



3-1967

On Reduction of Order of a System of Linear Nonhomogeneous Ordinary Differential Equations

Otis D. Shouse
Western Michigan University

Follow this and additional works at: https://scholarworks.wmich.edu/masters_theses



Part of the [Mathematics Commons](#)

Recommended Citation

Shouse, Otis D., "On Reduction of Order of a System of Linear Nonhomogeneous Ordinary Differential Equations" (1967). *Master's Theses*. 3282.

https://scholarworks.wmich.edu/masters_theses/3282

This Masters Thesis-Open Access is brought to you for free and open access by the Graduate College at ScholarWorks at WMU. It has been accepted for inclusion in Master's Theses by an authorized administrator of ScholarWorks at WMU. For more information, please contact maira.bundza@wmich.edu.



ON REDUCTION OF ORDER OF A SYSTEM OF
LINEAR NONHOMOGENEOUS ORDINARY
DIFFERENTIAL EQUATIONS

by

Otis D. Shouse

A Thesis
Submitted to the
Faculty of the School of Graduate
Studies in partial fulfillment
of the Degree of
Educational Specialist in Mathematics

Western Michigan University
Kalamazoo, Michigan
March 1967

ACKNOWLEDGEMENTS

In writing this thesis, I have become greatly indebted to Dr. Philip Po-Fang Hsieh for his advice, encouragement, and extreme patience. My thanks go to him, as well as the many others in the Department of Mathematics at Western Michigan University, who have always been quite willing to give me help. I am also grateful for the financial support of the National Science Foundation which allowed me to begin my graduate study, and to Western Michigan University for the additional assistance which enabled me to continue my work. Finally, I must gratefully acknowledge my wife, Kay, for giving me support and encouragement during a time which has been trying for her.

Otis D. Shouse

MASTER'S THESIS

M-1170

SHOUSE, Otis D.

ON REDUCTION OF ORDER OF A SYSTEM OF LINEAR
NONHOMOGENEOUS ORDINARY DIFFERENTIAL
EQUATIONS.

Western Michigan University, Ed.S., 1967
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

TABLE OF CONTENTS

SECTION	PAGE
1. INTRODUCTION	1
2. A BRIEF SKETCH OF I. M. MIU'S PROOF OF THEOREM 1	3
Part 1. A Special Case	4
Part 2. The General Case	6
3. PROOF OF THEOREM 2	9
4. CONCLUDING REMARKS	15
BIBLIOGRAPHY	17

1. INTRODUCTION

The reduction of order of a scalar n^{th} order linear homogeneous differential equation when $m(<n)$ linearly independent solutions are given is well known, and can be found in almost any ordinary differential equations text book (e.g. E. A. Coddington [1]*). Also the reduction of order of a system of n linear homogeneous equations when $m(<n)$ independent solutions are known has been well established (e.g. E. A. Coddington and N. Livinson [2] or P. Hartman [3]).

Recently I. M. Miu [4] proved the following theorem concerning the reduction of order of a scalar linear non-homogeneous differential equation of order n .

Theorem 1: Let the n^{th} order differential equation

$$(1.1) \quad y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(x)$$

be given where a_1, a_2, \dots, a_n and f are known continuous functions of x on an interval I . Assume $m(<n)$ linearly independent solutions y_1, y_2, \dots, y_m of the associated homogeneous equation

$$(1.2) \quad y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$$

* Numerals in [] refer to the bibliography.

are given. Then equation (1.1) can be reduced to a linear nonhomogeneous equation of order $n-m$.

In this paper I. M. Miu's result will be extended to a system of n linear nonhomogeneous differential equations under the assumption that $m(<n)$ linearly independent solutions of the associated system of homogeneous equations are known. The statement of this theorem is as follows:

Theorem 2: Let the system

$$(1.3) \quad \vec{y}' = A(t)\vec{y} + \vec{f}(t)$$

be given where \vec{y} is an n -dimensional vector valued function, $A(t)$ is a continuous $n \times n$ matrix, and $\vec{f}(t)$ is a continuous n -dimensional vector on the t -interval $[a,b]$.

