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# Edge Induced Weightings of Uniform Hypergraphs and Related Problems

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# Edge Induced Weightings of Uniform Hypergraphs and Related Problems

by

Laars C. Helenius

A dissertation submitted to the Graduate College  
in partial fulfillment of the requirements  
for the degree of Doctor of Philosophy  
Mathematics  
Western Michigan University  
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# Edge Induced Weightings of Uniform Hypergraphs and Related Problems

Laars C. Helenius, Ph.D.

Western Michigan University, 2018

The starting point of the research is the so called *1-2-3 Conjecture* formulated in 2004 by Karoński, Łuczak, and Thomason. Roughly speaking it says that the edges of any graph can be weighted from  $\{1, 2, 3\}$  so that the induced vertex coloring (as the sum of weights adjacent to a given vertex) is proper. The conjecture has attracted a lot of interest from researchers over the last decade but is still unanswered. More recently, the conjecture has been studied for hypergraphs.

The main result of this dissertation shows in particular that an analogous conjecture holds for almost all uniform hypergraphs. Additionally, it also studies how other sets with binary operations, e.g. finite abelian groups, can be used to color hypergraph edges, how hypergraphs are connected by a certain type of path structure, and calculates the threshold probability in a random hypergraph for the appearance of the related cycle structure.

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# Acknowledgements

This work is the culmination of a nearly 20 year journey for me. I finished my M.A. in 1998 after getting married the previous year, found a job, started a family, found a new job, and worked to improve our lives together. I always regretted not finishing my Ph.D. though. I had worked as both an actuarial analyst and a high school math teacher, but I missed the challenge of the more advanced mathematics from my graduate studies. I am eternally grateful that I eventually found a way to return to my studies and still be able to support my family.

Dr. Andrzej Dudek has been an advisor and a friend throughout my studies and he is the primary reason I was able to finish this. When I took his graph theory class, the information on random graphs was fascinating and I knew that I wanted to study more of that. His support and guidance throughout all phases of this project have been invaluable. Thank you for taking me on as your student!

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I first met Dr. David Galvin at the 1<sup>st</sup> *Lake Michigan Workshop on Combinatorics and Graph Theory*. He desperately needed coffee before presenting that first morning of the workshop, so I took him to a nearby coffee shop. During the trip, we made small talk, but I was impressed how personable he was. Besides, anyone who takes his coffee that seriously is worth paying attention to!

Graph theory and combinatorics is a vast field of study. It was Dr. Allen Schwenk who first introduced these ideas to me and I have been enamored ever since. My very first research project was guided by Allen. Thank you so much for showing me what is possible.

It was 14 years from the end of my M.A. to the start of my Ph.D.. I really struggled to find my mathematical footing that first semester back and contemplated quitting. It seemed

like I had forgotten so much and that perhaps I was fooling myself in the attempt to finish my Ph.D.. The only reason I did not quit was because of the patience and understanding from Dr. Ping Zhang, who was my professor for my first semester of graph theory. She encouraged me to keep working through the material and that I would eventually find my way. She was right and I am ever grateful for her guidance.

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Laars C. Helenius

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# Chapter 1

## Introduction

Let us begin with a few basic definitions. We say  $G = (V, E)$  is a (simple) graph on a *vertex set*  $V = V(G)$  and an *edge set*  $E = E(G) \subseteq \binom{V(G)}{2}$ . If  $|V(G)| = n$  and  $|E(G)| = m$  then  $G$  is said to be a graph of *order*  $n$  and *size*  $m$ .

One way we can explore the structure of  $G$  is by assigning colors (weights) to either the vertices or the edges of  $G$ . When we assign colors (weights) to the vertices of  $G$  so that any two adjacent vertices have distinct colors, we have a *proper vertex coloring (weighting) of  $G$* . Incidentally, the fewest number of colors that can be used to properly color the vertices of  $G$  is called the *chromatic number of  $G$*  and is denoted  $\chi(G)$ . Similarly, when we assign colors (weights) to the edges of  $G$  so that all the edges incident with any vertex have distinct colors (weights), we have a *proper edge coloring (weighting) of  $G$* .

Formally, a  *$k$ -vertex weighting* is a function  $\omega : V(G) \rightarrow [k]$  such that the weight of any  $v \in V(G)$  is given as  $w(v) = \ell$  for some  $1 \leq \ell \leq k$  and a  *$k$ -edge-weighting* is a function  $\omega : E(G) \rightarrow [k]$  such that the weight of any  $e \in E(G)$  is given as  $\omega(e) = \ell$  for some  $1 \leq \ell \leq k$ .

The number of edges incident with a vertex  $v$  is the *degree of  $v$*  and is denoted  $\deg(v)$ . We also have  $\delta(G) = \min_{v \in V(G)} \{\deg(v)\}$ , which denotes the minimum degree of a graph,  $\Delta(G) = \max_{v \in V(G)} \{\deg(v)\}$ , which denotes the maximum degree of the graph, and we say  $G$  is  *$k$ -regular* if  $\delta(G) = \Delta(G) = k$ .

A natural question to ask is can we have an *irregular* graph. An irregular graph would be one that is the complete opposite of  $k$ -regular, i.e. a graph such that for all pairs  $u, v \in V(G)$ ,  $\deg(u) \neq \deg(v)$ . It is easily seen that this is impossible. Indeed, if  $G$  were

irregular there would be two vertices  $u, v \in V$  such that  $\deg(u) = 0$  and  $\deg(v) = n - 1$ . So  $u$  is adjacent to no other vertex in  $G$  while  $v$  is adjacent to all other vertices, a contradiction.

Since no graph can be irregular, we would like to understand “how close” a graph might be to being irregular. A first step in this direction was taken by Harary and Plantholt [24] when they introduced *vertex-distinguishing proper edge-colorings*. A vertex-distinguishing proper edge-coloring is an edge coloring such that for each pair of adjacent vertices  $u, v \in V(G)$ , the list of colors used to color the edges incident with  $u$  is distinct from the list of colors used to color the edges incident with  $v$ .

Then Chartrand et al. [10] considered how one might induce a vertex-distinguishing weighting from a given edge weighting and introduced the concept of the *irregularity strength of  $G$* , denoted  $s(G)$ , which is the smallest positive integer  $k \geq 2$  for which there exists an edge weighting  $\omega : E(G) \rightarrow [k]$  such that  $\sum_{e \ni u} \omega(e) \neq \sum_{e \ni v} \omega(e)$  for all distinct  $u, v \in V(G)$ . Since no simple graph is irregular, it follows that  $s(G) \geq 2$  for all simple graphs.

A lot of work was done trying to find a tight upper bound on  $s(G)$ . It was conjectured by Aigner and Triesch in [3] that  $s(G) \leq |V(G)| - 1$  for all graphs and this was subsequently confirmed by Nierhoff in [34]. This upper bound was improved by Kalkowski, Karoński, and Pfender in [29] to  $s(G) \leq \left\lceil \frac{6n}{\delta(G)} \right\rceil$ .

From here, more generalizations of how one can induce various vertex weightings from given edge weightings were explored. In particular, researchers began exploring how non-proper edge weightings could induce various vertex weightings and what the minimum number of edge weights are required to achieve a given type of vertex weighting.

We say that a graph  $G$  is *nice* if it has no isolated edges and we say a nice graph  $G$  has a  *$k$ -neighbor distinguishing weighting* when there exists an edge weighting  $\omega : E(G) \rightarrow [k]$  such that for all *adjacent* pairs  $u, v \in V(G)$  we have  $\sum_{e \ni u} \omega(e) \neq \sum_{e \ni v} \omega(e)$ . Further, if  $k$  is the minimum integer such that a graph  $G$  that has a  $k$ -neighbor-distinguishing weighting, then we say  $G$  is  *$k$ -weighted*.

**1-2-3 Conjecture.** *If  $G$  is nice, then  $G$  is 3-weighted.*

Trivially, if a graph has an isolated edge, then there can be no hope of having a neighbor-distinguishing weighting because any weight assigned to the isolated edge will induce the same weight on both of the vertices the isolated edge connects. Further, there are infinitely many graphs that require exactly 3 weights to achieve a neighbor-distinguishing weighting, e.g. any clique of order 3 or greater. So if the conjecture is true, 3 weights is a sharp upper bound.

In a lot of ways, this is a remarkable conjecture. Karoński et al. [31] offered some evidence that their conjecture might be true by examining all graphs up to order 10. They were able to show that for any abelian group  $\Gamma$  of odd order and for any graph  $G$  with  $\chi(G) = |\Gamma|$ , then  $G$  was  $|\Gamma|$ -weighted. But it is well known that  $\chi(G)$  is generally unbounded as the order of  $G$  increases. And while they were able to show that there always existed a finite set  $S$  of *real* weights such that  $G$  was  $|S|$ -weighted, there was no proof that a finite set of *integers* sufficed. Other researchers immediately began searching for some kind of finite integer upper bound for  $k$ -weighted graphs. Their efforts were quickly rewarded.

**If  $G$  is nice, then:**

- $G$  has a 30-weighting (Addario-Berry, Dalal, McDiarmid, Reed, Thomason [1]),
- $G$  has a 16-weighting (Addario-Berry, Dalal, Reed [2]),
- $G$  has a 13-weighting (Wang, Yu [38]),
- $G$  has a 6-weighting (Kalkowski, Karoński, Pfender [27]),
- $G$  has a 5-weighting (Kalkowski, Karoński, Pfender [28]).

The proof of the best known bound is algorithmic, with an elegance and brevity that is seemingly straight from *The Book* [4].

Other researchers have turned to random graph theory for further evidence that supports the plausibility of the 1-2-3 Conjecture. In particular, Addario-Berry et al. [2] that almost all graphs satisfy the 1-2-3 Conjecture.

Recently, Kalkowski, Karoński, and Pfender [30] studied similar problems for *hypergraphs*, which are natural generalizations of graphs. In this dissertation we continue this type of research. In Chapter 2 we study the behavior of neighbor-distinguishing edge-weightings in hypergraphs and find evidence supporting a particular version of the 1-2-3 Conjecture for hypergraphs. In Chapter 3 we study the behavior of certain cycle structures that can arise in hypergraphs and conclude that an abelian group with  $\chi(H)$  elements can always give rise to a neighbor-distinguishing edge weighting in a hypergraph. In Chapter 4 we provide a quick reference for the reader to all of the basic techniques used throughout this dissertation.

# Chapter 2

## 1-2-3 Conjecture for $r$ -Uniform Hypergraphs

### 2.1 Introduction

The main results of this chapter were published as a joint work with Bennett, Dudek, Frieze, and Helenius in [8].

A hypergraph  $H$  is a natural generalization of a graph. In a graph  $G$ ,  $E(G) \subseteq \binom{V(G)}{2}$ , i.e. there are always exactly two vertices associated with any edge. But in a hypergraph, we relax this requirement and allow  $E(H) \subseteq 2^{V(H)}$  so that an edge in  $H$  can associate any number of vertices, see, e.g., Figure 2.1(a). A hypergraph is said to be *r-uniform* if every edge contains exactly  $r$  vertices, see, e.g., Figure 2.1(b). In hypergraphs, there are several notions to describe vertex colorings.

A *weak* vertex coloring of  $H$  is one in which all the vertices in each edge are not all the same color. If there exists a weak vertex coloring using  $k$  colors, then  $H$  is *weakly k-colorable* and the coloring is a *weak k-coloring* of  $H$ , see, e.g., Figure 2.2(a).

A *strong* vertex coloring of  $H$  is one in which all the vertices in each edge have distinct colors within that edge. An edge with this property is said to be *rainbow*. If there exists a strong vertex coloring using  $k$  colors, then  $H$  is *strongly k-colorable* and the coloring is a *strong k-coloring* of  $H$ , see, e.g., Figure 2.2(b). Observe that all strong colorings are also weak colorings.

Similar to our situation with graphs, we can let an edge weighting *induce* a vertex

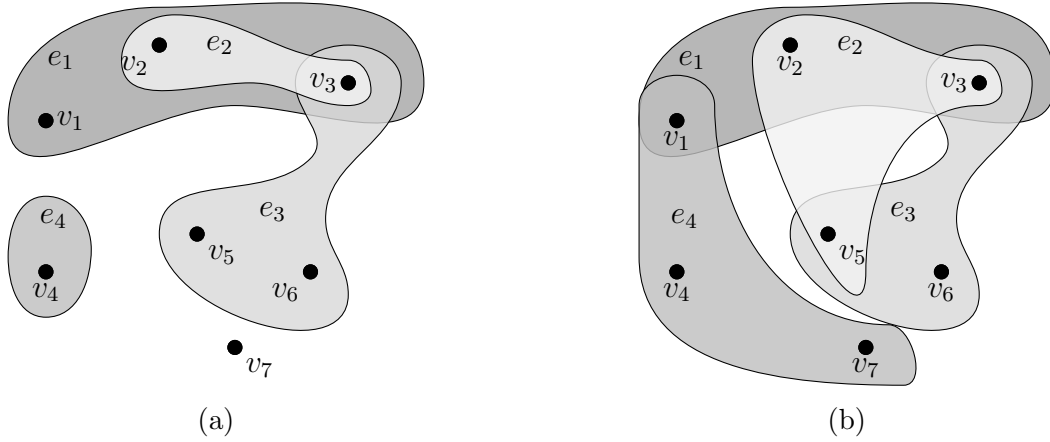


Figure 2.1: (a) A hypergraph on seven vertices  $\{v_1, \dots, v_7\}$  and four edges  $\{e_1, e_2, e_3, e_4\}$  and (b) a 3-uniform hypergraph on the same vertex set.

coloring. So if we let  $k$  be the smallest integer such that  $\omega : E(H) \rightarrow [k]$  is an edge weighting with  $c : V(H) \rightarrow \mathbb{N}$  such that

$$c(v) = \sum_{e \ni v} \omega(e),$$

such that  $c$  is a strong coloring, then we say that  $H$  is *strongly  $k$ -weighted*. If  $c$  is a weak coloring, then we say that  $H$  is *weakly  $k$ -weighted*.

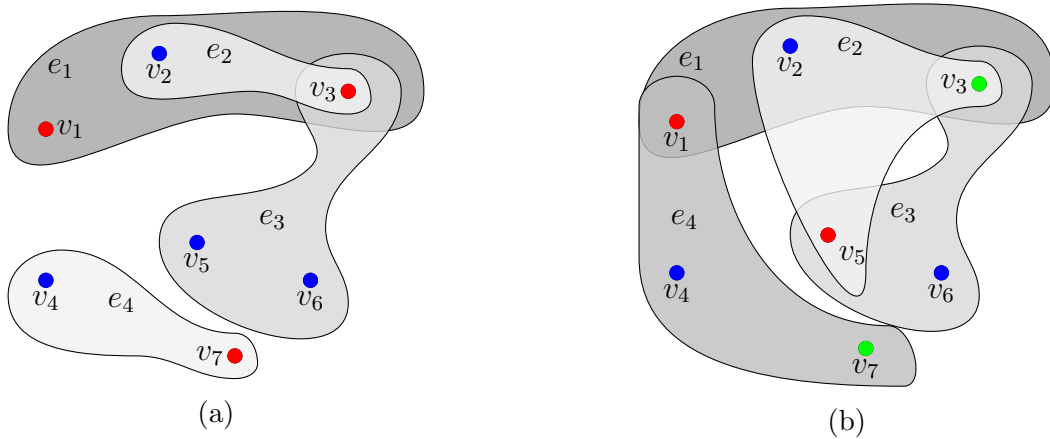


Figure 2.2: (a) A weak 2-coloring of a hypergraph on the vertex set  $\{v_1, \dots, v_7\}$  and (b) a strong 3-coloring of a 3-uniform hypergraph on the same vertex set.

We say that a hypergraph is *nice* if there is no pair of vertices  $u$  and  $v$  such that the set of edges containing  $u$  is the same as the set of edges containing  $v$ . Note that only nice

hypergraphs can be strongly weighted.

It is an immediate consequence of the work by Kalkowski et al. [30] that any  $r$ -uniform hypergraph is weakly  $\max\{5, r + 1\}$ -weighted and based on the work in this dissertation, we conjecture

**1-2-3 Conjecture for Hypergraphs.** *All  $r$ -uniform hypergraphs are weakly 3-weighted.*

The techniques used in this dissertation revolve around the study of *random graphs and hypergraphs*. The *binomial random graph*, denoted  $\mathbb{G}(n, p)$ , is a graph of order  $n$  such that each of the possible  $\binom{n}{2}$  edges appear independently with probability  $p \in [0, 1]$ . Similarly, the *binomial  $r$ -uniform random hypergraph*, denoted  $\mathbb{H}^{(r)}(n, p)$ , is a hypergraph of order  $n$  such that each of the possible  $\binom{n}{r}$  edges appear independently with probability  $p \in [0, 1]$ . Clearly,  $\mathbb{G}(n, p) = \mathbb{H}^{(2)}(n, p)$ . We also say that a sequence of events  $\{E_n\}_{n=1}^{\infty}$  occurs *with high probability*, or *w.h.p.* for brevity, if  $\lim_{n \rightarrow \infty} \Pr(E_n) = 1$ .

