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On Reduction of Order of a System of Linear Nonhomogeneous Ordinary Differential Equations

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ON REDUCTION OF ORDER OF A SYSTEM OF LINEAR NONHOMOGENEOUS ORDINARY DIFFERENTIAL EQUATIONS

by

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1. INTRODUCTION

The reduction of order of a scalar n^{th} order linear homogeneous differential equation when $m(<n)$ linearly independent solutions are given is well known, and can be found in almost any ordinary differential equations text book (e.g. E. A. Coddington [1]*). Also the reduction of order of a system of n linear homogeneous equations when $m(<n)$ independent solutions are known has been well established (e.g. E. A. Coddington and N. Livinson [2] or P. Hartman [3]).

Recently I. M. Miu [4] proved the following theorem concerning the reduction of order of a scalar linear non-homogeneous differential equation of order n.

<u>Theorem</u> 1: <u>Let the n^{th} order differential equation</u>

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(x) \tag{1.1}$$

<u>be given where</u> $a_1, a_2, \dots, a_n$ <u>and</u> f <u>are known continuous functions of</u> x <u>on an interval</u> I. <u>Assume</u> $m(<n)$ <u>linearly independent solutions</u> $y_1, y_2, \dots, y_m$ <u>of the associated homogeneous equation</u>

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0 \tag{1.2}$$

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*Numerals in [ ] refer to the bibliography.

are given. Then equation (1.1) can be reduced to a linear nonhomogeneous equation of order n-m.

In this paper I. M. Miu's result will be extended to a system of n linear nonhomogeneous differential equations under the assumption that $m(<n)$ linearly independent solutions of the associated system of homogeneous equations are known. The statement of this theorem is as follows:

Theorem 2: Let the system

$$\vec{y}' = A(t)\vec{y} + \vec{f}(t) \tag{1.3}$$

be given where $\vec{y}$ is an n-dimensional vector valued function, $A(t)$ is a continuous $n \times n$ matrix, and $\vec{f}(t)$ is a continuous n-dimensional vector on the t-interval $[a,b]$. Assume that $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m (m<n)$ are linearly independent solutions of

$$\vec{y}' = A(t)\vec{y}. \tag{1.4}$$

Denote the components of $\vec{y}_j (j = 1,2, \ldots, m)$ by $y_{ij} (i = 1,2, \ldots, n)$ and consider the $n \times m$ matrix $Y_m(t) = (y_{ij})$ $(i = 1,2, \ldots, n$ and $j = 1,2, \ldots, m)$. Let $Y_{m1}(t) = (y_{ij})$ $(i,j = 1,2, \ldots, m)$ and assume $\det Y_{m1}(t) \neq 0$ for some t_o in $[a,b]$. Then we can reduce system (1.3) to a system

$$\vec{z}' = B(t)\vec{z} + \vec{g}(t) \tag{1.5}$$

where $\vec{z}$ is an (n-m)-dimensional vector valued function, B(t) <u>is an</u> (n-m) x (n-m) <u>known continuous matrix, and</u> $\vec{g}(t)$ <u>is a known continuous</u> (n-m)-<u>dimensional vector on</u> [c,d], <u>a subinterval of</u> [a,b].

<u>Remark</u>: The motivation for Theorem 2 is to find a particular solution of the system (1.3) by finding a particular solution of the reduced equation (1.5). We will show that the assumption that det $Y_{m1}(t) \neq 0$ for some t_o in [a,b] places no restriction on the usefulness of the theorem in this regard.

A brief sketch of Miu's proof of Theorem 1 will be given in section 2 and the proof of Theorem 2 will be given in section 3. It is easily seen that the proof of Theorem 2 is more elegant than Miu's proof. Furthermore, in section 4, we will show that an n^{th} order scalar linear nonhomogeneous equation can be reduced to a system of type (1.3) in Theorem 2. Thus Theorem 2 can be regarded as a generalization of Theorem 1.

In proving Theorem 2 we will use the method of variation of constants. As a matter of fact, the variation-of-constants formula for a nonhomogeneous system can be obtained from the proof of Theorem 2 if we have m = n.

