A Study of Lommel Functions

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A STUDY OF
LOMMEL FUNCTIONS

by

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CHAPTER 0
INTRODUCTION

This paper is primarily a study of the Lommel functions which are solutions of the differential equation

$$\text{(E)} \qquad z^2y'' + zy' + (z^2 - \nu^2)y = kz^{\mu+1}, \quad ' = d/dz,$$

where z is a complex variable and ν, k, ~~and~~ μ are constants. Two observations should be made here. One is that the associated homogeneous equation of (E) is the Bessel equation with parameter ν. Hence the Bessel functions are important in the study of the Lommel functions. The second observation is that the only singular points of (E) are a regular singular point at $z = 0$ and an irregular one at $z = \infty$. Thus for a study of solutions of (E) for all z, the power series solution of (E) in powers of z and the asymptotic expansion of a solution of (E) for large $|z|$ are needed.

An interest in the asymptotic behavior of the Lommel functions and the Bessel functions is evidenced by papers published during the past few years. It has been shown that the Lommel functions are of importance in the solution of certain differential equations arising from physical problems involving thin toroidal shells.(e.g. see [3], [7], and [8]).

The first chapter is a brief discussion of nonhomogeneous differential equations. It includes an outline of two methods for solving differential equations, namely the Frobenius method and the method of variation of parameters. These two methods will be used in later chapters. The second chapter is a discussion of the Bessel functions, solutions of the Bessel equation. The properties of the Bessel functions discussed in this chapter are those related to the detailed development of the Lommel functions made in chapter three. Of special importance in chapter two is a discussion on the asymptotic behavior of the Bessel function which includes an outline of an elementary method developed by F. Brauer [2] for determining the first term of the asymptotic expansion of a Bessel function.

The third chapter is devoted to the development of the Lommel functions. First the power series solution of (E) in powers of z is developed by two methods, the Frobenius method and the method of variation of parameters. Then a second solution is developed by the Frobenius method. For this solution an asymptotic expansion is developed for large $|z|$; and through the use of Brauer's technique a method is presented for finding the first term of this asymptotic expansion.

CHAPTER I
NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

1. The Solution

The general nonhomogeneous ordinary linear differential equation of the second order can be written in the form

$$a_o(z)y'' + a_1(z)y' + a_2(z)y = f(z) \tag{1.1}$$

where $a_k(z)$ $(k = 0,1,2)$ and $f(z)$ are defined in a domain D in the z-plane and $a_o \neq 0$ in D. The associated homogeneous equation is

$$a_o(z)y'' + a_1(z)y' + a_2(z)y = 0. \tag{1.2}$$

The complete solution of equation (1.2) can be written in the form

$$y_h = c_1u_1(z) + c_2u_2(z) \tag{1.3}$$

where c_1 and c_2 are arbitrary constants and u_1 and u_2 are linearly independent solutions of equation (1.2) in D. Finally the complete solution of (1.1) can be written as

$$y(z) = y_h(z) + y_p(z) \tag{1.4}$$

where $y_p(z)$ is any particular solution of equation (1.1). We note here that if a_o is not 0 and $a_k(k = 1,2)$ are continuous in D, then a necessary and sufficient condition that u_1 and u_2 be linearly independent in D is that their Wronskian, (1.5) $$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} ,$$

is different from zero for all z in D.

2. The Frobenius Method

If (1.2) is written in the form

$$y'' + p_1(z)y' + p_2(z)y = 0 , \tag{2.1}$$

the behavior of the solutions near a point $z = z_o$ depends on the analyticity of $p_1(z)$ and $p_2(z)$ at $z = z_o$. If $p_1(z)$ and $p_2(z)$ are both holomorphic in a domain containing z_o, then the point z_o is said to be an ordinary point. Otherwise it is said to be a singular point. If z_o is a singular point and the products $(z-z_o)p_1(z)$ and $(z-z_o)^2 p_2(z)$ are holomorphic in a neighborhood of z_o, then z_o is said to be a regular singular point. Otherwise z_o is called an irregular singular point.

The Frobenius method is a method frequently used for finding a series solution of a differential equation at a regular singular point. This method applied to (2.1) consists of the following steps. First we let

$$y = (z-z_o)^s \sum_{k=0}^{\infty} A_k (z-z_o)^k , \qquad A_o \neq 0 \tag{2.2}$$

where s is a constant, real or complex, to be determined. Then we substitute (2.2) and its first two derivatives in (2.1). Since the right-hand side is zero, the coefficient of each power of $z-z_o$ is equal to zero. When we set the coefficient of the lowest such power equal to zero, we get an equation in s which is called the indicial equation. From this equation we get values for s which may or may not give us linear independent solutions of (2.1). Setting the coefficients of the higher degrees of $z-z_o$ equal to zero, we obtain relationships between the A_k's in (2.2). Thus we get one or two independent solutions of (2.1) in the form of (2.2), valid in their regions of convergence. If the indicial equation mentioned above gives us two distinct roots which do not differ by an integer, then (2.2) will yield two linearly independent solutions and thus the complete solution of (2.1). However, if the roots of the indicial equation are equal or differ by an integer, there may exist solutions of (2.1) not in the form of (2.2).

If the indicial equation gives two like roots, $s_1 = s_2$, the second solution is given by

$$(2.3) \qquad y_2(z) = \left[\frac{\partial y_s(z)}{\partial s}\right]_{s=s_1}$$

where

$$(2.4) \qquad y_s(z) = (z-z_o)^s \sum_{k=0}^{\infty} A_k(s)(z-z_o)^k .$$

When s_1 differs from s_2 by an integer and say $s_1 > s_2$, the second solution is given by

$$y_2(z) = \left\{ \frac{\partial}{\partial s} \left[(s-s_2) y_s(z) \right] \right\}_{s=s_2} . \tag{2.5}$$

To apply the Frobenius method to the type nonhomogeneous differential equation of interest to us, namely

$$y'' + p_1(z)y' + p_2(z)y = cz^m , \tag{2.6}$$

we proceed as before to find the solution of the associated homogeneous equation. Then to find a particular solution of (2.6), we alter the procedure slightly by letting $s = m+2$. This causes the least degree of $z-z_o$ on the left-hand side of (2.6) to be m which allows us to set its coefficient equal to c. The coefficients of the higher powers of $z-z_o$ are respectively equal to zero. Hence we get a particular solution in the form (2.2).

3. The Method of Variation of Parameters

A method for finding a particular solution of a nonhomogeneous differential equation when the complete solution of its associated homogeneous equation is known, is the variation of parameters method. To illustrate this method, we first consider the homogeneous equation (1.2) and its complete solution (1.3). Then we assume a particular solution of (1.1) in terms of the two linearly independent solutions of (1.2), u_1 and u_2, namely

$$y_p = v_1 u_1 + v_2 u_2 \tag{3.1}$$

where v_1 and v_2 are functions of z. Since u_1 and u_2 are known, we have two unknowns, v_1 and v_2, to determine. Therefore we need two conditions such that we can solve for these unknowns. One condition, that (3.1) is a solution of (1.1), is fixed. We obtain the other condition by taking the first derivative of (3.1),

$$y_p' = v_1 u_1' + v_2 u_2' + v_1' u_1 + v_2' u_2 \quad , \tag{3.2}$$

and setting

$$v_1' u_1 + v_2' u_2 = 0 \quad . \tag{3.3}$$

When we substitute y_p and its first two derivatives in (1.1) to satisfy our fixed condition, we use our condition (3.3) and the fact that u_1 and u_2 satisfy (1.2), to simplify the fixed condition to

(3.4) $$v_1'u_1' + v_2'u_2' = f(z) \quad .$$

Next we note that the coefficient determinant of our system of two equations, (3.3) and (3.6), is

(3.5) $$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \quad ,$$

which is the Wronskian of our two linearly independent solutions of (1.2). Since u_1 and u_2 are linearly independent solutions of (1.2), we know that W does not vanish. Cramer's rule then assures us of a solution for v_1' and v_2'. Having these solutions, integration gives us the v_1 and v_2 necessary to substitute in (3.1). As we are interested in a particular solution, the constants of integration are immaterial.

CHAPTER II
BESSEL FUNCTIONS

4. The Bessel Functions of the First Kind

The Bessel functions are solutions of the Bessel equation,

$$z^2y'' + zy' + (z^2-\nu^2)y = 0, \tag{4.1}$$

or

$$y'' + \frac{y'}{z} + \left(1 - \frac{\nu^2}{z^2}\right)y = 0, \tag{4.1'}$$

where ν is a constant. From (4.1)' we see that the Bessel equation has a regular singular point at z=0. Applying the Frobenius method to (4.1) at the point z=0, we get an indicial equation with the roots $\pm\nu$. The root ν yields the solution

$$y = A(z/2)^{\nu} \sum_{r=0}^{\infty} \frac{(-1)^r(z/2)^{2r}}{r!\,\Gamma(\nu+r+1)} \tag{4.2}$$

where A is an arbitrary constant. The infinite series in (4.2) converges uniformly in every circle $|z| \leq R$, for any positive R.

The Bessel functions of the first kind of order ν are defined to be

$$J_{\nu}(z) = (z/2)^{\nu} \sum_{r=0}^{\infty} \frac{(-1)^r(z/2)^{2r}}{r!\,\Gamma(\nu+r+1)}, \tag{4.3}$$

where $(z/2)^{\nu} = \exp(\nu \log(z/2))$ with $\log(z/2)$ having its principal value. Thus $(z/2)^{-\nu} J_{\nu}(z)$ is holomorphic over the finite plane. In a similar manner $-\nu$ yields a second solution

$$(4.4) \qquad J_{-\nu}(z) = (z/2)^{-\nu} \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r}}{r!\,\Gamma(-\nu+r+1)}$$

which is not necessarily independent of $J_{\nu}(z)$. The Wronskian of $J_{\nu}(z)$ and $J_{-\nu}(z)$,

$$(4.5) \qquad W(J_{\nu}(z)\ ,\ J_{-\nu}(z)\ ) = \frac{-2\sin(\nu\pi)}{\pi z}\ ,$$

shows us that the two solutions are linearly independent if and only if ν is not an integer. Thus if ν is not an integer, the complete solution of (1.2) is

$$(4.6) \qquad y = AJ_{\nu}(z) + BJ_{-\nu}(z)\ ,$$

when A and B are arbitrary constants.

5. The Recurrence Formulas

Four recurrence formulas which are useful in working with Bessel functions can be derived from the definition of $J_\nu(z)$. The last two of these formulas are obtained by finding respectively the sum and the difference of the first two. The formulas are

(5.1) $$J_{\nu-1}(z) + J_{\nu+1}(z) = 2\frac{\nu}{z} J_\nu(z),$$

(5.2) $$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z),$$

(5.3) $$\frac{\nu}{z} J_\nu(z) + J_\nu'(z) = J_{\nu-1}(z),$$

(5.4) $$\frac{\nu}{z} J_\nu(z) - J_\nu'(z) = J_{\nu+1}(z).$$

6. Bessel Functions of the Second and Third Kinds

As shown in section 4, $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent if and only if ν is not an integer. We shall outline a method for producing a second solution of (4.1) which is linearly independent of $J_\nu(z)$ for all ν.

Bessel's integral,

$$(6.1) \quad J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z \sin\theta)\, d\theta, \text{ (n an integer)},$$

can be developed from the definition of $J_\nu(z)$ given in (4.3), after applying Hankel's formula [5],

$$(6.2) \qquad \frac{1}{\Gamma(\nu+r+1)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t t^{-\nu-r-1} dt \quad ,$$

where the path of integration is taken along the lower edge of a cut in the z-plane from $-\infty$ to 0, around zero in the positive sense, and back to $-\infty$ along the upper edge of the cut, (i.e. $|\arg t| \leq \pi$). From (6.1) it can be shown that

$$(6.3) \quad J_n(z) = \frac{1}{2\pi i} \int_{-\pi i}^{\pi i} e^{z \sinh t - nt} dt, \text{ (n an integer)}.$$

The equation (6.3) will be used to find the Bessel functions of the third kind.

