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Graceful Colorings and Connection in Graphs

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Graceful Colorings and Connection in Graphs

by

Alexis D. Byers

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Graceful Colorings and Connection in Graphs

Alexis D. Byers, Ph.D.
Western Michigan University, 2018

For a graph $G$ of size $m$, a graceful labeling of $G$ is an injective function $f : V(G) \rightarrow \{0, 1, \ldots, m\}$ that gives rise to a bijective function $f' : E(G) \rightarrow \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$. A graph is graceful if it has a graceful labeling. Over the years, a number of variations of graceful labelings have been introduced, some of which have been described in terms of colorings.

A proper (vertex) coloring of a graph $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices are assigned distinct colors. The minimum number of colors required of a proper vertex coloring of $G$ is its chromatic number, $\chi(G)$. Similarly, a proper edge coloring of a graph $G$ is an assignment of colors to the edges of $G$ such that adjacent edges are assigned distinct colors. The minimum number of colors required of a proper edge coloring of $G$ is its chromatic index, $\chi'(G)$.

A proper vertex coloring $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ is called a graceful $k$-coloring if the induced edge coloring $c'$ defined by $c'(uv) = |c(u) - c(v)|$ is also proper. The minimum positive integer $k$ for which $G$ has a graceful $k$-coloring is its graceful chromatic number $\chi_g(G)$. These chromatic numbers are determined for several well-known classes of graphs, including cycles, wheels and caterpillars. An upper bound for the graceful chromatic number of trees is determined in terms of its maximum degree. We also present several other results and conjectures on this coloring concept.

A graph is edge-colored if each of its edges is assigned a color (where adjacent edges may be assigned the same color). Let $G$ be an edge-colored connected graph. A path $P$ in an edge-colored graph $G$ is a rainbow path of $G$ if no two edges of $P$ are colored the same. An edge coloring $c$ of a connected graph $G$ is a rainbow coloring of $G$ if every pair of distinct vertices of $G$ are connected by a rainbow path in $G$. In this case, $G$ is rainbow-connected. The minimum number of colors needed for a rainbow coloring of $G$ is referred to as the rainbow connection number of $G$ and is denoted by $rc(G)$. Analogously, we say that a path $P$ in an edge-colored connected graph $G$ is a proper path in $G$ if no two adjacent edges of $P$ are colored the same. An edge coloring $c$ of a connected graph $G$
is a proper-path coloring of $G$ if every pair of distinct vertices of $G$ are connected by a proper path in $G$. If $k$ colors are used, then $c$ is referred to as a proper-path $k$-coloring. The minimum $k$ for which $G$ has a proper-path $k$-coloring is called the proper connection number $pc(G)$ of $G$. Since rainbow and proper connection numbers were introduced in 2006 and 2009, respectively, these numbers have been studied by many researchers with a wide variety of applications.

A graph $G$ is Hamiltonian-connected if every pair of vertices of $G$ are connected by a Hamiltonian path, that is, every pair of vertices of $G$ are connected by a path containing every vertex of $G$. An edge coloring of a Hamiltonian-connected graph $G$ is a Hamiltonian-connected rainbow coloring if every two vertices of $G$ are connected by a rainbow Hamiltonian path. The minimum number of colors required of a Hamiltonian-connected rainbow coloring of $G$ is the rainbow Hamiltonian-connection number $hrc(G)$ of $G$. If $G$ has order $n$ and size $m$, then $n - 1 \leq hrc(G) \leq m$. The rainbow Hamiltonian-connection number is investigated for the Cartesian product of complete graphs and odd cycles with $K_2$. As a result of this, both the lower bound $n - 1$ and the upper bound $m$ for $hrc(G)$ are shown to be sharp. Furthermore, the rainbow Hamiltonian-connection numbers are investigated for several classes of Hamiltonian-connected graphs, including the join of graphs $G$ and $K_2$, where $G$ contains a Hamiltonian path, squares of Hamiltonian graphs, and Hamiltonian graphs of minimum size. Correspondingly, an edge coloring of a Hamiltonian-connected graph $G$ is a proper Hamiltonian-path coloring if every two vertices of $G$ are connected by a properly colored Hamiltonian path. The minimum number of colors in a proper Hamiltonian-path coloring of $G$ is the proper Hamiltonian-connection number of $G$, denoted by $hpc(G)$. Proper Hamiltonian-connection numbers are determined for several classes of Hamiltonian-connected graphs as well as two classes of Hamiltonian-connected graphs of minimum size. In particular, it is shown that $hpc(K_n) = 2$ for $n \geq 4$ and $hpc(C \square K_2) = 3$ for all prisms $C \square K_2$, where $C$ is an odd cycle.

Let $G$ be an edge-colored connected graph, where adjacent edges may be colored the same, and let $\ell$ be the length of a longest path in $G$. For an integer $k \geq 2$, a path $P$ in $G$ is a $k$-rainbow path if every subpath of $P$ having length at most $k$ is a rainbow path. In particular, every $k$-rainbow path of length at most $k$ is a rainbow path. An edge coloring $c$ of $G$ is a $k$-rainbow coloring if every pair of distinct vertices of $G$ are connected by a $k$-rainbow path in $G$. In this case, the graph $G$ is $k$-rainbow connected (with respect to $c$). If $j$ colors are used to produce a $k$-rainbow coloring of $G$, then $c$ is referred to
as a $k$-rainbow $j$-edge coloring (or simply a $k$-rainbow $j$-coloring). The minimum $j$ for which $G$ has a $k$-rainbow $j$-coloring is called the $k$-rainbow connection number $rc_k(G)$ of $G$. Hence, we have $rc_2(G) = pc(G)$, $rc_\ell(G) = rc(G)$ if $\ell$ is the length of a longest path in $G$, and $k$-rainbow colorings are intermediate to rainbow and proper-path colorings for all integers $k$ with $2 \leq k \leq \ell$. Therefore, for a nontrivial connected graph $G$ of size $m$ whose longest paths have length $\ell$,
\[
1 \leq pc(G) = rc_2(G) \leq rc_3(G) \leq \cdots \leq rc_\ell(G) = rc(G) \leq m.
\]

For an integer $k \geq 2$, a Hamiltonian path $P$ in $G$ is a $k$-rainbow Hamiltonian path if every subpath of $P$ having length at most $k$ is a rainbow path. An edge coloring of $G$ is a $k$-rainbow Hamiltonian-path coloring if every two vertices of $G$ are connected by a $k$-rainbow Hamiltonian path in $G$. The minimum number of colors in a $k$-rainbow Hamiltonian-path coloring of $G$ is the $k$-rainbow Hamiltonian-connection number of $G$. We investigate the $k$-rainbow Hamiltonian-path colorings in two well-known classes of Hamiltonian-connected graphs, namely the join $G \lor K_1$ of a Hamiltonian graph $G$ and the trivial graph $K_1$ and the prism $G \square K_2$ where $G$ is a Hamiltonian graph of odd order. Other results and open questions are also presented.
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# Table of Contents

ACKNOWLEDGMENTS ii  
LIST OF FIGURES v  

Chapter 1. Introduction 1  
1.1 Graph Colorings ................................................................. 1  
1.2 Graceful Graphs ................................................................. 2  
1.3 Edge-Graceful Graphs ......................................................... 8  
1.4 Modular Edge-Graceful Graphs ............................................. 10  
1.5 Rainbow Colorings ............................................................. 12  

Chapter 2. Graceful Colorings of Graphs 14  
2.1 Introduction ............................................................................ 14  
2.2 Graceful Chromatic Numbers of Graphs .................................. 14  
2.3 Cycles, Paths and Wheels ..................................................... 19  
2.4 Regular Complete Multipartite Graphs .................................. 22  
2.5 Caterpillars ............................................................................ 30  
2.6 An Upper Bound .................................................................... 35  
2.7 Some Problems Dealing with Graceful Colorings ..................... 38  

Chapter 3. Rainbow Connection in Graphs 40  
3.1 Introduction ............................................................................ 40  
3.2 The Graphs $K_n \square K_2$ ....................................................... 45  
3.3 Hamiltonian-Connected Prisms .............................................. 52  
3.4 The Square of Hamiltonian Graphs ...................................... 58  
3.5 Minimum Hamiltonian-Connected Graphs ............................ 61  

Chapter 4. Chromatic Connection in Graphs 66  
4.1 Introduction ............................................................................ 66  
4.2 Proper Hamiltonian-Path Colorings ...................................... 67
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3 Hamiltonian-Connected Prisms</td>
<td>71</td>
</tr>
<tr>
<td>4.4 Minimum Hamiltonian-Connected Graphs</td>
<td>72</td>
</tr>
<tr>
<td>4.5 $k$-Rainbow Connection in Graphs</td>
<td>82</td>
</tr>
<tr>
<td>4.6 On $k$-Rainbow Colorings of Wheels</td>
<td>84</td>
</tr>
<tr>
<td>4.7 On $k$-Rainbow Colorings in Prisms</td>
<td>90</td>
</tr>
<tr>
<td>Chapter 5. Topics for Further Study</td>
<td>97</td>
</tr>
<tr>
<td>5.1 Graceful Rainbow and Proper Colorings</td>
<td>97</td>
</tr>
<tr>
<td>3.2 Harmonious Rainbow and Proper Colorings</td>
<td>98</td>
</tr>
<tr>
<td><strong>BIBLIOGRAPHY</strong></td>
<td>99</td>
</tr>
</tbody>
</table>
List of Figures

1.1 A vertex-distinguishing edge coloring of the Petersen graph .......................... 2
1.2 A $K_3$-decomposition of $K_7$ ............................................................... 3
1.3 Three graphs that are not graceful ............................................................ 7
1.4 Two edge-graceful graphs ................................................................. 9
1.5 Two modular edge-graceful graphs ................................................... 10

2.1 Graceful colorings of $K_4$ and $C_4$ .................................................. 15
2.2 A graceful 5-coloring of $Q_3$ ............................................................ 17
2.3 Graceful colorings of five connected cubic graphs with diameter 2 .......... 18
2.4 Graceful colorings of $W_6, W_7, W_8$ ...................................................... 21
2.5 A graceful coloring of $K_{4(4)}$ .......................................................... 25
2.6 A graceful coloring of $K_{5,5,5}$ .......................................................... 27
2.7 A graph $G$ with $\chi_g(G) = \left\lceil \frac{5d}{3} \right\rceil$ ........................................... 29
2.8 Illustrating the coloring $c$ for $k = 1$ .................................................. 31
2.9 A tree $T_0$ with $\chi_g(T_0) = \Delta(T_0) + 3$ ........................................... 35
2.10 The trees $T_{3,2}$ and $T_{6,2}$ ............................................................. 38

3.1 An $n$-edge coloring $c$ of $W_n = C_n \lor K_1$ ........................................ 43
3.2 An $(n + 1)$-edge coloring $c$ of $P_n \lor \overline{K}_2$ ................................. 45
3.3 A Hamiltonian-connected rainbow coloring of $K_3 \square K_2$ using 7 colors .... 45
3.4 A rainbow Hamiltonian $u_1 - u_2$ path of $K_3 \square K_2$ ...................... 47
3.5 A step in the proof of Theorem 3.2.1 .................................................. 47
3.6 A Hamiltonian-connected rainbow 11-coloring of $W \square K_2$ for $n = 6$ .... 49
3.7 The set $X$ of edges in $C_7 \square K_2$ ..................................................... 54
3.8 The two Hamiltonian $u_1 - u_2$ paths in $C_7 \square K_2$ ............................ 55
3.9 The path $R_1$ in $C_7 \square K_2$ .......................................................... 56
List of Figures - Continued

3.10 The paths $R_3$ and $R_7$ in $C_7 \square K_2$ ........................................... 57
3.11 A Hamiltonian-connected rainbow coloring of $C_n^2$ using $n$ colors ............... 59
3.12 A Hamiltonian-connected rainbow coloring of $C_6^2$ using 5 colors .................... 60
3.13 Graphs $H_2, H_3$ and $H_k$ ................................................................. 61
3.14 The graph $H_3$ ..................................................................................... 62
3.15 A rainbow Hamiltonian coloring of $G$ ......................................................... 63
3.16 The graph $G^*$ ..................................................................................... 64
3.17 The digraph $D$ in the proof of Theorem 3.5.2 ................................................. 65

4.1 A proper 4-edge coloring and a proper Hamiltonian-path 2-coloring of $C_6^2$ .... 67
4.2 A proper 3-edge coloring of $K_3 \square K_2$ ..................................................... 68
4.3 Two Hamiltonian $u_1 - u_3$ paths in $C_9 \square K_2$ ......................................... 72
4.4 Two Hamiltonian $u_3 - u_5$ paths in $C_9 \square K_2$ ......................................... 73
4.5 Graphs $H_2, H_3$ and $H_k$ ................................................................. 74
4.6 Edge colorings of $H_k$ ............................................................................... 74
4.7 A step in the proof of Proposition 4.4.3 when $k$ is odd ................................. 75
4.8 Graphs $F_3, F_4$ and $F_k$ ................................................................. 76
4.9 Edge colorings of $F_k$ ............................................................................... 77
4.10 The colors of some edges of $P_k \square K_2$ in Case 1 for $k \in \{5,7\}$ ................. 79
4.11 The colors of some edges of $P_k \square K_2$ in Subcase 2.1 for $k \in \{6,8\}$ .......... 81
4.12 The colors of some edges of $P_k \square K_2$ in Subcase 2.2 ............................. 82
4.13 Three edge colorings of a graph $G$ .......................................................... 83
4.14 Proper Hamiltonian-path colorings of $W_n$ for $n = 4, 5, 6$ ............................. 88
4.15 The graph $C_3 \square K_2$ ................................................................. 91
4.16 A 4-rainbow Hamiltonian-path coloring of $C_9 \square K_2$ for $k = 4$ ................. 95
Chapter 1

Introduction

1.1 Graph Colorings

Graph coloring is one of the most popular research areas in graph theory. The most studied colorings are proper vertex colorings and proper edge colorings. A proper vertex coloring of a graph $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices are assigned distinct colors. The minimum number of colors required of a proper vertex coloring of $G$ is its chromatic number $\chi(G)$. And a proper edge coloring of a graph $G$ is an assignment of colors to the edges of $G$ such that adjacent edges are assigned distinct colors. The minimum number of colors required of a proper edge coloring of $G$ is its chromatic index $\chi'(G)$.

A vertex coloring (or labeling) of a graph $G$ is vertex-distinguishing if distinct vertices of $G$ are assigned distinct colors (or labels). There are numerous occasions when an edge coloring of a graph (not necessarily a proper coloring) gives rise to a vertex-distinguishing coloring (see [32, pp. 370-385] or [80], for example). An edge coloring (or labeling) of a graph $G$ is edge-distinguishing if distinct edges of $G$ are assigned distinct colors (or labels). There are also occasions when a vertex coloring of a graph (not necessarily a proper coloring) gives rise to an edge-distinguishing labeling (see [44, 73], [32, pp. 359-370] or [80, 81], for example).

One of best known examples of vertex-distinguishing colorings was introduced by Chartrand et al. in [23]. At the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne in 1986, a weighting of a connected graph $G$ was introduced for the purpose of producing a weighted graph whose degrees (obtained by adding the weights of the incident edges of each vertex) were distinct. Such a weighted graph was called irregular. This concept has also been looked at in another
manner. Let $E_v$ denote the set of edges of $G$ incident with a vertex $v$. For the set $\mathbb{N}$ of positive integers, an edge coloring $c : E(G) \rightarrow \mathbb{N}$, where adjacent edges may be colored the same, is said to be *vertex-distinguishing* if the coloring $s : V(G) \rightarrow \mathbb{N}$ induced by $c$ and defined by

$$s(v) = \sum_{e \in E_v} c(e),$$

has the property that $s(x) \neq s(y)$ for every two distinct vertices $x$ and $y$ of $G$. For example, the edge coloring of the Petersen graph with the colors $1, 2, \ldots, 5$ shown in Figure 1.1 is vertex-distinguishing, where the color $s(v)$ of each vertex $v$ is placed inside $v$.

![Figure 1.1: A vertex-distinguishing edge coloring of the Petersen graph](image)

The main emphasis of this research deals with minimizing the largest color assigned to the edges of a graph $G$ to produce an irregular graph. The largest such color is referred to as the *irregularity strength* of $G$. In fact, the irregularity strength of the Petersen graph is 5. Much work has been done in this area of research (see [2, 38, 41, 80], for example). In recent years, a variety of edge colorings have been introduced which induce, in a number of ways, vertex colorings possessing desirable properties (see [3, 19, 20, 21, 39], for example). There has also been a variety of vertex or edge colorings that induce other edge or vertex colorings, possessing several prescribed properties (see [4, 81], for example). We begin with vertex colorings or labelings that induce edge-distinguishing colorings or labelings of graphs.