Observe that  $\mathbb{H}^{(r)}(n, 1/2)$  defines a uniformly distributed probability space whose objects consist of all possible  $r$ -uniform hypergraphs of order  $n$ , i.e. the probability of selecting any graph from the space is  $2^{-\binom{n}{r}}$ . If a property  $\mathcal{P}$  holds in  $\mathbb{H}^{(r)}(n, 1/2)$  *w.h.p.*, then we say that  $\mathcal{P}$  holds for *almost all* graphs.

The main result of this chapter is that almost all  $r$ -uniform hypergraphs are strongly 2-weighted and consequently, almost all  $r$ -uniform hypergraphs are weakly 2-weighted. This extends the previous results of Addario-Berry et al. [2] that showed almost all graphs are 2-weighted.

On the one hand, we demonstrate that there is no absolute constant bound for strong weightings by constructing a class of  $r$ -uniform hypergraphs that require at least  $r^2 - r + 1$  weights, while on the other hand, we do find evidence supporting the plausibility of the conjecture that all  $r$ -uniform hypergraphs are weakly 3-weighted.

## 2.2 Lower Bounds for Strong and Weak Weightings

We begin our investigation using deterministic methods to find some lower bounds for both strong and weak weightings of  $r$ -uniform hypergraphs.

First we recall some basic properties of projective planes (see, e.g., [21]). A *projective plane*  $P(q)$  of order  $q$  is an incidence structure on a set  $P$  of points and a set  $L$  of lines such that: any two points lie in a unique line, and every line contains  $q + 1$  points, and every point lies on  $q + 1$  lines. It is known that for every prime power  $q$  such incidence structure  $P(q)$  exists with  $|P| = |L| = q^2 + q + 1$ . In other words,  $P(q)$  is a  $(q + 1)$ -regular  $(q + 1)$ -uniform hypergraph of order  $q^2 + q + 1$ .

### Theorem 2.2.1.

- (i) *Let  $r \geq 3$  be such that  $r - 1$  is a prime power. Then, there exists a nice  $r$ -uniform hypergraph that is not strongly  $(r^2 - r)$ -weighted. Furthermore, for all large  $r$  (with  $r - 1$  not necessarily a prime power), there exists a nice  $r$ -uniform hypergraph that is not strongly  $(r^2 - o(r^2))$ -weighted.*
- (ii) *For every  $r \geq 3$  there exists an  $r$ -uniform hypergraph that is not weakly 2-weighted.*

Kalkowski et al. [30] initially thought that perhaps two weights were sufficient to weakly color all  $r$ -uniform hypergraphs. Clearly, part (ii) of this theorem supplies a counterexample.

*Proof.* To prove part (i), suppose that a projective plane  $(P, L)$  of order  $q$  exists. We create a 2-regular  $(q + 1)$ -uniform hypergraph  $H = (V, E)$  as follows. First we blow up each point  $p \in P$  exactly  $q + 1$  times. Formally,

$$V = \{(p, \ell) : p \in P, \ell \in L, p \in \ell\}.$$

Clearly,  $|V| = (q + 1)(q^2 + q + 1)$ . Next we need to define two types of edges of  $H$ . Let  $p \in P$  and  $\ell_1, \dots, \ell_{q+1}$  be lines incident with  $p$ . Then,  $E_1$  consists of all edges of the form



$\{(p, \ell_1), (p, \ell_2), \dots, (p, \ell_{q+1})\}$ , which we denote by  $e(p)$ . Thus,

$$E_1 = \{e(p) = \{(p, \ell_1), \dots, (p, \ell_{q+1})\} : p \in P \text{ and } p \in \ell_i \text{ for all } 1 \leq i \leq q+1\}.$$

Similarly, we define

$$E_2 = \{f(\ell) = \{(p_1, \ell), \dots, (p_{q+1}, \ell)\} : \ell \in L \text{ and } p_i \in \ell \text{ for all } 1 \leq i \leq q+1\}.$$

Set  $E = E_1 \cup E_2$ . It is easy to see that  $H$  is 2-regular  $(q+1)$ -uniform hypergraph and nice.

We claim that  $H$  is not strongly  $(q^2 + q)$ -weighted. Assume for a contradiction that it is. Let  $\omega : E \rightarrow \{1, \dots, q^2 + q\}$  be such that the vertex-coloring  $c$  induced by  $\omega$  is strong. Since  $|E_1| = q^2 + q + 1 > |\omega(E)|$ , there are  $e(p_1)$  and  $e(p_2)$  in  $E_1$  such that  $\omega(e(p_1)) = \omega(e(p_2))$ . By properties of  $P(q)$  there is a line  $\ell$  incident with  $p_1$  and  $p_2$ . Thus,  $f(\ell)$  in  $E_2$  contains  $(p_1, \ell)$  and  $(p_2, \ell)$ . But

$$c((p_1, \ell)) = \omega(e(p_1)) + \omega(f(\ell)) = \omega(e(p_2)) + \omega(f(\ell)) = c((p_2, \ell))$$

and so  $f(\ell)$  is not rainbow, a contradiction.

So we set  $r = q + 1$  and observe that  $q^2 + q = r^2 - r$ . Thus for each  $r - 1$  that is a prime power, there exists an  $r$ -uniform hypergraph which is not strongly  $(r^2 - r)$ -weighted.

The remaining part of (i) is very similar. It is well-known that for large  $x$  there exists a prime number between  $x(1 - o(1))$  and  $x$  (see, e.g., [7]). Hence, for large  $r$  there is a prime number  $q$  such that  $(r - 1)(1 - o(1)) \leq q \leq r - 1$ . First, as above we start with  $P(q)$  and construct a 2-regular  $(q + 1)$ -uniform hypergraph  $H = (V, E)$  by blowing up each vertex of  $P(q)$ . If  $q = r - 1$ , then we are done. Otherwise, if  $q < r - 1$ , then we extend  $H$  to an  $r$ -uniform hypergraph  $I = (W, F)$ . Let  $W = V \cup U$ , where  $|U| = r + 1$ . For each  $e \in E$  we define a new edge in  $F$  by adding to  $e$  arbitrarily  $r - (q + 1)$  vertices from  $U$ . Finally, we add all possible  $\binom{r+1}{r}$  edges on  $U$ . The resulting hypergraph  $I$  is  $r$ -uniform, nice, and not

strongly  $(q^2 + q)$ -weighted. Since  $q^2 + q = r^2 - o(r^2)$ , the proof of (i) is finished.

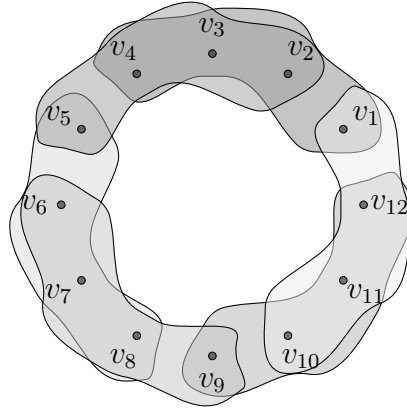


Figure 2.3: A 4-uniform hypergraph which is not weakly 2-weighted.

Now we prove (ii), which actually is an easy observation. Indeed, define an  $r$ -uniform hypergraph  $H = (V, E)$  on  $V = \{v_1, \dots, v_{3r}\}$  with edges  $e_1 = \{v_1, \dots, v_r\}$ ,  $e_2 = \{v_{r+1}, \dots, v_{2r}\}$ ,  $e_3 = \{v_{2r+1}, \dots, v_{3r}\}$ ,  $f_1 = \{v_2, \dots, v_{r+1}\}$ ,  $f_2 = \{v_{r+2}, \dots, v_{2r+1}\}$ , and  $f_3 = \{v_{2r+2}, \dots, v_{3r}, v_1\}$  (see Figure 2.3). If there is a weak coloring of  $H$  induced by some  $\omega : E \rightarrow \{1, 2\}$ , then for some  $i, j$  we must have  $\omega(e_i) = \omega(e_j)$  and consequently edge  $f_\ell \subseteq V(e_i) \cup V(e_j)$  is monochromatic, a contradiction.  $\square$

## 2.3 Weakly 2-Weighted $r$ -Uniform Hypergraphs

It is not difficult to see that we can have irregular hypergraphs and it is fairly obvious that any irregular  $r$ -uniform hypergraph is strongly (and weakly) 1-weighted. Indeed, the weighting  $c$  induced by weighting all edges 1 in  $H$  yields  $c(v) = \deg(v)$  for all  $v \in V(H)$ . Then since  $H$  is irregular, it follows that the vertices in any edge would have distinct colors because the vertices of each edge have distinct degree.

This section is devoted to showing a sufficient condition for a  $r$ -uniform hypergraph to be weakly 2-weighted. For a hypergraph  $H = (V, E)$  and two vertices  $u$  and  $v$  in  $V$  denote by  $\deg(u, v)$  the *pair degree* which is the number of edges in  $E$  containing  $\{u, v\}$ .

**Theorem 2.3.1.** *Let  $H = (V, E)$  be an  $r$ -uniform hypergraph of order  $n$  with maximum degree  $\Delta$  satisfying*

$$\Delta \cdot \sum_{\{x_1, \dots, x_k\} \in E(H)} \prod_{i=1}^r \frac{1}{\sqrt{\deg(x_i)}} = o(1),$$

and for any  $u, v \in V$

$$\deg(u, v) = o(\deg(u)),$$

where the asymptotic is taken in  $n$ . Then,  $H$  is weakly 2-weighted.

*Proof.* To each edge we assign 1 or 2 with probability  $1/2$ . Let  $c$  be the vertex coloring induced by this weighting. By the chain probability rule we get

$$\Pr(c(x_1) = \dots = c(x_r) = c) = \prod_{i=1}^r \Pr(c(x_i) = c \mid c(x_1) = \dots = c(x_{i-1}) = c),$$

and we wish to estimate each factor in the above product. Consider the first vertex  $x_1$ . Since each edge incident with  $x_1$  is weighted 1 or 2, then  $\deg(x_1) \leq c(x_1) \leq 2 \deg(x_1)$  and the actual value of  $c(x_1)$  is completely determined by the number  $t$  of edges weighted 2. Thus for any  $\deg(x_1) \leq c \leq 2 \deg(x_1)$  we have

$$\Pr(c(x_1) = c) = \binom{\deg(x_1)}{t} 2^{-\deg(x_1)}.$$

Now consider  $x_2$ . The actual value of  $c(x_2)$  is completely determined by  $t$ , the number of edges weighted 2 that aren't incident with  $x_1$  as well. Since by assumption there are  $o(\deg(x_2))$  of these, we have for any  $\deg(x_2) - o(\deg(x_2)) \leq c \leq 2(\deg(x_2) - o(\deg(x_2)))$  we have

$$\Pr(c(x_2) = c \mid c(x_1) = c) = \binom{\deg(x_2) - o(\deg(x_2))}{t} 2^{-\deg(x_2) + o(\deg(x_2))}.$$

Thus in general,

$$\begin{aligned} \Pr(c(x_i) = c \mid c(x_1) = \cdots = c(x_{i-1}) = c) &= \binom{\deg(x_i) - o(\deg(x_i))}{t} 2^{-\deg(x_i) + o(\deg(x_i))} \\ &\leq \binom{\deg(x_i) - o(\deg(x_i))}{\frac{\deg(x_i) - o(\deg(x_i))}{2}} 2^{-\deg(x_i) + o(\deg(x_i))} \end{aligned}$$

and using 4.3.3 we have

$$\begin{aligned} \Pr(c(x_i) = c \mid c(x_1) = \cdots = c(x_{i-1}) = c) &\leq \binom{\deg(x_i) - o(\deg(x_i))}{\frac{\deg(x_i) - o(\deg(x_i))}{2}} 2^{-\deg(x_i) + o(\deg(x_i))} \\ &\sim \frac{2^{\deg(x_i) - o(\deg(x_i)) + 1/2}}{\sqrt{\pi(\deg(x_i) - o(\deg(x_i)))}} \\ &= \frac{\sqrt{2/\pi}}{\sqrt{\deg(x_i) - o(\deg(x_i))}}. \end{aligned}$$

Hence,

$$\begin{aligned} \Pr(c(x_1) = \cdots = c(x_r) = c) &= \prod_{i=1}^r \Pr(c(x_i) = c \mid c(x_1) = \cdots = c(x_{i-1}) = c) \\ &\leq \prod_{i=1}^r \frac{\sqrt{2/\pi}}{\sqrt{\deg(x_i) - o(\deg(x_i))}}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}(|\{c : 0 \leq c \leq 2\Delta, \exists \{x_1, \dots, x_r\} \text{ such that } c(x_1) = \cdots = c(x_r) = c\}|) \\ &\leq \sum_{c=1}^{2\Delta} \sum_{\{x_1, \dots, x_r\} \in E(H)} \Pr(c(x_1) = \cdots = c(x_r) = c) \\ &\leq 2\Delta \sum_{\{x_1, \dots, x_r\} \in E(H)} \prod_{i=1}^r \frac{\sqrt{2/\pi}}{\sqrt{\deg(x_i) - o(\deg(x_i))}} = o(1), \end{aligned}$$

by assumption completing the proof. □

**Corollary 2.3.2.** *Let  $H = (V, E)$  be an  $r$ -uniform hypergraph of order  $n$  and size  $m$  with maximum degree  $\Delta$  and minimum degree  $\delta$  satisfying  $\Delta \cdot m \cdot (\frac{1}{\delta})^{r/2} = o(1)$  and  $\deg(u, v) =$*

$o(\deg(u))$  for any  $u, v \in V$ . Then,  $H$  is weakly 2-weighted.

Since trivially  $\Delta = O(n^{r-1})$  and  $m = O(n^r)$  we get for  $\delta = \Omega(n^4)$  that

$$\Delta \cdot m \cdot \left(\frac{1}{\delta}\right)^{r/2} = O(n^{r-1} \cdot n^r \cdot 1/n^{2r}) = o(1).$$

Consequently,

**Corollary 2.3.3.** *Let  $r \geq 5$ . Let  $H = (V, E)$  be an  $r$ -uniform hypergraph of order  $n$  with minimum degree  $\Omega(n^4)$  such that  $\deg(u, v) = o(\deg(u))$  for any  $u, v \in V$ . Then,  $H$  is weakly 2-weighted.*

## 2.4 Classification of Almost All Strongly Weighted $r$ -Uniform Hypergraphs

As observed in the previous section, if an  $r$ -uniform hypergraph  $H$  is irregular then it must be strongly 1-weighted. Thus if for every pair of vertices  $u, v \in V(H)$  we have  $\deg(u) \neq \deg(v)$ , then  $H$  is irregular. We wish to study the prevalence of pairs of vertices in  $\mathbb{H}^{(r)}(n, 1/2)$  with equal degree.

**Theorem 2.4.1.** *For any  $r \geq 6$  almost all nice  $r$ -uniform hypergraphs are strongly 1-weighted.*

*Proof.* Following our earlier observation, we show that for  $r \geq 6$  almost all nice  $r$ -uniform hypergraphs are irregular. Then since all irregular  $r$ -uniforms graphs are strongly 1-weighted, we are done.

To that end, suppose we have a nice  $r$ -uniform hypergraph  $H$  and two vertices  $u, v \in V(H)$ . If  $\deg(u) = \deg(v)$  then it must be true that the number of edges that contain  $u$  and not  $v$  is the same as the number of edge that contain  $v$  and not  $u$ , because the edges that contain both  $u$  and  $v$  contribute equally to  $\deg(u)$  and  $\deg(v)$ .

Let  $X_2^{(r)}$  count the number of pairs of vertices  $\{u, v\}$  in  $\mathbb{H}^{(r)}(n, 1/2)$  such that  $\deg(u) = \deg(v)$ . Observe that

$$\begin{aligned} \Pr(\deg(u) = \deg(v)) &= \sum_{a=0}^{\binom{n-2}{r-1}} \binom{\binom{n-2}{r-1}}{a}^2 2^{-2\binom{n-2}{r-1}} \\ &= \binom{2\binom{n-2}{r-1}}{\binom{n-2}{r-1}} 2^{-2\binom{n-2}{r-1}}. \end{aligned}$$

Thus, by 4.3.3

$$\begin{aligned} \mathbb{E}(X_2^{(k)}) &= \binom{n}{2} \binom{2\binom{n-2}{r-1}}{\binom{n-2}{r-1}} 2^{-2\binom{n-2}{r-1}} \\ &\sim \frac{n^2}{2} \cdot \frac{2^{2\binom{n-2}{r-1}+1/2}}{\sqrt{\pi 2^{\binom{n-2}{r-1}}}} \cdot 2^{-2\binom{n-2}{r-1}} \\ &\sim \frac{\sqrt{(r-1)!}}{2\sqrt{\pi}} n^{2-(r-1)/2} \end{aligned} \tag{2.1}$$

which is  $o(1)$  for any  $r \geq 6$ . Thus, the first moment method (see 4.1) yields the statement.  $\square$

While it is relatively easy to show that almost all  $r$ -uniform hypergraphs are irregular whenever  $r \geq 6$ , it is not nearly as obvious how one would actually construct such hypergraphs, but Gyárfás et al. [23] demonstrated a general method of constructing such objects after showing that almost all hypergraphs are irregular for  $r \geq 6$ . The irregularity part of the proof of Theorem 2.4.1 was identical to the proof presented by Gyárfás et al., but our proof was developed independently.