Assume that $\vec{y}_1, \vec{y}_2, \dots, \vec{y}_m$ ($m < n$) are linearly independent solutions of

$$(1.4) \quad \vec{y}' = A(t)\vec{y}.$$

Denote the components of \vec{y}_j ($j = 1, 2, \dots, m$) by

y_{ij} ($i = 1, 2, \dots, n$) and consider the $n \times m$ matrix

$Y_m(t) = (y_{ij})$ ($i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$).

Let $Y_{m1}(t) = (y_{ij})$ ($i, j = 1, 2, \dots, m$) and assume

$\det Y_{m1}(t) \neq 0$ for some t_0 in $[a,b]$. Then we can reduce

system (1.3) to a system

$$(1.5) \quad \vec{z}' = B(t)\vec{z} + \vec{g}(t)$$

where \vec{z} is an $(n-m)$ -dimensional vector valued function,

$B(t)$ is an $(n-m) \times (n-m)$ known continuous matrix, and $\vec{g}(t)$ is a known continuous $(n-m)$ -dimensional vector on $[c,d]$, a subinterval of $[a,b]$.

Remark: The motivation for Theorem 2 is to find a particular solution of the system (1.3) by finding a particular solution of the reduced equation (1.5). We will show that the assumption that $\det Y_{m1}(t) \neq 0$ for some t_0 in $[a,b]$ places no restriction on the usefulness of the theorem in this regard.

A brief sketch of Miu's proof of Theorem 1 will be given in section 2 and the proof of Theorem 2 will be given in section 3. It is easily seen that the proof of Theorem 2 is more elegant than Miu's proof. Furthermore, in section 4, we will show that an n^{th} order scalar linear nonhomogeneous equation can be reduced to a system of type (1.3) in Theorem 2. Thus Theorem 2 can be regarded as a generalization of Theorem 1.

In proving Theorem 2 we will use the method of variation of constants. As a matter of fact, the variation-of-constants formula for a nonhomogeneous system can be obtained from the proof of Theorem 2 if we have $m = n$.

2. A BRIEF SKETCH OF I. M. MIU'S PROOF OF THEOREM 1

We will divide this section into two parts. Part 1 is actually a proof for the special case of Theorem 1

when $m = n - 1$. We single out this case because it reduces the n^{th} order equation (1.1) to a first order equation which we know can be solved, hence this case takes on a special interest. In Part 2 we will give a proof for Theorem 1 in its general form.

Part 1. A Special Case.

Suppose $n-1$ linearly independent solutions y_1, y_2, \dots, y_{n-1} of (1.2) are known. Let

$$(2.1) \quad y_0 = c_1 y_1 + c_2 y_2 + \dots + c_{n-1} y_{n-1}$$

and differentiate y_0 n times after successively setting

$$(2.2) \quad \sum_{p=1}^{n-1} c'_p y_p^{(r)} = 0, \quad r = 0, 1, \dots, n-3,$$

$$\text{(where } y_p^{(0)} = y_p \text{)}$$

when these sums occur. Then we get,

$$y_0^{(s)} = \sum_{p=1}^{n-1} c_p y_p^{(s)}, \quad s = 1, 2, \dots, n-2,$$

$$y_0^{(n-1)} = \sum_{p=1}^{n-1} c_p y_p^{(n-1)} + \sum_{p=1}^{n-1} c'_p y_p^{(n-2)},$$

$$y_0^{(n)} = \sum_{p=1}^{n-1} c_p y_p^{(n)} + \sum_{p=1}^{n-1} c'_p y_p^{(n-1)} + \left[\sum_{p=1}^{n-1} c''_p y_p^{(n-2)} \right]'$$