### 1.2 Graceful Graphs

The best known example of an edge-distinguishing labeling is a graceful labeling. In 1968, Rosa [73] introduced a vertex-distinguishing labeling of a graph that induces an
edge-distinguishing labeling defined by subtracting the labels of the two vertices incident with each edge.

First, we present some historical background of this famous graph labeling and its connection with decompositions of certain complete graphs. The references [25, 32, 49] provide more information on this topic. It is known that the complete graph $K_7$ can be decomposed into seven triangles, that is, $K_7$ is $K_3$-decomposable. This is illustrated in Figure 1.2.

![Figure 1.2: A $K_3$-decomposition of $K_7$](image)

Other complete graphs can be decomposed into triangles. The following result describes all complete graphs that are $K_3$-decomposable (see [32], for example).

**Theorem 1.2.1** The complete graph $K_n$ of order $n \geq 3$ is $K_3$-decomposable if and only if $n \equiv 1 \pmod{6}$ or $n \equiv 3 \pmod{6}$.

Theorem 1.2.1 is actually a graph theoretic interpretation of those integers $n \geq 3$ for which there exists a Steiner triple system $S_n$ of order $n$. A Steiner triple system $S_n$ of order $n \geq 3$ is a collection of triples (3-element subsets) of the set $[n] = \{1, 2, \ldots, n\}$ having the property that every pair of elements of $[n]$ belongs to exactly one of these triples. In terms of Steiner triple systems, the preceding theorem is stated as follows.
**Theorem 1.2.2**  There is a Steiner triple system $S_n$ of order $n \geq 3$ if and only if $n \equiv 1 \pmod{6}$ or $n \equiv 3 \pmod{6}$.

Steiner triple systems are named for the Swiss mathematician Jacob Steiner. This theorem was not proved by Steiner, however. It was proved by Thomas Kirkman [58]. The origin of this theorem goes back many years. A popular journal during the 1840s was the *Lady's and Gentleman's Diary*. In 1844 the editor Wesley Woolhouse stated what was called Prize Question No. 1733 in the magazine:

*Determine the number of combinations that can be made of $n$ symbols, $p$ symbols in each; with this limitation, that no combination of $q$ symbols which may appear in any one of them shall be repeated in any other.*

The 1845 volume of the journal contained many attempted, but ultimately unsuccessful, solutions to this rather awkwardly worded problem. After a year, the problem was replaced by the special case where $p = 3$ and $q = 2$. The editor drew attention to the difficulties even with this problem, pointing out that when $n = 10$, it is impossible to find a system of triples ($p = 3$) in which each pair ($q = 2$) occurs exactly once.

On 15 December 1846, Thomas Kirkman presented a paper dealing with this substitute Prize Question to the Literary and Philosophical Society of Manchester. Shortly afterwards (in 1847), an article by him [58] was published in the *Cambridge and Dublin Mathematical Journal*. In it, he addressed the following problem:

*How many triples can be formed with $x$ symbols in such a way that no pair of symbols occurs more than once in the triple?*

In fact, the previously stated theorem dealing with Steiner triple systems was proved by Kirkman in 1847. Seven years later, an article by Steiner appeared in a different journal, asking conditions under which such systems (Steiner triple systems) exist. Steiner was obviously unaware of Kirkman’s theorem. Six years after Steiner’s paper appeared, a paper by M. Reiss was published that answered Steiner’s question (and reproved Kirkman’s theorem). Despite the fact that Steiner proved none of these results, these triples became known as Steiner triples.

Let’s return to the problem that appeared in the 1844 issue of the *Lady’s and Gentleman’s Diary* and which led to Kirkman’s theorem and Steiner triple systems. We saw that the solution to this problem when $n = 7$, $p = 3$, $q = 2$ is 7. That is, there are seven distinct 3-element subsets of $\{7\} = \{1, 2, \ldots, 7\}$ such that each 2-element subset
of \([7]\) belongs to \textit{exactly one} of these 3-elements subsets. Suppose that we increase the values of \(n, p, q\) by 1 each, obtaining \(n = 8, p = 4, q = 3\). In this case, we are being asked to determine the number of combinations (4-element subsets in this case) of the set \([8] = \{1, 2, \ldots, 8\}\) such that each 3-element subset of \([8]\) belongs to at most one of these 4-elements subsets. Of the \(\binom{8}{4} = 70\) distinct 4-element subsets of \([8]\), there are 14 of these subsets such that each of the \(\binom{8}{3} = 56\) distinct 3-element subsets of \([8]\) belongs to \textit{exactly one} of these 4-element subsets:

\[
\begin{align*}
1234 & \quad 1256 & \quad 1278 & \quad 1357 & \quad 1368 & \quad 1458 & \quad 1467 \\
2358 & \quad 2367 & \quad 2457 & \quad 2468 & \quad 3456 & \quad 3478 & \quad 5678.
\end{align*}
\]

After the publication of Kirkman's 1847 paper [58], he noticed that there was a Steiner triple system \(S_{15}\) that had an additional property. Kirkman observed that from a set of 15 symbols, there are 35 triples that can be divided into seven sets of five triples in a such a way that each of the 15 symbols occurs once in each set of five triples:

\[
\begin{align*}
123 & \quad 145 & \quad 167 & \quad 357 & \quad 346 & \quad 256 & \quad 247 \\
4ae & \quad 2bd & \quad 2ac & \quad 1eb & \quad 1cd & \quad 1ef & \quad 1gh \\
5cg & \quad 3fh & \quad 3eg & \quad 2fg & \quad 2eh & \quad 3bc & \quad 3ad \\
6bh & \quad 6ag & \quad 4bf & \quad 4ch & \quad 5af & \quad 4dg & \quad 5be \\
7df & \quad 7ce & \quad 5dh & \quad 6de & \quad 7bg & \quad 7ah & \quad 6cf
\end{align*}
\]

Kirkman placed his observation in a unique setting in an 1850 issue of the \textit{Lady's and Gentleman's Diary} in which he challenged readers to discover the existence of such an arrangement for themselves.

\textit{Fifteen young ladies in a school walk out three abreast for seven days in succession; it is required to arrange them daily, so that no two shall walk twice abreast.}

Norman Biggs wrote:

\textit{It is unfortunate that such a trifle should overshadow the many more significant contributions which its author was to make to mathematics. Nevertheless, it is his most lasting memorial.}

In any case, Steiner triple systems with this additional property gave rise to a concept that was named after Kirkman. A \textit{Kirkman triple system} of order \(n\) is a Steiner triple
system $S_n$ with the added property that the triples in $S_n$ can be partitioned into subsets so that each of the $n$ symbols appears exactly once in each subset. Thus, a solution to Kirkman’s Schoolgirl Problem constitutes a Kirkman triple system. We know that there is a Steiner triple system of order $n$ if and only if $n \equiv 1 \pmod{6}$ or $n \equiv 3 \pmod{6}$. It is not difficult to show that if a Steiner triple system is also a Kirkman triple system, then $n \equiv 3 \pmod{6}$ (see [58]). Over a hundred years later, in 1971, Dijen Ray-Chaudhuri and Richard Wilson [71] showed that there exists a Kirkman triple system of order $n$ if and only if $n \equiv 3 \pmod{6}$.

**Theorem 1.2.3** [58, 71]  *There exists a Kirkman triple system of order $n \geq 3$ if and only if $n \equiv 3 \pmod{6}$.***

For $n = 3r$ where $r$ is an odd integer, showing the existence of a Kirkman triple system of order $n$ is equivalent to verifying the graph theoretic fact that $K_n$ is $rK_3$-decomposable. After Kirkman, many graph theorists have shown an interest in decompositions of complete graphs (see [25, 32] for example). In particular, Ringel and Kotzig [72] posted the following conjecture in 1963.

**Conjecture 1.2.4** [72]  *For every tree $T$ of size $m$, the complete graph $K_{2m+1}$ is cyclically $T$-decomposable.*

The interest in Steiner triple systems led to graph decompositions, which then led to an interest in graph labelings. Alexander Rosa, one of Kotzig’s students, became interested in Kirkman’s Schoolgirl Problem, which led to his interest in Steiner triple systems, which led to graph decompositions, which then led to an interest in graph labelings. During July 5-8, 1966, Rosa attended, with his advisor Kotzig, the Theory of Graphs International Symposium in Rome and spoke on “On certain valuations of the vertices of a graph.” One such valuation was the so-called $\beta$-valuation. Specifically, for a graph $G$ of size $m$, a vertex labeling called a $\beta$-valuation by Rosa is an injective function $f : V(G) \to \{0, 1, \ldots, m\}$

for which the induced edge labeling $f' : E(G) \to \{1, 2, \ldots, m\}$

defined by

$$f'(uv) = |f(u) - f(v)| \quad (1.1)$$
is bijective. In 1972, Golomb [46] referred to a $\beta$-valuation as a \textit{graceful labeling} and a graph possessing a graceful labeling as a \textit{graceful graph}. It is this terminology that has become standard. Rosa proved the following result in connection with the Ringel and Kotzig conjecture (Conjecture 1.2.4).

\textbf{Theorem 1.2.5} [73] \textit{If $T$ is a graceful tree of size $m$, then $K_{2m+1}$ is cyclically $T$-decomposable.}

Over the past few decades the subject of graph labelings has grown in popularity. Gallian [43] has compiled a periodically updated survey of many kinds of labelings and numerous results, obtained from well over a thousand referenced research articles. A major problem in this area involves determining which graphs are graceful. Among the results obtained on graceful graphs are the following:

1. The cycle $C_n$ is graceful if and only if $n \equiv 0 \pmod{4}$ or $n \equiv 3 \pmod{4}$.
2. The complete graph $K_n$ is graceful if and only if $n \leq 4$.
3. The graph $K_{s,t}$ is graceful for all positive integers $s$ and $t$.
4. The $n$-cube $Q_n$ is graceful for all positive integers $n$.
5. The path $P_n$ is graceful for all positive integers $n$.
6. The grid $P_s \square P_t$ is graceful for all positive integers $s$ and $t$.
7. Every caterpillar is graceful.
8. Every tree with at most four end-vertices is graceful.
9. Every tree of order at most 27 is graceful.

While the three graphs shown in Figure 1.3 are the only connected graphs of order 5 that are not graceful, it has been shown that almost all graphs are not graceful [36].

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
$G_1$ : & $G_2$ : & $G_3$ : \\
\end{tabular}
\caption{Three graphs that are not graceful}
\end{figure}
One of the best known conjectures dealing with graceful graphs involves trees and is due to Kotzig and Ringel (see [43]).

**The Graceful Tree Conjecture**  
*Every nontrivial tree is graceful.*

The *gracefulness* $grac(G)$ of a graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the smallest positive integer $k$ for which it is possible to label the vertices of $G$ with distinct elements of the set $\{0, 1, 2, \ldots, k\}$ in such a way that distinct edges receive distinct labels, where each edge is labeled as described in (1.1). The gracefulness of every such graph is defined, for if we label $v_i$ by $2^{i-1}$ for $1 \leq i \leq n$, then a vertex labeling with this property exists. Thus, if $G$ is a graph of order $n$ and size $m$, then

$$m \leq grac(G) \leq 2^{n-1}.$$ 

If $grac(G) = m$, then $G$ is graceful. The gracefulness of a graph $G$ can therefore be considered as a measure of how close $G$ is to being graceful – the closer the gracefulness is to $m$, the closer the graph is to being graceful. The exact values of $grac(K_n)$ were determined for $1 \leq n \leq 10$ in [46]. For example, $grac(K_4) = 6$, $grac(K_5) = 11$ and $grac(K_6) = 17$. The exact value of $grac(K_n)$ is not known in general, however. On the other hand, Erdős showed that $grac(K_n) \sim n^2$ (see [46]).

### 1.3 Edge-Graceful Graphs

In 1985, Lo [61] introduced a dual type of graceful labeling – this one dealing with edge labelings. Let $G$ be a connected graph of order $n \geq 2$ and size $m$. For a vertex $v$ of $G$, let $N(v)$ denote the neighborhood of $v$. An *edge-graceful labeling* of $G$ is a bijective function $f : E(G) \to \{1, 2, \ldots, m\}$ that gives rise to a bijective function

$$f' : V(G) \to \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$$

given by

$$f'(v) = \sum_{u \in N(v)} f(uv),$$

where the sum is computed in $\mathbb{Z}_n$. Thus, if $u$ and $v$ are two distinct vertices of $G$, then $f'(u) \neq f'(v)$. A graph that admits an edge-graceful labeling is called an *edge-graceful graph*. Figure 1.4 shows two edge-graceful graphs $C_5$ and $K_{1,4}$ together with an edge-graceful labeling for each of them. It is well known that $C_n$ is graceful if and only if $n \equiv 0, 3 \pmod{4}$ and so $C_5$ is not graceful.
It was observed in [61] that if $G$ is an edge-graceful graph of order $n$ and size $m$, then

\[
\binom{n}{2} \equiv 2 \binom{m + 1}{2} \pmod{n}.
\] (1.2)

Since $\binom{n}{2} = 2 \binom{m + 1}{2} \pmod{n}$ if $G$ is a tree of odd order $n$ (and so $m = n - 1$), a tree satisfies (1.2) if and only if its order is odd. It is known that the path $P_n$ of odd order $n$ is edge-graceful. It was conjectured by Lee [60] that every nontrivial tree of odd order is edge-graceful. In fact, it was also conjectured by Lee [60] that every connected graph of order $n$ with $n \not\equiv 2 \pmod{4}$ is edge-graceful. Among the results obtained on edge-graceful graphs are the following:

1. The complete graph $K_n$ is edge-graceful if and only if $n \not\equiv 2 \pmod{4}$.

2. Every odd cycle is edge-graceful.

3. The Cartesian product $C_m \square C_n$ is edge-graceful if and only if $m$ and $n$ are both odd.

It was observed in [49] that in the definition of an edge-graceful labeling of a connected graph $G$ of order $n \geq 2$ and size $m$, the edge labeling $f$ is required to be one-to-one. Since the induced vertex labels $f'(v)$ are obtained by addition in $\mathbb{Z}_n$, the function $f$ is actually a function from $E(G)$ to $\mathbb{Z}_n$ and in general is not one-to-one. Dividing $m$ by $n$, we obtain $m = nq + r$, where $q = \lfloor m/n \rfloor$ and $0 \leq r \leq n - 1$. Hence, in an edge-graceful labeling of $G$, $q + 1$ edges are labeled $i$ for each $i$ with $1 \leq i \leq r$ and $q$ edges are labeled $i$ for each $i$ with $r + 1 \leq i \leq n$ (in $\mathbb{Z}_n$). Thus, this edge labeling $f : E(G) \rightarrow \mathbb{Z}_n$ is a one-to-one function only when $m = n - 1$ or $m = n$. This observation gives rise to another concept (see [49]).
1.4 Modular Edge-Graceful Graphs

Let $G$ be a connected graph of order $n \geq 3$ and let $f : E(G) \rightarrow \mathbb{Z}_n$, where $f$ need not be one-to-one. Let $f' : V(G) \rightarrow \mathbb{Z}_n$ be defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If $f'$ is one-to-one, then $f$ is called a modular edge-graceful labeling and $G$ is a modular edge-graceful graph. Consequently, every edge-graceful graph is a modular edge-graceful graph. This concept was introduced in 1991 by Jothi [45] under the terminology of line-graceful graphs (also see [43]). The graphs $G_1 = C_4$ and $G_2$ in Figure 1.5 are both modular edge-graceful, with a modular edge-graceful labelings shown for each one as well. In fact, the graph $G_2$ is neither graceful nor edge-graceful.

\[ G_1 = C_4 : \begin{array}{ccc} 1 & 3 & 0 \\ 3 & 0 & 1 \end{array} \quad G_2 : \begin{array}{ccc} 1 & 0 & 3 \\ 4 & 2 & 0 \end{array} \]

Figure 1.5: Two modular edge-graceful graphs

It is known that if $G$ is a connected graph of order $n \geq 3$ for which $n \equiv 2 \pmod{4}$, then $G$ is not modular edge-graceful. Furthermore, it was conjectured that if $T$ is a tree of order $n \geq 3$ for which $n \not\equiv 2 \pmod{4}$, then $T$ is modular edge-graceful (see [43]). This conjecture was verified in [54]. In fact, the conjecture holds not only for trees but for all connected graphs (see [54]).