For any  $r$ -uniform hypergraph  $H$ , it can be shown using the second moment method (see 4.1) that when  $r = 3, 4$ , or  $5$ , then *w.h.p.* there exists a pair of vertices with the same degree. So clearly for these cases, we must introduce a second edge weight in order to have any hope of inducing a strong vertex coloring of  $H$ . Therefore, we need the following two lemmas to help us figure out how to distribute these weights in order to produce a strong 2-weighting.

**Lemma 2.4.2.** *Let  $r \geq 3$ . Then, w.h.p. each pair of vertices of  $\mathbb{H}^{(r)}(n, 1/2)$  is contained in an edge.*

*Proof.* The probability that a fixed pair of vertices  $u$  and  $v$  is not contained in any edge is  $2^{-\binom{n-2}{r-2}} = 2^{-\Omega(n^{r-2})}$ . Thus, by the union bound (cf. 4.2) we get that the probability that there exists a pair of vertices which is not contained in any edge is at most  $\binom{n}{2}2^{-\Omega(n^{r-2})} = o(1)$ .  $\square$

The next lemma shows how to compute the probability that an arbitrary group of vertices  $U = \{v_1, v_2, \dots, v_k\}$  all have the same degree and it shows that this probability is determined by edges that contain a single vertex  $v_i \in U$  and none of the other vertices in  $U$ . If we restrict the probability space to only have edges of this type, the probability of all the vertices in  $U$  having the same degree would be exactly

$$\Pr(\deg(v_1) = \dots = \deg(v_k)) = 2^{-k\binom{n-k}{r-1}} \sum_{a=0}^{\binom{n-k}{r-1}} \binom{\binom{n-k}{r-1}}{a}^k.$$

But  $\mathbb{H}^{(r)}(n, 1/2)$  has edges that have more than one vertex from  $U$ , e.g. edges that include  $v_1$  and  $v_2$  and no others. Observe that such edges contribute only to the degree of  $v_1$  and  $v_2$  and no others.

**Lemma 2.4.3.** *Let  $r \geq 3$  and  $k_1, \dots, k_\alpha \geq 2$  be integers. Let  $k = k_1 + \dots + k_\alpha$ . Then, w.h.p. for each  $\{v_{1,1}, \dots, v_{1,k_1}, v_{2,1}, \dots, v_{2,k_2}, \dots, v_{\alpha,1}, \dots, v_{\alpha,k_\alpha}\} \subseteq [n]$  in  $\mathbb{H}^{(r)}(n, 1/2)$ ,*

$$\Pr\left(\bigcap_{i=1}^{\alpha} \deg(v_{i,1}) = \dots = \deg(v_{i,k_i})\right) \sim 2^{-k\binom{n-k}{r-1}} \prod_{i=1}^{\alpha} \sum_{a=0}^{\binom{n-k}{r-1}} \binom{\binom{n-k}{r-1}}{a}^{k_i} \quad (2.2)$$

$$= O(n^{-(r-1)(k-\alpha)/2}). \quad (2.3)$$

*Proof.* Let  $U = \{v_{1,1}, \dots, v_{1,k_1}, v_{2,1}, \dots, v_{2,k_2}, \dots, v_{\alpha,1}, \dots, v_{\alpha,k_\alpha}\}$  with  $k = |U|$ . For a fixed  $S \subseteq U$  with  $1 \leq |S| \leq r$ , let  $x_S$  be the random variable that counts the number of edges containing  $S$  and  $r - |S|$  other vertices from  $[n] \setminus U$ . Using Chernoff's bound (cf. 4.2) and

the union bound we can show *w.h.p.* that

$$\left| x_S - \binom{n-k}{r-|S|}/2 \right| = O(n^{(r-|S|)/2} \sqrt{\log n}) = O(n^{(r-1)/2} \sqrt{\log n}). \quad (2.4)$$

Let  $v \in U$ . Then,

$$\deg(v) = \sum_{v \in S \subseteq U} x_S$$

and the expected value is  $\sum_{1 \leq s \leq r} \binom{k-1}{s-1} \binom{n-k}{r-s}/2$ . Thus, *w.h.p.* for any  $U$  and  $v \in U$ ,

$$\left| \deg(v) - \sum_{1 \leq s < (r+1)/2} \binom{k-1}{s-1} \binom{n-k}{r-s}/2 \right| = O(n^{(r-1)/2} \sqrt{\log n}).$$

Conditioning on  $x_S = \beta_S$  for  $2 \leq |S| \leq r$  and  $S \subseteq U$ , we get that the probability that  $\deg(v) = a$  is

$$\left( a - \sum_{S \ni v} \beta_S \right)^{\binom{n-k}{r-1}}.$$

Due to (2.4) we may assume that  $\beta_S = \binom{n-k}{r-|S|}/2 + \ell_S$  and

$$\left| a - \sum_{1 \leq s < (r+1)/2} \binom{k-1}{s-1} \binom{n-k}{r-s}/2 \right| = O(n^{(r-1)/2} \sqrt{\log n}),$$

where  $|\ell_S| = O(n^{(r-|S|)/2} \sqrt{\log n}) = O(n^{(r-2)/2} \sqrt{\log n})$ .

By 4.3.1, applied with  $m = \binom{n-k}{r-1}$ ,  $p = a - \sum_{S \ni v} \binom{n-k}{r-|S|}/2$ , and  $\ell = \sum_{S \ni v} \ell_S$ , we obtain that

$$\begin{aligned} \left( a - \sum_{S \ni v} \beta_S \right)^{\binom{n-k}{r-1}} &= \left( a - \sum_{S \ni v} \binom{n-k}{r-|S|}/2 - \sum_{S \ni v} \ell_S \right)^{\binom{n-k}{r-1}} \\ &\sim \left( a - \sum_{S \ni v} \binom{n-k}{r-|S|}/2 \right)^{\binom{n-k}{r-1}} \left( \frac{\binom{n-k}{r-1} - \left( a - \sum_{S \ni v} \binom{n-k}{r-|S|}/2 \right)}{a - \sum_{S \ni v} \binom{n-k}{r-|S|}/2} \right)^{\sum_{S \ni v} \ell_S}. \end{aligned}$$



Now observe that

$$\begin{aligned} a - \sum_{S \ni v} \binom{n-k}{r-|S|} / 2 &= a - \sum_{2 \leq s \leq r} \binom{k-1}{s-1} \binom{n-k}{r-s} / 2 \\ &= \binom{n-k}{r-1} / 2 \pm O(n^{(r-1)/2} \sqrt{\log n}). \end{aligned}$$

This implies that

$$\frac{\binom{n-k}{r-1} - \left( a - \sum_{S \ni v} \binom{n-k}{r-|S|} / 2 \right)}{a - \sum_{S \ni v} \binom{n-k}{r-|S|} / 2} \leq \frac{\binom{n-k}{r-1} / 2 + O(n^{(r-1)/2} \sqrt{\log n})}{\binom{n-k}{r-1} / 2 - O(n^{(r-1)/2} \sqrt{\log n})} = 1 + O\left(\frac{\sqrt{\log n}}{n^{(r-1)/2}}\right)$$

and

$$\frac{\binom{n-k}{r-1} - \left( a - \sum_{S \ni v} \binom{n-k}{r-|S|} / 2 \right)}{a - \sum_{S \ni v} \binom{n-k}{r-|S|} / 2} \geq \frac{\binom{n-k}{r-1} / 2 - O(n^{(r-1)/2} \sqrt{\log n})}{\binom{n-k}{r-1} / 2 + O(n^{(r-1)/2} \sqrt{\log n})} = 1 - O\left(\frac{\sqrt{\log n}}{n^{(r-1)/2}}\right).$$

Hence, since  $|\sum_{S \ni v_i} \ell_S| = O(n^{(r-2)/2} \sqrt{\log n})$ , we get that

$$\left( \frac{\binom{n-k}{r-1} - \left( a - \sum_{S \ni v} \binom{n-k}{r-|S|} / 2 \right)}{a - \sum_{S \ni v} \binom{n-k}{r-|S|} / 2} \right)^{\sum_{S \ni v} \ell_S} \sim 1,$$

and consequently,

$$\left( a - \sum_{S \ni v} \binom{n-k}{r-|S|} \beta_S \right) \sim \left( a - \sum_{S \ni v} \binom{n-k}{r-|S|} / 2 \right).$$

Thus, conditioning on  $\beta_S$ 's for  $S \subseteq U$  with  $2 \leq |S| \leq r$  the probability that  $\deg(v_{i,1}) = \dots = \deg(v_{i,k_i}) = a_i$  for each  $1 \leq i \leq \alpha$  equals

$$\begin{aligned} 2^{-k \binom{n-k}{r-1}} \cdot \prod_{i=1}^{\alpha} \prod_{j=1}^{k_i} \binom{n-k}{a_i - \sum_{S \ni v_{i,j}} \beta_S} &= 2^{-k \binom{n-k}{r-1}} \cdot \prod_{i=1}^{\alpha} \binom{n-k}{a_i - \sum_{S \ni v_{i,1}} \beta_S}^{k_i} \\ &\sim 2^{-k \binom{n-k}{r-1}} \cdot \prod_{i=1}^{\alpha} \binom{n-k}{a_i - \sum_{S \ni v_{i,1}} \binom{n-k}{r-|S|} / 2}^{k_i} \end{aligned}$$

and further the probability that  $\deg(v_{i,1}) = \dots = \deg(v_{i,k_i})$  for each  $1 \leq i \leq \alpha$  (still

conditioning on  $\beta_S$ 's) asymptotically equals

$$\begin{aligned}
& 2^{-k \binom{n-k}{r-1}} \cdot \sum_{a_1, \dots, a_\alpha} \left( a_1 - \sum_{S \ni v_{1,1}} \binom{n-k}{r-|S|} / 2 \right)^{k_1} \cdots \left( a_\alpha - \sum_{S \ni v_{k_\alpha,1}} \binom{n-k}{r-|S|} / 2 \right)^{k_\alpha} \\
&= 2^{-k \binom{n-k}{r-1}} \\
&\quad \cdot \left( \sum_{a_1} \left( a_1 - \sum_{S \ni v_{1,1}} \binom{n-k}{r-|S|} / 2 \right)^{k_1} \right) \cdots \left( \sum_{a_\alpha} \left( a_\alpha - \sum_{S \ni v_{k_\alpha,1}} \binom{n-k}{r-|S|} / 2 \right)^{k_\alpha} \right),
\end{aligned}$$

where the summations are taken over all possible values of  $a_1, \dots, a_k$  satisfying

$$\left| a_i - \sum_{1 \leq s < (r+1)/2} \binom{k-1}{s-1} \binom{n-k}{r-s} / 2 \right| = O(n^{(r-1)/2} \sqrt{\log n}).$$

Since

$$\sum_{a_i} \left( a_i - \sum_{S \ni v_{i,1}} \binom{n-k}{r-|S|} / 2 \right)^{k_i} \sim \sum_{a=0}^{\binom{n-k}{r-1}} \binom{\binom{n-k}{r-1}}{a}^{k_i},$$

we get

$$\begin{aligned}
& 2^{-k \binom{n-k}{r-1}} \cdot \sum_{a_1, \dots, a_\alpha} \left( a_1 - \sum_{S \ni v_{1,1}} \binom{n-k}{r-|S|} / 2 \right)^{k_1} \cdots \left( a_\alpha - \sum_{S \ni v_{k_\alpha,1}} \binom{n-k}{r-|S|} / 2 \right)^{k_\alpha} \\
&\quad \sim 2^{-k \binom{n-k}{r-1}} \prod_{i=1}^{\alpha} \sum_{a=0}^{\binom{n-k}{r-1}} \binom{\binom{n-k}{r-1}}{a}^{k_i},
\end{aligned}$$

and finally by the law of total probability,

$$\begin{aligned}
& \Pr \left( \bigcap_{i=1}^{\alpha} \deg(v_{i,1}) = \dots = \deg(v_{i,k_i}) \right) \\
&\quad \sim \sum_{\beta_S^*} \Pr \left( \bigcap_S y_S = \beta_S \right) \cdot 2^{-k \binom{n-k}{r-1}} \prod_{i=1}^{\alpha} \sum_{a=0}^{\binom{n-k}{r-1}} \binom{\binom{n-k}{r-1}}{a}^{k_i},
\end{aligned}$$

where  $\beta_S^*$  denotes the summation over all possible values of  $\beta_S$  such that

$$2 \leq |S| \leq r \quad \text{and} \quad \left| \beta_S - \binom{n-k}{r-|S|} / 2 \right| = O(n^{(r-|S|)/2} \sqrt{\log n}).$$

This completes the proof of (2.2), since  $\sum_{\beta_S^*} \Pr(\bigcap_S y_S = \beta_S) \sim 1$ .

Now (2.3) easily follows from 4.3.2. Indeed, since

$$2^{-k_i \binom{n-k}{r-1}} \sum_{a=0}^{\binom{n-k}{r-1}} \binom{\binom{n-k}{r-1}}{a}^{k_i} = O(n^{-(r-1)(k_i-1)/2}),$$

we obtain

$$2^{-k \binom{n-k}{r-1}} \prod_{i=1}^{\alpha} \sum_{a=0}^{\binom{n-k}{r-1}} \binom{\binom{n-k}{r-1}}{a}^{k_i} = O\left(\prod_{i=1}^{\alpha} n^{-(r-1)(k_i-1)/2}\right) = O(n^{-(r-1)(k-\alpha)/2}).$$

□

Recall that Theorem 2.4.1 shows that almost all  $r$ -uniform hypergraphs are strongly 1-weighted for  $r \geq 6$ . It remains to show what the weightedness of  $r$ -uniform hypergraphs for  $r = 3, 4$  and  $5$  are.

Of these remaining cases,  $r = 5$  is the most interesting. When we evaluate (2.1) we see that the expected number of vertex pairs converges asymptotically to  $\sqrt{6/\pi}$ . This indicates that the family of 5-uniform hypergraphs have a distinct asymptotic split between irregular hypergraphs and non-irregular hypergraphs. Using the method of moments (see 4.1) and the results in Lemma 2.4.3, we show that

**Theorem 2.4.4.** *The probability that a hypergraph chosen uniformly at random from the space of all 5-uniform hypergraphs of order  $n$  is strongly 1-weighted is  $e^{-\sqrt{6/\pi}} + o(1)$  and that it is strongly 2-weighted (but not 1-weighted) is  $1 - e^{-\sqrt{6/\pi}} + o(1)$ .*

*Proof.* Observe that  $(X_2^{(5)})_k$  consists of  $\binom{n}{2} \binom{n-2}{2} \cdots \binom{n-2k+2}{2}$  terms of  $k$  vertex-disjoint pairs and  $O(n^{2k-1})$  remaining terms. The pairs in the remaining terms are not vertex disjoint.

Let  $U$  be the union over all vertices in such  $k$  pairs. Clearly,  $k \leq |U| \leq 2k - 1$ , where the lower bound,  $k$ , corresponds to a  $k$ -cycle. Vertices in  $U$  can be divided into  $\alpha$  groups (each of size  $k_i \geq 2$ ) in such a way that in each group all vertices have the same degrees. Observe that  $k_1 + \dots + k_\alpha = |U|$  and since  $|U| < 2k$ ,  $1 \leq \alpha < |U|/2$ . Due to (2.3) the probability of occurrence of this degree sequence is at most  $O(n^{-2(|U|-\alpha)}) = o(n^{-|U|})$ . Thus,  $O(n^{|U|} \cdot n^{-2(|U|-\alpha)}) = o(1)$  and

$$\begin{aligned} \mathbb{E} \left( (X_2^{(5)})_k \right) &\sim \binom{n}{2} \binom{n-2}{2} \dots \binom{n-2k+2}{2} 2^{-2k \binom{n-2k}{4}} \prod_{i=1}^k \sum_{a=0}^{\binom{n-2k}{4}} \binom{\binom{n-2k}{4}}{a}^2 \\ &= \binom{n}{2} \binom{n-2}{2} \dots \binom{n-2k+2}{2} 2^{-2k \binom{n-2k}{4}} \left( 2 \frac{\binom{n-2k}{4}}{\binom{n-2k}{4}} \right)^k \\ &\sim \left( \mathbb{E}(X_2^{(5)}) \right)^k \sim \lambda^k, \end{aligned}$$

since by 4.3.3

$$2^{-2 \binom{n-2}{4}} \binom{2 \binom{n-2}{4}}{\binom{n-2}{4}} \sim 2^{-2 \binom{n-2k}{4}} \binom{2 \binom{n-2k}{4}}{\binom{n-2k}{4}}.$$

Hence, the method of moments implies that  $X_2^{(5)} \sim \text{Po}(\lambda)$  and consequently

$$\Pr(\mathbb{H}^{(5)}(n, 1/2) \text{ is strongly 1-weighted}) = \Pr(X_2^{(5)} = 0) \sim e^{-\sqrt{6/\pi}}. \quad (2.5)$$

It remains to show that *w.h.p.* the random hypergraph  $\mathbb{H}^{(5)}(n, 1/2)$  is strongly 2-weighted.