2. A BRIEF SKETCH OF I. M. MIU'S PROOF OF THEOREM 1

We will divide this section into two parts. Part 1 is actually a proof for the special case of Theorem 1 when m = n - 1. We single out this case because it reduces the n^{th} order equation (1.1) to a first order equation which we know can be solved, hence this case takes on a special interest. In Part 2 we will give a proof for Theorem 1 in its general form.

Part 1. A Special Case

Suppose n-1 linearly independent solutions $y_1, y_2, \ldots, y_{n-1}$ of (1.2) are known. Let

(2.1) $$y_o = c_1 y_1 + c_2 y_2 + \ldots + c_{n-1} y_{n-1}$$

and differentiate y_o n times after successively setting

(2.2) $$\sum_{p=1}^{n-1} c_p' \, y_p^{(r)} = 0, \qquad r = 0, 1, \ldots, n-3,$$

$$\text{(where } y_p^{(o)} = y_p\text{)}$$

when these sums occur. Then we get,

$$y_o^{(s)} = \sum_{p=1}^{n-1} c_p y_p^{(s)}, \qquad s = 1, 2, \ldots, n-2,$$

$$y_o^{(n-1)} = \sum_{p=1}^{n-1} c_p y_p^{(n-1)} + \sum_{p=1}^{n-1} c_p' y_p^{(n-2)},$$

$$y_o^{(n)} = \sum_{p=1}^{n-1} c_p y_p^{(n)} + \sum_{p=1}^{n-1} c_p' y_p^{(n-1)} + \left[\sum_{p=1}^{n-1} c_p' y_p^{(n-2)}\right]'.$$

Substituting y_o and its derivatives into (1.1), we obtain

$$(2.3)\quad c_1' Y_1^{(n-2)} + c_2' Y_2^{(n-2)} + \ldots + c_{n-1}' Y_{n-1}^{(n-2)} = f(x) - U'$$

where

$$(2.4)\qquad U = \sum_{p=1}^{n-1} c_p' y_p^{(n-2)}$$

and

$$Y_p^{(n-2)} = y_p^{(n-1)} + a_1 y_p^{(n-2)}, \; p = 1, 2, \ldots, n-1.$$

We can now find c_1', c_2', ..., c_{n-1}' in terms of U by using (2.2) and (2.4). Substituting these values into (2.3) we get

$$(2.5)\qquad U' + \frac{A_{n-1}}{W_{n-1}} U = f(x)$$

where W_{n-1} is the Wronskian determinant of the independent solutions y_1, y_2, ..., y_{n-1} and

$$A_{n-1} = \begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} \\ y_1' & y_2' & \cdots & y_{n-1}' \\ \cdot & \cdot & \cdots & \cdot \\ y_1^{(n-3)} & y_2^{(n-3)} & \cdots & y_{n-1}^{(n-3)} \\ Y_1^{(n-2)} & Y_2^{(n-2)} & \cdots & Y_{n-1}^{(n-2)} \end{vmatrix} .$$

Since A_{n-1} and W_{n-1} are known quantities, the equation (1.1) is reduced to the first order equation (2.5).

We now observe that if a particular solution U_o of equation (2.5) is known, we can use (2.2) and (2.4) with U replaced by U_o to find $c_1, c_2, \ldots, c_{n-1}$ in terms of known functions. Substituting these values into (2.1) we obtain a particular solution of (1.1).

Part 2. The General Case

We now assume that $m(<n)$ linearly independent solutions $y_1, y_2, \ldots, y_m$ of (1.2) are known. Let

(2.6) $$y_o = c_1y_1 + c_2y_2 + \ldots + c_my_m$$

and again differentiate y_o n times after successively setting

(2.7) $$\sum_{p=1}^{m} c_p' y_p^{(r)} = 0, \qquad r = 0,1,2, \ldots, m-2.$$

Then

$$y_o^{(k)} = \sum_{p=1}^{m} c_p y_p^{(k)}, \; k = 1,2, \ldots, m-1,$$

$$y_o^{(m)} = \sum_{p=1}^{m} c_p y_p^{(m)} + U_1,$$

$$y_o^{(m+1)} = \sum_{p=1}^{m} c_p y_p^{(m+1)} + U_1' + U_2,$$

. . .