First it is assumed that there exists a solution of (4.1) of the form

$$(6.4) \qquad y = \int_a^b e^{z \sinh t - \nu t} dt, \quad (\nu \text{ a constant}).$$

Substitution of (6.4) and its first two derivatives in (4.1) leads to the functions

$$(6.5) \qquad \int_{-\infty + \alpha i}^{\infty - (\pi + \alpha) i} e^{z \sinh t - \nu t} dt,$$

$$(6.6) \qquad \int_{-\infty + \alpha i}^{\infty + (\pi - \alpha) i} e^{z \sinh t - \nu t} dt,$$

which are solutions of (4.1), valid when z lies in the sector $-\pi/2 + \alpha + \delta \leq \arg z \leq \pi/2 + \alpha - \delta$, where α is a real constant and δ is an arbitrary positive number. The Bessel functions of the third kind of order ν, also called Hankel functions of order ν, are defined to be

$$(6.7) \qquad H_\nu^{(1)}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty + \pi i} e^{z \sinh t - \nu t} dt,$$

$$(6.8) \quad H_{\nu}^{(2)}(z) = \frac{-1}{\pi i} \int_{-\infty}^{\infty - \pi i} e^{z \sinh t - \nu t} dt, \ |\arg z| < \pi/2 .$$

These functions are obtained from (6.6) by letting $\alpha = 0$ and thus are solutions of (4.1), valid for general ν. The Hankel functions can be continued analytically over the z-plane through the use of (6.6) by specializing α.

The Hankel and Bessel functions can be shown to be related by the equations

$$(6.9) \qquad 2J_{\nu}(z) = H_{\nu}^{(1)}(z) + H_{\nu}^{(2)}(z) \quad ,$$

$$(6.10) \qquad 2J_{-\nu}(z) = e^{\nu \pi i} H_{\nu}^{(1)}(z) + e^{-\nu \pi i} H_{\nu}^{(2)}(z) \quad ,$$

provided that $|\arg z| < \pi$. Finally it is easily seen from the Wronskian of the Hankel functions,

$$(6.11) \qquad W(H_{\nu}^{(1)}(z) , H_{\nu}^{(2)}(z)) = \frac{4}{\pi i z} \quad ,$$

that the Hankel functions are linearly independent, and thus

$$(6.12) \qquad y = AH_{\nu}^{(1)}(z) + BH_{\nu}^{(2)}(z)$$

is a complete solution of (4.1). However we wish to retain $J_{\nu}(z)$ as a basic Bessel function because of its simple behavior near the origin. Therefore we define the Bessel function of the second kind to be

$$Y_\nu(z) = \frac{1}{2i}\left\{H_\nu^{(1)}(z) - H_\nu^{(2)}(z)\right\} \quad . \tag{6.13}$$

Comparing (6.13) with (6.9), we see that $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for general ν. Thus we have a general solution for (4.1),

$$y = AJ_\nu(z) + BY_\nu(z) \quad , \tag{6.14}$$

which contains $J_\nu(z)$ as one of the independent solutions.

7. The Asymptotic Expansion of the Bessel Functions

Section 6 provided an outline for the development of the complete solution of Bessel's equation in terms of series which converge for all $|z| < \infty$. However these series converge so slowly for large $|z|$ that the initial terms of the series give little information regarding its sum. Therefore we wish to outline a development of a series which is asymptotic to the Bessel functions for large $|z|$.

In order to find the asymptotic expansion of the Bessel functions we shall use the following lemma of Watson [10] :

Lemma. _Let us suppose that_ $f(t)$ _is a holomorphic function, save possibly for a branch-point at the origin, when_ $|t| \leq a + \delta$, _where_ a _and_ δ _are positive, and let_

$$f(t) = \sum_{m=1}^{\infty} a_m t^{(m/r)-1}$$

when $|t| \leq a$, r _being positive. Let us suppose further, that, when_ t _is positive and_ $t \geq a$,

$$|f(t)| < Ke^{bt}$$

where K _and_ b _are positive numbers independent of_ t. _Then_

$$F(z) \equiv \int_0^{\infty} e^{-zt} f(t)\,dt \sim \sum_{m=1}^{\infty} a_m \Gamma(m/r) z^{-m/r}$$

when $|z|$ is large and $|\arg z| \leq \frac{\pi}{2} - \Delta$, where Δ is an arbitrary positive number.

To use this lemma for our purposes, we need an integral that is the solution of (4.1) and has the properties listed above. We shall use a form of Hankel's function, c.f.(6.6),

$$(7.1) \qquad H_{\nu}^{(1)}(z) = \frac{1}{\pi i} \int_{-\infty + \alpha i}^{\infty + (\pi - \alpha) i} e^{z \sinh t - \nu t}\,dt,$$

where $|\arg(ze^{-\alpha i})| < \pi/2$.

Since (7.1) is not in the form we need, we proceed to put it in the desired form. To do this we use a method developed by Debye [4], called the method of steepest descents. First we find a zero of d/dt (sinh t), which lies on the contour of (7.1), to be $t = \frac{\pi i}{2}$. This is a saddle-point in the surface defined by the equation

$$(7.2) \qquad u(x,y) = \text{Rl} \sinh(x + iy)$$

in a space in which (x,y,u) are Cartesian coordinates. We wish to begin our contour at $t = \frac{\pi i}{2}$. Making the substitutions

$$(7.3) \qquad t = \frac{\pi i}{2} + w, \quad z = \zeta e^{\alpha i}, \quad \beta = \alpha + \frac{\pi}{2},$$

in (7.1) we get

(7.4) $$\pi i e^{\frac{1}{2}\nu\pi i} H_{\nu}^{(1)}(\zeta e^{\alpha i})$$

$$= 2 \int_{0}^{\infty + (\pi-\beta)i} \exp\left\{\zeta e^{\beta i} \cosh w\right\} \cosh \nu w \, dw, \; |\arg\zeta| < \pi/2$$

where w = 0 is the point at which our path of steepest descent is to begin. Keeping the imaginary part of $e^{\beta i}$ cosh w constant, we get the path of steepest descent from our saddle-point. Therefore since

(7.5) $$e^{\beta i} \cosh w = e^{\beta i}, \quad w = 0 ,$$

we have the following equation for our steepest paths :

(7.6) $$e^{\beta i} \cosh w = e^{\beta i} - \tau , \quad (\tau \text{ is real}).$$

We next solve (7.6) and find that the path of steepest descent occurs when τ is positive. Moreover, when $\tau \to +\infty$ one path of w moves in the direction of $w \to \infty + (\pi-\beta)i$, which coincides with the direction of the contour in (7.4). Thus without changing the value of the integral in (7.4) we get

(7.7) $$\pi i e^{\frac{1}{2}\nu\pi i} H_{\nu}^{(1)}(\zeta e^{\alpha i})$$

$$= \frac{2}{\nu} \exp(\zeta\, e^{\beta i}) \int_0^\infty e^{-\zeta \tau} \frac{d \sinh \nu w}{d\tau} \, d\tau .$$

Now we have (7.7) which is in the form needed for Watson's lemma. Since the function d sinh $\nu w/d\tau$ satisfies the conditions in Watson's lemma, we can apply the lemma to (7.7) to get

$$(7.8) \qquad H_\nu^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp\left[i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] \times$$

$$\times \left[1 + \sum_{r=1}^{\infty} \frac{(-1)^r(\nu,r)}{(2iz)^r} \right],$$

where

$$(\nu,r) = \frac{(4\nu^2-1^2)(4\nu^2-3^2) \cdots (4\nu^2-(2r-1)^2)}{2^{2r} r!} .$$

Using the relation

$$(7.9) \qquad H_\nu^{(2)}(z) = -e^{\nu\pi i} H_\nu^{(1)}(ze^{\pi i}) ,$$

we determine the asymptotic expansion of $H_\nu^{(2)}(z)$. Then using (6.9) and (6.13) we get

$$(7.10) \qquad J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \times\right.$$

$$\times \sum_{r=0}^{\infty} \frac{(-1)^r(\nu,2r)}{(2z)^{2r}} - \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \times$$

$$\times \sum_{r=0}^{\infty} \frac{(-1)^r(\nu,2r+1)}{(2z)^{2r+1}} \Big],$$

(7.11) $$Y_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \Big[\sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \times$$

$$\times \sum_{r=0}^{\infty} \frac{(-1)^r(\nu,2r)}{(2z)^{2r}} + \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \times$$

$$\times \sum_{r=0}^{\infty} \frac{(-1)^r(\nu,2r+1)}{(2z)^{2r+1}} \Big],$$

which hold for large values of $|z|$ provided that $|\arg z| < \pi$.

8. An Elementary Method for Calculating the First Term of the Asymptotic Expansion of $J_n(x)$

F. Brauer [2] developed an elementary method for evaluating the first term of the asymptotic expansion of $J_n(x)$ where n is a positive integer and x is real. In this section we shall give an outline of his procedure.

First we substitute $v = y\sqrt{x}$ in

$$(4.1)'' \qquad x^2y'' + xy' + (x^2 - n^2)y = 0$$

and get

$$(8.1) \qquad v'' + \left[1 + \frac{1/4 - n^2}{x^2}\right] v = 0 \quad .$$

Then by letting $v_1 = v$ and $v_2 = v_1'$, we get the system of equations

$$(8.2) \qquad v_1' = v_2 \quad , \qquad v_2' = \frac{(n^2 - 1/4 - x^2)v_1}{x^2} \quad .$$

By letting

$$(8.3) \qquad a(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \quad b(x) = \begin{pmatrix} 0 & 0 \\ \frac{n^2 - 1/4}{x^2} & 0 \end{pmatrix} ;$$

$$\vec{v}(x) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} ,$$

we get the vector equation

$$(8.4) \qquad \vec{v}' = [a(x) + b(x)]\ \vec{v} \quad .$$

Next we compare (8.4) with the vector equation

$$(8.5) \qquad \vec{u}' = a(x)\vec{u}$$

which has a fundamental matrix solution

$$(8.6) \qquad \Phi(x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \quad .$$

Therefore any solution of (8.5) is of the form

$$(8.7) \qquad \vec{u}(x) = \Phi(x)\vec{c}$$

where $\vec{c}$ is a constant vector. Now we assume a solution of (8.4) to be of the form

$$(8.8) \qquad \vec{v}(x) = \Phi(x)\vec{w}(x) \quad .$$

(Note here that if $\vec{w}(x)$ were a constant then $\vec{v}(x) = \vec{u}(x)$). Next we substitute (8.8) in (8.4) and see that (8.8) is a solution if and only if

$$(8.9) \qquad \vec{w}'(x) = \Phi^{-1}(x)b(x)\Phi(x)\vec{w}(x) \quad .$$

Applying lemmas 1 and 2 [1], to (8.9) we see that $\vec{w}(x) \to \vec{w}(\infty)$, a constant, as $x \to \infty$ and further that

$$(8.10) \qquad |\vec{w}(x) - \vec{w}(\infty)| \leq K/x$$

where K is a constant and x is large. Now comparing (8.7) and (8.8) we get

$$(8.11)\quad |\vec{v}(x) - \vec{u}(x)| = |\Phi(x)\vec{w}(x) - \Phi(x)\vec{w}(\infty)| \leq M/x ,$$

where M is a constant and x is large. This tells us that the difference between $\vec{v}$ and $\vec{u}$ is $O(x^{-1})$ and thus enables us to express the function $J_n(x)$ in the forms

$$(8.12)\quad \begin{aligned} J_n(x) &= A_n x^{\frac{1}{2}} \cos(x - \delta_n) + O(x^{-\frac{3}{2}}) , \\ J_n'(x) &= -A_n x^{\frac{1}{2}} \sin(x - \delta_n) + O(x^{-\frac{3}{2}}) . \end{aligned}$$

Using (8.12) and the recurrence formulas in Section 5, we find that A_n and δ_n can be expressed as follows:

$$(8.13)\quad A_n = A_o , \quad \delta_n = \delta_o + \frac{n\pi}{2} , \qquad n = 0,1,2,\cdots .$$

Next by comparing

$$(8.14)\quad x^{\frac{1}{2}}J_o(x) \sim (2/\pi)^{\frac{1}{2}} \cos(x - \pi/4) , \qquad x \to \infty$$

with (8.12), we get $A_o = (2/\pi)^{\frac{1}{2}}$ and $\delta_o = \pi/4$. It follows from (8.13) that

$$(8.15)\quad A_n = (2/\pi)^{\frac{1}{2}} \quad ; \quad \delta_n = (2n + 1) \pi/4, \qquad n = 0,1,2,\cdots .$$

Finally when we substitute (8.15) in (8.12) we get

$$(8.16) \quad J_n(x) = \frac{2}{\pi x}^{\frac{1}{2}} \cos\left[x - (2n+1)\frac{\pi}{4}\right] + O(x^{-\frac{3}{2}}), \quad n = 0,1,2,\cdots.$$

Thus we have the first term of the asymptotic expansion of $J_n(x)$. Similarly we can get

$$(8.17) \qquad Y_n(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left[x - (2n+1)\frac{\pi}{4}\right] + O(x^{-\frac{3}{2}}), n = 0,1,2,\cdots.$$

CHAPTER III
THE LOMMEL FUNCTIONS

9. The Function $s_{\mu,\nu}(z)$

In this chapter we shall study the particular solutions of the differential equation

$$(9.1) \qquad y'' + \frac{1}{z} y' + \left(1 - \frac{\nu^2}{z^2}\right) y = kz^{\mu-1} \quad ; \qquad ' = \frac{d}{dz}$$

where z is an independent complex variable and μ, ν, and k are constants, or

$$(9.1)' \qquad z^2 y'' + zy' + (z^2 - \nu^2) y = kz^{\mu+1}$$

which is equation (E) mentioned in the introduction. Since the only singularities of the associated homogeneous equation are $z = 0$ which is regular, and $z = \infty$ which is irregular, we shall find two different types of series solutions corresponding to each of these singularities.