Let  $X_3^{(r)}$  count the number of triples of vertices  $\{v_1, v_2, v_3\}$  in  $\mathbb{H}^{(r)}(n, 1/2)$  such that  $\deg(v_1) = \deg(v_2) = \deg(v_3)$ . Then, by Lemma 2.4.3 (applied with  $k = k_1 = 3$  and  $\alpha = 1$ )

$$\mathbb{E}(X_3^{(r)}) \sim \binom{n}{3} 2^{-3 \binom{n-3}{r-1}} \sum_{a=0}^{\binom{n-3}{r-1}} \binom{\binom{n-3}{r-1}}{a}^3 = \Theta(n^{4-r}), \quad (2.6)$$

which is  $o(1)$  for  $r = 5$ .

So far we have established the following properties of  $\mathbb{H}^{(5)}(n, 1/2)$ . By Markov's bound (cf. 4.2),  $\Pr(X_2^{(5)} \geq \log n) = o(1)$  and by (2.6) and the first moment method *w.h.p.* no three

vertices have the same degree.

Once the edges of  $\mathbb{H}^{(5)}(n, 1/2)$  are revealed we may assume that we have  $s$  vertex-disjoint pairs  $\{u_i, v_i\}$  such that  $\deg(u_i) = \deg(v_i)$ , where  $1 \leq i \leq s \leq \log n$ . (All other vertices have different degrees.) Let  $S = \bigcup_i \{u_i, v_i\}$ . We show that there is a matching saturating all  $u_i$ 's such that each matching edge contains one vertex from  $S$  and 4 vertices from  $[n] \setminus S$ . One can find such matching greedily. Assume that we already chose matching edges  $e_1, \dots, e_k$  with  $u_i \in e_i$ . The number of edges incident to  $u_{k+1}$  that are not disjoint with  $e_1, \dots, e_k$  or  $S \setminus \{u_{k+1}\}$  is at most  $O(n^3 \log n)$  but  $\deg(u_i) = \Omega(n^4)$ . Hence, we can extend the matching by a new edge containing  $u_{k+1}$ .

Finally, we are ready to define a 2-weighting. First we assign to each edge weight 2. Now the colors of all vertices are even and every vertex has a different color except the  $u_i$  and  $v_i$ . Next we replace the weights of matching edges by 1 so now all vertices have different colors, since subtracting 1 from a number of vertices will not create new equalities and all old equalities are broken up.  $\square$

The next two theorems complete the classification of the strong-weightedness of  $r$ -uniform hypergraphs. We will use the second moment method and Lemma 2.4.3 in these proofs.

**Theorem 2.4.5.** *Almost all 4-uniform hypergraphs are strongly 2-weighted.*

*Proof.* By (2.1),  $E(X_2^{(4)}) = \Theta(\sqrt{n})$  and let  $X_2^{(4)} = \sum_{\{i,j\} \subseteq \binom{[n]}{2}} X_{i,j}$ , where  $X_{i,j}$  is an indicator random variable such that if  $X_{i,j} = 1$ , then  $\deg(v_i) = \deg(v_j)$ . Thus,

$$(X_2^{(4)})^2 = X_2^{(4)} + \sum_{\{i,j,k\}} X_{i,j} X_{j,k} + \sum_{\{i,j\} \cap \{k,\ell\} = \emptyset} X_{i,j} X_{k,\ell}.$$

By Lemma 2.4.3 (applied with  $r = 4$ ,  $k = k_1 = 3$ , and  $\alpha = 1$ ),

$$\Pr(X_{i,j} X_{j,k} = 1) = \Pr(\deg(v_i) = \deg(v_j) = \deg(v_k)) = O(n^{-3}).$$

Again by Lemma 2.4.3 (applied with  $r = 4$ ,  $k = 4$ ,  $k_1 = k_2 = 2$  and  $\alpha = 2$ )

$$\begin{aligned} \Pr(X_{i,j}X_{k,\ell} = 1) &= \Pr(\deg(v_i) = \deg(v_j) \text{ and } \deg(v_k) = \deg(v_\ell)) \\ &\sim 2^{-4\binom{n-4}{3}} \left( \sum_{a=0}^{\binom{n-4}{3}} \binom{\binom{n-4}{3}}{a} \right)^2 = 2^{-4\binom{n-4}{3}} \left( \frac{2^{\binom{n-4}{3}}}{\binom{n-4}{3}} \right)^2. \end{aligned}$$

Thus,

$$\sum_{\{i,j,k\}} \mathbb{E}(X_{i,j}X_{j,k}) = O(n^3 \cdot n^{-3}) = O(1)$$

and

$$\sum_{\{i,j\} \cap \{k,\ell\} = \emptyset} \mathbb{E}(X_{i,j}X_{k,\ell}) \sim \binom{n}{2} \binom{n-2}{2} 2^{-4\binom{n-4}{3}} \left( \frac{2^{\binom{n-4}{3}}}{\binom{n-4}{3}} \right)^2 \sim \mathbb{E}(X_2^{(4)})^2,$$

where the last asymptotic equality follows from 4.3.3. Consequently,

$$\mathbb{E}((X_2^{(4)})^2) \sim \mathbb{E}(X_2^{(4)}) + \mathbb{E}(X_2^{(4)})^2$$

and hence  $\text{Var}(X_2^{(4)}) \sim \mathbb{E}(X_2^{(4)}) = o(\mathbb{E}(X_2^{(4)})^2)$ . Thus, the second moment method together with Lemma 2.4.2 implies that *w.h.p.* there is a pair of vertices of the same degrees which is contained in an edge (so the hypergraph is not strongly 1-weighted).

Now we show that *w.h.p.*  $\mathbb{H}^{(4)}(n, 1/2)$  is strongly 2-weighted. We already observed that  $\mathbb{E}(X_2^{(4)}) = O(\sqrt{n})$ . Thus, by Markov's bound,  $\Pr(X_2^{(4)} \geq \sqrt{n} \log n) = o(1)$ . By (2.6),  $\mathbb{E}(X_3^{(4)}) = O(1)$ . Thus, again Markov's bound implies that  $\Pr(X_3^{(4)} \geq \log n) = o(1)$ . Lemma 2.4.3 implies that the probability that vertices  $v_1, v_2, v_3$  and  $v_4$  in  $\mathbb{H}^{(4)}(n, 1/2)$  have the same degrees is asymptotically equal to

$$2^{-4\binom{n-4}{3}} \sum_{a=0}^{\binom{n-4}{3}} \binom{\binom{n-4}{3}}{a}^4 = O(n^{-9/2}).$$

Hence, by the union bound (taken over all  $\binom{n}{4}$  quadruples) we get that *w.h.p.* there are no such four vertices. Similarly, (with essentially the same proof as Lemma 2.4.3) we get that *w.h.p.* there are no four vertices  $v_1, v_2, v_3$  and  $v_4$  in  $\mathbb{H}^{(4)}(n, 1/2)$  such that  $\deg(v_1) = \deg(v_2) = \deg(v_3) = \deg(v_4) - 1$ .

Once the edges of  $\mathbb{H}^{(4)}(n, 1/2)$  are revealed we may assume that there are  $a$  vertex-disjoint pairs  $\{u_i, v_i\}$  and  $b$  triples  $\{x_j, y_j, z_j\}$  such that  $\deg(u_i) = \deg(v_i)$ ,  $\deg(x_j) = \deg(y_j) = \deg(z_j)$ , and  $1 \leq i \leq a \leq \sqrt{n} \log n$  and  $1 \leq j \leq b \leq \log n$ . (All other vertices have different degrees.) Let  $S_1 = \bigcup_i \{u_i, v_i\}$  and  $S_2 = \bigcup_j \{x_j, y_j, z_j\}$  and  $T = [n] \setminus (S_1 \cup S_2)$ . We show that there are edges  $e_i, f_j, f'_j$ , and  $g_j$  such that  $u_i \in e_i$ ,  $\{x_j\} = f_j \cap f'_j$ ,  $y_j \in g_j$  and  $e_i, f_j, f'_j$ , and  $g_j$  contain no other vertices from  $S_1 \cup S_2$  and  $e_i \cap T, f_j \cap T, f'_j \cap T$ , and  $g_j \cap T$  are pairwise vertex-disjoint. Similarly as in the previous section one can find such edges greedily. First assume that we already found  $e_1, \dots, e_k$  edges so that  $u_i \in e_i$ . The number of edges incident to  $u_{k+1}$  that are not disjoint with  $e_1, \dots, e_k$  or  $S_1 \cup S_2 \setminus \{u_{k+1}\}$  is at most  $O(n^2 \sqrt{n} \log n)$  but  $\deg(u_i) = \Omega(n^3)$ . Hence, we can extend  $e_1, \dots, e_k$  by a new edge  $e_{k+1}$  that contains  $u_{k+1}$ . Similarly we find edges  $f_j, f'_j$ , and  $g_j$ .

Now we define a 2-weighting. First we assign to each edge weight 2. Now the colors of all vertices are even and only vertices  $\{u_i, v_i\} \subseteq S_1$  and  $\{x_j, y_j, z_j\} \subseteq S_2$  have the same color. Furthermore, there is no vertex  $w$  such that for some  $j$  we have  $\deg(x_j) = \deg(y_j) = \deg(z_j) = \deg(w) - 1$  or  $\deg(x_j) = \deg(y_j) = \deg(z_j) = \deg(w)$ . Next we replace the weights of all  $e_i, f_j, f'_j$ , and  $g_j$  edges by 1 yielding all vertices to have different colors.  $\square$

**Theorem 2.4.6.** *Almost all 3-uniform hypergraphs are strongly 2-weighted.*

*Proof.* As in Theorem 2.4.5 the second moment method and Lemma 2.4.2 imply that *w.h.p.* there is a pair of vertices of the same degrees which is contained in an edge. Thus, *w.h.p.*  $\mathbb{H}^{(3)}(n, 1/2)$  is not strongly 1-weighted. As a matter of fact we will see later that *w.h.p.*  $\mathbb{H}^{(3)}(n, 1/2)$  is not even weakly 1-weighted.

Now we show that *w.h.p.*  $\mathbb{H}^{(3)}(n, 1/2)$  is strongly 2-weighted. Here our proof method differs from our other proofs that hypergraphs are 2-weighted. Since the expected number of

pairs of vertices with the same degree is linear, we cannot simply give all edges weight 2 and then alter the weighting by flipping the weights of a few edges. We use a more complicated argument which we will outline after some lemmas.

First we need some auxiliary results. For  $G = (V, E)$  and  $S \subseteq V$ , let  $N(S)$  denote the *neighborhood of  $S$* , i.e., the set of all vertices in  $V$  adjacent to some element of  $S$ .

**Lemma 2.4.7.** *There exists a positive constant  $\gamma$  such that with probability  $1 - o(1/n)$  the random bipartite graph  $\mathbb{G}(n, n, 1/2)$  contains at least  $\gamma n$  edge disjoint perfect matchings.*

This lemma is a weaker version of a more general result of Frieze and Krivelevich [20] (where the authors obtained an optimal constant  $\gamma = 1/2 - o(1)$ ). For the sake of completeness we show here a simple proof.

*Proof.* Let  $G = \mathbb{G}(n, n, 1/2)$  be a random bipartite graph on the set of vertices  $A \cup B$ , where  $|A| = |B| = n$ . Set  $\gamma = 1/10$ .

First observe that since for any  $u, v \in A$  and  $u, v \in B$  we have  $|N(v)| \sim \text{Bin}(n, 1/2)$  and  $|N(\{u, v\})| \sim \text{Bin}(n, 3/4)$ , so Chernoff's bound yields that with probability  $1 - o(1/n)$  for any  $u, v \in A$  and  $u, v \in B$ ,

$$|N(v)| \geq n/2 - O(\sqrt{n \log n}) \quad \text{and} \quad |N(\{u, v\})| \geq 3n/4 - O(\sqrt{n \log n}).$$

Assume that we already found a collection  $\mathcal{M}_i = \{M_1, \dots, M_i\}$  of perfect matchings in  $G$ . We show that  $G_{i+1} = G \setminus \mathcal{M}_i$  also contains a perfect matching  $M_{i+1}$ . It suffices to show that if  $i < \gamma n$ , then the Hall condition holds, i.e.,

$$\text{if } S \subseteq A \text{ and } |S| \leq n/2, \text{ then } |N_{G_{i+1}}(S)| \geq |S|, \tag{2.7}$$

and

$$\text{if } T \subseteq B \text{ and } |T| \leq n/2, \text{ then } |N_{G_{i+1}}(T)| \geq |T|. \tag{2.8}$$



Indeed, if  $S = \{v\}$ , then

$$|N_{G_{i+1}}(S)| = |N_{G_{i+1}}(v)| = |N_G(v)| - i \geq n/2 - O(\sqrt{n \log n}) - \gamma n \geq 1 = |S|.$$

Therefore, we may assume that  $2 \leq |S| \leq n/2$ . Let  $\{u, v\} \subseteq S$ . Then,

$$|N_{G_{i+1}}(S)| \geq |N_{G_{i+1}}(\{u, v\})| \geq |N_G(\{u, v\})| - 2i \geq 3n/4 - O(\sqrt{n \log n}) - 2\gamma n \geq n/2 \geq |S|$$

and (2.7) holds. Similarly, (2.8) holds, too.  $\square$

**Lemma 2.4.8.** *Let  $\mathbb{H}$  be a 3-partite 3-uniform random hypergraph on vertex set  $V_1 \cup V_2 \cup V_3$  with  $|V_1| = |V_2| = |V_3| = n$  and probability  $1/2$ . Then, there exists a positive constant  $\gamma$  such that *w.h.p.*  $\mathbb{H}$  contains at least  $\gamma n^2$  edge disjoint perfect matchings.*

*Proof.* Consider first a complete bipartite graph  $F$  on  $V_1 \cup V_2$ . Then, since  $F$  is  $n$ -regular bipartite graph, it can be decomposed into  $n$  edge disjoint perfect matchings, say  $F = M_1 \cup \dots \cup M_n$ . Let  $H_i$  be a 3-partite 3-uniform hypergraph on  $V_1 \cup V_2 \cup V_3$  and the set of edges

$$E_i = \{e \cup v : e \in M_i \text{ and } v \in V_3\}.$$

Thus, the complete 3-partite 3-uniform hypergraph on  $V_1 \cup V_2 \cup V_3$  is the edge disjoint union of  $H_i$ 's over all  $1 \leq i \leq n$ . We generate a random 3-partite 3-uniform hypergraph by revealing edges in each  $H_i$ . It suffices to show that the random hypergraph induced by  $H_i$  contains with probability  $1 - o(1/n)$  at least  $\gamma n$  edge disjoint perfect matchings, where  $\gamma$  is a constant from Lemma 2.4.7. The latter is obviously true since the random graph induced by  $H_i$  can be viewed as  $G = \mathbb{G}(n, n, 1/2)$  on  $M \cup V_3$  (since  $|M| = |V_3| = n$ ) and any perfect matching in  $G$  yields a perfect matching in the 3-partite hypergraph.  $\square$

Now we are ready to show that *w.h.p.*  $\mathbb{H}^{(3)}(n, 1/2)$  is strongly 2-weighted. We start with an outline. First we consider an equipartition  $V_1 \cup V_2 \cup V_3$  of the vertex set  $V = [n]$ .

Then we reveal the random hypergraph  $\mathbb{H}^{(3)}(n, 1/2)$ . Due to Lemma 2.4.8 (assuming that  $n$  is divisible by 3) there is *w.h.p.* a family  $\mathcal{M}$  of disjoint perfect matchings such that  $|\mathcal{M}| = \lfloor \gamma n^2 \rfloor$  and each edge in each matching has one vertex in each part of the partition. We then randomly label the vertices in  $V_1$  with real number labels in  $[0, 1/9]$ , the vertices in  $V_2$  with labels in  $[1/9, 2/9]$ , and the vertices in  $V_3$  with labels in  $[2/9, 1/3]$ . Let  $\ell(v)$  denote the label of vertex  $v$ . We then randomly weight the edges, where any non-matching edge  $\{u, v, w\} \in \binom{[n]}{3}$  gets weight 2 with probability  $\ell(u) + \ell(v) + \ell(w)$ , and any matching edge gets weight 2 with probability  $\frac{1}{2}$ . We will show that

$$\begin{aligned} & \mathbb{E}(|\{c : 0 \leq c \leq n^2, \exists x, y \text{ such that } |c(x) - c| \leq 2 \text{ and } |c(y) - c| \leq 2\}|) \\ & \leq \sum_{c=0}^{n^2} \mathbb{E}(|\{\{x, y\} : |c(x) - c| \leq 2 \text{ and } |c(y) - c| \leq 2\}|) = O(\log^4 n). \end{aligned}$$

We will call pairs  $\{x, y\}$  such that  $|c(x) - c| \leq 2$  and  $|c(y) - c| \leq 2$  for some  $c$  *dangerous* and pairs such that  $c(x) = c(y)$  *bad*. We will also show that

$$\mathbb{E}(|\{c : 0 \leq c \leq n^2, \exists x, y, z \text{ such that } |c(x) - c| \leq 2, |c(y) - c| \leq 2, |c(z) - c| \leq 2\}|)$$

is  $o(1)$ . Finally, we will alter the weights on certain edges so that we kill all bad pairs (i.e. after changing the weights no two vertices will have the same color) which will complete the proof.