**Theorem 1.4.1** A connected graph of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod{4}$.

For every connected graph $G$ of order $n$, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \rightarrow \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \rightarrow \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. The number $k$ is called the modular edge-gracefulness $\text{meg}(G)$ of $G$. Thus, $\text{meg}(G) \geq n$ and $\text{meg}(G) = n$ if and only if $G$ is a modular edge-graceful graph of order $n$. If $G$ is not modular edge-graceful, then $\text{meg}(G) \geq n + 1$. As with the gracefulness of a graph, the modular edge-gracefulness of a graph $G$ is a measure of how close $G$ is to being modular edge-graceful. The number $\text{meg}(G)$ was determined for every connected graph $G$ in [54].
Theorem 1.4.2  If $G$ is a nontrivial connected graph of order $n \geq 6$ that is not modular edge-graceful, then $\text{meg}(G) = n + 1$.

If $G$ is a modular edge-graceful spanning subgraph of a graph $H$ where $G$ and $H$ are connected, then a modular edge-graceful labeling of $G$ can be extended to a modular edge-graceful labeling of $H$ by assigning the label 0 to each edge of $H$ that does not belong to $G$. Modular edge-graceful labelings of a graph that assigns the label 0 to some edges of the graph play an important role in establishing Theorems 1.4.1 and 1.4.2. For this reason, those modular edge-graceful labelings in which 0 is not permitted were investigated in [55]. This gives rise to another concept and to other problems. More formally, for a connected graph $G$ of order $n \geq 3$, let $f : E(G) \to \mathbb{Z}_n - \{0\}$, where $f$ need not be one-to-one and let $f' : V(G) \to \mathbb{Z}_n$ be defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_n$. If $f'$ is one-to-one, then $f$ is called a nowhere-zero modular edge-graceful labeling and $G$ is a nowhere-zero modular edge-graceful graph. A characterization of connected nowhere-zero modular edge-graceful graphs was established in [55].

Theorem 1.4.3  A connected graph $G$ of order $n \geq 3$ is nowhere-zero modular edge-graceful if and only if

(i) $n \not\equiv 2 \pmod{4}$,

(ii) $G \neq K_3$ and

(iii) $G$ is not a star of even order.

For every connected graph $G$ of order $n$, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k - \{0\}$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by $f'(v) = \sum_{u \in N(v)} f(uv)$, where the sum is computed in $\mathbb{Z}_k$, is one-to-one. This number $k$ is referred to as the nowhere-zero modular edge-gracefulness of $G$ and is denoted by $\text{nzg}(G)$. Thus, $\text{nzg}(G) = n$ if and only if $G$ is nowhere-zero modular edge-graceful and so $\text{nzg}(G) \geq n + 1$ if $G$ is not nowhere-zero modular edge-graceful. For a connected graph $G$ of order $n \geq 3$ with $n \not\equiv 2 \pmod{4}$ that is not nowhere-zero modular edge-graceful, the exact value of $\text{nzg}(G)$ has been determined (see [55]).

Theorem 1.4.4  If $G$ is a connected graph of order $n \geq 3$ that is not nowhere-zero modular edge-graceful, then $\text{nzg}(G) \in \{n + 1, n + 2\}$. Furthermore,
(i) if \( n \not\equiv 2 \pmod{4} \), then \( nzg(G) = n + 1 \) if and only if \( G = K_3 \) and \( nzg(G) = n + 2 \) if and only if \( G \) is a star of even order.

(ii) if \( n \equiv 2 \pmod{4} \), then \( nzg(G) = n + 2 \) if and only if \( G \) is a star.

1.5 Rainbow Colorings

Graceful labelings have also been looked at in terms of colorings. A rainbow vertex coloring of a graph \( G \) of size \( m \) is an assignment \( f \) of distinct colors to the vertices of \( G \). If the colors are chosen from the set \( \{0, 1, \ldots, m\} \), resulting in each edge \( uv \) of \( G \) being colored \( f'(uv) = |f(u) - f(v)| \) such that the colors assigned to the edges of \( G \) are also distinct, then this rainbow vertex coloring results in a rainbow edge coloring \( f' : E(G) \to \{1, 2, \ldots, m\} \). So, such a rainbow vertex coloring is a graceful labeling of \( G \).

A well-studied concept of rainbow colorings was introduced and studied by Chartrand, Johns, McKeon and Zhang in 2006. A rainbow coloring of a connected graph \( G \) (where adjacent edges may be colored the same) is an edge coloring \( c \) of \( G \) with the property that for every two vertices \( u \) and \( v \) of \( G \), there exists a \( u-v \) rainbow path (no two edges of the path are colored the same). In this case, \( G \) is said to be rainbow-connected (with respect to \( c \)). The minimum number of colors needed for a rainbow coloring of \( G \) is referred to as the rainbow connection number of \( G \) and is denoted by \( rc(G) \). The first paper [24] on this topic was published in 2008. In recent years, this topic has been studied by many and, in fact, there is a book [77] on rainbow colorings, published in 2012.

This also relates to a topic of great importance to our national security, that of the secure transfer of classified information (see [37]). Protection of sensitive information is crucial, but often it is just as important for that information to be efficiently shared among appropriate parties, like law enforcement and intelligence agencies. Ideally, we need a procedure that assigns information-transfer paths between approved agencies that requires a large enough number of passwords and firewalls to prohibit intruders, yet is still manageable for the agencies to implement. It is possible to represent this situation with a graph in which the vertices are the agencies and these vertices (agencies) are adjacent if there is direct access between them. If distinct passwords are represented by distinct colors, then an edge-coloring of this graph can be used to study the information-transfer paths between agencies.

Rainbow colorings of graphs can model an extremely secure method of information
transfer between agencies of a communications network. However, if we were to reduce
the restrictions on the transfer of information, in the way of number of passwords and
arrangement of passwords, then one might want to consider information-transfer paths
that require only sequential passwords to be distinct. This would be well represented
using proper edge-colorings of $G$ instead of rainbow edge-colorings.

Inspired by graceful labelings and proper colorings in graphs, another type of vertex
coloring was introduced by Gary Chartrand (see [11]) that induces an edge coloring,
where both colorings are proper rather than rainbow. This is one of the main topics
in this work. The other main topic we study here involves rainbow colorings in some
well-known classes of graphs. We refer to the book [32] for graph theory notation and
terminology not described in this paper.
Chapter 2
Graceful Colorings of Graphs

2.1 Introduction

As we mentioned in Chapter 1, graceful labelings have also been looked at in terms of colorings. For example, let $G$ be a graph of size $m$ with a vertex coloring $f : V(G) \rightarrow \{0,1,\ldots,m\}$ resulting in the edge coloring $f' : E(G) \rightarrow [m]$ defined by $f'(uv) = |f(u) - f(v)|$. If the colors assigned to the vertices and edges of $G$ are both distinct, then this rainbow vertex coloring results in a rainbow edge coloring. So, such a rainbow vertex coloring is a graceful labeling of $G$ (see [80]).

However, as previously discussed, the colorings of graphs that have received the most attention are proper vertex colorings and proper edge colorings. In such colorings of a graph $G$, every pair of adjacent vertices, or every pair of adjacent edges, are assigned distinct colors. The minimum number of colors needed in a proper vertex coloring of $G$ is its chromatic number, denoted $\chi(G)$, while the minimum number of colors needed in a proper edge coloring of $G$ is its chromatic index, denoted $\chi'(G)$.

Combining the concepts of graceful labelings and proper colorings, we now introduce a new type of vertex coloring that induces an edge coloring, both of which are proper rather than rainbow.

2.2 Graceful Chromatic Numbers of Graphs

It is useful to describe some notation for certain intervals of integers. Recall from Chapter 1 that $[b] = 1,2,\ldots,b$. In general, for positive integers $a,b$ with $a \leq b$, let $[a,b] = \{a,a+1,\ldots,b\}$ and then $[b] = [1, b]$. A graceful $k$-coloring of a nonempty graph $G$ is a proper vertex coloring $c : V(G) \rightarrow [k]$, where $k \geq 2$, that induces a proper edge coloring $c' : E(G) \rightarrow [k-1]$ defined by $c'(uv) = |c(u) - c(v)|$. A vertex coloring $c$ of a graph $G$
is a *graceful coloring* if $c$ is a graceful $k$-coloring for some $k \in \mathbb{N}$. The minimum $k$ for which $G$ has a graceful $k$-coloring is called the *graceful chromatic number* of $G$, denoted by $\chi_g(G)$. A graceful $\chi_g(G)$-coloring of $G$ is a *minimum graceful coloring*. This concept was introduced by Gary Chartrand in 2015 and first studied in [11].

Note that in a graceful labeling of a nonempty graph of size $m$, the colors are chosen from the set $\{0, 1, \ldots, m\}$ and so the color 0 could be used; while in a graceful coloring, each color is a positive integer. There are immediate lower and upper bounds for the graceful chromatic number of a graph.

**Observation 2.2.1** If $G$ is a nonempty graph of order $n$, then $\chi_g(G)$ exists and

$$\chi(G) \leq \chi_g(G) \leq \text{grac}(G) + 1 \leq 2^{n-1} + 1.$$  

Figure 2.1 shows two graceful graphs $K_4$ and $C_4$ together with a graceful coloring for each of these two graphs. In fact, $\chi_g(K_4) = 5 < \text{grac}(K_4) = 6$ and $\chi_g(C_4) = \text{grac}(C_4) = 4.$

![Graceful Coloring of K4 and C4](image)

**Figure 2.1: Graceful colorings of $K_4$ and $C_4$**

We make some additional useful observations. For a graceful $k$-coloring $c$ of a graph $G$, the *complementary coloring* $\overline{c} : V(G) \to [k]$ of $G$ is a $k$-coloring defined by $\overline{c}(v) = k + 1 - c(v)$ for each vertex $v$ of $G$. If $xy \in E(G)$, then the color $\overline{c}'(xy)$ of $xy$ induced by $\overline{c}$ is

$$\overline{c}'(xy) = |\overline{c}(x) - \overline{c}(y)| = |((k + 1) - c(x)) - ((k + 1) - c(y))| = |c(x) - c(y)| = c'(xy).$$

This results in the following observation.

**Observation 2.2.2** The complementary coloring of a graceful coloring of a graph is also graceful.

If $c$ is a graceful $k$-coloring of a graph $G$, then the restriction of $c$ to a subgraph $H$ of $G$ is also a graceful coloring. Thus, we have the following observation.
Observation 2.2.3 If $H$ is a subgraph of a graph $G$, then
\[ \chi_g(H) \leq \chi_g(G). \]

If $G$ is a disconnected graph having $p$ components $G_1, G_2, \ldots, G_p$ for some integer $p \geq 2$, then $\chi_g(G) = \max\{\chi_g(G_i) : 1 \leq i \leq p\}$. Thus, it suffices to consider only nontrivial connected graphs. If $c$ is a graceful coloring of a nontrivial connected graph $G$ and $v \in V(G)$, then $c$ must assign distinct colors to the vertices in the closed neighborhood $N[v]$ of $v$. Thus, if $u, w \in V(G)$ such that $u \neq w$ and $d(u, w) \leq 2$, then $c(u) \neq c(w)$. Furthermore, if $(x, y, z)$ is an $x-z$ path in $G$, where say $c(x) > c(z)$, then $c(x) - c(y) \neq c(y) - c(z)$ and so $c(y) \neq \frac{c(x) + c(z)}{2}$. We state these observations next.

Observation 2.2.4 Let $c : V(G) \to [k]$, $k \geq 2$, be a coloring of a nontrivial connected graph $G$. Then $c$ is a graceful coloring of $G$ if and only if

(i) for each vertex $v$ of $G$, the vertices in the closed neighborhood $N[v]$ of $v$ are assigned distinct colors by $c$ and

(ii) for each path $(x, y, z)$ of order 3 in $G$, $c(y) \neq \frac{c(x) + c(z)}{2}$.

As a consequence of condition (i) in Observation 2.2.4, it follows that if $G$ is a nontrivial connected graph, then
\[ \chi_g(G) \geq \Delta(G) + 1. \tag{2.1} \]

As an illustration, we determine $\chi_g(Q_3)$. Figure 2.2 shows a graceful 5-coloring of $Q_3$ and so $\chi_g(Q_3) \leq 5$. By (2.1), $\chi_g(Q_3) \geq 4$. Therefore, either $\chi_g(Q_3) = 4$ or $\chi_g(Q_3) = 5$. We show that $\chi_g(Q_3) \neq 4$. Assume, to the contrary, that $Q_3$ has a graceful 4-coloring using colors from the set $[4]$. By Observation 2.2.4, the four vertices in a 4-cycle in $Q_3$ must be colored differently. Thus, some vertex $v$ of $Q_3$ is colored 3. However then, the three neighbors of $v$ must be colored 1, 2, 4, which implies that two incident edges of $v$ are colored 1. Since this is impossible, $\chi_g(Q_3) = 5$. This example also illustrates the following observation.

Observation 2.2.5 If $G$ is an $r$-regular graph where $r \geq 2$, then
\[ \chi_g(G) \geq r + 2. \]
Figure 2.2: A graceful 5-coloring of $Q_3$

Since $\chi_g(K_{1,n-1}) = n = \Delta(K_{1,n-1}) + 1$, the bound in (2.1) is attained for all stars and consequently, the bound is sharp. By Brooks’ theorem [18], $\chi(G) \leq \Delta(G) + 1$ for every graph $G$ and, when $G$ is connected, $\chi(G) = \Delta(G) + 1$ if and only if $G$ is a complete graph or an odd cycle. Furthermore, by Vizing’s theorem [79], $\chi'(G) \leq \Delta(G) + 1$ for every nonempty graph $G$. Thus, $\chi_g(G) \geq \max\{\chi(G), \chi'(G)\}$. These observations together with Observation 2.2.5 yield the following.

**Proposition 2.2.6**  If $G$ is a nontrivial connected graph of order at least 3, then

$$\chi_g(G) \geq \max\{\chi(G), \chi'(G)\} + 1.$$  

The diameter $\text{diam}(G)$ of a connected graph $G$ is the largest distance between any two vertices of $G$. The following result is also a consequence of Observation 2.2.4.

**Corollary 2.2.7**  If $G$ is a connected graph of order $n \geq 3$ with diameter at most 2, then $\chi_g(G) \geq n$.

Figure 2.3 shows all five connected cubic graphs with diameter 2. The graceful chromatic number of each of these graphs equals its order. Figure 2.3 also shows a graceful coloring for each of these five graphs. This example illustrates a conjecture concerning the graceful chromatic numbers of connected cubic graphs.

**Conjecture 2.2.8**  If $G$ is a connected cubic graph, then $5 \leq \chi_g(G) \leq 10$.

While the star $K_{1,n-1}$, $n \geq 3$, is a graph of order $n$ and diameter 2 having graceful chromatic number $n$, there are other infinite classes of connected graphs having diameter 2 whose graceful chromatic number is its order.
Figure 2.3: Graceful colorings of five connected cubic graphs with diameter 2

Proposition 2.2.9 If $G$ is a complete bipartite graph of order $n \geq 3$, then

$$\chi_g(G) = n.$$  

Proof. Let $G = K_{s,t}$ be a complete bipartite graph of order $n = s + t$ with partite sets $U$ and $W$, where $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. Since the diameter of $G$ is 2, it follows by Corollary 2.2.7 that $\chi_g(G) \geq n$. Next, consider a proper coloring $c : V(G) \to [n]$ defined by $c(u_i) = i$ for $1 \leq i \leq s$ and $c(w_j) = s + j$ for $1 \leq j \leq t$. Thus, $c'(u_iw_j) = |s + (j - i)|$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. If $i$ is fixed and $1 \leq j_1 \neq j_2 \leq t$, then $|s + (j_1 - i)| \neq |s + (j_2 - i)|$ and similarly, if $j$ is fixed and $1 \leq i_1 \neq i_2 \leq s$, then $|s + (j - i_1)| \neq |s + (j - i_2)|$. Hence, $c'$ is a proper edge coloring and $c$ is a graceful $n$-coloring. Therefore, $\chi_g(G) = n$.  

In fact, there are also infinite classes of connected graphs $G$ of order $n$ such that $\text{diam}(G) = 2$ and $\chi_g(G) > n$.

Proposition 2.2.10 If $G$ is a nontrivial connected graph of order $n$ such that $\delta(G) > n/2$, then $\chi_g(G) > n$.