First we shall use the Frobenius method to find a series solution in powers of z. Considering (9.1) and the explanation given in the last paragraph of Section 2, we begin by assuming a solution of (9.1)' to be

$$(9.2) \qquad y = \sum_{r=0}^{\infty} a_r z^{r+\mu+1} .$$

Substituting (9.2) and its first two derivatives in (9.1)' we get the identity

(9.3) $$\sum_{r=0}^{\infty}\left[(r+\mu+1)^2-\nu^2\right]a_r z^{r+\mu+1}+\sum_{r=0}^{\infty}a_r z^{r+\mu+3}$$

$$= kz^{\mu+1} \quad .$$

Setting coefficients of like powers of z equal we get

(i) $$a_o\left[(\mu+1)^2-\nu^2\right]=k$$

(9.4) (ii) $$a_1\left[(\mu+2)^2-\nu^2\right]=0$$

(iii) $$a_r=\frac{-a_{r-2}}{(r+\mu+1)^2-\nu^2}\quad , \qquad r=2,3,4,\cdots \; .$$

If neither $\mu+\nu$ nor $\mu-\nu$ is a negative integer, the equations (ii) and (iii) show us that $a_r=0$, $r=1,3,5,\cdots$. Substituting (9.4) in (9.2) we get

(9.5) $$y=k\left[\frac{z^{\mu+1}}{(\mu+1)^2-\nu^2}-\frac{z^{\mu+3}}{\left[(\mu+1)^2-\nu^2\right]\left[(\mu+3)^2-\nu^2\right]}+\cdots\right]$$

$$=kz^{\mu-1}\left[\frac{z^2}{(\mu+\nu+1)(\mu-\nu+1)}-\right.$$

$$\left.-\frac{z^4}{(\mu+\nu+1)(\mu+\nu+3)(\mu-\nu+1)(\mu-\nu+3)}+\cdots\right]$$

$$= kz^{\mu-1}\left[\frac{(z/2)^2}{(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})} - \right.$$

$$\left. - \frac{(z/2)^4}{(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})(\frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{2})(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})(\frac{\mu}{2}-\frac{\nu}{2}+\frac{3}{2})} + \cdots\right]$$

therefore,

$$(9.5)' \qquad y = kz^{\mu-1}\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{2r+2}}{\left[\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}\right]_{r+1}\left[\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}\right]_{r+1}}$$

$$= kz^{\mu-1}\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{2r+2}\,\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+r+\frac{3}{2})\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+r+\frac{3}{2})}$$

where

$$\left[(\alpha)_{r+1}\right] = (\alpha)(\alpha+1)\cdots(\alpha+r) = \frac{\Gamma(\alpha+r+1)}{\Gamma(\alpha)} .$$

For the sake of conciseness we let

$$(9.6) \quad s_{\mu,\nu}(z) = z^{\mu-1}\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{2r+2}\,\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+r+\frac{3}{2})\,\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+r+\frac{3}{2})} .$$

This is our first Lommel function. It is not defined for $\mu \pm \nu$ = an odd negative integer since $\Gamma(-n)$, n = 0,1,2, $\cdots$ are not defined. Thus for $\mu \pm \nu$ not equal to -n, n = 1,3,5, $\cdots$ we see that the complete solution of (9.1) is

$$(9.7)\quad y = AJ_{\nu}(z) + BJ_{-\nu}(z) + ks_{\mu,\nu}(z), \qquad \nu \neq \text{an integer}$$

$$(9.7)'\quad y = AJ_{\nu}(z) + BY_{\nu}(z) + ks_{\mu,\nu}(z), \qquad \nu \text{ a constant.}$$

In order to get an integral representation of $s_{\mu,\nu}(z)$, we use the method of variation of parameters to find a solution of (9.1). For ν not an integer, $J_{\nu}(z)$ and $J_{-\nu}(z)$ are linearly independent and we assume a solution of (9.1) to be

$$(9.8)\quad y = A(z)J_{\nu}(z) + B(z)J_{-\nu}(z), \qquad \nu \neq \text{ an integer.}$$

Then our two conditions are

$$(9.9)\quad \begin{array}{ll} \text{(i)} & A(z)'\,J_{\nu}(z) + B(z)'\,J_{-\nu}(z) = 0 \\ \text{(ii)} & A(z)'\,J_{\nu}(z)' + B(z)'\,J_{-\nu}(z)' = kz^{\mu-1} \end{array}.$$

Solving for $A(z)'$ we get

$$(9.10)\qquad A(z)' = \frac{\begin{vmatrix} 0 & J_{-\nu}(z) \\ kz^{\mu-1} & J_{-\nu}'(z) \end{vmatrix}}{\begin{vmatrix} J_{\nu}(z) & J_{-\nu}(z) \\ J_{\nu}'(z) & J_{-\nu}'(z) \end{vmatrix}}.$$

Noting that the denominator in (9.10) is the Wronskian of $J_{\nu}(z)$ and $J_{-\nu}(z)$, we use (4.5) to get

$$(9.11)\qquad A'(z) = -kz^{\mu-1} J_{-\nu}(z)\ \frac{-\pi z}{2\sin(\nu\pi)}$$

$$= \frac{k\pi z^{\mu} J_{-\nu}(z)}{2\sin(\nu\pi)} \ .$$

And finally after integrating we get

$$(9.12)\qquad A(z) = \frac{k\pi}{2\sin(\nu\pi)} \int^{z} z^{\mu} J_{-\nu}(z)dz \ .$$

In a similar manner B(z) is found to be

$$(9.13)\qquad B(z) = \frac{-k\pi}{2\sin(\nu\pi)} \int^{z} z^{\mu} J_{\nu}(z)dz \ .$$

Therefore

$$(9.14)\qquad y = \frac{k\pi}{2\sin(\nu\pi)} \left[J_{\nu} \int^{z} z^{\mu} J_{-\nu}(z)dz - J_{-\nu} \int^{z} z^{\mu} J_{\nu}(z)dz \right]$$

if ν is not an integer. In a similar manner we can get an expression valid for all ν, of the form

$$(9.14)'\qquad y = \frac{k\pi}{2}\left[Y_{\nu}(z) \int^{z} z^{\mu} J_{\nu}(z)dz - J_{\nu}(z) \int^{z} z^{\mu} Y_{\nu}(z)dz \right] \ .$$

In (9.14) and (9.14)', if both the numbers $\mu \pm \nu + 1$ have positive real parts, the lower limits may be taken to be zero. With the above restrictions on $\mu \pm \nu$ we have their sums and differences not equal to an odd negative integer. Upon expanding (9.14) as a series in ascending powers of z, we see that the powers of z are $\mu + 1$, $\mu + 3$, $\mu + 5$, $\cdots$, the same powers as those in (9.5). Further we see that the coefficient of $z^{\mu+1}$ in each case is the same. Therefore we have

$$(9.15) \qquad s_{\mu,\nu}(z) = \frac{\pi}{2\sin(\nu\pi)}\left[J_{\nu}\int_0^z z^{\mu} J_{-\nu}(z)dz - J_{-\nu}\int_0^z z^{\mu} J_{\nu}(z)dz\right], \qquad \nu \neq \text{an integer.}$$

Similarly it can be shown that

$$(9.16) \qquad s_{\mu,\nu}(z) = \frac{\pi}{2}\left[Y_{\nu}(z)\int_0^z z^{\mu} J_{\nu}(z)dz - J_{\nu}(z)\int_0^z z^{\mu} Y_{\nu}(z)dz\right]$$

for general ν.

10. The Lommel Function $S_{\mu,\nu}(z)$

In Section 9 we developed a solution of (9.1)' corresponding to the regular singular point $z = 0$. In this section we shall develope a solution corresponding to the irregular singular point $z = \infty$. For the associated homogeneous equation as shown in Ince [6], the solution can be expressed in the form

$$y = e^{\phi(z)}u(z) \tag{10.1}$$

where ϕ, the determining factor, is a polynomial in z; and u is an infinite series in $1/z$ which, though divergent, is asymptotic. However, as we are interested in getting a particular solution we do not need the determining factor ϕ. We shall develope a formal solution for (9.1) which is a divergent series except when it terminates. This series will terminate for certain values of the parameters μ and ν. For these values we shall develope the function $S_{\mu,\nu}(z)$. In a later section we shall show that $S_{\mu,\nu}(z)$, through a limiting process, can be defined when $s_{\mu,\nu}(z)$ is not defined (i.e. when $\mu \pm \nu$ is a negative integer).

We begin the development of $S_{\mu,\nu}(z)$ by assuming a solution of (9.1)' to be

$$y = \sum_{r=0}^{\infty} a_r z^{c-r} \tag{10.1'}$$

Formally differentiating and substituting (10.1)' and its first two derivatives in (9.1)' we get

$$(10.2)\qquad \sum_{r=2}^{\infty} a_{r-2}\left[(c+2-r)^2 - \nu^2\right] z^{c+2-r} +$$

$$+ \sum_{r=0}^{\infty} a_r z^{c+2-r} = kz^{\mu+1} .$$

Letting

$$(10.3)\qquad c + 2 = \mu + 1$$

we get

$$(10.4)\qquad c = \mu - 1$$

and the substitution of (10.4) in (10.2) gives us

$$(10.5)\qquad \sum_{r=2}^{\infty} a_{r-2}\left[(\mu + 1 - r)^2 - \nu^2\right] z^{\mu+1-r} +$$

$$+ \sum_{r=0}^{\infty} a_r z^{\mu+1-r} = kz^{\mu+1}$$

which, when we consider the coefficients of the different powers of z, gives

$$(10.5)' \quad \begin{aligned} &\text{(i)} \quad a_o = k \\ &\text{(ii)} \quad a_1 = 0 \\ &\text{(iii)} \quad a_r = -a_{r-2}\left[(\mu + 1 - r)^2 - \nu^2\right], \ r = 2,3,4,\cdots . \end{aligned}$$

From (ii) and (iii) above we see that $a_r = 0$, $r = 1,3,5,\cdots$. Therefore the formal solution of (9.1)' may be written in the form

$$(10.6) \qquad y = kz^{\mu-1}\left[1 - \frac{(\mu-1)^2 - \nu^2}{z^2} + \right.$$

$$\left. + \frac{[(\mu-1)^2 - \nu^2][(\mu-3)^2 - \nu^2]}{z^4} - \cdots\right].$$

Factoring the numerators of terms after the first we see that the factors are of the form $\mu \pm \nu - (2p + 1)$, $p = 0,1,2,\cdots$. Thus the series terminates when $\mu - \nu$ or $\mu + \nu$ is an odd positive integer. Otherwise the series diverges. Therefore (10.4) is a solution of (9.1)' if and only if the series terminates. In this case we denote the right-hand side of (10.6) by $kS_{\mu,\nu}(z)$, and $S_{\mu,\nu}(z)$ is called the second Lommel function.