In this paragraph and the next we are revealing only the random hypergraph  $\mathbb{H} = \mathbb{H}^{(3)}(n, 1/2)$ , and all probabilities are with respect to this distribution. The degrees of vertices are concentrated by Chernoff's bound. More specifically, for any vertex  $v$ , and any distinct parts  $V_i$  and  $V_j$  there are  $m = n^2/9 + O(n)$  triples containing  $v$  together with one additional vertex from  $V_i$  and one from  $V_j$  (the  $O(n)$  term is to account for the possibility that  $v$  is in one of  $V_i$  or  $V_j$ ). Each of these triples has probability  $1/2$  of being an edge of  $\mathbb{H}$ , independently, and so if we let  $\deg_{i,j}(v)$  denote the number of these edges present in  $\mathbb{H}$ , then Chernoff's

bound tells us that

$$\Pr(|d_{i,j}(v) - m/2| > n \log n) \leq 2 \exp\left(-\frac{(n \log n)^2}{3m/2}\right) = \exp(-\Omega(\log^2 n))$$

and so the union bound gives us that the probability there exists  $v, i, j$  such that  $|d_{i,j}(v) - m/2| > n \log n$  is at most  $3n \cdot \exp\{-\Omega(\log^2 n)\} = o(1)$ . Hence, *w.h.p.* for every vertex  $v$  and distinct parts  $V_i, V_j$  there are  $n^2/18 + O(n \log n)$  edges containing  $v$  and additionally one vertex from each of  $V_i, V_j$ . Similarly (using Chernoff's bound and union bounds), *w.h.p.* for any vertex  $v$  and part  $V_i$  there are  $n^2/36 + O(n \log n)$  edges containing  $v$  and additionally two vertices from  $V_i$ .

Until the last paragraph of this proof will assume that  $n$  is divisible by 3. Due to Lemma 2.4.8 there is *w.h.p.* a family  $\mathcal{M}$  of disjoint perfect matchings where  $|\mathcal{M}| = \lfloor \gamma n^2 \rfloor$  and each edge in each matching goes between the partition (i.e. has one vertex in each  $V_i$ ).

Henceforth we assume that all the edges of  $\mathbb{H}$  were revealed obtaining a hypergraph  $H$  that has the degree properties mentioned in the previous paragraphs, and the family of matchings  $\mathcal{M}$ .

Next we reveal the vertex labels and the non-matching edge weights. The weight of edge  $\{x, y, z\}$  is distributed as

$$w(x, y, z) = \begin{cases} 1 & \text{with probability } 1 - \ell(x) - \ell(y) - \ell(z) \\ 2 & \text{with probability } \ell(x) + \ell(y) + \ell(z) \end{cases}$$

so

$$\mathbb{E}(w(x, y, z) \mid \ell(x), \ell(y), \ell(z)) = 1 + \ell(x) + \ell(y) + \ell(z)$$

and therefore if by  $c_{H \setminus \mathcal{M}}(x)$  we denote the sum of the weights of non-matching edges containing  $x$ , and say we are given only the label  $\ell(x)$ , and the hypergraph  $H$  with family of

matchings  $\mathcal{M}$ , and  $x \in V_i$  then we have

$$\begin{aligned}
\mathbb{E}(c_{H \setminus \mathcal{M}}(x) \mid \ell(x), H) &= \sum_{\{x,v,w\} \in E(H) \setminus \mathcal{M}} \mathbb{E}(w(x,v,w) \mid \ell(x)) \\
&= \sum_{\{x,v,w\} \in E(H) \setminus \mathcal{M}} (1 + \ell(x) + \mathbb{E}(\ell(v)) + \mathbb{E}(\ell(w))) \\
&= (1 + \ell(x)) \deg_{H \setminus \mathcal{M}}(x) + \sum_{\{x,v,w\} \in E(H) \setminus \mathcal{M}} (\mathbb{E}(\ell(v)) + \mathbb{E}(\ell(w))) \\
&= \ell(x) \left( \frac{1}{4} - \gamma \right) n^2 + \Theta(n^2),
\end{aligned}$$

where on the last line we have used our estimate of  $\deg_{H \setminus \mathcal{M}}(x)$ . Note that the  $\Theta(n^2)$  term may depend on  $i$  (recall  $x \in V_i$ ) but does not otherwise depend on  $x$  or  $\ell(x)$  (by the fact that the degree of each vertex  $v$  into sets  $V_i, V_j$  etc. is concentrated).

Now since  $c_{H \setminus \mathcal{M}}(x) = \sum_{\{x,v,w\} \in E(H) \setminus \mathcal{M}} w(x,v,w)$ , and the random variables  $w(x,v,w)$  are independent (given the vertex labels), we can apply Bernstein's bound (cf. 4.2). For our application we use  $m = \deg_{H \setminus \mathcal{M}}(x)$ . We can easily use  $C = 2$ , and put  $\text{Var}(w(x,v,w)) \leq \mathbb{E}(w(x,v,w)^2) \leq 4$ . We will set  $\gamma = n \log n$ . Then we get

$$\begin{aligned}
&\Pr \left( \left| c_{H \setminus \mathcal{M}}(x) - \mathbb{E}(c_{H \setminus \mathcal{M}}(x) \mid \ell(x), H) \right| > n \log n \right) \\
&\leq 2 \exp \left( - \frac{\frac{1}{2} n^2 \log^2 n}{4 \deg_{H \setminus \mathcal{M}}(x) + \frac{1}{3} \cdot 2n \log n} \right) = \exp \{ -\Omega(\log^2 n) \}
\end{aligned}$$

and so by the union bound, *w.h.p.* for each vertex  $x$ ,  $c_{H \setminus \mathcal{M}}(x)$  is within  $n \log n$  of its expectation.

Now for any fixed integer  $a$  and fixed vertex  $x$ ,

$$\Pr \left( \left| \ell(x) \left( \frac{1}{4} - \gamma \right) n^2 - a \right| \leq n \log n \right) = O(\log n / n),$$

since this is the probability that  $\ell(x)$  falls within an interval of length  $O(\log n / n)$ . Since the vertex labels are independent, the probability that there are  $\log^2 n$  many vertices  $x$  with

$|\ell(x) \left(\frac{1}{4} - \gamma\right) n^2 - a| \leq n \log n$  is at most

$$\begin{aligned} \binom{n}{\log^2 n} \cdot (O(\log n/n))^{\log^2 n} &\leq \left(\frac{ne}{\log^2 n}\right)^{\log^2 n} (O(\log n/n))^{\log^2 n} \\ &= \left(O\left(\frac{1}{\log n}\right)\right)^{\log^2 n} = o(n^2). \end{aligned}$$

Therefore, by the union bound over integers  $c$  from 0 to  $n^2$ , we have that for all such  $c$  the number of vertices  $x$  with  $|c_{H \setminus \mathcal{M}}(x) - c| \leq n \log n$  is at most  $\log^2 n$ . Henceforth we assume that the labels and non-matching edge weights have been revealed and they satisfy this property.

Now we reveal the matching edge weights. These are 1 or 2 with probability  $1/2$ . We will denote by  $c_{\mathcal{M}}(x)$  the sum of the weights of matching edges adjacent to  $x$ ; note that  $c(x) = c_{\mathcal{M}}(x) + c_{H \setminus \mathcal{M}}(x)$ . The key fact we will use here is that since  $|\mathcal{M}| = \lfloor \gamma n^2 \rfloor$ ,  $c_{\mathcal{M}}(x)$  is not likely to be any one particular value. Indeed, since  $c_{\mathcal{M}}(x) - |\mathcal{M}| \sim \text{Bin}(|\mathcal{M}|, 1/2)$ , the mode of  $c_{\mathcal{M}}(x)$  occurs with probability

$$\binom{|\mathcal{M}|}{\lfloor |\mathcal{M}|/2 \rfloor} 2^{-|\mathcal{M}|} \sim \frac{\sqrt{2/\gamma\pi}}{n} = O(1/n),$$

where the latter follows from 4.3.3. Also, *w.h.p.* for all  $x$  we have  $|c_{\mathcal{M}}(x) - \frac{3}{2}\gamma n^2| \leq n \log n$ .

Therefore,

$$\begin{aligned} \sum_{c=0}^{n^2} \mathbb{E}(|\{\{x, y\} : |c(x) - c| \leq 2 \text{ and } |c(y) - c| \leq 2\}|) \\ = O(n^2 \cdot (\log^2 n)^2 \cdot (1/n)^2) = O(\log^4 n), \end{aligned}$$

where on the last line we get the  $(1/n)^2$  in the big- $O$  term since the probability that  $c(x) = c$  is  $O(1/n)$ , and conditioning on that event, the event that  $c(y) = c$  still has probability  $O(1/n)$  (since revealing  $c(x)$  only reveals the weights of  $O(n)$  of the  $\lfloor \gamma n^2 \rfloor$  matching edges

containing  $y$ ,  $c(y)$  still essentially has the same distribution as it did before conditioning) and similarly for the conditional probability that  $c(z) = c$ . Similarly,

$$\begin{aligned} \mathbb{E}(|\{c : 0 \leq c \leq n^2, \exists x, y, z \text{ such that } |c(x) - c| \leq 2, |c(y) - c| \leq 2, |c(z) - c| \leq 2\}|) \\ \leq O(n^2 \cdot (\log^2 n)^3 \cdot (1/n)^3) = o(1). \end{aligned}$$

Thus, by Markov's bound *w.h.p.* there are at most  $\log^5 n$  dangerous pairs, and there is no triple  $x, y, z$  such that  $|c(x) - c| \leq 2$ ,  $|c(y) - c| \leq 2$ , and  $|c(z) - c| \leq 2$  for any  $c$ . Suppose there are  $b \leq \log^5 n$  bad pairs and let  $\{\{u_i, v_i\} : 1 \leq i \leq b\}$  be the set of bad pairs. For each  $i$  we will choose an edge  $e_i$  such that  $u_i \in e_i$  and  $e_i$  does not intersect any other dangerous pair and all edges  $e_i$  are vertex-disjoint. As in the previous sections this is easy since each vertex has degree  $\Theta(n^2)$ , the number of edges containing any two vertices is  $O(n)$ , and there are at most  $\log^5 n$  dangerous pairs. Now for each bad pair  $\{u_i, v_i\}$  we flip the weight of  $e_i$  (from 1 to 2 or from 2 to 1). Thus  $u_i$  no longer has the same color as  $v_i$ . Also note that now no two vertices in the entire hypergraph can have the same color, since the new color of any vertex can only differ from its old color by at most 1, the only vertices in dangerous pairs whose colors are changed are the vertices that are in bad pairs (and they are only changed to make them not bad anymore), and there is no triple  $x, y, z$  such that  $|c(x) - c| \leq 2$ ,  $|c(y) - c| \leq 2$ , and  $|c(z) - c| \leq 2$  for any  $c$ . This completes the proof assuming that 3 divides  $n$ .

If  $n$  is not divisible by 3, then we can still use an equipartition  $V_1, V_2, V_3$ , and a family of (not quite perfect) matchings  $\mathcal{M}$  where  $|\mathcal{M}| = \gamma n^2$ , each  $M \in \mathcal{M}$  has size  $\lfloor n/3 \rfloor$ , and each vertex is covered by  $\gamma n^2 - O(n)$  matchings in  $\mathcal{M}$ . It is relatively straightforward to check that with this small change, the same proof still works.  $\square$

## 2.5 Classification of Almost All Weakly Weighted $r$ -Uniform Hypergraphs

On the one hand, it is clear that any strong weighting of  $H$  is also a weak weighting and since we have shown that almost all  $H$  are strongly 2-weighted, then almost all  $H$  are weakly 2-weighted. On the other hand, because there are weak weightings that are not strong, we might be able to do better, which, indeed, is the case.

**Theorem 2.5.1.** *For any  $r \geq 4$  almost all  $r$ -uniform hypergraphs are weakly 1-weighted.*

*Proof.* Now let  $X_4^{(r)}$  counts the number of quadruples of vertices  $\{v_1, v_2, v_3, v_4\}$  in  $\mathbb{H}^{(r)}(n, 1/2)$  such that  $\deg(v_1) = \deg(v_2) = \deg(v_3) = \deg(v_4)$ . Then, by Lemma 2.4.3 (applied with  $k = k_1 = 4$  and  $\alpha = 1$ )

$$\mathbb{E}(X_4^{(r)}) \sim \binom{n}{4} 2^{-4 \binom{n-4}{r-1}} \sum_{a=0}^{\binom{n-4}{r-1}} \binom{\binom{n-4}{r-1}}{a}^4 = \Theta(n^{4-3(r-1)/2}), \quad (2.9)$$

which is  $o(1)$  for  $r \geq 4$ . □

**Theorem 2.5.2.** *Almost all 3-uniform hypergraphs are weakly 2-weighted.*

*Proof.* By Theorem 2.4.6 *w.h.p.*  $\mathbb{H}^{(3)}(n, 1/2)$  is strongly 2-weighted which obviously implies that it is also weakly 2-weighted. Hence, we only need to show that *w.h.p.*  $\mathbb{H}^{(3)}(n, 1/2)$  is not weakly 1-weighted. We do it by applying the second moment method as in Theorems 2.4.5 and 2.4.6.

Let  $X_3$  counts the number of edges  $\{v_1, v_2, v_3\}$  in  $\mathbb{H}^{(3)}(n, 1/2)$  such that  $\deg(v_1) = \deg(v_2) = \deg(v_3)$ . By Lemma 2.4.3 we get

$$\Pr(\deg(v_1) = \deg(v_2) = \deg(v_3), \{v_1, v_2, v_3\} \in E) \sim \frac{1}{2} \sum_{a=0}^{\binom{n-3}{2}} \binom{\binom{n-3}{2}}{a}^3 2^{-3 \binom{n-3}{2}} \sim \frac{2}{\pi \sqrt{3n^2}},$$

where the latter follows from 4.3.2 (applied with  $m = \binom{n-3}{2}$  and  $k = 3$ ). Thus,

$$\mathbb{E}(X_3) \sim \binom{n}{3} \frac{2}{\pi\sqrt{3}n^2},$$

which goes to infinity together with  $n$ .

We show that  $\mathbb{E}(X_3^2) \sim (\mathbb{E}(X_3))^2$ . For  $e \in \binom{[n]}{3}$ , let  $X_e$  be an indicator random variable which is equal to 1 if all three vertices in  $e$  have the same degree. Thus,

$$X_3^2 = X_3 + \sum_{|e \cap f|=0} X_e X_f + \sum_{|e \cap f|=1} X_e X_f + \sum_{|e \cap f|=2} X_e X_f.$$

If  $|e \cap f|=2$ , then we may assume that  $e \cup f = \{v_1, \dots, v_4\}$  and by Lemma 2.4.3

$$\begin{aligned} \Pr(X_e X_f = 1) &= \Pr(\deg(v_1) = \dots = \deg(v_4) \text{ and } e, f \in E) \\ &\sim \frac{1}{4} \sum_a \binom{\binom{n-4}{2}}{a}^4 2^{-4\binom{n-4}{2}} = O\left(\frac{1}{n^3}\right) \end{aligned}$$

and consequently,

$$\sum_{|e \cap f|=2} \mathbb{E}(X_e X_f) = O\left(n^4 \cdot \frac{1}{n^3}\right) = O(n) = o((\mathbb{E}(X_3))^2).$$

Similarly,

$$\sum_{|e \cap f|=1} \mathbb{E}(X_e X_f) = O\left(n^5 \cdot \frac{1}{n^4}\right) = O(n) = o((\mathbb{E}(X_3))^2).$$

Finally, if  $|e \cap f|=0$ , then Lemma 2.4.3 (applied with  $r = 3$ ,  $k = 6$ ,  $k_1 = k_2 = 3$ , and  $\alpha = 2$ ) implies that

$$\sum_{|e \cap f|=0} \mathbb{E}(X_e X_f) \sim \frac{1}{4} \binom{n}{3} \binom{n-3}{3} 2^{-5\binom{n-6}{2}} \left( \sum_a \binom{\binom{n-6}{2}}{a}^3 \right)^2 \sim (\mathbb{E}(X_3))^2.$$

Thus, we are done by the second moment method.  $\square$



## 2.6 NP-completeness of 2-WEIGHTED<sup>(r)</sup>

While it is relatively easy to determine if a given edge weighting will produce a strong weighting in an  $r$ -uniform hypergraph, it is a much more difficult problem to decide if a given  $r$ -uniform hypergraph is strongly 2-weighted. We formalize this idea and prove that this decision problem is *NP-complete*. (For more details on the complexity theory see, e.g., [37].)