$$y_o^{(n)} = \sum_{p=1}^{m} c_p y_p^{(n)} + U_1^{(n-m)} + U_2^{(n-m-1)}$$

$$+\dots+ U'_{n-m} + U_{n-m+1},$$

where

(2.8) $$\sum_{p=1}^{m} c'_p y_p^{(s)} = U_{s-m+2}, \qquad s = m-1,\ m,\ m+1,\ \dots,\ n-1.$$

Substituting these derivatives of y_o into (1.1) we obtain

(2.9) $$U_1^{(n-m)} + U_2^{(n-m-1)} +\dots+ U'_{n-m} + U_{n-m+1}$$

$$+ a_1\ (U_1^{(n-m-1)} +\dots+ U'_{n-m+1} + U_{n-m})$$

$$+\dots+ a_{n-m-1}\ (U'_1 + U_2) + a_{n-m}U_1 = f(x).$$

We now observe from equations (2.7), the first equation in (2.8), and one at a time from the remaining equations in (2.8) that we can express U_2, U_3, ..., U_{n-m+1} in terms of U_1; namely,

$$U_j = \frac{W_m^j}{W_m} U_1 = V_j U_1, \qquad j = 2,3,\ \dots,\ n-m+1$$

where W_m is the Wronskian determinant of the independent solutions y_1, y_2, ..., y_m,

$$W_m^j = \begin{vmatrix} y_1 & y_2 & \cdots & y_m \\ y_1' & y_2' & \cdots & y_m' \\ \cdot & \cdot & \cdots & \cdot \\ y_1^{(m-2)} & y_2^{(m-2)} & \cdots & y_m^{(m-2)} \\ y_1^{(m+j-2)} & y_2^{(m+j-2)} & \cdots & y_m^{(m+j-2)} \end{vmatrix},$$

and for the sake of simplicity, we use the notation

$$V_j = \frac{W_m^j}{W_m}, \qquad j = 2,3, \ldots, n-m+1.$$

Now substituting these U_j's into (2.9) we get

$$\begin{aligned} (2.10) \quad & U_1^{(n-m)} + (V_2U_1)^{(n-m-1)} + \ldots + (V_{n-m}U_1)' + V_{n-m+1}U_1 \\ & + a_1 \, [U_1^{(n-m-1)} + (V_2U_1)^{(n-m-2)} \\ & + \ldots + (V_{n-m-1}U_1)' + V_{n-m}U_1] \\ & + \ldots + a_{n-m-1} \, [U_1' + V_2U_1] + a_{n-m}U_1 = f(x). \end{aligned}$$

Since the V_j's ($j = 2,3, \ldots, n-m+1$) are known functions of x, the equation (1.1) has been reduced to the equation (2.10) which is of order $n-m$. This completes the proof of Theorem 1 in its general form.

As in Part 1 we again observe that if U_{o1} is a particular solution of (2.10), then $c_1, c_2, \ldots, c_m$ can be obtained from (2.7) and the first of equations (2.8) with U_1 replaced by U_{o1}. Thus a particular solution of (1.1) is obtained by substituting these values of $c_1, c_2, \ldots, c_m$ into (2.6).

3. PROOF OF THEOREM 2

We are now ready to prove Theorem 2 which is the primary purpose of this paper. For convenience of notation we will, in this section, write x, y, and z in place of $\vec{x}$, $\vec{y}$, and $\vec{z}$, since it will be clear that we are dealing with vectors.

First we observe that $Y_m(t)$, as defined in Theorem 2, is a solution of the matrix differential equation

$$Y' = A(t)Y \tag{3.1}$$

where Y is an n x m matrix valued function on [a,b]. Now using the assumption that det $Y_{m1}(t) \neq 0$ for some t_o in [a,b] and the fact that det $Y_{m1}(t)$ is continuous in [a,b], we know that det $Y_{m1}(t) \neq 0$ for all t in some sub-interval [c,d] of [a,b], containing t_o. We now write