Proof. Since $\delta(G) > n/2$, it follows that $\text{diam}(G) \leq 2$. Assume, to the contrary, that there is a graceful $n$-coloring $c$ of $G$. By Observation 2.2.4, all vertices are assigned distinct colors by $c$ and so there is a vertex $v$ of $G$ such that $c(v) = \lceil \frac{n}{2} \rceil$. Let $S = [1, \lceil \frac{n}{2} \rceil - 1]$ and $T = [\lceil \frac{n}{2} \rceil + 1, n]$, where then $|S| \leq |T| = n - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor$. By Observation 2.2.4, at
most one element in each set \(\lceil \frac{n}{2} \rceil - i, \lceil \frac{n}{2} \rceil + i\), \(1 \leq i \leq \lceil \frac{n}{2} \rceil - 1\), can be used to color the vertices in \(N(v)\). Hence, there are at most \(\lceil \frac{n}{2} \rceil\) colors that are available for the vertices in \(N(v)\). Since \(\deg v > n/2 \geq \lceil \frac{n}{2} \rceil\), this is impossible. Therefore, \(\chi_g(G) > n\).

\[
\chi_g(\mathcal{C}_n) = \begin{cases} 4 & \text{if } n \neq 5 \\ 5 & \text{if } n = 5. \end{cases}
\]

**Proof.** Let \(C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)\) be a cycle of order \(n \geq 4\) where \(e_i = v_iv_{i+1}\) for \(i = 1, 2, \ldots, n\). First, suppose that \(n = 5\). Since \(\text{diam}(C_5) = 2\), it follows by Corollary 2.2.7 that \(\chi_g(C_5) \geq 5\). Define a vertex coloring \(c\) such that \(s_c = (1, 5, 3, 4, 2)\).

Then the induced edge coloring \(c'\) satisfies \(s_{c'} = (4, 2, 1, 2)\). Thus \(c\) is a graceful 5-coloring and so \(\chi_g(C_n) = 5\).

Next, suppose that \(n \neq 5\). First, we show that \(\chi_g(C_n) \geq 4\). Assume, to the contrary, that there is a graceful 3-coloring \(c\) of \(C_n\), say \(c(v_1) = 1\). Since \(c\) is a graceful coloring, \(\{c(v_2), c(v_n)\} = \{2, 3\}\), say \(c(v_2) = 2\) and \(c(v_n) = 3\). However then, \(c(v_3) = 3\) and so \(c'(v_1v_2) = c'(v_2v_3) = 1\), which is impossible. Hence, \(\chi_g(C_n) \geq 4\). Note: Observation 2.2.5 also implies that \(\chi_g(C_n) \geq 4\). It remains to define a graceful 4-coloring \(c\) of \(C_n\).

- \(n \equiv 0 \pmod{4}\). For \(n = 4\), let \(s_c = (1, 2, 4, 3)\). Then \(s_{c'} = (1, 2, 1, 2)\).

For \(n \geq 8\), let \(s_c = (1, 2, 4, 3, \ldots, 1, 2, 4, 3)\). Then \(s_{c'} = (1, 2, \ldots, 1, 2)\).
• \( n \equiv 1 \pmod{4} \). For \( n = 9 \), let \( s_c = (1, 2, 4, 1, 2, 4, 1, 2, 4) \).

So \( s_{c'} = (1, 2, 3, 1, 2, 3, 1, 2, 3) \).

For \( n \geq 13 \), let \( s_c = (1, 2, 4, 3, \ldots, 1, 2, 4, 3, 1, 2, 4, 1, 2, 4) \).

Then \( s_{c'} = (1, 2, 1, 2, \ldots, 1, 2, 1, 2, 3, 1, 2, 3) \).

• \( n \equiv 2 \pmod{4} \). For \( n = 6 \), let \( s_c = (1, 2, 4, 1, 2, 4) \).

Then \( s_{c'} = (1, 2, 3, 1, 2, 3) \).

For \( n \geq 10 \), let \( s_c = (1, 2, 4, 3, \ldots, 1, 2, 4, 3, 1, 2, 4, 1, 2, 4) \).

Then \( s_{c'} = (1, 2, 1, 2, \ldots, 1, 2, 1, 2, 3, 1, 2, 3) \).

• \( n \equiv 3 \pmod{4} \). In this case, \( n \geq 7 \).

Let \( s_c = (1, 2, 4, 3, \ldots, 1, 2, 4, 3, 1, 2, 4) \).

Then \( s_{c'} = (1, 2, 1, 2, \ldots, 1, 2, 1, 2, 3) \).

In each case, there is a graceful 4-coloring of \( C_n \). Therefore, \( \chi_g(C_n) = 4 \) when \( n \neq 5 \). \(\square\)

It is easy to see that \( \chi_g(P_4) = 3 \). For \( n \geq 5 \), the following is a consequence of Proposition 2.3.1.

**Proposition 2.3.2**  For each integer \( n \geq 5 \), \( \chi_g(P_n) = 4 \).

**Proof.** Let \( P_n = (v_1, v_2, \ldots, v_n) \) where \( n \geq 5 \). For \( n = 5 \), a graceful 4-coloring \( c^* \) of \( P_5 \) is defined by

\[
(c^*(v_1), c^*(v_2), c^*(v_3), c^*(v_4), c^*(v_5)) = (1, 2, 4, 1, 2)
\]

and so \( \chi_g(P_5) \leq 4 \). For \( n \geq 6 \), since \( P_n \) is a subgraph of \( C_n \), it follows by Observation 2.2.3 and Proposition 2.3.1 that \( \chi_g(P_n) \leq 4 \). We show that \( \chi_g(P_n) \neq 3 \). Suppose that there is a graceful 3-coloring \( c \) of \( P_n \). Necessarily, \( c(v_3) \neq 2 \) and so we may assume that \( c(v_3) = 1 \). Thus, \( \{c(v_2), c(v_4)\} = \{2, 3\} \), say \( c(v_2) = 2 \). However then, \( c(v_1) = 3 \) and so \( c'(v_1v_2) = c'(v_2v_3) = 1 \), which is impossible. Therefore, \( \chi_g(P_n) = 4 \). \(\square\)

We now turn our attention to wheels \( W_n \) of order \( n \geq 6 \), constructed by joining a new vertex to every vertex of an \((n-1)\)-cycle.

**Theorem 2.3.3**  If \( W_n \) is the wheel of order \( n \geq 6 \), then \( \chi_g(W_n) = n \).
Proof. Let $G = W_n$, where $C_{n-1} = (v_1, v_2, \ldots, v_{n-1}, v_1)$ and whose central vertex is $v_0$. By Corollary 2.2.7, $\chi_g(G) \geq n$. Thus, it suffices to show that $G$ has a graceful $n$-coloring. Figure 2.4 shows a graceful $n$-coloring of $W_n$ for $n = 6, 7, 8$, where the central vertex is colored 1 and the graceful $n$-coloring of $W_n$ for $n = 7, 8$ is obtained from the graceful $(n - 1)$-coloring of $W_{n-1}$ by inserting a new vertex into the cycle $C_{n-2}$ of $W_{n-1}$, joining this vertex to the central vertex and then assigning the color $n$ to this vertex.

![Figure 2.4: Graceful colorings of $W_6, W_7, W_8$](image)

Next, we show that for a given graceful $(n - 1)$-coloring of $W_{n-1}$ for some integer $n \geq 7$, in which the central vertex is colored 1, there is an edge $xy$ on the $(n - 2)$-cycle $C_{n-2}$ of $W_{n-1}$ such that (1) a new vertex $v$ can be inserted into the edge $xy$ and joined to the central vertex $v_0$ of $W_{n-1}$ to produce $W_n$ and (2) the color $n$ can be assigned to $v$ to produce a graceful $n$-coloring of the resulting graph $W_n$. Now, let there be given a graceful $(n - 1)$-coloring $c$ of $W_{n-1}$ for some integer $n \geq 7$, in which the central vertex is colored 1. It suffices to show that there exists an edge $xy$ on $C_{n-2}$ such that $c(x)$ and $c(y)$ satisfy the following two conditions:

(i) $c(x) \neq \frac{n+1}{2}$ and $c(y) \neq \frac{n+1}{2}$.

(ii) If $(x', x, y, y')$ is a path on $C_{n-2}$, then $c(x) \neq \frac{c(x')+n}{2}$ and $c(y) \neq \frac{c(y')+n}{2}$.

Let $C_{n-2} = (v_1, v_2, \ldots, v_{n-2}, v_1)$. Since the diameter of $W_{n-1}$ is 2, all vertices of $W_{n-1}$ are assigned different colors by $c$. Hence, if $c(v_{i+1}) = \frac{c(v_{i+2})+n}{2}$ for some $i$, then $c(v_j) \neq \frac{c(v_{i+2})+n}{2}$ for all $j \neq i + 1$ (where the subscripts are expressed as integers modulo $n - 2$). We consider two cases.

Case 1. $n$ is odd. Suppose that $c(v_{i+1}) = \frac{c(v_{i+2})+n}{2}$ for some $i$, in which case the edge $v_iv_{i+1}$ fails to satisfy condition (ii). Since $n = 2c(v_{i+1}) - c(v_{i+2})$ is odd, it follows that $c(v_{i+2})$ is odd. Because there are $\frac{n-3}{2}$ vertices of $C_{n-2}$ that are assigned odd colors
by \( c \) (as the central vertex is colored 1), at most \( \frac{n-3}{2} \) edges on \( C_{n-2} \) fail to satisfy condition (ii). Hence, there are at least \((n-2) - \frac{n-3}{2} = \frac{n-1}{2} \geq 3\) edges on \( C_{n-2} \) that satisfy condition (ii). Among these edges that satisfy condition (ii), at most two of them fail to satisfy condition (i). Thus, there is at least one edge \( xy \) on \( C_{n-2} \) such that \( c(x) \) and \( c(y) \) satisfy both (i) and (ii).

Case 2. \( n \) is even. Suppose that \( c(v_{i+1}) = \frac{c(v_{i+2})+n}{2} \) for some \( i \). Since \( n = 2c(v_{i+1}) - c(v_{i+2}) \) is even, it follows that \( c(v_{i+2}) \) is even. Because there are \( \frac{n-2}{2} \) vertices on \( C_{n-2} \) that are assigned even colors by \( c \), at most \( \frac{n-2}{2} \) edges fail to satisfy condition (ii). Hence, there are at least \((n-2) - \frac{n-2}{2} = \frac{n-2}{2} \geq 4\) edges that satisfy condition (ii). Since \((n+1)/2\) is not an integer, all of these edges satisfy condition (i). Therefore, there is at least one edge \( xy \) such that \( c(x) \) and \( c(y) \) satisfy both (i) and (ii).

\[\begin{cases} 
\chi_g(K_{p,p}) = 2p & \text{if } k \text{ is even} \\
\left( \frac{2}{k-2} - 3 \right) p - 2 + 1 & \text{if } k \text{ is odd} 
\end{cases}\]

\(\chi_g(K_{k(p)}) \leq \)

\(\left( \frac{2}{k-2} - 2 \right) p - 2 + 1 \)

\(\left( \frac{2}{k-2} - 3 \right) p - 2 + 1 \)

\(\text{Proof.} \) For integers \( p \geq 2 \) and \( k \geq 3 \), let \( G = K_{k(p)} \) with partite sets \( V_1, V_2, \ldots, V_k \), where \( V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,p}\} \) for \( 1 \leq i \leq k \). Define a proper coloring \( c : V(G) \to \mathbb{N} \) of \( G \) by assigning colors recursively to the vertices of \( V_1, V_2, \ldots, V_k \) (in this order) as follows.
For $1 \leq j \leq p$, let

\[
\begin{align*}
c(v_{1,j}) &= j \\
c(v_{2,j}) &= c(v_{1,p}) + j = p + j \\
c(v_{3,j}) &= 2c(v_{2,p}) + (j-1) = 4p + (j-1).
\end{align*}
\]

In general, for $i \geq 4$ and $1 \leq j \leq p$, let

\[
\begin{align*}
c(v_{i,j}) &= c(v_{i-1,p}) + j \quad \text{if } i \text{ is even} \\
c(v_{i,j}) &= 2c(v_{i-1,p}) + (j-1) \quad \text{if } i \text{ is odd}.
\end{align*}
\]

Thus, $c(V_1) = [p]$, $c(V_2) = [p+1, 2p]$ and $c(V_3) = [4p, 5p-1]$. In general, for each $\ell \geq 2$,

\[
\begin{align*}
c(V_{2\ell}) &= \{ (2^{\ell+1} - 3)p - 2^{\ell-1} + 1 : 1 \leq j \leq p \} \\
c(V_{2\ell+1}) &= \{ (2^{\ell+2} - 4)p - 2^{\ell+1} + (j-1) : 1 \leq j \leq p \}.
\end{align*}
\]

Hence, if $k$ is even, then $c(v) \in \left[ \left( 2^{\frac{k+3}{2}} - 2 \right) p - 2^{\frac{k-1}{2}} + 1 \right]$ for each $v \in V(G)$; while if $k$ is odd, then $c(v) \in \left[ \left( 2^{\frac{k+3}{2}} - 3 \right) p - 2^{\frac{k-1}{2}} + 1 \right]$ for each $v \in V(G)$.

Next, we show that $c$ is a graceful coloring of $G$. Since all vertices of $G$ are assigned distinct colors by $c$, it remains to show that the induced edge coloring is proper. Suppose that $(x, z, y)$ is a path of order 3 in $G$. We show that

\[
\frac{c(x) + c(y)}{2} \neq c(z). \quad (2.2)
\]

Let $x \in V_i$, $y \in V_j$, $z \in V_t$, where $1 \leq i, j, t \leq k$ and $t \neq i, j$. We may assume that $j \leq i$. First, suppose that $t < j$ or $t > i$. Since $c(v_{j,1}) < \frac{c(x) + c(y)}{2} < c(v_{i,p})$, it follows that either $c(z) < c(v_{j,1})$ or $c(z) > c(v_{i,p})$. Hence, (2.2) holds. Next, suppose that $j < t < i$. We consider two cases, according to whether $i$ is even or $i$ is odd.

\textbf{Case 1.} $i$ is even. Then $i = 2\ell$ for some integer $\ell \geq 2$ and $j \leq 2\ell - 2$. Next, we show that

\[
c(v_{2\ell-2,p}) < \frac{c(x) + c(y)}{2} < c(v_{2\ell-1,1}). \quad (2.3)
\]

Since $x \in V_{2\ell}$, it follows that

\[
c(x) \geq c(v_{2\ell,1}) = (2^{\ell+1} - 3)p - 2^{\ell-1} + 2.
\]
Thus, \[
\frac{c(x) + c(y)}{2} \geq \frac{c(v_2\ell,1) + 1}{2} = \frac{[(2\ell+1 - 3)p - 2^{\ell-1} + 2] + 1}{2}
= \left(\frac{2\ell - 3}{2}\right)p - 2^{\ell-2} + \frac{3}{2}
> (2\ell - 2)p - 2^{\ell-2} + 1 = c(v_{2\ell-2,p}).
\]

Similarly, since \(x \in V_{2\ell}\) and \(y \in V_j\) \((j \leq 2\ell - 2)\), it follows that \(c(x) \leq c(v_{2\ell,p})\) and \(c(y) \leq c(v_{2\ell-2,p})\). Hence,

\[
\frac{c(x) + c(y)}{2} \leq \frac{c(v_{2\ell,p}) + c(v_{2\ell-2,p})}{2}
= \frac{[(2\ell+1 - 2)p - 2^{\ell-1} + 1] + [(2\ell - 2)p - 2^{\ell-2} + 1]}{2}
= (2\ell + 2\ell-1 - 2)p - 2^{\ell-2} - 2^{\ell-3} + 1
< (2\ell+1 - 4)p - 2^{\ell-1} + 2 = c(v_{2\ell-1,1}) \text{ since } \ell \geq 2.
\]

Hence, (2.3) holds. Since no vertex is assigned a color from the set \([c(v_{2\ell-2,p}) + 1, c(v_{2\ell-1,1}) - 1]\), it follows that (2.2) holds.

**Case 2. i is odd.** Then \(i = 2\ell + 1\) for some integer \(\ell \geq 1\). Observe that

\[
\frac{c(x) + c(y)}{2} \geq \frac{c(v_{2\ell+1,1}) + 1}{2} = \frac{[(2\ell+2 - 4)p - 2^\ell + 2] + 1}{2}
= (2\ell+1 - 2)p - 2^{\ell-1} + \frac{3}{2}
> (2\ell+1 - 2)p - 2^{\ell-1} + 1 = c(v_{2\ell,p}) \geq c(z).
\]

Thus, (2.2) holds and so \(c\) is a graceful coloring of \(G\). 

For example, \(\chi_g(K_{4(4)}) \leq 23\) by Theorem 2.4.1. A graceful coloring \(c : V(K_{4(4)}) \to [23]\) of \(K_{4(4)}\) using colors from the set \([23]\) is shown in Figure 2.5. Note that the seven colors \(9, 10, \ldots, 15 \in [23]\) are not used in this coloring.
The proof of Theorem 2.4.1 provides us with the following recursive upper bounds for the graceful chromatic numbers of regular complete multipartite graphs. For each positive integer $\ell$,

$$
\chi_g(K_{(2\ell+1)(p)}) \leq 2\chi_g(K_{2\ell(p)}) + (p - 1)
$$

$$
\chi_g(K_{2\ell(p)}) \leq \chi_g(K_{(2\ell+1)(p)}) + p.
$$

The upper bound for $\chi_g(K_{k(p)})$ presented in Theorem 2.4.1 is almost certainly not sharp. While $\chi_g(K_{p,p,p}) \leq 5p - 1$ for $p \geq 2$ according to Theorem 2.4.1, the following result gives an improved upper bound in this case. First, we introduce some useful notation. For a vertex coloring $c$ of a graph $G$ and a set $X$ of vertices of $G$, let $c(X) = \{c(x) : x \in X\}$ be the set of colors of the vertices of $X$.