Next we wish to express $S_{\mu,\nu}(z)$ in terms of $s_{\mu,\nu}(z)$ and Bessel functions. Letting $\mu - \nu = 2p + 1$ we have

$$(10.6)' \quad s_{\mu,\nu}(z) = z^{\mu-1}\left[1 + \sum_{m=1}^{p} \frac{(-1)^m}{z^{2m}} \times \right.$$

$$\times (\mu-\nu-1)(\mu-\nu-3) \cdots [\mu-\nu-(2m-1)] \times$$

$$\left. \times (\mu+\nu-1) \cdots \mu+\nu-(2m-1)\right]$$

$$= z^{\mu-1}\left[1 + \sum_{m=1}^{p} \frac{(-1)^m}{\frac{z}{2}^{2m}} \times \right.$$

$$\left. \times \left(\frac{\mu-\nu-1}{2}\right) \cdots \left(\frac{\mu-\nu+1-2m}{2}\right)\left(\frac{\mu+\nu-1}{2}\right) \cdots \left(\frac{\mu+\nu+1-2m}{2}\right)\right]$$

$$= z^{\mu-1} \sum_{m=0}^{p} \frac{(-1)^m}{\frac{z}{2}^{2m}} \times$$

$$\times \frac{\Gamma\left(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2}\right) \Gamma\left(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{\mu}{2} - \frac{\nu}{2} - m + \frac{1}{2}\right) \Gamma\left(\frac{\mu}{2} + \frac{\nu}{2} - m + \frac{1}{2}\right)}$$

$$= z^{\mu-1} \sum_{m=0}^{p} \frac{(-1)^m \Gamma(p+1)\, \Gamma(p+\nu+1)}{(z/2)^{2m}\, \Gamma(p-m+1)\, \Gamma(p+\nu-m+1)} \quad .$$

Next we replace m by p - m and get

$$s_{\mu,\nu}(z) = (-1)^p z^{\mu-1} \sum_{m=0}^{p} \frac{(-1)^m (z/2)^{2m-2p}\,\Gamma(p+1)\,\Gamma(p+\nu+1)}{\Gamma(m+1)\,\Gamma(m+\nu+1)}$$

$$= (-1)^p z^{\mu-1} \left[\sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu-\mu+1}}{\Gamma(m+1)\,\Gamma(m+\nu+1)} \right] \times$$

$$\times \Gamma(p+1)\,\Gamma(p+\nu+1) - (-1)^p z^{\mu-1} \times$$

$$\times \sum_{m=p+1}^{\infty} \frac{(-1)^m (z/2)^{2m-2p}\,\Gamma(p+1)\,\Gamma(p+\nu+1)}{\Gamma(m+1)\,\Gamma(m+\nu+1)}$$

$$= (-1)^p 2^{\mu-1}\,\Gamma\left(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2}\right) \Gamma\left(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2}\right) \times$$

$$\times J_\nu(z) - (-1)^p z^{\mu-1} \times$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^{m+p+1} (z/2)^{2m+2}\,\Gamma\left(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2}\right) \Gamma\left(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{\mu}{2} - \frac{\nu}{2} + m + \frac{3}{2}\right) \Gamma\left(\frac{\mu}{2} + \frac{\nu}{2} + m + \frac{3}{2}\right)}$$

$$= (-1)^p 2^{\mu-1}\,\Gamma\left(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2}\right) \Gamma\left(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2}\right) \times$$

$$\times J_\nu(z) + s_{\mu,\nu}(z)$$

$$= -2^{\mu-1}\,\Gamma\left(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2}\right) \Gamma\left(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2}\right) \times$$

$$\times \sin\left[-(p+\tfrac{1}{2})\pi\right] J_{\nu}(z) + s_{\mu,\nu}(z)$$

$$= -2^{\mu-1}\,\Gamma(\tfrac{\mu}{2}-\tfrac{\nu}{2}+\tfrac{1}{2})\,\Gamma(\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac{1}{2})\times$$

$$\times \frac{\cos\frac{1}{2}(\mu+\nu)\pi}{\sin|\nu\pi|} J_{\nu}(z) + s_{\mu,\nu}(z)$$

where ν is not an integer. Therefore when ν is not an integer and $\mu-\nu$ is an odd positive integer we have

$$\text{(10.7)}\quad S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + \frac{2^{\mu-1}\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})}{\sin|\nu\pi|}\times$$

$$\times\left[\cos\tfrac{1}{2}(\mu-\nu)\pi\cdot J_{-\nu}(z) - \cos\tfrac{1}{2}(\mu+\nu)\pi\cdot J_{\nu}(z)\right],$$

since $\cos\frac{1}{2}(\mu-\nu)\pi$ vanishes under these conditions. $S_{\mu,\nu}(z)$ is an even function of ν, thus

$$\text{(10.8)}\qquad S_{\mu,\nu}(z) = S_{\mu,-\nu}(z)\ ,$$

and the requirement that $\mu-\nu$ be an odd positive integer becomes equivalent to the requirement that $\mu-(-\nu)$ be an odd positive integer. Therefore $S_{\mu,\nu}(z)$ is defined as a terminating series if either $\mu+\nu$ or $\mu-\nu$ is an odd positive integer and if ν is not an integer. When ν is an integer we use the equivalent form

(10.9) $$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1}\,\Gamma(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2})\times$$

$$\times\Gamma(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2})\quad\left[\sin\frac{1}{2}(\mu-\nu)\pi \cdot J_{\nu}(z) - \right.$$

$$\left. - \cos\frac{1}{2}(\mu-\nu)\pi \cdot Y_{\nu}(z)\right] .$$

We adopt (10.7) and (10.9) as the general definitions of the second Lommel function $S_{\mu,\nu}(z)$.

11. The Recurrence Formulas

In this section we shall develope recurrence formulas for both the functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$. First we observe that by using (9.5) we get

$$(11.1)\qquad s_{\mu+2,\nu}(z) = \frac{z^{\mu+3}}{(\mu+3)^2-\nu^2} - \frac{z^{\mu+5}}{[(\mu+3)^2-\nu^2][(\mu+5)^2-\nu^2]} + \cdots .$$

From (11.1) we see that

$$(11.2)\qquad \frac{z^{\mu+1}}{(\mu+1)^2-\nu^2} - \frac{s_{\mu+2,\nu}(z)}{(\mu+1)^2-\nu^2} = s_{\mu,\nu}(z) .$$

We may write (11.2) as

$$(11.2)'\qquad s_{\mu+2,\nu}(z) = z^{\mu+1} - \left[(\mu+1)^2-\nu^2\right] s_{\mu,\nu}(z) .$$

Next we wish to consider

$$(11.3)\qquad \frac{d}{dz}\left[z^{\nu} s_{\mu,\nu}(z)\right] = \nu z^{\nu-1} s_{\mu,\nu}(z) + z^{\nu} s_{\mu,\nu}'(z) .$$

From (9.5) we see that

$$(11.4)\qquad z^{\nu} \frac{d}{dz}\left[s_{\mu,\nu}(z)\right] = z^{\nu}\left[\frac{(\mu+1)z^{\mu}}{(\mu+1)^2-\nu^2} - \right.$$

$$- \frac{(\mu+3)z^{\mu+2}}{(\quad +1)^2 - \quad ^2 \quad (\quad +3)^2 - \quad ^2} + \cdots \Bigg] .$$

We also have

$$(11.5) \quad \nu z^{\nu-1} s_{\mu,\nu}(z) = z^{\nu} \Bigg[\frac{\nu z^{\mu}}{(\mu+1)^2 - \nu^2} -$$

$$- \frac{\nu z^{\mu+2}}{[(\mu+1)^2 - \nu^2][(\mu+3)^2 - \nu^2]} + \cdots \Bigg] .$$

Combining (11.4) and (11.5) we find the right-hand side of (11.3) to be

$$(11.6) \qquad \nu z^{\nu-1} s_{\mu,\nu}(z) + z^{\nu} s_{\mu,\nu}'(z)$$

$$= z^{\nu} (\mu+\nu-1) s_{\mu-1,\nu-1}(z) .$$

Thus we have

$$(11.6)' \qquad s_{\mu,\nu}'(z) + \frac{\nu}{z} s_{\mu,\nu}(z)$$

$$= (\mu+\nu-1) s_{\mu-1,\nu-1}(z) ,$$

and similarly

$$(11.7)' \qquad s_{\mu,\nu}'(z) - \frac{\nu}{z} s_{\mu,\nu}(z)$$

$$= (\mu-\nu-1) s_{\mu-1,\nu+1}(z) .$$

Adding and subtracting (11.6) and (11.7) we get

$$(11.8)' \qquad 2s_{\mu,\nu}'(z) = (\mu+\nu-1)s_{\mu-1,\nu-1}(z) + (\mu-\nu-1)s_{\mu-1,\nu+1}(z) \quad ,$$

$$(11.9)' \qquad \frac{2\nu}{z} s_{\mu,\nu}(z) = (\mu+\nu-1)s_{\mu-1,\nu-1}(z) - (\mu-\nu-1)s_{\mu-1,\nu+1}(z) \quad .$$

Next we show that the recurrence formulas for $S_{\mu,\nu}(z)$ can be obtained from the above formulas by replacing functions of the type $s_{\mu,\nu}(z)$ by functions of the type $S_{\mu,\nu}(z)$. From (10.7) we have

$$(11.10)' \quad S_{\mu+2,\nu}(z) = z^{\mu+1} - \left[(\mu+1)^2 - \nu^2\right] s_{\mu,\nu}(z) - \left[(\mu+1)^2 - \nu^2\right] 2^{\mu-1} \times \frac{\Gamma(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2})\ \Gamma(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2})}{\sin(\nu\pi)} \times \left[\cos \tfrac{1}{2}(\mu-\nu)\pi \cdot J_{-\nu}(z) - \cos \tfrac{1}{2}(\mu+\nu)\pi \cdot J_{\nu}(z)\right]$$

$$= z^{\mu+1} - \left[(\mu+1)^2 - \nu^2\right] S_{\mu,\nu}(z) \quad .$$

We consider

(11.11) $$S_{\mu,\nu}'(z) = s_{\mu,\nu}'(z) +$$

$$+ \frac{2^{\mu-1}\,\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})}{\sin(\nu\pi)} \times$$

$$\times \left[\cos\tfrac{1}{2}(\mu-\nu)\pi \cdot J_{-\nu}'(z) - \cos\tfrac{1}{2}(\mu+\nu)\pi \cdot J_{\nu}'(z)\right]$$

$$= -\frac{\nu}{z} s_{\mu,\nu}(z) + (\mu+\nu-1)s_{\mu-1,\nu-1}(z) +$$

$$+ \frac{2^{\mu-1}\,\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})}{\sin(\nu\pi)} \times$$

$$\times \left[\cos\tfrac{1}{2}(\mu-\nu)\pi\left[-\frac{\nu}{z}J_{-\nu}(z) - J_{-(\nu-1)}(z)\right] -\right.$$

$$\left. - \cos\tfrac{1}{2}(\mu+\nu)\pi\left[-\frac{\nu}{z}\,J_{\nu}(z) + J_{\nu-1}(z)\right]\right]$$

$$= -\frac{\nu}{z} S_{\mu,\nu}(z) + (\mu+\nu-1)s_{\mu-1,\nu-1}(z) +$$

$$+ \frac{2^{\mu-1}\,\Gamma(\frac{\mu-1}{2}-\frac{\nu-1}{2}+\frac{1}{2})\,\Gamma(\frac{\mu-1}{2}+\frac{\nu-1}{2}+\frac{1}{2})\frac{1}{2}(\mu+\nu-1)}{-\sin(\nu-1)\pi} \times$$

$$\times \left[\cos\tfrac{1}{2}\left[(\mu-1)-(\nu-1)\right]\pi \cdot \left[-J_{-(\nu-1)}(z)\right] -\right.$$

$$\left. - \cos\tfrac{1}{2}\left[(\mu-1)+(\nu-1)\right]\pi \cdot \left[-J_{\nu-1}(z)\right]\right]$$

$$= -\frac{\nu}{z} S_{\mu,\nu}(z) + (\mu+\nu-1)S_{\mu-1,\nu-1}(z) .$$

Now we proceed as we did for $s_{\mu,\nu}(z)$ to get the recurrence formulas,

(11.12)' $S_{\mu,\nu}'(z) + (\nu/z)S_{\mu,\nu}(z) = (\mu+\nu-1)S_{\mu-1,\nu-1}(z)$,

(11.13)' $S_{\mu,\nu}'(z) - (\nu/z)S_{\mu,\nu}(z) = (\mu-\nu-1)S_{\mu-1,\nu+1}(z)$,

(11.14)' $(2\nu/z)S_{\mu,\nu}(z) = (\mu+\nu-1)S_{\mu-1,\nu-1}(z) -$

$- (\mu-\nu-1)S_{\mu-1,\nu+1}(z)$,

(11.15)' $2S_{\mu,\nu}(z) = (\mu+\nu-1)S_{\mu-1,\nu-1}(z) +$

$+ (\mu-\nu-1)S_{\mu-1,\nu+1}(z)$.