For any  $r \geq 2$  we define

$$2\text{-WEIGHTED}^{(r)} = \{H : H \text{ is a strongly 2-weighted } r\text{-uniform hypergraph}\}.$$

Note that  $2\text{-WEIGHTED}^{(r)}$  is clearly in NP, since for a given hypergraph  $H = (V, E)$  and  $\omega : E \rightarrow \{1, 2\}$  one can verify in polynomial time whether a vertex-coloring induced by  $\omega$  is strong.

It is known that for graphs  $2\text{-WEIGHTED}^{(2)}$  is NP-complete as it was proven independently by Dehghan, Sadeghi and Ahadi [11], and Dudek and Wajc [16].

In order to prove that  $2\text{-WEIGHTED}^{(r)}$  is NP-complete, we show a reduction from  $2\text{-WEIGHTED}^{(2)}$  to  $2\text{-WEIGHTED}^{(r)}$ . To this end, we define a polynomial time reduction  $h$ , such that  $G \in 2\text{-WEIGHTED}^{(2)}$  if and only if  $h(G) \in 2\text{-WEIGHTED}^{(r)}$ .

We will need an auxiliary gadget. First we define an  $r$ -partite  $r$ -uniform hypergraph  $T = (V, E)$ . Let  $V = V_1 \cup \dots \cup V_r$ , where  $|V_i| = i$  for each  $1 \leq i \leq r$ , and  $E$  be the set of edges consisting of all possible edges containing exactly one vertex from each  $V_i$ . Observe that if  $v \in V_i$ , then  $\deg(v) = r!/i$ . Consequently  $T$  is strongly 1-weighted (and so nice). We refer to the unique vertex  $v \in V_1$  as a *root*. Let  $T(k)$  be a union of  $k$  copies of  $T$  with the same root which we refer as the root of  $T(k)$  (any two copies only share its roots). Clearly,  $T(k)$  is still strongly 1-weighted and if  $v$  is its root, then  $\deg_{T(k)}(v) = k \cdot r!$ .

Now we are ready to show a reduction from  $2\text{-WEIGHTED}^{(2)}$  to  $2\text{-WEIGHTED}^{(r)}$ ,  $h$ , such that  $G \in 2\text{-WEIGHTED}^{(2)}$  if and only if  $h(G) \in 2\text{-WEIGHTED}^{(r)}$ . Let  $G = (V, E)$

be a graph of order  $n$ . We construct a nice  $r$ -uniform hypergraph  $h(G) = (W, F)$  as follows. For each edge  $e = \{x, y\}$  in  $G$  we define an edge  $\{x, y, v_1, \dots, v_{r-2}\}$  in  $h(G)$ , where all  $v_i$ 's are different for all  $e$  and  $i$ . Now to each  $v_i$  (for  $1 \leq i \leq r-2$ ) we attach a copy of  $T(2in)$  on the new set of vertices with  $v_i$  as its root.

Let us assume that  $G = (V, E) \in 2\text{-WEIGHTED}^{(2)}$ . Thus, there is  $\omega_G : E \rightarrow \{1, 2\}$  such that the vertex-coloring  $c_G$  induced by  $\omega_G$  is proper. Let  $H = h(G) = (W, F)$ . We define a weight function  $\omega_H : F \rightarrow \{1, 2\}$  as follows. To each edge  $\{x, y, v_1, \dots, v_{r-2}\} \in F$  derived from  $\{x, y\} \in E$ , we assign  $\omega_G(\{x, y\})$ ; otherwise we assign weight 1. Now we claim that the vertex-coloring  $c_H : W \rightarrow \mathbb{N}$  induced by  $\omega_H$  is strong. By construction, any edge of  $H$  which is contained in a copy of  $T(2in)$  is rainbow. We show that this also holds for  $\{x, y, v_1, \dots, v_{r-2}\} \in F$  derived from  $G$ . Observe that

$$c_H(v_i) = c_G(v_i) + 2in \cdot r!,$$

and consequently,  $2n < c_H(v_1) < \dots < c_H(v_{r-2})$ . Moreover, since  $c_G(x) \neq c_G(y) \leq 2(n-1)$ , we get that  $\{x, y, v_1, \dots, v_{r-2}\}$  is rainbow. Hence,  $h(G) \in 2\text{-WEIGHTED}^{(r)}$ .

Now suppose that  $G = (V, E) \notin 2\text{-WEIGHTED}^{(2)}$ . We show that  $H = h(G) = (W, F) \notin 2\text{-WEIGHTED}^{(r)}$ . Assume not. That means there exists a weight function  $\omega_H : W \rightarrow \{1, 2\}$  such that the vertex-coloring  $c_H$  induced by  $\omega_H$  is strong. Now let  $\omega_G : V \rightarrow \{1, 2\}$  be such that  $\omega_G(\{x, y\}) = \omega_H(\{x, y, v_1, \dots, v_{r-2}\})$ . Since  $c_G(x) = c_H(x)$  and  $c_H$  is strong, we conclude that  $c_G$  is proper, a contradiction.

## 2.7 Concluding Remarks

In this chapter, we:

- constructed a class of  $r$ -uniform hypergraphs that require at least  $r^2 - r + 1$  edge weights to be strongly weighted;

- constructed a class of  $r$ -uniform hypergraphs which are at least weakly 3-weighted;
- gave a sufficient condition for  $r$ -uniform hypergraphs to be weakly 2-weighted;
- demonstrated that almost all  $r$ -uniform hypergraphs are strongly (and weakly) 2-weighted;
- showed that deciding a given  $r$ -uniform hypergraph is strongly 2-weighted is NP-complete.

There are still several questions that we can pursue. For example, it might be of some interest to consider the weightedness of  $\mathbb{H}^{(r)}(n, p)$  for an arbitrary  $p \in (0, 1)$ . We could also further explore the lower bound on strong colorings. Since we know there exists an infinite class of  $r$ -uniform hypergraphs that require at least  $r^2 - r + 1$  weights, it would be nice to decide if this lower bound is optimal for all  $r$ -uniform hypergraphs.

# Chapter 3

## Hamiltonicity of Path Structures for Group Labelings

### 3.1 Introduction

The main results of this chapter were published as a joint work with Dudek and Helenius in [15].

Karoński et al. [31] showed that for any abelian group  $\Gamma$  of odd order and for any graph  $G$  such that  $\chi(G) = |\Gamma|$  then  $G$  was  $|\Gamma|$ -weighted. In this chapter we study a similar problem for hypergraphs. First we define  *$\ell$ -offset Hamiltonian cycles* in  $r$ -uniform hypergraphs, which are closely related what are known as  *$\ell$ -overlapping Hamiltonian cycles* in  $r$ -uniform hypergraphs. These cycle structures are both natural generalizations of Hamiltonian cycles in simple graphs.

We will give a definition of  *$\ell$ -overlapping Hamiltonian cycles* in the next section, Suppose that  $1 \leq \ell < r$ . An  *$\ell$ -overlapping Hamiltonian cycle*  $C$  in a  $r$ -uniform hypergraph  $H = (V, \mathcal{E})$  on  $n$  vertices is a collection of  $m_\ell = n/(r - \ell)$  edges of  $H$  such that for some cyclic order of  $[n]$  every edge consists of  $r$  consecutive vertices and for every pair of consecutive edges  $E_{i-1}, E_i$  in  $C$  (in the natural ordering of the edges) we have  $|E_{i-1} \cap E_i| = \ell$ . Thus, in every  $\ell$ -overlapping Hamiltonian cycle the sets  $C_i = E_i \setminus E_{i-1}$ ,  $i = 1, 2, \dots, m_\ell$ , are a partition of  $V$  into sets of size  $r - \ell$ . Hence,  $m_\ell = n/(r - \ell)$ . Thus,  $r - \ell$  divides  $n$ . In the literature, when  $\ell = r - 1$  we have a *tight* Hamiltonian cycle and when  $\ell = 1$  we have a *loose* Hamiltonian cycle.

A  $r$ -uniform hypergraph is said to be  $\ell$ -Hamiltonian when it contains an  $\ell$ -overlapping Hamiltonian cycle. The distinction between loose and tight cycles is lost when we go from hypergraphs to graphs. The sharp threshold for the existence of Hamiltonian cycles in the random graph  $\mathbb{G}(n, p)$  has been known for many years (see, e.g., [5], [9] and [32]), but until recently nothing was known about the threshold of  $\ell$ -Hamiltonian random hypergraphs. Results on loose hamiltonicity of  $\mathbb{H}^{(r)}(n, p)$  were obtained by Frieze [18] (for  $r = 3$ ), Dudek and Frieze [12] (for  $r \geq 4$  and  $2(r - 1)|n$ ), and by Dudek, Frieze, Loh and Speiss [14] (for  $r \geq 3$  and  $(r - 1)|n$ ).

**Theorem 3.1.1** ([18, 12, 14]). *There exists an absolute constant  $c > 0$  such that if  $p \geq c(\log n)/n^2$ , then  $\mathbb{H}^{(3)}(n, p)$  contains a loose Hamiltonian cycle w.h.p. provided that  $2|n$ . Furthermore, for every  $r \geq 4$  if  $p \geq \omega(\log n)/n^{r-1}$ , then  $\mathbb{H}^{(r)}(n, p)$  contains a loose Hamiltonian cycle provided that  $(r - 1)|n$ .*

These results are basically optimal since if  $p \leq (1 - \varepsilon)(r - 1)!(\log n)/n^{r-1}$  and  $\varepsilon > 0$  is constant, then  $\mathbb{H}^{(r)}(n, p)$  contains isolated vertices w.h.p.. More recently Ferber [17] simplified some of the proofs of Theorem 3.1.1 and Dudek and Frieze [13] were able to extend these to an arbitrary  $\ell \geq 2$ .

**Theorem 3.1.2** ([13]). (i) *For integers  $r > \ell \geq 2$  and fixed  $\varepsilon > 0$ , if  $p \leq (1 - \varepsilon)e^{r-\ell}/n^{r-\ell}$ , then  $\mathbb{H}^{(r)}(n, p)$  is not  $\ell$ -Hamiltonian w.h.p..*

(ii) *For all integers  $r > \ell \geq 3$ , there exists a constant  $c = c(r)$  such that if  $p \geq c/n^{r-\ell}$  and  $n$  is a multiple of  $r - \ell$ , then  $\mathbb{H}^{(r)}(n, p)$  is  $\ell$ -Hamiltonian w.h.p..*

(iii) *If  $r > \ell = 2$  and  $p \geq \omega/n^{r-2}$  and  $n$  is a multiple of  $r - 2$ , then  $\mathbb{H}^{(r)}(n, p)$  is 2-Hamiltonian w.h.p..*

(iv) *For a fixed  $\varepsilon > 0$ , if  $r \geq 4$  and  $p \geq (1 + \varepsilon)e/n$ , then  $\mathbb{H}^{(r)}(n, p)$  contains a tight Hamiltonian cycle w.h.p..*

This theorem shows, in particular, that  $e/n$  is the sharp threshold for the existence of a tight Hamiltonian cycle in a  $r$ -uniform hypergraph, when  $r \geq 4$ .

Finally Poole [36] considered *weak (Berge) Hamiltonian cycles*  $C$  in  $r$ -uniform hypergraphs  $H$  on  $n$  vertices which are collections of edges of  $H$  such that for some cyclic order of  $[n]$  every pair of consecutive vertices belong to an edge from  $C$  and these edges are not necessarily distinct. Notice that loose Hamiltonian cycles are weak Hamiltonian cycles, too. In particular,

**Theorem 3.1.3** ([36]). *Let  $r \geq 3$ . Then,  $p = (r - 1)! (\log n) / n^{r-1}$  is a sharp threshold for the existence of the weak Hamiltonian cycle in  $\mathbb{H}^{(r)}(n, p)$ .*

Now let us define  $\ell$ -offset Hamiltonian cycles which are the main object of interest in this chapter. Let  $1 \leq \ell \leq r/2$  and define an  $\ell$ -offset Hamiltonian cycle  $C$  in a  $r$ -uniform hypergraph  $H$  on  $n$  vertices as a collection of  $m$  edges of  $H$  such that for some cyclic order of  $[n]$  and when  $i$  is odd and taking  $E_0 = E_m$  we have every pair of consecutive edges  $E_{i-1}, E_i$  in  $C$  (in the natural ordering of the edges) satisfying  $|E_{i-1} \cap E_i| = \ell$  and every pair of consecutive edges  $E_i, E_{i+1}$  in  $C$  satisfying  $|E_i \cap E_{i+1}| = r - \ell$ . When  $i$  is even, every pair of consecutive edges  $E_{i-1}, E_i$  in  $C$  satisfies  $|E_{i-1} \cap E_i| = r - \ell$  and every pair of consecutive edges  $E_i, E_{i+1}$  in  $C$  satisfies  $|E_i \cap E_{i+1}| = \ell$ . (see Figure 3.1).

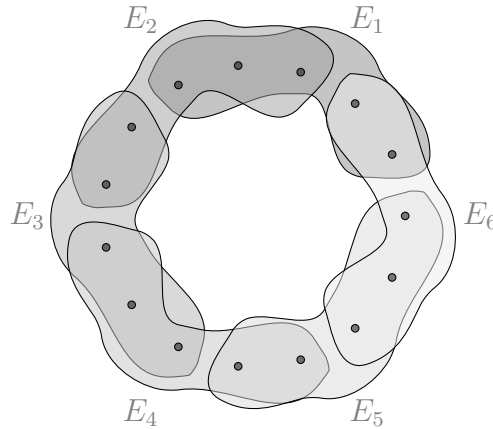


Figure 3.1: A 2-offset Hamiltonian cycle in a 5-uniform hypergraph with 6 edges.

Since every  $\ell$ -offset Hamiltonian cycle consists of two perfect matching of size  $n/r$ , we

have  $m = 2n/r$  and we always assume that  $r$  divides  $n$  when discussing  $\ell$ -offset Hamiltonian cycles. A  $r$ -uniform hypergraph is said to be  $\ell$ -offset Hamiltonian when it contains an  $\ell$ -offset Hamiltonian cycle. While it follows from a result of Parczyk and Person [35] that if  $p = \omega/n^{r/2}$ , then  $\mathbb{H}^{(r)}(n, p)$  is  $\ell$ -offset Hamiltonian *w.h.p.* for any  $r \geq 4$  and  $\ell \geq 2$ , in Theorem 3.2.1 we replace this asymptotic threshold by the sharp one for  $r \geq 6$  and  $\ell \geq 3$ .

In the next sections, we will study the properties of  $\ell$ -offset Hamiltonian cycles and we will see that the structures captured by these cycles arise in a natural way in a problem related to the 1-2-3 Conjecture for Hypergraphs.

## 3.2 Properties of $\ell$ -Offset Hamiltonian Cycles

In order to understand the prevalence of hypergraphs with offset cycles, we establish the sharp threshold for  $\ell$ -offset Hamiltonian cycles in  $\mathbb{H}^{(r)}(n, p)$ .

**Theorem 3.2.1.** *Let  $\varepsilon > 0$ . Then:*

(i) *For all integers  $r \geq 3$  and  $1 \leq \ell \leq \frac{r}{2}$ , if  $p = (1 - \varepsilon)\sqrt{e^r \ell! (r - \ell)! / n^r}$ , then  $\mathbb{H}^{(r)}(n, p)$  is not  $\ell$ -offset Hamiltonian *w.h.p.*.*

(ii) *For all integers  $r \geq 6$  and  $3 \leq \ell \leq \frac{r}{2}$ , if  $p = (1 + \varepsilon)\sqrt{e^r \ell! (r - \ell)! / n^r}$ , then  $\mathbb{H}^{(r)}(n, p)$  is  $\ell$ -offset Hamiltonian *w.h.p.*.*

(iii) *For all integers  $r \geq 4$  and  $\ell = 2$  and if  $p = \frac{\omega}{n^{r/2}}$ , then  $\mathbb{H}^{(r)}(n, p)$  is 2-offset Hamiltonian *w.h.p.*.*

*Proof.* Let  $X$  be the random variable that counts the number of  $\ell$ -offset Hamiltonian cycles. Observe that the number of cycles in the complete  $r$ -uniform hypergraph is:

$$\gamma_n := \frac{n!}{2n} \cdot \frac{r - \ell}{(\ell! (r - \ell)!)^{n/r}}.$$

Indeed, first we order all vertices into a cycle, next we add  $2n/r$  edges, which can be shifted

by at most  $r - \ell$  positions, and finally we need to correct this by permuting all vertices in any two consecutive edges.