Substituting y_0 and its derivatives into (1.1), we obtain

$$(2.3) \quad c_1' y_1^{(n-2)} + c_2' y_2^{(n-2)} + \dots + c_{n-1}' y_{n-1}^{(n-2)} = f(x) - U'$$

where

$$(2.4) \quad U = \sum_{p=1}^{n-1} c_p' y_p^{(n-2)}$$

and

$$y_p^{(n-2)} = y_p^{(n-1)} + a_{1p} y_p^{(n-2)}, \quad p = 1, 2, \dots, n-1.$$

We can now find c_1' , c_2' , ..., c_{n-1}' in terms of U by using (2.2) and (2.4). Substituting these values into (2.3) we get

$$(2.5) \quad U' + \frac{A_{n-1}}{W_{n-1}} U = f(x)$$

where W_{n-1} is the Wronskian determinant of the independent solutions y_1, y_2, \dots, y_{n-1} and

$$A_{n-1} = \begin{vmatrix} y_1 & y_2 & \dots & y_{n-1} \\ y_1' & y_2' & \dots & y_{n-1}' \\ \cdot & \cdot & \dots & \cdot \\ y_1^{(n-3)} & y_2^{(n-3)} & \dots & y_{n-1}^{(n-3)} \\ y_1^{(n-2)} & y_2^{(n-2)} & \dots & y_{n-1}^{(n-2)} \end{vmatrix}.$$

Since A_{n-1} and W_{n-1} are known quantities, the equation (1.1) is reduced to the first order equation (2.5).

We now observe that if a particular solution U_0 of equation (2.5) is known, we can use (2.2) and (2.4) with U replaced by U_0 to find c_1, c_2, \dots, c_{n-1} in terms of known functions. Substituting these values into (2.1) we obtain a particular solution of (1.1).

Part 2. The General Case

We now assume that $m(<n)$ linearly independent solutions y_1, y_2, \dots, y_m of (1.2) are known. Let

$$(2.6) \quad y_0 = c_1 y_1 + c_2 y_2 + \dots + c_m y_m$$

and again differentiate y_0 n times after successively setting

$$(2.7) \quad \sum_{p=1}^m c'_p y_p^{(r)} = 0, \quad r = 0, 1, 2, \dots, m-2.$$

Then

$$y_0^{(k)} = \sum_{p=1}^m c_p y_p^{(k)}, \quad k = 1, 2, \dots, m-1,$$

$$y_0^{(m)} = \sum_{p=1}^m c_p y_p^{(m)} + U_1,$$

$$y_0^{(m+1)} = \sum_{p=1}^m c_p y_p^{(m+1)} + U_1' + U_2,$$

...

$$y_0^{(n)} = \sum_{p=1}^m c_p y_p^{(n)} + U_1^{(n-m)} + U_2^{(n-m-1)} \\ + \dots + U'_{n-m} + U_{n-m+1},$$

where

$$(2.8) \quad \sum_{p=1}^m c_p' y_p^{(s)} = U_{s-m+2}, \quad s = m-1, m, m+1, \dots, n-1.$$

Substituting these derivatives of y_0 into (1.1)

we obtain

$$(2.9) \quad U_1^{(n-m)} + U_2^{(n-m-1)} + \dots + U'_{n-m} + U_{n-m+1} \\ + a_1 (U_1^{(n-m-1)} + \dots + U'_{n-m+1} + U_{n-m}) \\ + \dots + a_{n-m-1} (U_1' + U_2) + a_{n-m} U_1 = f(x).$$