12. Lommel's Function When $\mu \pm \nu$ is an Odd Negative Integer

We have seen that $ks_{\mu,\nu}(z)$ and $kS_{\mu,\nu}(z)$ are solutions of (4.1) when $\mu \pm \nu$ is not an odd negative integer. Now we shall use $S_{\mu,\nu}(z)$ to develope a solution when $\mu \pm \nu$ is an odd negative integer. Since $S_{\mu,\nu}(z)$ is an even function of ν, it will suffice to consider the case when $\mu - \nu$ is a negative odd integer. First we shall express $S_{\nu-2p-1,\nu}(z)$ in terms of $S_{\nu-1,\nu}(z)$, for any positive integer p. Using the recurrence formula (11.10)' we see that if neither $(1-\nu)_p$ nor p vanish, then

$$(12.1)\quad S_{\nu-2p-1,\nu}(z) = \frac{-S_{\nu-2p+1,\nu}(z)}{(\nu-2p)^2-\nu^2} + \frac{z^{\nu-2p}}{(\nu-2p)^2-\nu^2}$$

$$= \frac{z^{\nu-2p}}{(-2p)(2\nu-2p)} - $$

$$- \frac{z^{\nu-2p+2} - S_{\nu-2p+3,\nu}(z)}{(-2p)(2\nu-2p)(-2p+2)(2\nu-2p+2)}$$

$$= \frac{z^{\nu-2p}}{2^2(-p)(\nu-p)} - $$

$$- \frac{z^{\nu-2p+2}}{2^{2+2}(-p)(-p+1)(\nu-p)(\nu-p+1)} +$$

$$+ \frac{(-1)^2 S_{\nu-2p+3,\nu}(z)}{2^{2+2}(-p)_2(\nu-p)_2}$$

$$= \sum_{m=0}^{p-1} \frac{(-1)^m z^{\nu-2p+2m}}{2^{2+2m}(-p)_{m+1}(\nu-p)_{m+1}} + \frac{(-1)^p S_{\nu-1,\nu}(z)}{2^{2p}(-p)_p(\nu-p)_p}$$

$$= \sum_{m=0}^{p-1} \frac{(-1)^m z^{\nu-2p+2m}}{2^{2+2m}(-p)_{m+1}(\nu-p)_{m+1}} +$$

$$+ \frac{(-1)^p S_{\nu-1,\nu}(z)}{2^{2p}(-p)(-p+1)\cdots(-1)(\nu-p)\cdots(\nu-1)}$$

$$= \sum_{m=0}^{p-1} \frac{(-1)^m z^{\nu-2p+2m}}{2^{2+2m}(-p)_{m+1}(\nu-p)_{m+1}} + \frac{(-1)^p S_{\nu-1,\nu}(z)}{2^{2p} p!(1-\nu)_p} .$$

Therefore if the above conditions on ν and p hold, we know that for any $S_{\mu,\nu}(z)$ for which $\mu-\nu$ is an odd negative integer, we can write an equivalent expression in terms of $S_{\mu-1,\nu}(z)$. Next we consider $S_{\nu-1,\nu}(z)$. Using (11.10)' we see that

$$(12.2) \quad S_{\mu,\nu}(z) = \frac{z^{\mu+1} - S_{\mu+2,\nu}(z)}{(\mu-\nu+1)(\mu+\nu+1)} , \qquad \mu = \nu - 1$$

is indeterminate. However, since the equation (10.7) is holomorphic in μ, $S_{\mu+2,\nu}(z)$ is a holomorphic function of μ near $\mu = \nu - 1$. Therefore we define $S_{\nu-1,\nu}(z)$ in the following manner:

$$(12.3)\quad S_{\nu-1,\nu}(z) = \lim_{\mu\to\nu-1}\left[\frac{z^{\mu+1} - S_{\mu+2,\nu}(z)}{(\mu-\nu+1)(\mu+\nu+1)}\right] ,$$

and use L'Hospital's theorem to get

$$(12.4)\quad S_{\nu-1,\nu}(z) = \left[\frac{z^{\mu+1}\log z - \frac{\partial}{\partial\mu}\left(S_{\mu+2,\nu}(z)\right)}{2\mu+2}\right]_{\mu=\nu-1}$$

$$= \frac{1}{2\nu}\left[z^{\nu}\log z - \frac{\partial}{\partial\mu}\left(S_{\mu+2,\nu}(z)\right)\right]_{\mu=\nu-1} .$$

From (10.9) we see that $S_{\mu+2,\nu}(z)$ can be expressed as the sum of three terms. Differentiating each of these with respect to μ we get

$$(12.5)\quad \frac{\partial}{\partial\mu}\left[S_{\mu+2,\nu}(z)\right]_{\mu=\nu-1} = \frac{\partial}{\partial\mu}\left[z^{\mu+1} \times\right.$$

$$\left.\times \sum_{m=0}^{\infty}\frac{(-1)^m (z/2)^{2m+2}}{\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{3}{2}\right)_{m+1}\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{2}\right)_{m+1}}\right]_{\mu=\nu-1}$$

$$= \left\{z^{\nu}\log z\sum_{m=0}^{\infty}\frac{(-1)^m (z/2)^{2m+2}}{(m+1)!(\nu+1)\cdots(\nu+m+1)} + \right.$$

$$+ z^{\nu}\sum_{m=0}^{\infty}\frac{(-1)^m (z/2)^{2m+2}}{\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{3}{2}\right)_{m+1}\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{2}\right)_{m+1}} \times$$

$$\times \left(\frac{\mu}{2} - \frac{\nu}{2} + \frac{3}{2}\right)_{m+1} \left(\frac{\mu}{2} + \frac{\nu}{2} + \frac{3}{2}\right)_{m+1} \times$$

$$\times \frac{\partial}{\partial \mu}\left[\left(\frac{\mu}{2} - \frac{\nu}{2} + \frac{3}{2}\right)^{-1}_{m+1} \left(\frac{\mu}{2} + \frac{\nu}{2} + \frac{3}{2}\right)^{-1}_{m+1} \right]\Bigg\}_{\mu = \nu - 1}$$

$$= z^{\nu} \log z \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+2}\,\Gamma(\nu+1)}{(m+1)!\,\Gamma(\nu+m+2)} +$$

$$+ z^{\nu} \sum_{m=0}^{\infty} \left\{ \frac{(-1)^m (z/2)^{2m+2}\,\Gamma(\nu+1)}{(m+1)!\,\Gamma(\nu+m+2)} \times \right.$$

$$\times \frac{\Gamma(\frac{\mu}{2} - \frac{\nu}{2} + \frac{3}{2} + m + 1)}{\Gamma(\frac{\mu}{2} - \frac{\nu}{2} + \frac{3}{2})} \times \frac{\Gamma(\frac{\mu}{2} + \frac{\nu}{2} + \frac{3}{2} + m + 1)}{\Gamma(\frac{\mu}{2} + \frac{\nu}{2} + \frac{3}{2})} \times$$

$$\left. \times \frac{\partial}{\partial \mu}\left[\left(\frac{\Gamma(\frac{\mu}{2} - \frac{\nu}{2} + \frac{3}{2} + m + 1)\,\Gamma(\frac{\mu}{2} + \frac{\nu}{2} + \frac{3}{2} + m + 1)}{\Gamma(\frac{\mu}{2} - \frac{\nu}{2} + \frac{3}{2})\,\Gamma(\frac{\mu}{2} + \frac{\nu}{2} + \frac{3}{2})} \right)^{-1} \right] \right\}_{\mu = \nu - 1}$$

$$= z^{\nu}\,\Gamma(\nu+1) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+2}}{(m+1)!\,\Gamma(\nu+m+2)} \times$$

$$\times \left\{ \log z - \frac{\partial}{\partial \mu}\left[\log \Gamma(\frac{\mu}{2} - \frac{\nu}{2} + \frac{3}{2} + m + 1) + \right. \right.$$

$$+ \log \Gamma(\tfrac{\mu}{2} + \tfrac{\nu}{2} + \tfrac{3}{2} + m + 1) - \log \Gamma(\tfrac{\mu}{2} - \tfrac{\nu}{2} + \tfrac{3}{2}) -$$

$$- \log \Gamma(\tfrac{\mu}{2} + \tfrac{\nu}{2} + \tfrac{3}{2})\Big] \Big\}_{\mu = \nu - 1} .$$

Now by the use of the notation

$$\Psi(\mu) = \Gamma'(\mu)/\Gamma(\mu)$$

we have

$$(12.6) \quad \frac{\partial}{\partial\mu}\Big[s_{\mu+2,\nu}(z)\Big]_{\mu=\nu-1} = (z/2)^{\nu}\ \Gamma(\nu+1) \times$$

$$\times \sum_{m=1}^{\infty} \frac{(-1)^{m-1}(z/2)^{2m}}{m!\,\Gamma(\nu+m+1)} \Big[2 \log z + \Psi(\nu+1) + \Psi(1) -$$

$$- \Psi(\nu+m+1) - \Psi(m+1)\Big] .$$

Next we need

$$(12.7) \quad -\frac{\partial}{\partial\mu}\Big[2^{\mu+1}\Gamma(\tfrac{\mu}{2} - \tfrac{\nu}{2} + \tfrac{3}{2})\ \Gamma(\tfrac{\mu}{2} + \tfrac{\nu}{2} + \tfrac{3}{2}) \times$$

$$\times \cos \tfrac{1}{2}(\mu - \nu + 2)\pi \cdot Y_{\nu}(z)\Big]_{\mu=\nu-1}$$

$$= 2^{\nu}\ \Gamma(\nu+1)\ \frac{\pi}{2} Y_{\nu}(z)$$

$$= 2^{\nu-1}\ \Gamma(\nu+1) Y_{\nu}(z)$$

and

(12.8) $$\frac{\partial}{\partial\mu}\Big[2^{\mu+1}\,\Gamma(\tfrac{\mu}{2}-\tfrac{\nu}{2}+\tfrac{3}{2})\,\Gamma(\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac{3}{2})\times$$

$$\times\sin\tfrac{1}{2}(\mu-\nu+2)\pi\cdot J_\nu(z)\Big]_{\mu=\nu-1}$$

$$=\Big\{2^\nu\,\Gamma(\nu+1)\log 2+2^\nu\,\Gamma(\tfrac{\mu}{2}-\tfrac{\nu}{2}+\tfrac{3}{2})\times$$

$$\times\,\Gamma(\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac{3}{2})\times$$

$$\times\;\frac{\frac{\partial}{\partial\mu}\Big[\Gamma(\tfrac{\mu}{2}-\tfrac{\nu}{2}+\tfrac{3}{2})\,\Gamma(\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac{3}{2})\Big]}{\Gamma(\tfrac{\mu}{2}-\tfrac{\nu}{2}+\tfrac{3}{2})\,\Gamma(\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac{3}{2})}\Big\}_{\mu=\nu-1}\;J_\nu(z)$$

$$=\;2^\nu\,\Gamma(\nu+1)\sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{2m+\nu}}{m!\,\Gamma(\nu+m+1)}\times$$

$$\times\Big[\log 2+\tfrac{1}{2}\Psi(1)\;+\;\tfrac{1}{2}\Psi(\nu+1)\Big]\;.$$

Putting (12.6), (12.7), and (12.8) into (12.4) we get

(12.9) $$S_{\nu-1,\nu}(z)=\frac{1}{2\nu}\Big\{z^\nu\;\log z-\Big[\tfrac{1}{2}\,z^\nu\,\Gamma(\nu+1)\times$$

$$\times\sum_{m=1}^{\infty}\frac{(-1)^{m-1}(z/2)^{2m}}{m!\,\Gamma(\nu+m+1)}\;(2\log z+\Psi(1)+\Psi(\nu+1)\;-$$

$$- \Psi(\nu+m+1) - \Psi(m+1) \,) + 2^{\nu-1}(z/2)^{\nu}\,\Gamma(\nu+1)\times$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m \, (z/2)^{2m}}{m!\,\Gamma(\nu+m+1)} \quad (2\log 2 + \Psi(1) + \Psi(\nu+1)\,) +$$

$$+ \; 2^{\nu-1}\pi\Gamma(\nu+1)Y_{\nu}(z)\Big]\Big\}$$

$$= \frac{z^{\nu}\log z}{2\nu} + \frac{1}{4\nu}\; z^{\nu}\,\nu\,\Gamma(\nu)\times$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!\,\Gamma(\nu+m+1)} \quad (2\log\frac{z}{2} - \Psi(m+1) -$$

$$- \Psi(\nu+m+1)\,) \;-\frac{1}{4\nu}\; z^{\nu}\; 2\log z - \frac{1}{2\nu}\,2^{\nu-1}\times$$

$$\times\pi\,\nu\,\Gamma(\nu)Y_{\nu}(z)$$

$$= \frac{1}{4}\; z^{\nu}\;\Gamma(\nu) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!\,\Gamma(\nu+m+1)} \;(2\log(z/2) -$$

$$- \Psi(m+1) - \Psi(\nu+m+1)\,) \;\; - 2^{\nu-2}\pi\,\Gamma(\nu)Y_{\nu}(z).$$

This formula holds when ν is neither zero nor a negative integer. We note that $S_{\mu,\nu}(z)$ is an even function, and use (12.9) to get