Using Stirling's formula we have

$$E(X) = \gamma_n \cdot p^{2n/r} = (1 + o(1))(r - \ell) \sqrt{\frac{\pi}{2n}} \cdot \left( \frac{n}{e} \cdot \left( \frac{p^2}{\ell! (r - \ell)!} \right)^{1/r} \right)^n$$

and letting

$$p = (1 - \varepsilon) \sqrt{e^r \ell! (r - \ell)! / n^r},$$

we have

$$\begin{aligned} E(X) &= (1 + o(1))(r - \ell) \sqrt{\frac{\pi}{2n}} \cdot \left( \frac{n}{e} \cdot \left( \frac{\left( (1 - \varepsilon) \sqrt{e^r \ell! (r - \ell)! / n^r} \right)^2}{\ell! (r - \ell)!} \right)^{1/r} \right)^n \\ &= (1 + o(1))(r - \ell) \sqrt{\frac{\pi}{2n}} (1 - \varepsilon)^{2n/r} = o(1). \end{aligned}$$

This verifies part (i).

Now we let

$$p = (1 + \varepsilon) \sqrt{e^r \ell! (r - \ell)! / n^r}$$

and let  $H$  be a fixed  $\ell$ -offset Hamiltonian cycle. Observe that

$$E(X) = (1 + o(1))(r - \ell) \sqrt{\frac{\pi}{2n}} (1 + \varepsilon)^{2n/r} = \infty.$$

Let  $N(b, a)$  be the number of  $H'$   $\ell$ -offset Hamiltonian cycles such that  $|E(H) \cap E(H')| = b$



and  $E(H) \cap E(H')$  consists of  $a$  edge disjoint paths. Since trivially  $N(0, 0) \leq \gamma_n$ , we obtain

$$\begin{aligned} \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} &= \frac{\gamma_n N(0, 0) p^{4n/r}}{\mathbb{E}(X)^2} + \sum_{b=1}^{2n/r} \sum_{a=1}^{\min\{b, n/r\}} \frac{\gamma_n N(b, a) p^{4n/r-b}}{\mathbb{E}(X)^2} \\ &\leq 1 + \sum_{b=1}^{2n/r} \sum_{a=1}^{\min\{b, n/r\}} \frac{\gamma_n N(b, a) p^{2n/r-b}}{\mathbb{E}(X)}. \end{aligned}$$

It remains to show that

$$\sum_b \sum_a \frac{N(b, a) p^{2n/r-b}}{\mathbb{E}(X)} = o(1)$$

so we can use Chebyshev's bound (see 4.2) to imply that

$$\Pr(X = 0) \leq \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} - 1 = o(1), \quad (3.1)$$

as required.

To find an upper bound on  $N(b, a)$  we first consider how many ways we can find paths  $P_1, P_2, \dots, P_a$  with a total of  $b$  edges. To begin, for each  $1 \leq i \leq a$  choose vertices  $v_i$  on  $V(H)$ . We have at most

$$n^a \quad (3.2)$$

choices. Let

$$b_1 + b_2 + \dots + b_a = b,$$

where  $b_i \geq 1$  is an integer for every  $1 \leq i \leq a$ . Note that this equation has exactly

$$\binom{b-1}{a-1} \quad (3.3)$$

solutions. So for every  $i$ , we choose a path of length  $b_i$  in  $H$  which starts at  $v_i$  and it moves

clockwise. Thus we (3.2) and (3.3) tell us we have at most

$$\binom{b-1}{a-1} n^a \tag{3.4}$$

ways to choose our paths.

Now we count the number of  $H'$  containing  $P_1, \dots, P_a$ . For each even path (that means with even number of edges)

$$|V(P_i)| = \frac{b_i r}{2} + \ell \quad \text{or} \quad |V(P_i)| = \frac{b_i r}{2} + (r - \ell)$$

and for each odd path

$$|V(P_i)| = \frac{b_i r}{2} + \frac{r}{2}.$$

Since  $\ell \leq r/2$  then for all paths we have

$$|V(P_i)| \geq \frac{b_i r}{2} + \ell.$$

Then

$$\sum_i |V(P_i)| \geq \sum_i \left( \frac{b_i r}{2} + \ell \right) = br/2 + a\ell.$$

Thus, we have at most  $n - br/2 - a\ell$  vertices not in  $\bigcup_{i=1}^a V(P_i)$ . Observe that  $H'$  is uniquely determined by the sequence of  $2n/r$  subsets each of sizes alternating from  $r - \ell$  to  $\ell$ . For each  $V(P_i)$ , if  $b_i = 1$  then we need to divide  $|V(P_i)| = r$  vertices into 2 subsets of size  $r - \ell$  and  $\ell$ . The number of ways these paths can be split into alternating subsets is at most

$$\binom{r}{\ell}^a \leq \binom{r}{r/2}^a < 2^{ra}. \tag{3.5}$$

(If  $b_i > 1$ , then there is nothing to do.)

Next we divide the vertices in  $V(H) \setminus (V(P_1) \cup \dots \cup V(P_a))$  into subsets of size  $\ell$  and  $r - \ell$  to obtaining a cycle of alternating subsets. Let  $b'_i$  be the number of edges in  $H'$  that lie between  $P_i$  and  $P_{i+1}$  and connect  $P_i$  with  $P_{i+1}$ . Then there are exactly  $b'_i - 1$  alternating subsets between  $P_i$  and  $P_{i+1}$  in  $H'$ . Thus, we have at least  $(b'_i - 2)/2$  groups of size  $\ell$  and of size  $r - \ell$  between  $P_i$  and  $P_{i+1}$ . Since

$$\sum_{i=1}^a (b'_i - 2)/2 = \left( \frac{2n}{r} - b \right) / 2 - a = n/r - b/2 - a,$$

we conclude that we have at least  $(n/r - b/2 - a)$  groups of size  $\ell$  and at least  $(n/r - b/2 - a)$  groups of size  $r - \ell$  on  $V(H) \setminus (V(P_1) \cup \dots \cup V(P_a))$ . Consequently, we can divide  $V(H) \setminus (V(P_1) \cup \dots \cup V(P_a))$  into alternating groups in at most

$$\frac{(n - br/2 - a\ell)!}{2(n - br/2 - a\ell)} \cdot \frac{1}{(\ell!(r - \ell)!)^{n/r - b/2 - a}} = (n - br/2 - a\ell - 1)! \cdot \frac{1}{2(\ell!(r - \ell)!)^{n/r - b/2 - a}} \quad (3.6)$$

choices.

Now mark  $a$  positions to insert  $P_i$ 's. We can trivially do it in

$$(n - br/2 - a\ell)^a \leq (n - br/2 - a\ell) \cdot n^{a-1} \quad (3.7)$$

ways.

Using (3.5), (3.6) and (3.7), the number  $H'$ 's containing  $P_1, P_2, \dots, P_a$  is smaller than

$$2^{ra} \cdot (n - br/2 - a\ell)! \frac{1}{(\ell!(r - \ell)!)^{n/r - b/2 - a}} \cdot n^{a-1}. \quad (3.8)$$

Thus, by (3.4) and (3.8) we obtain

$$N(b, a) < \binom{b-1}{a-1} \cdot 2^{ra} \cdot (n - br/2 - a\ell)! \frac{1}{(\ell!(r - \ell)!)^{n/r - b/2 - a}} \cdot n^{2a-1}$$

and so

$$\begin{aligned} \frac{N(b, a)p^{2n/r-b}}{\mathbb{E}(X)} &\leq \binom{b-1}{a-1} \frac{2^{ra} \cdot (n - br/2 - a\ell)! \cdot n^{2a-1} \cdot p^{2n/r-b} \cdot 2n \cdot (\ell! (r - \ell)!)^{n/r}}{(\ell! (r - \ell)!)^{n/r-b/2-a} \cdot n! \cdot (r - \ell)p^{2n/r}} \\ &= \binom{b-1}{a-1} \frac{(\ell! (r - \ell)!)^{b/2} \cdot (n - br/2 - a\ell)!}{n! \cdot p^b} \cdot \frac{(\ell! (r - \ell)!)^a \cdot 2^{ra+1} \cdot n^{2a}}{(r - \ell)}. \end{aligned}$$

Using Stirling's approximation, letting  $p = (1 + \varepsilon)\sqrt{e^r \ell! (r - \ell)! / n^r}$ , and observing that  $n - br/2 - a\ell \leq n$  we have

$$\begin{aligned} \frac{N(b, a)p^{2n/r-b}}{\mathbb{E}(X)} &\leq \binom{b-1}{a-1} \frac{(\ell! (r - \ell)!)^{b/2} \cdot \left(\frac{n-br/2-a\ell}{e}\right)^{(n-br/2-a\ell)}}{\left(\frac{n}{e}\right)^n \cdot (1 + \varepsilon)^b \cdot \frac{e^{br/2}(\ell!(r-\ell)!)^{b/2}}{n^{br/2}}} \cdot \frac{(\ell! (r - \ell)!)^a \cdot 2^{ra+1} \cdot n^{2a}}{(r - \ell)} \\ &\leq \binom{b-1}{a-1} \frac{(\ell! (r - \ell)!)^{b/2} \cdot \left(\frac{n}{e}\right)^{(n-br/2-a\ell)}}{\left(\frac{n}{e}\right)^n \cdot (1 + \varepsilon)^b \cdot \frac{e^{br/2}(\ell!(r-\ell)!)^{b/2}}{n^{br/2}}} \cdot \frac{(\ell! (r - \ell)!)^a \cdot 2^{ra+1} \cdot n^{2a}}{(r - \ell)} \\ &= \frac{2}{r - \ell} \left(\frac{1}{1 + \varepsilon}\right)^b \binom{b-1}{a-1} \left(\frac{2^r (e^\ell \ell! (r - \ell)!)^a}{n^{\ell-2}}\right)^a. \end{aligned}$$

This implies that

$$\sum_b \sum_a \frac{N(b, a)p^{2n/r-b}}{\mathbb{E}(X)} \leq \frac{2}{r - \ell} \sum_b \left(\frac{1}{1 + \varepsilon}\right)^b \sum_a \binom{b-1}{a-1} \left(\frac{2^r e^\ell \ell! (r - \ell)!}{n^{\ell-2}}\right)^a.$$

Since

$$\begin{aligned} \sum_{a=1}^b \binom{b-1}{a-1} \left(\frac{2^r e^\ell \ell! (r - \ell)!}{n^{\ell-2}}\right)^a &= \frac{2^r e^\ell \ell! (r - \ell)!}{n^{\ell-2}} \sum_{a=1}^b \binom{b-1}{a-1} \left(\frac{2^r e^\ell \ell! (r - \ell)!}{n^{\ell-2}}\right)^{a-1} \\ &= \frac{2^r e^\ell \ell! (r - \ell)!}{n^{\ell-2}} \left(1 + \frac{2^r e^\ell \ell! (r - \ell)!}{n^{\ell-2}}\right)^{b-1} \\ &\leq \frac{O(1)}{n^{\ell-2}} \left(1 + \frac{O(1)}{n^{\ell-2}}\right)^b, \end{aligned}$$

we get for  $\ell \geq 3$  and  $\varepsilon > 0$  that

$$\sum_b \sum_a \frac{N(b, a)p^{2n/r-b}}{\mathbb{E}(X)} \leq \frac{O(1)}{n^{\ell-2}} \sum_b \left(\frac{1 + \frac{O(1)}{n^{\ell-2}}}{1 + \varepsilon}\right)^b \leq \frac{O(1)}{n^{\ell-2}} \cdot O(\varepsilon) = o(1).$$

This proves part (ii).

Furthermore, if  $\ell = 2$  and  $p = \frac{\omega}{n^{r/2}}$  (that means  $\varepsilon = \omega$ ), then

$$\sum_b \sum_a \frac{N(b, a) p^{2n/r-b}}{E(X)} \leq O(1) \sum_{b \geq 1} \left( \frac{1 + O(1)}{\omega} \right)^b \leq O(1) \cdot \frac{1 + O(1)}{\omega} = o(1).$$

This proves part (iii) and completes the proof of Theorem 3.2.1. □

### 3.3 $\ell$ -Offset Hamiltonian Cycles and Group Colorings

When the 1-2-3 Conjecture was introduced by Karoński et al. [31] they showed that:

**Theorem 3.3.1** ([31]). *Let  $\Gamma$  be a finite abelian group of odd order and let  $G$  be a non-trivial  $|\Gamma|$ -colorable graph. Then there is a weighting of the edges of  $G$  with the elements of  $\Gamma$  such that the resultant vertex weighting is a proper coloring.*

It was our attempts to extend this idea to  $r$ -uniform hypergraphs that led us to consider the concept of  $\ell$ -offset Hamiltonian cycles. So for  $u, v \in V(H)$  let  $T$  be a  $uv$ -trail (that means a sequence of vertices with repeated vertices allowed) in  $H$  such that

- (i) the first two edges have  $r - 1$  vertices in common not including  $u$ , and
- (ii) the last two edges have  $r - 1$  vertices in common not including  $v$ , and
- (iii) any two successive edges in the trail have either 1 or  $r - 1$  vertices in common in an alternating fashion.

Any trail that fits this pattern will be called *1-offset*. Thus the vertices along  $T$  are subdivided into alternating groups of size 1 and  $r - 1$ , starting and ending with subdivisions of 1. Observe that this condition implies the number of edges in  $T$  must be even. Also, since  $T$  is a trail, vertices can get used in multiple edges, including  $u$  and  $v$ , but the first two edges must start with  $u$  as a singleton and the last two edges must finish with  $v$  as a singleton (see

Figure 3.2). Let  $\mathcal{T}$  be the hypergraph property that for all pairs of vertices  $u, v \in V(H)$  there exists a 1-offset  $uv$ -trail. If hypergraph  $H \in \mathcal{T}$ , then we say that  $H$  is  $\mathcal{T}$ -connected.

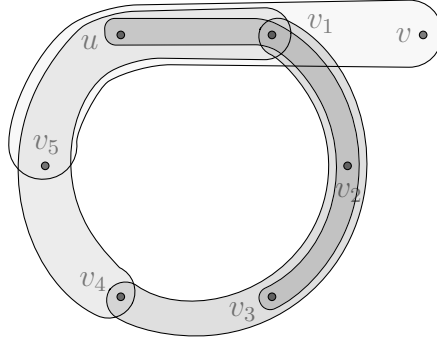


Figure 3.2: A 1-offset  $uv$ -trail,  $u, v_1, v_2, v_3, v_4, v_5, u, v_1, v$ , in a 4-uniform hypergraph that consists of 4 edges.

Hypergraph property  $\mathcal{T}$  is what allows us to state and prove a result analogous to Theorem 3.3.1 with only slight modification of the proof as presented in [31].

**Theorem 3.3.2.** *Let  $\Gamma$  be a finite abelian group of order  $w$  and let  $H$  be a  $r$ -uniform hypergraph that is  $\mathcal{T}$ -connected and strongly (weakly)  $w$ -colorable. Furthermore, let  $\gcd(w, r) = 1$ . Then  $H$  is strongly (weakly)  $w$ -weighted by the elements of  $\Gamma$ .*

We will need a simple fact:

**Observation 3.3.3.** *Let  $\Gamma$  be an additive finite abelian group of order  $w$  and let  $r$  be a positive integer with  $\gcd(w, r) = 1$ . Then for every  $g \in \Gamma$  there exists  $h \in \Gamma$  such that  $g = rh$ .*

*Proof.* Define the group homomorphism  $\varphi : \Gamma \rightarrow \Gamma$  as  $\varphi(g) = rg$ . Then  $\ker \varphi$  consists of the identity and all group elements of order  $r$ . But  $\gcd(w, r) = 1$ , so there are no groups elements of order  $r$  by Lagrange's Theorem. Thus  $\ker \varphi$  is trivial and since  $\Gamma$  is finite, we conclude that  $\varphi$  is an isomorphism. Thus for any  $g \in \Gamma$  there exists  $h \in \Gamma$  such that  $g = rh$ .  $\square$

*Proof of Theorem 3.3.2.* Fix a strong (weak) vertex coloring  $c : V \rightarrow \Gamma$  of  $H = (V, \mathcal{E})$ . Then

by Observation 3.3.3, we know that there exists  $h \in \Gamma$  such that

$$\sum_{v \in V} c(v) = rh.$$

Now select an arbitrary  $f \in \mathcal{E}$  and let it have weight  $h$  with all other edges given weight 0. This induces a vertex coloring  $c'$  and if for all vertices  $v$  we have  $c(v) = c'(v)$ , then there is nothing else to do. So we may assume that there exists a vertex  $x$  such that  $c(x) \neq c'(x)$ . Then there must be another vertex  $y \neq x$  such that  $c(y) \neq c'(y)$ . This immediately follows from the fact that  $\sum_v c(v) = \sum_v c'(v)$ .

Then let  $T$  be the trail between  $x$  and  $y$  guaranteed to us by property  $\mathcal{T}$  and let  $g = c'(x) - c(x)$ . By alternately subtracting and adding  $g$  along each edge of  $T$ , we redefine  $c'$  and end up with  $c(x) = c'(x)$ . Furthermore, since  $T$  has an even length, the colors induced by  $c'$  of all the other vertices of  $T$  remain unchanged except possibly  $y$ . More importantly, equality  $\sum_v c(v) = \sum_v c'(v)$  still holds.