**Theorem 2.4.2** For each integer $p \geq 2$,

$$
\chi_g(K_{p,p,p}) \leq \begin{cases} 
4p - 1 & \text{if } p \text{ is even} \\
4p & \text{if } p \text{ is odd.}
\end{cases}
$$

**Proof.** Let $G = K_{p,p,p}$ with partite sets $V_1, V_2, V_3$, where $|V_i| = p$ for $1 \leq i \leq 3$. First, suppose that $p$ is even. Define a proper coloring $c : V(G) \to [4p - 1]$ of $G$ such that $c(V_1) = [p]$, $c(V_2) = [p + 1, 2p - \frac{p}{2}] \cup [2p + \frac{p}{2}, 3p - 1]$ and $c(V_3) = [3p, 4p - 1]$. To show that $c$ is a graceful coloring of $G$, it suffices to show that if $(x, z, y)$ is a path of order 3 in $G$, then

$$
\frac{c(x) + c(y)}{2} \neq c(z). \tag{2.4}
$$

Figure 2.5: A graceful coloring of $K_{4(4)}$
Let \( x \in V_i, y \in V_j, z \in V_t, \) where \( 1 \leq i, j, t \leq 3 \) and \( t \neq i, j. \) We may assume that \( j \leq i \) and \( c(y) \leq c(x). \)

\* If \( t < j, \) then \( c(z) < c(y) \leq c(x) \).

\* If \( t > i, \) then \( c(x) < c(y) \leq c(z) \).

Hence, we may assume that \( j < t < i \) and so \( j = 1, t = 2 \) and \( i = 3. \) Observe that
\[
\frac{c(x) + c(y)}{2} \geq \frac{3p + 1}{2} = 2p - \frac{p - 1}{2} > 2p - \frac{p}{2}
\]
\[
\frac{c(x) + c(y)}{2} \leq \frac{p + 4p - 1}{2} = 2p + \frac{p - 1}{2} < 2p + \frac{p}{2}.
\]

Thus, (2.4) holds.

Next, suppose that \( p \) is odd. A proper coloring \( c : V(G) \rightarrow [4p] \) of \( G \) is defined by \( c(V_1) = [p], c(V_2) = [p + 1, 2p - \frac{p}{2}] \cup [2p + \frac{p}{2}, 3p] \) and \( c(V_3) = [3p + 1, 4p]. \) Let \( (x, z, y) \) be a path of order 3 in \( G. \) Suppose that \( x \in V_i, y \in V_j, z \in V_t, \) where \( 1 \leq i, j, t \leq 3 \) and \( t \neq i, j. \) By an argument similar to the one used in Case 1, we may assume that \( j = 1, t = 2 \) and \( i = 3. \) Observe that
\[
\frac{c(x) + c(y)}{2} \geq \frac{(3p + 1) + 1}{2} > \frac{3p + 1}{2} = 2p - \frac{p - 1}{2} > 2p - \left\lfloor \frac{p}{2} \right\rfloor
\]
\[
\frac{c(x) + c(y)}{2} \leq \frac{p + 4p - 1}{2} < 2p + \frac{p + 1}{2} = 2p + \left\lfloor \frac{p}{2} \right\rfloor.
\]

Thus, (2.4) holds.

Indeed, there is reason to believe that the upper bound for \( \chi_g(K_{p,p,p}) \) presented in Theorem 2.4.2 is the actual value of \( \chi_g(K_{p,p,p}) \) for every integer \( p \geq 2. \)

**Conjecture 2.4.3** For each integer \( p \geq 2, \)
\[
\chi_g(K_{p,p,p}) = \begin{cases} 
4p - 1 & \text{if } p \text{ is even} \\
4p & \text{if } p \text{ is odd.}
\end{cases}
\]

For example, if \( p = 5, \) then \( \chi_g(K_{5,5,5}) \leq 20 \) by (2.4.2). A graceful coloring of \( K_{5,5,5} \) using colors from the set \([20]\) is shown in Figure 2.6. In fact, if \( c : V(K_{5,5,5}) \rightarrow [20] \) is a graceful coloring of \( K_{5,5,5}, \) then \( c \) cannot assign any of \( 8, 9, 10, 11, 12 \) as a color to a vertex of \( K_{5,5,5}. \)

Conjecture 2.4.3 has been verified when \( 2 \leq p \leq 6. \) As an illustration, we verify this for \( p = 3 \) and \( p = 4. \)
Proposition 2.4.4 \( \chi_9(K_{3,3,3}) = 12 \).

**Proof.** By Theorem 2.4.2, \( \chi_9(K_{3,3,3}) \leq 12 \). Hence, it remains to show that there is no graceful 11-coloring of \( G = K_{3,3,3} \). Let \( V_1, V_2, V_3 \) be the partite sets of \( G \). Assume, to the contrary, that \( G \) has a graceful coloring \( c : V(G) \to [11] \). Since \( \text{diam}(G) = 2 \), no two vertices of \( G \) are assigned the same color. First, we claim that the color 6 cannot be used; for otherwise, say \( 6 \in c(V_1) \). Then at least one color in each of the five sets \( \{i, 12-i\} \) (1 \( \leq i \leq 5 \)) is either not used by \( c \) or is in \( c(V_1) \). Since \( |c(V_1)| = 3 \) and exactly two colors in \([11]\) are not used by \( c \), this is impossible. Thus, 6 is not used and so exactly nine of the ten colors in \([11] - \{6\} \) are used by \( c \). We consider two cases.

**Case 1.** 5,7 \( \in c(V_1) \). If 5,7 \( \in c(V_i) \) for some \( i = 1, 2, 3 \), say \( i = 1 \), then one color in each of the four sets \( \{1, 9\}, \{2, 8\}, \{3, 11\}, \{4, 10\} \) is either in \( c(V_1) \) or is not used by \( c \). Since \( |c(V_1)| = 3 \) and exactly one color in \([11] - \{6\} \) is not used \( c \), this is impossible. Thus, we may assume that 5 \( \in c(V_1) \) and 7 \( \in c(V_2) \). Then the color 3 is either not used or is in \( c(V_1) \) and the color 9 is either not used or is in \( c(V_2) \). We may assume that 3 \( \in c(V_1) \) and so the color 4 is either not used or is in \( c(V_1) \).

**Subcase 1.1.** 9 \( \in c(V_2) \). Then the color 8 is either not used or is in \( c(V_2) \). We saw that the color 4 is either not used or is in \( c(V_1) \). By symmetry, we may assume that 4 \( \in c(V_1) \). Then each of 10 and 11 is either not used or is in \( c(V_2) \). Therefore, each of the three colors 8, 10, 11 is either not used or is in \( c(V_2) \). Since (i) 7,9 \( \in c(V_2) \), (ii) at most one of 8, 10, 11 belongs to \( c(V_2) \) and (iii) at most one of 8, 10, 11 is not used by \( c \), it follows that at least one of 8, 10, 11 is in \( c(V_3) \), a contradiction.

**Subcase 1.2.** 9 is not used. Then the colors used by \( c \) are 1, 2, 3, 4, 5, 7, 8, 10, 11.
Since $3, 4, 5 \in c(V_1)$, it follows that $c(V_1) = \{3, 4, 5\}$. Because $2, 8 \not\in c(V_1)$, the vertex colored 5 is incident with two edges colored 3, a contradiction.

**Case 2.** *Exactly one of 5 and 7 is used by c, say 5.* Then the colors used by $c$ are $1, 2, 3, 4, 5, 8, 9, 10, 11$. We may assume that $5 \in c(V_1)$. Thus, at least one color in $\{2, 8\}$ and at least one color in $\{1, 9\}$ belong to $c(V_1)$. Assume that $2 \in c(V_1)$. Thus, exactly one color in $\{1, 3\}$ belongs to $c(V_1)$. Since at least one color in $\{1, 9\}$ belongs to $c(V_1)$, it follows that $1 \in c(V_1)$ and so $c(V_1) = \{1, 2, 5\}$. However then, $3 \in c(V_2 \cup V_3)$ and the vertex colored 3 is incident with two edges colored 2, a contradiction. Thus, $2 \not\in c(V_1)$ and so $8 \in c(V_1)$.

Next, suppose that $1 \in c(V_1)$. Thus, $c(V_1) = \{1, 5, 8\}$. However then, $3 \in c(V_2 \cup V_3)$ and the vertex colored 3 is incident with two edges colored 2, a contradiction. Thus, $1 \not\in c(V_1)$ and so $9 \in c(V_1)$. Hence, $c(V_1) = \{5, 8, 9\}$. We may assume that $1 \in c(V_2)$. Since $5 \in c(V_1)$, it follows that $3 \in c(V_2)$ and so $2 \in c(V_2)$. Thus, $c(V_2) = \{1, 2, 3\}$ and $c(V_3) = \{4, 10, 11\}$. However then, the vertex colored 4 is incident with two edges colored 1, producing a contradiction.

The proof of Proposition 2.4.4 shows not only that $\chi_g(K_{3,3,3}) = 12$ but that there is a proper vertex coloring $c : V(G) \to [11]$ of $G = K_{3,3,3}$ for which $c(V_1) = \{5, 8, 9\}$, $c(V_2) = \{1, 2, 3\}$ and $c(V_3) = \{4, 10, 11\}$, whose induced edge coloring $c'$ results only in one pair of adjacent edges having the same color.

**Proposition 2.4.5** $\chi_g(K_{4,4,4}) = 15$.

**Proof.** By (2.4.2), $\chi_g(K_{4,4,4}) \leq 15$. Hence, it remains to show that there is no graceful 14-coloring of $G = K_{4,4,4}$. Assume, to the contrary, that $G$ has a graceful coloring $c : V(G) \to [14]$. Since diam$(G) = 2$, no two vertices of $G$ are assigned the same color. Thus, 12 colors from the set $[14]$ are used in this coloring.

First, we show that no vertex of $G$ is assigned the color 7 or 8. Assume, to the contrary, that some vertex of $G$ is assigned one of these colors. Since the complementary coloring of $c$ is also graceful coloring, we may assume that $c(v) = 7$ for some vertex $v$ of $G$, say $v \in V_1$, one of the three partite sets of $G$. Now consider the six 2-element sets $\{6, 8\}$, $\{5, 9\}$, $\{4, 10\}$, $\{3, 11\}$, $\{2, 12\}$, $\{1, 13\}$. Since at most three of these sets contain a color assigned to a vertex in $V_1$ and at most two of these sets contain a color not used by the coloring $c$, there is a 2-element set each of whose colors is assigned to a vertex not in $V_1$. However then, two edges incident with $v$ are assigned the same color, which is impossible. Hence, no vertex of $G$ is assigned the color 7 or 8.
Therefore, the vertices of $G$ are assigned colors from the set $[6] \cup [9, 14]$. Let $V_1, V_2, V_3$ be the partite sets of $G$. We may assume that $6 \in c(V_1)$. Necessarily, one element from each of the 2-element sets $\{3, 9\}, \{2, 10\}$ and $\{1, 11\}$ belongs to $c(V_1)$. We consider two cases, according to whether $3 \in c(V_1)$ or $9 \in c(V_1)$.

**Case 1.** $3 \in c(V_1)$. Thus, one element from each of the two sets $\{2, 4\}$ and $\{1, 5\}$ belongs to $c(V_1)$. Since $6 \in c(V_1)$, one element from each of the sets $\{2, 10\}$ and $\{1, 11\}$ belongs to $c(V_1)$. Therefore, $c(V_1) = \{1, 2, 3, 6\}$. Since $4 \in c(V_i)$ for $i = 2, 3$, the vertex colored 4 is incident with two edges colored 2, which is impossible.

**Case 2.** $9 \in c(V_1)$. Hence, $\{6, 9\} \subseteq c(V_1)$. Since $6 \in c(V_1)$, one element from each of the sets $\{2, 10\}$ and $\{1, 11\}$ belongs to $c(V_1)$. Similarly, since $9 \in c(V_1)$, one element from each of the sets $\{5, 13\}$ and $\{4, 14\}$ belongs to $c(V_1)$. This is impossible.

A lower bound for the graceful chromatic number of a connected graph was established in [35] in terms of its minimum degree.

**Theorem 2.4.6** If $G$ is a connected graph with minimum degree $\delta(G) \geq 2$, then

$$\chi_g(G) \geq \left\lceil \frac{5\delta(G)}{3} \right\rceil.$$  

It was observed in [35] that the lower bound for the graceful chromatic number of a graph presented in Theorem 2.4.6 is best possible. For example, the graph $G$ of Figure 2.7 has $\delta(G) = \delta = 2$ and graceful chromatic number $\chi_g(G) = \left\lceil \frac{5\delta}{3} \right\rceil = 4$. A graceful 4-coloring of $G$ is shown in the figure.

\[G:\]

Figure 2.7: A graph $G$ with $\chi_g(G) = \left\lceil \frac{5\delta}{3} \right\rceil$

The graph $G$ of Figure 2.7 gives rise to the following question.

**Problem 2.4.7** Is there an infinite class of connected graphs $G$ such that $\delta(G) \geq 2$ and

$$\chi_g(G) = \left\lceil \frac{5\delta(G)}{3} \right\rceil?$$
2.5 Caterpillars

We now determine the graceful chromatic numbers of some well-known trees. A caterpillar is a tree $T$ of order 3 or more, such that the removal of all vertices of degree one (leaves) from $T$ results in a path (called the spine of $T$). Thus, every path, every star (of order at least 3) and every double star (a tree of diameter 3) is a caterpillar.

**Theorem 2.5.1** Let $T$ be a caterpillar with maximum degree $\Delta \geq 2$. If $T$ has a vertex of degree $\Delta$ that is adjacent to two vertices of degree $\Delta$ in $T$, then $\chi_g(T) = \Delta + 2$.

**Proof.** Since the theorem holds when $\Delta = 2$ by Proposition 2.3.2, we may assume that $\Delta \geq 3$. First, we show that $\chi_g(T) \geq \Delta + 2$. Assume, to the contrary, that $\chi_g(T) \leq \Delta + 1$. It then follows by (2.1) that $\chi_g(T) = \Delta + 1$ and so $T$ has a graceful coloring $c$ using colors from $[\Delta + 1]$. Let $v \in V(G)$ with $\deg v = \Delta$. Suppose that $c(v) = a$. If $1 < a < \Delta + 1$, then there are two neighbors $u$ and $w$ of $v$ such that $c(u) = a + 1$ and $c(w) = a - 1$. However then, $c'(uw) = c'(vw) = 1$, which is impossible. Since $c(N[v]) = [\Delta + 1]$, there is $u \in N(v)$ such that $c(u) = a + \Delta$ or $c(u) = a - \Delta$. Because $a \leq \Delta + 1$, either $a = 1$ or $a = \Delta + 1$. Hence, every vertex of degree $\Delta$ is colored 1 or $\Delta + 1$. However, $T$ has a vertex of degree $\Delta$ that is adjacent to two vertices of degree $\Delta$ in $T$ and so $c$ is not proper, which is a contradiction. Therefore, $\chi_g(T) \geq \Delta + 2$.

To verify that $\chi_g(T) \leq \Delta + 2$, it suffices to show that there is a graceful coloring $c$ of $T$ using colors in $[\Delta + 2]$. First, we consider a caterpillar $T^*$ with $\Delta(T^*) = \Delta$ such that each non-end-vertex of $T^*$ has degree $\Delta$ and the spine of $T^*$ is a path $(v_1, v_2, \ldots, v_{3k+1})$ of order $3k + 1$. Thus, $\deg_{T^*} v_i = \Delta$ for each $i$ with $1 \leq i \leq 3k + 1$. We show that $\chi_g(T^*) \leq \Delta + 2$. Define a proper coloring $c : V(T^*) \rightarrow [\Delta + 2]$ of $T^*$ as follows. First, let $c(v_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \\ \Delta + 1 & \text{if } i \equiv 0 \pmod{3}. \end{cases}$

Let $e_i = v_iv_{i+1}$ for $1 \leq i \leq 3k$. The induced edge coloring $c'$ satisfies $(c'(e_1), c'(e_2), \ldots, c'(e_n)) = (1, \Delta - 1, \Delta, 1, \Delta - 1, \Delta, \ldots, 1, \Delta - 1, \Delta)$. For each integer $i$ with $1 \leq i \leq 3k + 1$, let $L(v_i)$ be the set of leaves that are adjacent to $v_i$.