(12.10) $\quad S_{-\nu-1,\nu}(z) \;=\; S_{-\nu-1,-\nu}(z)$

$$= \frac{1}{4} z^{-\nu} \Gamma(-\nu) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!\,\Gamma(-\nu+m+1)} \left(2 \log \frac{z}{2} - \right.$$

$$\left. - \Psi(m+1) - \Psi(-\nu+m+1) \right) - 2^{-\nu-2} \pi \Gamma(-\nu) Y_{-\nu}(z) \quad .$$

Thus if ν is positive we use (12.9) and if ν is negative we use (12.10).

Finally we look at the case where $\nu = 0$. From (12.3) we have

$$(12.11) \quad s_{-1,0}(z) = \lim_{\mu \to -1} \left[\frac{z^{\mu+1} - s_{\mu+2,0}(z)}{(\mu+1)^2} \right]$$

$$= \frac{1}{2} \frac{\partial^2}{\partial \mu^2} \left[z^{\mu+1} - s_{\mu+2,0}(z) \right]_{\mu=-1} \quad .$$

By a procedure similar to that used in the development of (12.9) we get

$$(12.12) \quad s_{-1,0}(z) = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{(m!)^2} \left[\left(\log \frac{z}{2} - \Psi(m+1)\right)^2 - \right.$$

$$\left. - \frac{1}{2} \Psi'(m+1) + \frac{\pi^2}{4} \right] \quad .$$

13. The Asymptotic Expansion of $S_{\mu,\nu}(z)$

We shall find the asymptotic expansion of $S_{\mu,\nu}(z)$ as $|z|$ tends to ∞ in an appropriate sector of the z-plane using Barne's method. First we shall consider

$$(13.1) \qquad I(z) = \frac{-z^{\mu-1}}{2\pi i} \times$$

$$\times \int_{-\infty i - p + \frac{1}{2}}^{\infty i - p + \frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{\mu}{2} + \frac{\nu}{2} - s)\, \Gamma(\frac{1}{2} - \frac{\mu}{2} - \frac{\nu}{2} - s)\, \pi (\frac{z}{2})^{2s}}{\Gamma(\frac{1}{2} - \frac{\mu}{2} + \frac{\nu}{2})\, \Gamma(\frac{1}{2} - \frac{\mu}{2} - \frac{\nu}{2}) \sin(s\pi)}\, ds$$

with p chosen a large enough integer such that the only poles to the left of the contour are ones involving $\sin(s\pi)$. We shall show that

$$(13.2) \qquad I(z) = O(z^{\mu-2p})$$

and that

$$(13.3) \qquad I(z) = z^{\mu-1} \left[1 - \frac{(\mu-1)^2 - \nu^2}{z^2} + \cdots + \right.$$

$$+ (-1)^{p-1} \frac{[(\mu-1)^2 - \nu^2] \cdots [[\mu-(2p-3)]^2 - \nu^2]}{z^{2(p-1)}} -$$

$$- S_{\mu,\nu}(z) \quad ,$$

thus showing that the difference between $S_{\mu,\nu}(z)$ and the sum of the first m terms of the series

$$(13.4) \quad z^{\mu-1}\left[1 - \frac{(\mu-1)^2-\nu^2}{z^2} + \frac{[(\mu-1)^2-\nu^2][(\mu-3)^2-\nu^2]}{} \ldots\right]$$

(for $m \geq p$) is $O(z^{\mu-2m})$ whereas the least degree of these first m terms is $z^{\mu-2m+1}$.

For convenience we let $\frac{1}{2} - \frac{\mu}{2} + \frac{\nu}{2} = a$ and $\frac{1}{2} - \frac{\mu}{2} - \frac{\nu}{2} = b$. Under this substitution our integrand becomes

$$(13.5) \qquad \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi\,(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin(s\pi)} \quad .$$

Next we consider the order of the integrand as $|s|$ becomes large. We let

$$(13.6) \qquad s = Ri - p + \frac{1}{2} \quad ;$$

and we let R tend to ∞ or $-\infty$ depending on which is applicable. Since

$$(13.7) \quad |\sin(s\pi)| = \frac{e^{i(Ri-p+\frac{1}{2})\pi} - e^{-i(Ri-p+\frac{1}{2})\pi}}{2i} \leq \frac{e^{-R\pi} + e^{R\pi}}{2} \; ,$$

we have

$$(13.7)' \qquad |\sin(s\pi)| = O(\exp(\pi|R|)) \quad .$$

Also

$$(13.8)\quad |(z/2)^{2s}| = |(z/2)^{2Ri-2p+1}|$$

$$= |e^{(2Ri-2p+1)(\log|z/2| + i\arg(z))}|$$

$$= e^{(-2p+1)\log|z/2| - 2R\arg(z)}$$

gives

$$(13.8)'\quad |(z/2)^{2s}| = O(\exp(-2R\arg(z)))\quad .$$

Using the definition(see Whittaker and Watson [11]),

$$(13.9)\quad |\Gamma(c+s)| = \exp\Big[(s + c - \tfrac{1}{2})(\log|s| + i\arg(s)) -$$

$$- s + \tfrac{1}{2}\log 2\pi\Big]\quad ,$$

where c is a constant; $|\arg(s)| \leq \pi - \delta, \delta > 0$; and $|s|$ tends to ∞ ; we get

$$(13.10)\quad |\Gamma(a-s)| = (2\pi)^{\frac{1}{2}}\ \Big|\exp\Big[(-s + a - \tfrac{1}{2})(\log|-s| + i\arg(-s)) + s\Big]\Big|$$

$$= (2\pi)^{\frac{1}{2}}\ \Big|\exp\Big[(-Ri + p + a - 1)(\log\sqrt{R^2 + \left(p - \tfrac{1}{2}\right)^2} +$$

$$+ i\arg(-s)) + Ri - p + \tfrac{1}{2}\Big]\Big|\quad .$$

Therefore

$$|\Gamma(a-s)| = O\left[\exp\left((p+a-1)\log\sqrt{R^2+\left(p-\frac{1}{2}\right)^{\frac{1}{2}}} + \right.\right.$$

$$\left.\left. + R\arg(-s) - p + \frac{1}{2}\right)\right]$$

$$= O\left[\,|R|^{p+a-1}\exp(R\arg(-s))\,\right] \quad .$$

Thus we have

$$(13.11) \qquad \left|\frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin(s\pi)}\right| = O\left[|R|^{2p+a+b-2} \times \right.$$

$$\left. \times \exp(-2R\arg(z) - \pi|R| + 2R\arg(-s))\right]$$

where the last two terms in the exponent are negative for large $|s|$.

Now we shall show that, if $|\arg(z)| < \pi$, then

$$(13.12) \qquad \left|\frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin(2\pi)}\right| < \frac{1}{|R|^2} \qquad , \qquad |R| < R_o$$

where R_o is a positive number. First we consider the order of (13.11) as R tends to ∞. We let $s = R_1 e^{i\theta}$. Then we note that as R tends to ∞, R_1 tends to ∞ and θ tends to $\frac{\pi}{2}$. When R tends to $-\infty$, R_1 tends to ∞ and θ tends to $-\frac{\pi}{2}$. Considering the term $2R\arg(-s)$, we see that if R tends to ∞, $\arg(-s)$ tends to $-\frac{\pi}{2}$ and $2R\arg(-s)$ tends to $-\pi R$. If R tends to $-\infty$, then $2R\arg(-s)$ tends to $-\pi R$. On the other hand, if we maximize $-2R\arg(z)$, we get $2|R|(\pi-\delta)$ where δ is a fixed positive number. The above implies that

$$(13.13) \quad \left| \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi\,(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin(s\pi)} \right| = O\left[\,|R|^{2p+a+b-2}\exp(-2\,|R|\,\delta\,)\right]$$

for large $|R|$. From (13.13) we see that for large $|R|$ we have (13.12). But the fact that

$$\int_{R_o}^{\infty} \frac{1}{|R|^2}\,dr$$

converges implies that our integral converges uniformly by the M-test.

Since I(z) converges uniformly for $|\arg z| < \pi$, we have

$$(13.14) \quad |I(z)| \leq \left|\frac{-z^{\mu-1}}{2}\right| \lim_{R\to\infty} \int_{-Ri-p+\frac{1}{2}}^{Ri-p+\frac{1}{2}} \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi}{\Gamma(a)\,\Gamma(b)\,\sin(s\pi)} \times$$

$$\times \left| e^{(2Ri-2p+1)(\log\,|(z/2)|\,+\,i\,\arg(z))} \right| |ds|$$

$$\leq \left|\frac{z^{\mu-1}}{2\pi}\right| \left|\frac{z}{2}\right|^{-2p+1} \times$$

$$\times \left| \lim_{R\to\infty} \int_{-Ri-p+\frac{1}{2}}^{Ri-p+\frac{1}{2}} \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi}{\Gamma(a)\,\Gamma(b)\,\sin(s\pi)}\, e^{-2R(\pi-\delta)}\, ds \right|$$

where the limit factor converges. Therefore we have

$$(13.15) \qquad I(z) = O(z^{\mu-2p}) \quad .$$

Next we wish to evaluate I(z) using the Residue theorem. First we note that we have a finite number of poles in each unit interval to the right of our contour and that the poles in each interval are respectively one unit distance from the ones in the preceding interval. Therefore we choose a radius R and center $-p + \frac{1}{2}$ such that a semicircle drawn to the right of our contour passes through no pole of the integrand. We let R tend to ∞ by increasing it one unit at a time. We next consider the integral around the closed contour as R tends to ∞ and let c_R represent the path along the semicircle from $-p + \frac{1}{2} - Ri$ to $-p + \frac{1}{2} + Ri$. Then

$$(13.16) \qquad \left[I(z) + \frac{z^{\mu-1}}{2\pi i} \int_{c_R} \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi\,(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin(s\pi)}\, ds \right] \Big/ \, z^{\mu-1}$$

is equal to the sum of the residues of the poles of the integrand inside the closed contour. We shall show that the integral over the semicircle contributes nothing to the integral over the closed path when R tends to ∞.