We now observe from equations (2.7), the first equation in (2.8), and one at a time from the remaining equations in (2.8) that we can express $U_2, U_3, \dots, U_{n-m+1}$ in terms of U_1 ; namely,

$$U_j = \frac{W_m^j}{W_m} U_1 = V_j U_1, \quad j = 2, 3, \dots, n-m+1$$

where W_m is the Wronskian determinant of the independent solutions y_1, y_2, \dots, y_m ,

$$W_m^j = \begin{vmatrix} y_1 & y_2 & \dots & y_m \\ y_1' & y_2' & \dots & y_m' \\ \cdot & \cdot & \dots & \cdot \\ y_1^{(m-2)} & y_2^{(m-2)} & \dots & y_m^{(m-2)} \\ y_1^{(m+j-2)} & y_2^{(m+j-2)} & \dots & y_m^{(m+j-2)} \end{vmatrix},$$

and for the sake of simplicity, we use the notation

$$V_j = \frac{W_m^j}{W_m}, \quad j = 2, 3, \dots, n-m+1.$$

Now substituting these U_j 's into (2.9) we get

$$\begin{aligned} (2.10) \quad & U_1^{(n-m)} + (V_2 U_1)^{(n-m-1)} + \dots + (V_{n-m} U_1)' + V_{n-m+1} U_1 \\ & + a_1 [U_1^{(n-m-1)} + (V_2 U_1)^{(n-m-2)} \\ & + \dots + (V_{n-m-1} U_1)' + V_{n-m} U_1] \\ & + \dots + a_{n-m-1} [U_1' + V_2 U_1] + a_{n-m} U_1 = f(x). \end{aligned}$$

Since the V_j 's ($j = 2, 3, \dots, n-m+1$) are known functions of x , the equation (1.1) has been reduced to the equation (2.10) which is of order $n-m$. This completes the proof

of Theorem 1 in its general form.

As in Part 1 we again observe that if U_{o1} is a particular solution of (2.10), then c_1, c_2, \dots, c_m can be obtained from (2.7) and the first of equations (2.8) with U_1 replaced by U_{o1} . Thus a particular solution of (1.1) is obtained by substituting these values of c_1, c_2, \dots, c_m into (2.6).

3. PROOF OF THEOREM 2

We are now ready to prove Theorem 2 which is the primary purpose of this paper. For convenience of notation we will, in this section, write $x, y,$ and z in place of $\vec{x}, \vec{y},$ and $\vec{z},$ since it will be clear that we are dealing with vectors.

First we observe that $Y_m(t),$ as defined in Theorem 2, is a solution of the matrix differential equation

$$(3.1) \quad Y' = A(t)Y$$

where Y is an $n \times m$ matrix valued function on $[a,b].$

Now using the assumption that $\det Y_{m1}(t) \neq 0$ for some t_0 in $[a,b]$ and the fact that $\det Y_{m1}(t)$ is continuous in $[a,b],$ we know that $\det Y_{m1}(t) \neq 0$ for all t in some subinterval $[c,d]$ of $[a,b],$ containing $t_0.$ We now write

$$(3.2) \quad Y_m(t) = \begin{pmatrix} Y_{m1}(t) \\ Y_{m2}(t) \end{pmatrix}$$

where $Y_{m1}(t)$ is the $m \times m$ matrix described in Theorem 2 and $Y_{m2}(t) = (y_{ij})$ ($i = m+1, \dots, n$, and $j = 1, 2, \dots, m$). Next we define the $n \times m$ matrix

$$(3.3) \quad Z(t) = \begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & I_{n-m} \end{pmatrix}$$

where 0 is the $m \times (n-m)$ zero matrix and I_{n-m} is the $(n-m) \times (n-m)$ identity matrix. Certainly $Z(t)$ is non-singular on $[c, d]$, since $\det Z(t) = \det Y_{m1}(t) \neq 0$ for t in $[c, d]$. It is quite easy to see that

$$(3.4) \quad Z^{-1}(t) = \begin{pmatrix} Y_{m1}^{-1}(t) & 0 \\ Z_{m2}(t) & I_{n-m} \end{pmatrix}$$

where $Z_{m2}(t)$ is the $(n-m) \times m$ matrix

$$(3.5) \quad Z_{m2}(t) = -Y_{m2}(t)Y_{m1}^{-1}(t).$$

We also observe that $Z'(t) = (Y'_m(t) \ 0)$. Thus

$$Z'(t) = (A(t)Y_m(t) \ 0) = A(t) (Y_m(t) \ 0)$$

since $Y_m(t)$ is a solution of (3.1). Hence we have

$$(3.6) \quad Z'(t) = A(t) \begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & 0 \end{pmatrix}.$$