* Let $c(L(v_1)) = [3, \Delta + 1]$ and let $c(L(v_{3k+1})) = [2, \Delta]$. 

30
**If** \( i \equiv 1 \pmod{3} \) \( i \neq 1, 3k + 1 \), let \( c(L(v_i)) = [3, \Delta] \).

**If** \( i \equiv 2 \pmod{3} \), let \( c(L(v_i)) = [4, \Delta] \cup \{\Delta + 2\} \).

**If** \( i \equiv 0 \pmod{3} \), let \( c(L(v_i)) = [3, \Delta] \).

This is illustrated in Figure 2.8 for \( k = 1 \). For each vertex \( v_i \) \( (1 \leq i \leq 3k + 1) \), let \( E(v_i) \) be the set of edges incident with \( v_i \). Then \( c'(E(v_i)) = [\Delta] \). Hence, \( c' \) is proper and so \( c \) is a graceful \((\Delta + 2)\)-coloring of \( T^* \). Therefore, \( \chi_g(T^*) = \Delta + 2 \).

![Figure 2.8: Illustrating the coloring c for k = 1](image)

Next, let \( T \) be a caterpillar with maximum degree \( \Delta \geq 3 \) such that some vertex of degree \( \Delta \) in \( T \) is adjacent to at least two vertices of degree \( \Delta \) in \( T \). Then there is a caterpillar \( T^* \) with \( \Delta(T^*) = \Delta \) having the structure as described above such that \( T \) is a subtree of \( T^* \). By Observation 2.2.3, \( \chi_g(T) \leq \chi_g(T^*) = \Delta + 2 \). Therefore, \( \chi_g(T) = \Delta + 2 \).

By Theorem 2.5.1, if \( T \) is a caterpillar with maximum degree 3 containing a vertex of degree 3 adjacent to two vertices of degree 3 in \( T \), then \( \chi_g(T) = 5 \). If \( T \) has no such vertex, then we show that \( \chi_g(T) = 4 \).

**Theorem 2.5.2** Let \( T \) be a caterpillar with maximum degree 3. If every vertex of degree 3 in \( T \) is adjacent to at most one vertex of degree 3 in \( T \), then \( \chi_g(T) = 4 \).

**Proof.** Let \( P = (v_1, v_2, \ldots, v_t) \) be the spine of \( T \). Thus, every vertex of \( P \) has degree 2 or 3 in \( T \) and every vertex of \( T \) not on \( P \) is an end-vertex of \( T \). Since the result holds for a star by Proposition 2.2.9, we may assume that \( t \geq 2 \) and at least one vertex of \( P \) has degree 3 in \( T \). If \( P \) has only vertices of degree 3 in \( T \), then \( t = 2 \) and both \( v_1 \) and \( v_2 \) have degree 3 in \( T \). Color one of these vertices 1 and the other 4. We may assume that \( P \) contains a vertex of degree 3 in \( T \) immediately followed by a vertex of degree 2 in \( T \) (otherwise, we may let \( P = (v_t, v_{t-1}, \ldots, v_1) \)).
Let \( v_a \) be the first vertex of degree 3 on \( P \) immediately followed by a vertex of degree 2 in \( T \). Assign \( v_a \) the color 1 or 4. If \( v_{a-1} \) also has degree 3 in \( T \), then assign this vertex a color such that \( v_{a-1}, v_a \) are colored 1, 4 or 4, 1. If no vertex of \( P \) following \( v_a \) has degree 3, then color the vertices \( v_{a+1}, v_{a+2}, \ldots, v_t \) with 2, 4, 3, 1, 2, 4, 3, 1, \ldots if \( v_a \) is colored 1; while \( v_{a+1}, v_{a+2}, \ldots, v_t \) are colored with 3, 1, 2, 4, 3, 1, 2, 4, \ldots if \( v_a \) is colored 4. If all vertices prior to \( v_a \) on \( P \) have degree 2 in \( T \), then color \( v_{a-1}, v_{a-2}, \ldots, v_1 \) with 3, 4, 2, 1, 3, 4, 2, 1, \ldots if \( v_a \) is colored 1; or 2, 1, 3, 4, 2, 1, 3, 4, \ldots if \( v_a \) is colored 4. If \( v_{a-1} \) has degree 3 and is colored 1, then \( v_{a-2}, v_{a-3}, \ldots, v_1 \) are colored 3, 4, 2, 1, 3, 4, 2, 1, \ldots; while if \( v_{a-1} \) has degree 3 and is colored 4, then \( v_{a-2}, v_{a-3}, \ldots, v_1 \) are colored 2, 1, 3, 4, 2, 1, 3, 4, \ldots.

Thus, we may now assume that there are one or more vertices following \( v_a \) that have degree 3 in \( T \). Let \( v_b \) be the first vertex following \( v_a \) that has degree 3 in \( T \). We now consider two cases.

Case 1. \( a \) and \( b \) are of the same parity. Then \( v_{a+1}, v_{a+2}, \ldots, v_b \) are colored 2, 4, 3, 1, 2, 4, 3, 1, \ldots if \( v_a \) is colored 1; while \( v_{a+1}, v_{a+2}, \ldots, v_b \) are colored 3, 1, 2, 4, 3, 1, 2, 4, \ldots if \( v_a \) is colored 4.

Case 2. \( a \) and \( b \) are of opposite parity. If \( v_{a-1} \) also has degree 3, then \( v_{a+1}, v_{a+2}, \ldots, v_b \) are colored with 3, 2, 4, 3, 1, 2, 4, 3, 1, \ldots if \( v_a \) is colored 1; while \( v_{a+1}, v_{a+2}, \ldots, v_b \) are colored 2, 3, 1, 2, 4, 3, 1, 2, 4, \ldots if \( v_a \) is colored 4. If \( a = 1 \) or \( v_{a-1} \) has degree 2 in \( T \), then \( v_{a+1}, v_{a+2}, \ldots, v_b \) are colored with 4, 3, 1, 2, 4, 3, 1, 2, 4, \ldots if \( v_a \) is colored 1; while \( v_{a+1}, v_{a+2}, \ldots, v_b \) are colored 1, 2, 4, 3, 1, 2, 4, 3, 1, \ldots if \( v_a \) is colored 4.

If there are vertices of degree 2 and 3 in \( T \) following \( v_b \) on \( P \), then relabel \( v_b \) as \( v_a \) if \( v_{b+1} \) has degree 2 in \( T \) or relabel \( v_{b+1} \) as \( v_a \) if \( v_{b+1} \) has degree 3 in \( T \). We then proceed as above.

Since each vertex of degree 3 is colored 1 or 4, there is a color available for each end-vertex of \( T \) that results in both a proper vertex coloring of \( T \) and a proper induced edge coloring of \( T \). Thus, \( \chi_g(T) = 4. \)

Next, we show that a theorem analogous to Theorem 2.5.2 holds when \( \Delta \geq 4 \).

**Theorem 2.5.3** Let \( T \) be a caterpillar with maximum degree \( \Delta \geq 4 \). If no vertex of degree \( \Delta \) in \( T \) is adjacent to two vertices of degree \( \Delta \) in \( T \), then

\[
\chi_g(T) = \Delta + 1.
\]

**Proof.** Since \( \chi_g(T) \geq \Delta + 1 \), it remains to show that \( \chi_g(T) \leq \Delta + 1 \). Let \( T \) be a caterpillar with maximum degree \( \Delta \geq 4 \) in which
(i) no vertex of degree $\Delta$ in $T$ is adjacent to two vertices of degree $\Delta$ in $T$.

Adding leaves to the tree $T$ if necessary, we may further assume that

(ii) each vertex on the spine of $T$ has degree $\Delta$ and $\Delta - 1$ in $T$ and that no vertices of degree $\Delta - 1$ is adjacent to two vertices of degree $\Delta - 1$ in $T$.

We establish the following stronger statement:

If $T$ is a caterpillar with maximum degree $\Delta \geq 4$ that satisfies (i) and (ii), then $T$ has a graceful $(\Delta + 1)$-coloring $c$ such that a vertex $v$ on the spine of $T$ is colored 1 or $\Delta + 1$ by $c$ if and only if $\deg_T v = \Delta$.

We proceed by induction on the order $\ell$ of the spine of a tree. If $\ell = 1$, then $T$ is a star and the statement is true by Proposition 2.2.9. If $\ell = 2$, then $T$ is a double star. Let $v_1$ and $v_2$ be the two central vertices (non-end-vertices) of $T$. First, suppose that $\deg_T v_1 = \deg_T v_2 = \Delta$. Assign the color 1 to $v_1$, the color $\Delta + 1$ to $v_2$ and the colors in $[2, \Delta]$ to the vertices in $N(v_1) - \{v_2\}$ and to the vertices in $N(v_2) - \{v_1\}$. Next, suppose that exactly one of $v_1$ and $v_2$ has degree $\Delta$, say $v_1$. Assign the color 1 to $v_1$, the color 2 to $v_2$, the colors in $[3, \Delta + 1]$ to the vertices in $N(v_1) - \{v_2\}$ and the colors in $[4, \Delta + 1]$ to the vertices in $N(v_1) - \{v_2\}$. In each case, $T$ has a graceful $(\Delta + 1)$-coloring with the desired property. This establishes the base step.

Assume that if $T'$ is a tree of maximum degree $\Delta' \geq 4$, the length of whose spine is $\ell - 1$ for some $\ell \geq 3$ such that $T'$ satisfies (i) and (ii), then $T'$ has a graceful $(\Delta' + 1)$-coloring such that a vertex on the spine of $T'$ is colored 1 or $\Delta' + 1$ if and only if $\deg_{T'} v = \Delta'$. Let $T$ be a tree of maximum degree $\Delta \geq 4$ the length of whose spine is $\ell$ such that $T$ satisfies (i) and (ii). Let $P = (v_1, v_2, \ldots, v_\ell)$ be the spine of $T$. We may assume that there is $i \in \{2, 3, \ldots, \ell\}$ such that $\deg_T v_i = \Delta$; for otherwise, $v_1$ is the only vertex of degree $\Delta$ in $T$ and let $P = (v_\ell, v_{\ell - 1}, \ldots, v_1)$. Let $T'$ be the caterpillar obtained from $T$ by removing all leaves adjacent to $v_1$. Then $T'$ is a caterpillar of maximum degree $\Delta \geq 4$, whose spine is $P' = (v_2, v_3, \ldots, v_\ell)$ of length $\ell - 1$. Since $\deg_{T'} v_i = \deg_T v_i$ for $2 \leq i \leq \ell$, it follows that $T'$ satisfies (i) and (ii). By the induction hypothesis, $T'$ has a graceful $(\Delta + 1)$-coloring $c$ such that a vertex $v$ on $P'$ is colored 1 or $\Delta + 1$ by $c$ if and only if $\deg_{T'} v = \Delta$.

Next, we show that that $T$ has a graceful $(\Delta + 1)$-coloring $c_T$ such that

a vertex $v$ on $P$ is colored 1 or $\Delta + 1$ by $c_T$ if and only if $\deg_T v = \Delta$.  \hfill (2.5)
We consider four cases, according to the degrees of $v_1$ and $v_2$.

**Case 1.** $\deg_T v_1 = \deg_T v_2 = \Delta$. Let $N_T(v_1) = \{v_2, x_1, x_2, \ldots, x_{\Delta-1}\}$. Then $\deg_T v_3 = \Delta - 1$. By the induction hypothesis, $c(v_2) \in \{1, \Delta + 1\}$ and $c(v_3) \notin \{1, \Delta + 1\}$. By Observation 2.2.2, we may assume $c(v_2) = \Delta + 1$. Thus, one of the leaves adjacent to $v_2$, say $v_1$, is colored 1. Define a vertex coloring $c_T$ of $T$ by $c_T(v) = c(v)$ if $v \in V(T')$ and $c_T(x_i) = i + 1$ for $x_i \in N(v_1)$. Then $c_T$ is a graceful $(\Delta + 1)$-coloring of $T$ that satisfies (2.5).

**Case 2.** $\deg_T v_1 = \Delta$ and $\deg_T v_2 = \Delta - 1$. Let $N_T(v_1) = \{v_2, x_1, x_2, \ldots, x_{\Delta-1}\}$. By the induction hypothesis, $c(v_2) \notin \{1, \Delta + 1\}$. Furthermore, we may assume that $c(v_3) \neq 1$ by Observation 2.2.2. If $v_2$ is adjacent to a leaf that is colored 1, then we may assume that this leaf is $v_1$; while if $v_2$ is not adjacent to a leaf colored 1, then $c(v_2) = 2$ and there exists a leaf adjacent to $v_2$ colored 3. In this case, we may assume that this leaf is $v_1$ and change the color of $v_1$ to 1 such that the resulting coloring is still graceful. Define a vertex coloring $c_T$ of $T$ by $c_T(v) = c(v)$ if $v \in V(T')$ and $c_T(x_i) = i + 2$ for $x_i \in N_T(v_1)$. Then $c_T$ is a graceful $(\Delta + 1)$-coloring of $T$ satisfying (2.5).

**Case 3.** $\deg_T v_1 = \Delta - 1$ and $\deg_T v_2 = \Delta$. Let $N_T(v_1) = \{v_2, x_1, x_2, \ldots, x_{\Delta-2}\}$. Then $c(v_2) \in \{1, \Delta + 1\}$. By Observation 2.2.2, we may assume that $c(v_3) \neq 2$. Hence, $v_2$ is adjacent to a leaf adjacent that is colored 2, say $v_1$. Define a vertex coloring $c_T$ of $T$ by $c_T(v) = c(v)$ if $v \in V(T')$, and $c_T(x_i) = i + 2$ for each $x_i \in N_T(v_1) - \{x_1\}$. If $c(v_2) = 1$, then let $c_T(x_1) = \Delta + 1$; while if $c(v_2) = \Delta + 1$, then let $c_T(x_1) = 3$. Then $c_T$ is a graceful $(\Delta + 1)$-coloring of $T$ satisfying (2.5).

**Case 4.** $\deg_T v_1 = \deg_T v_2 = \Delta - 1$. Let $N_T(v_1) = \{v_2, x_1, x_2, \ldots, x_{\Delta-2}\}$. Since no vertex of degree $\Delta - 1$ is adjacent to two vertices of degree $\Delta - 1$, it follows that $\deg_T v_3 = \Delta$. Thus, $c(v_3) \in \{1, \Delta + 1\}$. By Observation 2.2.2, we may assume $c(v_2) \neq 2$. Then $v_2$ is adjacent to a leaf colored 2, say $v_1$. Define a vertex coloring $c_T$ of $T$ by $c_T(v) = c(v)$ if $v \in V(T')$, and $c_T(x_i) = i + 2$ for $1 \leq i \leq \Delta - 3$ and $c_T(x_{\Delta-2}) = \Delta + 1$. Then $c_T$ is a graceful $(\Delta + 1)$-coloring of $T$ satisfying (2.5).

By the Principle of Mathematical Induction, if $T$ is a caterpillar with maximum degree $\Delta \geq 4$ that satisfies (i) and (ii), then $T$ has a graceful $(\Delta + 1)$-coloring $c$ such that a vertex $v$ on the spine of $T$ is colored 1 or $\Delta + 1$ by $c$ if and only if $\deg_T v = \Delta$. Since every caterpillar with maximum degree $\Delta$ satisfying (i) is a subtree of a caterpillar with maximum degree $\Delta$ that satisfies (i) and (ii), the result follows by Observation 2.2.3.
By Theorems 2.5.1–2.5.3, the following result provides the graceful chromatic number of every caterpillar.

**Theorem 2.5.4** If $T$ is a caterpillar with maximum degree $\Delta \geq 2$, then

$$\Delta + 1 \leq \chi_g(T) \leq \Delta + 2.$$  

Furthermore, $\chi_g(T) = \Delta + 2$ if and only if $T$ has a vertex of degree $\Delta$ that is adjacent to two vertices of degree $\Delta$ in $T$.

### 2.6 An Upper Bound

We have seen examples of trees $T$ for which $\chi_g(T) = \Delta(T) + 1$ and trees $T$ for which $\chi_g(T) = \Delta(T) + 2$. This brings up the question of whether there exists a tree $T$ such that $\chi_g(T) = \Delta(T) + 3$. To answer this question, we consider the tree $T_0$ shown in Figure 2.9.