In order to do this, we shall investigate the order of the integrand on the semicircle when

$$(13.17)\qquad \text{(i)}\quad s = R_1 e^{i\theta} \qquad \text{(ii)}\quad = \xi + \eta i$$

and

$$(13.18)\qquad z = ke^{i\Psi} \quad .$$

Using (13.17)(i) we have

$$(13.19)\quad |\sin s\pi| = \left| \frac{e^{iR_1(\cos\theta + i\sin\theta)\pi} - e^{-iR_1(\cos\theta + i\sin\theta)\pi}}{2i} \right|$$

$$\leq \left| \frac{e^{-R_1 \pi \sin\theta} + e^{R_1 \pi \sin\theta}}{2} \right| \quad .$$

Therefore

$$(13.19)'\qquad |\sin s\pi| = O(\exp(R_1\pi)) \quad .$$

When we consider $|(z/2)^{2s}|$, we use (13.17)(ii) and (13.18) to get

$$(13.20)\quad |(z/2)^{2s}| = |\exp(2\xi + 2i\eta)(\log|k/2| + i\arg(z))|$$

$$= O(|k/2|^{2\xi} \exp(-2\eta\Psi)) \quad ,$$

and we use (13.17) and (13.18) to get

$$(13.21)\quad |\Gamma(a-s)| = (2\pi)^{\frac{1}{2}} \exp\Big[(-\xi - i\eta + a - \tfrac{1}{2})(\log R_1 +$$

$$+ i \arg(-s)) + \xi + \eta i \Big]$$

$$= O\left[R_1^{-\xi + a - \frac{1}{2}} \exp(\eta \arg(-s) + \xi)\right], \ |\arg(-s)| < \pi.$$

Therefore the order of the integrand on the semicircle, except where $|\arg(-s)| = \pi$, is

(13.22) $$\left|\frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin(s\pi)}\right|$$

$$= O\Big[R_1^{-2\xi + a + b - 1} |k/2|^{2\xi} e^{2\xi} \exp(2\eta \arg(-s) -$$

$$- 2\eta\Psi - R_1\pi) \Big] .$$

As R tends to ∞, R_1 tends to ∞ and we have three cases:

(13.23)
(i) ξ tends to ∞, $|\eta|$ is finite,
(ii) $|\eta|$ tends to ∞, $|\xi|$ is finite,
(iii) $|\eta|$ tends to ∞, $|\xi|$ tends to ∞ .

Now we shall examine (13.23) case by case. In case (i) since $|\eta|$ is finite and $|\xi|$ is infinite we have

(13.24) $$\left|\frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin(s\pi)}\right| = O\left[\frac{(M)^{\xi}}{(R_1)^{2\xi + c}}\right] ,$$

where R_1 tends to ∞, ξ tends to ∞, $c = a+b-1$, and $M = (|k/2|e)^2$. In case (ii) where $|\eta|$ is infinite and $|\xi|$ is finite we have

$$(13.25) \quad \left|\frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi\,(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\ \sin|s\pi|}\right| = 0(R_1^{\ c_1} \ \exp(2\,\eta\,\arg(-s) - $$

$$- 2\eta\Psi - R_1\pi) \ , \qquad c_1 = -2\,\xi + a + b - 1 \ .$$

Since $R_1 < 2\eta$ and $\eta < R_1$ we have

$$(13.26) \quad \left|\frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi\,(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\ \sin|s\,\pi|}\right| = 0(\,\eta^{\ c_1} \exp(2\,\eta\,\arg(-s) - $$

$$- 2\eta\,\Psi - \pi|\eta|)\,) \ .$$

Since η and arg(-s) always have opposite sign, the term $2\,\eta\,\arg(-s)$ is always negative. As R tends to ∞, the minimum value of $|\arg(-s)|$ tends to $\frac{\pi}{2}$. Therefore the maximum value of $2\,\eta\,\arg(-s)$ tends to $-\pi|\eta|$ as R tends to ∞. Since $|\arg(z)| < \pi$ we have the max value of $-2\eta\Psi = 2\,|\eta|(\pi - \delta)$, $\delta > 0$. Therefore

$$(13.27) \quad \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi\,(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\ \sin|s\,\pi|} = 0(\eta^{\ c_1} \ \exp(-2\,|\eta|\,\delta\,)\,)$$

$$= 0(\eta^{\ c_1}/e^{2|\eta|\delta}\,), \quad \eta \text{ tends to } \infty .$$

In case (iii) when both ξ and $|\eta|$ are infinite we consider cases (i) and (ii) and get

$$(13.28) \quad \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin|s\pi|} = O\left(\frac{(M)^{\xi}}{(R_1)^{2\xi+c}} \cdot \frac{\eta^{c_1}}{e^{2|\eta|\delta}} \right),$$

where M, c, c_1, δ, are constants; and ξ, $|\eta|$, R_1 all tend to ∞. Next we consider the integral

$$(13.29) \quad \left| \int_{c_R} \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin|s\pi|}\, ds \right|$$

$$\leq \pi R\Big(\max_{s \in c_R} \left| \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin|s\pi|} \right| \Big) .$$

But the right-hand side of (13.28) tends to zero as R tends to ∞. Therefore we have shown that the integral over the semicircle contributes nothing to the integral over the closed path when R tends to ∞ except possibly in a neighborhood of the point where the $|\arg(-s)| = \pi$. Since the contour does not pass through a pole, we can bound the integrand in a neighborhood of $s = R_1$,

$$(13.30) \quad \left| \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin|s\pi|} \right| < M ,$$

for any given radius. Then we have

$$(13.31) \quad \left| \int_{\Delta s} \frac{\Gamma(a-s)\,\Gamma(b-s)\,\pi(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin|s\pi|}\, ds \right| \leq \bar{M} \cdot \Delta s$$

where Δs tends to zero. Therefore we prove that the integral over the semicircle tends to zero when R tends to ∞.

Since the integral contributes nothing on the semicircle, we know that I(z) is equal to the sum of the residues of

$$(13.32) \qquad z^{\mu-1} \frac{\Gamma(a-s)\Gamma(b-s)\pi(z/2)^{2s}}{\Gamma(a)\Gamma(b)\sin(s\pi)}$$

at the poles

$$(13.33) \qquad \begin{array}{l} 0, -1, \cdots, -(p-1) \\ 1, 2, 3, \cdots \\ a, a+1, a+2, \cdots \\ b, b+1, b+2, \cdots \quad . \end{array}$$

Since we integrate in the positive sense, we consider

$$(13.34) \quad I(z) = \frac{-1}{2\pi i} \int_{-\infty i-p+\frac{1}{2}}^{\infty i-p+\frac{1}{2}} z^{\mu-1} \frac{\Gamma(a-s)\Gamma(b-s)\pi(z/2)^{2s}}{\Gamma(a)\Gamma(b)\sin(s\pi)} ds$$

$$= \frac{1}{2\pi i} \int_{\infty i-p+\frac{1}{2}}^{-\infty i-p+\frac{1}{2}} F(s)\, ds$$

which is equal to the sum of the residues of F(s) at the poles above, where F(s) is the integrand of (13.34). In the first set of (13.33) we have simple poles $-m$ where $m = 0, 1, \cdots, (p-1)$. Therefore

$$(13.35) \quad \mathrm{Res}\,(F(s),-m) = z^{\mu-1} \frac{\Gamma(a+m)\Gamma(b+m)\pi(z/2)^{-2m}}{\Gamma(a)\Gamma(b)\,\pi\cos(-m\pi)} ,$$

$$(13.36) \quad \sum_{m=0}^{p-1} \mathrm{Res}\,(F(s),-m) = z^{\mu-1}\Big[1 - \frac{\Gamma(a+1)\Gamma(b+1)}{(z/2)^2\Gamma(a)\Gamma(b)} +$$

$$+ \cdots + (-1)^{p-1} \frac{\Gamma(a+p-1)\Gamma(b+p-1)}{(z/2)^{2p-2}\Gamma(a)\Gamma(b)}\Big]$$

$$= z^{\mu-1}\Big[1-\frac{ab}{(z/2)^2} + \cdots + (-1)^{p-1} \times$$

$$\times \frac{(ab)\cdots((a+p-2)(b+p-2))}{(z/2)^{2p-2}}\Big]$$

$$= z^{\mu-1}\Big[1- \frac{(1-\mu+\nu)(1-\mu-\nu)}{z^2} + \cdots +$$

$$+ (-1)^{p-1} \frac{[(\mu-1)^2-\nu^2]\cdots[(\mu-(2p-3))^2-\nu^2]}{z^{2p-2}}\Big] .$$

For the second set of poles, $m = 1, 2, 3, \cdots$, we have

$$(13.37) \quad \sum_{m=1}^{\infty} \mathrm{Res}\,(F(s),m) = z^{\mu-1} \sum_{m=1}^{\infty} \frac{\Gamma(a-m)\Gamma(b-m)\pi(z/2)^{2m}}{\Gamma(a)\Gamma(b)\,\pi\cos(m\pi)} .$$

Using $\Gamma(p)\Gamma(1-p) = \pi\csc\pi p$ we get

$$(13.38) \quad \sum_{m=1}^{\infty} \text{Res }(F(s),m) = z^{\mu-1} \sum_{m=1}^{\infty} \frac{(\frac{z}{2})^{2m}}{(-1)^m} \times$$

$$\times \frac{\Gamma(\frac{1}{2} - \frac{\mu}{2} + \frac{\nu}{2} - m)\,\Gamma(\frac{1}{2} - \frac{\mu}{2} - \frac{\nu}{2} - m)}{\Gamma(\frac{1}{2} - \frac{\mu}{2} + \frac{\nu}{2})\,\Gamma(\frac{1}{2} - \frac{\mu}{2} - \frac{\nu}{2})}$$

$$= z^{\mu-1} \sum_{m=1}^{\infty} \frac{(-1)^m (\frac{z}{2})^{2m}\, \Gamma(\frac{1}{2} + \frac{\mu}{2} - \frac{\nu}{2})}{\Gamma(\frac{1}{2} + \frac{\mu}{2} - \frac{\nu}{2} + m)} \times$$

$$\times \frac{\Gamma(\frac{1}{2} + \frac{\mu}{2} + \frac{\nu}{2})}{\Gamma(\frac{1}{2} + \frac{\mu}{2} + \frac{\nu}{2} + m)}$$

since

$$(13.39) \qquad \frac{\pi \csc[(a-m)\pi]\ \pi \csc[(b-m)\pi]}{\pi \csc(a\pi)\ \pi \csc(b\pi)} = 1 \quad .$$

Then replacing m by $m+1$ we get

$$(13.40) \quad \sum_{m=1}^{\infty} \text{Res }(F(s),m) = -z^{\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{2m+2}\, \Gamma(\frac{1}{2} + \frac{\mu}{2} - \frac{\nu}{2})}{\Gamma(\frac{3}{2} + \frac{\mu}{2} - \frac{\nu}{2} + m)} \times$$

$$\times \frac{\Gamma(\frac{1}{2} + \frac{\mu}{2} + \frac{\nu}{2})}{\Gamma(\frac{3}{2} + \frac{\mu}{2} + \frac{\nu}{2} + m)}$$

$$= -s_{\mu,\nu}(z) \quad .$$

The sum of the residues at the poles, $a \pm m$, $m = 0,1,2,\cdots$, is given by

$$(13.41) \qquad \sum_{m=0}^{\infty} \operatorname{Res}\,(F(s), a+m)$$

$$= z^{\mu-1} \sum_{m=0}^{\infty} \lim_{s\to a+m} \frac{(s-(a+m))\,\Gamma(a-s)\,\Gamma(b-s)\,\pi\,(z/2)^{2s}}{\Gamma(a)\,\Gamma(b)\,\sin(s\pi)}$$

$$= z^{\mu-1} \sum_{m=0}^{\infty} \lim_{s\to a+m} \frac{(s-(a+m))\,\Gamma((a+m+1)-s)\,\Gamma(b-s)\,\pi\,(z/2)^{2s}}{(a-s)(a+1-s)\cdots(a+m-s)\,\Gamma(a)\,\Gamma(b)\,\sin(s\pi)}$$

$$= -z^{\mu-1} \sum_{m=0}^{\infty} \frac{\Gamma(1)\,\Gamma(-\nu-m)}{(-1)^m\, m!\,\Gamma(\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2})\,\Gamma(\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2})} \times$$

$$\times \frac{\pi(z/2)^{1-\mu+\nu+2m}}{\sin(\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2})\pi \cdot \cos(m\pi)}$$

$$= \frac{-2^{\mu-1}\,\pi\,\Gamma(\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2})}{\Gamma(\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2})\,\pi\,\csc(\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2})\pi \cdot \sin(\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2})\pi} \times$$

$$\times \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}\,\Gamma(-\nu-m)\,\pi}{m!}$$

$$= \frac{-2^{\mu-1}\Gamma(\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2})}{\Gamma(\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2})} \sum_{m=0}^{\infty} \frac{\pi(z/2)^{\nu+2m}}{m!\,\Gamma(1+\nu+m)\,\sin(-\nu\pi)\,\cos(-m\pi)}$$

$$= \frac{2^{\mu-1}\pi\,\Gamma(\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2})}{\Gamma(\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2})\,\sin(\nu\pi)} \times J_{\nu}(z) .$$

