). In addition we can
write \( \sigma(n) = \sigma(p^\beta)\sigma(m) \). Recall that \( \sigma(p^\beta) = \frac{p^{\beta+1} - 1}{p - 1} \).
Thus \( \frac{S(1,p,n)}{2} = \frac{\sigma(n)}{\sigma(p^\beta)} = \frac{(p - 1)\sigma(n)}{(p^{\beta+1} - 1)} \).

Example: Consider \( \frac{S(1,3,12)}{2} \), the sum of the divisors \( d \)
of \( 36 \) such that \( (d,3) = 1 \). By the above theorem \( \frac{S(1,3,12)}{2} = \sigma(4) = 1 + 2 + 4 = 7 \), or equivalently, \( \frac{S(1,3,12)}{2} = \frac{3^2 - 1}{3 - 1} \sigma(12) = 1/4 \times 28 = 7 \). Enumerating all the divisors
of \( 36: 1, 2, 3, 4, 6, 9, 12, 18, \) and \( 36 \), those divisors
incongruent to zero \( \text{(mod 3)} \) are just \( 1, 2, \) and \( 4 \), and
their sum is 7.

We wish now to exhibit the Dirichlet generator,
\( s_1(n) \), of \( \frac{S(1,p,n)}{2} \).

Theorem 7: Let the Dirichlet generator
\( s_1(n) = \sum_{d|n} \mu(n/d) \frac{S(1,p,d)}{2} \). Then \( s_1(n) = n \) if \( p \nmid n \),
where \( p \) is the given prime of \( \frac{S(1,p,n)}{2} \), and \( s_1(n) = 0 \)
otherwise.

Proof: We recall from the proof of Theorem 4.
that \( s_1 \) is multiplicative. Hence evaluating \( s_1 \) at the prime powers we first let \( q \) be a prime different from the given prime \( p \) and \( S(q^\alpha) = \frac{S(1, p, q^\alpha)}{2} \). Then

\[
S_1(q^\alpha) = S(q^\alpha)\mu(1) + S(q^{\alpha-1})\mu(q)
\]

\[
= \frac{q^{\alpha+1} - 1}{q - 1} - \frac{q^\alpha - 1}{q - 1}
\]

\[
= \frac{q^{\alpha+1} - q^\alpha}{q - 1} = \frac{q^\alpha(q - 1)}{q - 1} - q^\alpha.
\]

For the given prime \( p \),

\[
S_1(p^\alpha) = S(p^\alpha)\mu(1) + S(p^{\alpha-1})\mu(p)
\]

\[
= \frac{p^{\alpha+1} - 1}{p - 1} \cdot \frac{p - 1}{p^{\alpha+1} - 1} - \frac{p^{\alpha-1} + 1}{p - 1} \cdot \frac{p - 1}{p^{\alpha-1} + 1 - 1}
\]

\[
= 1 - 1 = 0.
\]

By the multiplicative property of \( s_1 \), the Dirichlet generator of \( S(1, p, n) \) is now completely determined.

The sum of the \( j \)th powers of divisors of \( n \) is given by \( \sigma_j(n) = \sum_{d|n} d^j \) and hence \( \sigma(n) = \sigma_1(n) \). It is proved in [3, p.238] that \( \sigma_j \) is multiplicative and if \( N = \prod_{i=1}^{r} p_i^{a_i} \)

\[
\sigma_j(N) = \prod_{i=1}^{r} \frac{(p_i^{a_i+1})^j - 1}{p_i^j - 1}.
\]

In like manner we
generalize our function $S(1,p,n)$. 

**Theorem 8**: Let $S(1,p,n)$ denote the sum of the $j$th powers of divisors $d$ of $pn$ such that $(d,p) = 1$. Then $S(1,p,n)$ is multiplicative and 

$$
S(1,p,n) = \frac{p^j - 1}{p^j(p+1) - 1} \sigma_j(n)
$$

where $n = p^\sigma \prod_{i=1}^s q_i^{a_i}$. 

**Proof**: As in Theorem 7, we note that the divisors incongruent to zero (mod $p$) are those $d$ such that $d \mid m = \prod_{i=1}^s q_i^{a_i}$. Thus $S(1,p,n) = \sigma_j(m)$. Hence our function is multiplicative and since \( \sigma_j(n) = \frac{p^j(p+1) - 1}{p^j - 1} \sigma_j(m) \), the assertion now follows. 

**Example**: Consider $S(2,3,12)$, the sum of the squares of divisors $d$ of $36$ such that $(d,3) = 1$. By the previous example these divisors are $1$, $2$, and $4$. The sum of their squares is then $1^2 + 2^2 + 4^2 = 21$. By the above theorem $S(2,3,12) = \sigma_2(4) = 21$. Also we can write 

$$
S(2,3,12) = \frac{3^2 - 1}{3^2(3) - 1} \sigma_2(12) = \frac{8}{81 - 1} \cdot 210 = 21.
$$
Remark: It is clear that Theorem 6 is a special case of Theorem 8 for \( j = 1 \).

We continue the characterization of \( \frac{S(j, p, n)}{2} \) by determining the Dirichlet generator, which we denote \( s_j \).

**Theorem 2:** Let the Dirichlet generator \( s_j(n) \)

\[
= \sum_{d|n} \mu(n/d) \frac{S(j, p, d)}{2} .
\]

Then \( s_j(n) = n^j \) if \( p \nmid n \) and \( s_j(n) = 0 \) otherwise.