Notice that with $\Delta(T_0) = 4$. First, we show that there is no graceful 6-coloring of $T_0$. Assume to the contrary that there is such a coloring $c : V(T_0) \to [6]$. The vertices in $N[u]$ must be colored with five colors from the set $[6]$. If $c(u) = 3$, then no two vertices in $N(u)$ can be colored both 2 and 4 or both 1 or 5 by Observation 2.2.4. Similarly, it is impossible that $c(u) = 4$. Thus, $c(u) \in \{1, 2, 5, 6\}$. The same can be said of $v, w, x$ and $y$. This implies that two vertices of $N[u]$ are colored the same, which is impossible. Since the 7-coloring of $T_0$ shown in Figure 2.9 is a graceful coloring, it follows that $\chi_g(T_0) = 7 = \Delta(T_0) + 3$.

![Figure 2.9: A tree $T_0$ with $\chi_g(T_0) = \Delta(T_0) + 3$](image)

For the tree $T_0$ in Figure 2.9, observe that $\chi_g(T_0) = 7 = \left\lceil \frac{5\Delta(T_0)}{3} \right\rceil$. Indeed, for every tree $T$ with maximum degree $\Delta$, the graceful chromatic number of $T$ can never exceed $\left\lceil \frac{5\Delta}{3} \right\rceil$, as we now show.
Theorem 2.6.1  If $T$ is a nontrivial tree with maximum degree $\Delta$, then

$$\chi_g(T) \leq \left\lceil \frac{5\Delta}{3} \right\rceil.$$  

Proof.  Let $S_1 = \left\lfloor \frac{2\Delta}{3} \right\rfloor$, $S_2 = \left\lfloor \Delta + 1, \frac{5\Delta}{3} \right\rfloor$ and $S = S_1 \cup S_2$. In order to show that $T$ has a graceful coloring using the colors in $S$, we first verify the following claim.

Claim.  For each $a \in S$, there are at least $\Delta$ distinct elements $a_1, a_2, \ldots, a_\Delta \in S - \{a\}$ such that all of the $\Delta$ integers $|a - a_1|, |a - a_2|, \ldots, |a - a_\Delta|$ are distinct.

We consider three cases, according to the values of $\Delta$ modulo 3.

Case 1.  $\Delta \equiv 0 \pmod{3}$.  Let $\Delta = 3k$ for some positive integer $k$. Then $\left\lfloor \frac{2\Delta}{3} \right\rfloor = 2k$ and so $S_1 = [2k]$ and $S_2 = [3k + 1, 5k]$.  Let $a \in S$.  By Observation 2.2.2, we may assume that $a \in S_1$.  For each $i \in \{1, 2, \ldots, 2k\}$, let $a_i = 3k + i$.  Then all of $|a - a_1|, |a - a_2|, \ldots, |a - a_{2k}|$ are distinct and

$$|a - a_i| = 3k + i - a \geq 3k + i - 2k = k + i \geq 1$$

for $1 \leq i \leq 2k$.  If $a \leq k$, then choose $a_{2k+j} = a + j$ for $1 \leq j \leq k$; while if $a \geq k + 1$, then choose $a_{2k+j} = a - j$ for $1 \leq j \leq k$.  Then all of $|a - a_{2k+1}|, |a - a_{2k+2}|, \ldots, |a - a_{3k}|$ are distinct and $|a - a_{2k+j}| = j \leq k$.  Since $|a - a_i| \geq k + 1$ for $1 \leq i \leq 2k$ and $|a - a_i| \leq k$ for $2k + 1 \leq i \leq 3k$, it follows that $|a - a_1|, |a - a_2|, \ldots, |a - a_{3k}|$ are distinct.

Case 2.  $\Delta \equiv 1 \pmod{3}$.  Let $\Delta = 3k + 1$ for some nonnegative integer $k$.  Then $\left\lfloor \frac{2\Delta}{3} \right\rfloor = 2k + 1$ and so $S_1 = [2k + 1]$ and $S_2 = [3k + 2, 5k + 2]$.  Let $a \in S$.  As observed in Case 1, we may assume that $a \in S_1$.  For each $i \in \{1, 2, \ldots, 2k + 1\}$, let $a_i = 3k + 1 + i$.  Then all of $|a - a_1|, |a - a_2|, \ldots, |a - a_{2k+1}|$ are distinct and

$$|a - a_i| = 3k + 1 + i - a \geq 3k + 1 + i - (2k + 1)$$

$$= k + i \geq k + 1$$

for $1 \leq i \leq 2k + 1$.  If $a \leq k$, then choose $a_{2k+1+j} = a + j$ for $1 \leq j \leq k$; while if $a \geq k + 1$, then choose $a_{2k+1+j} = a - j$ for $1 \leq j \leq k$.  Then all of

$$|a - a_{2k+2}|, |a - a_{2k+3}|, \ldots, |a - a_{3k+1}|$$
are distinct and $|a - a_{2k+1+j}| = j \leq k$. Since $|a - a_i| \geq k + 1$ for $1 \leq i \leq 2k + 1$ and $|a - a_i| \leq k$ for $2k + 2 \leq i \leq 3k + 1$, it follows that $|a - a_1|, |a - a_2|, \ldots, |a - a_{3k+1}|$ are distinct.

**Case 3.** $\Delta \equiv 2 \pmod{3}$. Let $\Delta = 3k + 2$ for some nonnegative integer $k$. Then $\lfloor \frac{2\Delta}{3} \rfloor = 2k + 2$ and so $S_1 = [2k + 2]$ and $S_2 = [3k + 2, 5k + 4]$. The argument is similar to the one in Case 2.

Therefore, the claim holds. It remains to construct a graceful coloring $c$ of $T$ using the colors in $S$. Let $v \in V(T)$ such that $\deg v = \Delta$ and let $V_i = \{w \in V(T) : d(v, w) = i\}$ for $0 \leq i \leq e(v)$, where $e(v)$ is the eccentricity of $v$. Thus, $V_0 = \{v\}$ and $V_1 = N(v)$. Let $c(v) = a$ for some $a \in S$ and let $a_1, a_2, \ldots, a_\Delta \in S - \{a\}$ for which $|a - a_1|, |a - a_2|, \ldots, |a - a_\Delta|$ are distinct. Color the vertices of $V_1$ such that
\[
\{c(w) : w \in V_1\} = \{a_1, a_2, \ldots, a_\Delta\}.
\]
Thus, each vertex in $V_0 \cup V_1$ has been assigned a color from $S$ such that all vertices and edges of the tree $T_1 = T[V_0 \cup V_1]$ are properly colored. Suppose then, for some integer $i$ where $1 \leq i < e(v)$, that the colors of the vertices in the tree $T_i = T \bigcup_{j=0}^{i} V_j$ have been chosen from $S$ so that all vertices and edges of $T_i$ are properly colored. Next, we define the colors of the vertices in $V_{i+1}$. Let $w \in V_i$ that is not an end-vertex of $T$. Suppose that $\deg w = t \leq \Delta$ and $c(w) = b \in S$. Choose $b_1, b_2, \ldots, b_\Delta \in S - \{b\}$ such that $|b - b_1|, |b - b_2|, \ldots, |b - b_\Delta|$ are distinct. Let $u \in V_{i-1}$ such that $uw \in E(T)$. We may assume, without loss of generality, that $b_j \neq c(u)$ and $b_j \neq 2c(w) - c(u)$ for $1 \leq j \leq t - 1 \leq \Delta - 1$. Color the vertices in $N(w) - \{u\} \subseteq V_{i+1}$ such that
\[
\{c(w) : w \in N(w) - \{u\}\} = \{b_1, b_2, \ldots, b_{t-1}\}.
\]
Continue this procedure for each non-end-vertex in $V_i$ to define the color of each vertex in $V_{i+1}$. Therefore, $T$ has a graceful coloring using colors from the set $S \subseteq \lceil \frac{5\Delta}{3} \rceil$ and so $\chi_g(T) \leq \lceil \frac{5\Delta}{3} \rceil$.

The upper bound is Theorem 2.6.1 is best possible, as demonstrated with the tree $T_0$ in Figure 2.9. In addition, the graceful chromatic numbers of trees belonging to a particular class of trees, which achieve this bound, were investigated in [35]. For each integer $\Delta \geq 2$, let $T_{\Delta, 1}$ be the star $K_{1, \Delta}$. The **central vertex** of $T_{\Delta, 1}$ is denoted by $v$. Thus, $\deg v = \Delta$ and all other vertices of $T_{\Delta, 1}$ have degree 1. For each integer $h \geq 2$, let $T_{\Delta, h}$ be the tree obtained from $T_{\Delta, h-1}$ by identifying each end-vertex with the central
vertex of the star $K_{1,\Delta-1}$. The tree $T_{\Delta,h}$ is therefore a rooted tree (with root $v$) having height $h$. The vertex $v$ is then the central vertex of $T_{\Delta,h}$. In $T_{\Delta,h}$, every vertex of distance less than $h$ from $v$ has degree $\Delta$; while all remaining vertices are leaves and are of distance $h$ from $v$. Thus, $T_{2,2} = P_5$, while $T_{3,2}$ and $T_{6,2}$ are shown in Figure 2.10. For integers $\Delta$ and $h$ with $\Delta \geq 2$ and $h \in \{2, 3, 4\}$, the graceful chromatic numbers of the trees $T_{\Delta,h}$ were determined in [35].

![Figure 2.10: The trees $T_{3,2}$ and $T_{6,2}$](image)

**Theorem 2.6.2** For each integer $\Delta \geq 2$,

(i) $\chi_g(T_{\Delta,2}) = \left\lceil \frac{3\Delta+1}{2} \right\rceil$.

(ii) $\chi_g(T_{\Delta,3}) = \left\lceil \frac{13\Delta+1}{8} \right\rceil$.

(iii) $\chi_g(T_{\Delta,4}) = \left\lceil \frac{53\Delta+1}{32} \right\rceil$.

The results obtained in Theorem 2.6.2 on $T_{\Delta,h}$ for $\Delta \geq 2$ and $h \in \{2, 3, 4\}$ suggest the following conjecture (see [35]).

**Conjecture 2.6.3** For an integer $h \geq 2$, let $\sigma_h = 2^{2h-3} + \sum_{i=2}^{h} 2^{2i-4}$. Then

$$\chi_g(T_{\Delta,h}) = \left\lceil \frac{\sigma_h \Delta + 1}{2^{2h-3}} \right\rceil.$$ 

### 2.7 Some Problems Dealing with Graceful Colorings

We saw that if $G$ is a nontrivial connected graph, then

$$\chi_g(G) \geq \max\{\chi(G), \chi'(G)\} + 1.$$
Since there appear to be relatively few graphs \( G \) for which  
\[
\chi_g(G) = \chi(G) + 1 \text{ or } \chi_g(G) = \chi'(G) + 1,
\]
this gives rise to the following natural question.

**Problem 2.7.1** Under what conditions does a connected graph \( G \) satisfy \( \chi_g(G) = \chi(G) + 1 \) or \( \chi_g(G) = \chi'(G) + 1 \)?

We saw that if \( G \) is a connected graph of order \( n \) with diameter 2, then \( \chi_g(G) \geq n \). Thus, we have the following question.

**Problem 2.7.2** For each integer \( k \in \mathbb{N} \), does there exist a connected graph \( G \) of order \( n \) such that \( \chi_g(G) = n + k \)?

Since it is evidently challenging to determine the exact value of the graceful chromatic number of a given graph, it appears to be more practical to establish bounds for this parameter in terms of other well-known graphical parameters. We conclude with two questions related to the graceful chromatic number of a connected graph or a graceful graph in terms of the size of the graph.

**Problem 2.7.3** Let \( G \) be a connected graph of size \( m \). Is there a function \( f(m) \) such that \( \chi_g(G) \leq f(m) \)?

**Problem 2.7.4** Let \( G \) be a graceful graph of order \( n \). Is there a function \( g(n) \) such that \( \chi_g(G) \leq g(n) \)?
Chapter 3

Rainbow Connection in Graphs

3.1 Introduction

Recall that a rainbow coloring of a connected graph $G$ (where adjacent edges may be colored the same) is an edge coloring $c$ of $G$ with the property that for every two vertices $u$ and $v$ of $G$, there exists a $u - v$ rainbow path (no two edges of the path are colored the same). In this case, $G$ is said to be rainbow-connected (with respect to $c$). The minimum number of colors needed for a rainbow coloring of $G$ is referred to as the rainbow connection number of $G$, denoted by $rc(G)$. In a rainbow coloring of a connected graph $G$, every two vertices $u$ and $v$ of $G$ are connected by a rainbow $u - v$ path, but there is no condition on what the length of such a path must be. For certain graphs $G$, however, it is natural to ask whether there may exist an edge coloring of $G$ using a certain number of colors such that every two vertices of $G$ are connected by a rainbow path of a prescribed length.

A Hamiltonian cycle in a graph $G$ is a cycle containing every vertex of $G$, and a graph having a Hamiltonian cycle is a Hamiltonian graph. A path containing every vertex of a graph $G$ is a Hamiltonian path in $G$. A graph $G$ is Hamiltonian-connected if $G$ contains a Hamiltonian $u - v$ path for every pair $u, v$ of distinct vertices of $G$. Every Hamiltonian-connected graph of order at least 3 is Hamiltonian. Yet, the converse is not true. For example, a cycle $C_n$ of order $n \geq 4$ is Hamiltonian but not Hamiltonian-connected. For a graph $G$, the minimum and maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a nontrivial graph $G$, let $\deg w$ be the degree of a vertex $w$ in $G$, and let

$$\sigma_2(G) = \min\{\deg u + \deg v : uv \notin E(G)\}.$$ 

Ore [77] proved the following results in 1963.

40
Theorem 3.1.1  If $G$ is a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n + 1$, then $G$ is Hamiltonian-connected.

There is an immediate corollary that gives a sufficient condition for a graph to be Hamiltonian-connected.

Corollary 3.1.2  If $G$ is a graph of order $n \geq 4$ such that $\delta(G) \geq (n + 1)/2$, then $G$ is Hamiltonian-connected.

For a Hamiltonian-connected graph $G$, an edge coloring $c : E(G) \to [k] = \{1, 2, \ldots, k\}$ is called a Hamiltonian-connected rainbow $k$-coloring if every two vertices of $G$ are connected by a rainbow Hamiltonian path in $G$. An edge coloring $c$ is a Hamiltonian-connected rainbow coloring if $c$ is a Hamiltonian-connected rainbow $k$-coloring for some positive integer $k$. The minimum $k$ for which $G$ has a Hamiltonian-connected rainbow $k$-coloring is the rainbow Hamiltonian-connection number of $G$, denoted by $hrc(G)$. This concept was introduced by Gary Chartrand, first studied in [13] and studied further in [15].

If $H$ is a Hamiltonian-connected spanning subgraph of a graph $G$ and $c$ is a Hamiltonian-connected rainbow coloring of $H$, then the coloring $c$ can be extended to a Hamiltonian-connected rainbow coloring of $G$ by assigning any color used by $c$ to each edge in $E(G) - E(H)$. Thus, we have the following observation.

Observation 3.1.3  If $H$ is a Hamiltonian-connected spanning subgraph of a graph $G$, then

\[ hrc(G) \leq hrc(H). \]

Let $G$ be a Hamiltonian-connected graph of order $n \geq 4$. Since (1) every Hamiltonian-connected rainbow coloring of $G$ is a rainbow coloring, (2) there is no Hamiltonian-connected rainbow coloring of $G$ using less than $n - 1$ colors, and (3) the edge coloring that assigns distinct colors to distinct edges of $G$ is a Hamiltonian-connected rainbow coloring, we have the following observation.

Observation 3.1.4  If $G$ is a Hamiltonian-connected graph of order $n \geq 4$ and size $m$, then

\[ \max\{rc(G), n - 1\} \leq hrc(G) \leq m. \]
If \( G \) is a Hamiltonian-connected graph of order \( n \geq 3 \) and size at most \( 2n - 3 \), then \( \text{hrc}(G) \neq n - 1 \), for otherwise, each \((n - 1)\)-edge coloring of \( G \) results in some edge \( e = uv \) of \( G \) having the property that \( e \) is the only edge possessing the color assigned to it. However then, there is no rainbow Hamiltonian \( u - v \) path \( P \) in \( G \). This gives rise to the following observation.

**Observation 3.1.5** If \( G \) is a Hamiltonian-connected graph of order \( n \geq 3 \) and size at most \( 2n - 3 \), then \( \text{hrc}(G) \geq n \).

We now present infinite classes of Hamiltonian-connected graphs \( G \) such that \( \text{hrc}(G) = |V(G)| - 1 \). For two vertex-disjoint graphs \( F \) and \( H \), let \( F \lor H \) denote the join of \( F \) and \( H \), which is the graph with vertex set \( V(F) \cup V(H) \) and edge set

\[
E(F) \cup E(H) \cup \{uv : u \in E(F), v \in E(H)\}.
\]

For an integer \( n \geq 3 \), the wheel \( W_n = C_n \lor K_1 \) of order \( n + 1 \) is Hamiltonian-connected.