In the same manner we could show that

$$(13.42) \quad \sum_{m=0}^{\infty} \mathrm{Res}\,(F(s), b+m) = \frac{-2^{\mu-1}\pi\;\Gamma(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2})}{\Gamma(\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2})\,\sin(\nu\pi)} \times J_{-\nu}(z).$$

Adding (13.41) and (13.42) and factoring

$$\frac{-2^{\mu-1}\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})}{\sin(\nu\pi)}$$

out of each term we get

$$(13.43) \quad \frac{-2^{\mu-1}\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})}{\sin(\nu\pi)} \times$$

$$\times \left[\frac{J_{-\nu}(z)\,\pi}{\Gamma(1-(\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2}))\,\Gamma(\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2})} - \right.$$

$$\left. - \frac{J_{\nu}(z)\,\pi}{\Gamma(1-(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2}))\,\Gamma(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2})} \right]$$

$$= \frac{-2^{\mu-1}\Gamma(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2})\,\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2})}{\sin(\nu\pi)} \times$$

$$\times\left[\frac{J_{-\nu}(z)\ \pi\sin(\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2})\pi}{\pi} - \frac{J_{\nu}(z)\ \pi\sin(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2})\pi}{\pi}\right]$$

$$= \frac{-2^{\mu-1}\Gamma(\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2})\,\Gamma(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2})}{\sin(\nu\pi)} \times$$

$$\times\left[J_{-\nu}(z)\ \cos(\tfrac{1}{2}(\mu-\nu)\pi) - J_{\nu}(z)\ \cos(\tfrac{1}{2}(\mu+\nu)\pi\right] .$$

But the sum of $-s_{\mu,\nu}(z)$ and (13.43) is $-S_{\mu,\nu}(z)$ and thus we have

(13.44) $$S_{\mu,\nu}(z) \sim z^{\mu-1}\left[1 - \frac{(\mu-1)^2-\nu^2}{z^2} + \right.$$

$$\left. + \frac{((\mu-1)^2-\nu^2)\ ((\mu-3)^2-\nu^2)}{z^4} - \cdots\right] ,$$

for large $|z|$ when $|\arg(z)| < \pi$.

14. A Method for Determining the First Term of the Asymptotic Expansion of $S_{\mu,\nu}(z)$

The method developed in this section is based in part on Brauer's method which was outlined in Section 8. First we make the substitution $v = y\sqrt{z}$ in

$$z^2y'' + zy' + (z^2 - \nu^2)y = kz^{\mu+1} \tag{4.1}$$

and get

$$v'' + \left[1 + (\tfrac{1}{4} - \nu^2)/z^2\right]v = kz^{\mu-\frac{1}{2}} \tag{14.1}$$

Next we let

$$v = v_1 \quad ; \quad v_1' = v_2 \tag{14.2}$$

which gives us the system of equations

$$\text{(14.3)} \qquad \begin{aligned} &\text{(i)} \quad v_1' = v_2 \\ &\text{(ii)} \quad v_2' = \left[-1 + (\nu^2 - \tfrac{1}{4})/z^2\right]v_1 + kz^{\mu-\frac{1}{2}} \end{aligned} \quad .$$

If we let

$$\text{(14.4)} \qquad a(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad b(z) = \begin{pmatrix} 0 & 0 \\ (\nu^2 - \frac{1}{4})/z^2 & 0 \end{pmatrix} ,$$

$$\vec{v}(z) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \vec{c}(z) = \begin{pmatrix} 0 \\ kz^{\mu-\frac{1}{2}} \end{pmatrix}$$

we get the equation

$$(14.5) \qquad \vec{v}\,'(z) = (a(z) + b(z))\,\vec{v}(z) + \vec{c}(z) \quad .$$

Using (7.10) and (7.11) from Chapter II on Bessel functions, we get a fundamental solution $\Phi(z)$ of

$$(14.6) \qquad \vec{v}\,'(z) = (a(z) + b(z))\,\vec{v}(z)$$

namely,

$$(14.7) \qquad \Phi(z)$$

$$= \begin{pmatrix} \sqrt{\frac{2}{\pi}}\cos(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) & \sqrt{\frac{2}{\pi}}\sin(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) \\ -\sqrt{\frac{2}{\pi}}\sin(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) & \sqrt{\frac{2}{\pi}}\cos(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) \end{pmatrix}$$

where the first line consists of the respective first terms of the asymptotic expansions of $J_\nu(z)$ and $Y_\nu(z)$. Since $\Phi(z)$ is a fundamental solution of (14.6), it has the property

$$(14.8) \qquad \Phi'(z) = (a(z) + b(z))\,\Phi(z) \quad .$$

Next we consider the solution of (14.5) to be

(14.9) $$\vec{v}_p(z) = \Phi(z) \int_{\infty}^{z} \left[\Phi(t)\right]^{-1} \vec{c}(t) \, dt ,$$

where the path of integration is taken along the ray emerging from the origin and passing through z. Since

(14.10) $$\left|\Phi(z)\right| = \frac{2}{\pi} \cos^2(z-(2\nu+1)\frac{\pi}{4}) + $$

$$+ \frac{2}{\pi} \sin^2(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1})$$

$$= \frac{2}{\pi} + O(z^{-1}) ,$$

which is different from zero in a neighborhood of $z = \infty$, $\Phi(z)$ has an inverse in that neighborhood. Using properties of asymptotic expansion (e.g. see Wasow [9]), we get

(14.11) $$(\Phi(z))^{-1} =$$

$$= \frac{\begin{pmatrix} \sqrt{\frac{2}{\pi}} \cos(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) & -\sqrt{\frac{2}{\pi}} \sin(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) \\ \sqrt{\frac{2}{\pi}} \sin(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) & \sqrt{\frac{2}{\pi}} \cos(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) \end{pmatrix}}{\frac{2}{\pi} + O(z^{-1})}$$

$$= \begin{pmatrix} \sqrt{\frac{\pi}{2}}\cos(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) & -\sqrt{\frac{\pi}{2}}\sin(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) \\ \sqrt{\frac{\pi}{2}}\sin(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) & \sqrt{\frac{\pi}{2}}\cos(z-(2\nu+1)\frac{\pi}{4}) + O(z^{-1}) \end{pmatrix} .$$

Next we note that (14.9) is a solution of (14.5). When we substitute $\vec{v}_p(z)$ of (14.9) into the left-hand side of (14.5), we get

$$(14.12) \qquad \left[\Phi(z)\right]' \int_{\infty}^{z} \left[\Phi(t)\right]^{-1} \vec{c}(t)\,dt + \vec{c}(z)$$

$$= (a(z) + b(z))\,\Phi(z) \int_{\infty}^{z} \left[\Phi(t)\right]^{-1} \vec{c}(t)\,dt + \vec{c}(z)$$

$$= (a(z) + b(z))\,\vec{v}_p(z) + \vec{c}(z)$$

(i.e. $\vec{v}_p(z)$ satisfies (14.5)). If we let

$$(14.13) \qquad f(z) = z-(2\nu+1)\frac{\pi}{4}$$

we have

$$(14.14) \quad \vec{v}_p(z) = \begin{pmatrix} \sqrt{\frac{2}{\pi}}\cos(f(z)) + O(z^{-1}) & \sqrt{\frac{2}{\pi}}\sin(f(z)) + O(z^{-1}) \\ -\sqrt{\frac{2}{\pi}}\sin(f(z)) + O(z^{-1}) & \sqrt{\frac{2}{\pi}}\cos(f(z)) + O(z^{-1}) \end{pmatrix} \times$$

$$\times\int_{\infty}^{z}\begin{pmatrix}\sqrt{\frac{\pi}{2}}\cos(f(t)) +O(t^{-1}) & -\sqrt{\frac{\pi}{2}}\sin(f(t)) +O(t^{-1}) \\ \sqrt{\frac{\pi}{2}}\sin(f(t)) +O(t^{-1}) & \sqrt{\frac{\pi}{2}}\cos(f(t)) +O(t^{-1})\end{pmatrix}\begin{pmatrix}0 \\ kt^{\mu-\frac{1}{2}}\end{pmatrix}dt$$

$$=\begin{pmatrix}\sqrt{\frac{2}{\pi}}\cos(f(z)) +O(z^{-1}) & \sqrt{\frac{2}{\pi}}\sin(f(z)) +O(z^{-1}) \\ -\sqrt{\frac{2}{\pi}}\sin(f(z)) +O(z^{-1}) & \sqrt{\frac{2}{\pi}}\cos(f(z)) +O(z^{-1})\end{pmatrix}\times$$

$$\times\int_{\infty}^{z}\begin{pmatrix}-k\sqrt{\frac{\pi}{2}}\, t^{\mu-\frac{1}{2}}\sin(f(t)) +O(t^{\mu-\frac{3}{2}}) \\ k\sqrt{\frac{\pi}{2}}\, t^{\mu-\frac{1}{2}}\cos(f(t)) +O(t^{\mu-\frac{3}{2}})\end{pmatrix}dt \quad .$$

Integrating by parts we get

$$(14.15)\quad \vec{v}_p(z) = \begin{pmatrix}\sqrt{\frac{2}{\pi}}\cos(f(z)) +O(z^{-1}) & \sqrt{\frac{2}{\pi}}\sin(f(z)) +O(z^{-1}) \\ -\sqrt{\frac{2}{\pi}}\sin(f(z)) +O(z^{-1}) & \sqrt{\frac{2}{\pi}}\cos(f(z)) +O(z^{-1})\end{pmatrix}\times$$

$$\times\begin{pmatrix}-k\sqrt{\frac{\pi}{2}}\,(-z^{\mu-\frac{1}{2}})\cos(f(z)) + O(z^{\mu-\frac{3}{2}}) \\ k\sqrt{\frac{\pi}{2}}\,(z^{\mu-\frac{1}{2}})\sin(f(z)) + O(z^{\mu-\frac{3}{2}})\end{pmatrix} .$$

Since

$$(14.16) \qquad \vec{v}_p(z) = \begin{pmatrix} v_{p_1}(z) \\ \\ v'_{p_2}(z) \end{pmatrix} ,$$

we have

$$(14.17) \quad v_p(z) = kz^{\mu-\frac{1}{2}} \cos^2(f(z)) + kz^{\mu-\frac{1}{2}} \sin^2(f(z)) + O(z^{\mu-\frac{3}{2}})$$

$$= kz^{\mu-\frac{1}{2}} + O(z^{\mu-\frac{3}{2}}) \quad .$$

But

$$(14.18) \qquad v_p(z) = z^{\frac{1}{2}} y_p(z)$$

implies that

$$(14.19) \qquad y_p(z) = kz^{\mu-1} + O(z^{\mu-2}) \quad .$$

Thus we obtain the first term of (13.4). We note here that by carrying one more term in each of the entries in the matrix $\Phi(z)$, we could get the more accurate formula

$$(14.20) \qquad y_p(z) = kz^{\mu-1} [1 + O(z^{-2})] \quad .$$

BIBLIOGRAPHY

1. F. Brauer, "Asymptotic equivalence and asymptotic behavior of linear systems," Michigan Math. J., 9 (1962), 33 - 43.

2. F. Brauer, "On the asymptotic behavior of Bessel functions," Amer. Math. Monthly, 70 (1963), 954 - 957.

3. R. A. Clark, "Asymptotic solutions of a nonhomogeneous differential equation with a turning point," Arch. Rational Mech. Anal., 12 (1963), 34 - 51.

4. P. Debye, "Näherungsformeln für die Zylinderfunktionen für grobe Werte des Arguments und unbeschränkt veränderliche Werte des Index," Math. Ann., 67 (1909), 535 - 558.

5. H. Hankel, Z. Math. Physik., 9 (1864), 7.

6. E. L. Ince, Ordinary differential equations, Dover, New York, 1962.

7. D. J. McGuinnes, "Nonhomogeneous differential equations with a second-order turning point," J. Math and Phys., 7 (1966), 1030 - 1037.

8. C. R. Steele, "On the asymptotic solutions of nonhomogeneous ordinary differential equations with a large parameter," Quart. Appl. Math., 23 (1965), 193 - 201.

9. W. R. Wasow, Asymptotic expansions for ordinary differential equations, Interscience, New York, (1965).

10. G. N. Watson, Theory of Bessel functions, Macmillan, New York, (1922).

11. E. T. Whitaker and G. N. Watson, Modern Analysis, University Press, Cambridge, (1958).