Repeated application of this process will eventually terminate in an edge weighting  $c'$  of  $H$  for which  $c(v) = c'(v)$  for all  $v \in V$ . This is because once a vertex has been corrected, it can only ever be an internal vertex for every future trail chosen, which leaves the corrected coloring unchanged for all future iterations and each iteration of this process leaves us with at least one less vertex with an incorrect induced color.  $\square$

### 3.4 $\mathcal{T}$ -Connectivity

In the previous section, we assume that  $H$  is  $\mathcal{T}$ -connected. We would like to know what the threshold for  $\mathcal{T}$ -connectivity in  $\mathbb{H}^{(r)}(n, p)$  might be. Heuristically, since the probability that a given 1-offset  $uv$ -trail of length 2 existing is  $p^2$ , then for fixed  $u$  and  $v$  the probability that *no* 1-offset  $uv$ -trail of length 2 existing is exactly

$$p' = (1 - p^2)^{\binom{n-2}{r-1}}.$$

If we assume that the choices for each pair of vertices are independent, then we should be able to model the threshold for  $\mathcal{T}$ -connectivity in  $\mathbb{H}^{(r)}(n, p)$  with connectivity in  $\mathbb{G}(n, 1 - p')$ , which we know to be connected when  $1 - p' = (\log n + \omega)/n$ . So

$$1 - p' = 1 - (1 - p^2)^{\binom{n-2}{r-1}} \approx p^2 \frac{n^{r-1}}{(r-1)!}$$

and that leaves us with probability  $\sqrt{(r-1)! (\log n)/n^r}$  as a target for the sharp threshold for  $\mathcal{T}$ -connectivity in  $\mathbb{H}^{(r)}(n, p)$ . This heuristic argument turns out to be accurate and  $\sqrt{(r-1)! (\log n)/n^r}$  is the sharp threshold for  $\mathcal{T}$ -connectivity.

**Theorem 3.4.1.** *Let  $r \geq 3$  be an integer and  $\varepsilon > 0$ . Then:*

(i) *If  $p \geq (1 + \varepsilon)\sqrt{(r-1)! (\log n)/n^r}$ , then  $\mathbb{H}^{(r)}(n, p) \in \mathcal{T}$  w.h.p..*

(ii) *If  $p \leq (1 - \varepsilon)\sqrt{(r-1)! (\log n)/n^r}$ , then  $\mathbb{H}^{(r)}(n, p) \notin \mathcal{T}$  w.h.p..*

*Proof.* First we show part (i). Let us divide  $V$  into two sets,  $S$  and  $V \setminus S$  such that  $|S| = s$  with  $1 \leq s \leq n/2$  and let  $X_S$  be the number of 1-offset trails of length 2 connecting the two sets. Since the trails must start in one set and end in another there should be  $\ell = s(n-s)\binom{n-2}{r-1}$  potential trails. Now suppose we enumerate each of the trails and let  $X_i$  be the indicator variable that the  $i$ th trail is present. Clearly,  $X_S = X_1 + X_2 + \dots + X_\ell$  and

$$\mu = \mathbb{E}(X_S) = s(n-s) \binom{n-2}{r-1} p^2 = (1 + o(1))(1 + \varepsilon)^2 s \frac{n-s}{n} \log n.$$

Observe that some of these trails share edges with the others, so the event that  $X_i = 1$  and  $X_j = 1$  is not necessarily independent for all  $i$  and  $j$ . So we write  $i \sim j$  if  $X_i$  and  $X_j$  share an edge, then

$$\Delta = \sum_{\substack{\{i,j\}: i \sim j \\ i \neq j}} \mathbb{E}(X_i X_j) = O \left( s(n-s) \binom{n-2}{r-1} n p^3 \right) = O \left( s (\log n)^{3/2} / n^{r/2-1} \right).$$



So by Janson's bound (cf. 4.2) we have

$$\Pr(X_S = 0) \leq e^{-\mu+\Delta} \leq e^{-(1+o(1))(1+\varepsilon)^2 s \frac{n-s}{n} \log n}$$

and the union bound taken over all sets  $S$  of size  $1 \leq s \leq n/2$  implies

$$\begin{aligned} \sum_{s=1}^{n/2} \binom{n}{s} \Pr(X_S = 0) &= \sum_{s=1}^{1/\varepsilon} \binom{n}{s} \Pr(X_S = 0) \\ &\quad + \sum_{s=1+1/\varepsilon}^{n/\log n} \binom{n}{s} \Pr(X_S = 0) + \sum_{s=1+n/\log n}^{n/2} \binom{n}{s} \Pr(X_S = 0). \end{aligned}$$

If  $1 \leq s \leq 1/\varepsilon$ , then  $(n-s)/n \approx 1$  and

$$\sum_{s=1}^{1/\varepsilon} \binom{n}{s} \Pr(X_S = 0) \leq \sum_{s=1}^{1/\varepsilon} n^s e^{-(1+o(1))(1+\varepsilon)^2 s \log n} \leq \sum_{s=1}^{1/\varepsilon} \frac{1}{n^{\varepsilon s}} \leq \frac{1}{\varepsilon} \cdot \frac{1}{n^\varepsilon} = o(1).$$

If  $1/\varepsilon \leq s \leq n/\log n$ , then again  $(n-s)/n \approx 1$  and

$$\sum_{s=1+1/\varepsilon}^{n/\log n} \binom{n}{s} \Pr(X_S = 0) \leq \sum_{s=1+1/\varepsilon}^{n/\log n} n^s e^{-(1+o(1))(1+\varepsilon)^2 s \log n} \leq \sum_{s=1+1/\varepsilon}^{n/\log n} \frac{1}{n^{\varepsilon s}} \leq \frac{n}{\log n} \cdot \frac{1}{n} = o(1).$$

Finally, if  $n/\log n \leq s \leq n/2$ , then  $(n-s)/n \geq 1/2$  and since  $\binom{n}{s} \leq (en/s)^s \leq (e \log n)^s$ , we get

$$\begin{aligned} \sum_{s=1+n/\log n}^{n/2} \binom{n}{s} \Pr(X_S = 0) &\leq \sum_{s=1+n/\log n}^{n/2} (e \log n)^s e^{-(1+o(1))(1+\varepsilon)^2 s (\log n)/2} \\ &= \sum_{s=1+n/\log n}^{n/2} e^{s+s \log \log n - (1+o(1))(1+\varepsilon)^2 s (\log n)/2} = o(1). \end{aligned}$$

This completes the proof of part (i).

Now we prove part (ii). Clearly it suffices to show that if  $p = (1-\varepsilon)\sqrt{(r-1)!(\log n)/n^r}$ , then  $\mathbb{H}^{(r)}(n, p)$  has a vertex which is not an endpoint of any 1-offset trail of length 2. For a

fixed vertex  $v$  let  $X_v$  counts the number of 1-offset trails of length 2 with  $v$  as an endpoint. Let  $Y_v$  be an indicator random variable which equal to 1 if  $X_v = 0$ . Let  $Y = \sum_v Y_v$ . Since there are  $\binom{n-1}{r} \binom{r}{r-1}$  potential trails with  $v$  as its endpoint, the FKG bound (cf. 4.2) implies that

$$\Pr(Y = 1) = \Pr(X_v = 0) \geq (1 - p^2)^{\binom{n-1}{r} \binom{r}{r-1}}$$

and so

$$\mathbb{E}(Y) \geq n(1 - p^2)^{\binom{n-1}{r} \binom{r}{r-1}} \approx ne^{-\binom{n-1}{r} \binom{r}{r-1} p^2} \rightarrow \infty. \quad (3.9)$$

Let  $v \neq w$ . Now we will use Janson's bound to estimate from above

$$\Pr(Y_v = Y_w = 1) = \Pr(X_v = X_w = 0) = \Pr(X_v + X_w = 0).$$

Let

$$X_v + X_w = \sum_i Z_i^v + \sum_i Z_i^w,$$

where  $Z_i^v$  and  $Z_i^w$  are potential 1-offset trails of length 2 with an endpoint  $v$  or  $w$ , respectively.

Clearly,

$$\mathbb{E}(X_v + X_w) = 2 \binom{n-1}{r} \binom{r}{r-1} p^2$$

and

$$\Delta = \sum_{\substack{\{i,j\}:i\sim j \\ i \neq j}} \mathbb{E}(Z_i^v Z_j^v) + \sum_{\substack{\{i,j\}:i\sim j \\ i \neq j}} \mathbb{E}(Z_i^w Z_j^w) + \sum_{\substack{\{i,j\}:i\sim j \\ i \neq j}} \mathbb{E}(Z_i^v Z_j^w).$$

Now each of these three sums is bounded by  $O(n^{r+1}p^3 + n^{r-2}p^2)$ . The first term counts all pairs of 1-offset trails of length 2 with exactly one edge in common. The second term counts 1-offset trails of length 2 with  $v$  and  $w$  as its endpoints. (Observe that  $Z_i^v$  and  $Z_i^w$  can be associated with the same trail.) Consequently,  $\Delta = o(1)$  and Janson's bound yields that

$$\Pr(X_v + X_w = 0) \leq e^{-2 \binom{n-1}{r} \binom{r}{r-1} p^2 + o(1)}$$

so that

$$\begin{aligned} \mathbb{E}(Y^2) &= \mathbb{E}(Y) + \sum_{v \neq w} \Pr(Y_v = Y_w = 1) \\ &= \mathbb{E}(Y) + \sum_{v \neq w} \Pr(X_v + X_w = 0) \leq \mathbb{E}(Y) + n^2 e^{-2 \binom{n-1}{r} p^{2+o(1)}}. \end{aligned}$$

Thus, due to (3.9), we have  $\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} \approx 1$  and then Chebyshev's bound implies that  $Y > 0$  *w.h.p.*, as required.  $\square$

### 3.5 Concluding Remarks

In this chapter we have studied the  $\ell$ -offset Hamiltonicity of random hypergraphs and why these structures are important in relation to the 1-2-3 Conjecture. The case when  $\ell = 1$  is not understood yet, but we conjecture that the asymptotic threshold for the existence of 1-offset Hamilton cycle in  $H_{n,p}^{(r)}$  is  $\sqrt{(\log n)/n^r}$ . In order to prove this, one can try to use a similar approach as in [18, 12]. This will require to show that  $H_{n,p}^{(r)}$  has a factor of 1-offset trails of length 2 *w.h.p.*, which is similar to a celebrated result of Johansson, Khan and Vu [26] about factors in hypergraphs.

One can also consider a directed version of Theorem 3.2.1. Let  $\vec{H}_{n,p}^{(r)}$  be a *directed random hypergraph*, where every ordered  $r$ -tuple appears independently with probability  $p$ . An immediate consequence of the General Clutter Percolation Theorem of McDiarmid [33] implies that all results in parts (iii) and (iv) of Theorem 3.2.1 also hold for  $\vec{H}_{n,p}^{(r)}$ . As a matter of fact an easy modification of the proof of Theorem 3.2.1 yields more accurate results. In particular, one can show that  $(e/n)^{r/2}$  is the sharp threshold for the existence of the directed  $\ell$ -offset Hamilton cycle for  $r \geq 6$  and  $\ell \geq 3$ .

# Chapter 4

## Probabilistic Tools

Throughout this dissertation, all logarithms are natural (base  $e$ ) and all asymptotics are taken in  $n$  and we used various tools and approximations that, for the sake of convenience and clarity, we summarize here (for more details see, e.g., [6, 19, 25]).

### 4.1 Techniques

**First Moment Method.** *Let  $X$  be a nonnegative integral random variable. If  $E(X) = o(1)$ , then w.h.p.  $X = 0$ .*

**Second Moment Method.** *Let  $X$  be a nonnegative integral random variable. If  $\text{Var}(X) = o(E(X)^2)$ , then w.h.p.  $X \geq 1$ .*

Let  $Po(\lambda)$  denote the random variable with Poisson distribution with mean  $\lambda$ . Moreover, let  $(X)_k := X(X-1)(X-2)\cdots(X-k+1)$  denote the factorial moment of the random variable  $X$ .

**Method of Moments.** *Let  $\lambda$  be a positive constant. Suppose that  $X_1, X_2, \dots$  are random variables such that for each fixed  $k$  we have  $E((X_n)_k) \rightarrow \lambda^k$  as  $n$  tends to infinity. Then,  $X_n \sim Po(\lambda)$ .*

## 4.2 Inequalities

**Bernstein's Bound.** Let  $X_1, \dots, X_m$  be independent random variables, and  $X = \sum_{i=1}^m X_i$ . Suppose that  $|X_i - \mathbb{E}(X_i)| \leq C$  always holds for all  $i$ , and  $\gamma > 0$ . Then,

$$\Pr(|X - \mathbb{E}(X)| \geq \gamma) \leq 2 \exp\left(-\frac{\frac{1}{2}\gamma^2}{\sum_{i=1}^m \text{Var}(X_j) + \frac{1}{3}C\gamma}\right).$$

**Chebyshev's Bound.** Let  $X$  be a nonnegative integral random variable then

$$P(X > 0) \geq \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2}.$$

**Chernoff's Bound.** Let  $\text{Bin}(n, p)$  denote the random variable with binomial distribution with number of trials  $n$  and probability of success  $p$ . If  $X \sim \text{Bin}(n, p)$  and  $0 < \gamma \leq \mathbb{E}(X)$ , then

$$\Pr(|X - \mathbb{E}(X)| \geq \gamma) \leq 2 \exp(-\gamma^2 / (3\mathbb{E}(X))).$$

**FKG Bound.** A function  $f : C_n = \{0, 1\}^{[n]} \rightarrow \mathbb{R}$  is said to be monotone increasing if whenever  $x = (x_1, x_2, \dots, x_n)$ ,  $y = (y_1, y_2, \dots, y_n) \in C_n$  and  $x \leq y \in C_n$  (i.e.  $x_j \leq y_j$ ,  $j = 1, 2, \dots, n$ ) then  $f(x) \leq f(y)$ . Similarly,  $f$  is said to be monotone decreasing if  $-f$  is monotone increasing. If  $f$  and  $g$  are monotone functions on  $C_n$ , then (cf. Theorem 21.5 [19])

$$\mathbb{E}(fg) \geq \mathbb{E}(f) \cdot \mathbb{E}(g).$$

**Janson's Bound.** Fix a family of  $n$  subsets  $D_i$ ,  $i \in [m]$ . Let  $R$  be a random subset of  $[m]$  such that for  $s \in [m]$  we have  $0 < \Pr(s \in R) = q_s < 1$ . The elements of  $R$  are chosen independently of each other and the sets  $D_i$ ,  $i = 1, 2, \dots, m$ . Let  $\mathcal{A}_i$  be the event that  $D_i$  is a subset of  $R$ . Moreover, let  $X_i$  be the indicator of the event  $\mathcal{A}_i$ . Note that,  $X_i$  and  $X_j$  are independent iff  $D_i \cap D_j = \emptyset$ . One can easily see that the  $X_i$ s are increasing.

We let

$$X = X_1 + X_2 + \cdots + X_m,$$

and

$$\mu = \mathbf{E}(X) = \sum_{i=1}^m \mathbf{E}(X_i).$$

We write  $i \sim j$  if  $D_i \cap D_j \neq \emptyset$  and set

$$\Delta = \sum_{\substack{\{i,j\}: i \sim j \\ i \neq j}} \mathbf{E}(X_i X_j).$$

Then (cf. Theorem 21.12 and Corollary 21.13 [19])

$$\Pr(X = 0) \leq e^{-\mu + \Delta}.$$

**Markov's Bound.** Let  $X$  be a nonnegative integral random variable and  $\gamma > 0$ . Then

$$\Pr(X \geq \gamma) \leq \mathbf{E}(X)/\gamma.$$

**Union Bound.** If  $E_1, \dots, E_m$  are events, then

$$\Pr\left(\bigcup_{i=1}^m E_i\right) \leq m \cdot \max\{\Pr(E_i) : i \in [m]\}.$$

### 4.3 Binomial Coefficient Estimates

Several times we will also need to estimate binomial coefficients (for more details see, e.g., Chapter 22 in [22]).

**Approximation 4.3.1.** Let  $p > 0$  and  $\ell$  be functions of  $m$  ( $\ell$  can be negative). Assume that

$\ell^2 = o(p)$  and  $\ell^2 = o(m - p)$  as  $m$  tends to infinity. Then,

$$\binom{m}{p + \ell} \sim \binom{m}{p} \left( \frac{m - p}{p} \right)^\ell.$$

**Approximation 4.3.2.** Let  $k \geq 1$  be a fixed integer. Then,

$$\sum_{i=0}^m \binom{m}{i}^k \sim \left( 2^m \sqrt{\frac{2}{\pi m}} \right)^k \sqrt{\frac{\pi m}{2k}} = \Theta(2^{km} m^{-(k-1)/2}).$$

**Approximation 4.3.3.**

$$\binom{m}{\lfloor m/2 \rfloor} \sim \frac{2^{m+1/2}}{\sqrt{\pi m}}.$$

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