$$Y_m(t) = \begin{pmatrix} Y_{m1}(t) \\ Y_{m2}(t) \end{pmatrix} \tag{3.2}$$

where $Y_{m1}(t)$ is the $m \times m$ matrix described in Theorem 2 and $Y_{m2}(t) = (y_{ij})$ $(i = m + 1, \ldots, n$, and $j = 1, 2, \ldots, m)$. Next we define the $n \times m$ matrix

$$Z(t) = \begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & I_{n-m} \end{pmatrix} \tag{3.3}$$

where 0 is the $m \times (n-m)$ zero matrix and I_{n-m} is the $(n-m) \times (n-m)$ identity matrix. Certainly $Z(t)$ is non-singular on $[c,d]$, since $\det Z(t) = \det Y_{m1}(t) \neq 0$ for t in $[c,d]$. It is quite easy to see that

$$Z^{-1}(t) = \begin{pmatrix} Y_{m1}^{-1}(t) & 0 \\ Z_{m2}(t) & I_{n-m} \end{pmatrix} \tag{3.4}$$

where $Z_{m2}(t)$ is the $(n-m) \times m$ matrix

$$Z_{m2}(t) = -Y_{m2}(t)Y_{m1}^{-1}(t). \tag{3.5}$$

We also observe that $Z'(t) = (Y_m'(t)\ 0)$. Thus

$$Z'(t) = (A(t)Y_m(t)\ 0) = A(t)\ (Y_m(t)\ 0)$$

since $Y_m(t)$ is a solution of (3.1). Hence we have

$$Z'(t) = A(t) \begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & 0 \end{pmatrix}. \tag{3.6}$$

Now let us have a change of variables

$$y = Z(t)z \tag{3.7}$$

in (1.3). We get

$$Z'(t)z + Z(t)z' = A(t)Z(t)z + f(t),$$

or

$$Z(t)z' = [A(t)Z(t) - Z'(t)]z + f(t).$$

Now using (3.6), we know that

$$z' = Z^{-1}(t)\left[A(t)Z(t) - A(t)\begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & 0 \end{pmatrix}\right] z + Z^{-1}(t)f(t),$$

or

$$z' = Z^{-1}(t)A(t)\left[\begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & I_{n-m} \end{pmatrix} - \begin{pmatrix} Y_{m1}(t) & 0 \\ Y_{m2}(t) & 0 \end{pmatrix}\right] z + Z^{-1}(t)f(t)$$

which finally yields

$$(3.8) \qquad z' = Z^{-1}(t)A(t)\begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix} z + Z^{-1}(t)f(t).$$

We now put A(t) into block form as follows:

$$(3.9) \qquad A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}$$

where $A_{11}(t)$, $A_{12}(t)$, $A_{21}(t)$, and $A_{22}(t)$ are $m \times m$, $m \times (n-m)$, $(n-m) \times m$, and $(n-m) \times (n-m)$ matrices respectively. We also write

$$(3.10) \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad \text{and} \qquad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

where z_1 and $f_1(t)$ are m-dimensional vectors, while z_2 and $f_2(t)$ are (n-m)-dimensional vectors. Now substituting (3.9) and (3.10) into (3.8) we get

$$(3.11) \qquad \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} Y_{m1}^{-1}(t) & 0 \\ Z_{m2}(t) & I_{n-m} \end{pmatrix}$$

$$\cdot \left[\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \right]$$

$$= \begin{pmatrix} Y_{m1}^{-1}(t) & 0 \\ Z_{m2}(t) & I_{n-m} \end{pmatrix} \left[\begin{pmatrix} 0 & A_{12}(t) \\ 0 & A_{22}(t) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \right]$$

$$= \begin{pmatrix} Y_{m1}^{-1}(t) & 0 \\ Z_{m2}(t) & I_{n-m} \end{pmatrix} \begin{pmatrix} A_{12}(t)z_2 + f_1(t) \\ A_{22}(t)z_2 + f_2(t) \end{pmatrix}$$

$$= \begin{pmatrix} Y_{m1}^{-1}(t)[A_{12}(t)z_2 + f_1(t)] \\ [Z_{m2}(t)A_{12}(t) + A_{22}(t)]z_2 + [Z_{m2}(t)f_1(t) + f_2(t)] \end{pmatrix}$$