Now let us have a change of variables

$$(3.7) \quad y = Z(t)z$$

in (1.3). We get

$$Z'(t)z + Z(t)z' = A(t)Z(t)z + f(t),$$

or,

$$Z(t)z' = [A(t)Z(t) - Z'(t)]z + f(t).$$

Now using (3.6), we know that

$$z' = Z^{-1}(t) \left[A(t)Z(t) - A(t) \begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & 0 \end{pmatrix} \right] z + Z^{-1}(t)f(t),$$

or,

$$z' = Z^{-1}(t)A(t) \left[\begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & I_{n-m} \end{pmatrix} - \begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & 0 \end{pmatrix} \right] z + Z^{-1}(t)f(t)$$

which finally yields

$$(3.8) \quad z' = Z^{-1}(t)A(t) \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix} z + Z^{-1}(t)f(t).$$

We now put $A(t)$ into block form as follows:

$$(3.9) \quad A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}$$

where $A_{11}(t)$, $A_{12}(t)$, $A_{21}(t)$, and $A_{22}(t)$ are $m \times m$, $m \times (n-m)$, $(n-m) \times m$, and $(n-m) \times (n-m)$ matrices respectively. We also write

$$(3.10) \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

where z_1 and $f_1(t)$ are m -dimensional vectors, while z_2 and $f_2(t)$ are $(n-m)$ -dimensional vectors. Now substituting (3.9) and (3.10) into (3.8) we get

$$(3.11) \quad \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} Y_{m1}^{-1}(t) & 0 \\ Z_{m2}(t) & I_{n-m} \end{pmatrix} \cdot \left[\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \right]$$

$$= \begin{pmatrix} Y_{m1}^{-1}(t) & 0 \\ Z_{m2}(t) & I_{n-m} \end{pmatrix} \left[\begin{pmatrix} 0 & A_{12}(t) \\ 0 & A_{22}(t) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \right]$$

$$= \begin{pmatrix} Y_{m1}^{-1}(t) & 0 \\ Z_{m2}(t) & I_{n-m} \end{pmatrix} \begin{pmatrix} A_{12}(t)z_2 + f_1(t) \\ A_{22}(t)z_2 + f_2(t) \end{pmatrix}$$

$$= \begin{pmatrix} Y_{m1}^{-1}(t)[A_{12}(t)z_2 + f_1(t)] \\ [Z_{m2}(t)A_{12}(t) + A_{22}(t)]z_2 + [Z_{m2}(t)f_1(t) + f_2(t)] \end{pmatrix}$$

and hence we conclude that

$$(3.12) \quad z'_1 = Y_{m1}^{-1}(t)[A_{12}(t)z_2 + f_1(t)]$$

and

$$(3.13) \quad z_2' = [Z_{m2}(t)A_{12}(t) + A_{22}(t)]z_2 \\ + [Z_{m2}(t)f_1(t) + f_2(t)]$$

where $Z_{m2}(t)A_{12}(t) + A_{22}(t)$ is a known continuous $(n-m) \times (n-m)$ matrix and $Z_{m2}(t)f_1(t) + f_2(t)$ is a known continuous $(n-m)$ -dimensional vector on $[c,d]$. Thus we now see that (3.13) has the required conditions of the reduced equation (1.5) that we set out to find in Theorem 2. Hence we have completed the proof.

We now observe that if a particular solution, z_2 , of (3.13) can be found, then z_1 can be obtained from (3.12) by integration. Thus we obtain a vector

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

which satisfies (3.11), and hence (3.8). It now follows that the vector $y = Z(t)z$ is a particular solution of (1.3).