**Proof:** Let \( p, j \) be arbitrary and fixed, and let
\[
S_j(q^a) = \frac{S(j, p, q^a)}{2}
\]
for brevity. Again from the assertion of Theorem 4, \( s_j \) need only be evaluated at the prime powers, and so for \( q \) different from the given prime \( p \) we obtain:

\[
s_j(q^a) = S_j(q^a)\mu(1) + S_j(q^{a-1})\mu(q)
\]

\[
= \frac{q^j(a+1) - 1}{q^j - 1} - \frac{q^j(a-1) + 1}{q^j - 1}
\]

\[
= \frac{q^j(a+1) - 1 - q^ja + 1}{q^j - 1} = \frac{q^ja - q^j}{q^j - 1}
\]

\[
= \frac{q^ja(q^{j-1} - 1)}{q^j - 1} = q^ja = (q^a)^j .
\]

For the given prime \( p \), \( s_j(p^a) = S_j(p^a)\mu(1) + S_j(p^{a-1})\mu(p) \)
= 1·1 - 1·1 = 0.

Remark: We note that Theorem 7 is a special case of Theorem 9 for $j = 1$. In addition observe that just as

$$\tau(n) = \sigma_0(n) \text{ so } T(p_1,n/2) = S(0,p_1,n).$$

Since the functions are equal, this implies their Dirichlet generators $s_0$ and $t$ must be equal. This is clear since $t(q^{\alpha}) = \tau_1(q^{\alpha}) = 1$. But then $s_0(q^{\alpha}) = (q^{\alpha})^0 = 1$. 

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ASYMPTOTIC APPROXIMATIONS OF SUMMATORIY FUNCTIONS OF T AND S

Just as it is useful to consider averages of the \( \tau \) and \( \sigma \) functions, so also do we wish to conclude our discussion of the functions we have defined by describing the behavior of certain special cases in terms of summatory functions. Interpreting the average order of \( \tau(n) \) as the number of lattice points \((d, d')\) lying in the first quadrant and on or below the hyperbola \( dd' = n \) and using the identity

\[
\sum_{n \leq x} \sum_{d \mid n} t(d) \mu^{-1}(n/d) = \sum_{d \leq x} t(d) \sum_{j \leq x/d} \mu^{-1}(j),
\]

which is verified in \([5, p.105]\), we determine an asymptotic approximation for the summatory function \( \sum_{n \leq x} \frac{T(p, n, 2)}{2} \).

**Theorem 10:** \( \sum_{n \leq x} \frac{T(p, n, 2)}{2} = (x - \frac{x}{p}) \log x + \frac{x}{p} \log p + (2\gamma - 1)(x - \frac{x}{p}) + O(\sqrt{x}), \) where \( \gamma \) is the Euler-Mascheroni constant.

**Proof:** \( \sum_{n \leq x} \frac{T(p, n, 2)}{2} = \sum_{n \leq x} t_0 \mu^{-1}(n) \)

\[= \sum_{n \leq x} \sum_{d \mid n} t(d) \mu^{-1}(n/d) \]

26
Dirichlet established in 1849 and the proof is reproduced in [3, p. 264] that
\[ \sum_{d=1}^{\lfloor \frac{n}{d} \rfloor} \tau(d) = n \log n + (2\gamma - 1)n + O(\sqrt{n}). \]
Hence it follows that
\[
\sum_{n \leq x} \frac{T(p, n, 2)}{\tau(n)} = \sum_{n \leq x} \tau(n) - \sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x})
\]
\[= x \log x + (2\gamma - 1)x + O(\sqrt{x}) - \frac{x}{p} \log \frac{x}{p} - (2\gamma - 1) \frac{x}{p} \]
\[= (x - \frac{x}{p}) \log x + \frac{x}{p} \log p + (2\gamma - 1)(x - \frac{x}{p}) + O(\sqrt{x}). \]

**Corollary:** The average order of \( \frac{T(p, n, 2)}{2} \) is given by:
\[
\frac{1}{x} \sum_{n \leq x} \frac{T(p, n, 2)}{2} \sim \frac{p - 1}{p} \log x + \frac{1}{p} \log p.
\]

**Example:**
\[
\sum_{n \leq x} \frac{T(2, n, 2)}{2} = \sum_{n \leq x} \tau(n) - \sum_{n \leq \frac{x}{2}} \tau(n)
\]
\[= \frac{x}{2} \log x + \frac{x}{2} \log 2 + \frac{x}{2} (2\gamma - 1) + O(\sqrt{x}). \]
The average order of \( \frac{T(2, n, 2)}{2} \) is given by:
\[
\frac{1}{x} \sum_{n \leq x} \frac{T(2, n, 2)}{2} \sim \frac{1}{2} (\log x + \log 2) = \log \sqrt{2x}.
\]

Using the known approximation \( \sum_{m=1}^{n} \sigma(m) = \frac{n^2}{12} \) + \( O(n \log n) \) also verified in [3, p.266], we can determine an asymptotic approximation for \( S(1, p, n) \).

**Theorem 11:** \( \sum_{n \leq x} S(1, p, n) \) = \( \frac{p-1}{12p} n^2 \pi^2 + O(x \log x) \).

**Proof:** \( \sum_{n \leq x} S(1, p, n) \) = \( \sum_{d \leq x} \sigma_1(d) \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d \leq x} \frac{x}{d} - \sum_{d \leq x} \frac{x}{d} = \sum_{n \leq x} \sigma(n) - p \sum_{n \leq \frac{x}{p}} \sigma(n) \).

Hence \( \sum_{n \leq x} S(1, p, n) \) = \( \frac{\pi^2}{12} x^2 + O(x \log x) - p \frac{\pi^2}{12} \left( \frac{x}{p} \right)^2 \)

\[
= \frac{p-1}{12p} \pi^2 x^2 + O(x \log x).
\]
REFERENCES