and hence we conclude that

$$(3.12) \qquad z_1' = Y_{m1}^{-1}(t)[A_{12}(t)z_2 + f_1(t)]$$

and

$$(3.13) \qquad z_2' = [Z_{m2}(t)A_{12}(t) + A_{22}(t)]z_2 + [Z_{m2}(t)f_1(t) + f_2(t)]$$

where $Z_{m2}(t)A_{12}(t) + A_{22}(t)$ is a known continuous (n-m) x (n-m) matrix and $Z_{m2}(t)f_1(t) + f_2(t)$ is a known continuous (n-m)-dimensional vector on [c,d]. Thus we now see that (3.13) has the required conditions of the reduced equation (1.5) that we set out to find in Theorem 2. Hence we have completed the proof.

We now observe that if a particular solution, z_2, of (3.13) can be found, then z_1 can be obtained from (3.12) by integration. Thus we obtain a vector

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

which satisfies (3.11), and hence (3.8). It now follows that the vector $y = Z(t)z$ is a particular solution of (1.3).

Next we return to the remark following the statement of Theorem 2 in the introduction. We will show how Theorem 2 can be used in finding a particular solution of (1.3). The important thing to observe in the following discussion is that the assumption det $Y_{m1}(t) \neq 0$ for some t_o in [a,b] is not restricting.

We recall that $Y_m(t) = (y_{ij})$ ($i = 1,2, \ldots, n$ and $j = 1,2, \ldots, m$) is made up of m linearly independent column vectors. Thus the rank of $Y_m(t)$ is m for every t in [a,b]. This means that for any t in [a,b] there exists a nonsingular $m \times m$ submatrix of $Y_m(t)$. Let t_o be any element of [a,b] and assume M(t) is the $m \times m$ nonsingular submatrix of $Y_m(t)$ corresponding to t_o. Again by the continuity of det M(t) we know there is some subinterval I of [a,b], containing t_o, such that det M(t) $\neq 0$ for all t in I. Let $M(t) = (y_{ij})$ ($i = i_1, i_2, \ldots, i_m$ and $j = 1, 2, \ldots, m$ with $1 \leq i_k \leq n$ for $k = 1, 2, \ldots, m$ and $i_k \neq i_\ell$ if $k \neq \ell$). Then it is well known that there exists a nonsingular $n \times n$ constant matrix T with the property that

$$TY_m(t) = \begin{pmatrix} M(t) \\ N(t) \end{pmatrix}$$

where N(t) is the $(n-m) \times m$ matrix made up of the $n-m$ rows of $Y_m(t)$ after the m rows $i_1, i_2, \ldots, i_m$ have been removed. If we now make the transformation $y = T^{-1}x$ in (1.3). We get

$$T^{-1}x' = A(t)T^{-1}x + f(t)$$

which yields

$$x' = TA(t)T^{-1}x + Tf(t). \tag{3.14}$$

It is now clear that $TY_m(t)$ has the very same properties in relation to (3.14) as we assumed $Y_m(t)$ had in relation to (1.3). Thus we may apply Theorem 2 to (3.14) and reduce it as we did (1.3). If we can now obtain a particular solution x of (3.14), we see that $y = T^{-1}x$ is a particular solution of (1.3). Thus we have shown that our assumption that the det $Y_{m1}(t) \neq 0$ at t_o does not affect the value of Theorem 2 when used to determine a particular solution of (1.3).

4. CONCLUDING REMARKS

We conclude this paper with the remark that I. M. Miu's theorem is now an immediate corollary to Theorem 2. This is easily seen since the n^{th} order differential equation (1.1) can be written as a system of first order linear nonhomogeneous differential equations. To see this let $y_1 = y$, $y_2 = y_1'$, ..., $y_n = y^{(n-1)}$, then (1.1) is equivalent to the system

$$y_1' = y_2$$

$$y_2' = y_3$$

$$\cdots$$

$$y_{n-1}' = y_n$$

$$y_n' = -a_1y_n - a_2y_{n-1} - \cdots - a_ny_1 + f(x).$$

This tells us (1.1) is equivalent to the vector equation

$$\vec{y}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{pmatrix} \vec{y} + \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ f(x) \end{pmatrix}$$

where

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}.$$

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