Next we return to the remark following the statement of Theorem 2 in the introduction. We will show how Theorem 2 can be used in finding a particular solution of (1.3). The important thing to observe in the following discussion is that the assumption $\det Y_{m1}(t) \neq 0$ for some t_0 in $[a,b]$ is not restricting.

We recall that $Y_m(t) = (y_{ij})$ ($i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$) is made up of m linearly independent

column vectors. Thus the rank of $Y_m(t)$ is m for every t in $[a,b]$. This means that for any t in $[a,b]$ there exists a nonsingular $m \times m$ submatrix of $Y_m(t)$. Let t_0 be any element of $[a,b]$ and assume $M(t)$ is the $m \times m$ nonsingular submatrix of $Y_m(t)$ corresponding to t_0 . Again by the continuity of $\det M(t)$ we know there is some subinterval I of $[a,b]$, containing t_0 , such that $\det M(t) \neq 0$ for all t in I . Let $M(t) = (y_{ij})$ ($i = i_1, i_2, \dots, i_m$ and $j = 1, 2, \dots, m$ with $1 \leq i_k \leq n$ for $k = 1, 2, \dots, m$ and $i_k \neq i_\ell$ if $k \neq \ell$). Then it is well known that there exists a nonsingular $n \times n$ constant matrix T with the property that

$$TY_m(t) = \begin{pmatrix} M(t) \\ N(t) \end{pmatrix}$$

where $N(t)$ is the $(n-m) \times m$ matrix made up of the $n-m$ rows of $Y_m(t)$ after the m rows i_1, i_2, \dots, i_m have been removed. If we now make the transformation $y = T^{-1}x$ in (1.3). We get

$$T^{-1}x' = A(t)T^{-1}x + f(t)$$

which yields

$$(3.14) \quad x' = TA(t)T^{-1}x + Tf(t).$$

It is now clear that $TY_m(t)$ has the very same properties in relation to (3.14) as we assumed $Y_m(t)$ had in relation

to (1.3). Thus we may apply Theorem 2 to (3.14) and reduce it as we did (1.3). If we can now obtain a particular solution x of (3.14), we see that $y = T^{-1}x$ is a particular solution of (1.3). Thus we have shown that our assumption that the $\det Y_{m1}(t) \neq 0$ at t_0 does not affect the value of Theorem 2 when used to determine a particular solution of (1.3).

4. CONCLUDING REMARKS

We conclude this paper with the remark that I. M. Miu's theorem is now an immediate corollary to Theorem 2. This is easily seen since the n^{th} order differential equation (1.1) can be written as a system of first order linear nonhomogeneous differential equations. To see this let $y_1 = y$, $y_2 = y_1'$, ..., $y_n = y^{(n-1)}$, then (1.1) is equivalent to the system

$$y_1' = y_2$$

$$y_2' = y_3$$

. . .

$$y_{n-1}' = y_n$$

$$y_n' = -a_1 y_n - a_2 y_{n-1} - \dots - a_n y_1 + f(x).$$

This tells us (1.1) is equivalent to the vector equation

$$\vec{y}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ f(x) \end{pmatrix}$$

where

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix} .$$

BIBLIOGRAPHY

- [1] Coddington, Earl A., An Introduction to Ordinary Differential Equations. Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1961. Pp. 118-119.
- [2] Coddington, Earl A., and Levinson, Norman, Theory of Ordinary Differential Equations. New York: McGraw-Hill, 1955. Pp. 71-74.
- [3] Hartman, Philip, Ordinary Differential Equations. New York: John Wiley and Sons, Inc., 1964. Pp. 49-51.
- [4] Miu, I. M., "Über die partikuläre Lösung der linearen inhomogenen Differentialgleichung." Buletinul Institutului Politehnic Bucuresti, XXV (1963) No. 2, 13-24.