**Theorem 3.1.6** For each integer \( n \geq 3 \), \( \text{hrc}(W_n) = n \).

**Proof.** Let \( W_n = C_n \lor K_1 \), where \( C_n = (v_1, v_2, \ldots, v_n, v_1) \) and the vertex \( v \in V(K_1) \) is adjacent to each vertex of \( C_n \). Since \( \text{hrc}(W_n) \geq n \) by Observation 3.1.4, it remains to show that \( \text{hrc}(W_n) \leq n \). Define an \( n \)-edge coloring \( c : E(W_n) \to [n] \) by \( c(v_i v_{i+1}) = c(v_i v) = i \) for \( 1 \leq i \leq n \), where \( v_{n+1} = v_1 \). We show that every two vertices \( x \) and \( y \) in \( W_n \) are connected by a rainbow Hamiltonian path in \( W_n \). We may assume, without loss of generality, that \( x = v_1 \). If \( y = v \), then \((v_1, v_2, \ldots, v_n, v)\) is a rainbow Hamiltonian path. If \( y = v_n \), then \((v_1, v_2, v_3, \ldots, v_{n-1}, v, v_n)\) is a rainbow Hamiltonian path. If \( y = v_i \) where \( 2 \leq i \leq n - 1 \), then \((v_1, v_2, v_3, \ldots, v_{i-1}, v, v_n, v_{n-1}, \ldots, v_i)\) is a rainbow Hamiltonian path. Thus, \( c \) is a Hamiltonian-connected rainbow coloring of \( W_n \) and so \( \text{hrc}(W_n) \leq n \). Therefore, \( \text{hrc}(W_n) = n \). \( \square \)

The following results are consequences of Observations 3.1.3 and 3.1.4 and Theorem 3.1.6.

**Corollary 3.1.7** If \( G \) is a Hamiltonian graph of order \( n \geq 3 \), then

\[
\text{hrc}(G \lor K_1) = n.
\]

**Corollary 3.1.8** For each integer \( n \geq 4 \), \( \text{hrc}(K_n) = n - 1 \).
The following result on complete $k$-partite graphs is known (see [25, pp. 155]).

**Observation 3.1.9** For an integer $k \geq 2$, let $G = K_{n_1,n_2,\ldots,n_k}$ be the complete $k$-partite graph with partite sets $V_1, V_2, \ldots, V_k$ where $|V_i| = n_i$ for $1 \leq i \leq k$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. Then

(a) $G$ is Hamiltonian-connected if and only if $n_k \leq (\sum_{i=1}^{k-1} n_i) - 1$,

(b) $G$ is Hamiltonian if and only if $n_k \leq \sum_{i=1}^{k-1} n_i$ and

(c) $G$ contains a Hamiltonian path if and only if $n_k \leq (\sum_{i=1}^{k-1} n_i) + 1$.

Therefore, no complete bipartite graph is Hamiltonian-connected.

**Proposition 3.1.10** If $G$ is a Hamiltonian-connected complete multipartite graph of order $n \geq 4$ having a partite set consisting of at most two vertices, then $\text{hrc}(G) = n - 1$.

**Proof.** Let $G = K_{n_1,n_2,\ldots,n_k}$ be a Hamiltonian-connected complete $k$-partite graph with partite sets $V_1, V_2, \ldots, V_k$ where $|V_i| = n_i$ for $1 \leq i \leq k$, $n_1 \in \{1, 2\}$ and $n_1 \leq n_2 \leq \cdots \leq n_k$. Then $k \geq 3$ and $n_k \leq (\sum_{i=1}^{k-1} n_i) - 1$. Let $n = n_1 + n_2 + \cdots + n_k$ be the order of $G$. Since $\text{hrc}(G) \geq n - 1$ by Observation 3.1.4, it suffices to show that $\text{hrc}(G) \leq n - 1$. If $n_1 = 1$, then $G - V_1$ is Hamiltonian by Observation 3.1.9 (where $n_3 = n_2$ if $k = 3$) and so $G$ contains $C_{n-1} \vee K_1$ as a spanning subgraph; while if $n_1 = 2$, then $G - V_1$ contains a Hamiltonian path by Observation 3.1.9 (where $n_3 = n_2$ or $n_3 = n_2 + 1$ if $k = 3$) and so $G$ contains $P_{n-2} \vee K_2$ as a spanning subgraph by Observation 3.1.9. It then follows by Observation 3.1.3 and Theorems 3.1.7 and 3.1.11 that $\text{hrc}(G) = n - 1$.  

Figure 3.1: An $n$-edge coloring $c$ of $W_n = C_n \vee K_1$
If $G$ is a graph of order $n \geq 3$ containing a Hamiltonian path, then the join $G \vee K_2$ of $G$ and $K_2$ is Hamiltonian-connected. In particular, $P_n \vee K_2$ is Hamiltonian-connected for $n \geq 3$. We determine the rainbow Hamiltonian-connection numbers graphs that are joins of graphs possessing a Hamiltonian path with $K_2$.

**Theorem 3.1.11** For each integer $n \geq 3$, $\text{hrc}(P_n \vee K_2) = n + 1$.

**Proof.** If $n = 3$, then $P_3 \square K_2$ is the wheel of order 5 and so $\text{hrc}(W_5) = 4$ by Theorem 3.1.6. Thus, we assume that $n \geq 4$. Let $G = P_n \vee K_2$ where $V(K_2) = \{u, v\}$ and $P_n = (x_1, x_2, \ldots, x_n)$. Since $G$ has order $n+2$, it follows that $\text{hrc}(G) \geq n+1$. Thus, it remains to show that $G$ has a Hamiltonian-connected rainbow $(n+1)$-coloring. Define the edge coloring $c : V(G) \rightarrow [n+1]$ of $G$ by

\[
\begin{align*}
\star c(ux_i) &= i & &\text{for } 1 \leq i \leq n \text{ and } c(x_i x_{i+1}) = i + 1 & &\text{for } 1 \leq i \leq n-1 \\
\star c(vx_1) &= 1 & &\text{and } c(vx_i) = i + 1 & &\text{for } 2 \leq i \leq n-2 \text{ and } c(x_{n-1}) = c(x_n) = n + 1.
\end{align*}
\]

This coloring is shown in Figure 3.2. To show that $c$ is a Hamiltonian-connected rainbow coloring of $G$, we illustrate a rainbow Hamiltonian path connecting each pair of the vertices of $G$ as listed below.

\[
\begin{align*}
u - v: & \quad (u, x_1, x_2, x_3, \ldots, x_n, v) \\
u - x_i: & \quad (u, x_{i+1}, x_{i+2}, \ldots, x_n, v, x_1, x_2, \ldots, x_i) \quad & &\text{for } 1 \leq i \leq n-1 \\
u - x_n: & \quad (u, x_1, x_2, x_3, \ldots, x_{n-2}, v, x_{n-1}, x_n) \\
v - x_i: & \quad (v, x_n, x_{n-1}, \ldots, x_{i+1}, u, x_1, x_2, \ldots, x_i) \quad & &\text{for } 1 \leq i \leq n-1 \\
v - x_n: & \quad (v, x_{n-1}, x_{n-2}, \ldots, x_1, u, x_n) \\
x_i - x_{i+1}: & \quad (x_i, x_{i+1}, \ldots, x_1, v, x_n, x_{n-1}, \ldots, x_{i+2}, u, x_{i+1}) \quad & &\text{for } 1 \leq i \leq n-2 \\
x_{n-1} - x_n: & \quad (x_{n-1}, v, x_{n-2}, x_{n-3}, \ldots, x_1, u, x_n) \\
x_i - x_j: & \quad (x_i, x_{i+1}, \ldots, x_1, u, x_{i+1}, \ldots, x_{j-1}, v, x_n, x_{n-1}, \ldots, x_j) \quad & &\text{for } 1 \leq i < j \leq n-1 \\
x_i - x_n: & \quad (x_i, x_{i+1}, \ldots, x_1, u, x_{i+1}, \ldots, x_{n-2}, v, x_{n-1}, x_n) \quad & &\text{for } 1 \leq i \leq n-3 \\
x_{n-2} - x_n: & \quad (x_{n-2}, x_{n-3}, \ldots, x_1, v, x_{n-1}, u, x_n).
\end{align*}
\]

Therefore, $c$ is a Hamiltonian-connected rainbow coloring of $G$ and so $\text{hrc}(G) = n + 1$. $lacksquare$

The following are consequences of Observations 3.1.3 and 3.1.4 and Theorem 3.1.11.
Figure 3.2: An \((n + 1)\)-edge coloring \(c\) of \(P_n \lor \overline{K}_2\)

**Corollary 3.1.12** If \(G\) is a graph of order \(n \geq 3\) containing a Hamiltonian path, then 
\[\text{hrc}(G \lor \overline{K}_2) = n + 1.\]

### 3.2 The Graphs \(K_n \square K_2\)

We now investigate the rainbow Hamiltonian-connection numbers of the Hamiltonian-connected graphs \(K_n \square K_2\) for several integers \(n \geq 3\). First, we determine the rainbow Hamiltonian-connection number of \(K_3 \square K_2\), the unique Hamiltonian-connected cubic graph of order 6.

**Theorem 3.2.1** \(\text{hrc}(K_3 \square K_2) = 7.\)

**Proof.** Since the 7-edge coloring of the graph \(G = K_3 \square K_2\) shown in Figure 3.3 is a Hamiltonian-connected rainbow 7-coloring, it follows that \(\text{hrc}(G) \leq 7.\)

Figure 3.3: A Hamiltonian-connected rainbow coloring of \(K_3 \square K_2\) using 7 colors

45
It remains to show that $hrc(G) \geq 7$. By Observation 3.1.5, $hrc(G) \geq 6$. Hence, either $hrc(G) = 6$ or $hrc(G) = 7$. We show that $hrc(G) \neq 6$; for suppose that there is a Hamiltonian-connected rainbow $6$-coloring $c : E(G) \to \{1, 2, \ldots, 6\}$ of $G$. For each pair $x, y$ of vertices of $G$, there are exactly two Hamiltonian $x - y$ paths. At least one of these two paths is necessarily a rainbow path. These $2\binom{6}{2} = 30$ Hamiltonian paths are shown below.

(1) $u_1 - u_2$ paths: $(u_1, v_1, v_2, v_3, u_2, u_1), (u_1, u_3, v_1, v_2, u_2, u_1)$
(2) $u_1 - u_3$ paths: $(u_1, u_2, v_2, v_1, v_3, u_3), (u_1, v_1, v_3, v_2, u_2, u_3)$
(3) $u_1 - v_1$ paths: $(u_1, u_2, u_3, v_2, v_1, u_1), (u_1, u_3, u_2, v_2, v_1, u_3)$
(4) $u_1 - v_2$ paths: $(u_1, v_1, v_3, u_2, v_2, u_1), (u_1, u_2, v_3, v_1, v_2, u_1)$
(5) $u_1 - v_3$ paths: $(u_1, v_1, v_2, u_2, u_3, v_3), (u_1, u_3, u_2, v_1, v_3, u_3)$
(6) $u_2 - u_3$ paths: $(u_2, u_1, v_1, v_2, u_3, v_3), (u_2, v_2, v_3, u_1, v_1, u_3)$
(7) $u_2 - v_1$ paths: $(u_2, u_1, v_3, v_2, v_1, u_2), (u_2, v_3, v_1, v_2, v_1, u_3)$
(8) $u_2 - v_2$ paths: $(u_2, u_1, u_3, v_3, v_1, v_2), (u_2, u_3, u_1, v_1, v_3, v_2)$
(9) $u_2 - v_3$ paths: $(u_2, v_2, v_1, u_1, u_3, v_3), (u_2, u_3, u_1, v_1, v_2, v_3)$
(10) $u_3 - v_1$ paths: $(u_3, v_3, v_2, u_2, u_1, v_1), (u_3, u_1, u_2, v_2, v_3, v_1)$
(11) $u_3 - v_2$ paths: $(u_3, v_3, v_1, u_1, u_2, v_2), (u_3, u_2, u_1, v_1, v_3, v_2)$
(12) $u_3 - v_3$ paths: $(u_3, u_1, u_2, v_2, v_1, v_3), (u_3, u_2, u_1, v_1, v_2, v_3)$
(13) $v_1 - v_2$ paths: $(v_1, u_1, u_2, u_3, v_3, v_2), (v_1, v_3, u_3, u_1, u_2, v_2)$
(14) $v_1 - v_3$ paths: $(v_1, v_2, u_2, u_1, u_3, v_3), (v_1, u_1, u_3, u_2, v_2, v_3)$
(15) $v_2 - v_3$ paths: $(v_2, u_2, u_3, u_1, v_1, v_3), (v_2, v_1, u_1, u_2, u_3, v_3)$

Because of the symmetry of two Hamiltonian $u_1 - u_2$ paths, we may assume that the Hamiltonian-connected rainbow $6$-coloring $c$ is such that the first of the paths in (1) is a rainbow path. Thus, we may assume that
\[ c(u_1v_1) = 1, c(v_1v_2) = 2, c(v_2v_3) = 3, c(v_3u_3) = 4, c(u_3u_2) = 5. \]
See Figure 3.4.

It remains, therefore, to determine the possible colors in $\{1, 2, \ldots, 6\}$ assigned to the remaining four edges of $G$. For this purpose, we first make a number of observations, beginning with the possible color of the edge $u_1u_2$. Since both Hamiltonian $u_3 - v_2$ paths in (11) contain both edges $u_1v_1$ and $u_1u_2$, the edge $u_1u_2$ cannot be colored 1. Similarly by (12), the edge $u_1u_2$ cannot be colored 2. According to (10), $u_1u_2$ cannot be colored 3. By (13), $u_1u_2$ cannot be colored 4. Consequently, this edge must be colored either 5 or 6. See Figure 3.5.
Next, we consider the possible color of the edge $u_1u_3$. Since the two Hamiltonian $u_2 - v_3$ paths in (9) contain both edges $u_1v_1$ and $u_1u_3$, it follows that $c(u_1u_3) \neq 1$. Also by (9), $c(u_1u_3) \neq 2$. By (7), $c(u_1u_3) \neq 3$ and $c(u_1u_3) \neq 4$. Therefore, either $c(u_1u_3) = 5$ or $c(u_1u_3) = 6$. In a similar manner, we see that $c(u_2v_2) \in \{1, 4\}$ and $c(v_1v_3) \in \{2, 3\}$. Again, see Figure 3.5. If $c(u_1u_3) = 5$, then no rainbow Hamiltonian $u_1 - v_3$ path can contain both $u_1u_3$ and $u_2v_3$. Thus, the only possible rainbow Hamiltonian $u_1 - v_3$ path contains the edges $u_1v_1, u_2v_2$ and $u_3v_3$ (see (5)). Since each of these three edges is colored 1 or 4, this is impossible. Therefore, $c(u_1u_3) = 6$. Similarly, if $c(u_1u_2) = 5$, then no rainbow Hamiltonian $u_1 - v_2$ path can contain both $u_1u_2$ and $u_2v_3$. Hence, the only possible rainbow Hamiltonian $u_1 - v_2$ path contains the edges $u_1v_1, u_2v_2$ and $u_3v_3$, again a contradiction, and so $c(u_1u_2) = 6$. Since no rainbow Hamiltonian path can contain both $u_1u_2$ and $u_1u_3$, the only possible rainbow Hamiltonian $u_2 - v_1$ path must contain the edges $u_1v_1, u_2v_2$ and $u_3v_3$, a contradiction.

Hence, $c$ is not a Hamiltonian-connected rainbow 6-coloring of $G$ and so $\text{hrc}(G) = 7$. ■

According to Theorem 3.2.1 then, for the graph $K_3 \square K_2$ of order $n = 6$ and size $m = 9$, we have $\text{hrc}(K_3 \square K_2) = 7 = n+1$. Next, we turn our attention to the graphs $K_n \square K_2$.
where \( n \geq 5 \). First, we determine \( hrc(W \Box K_2) \) for all wheels \( W \) of order 5 or more. Note that the wheel of order 4 is the complete graph \( K_4 \).

**Theorem 3.2.2** If \( W \) is a wheel of order \( n \geq 5 \), then \( hrc(W \Box K_2) = 2n - 1 \).

**Proof.** Let \( G = W \Box K_2 \) be obtained from two copies \( F \) and \( F' \) of the wheel \( W \) of order \( n \geq 5 \), where \( V(F) = \{ u, u_1, u_2, \ldots, u_{n-1} \} \) and \( V(F') = \{ v, v_1, v_2, \ldots, v_{n-1} \} \), by adding the \( n \) edges \( uv \) and \( u_iv_i \) for \( 1 \leq i \leq n-1 \). Furthermore, assume that \( F = C_{n-1} \lor K_1 \) where \( C_{n-1} = (u_1, u_2, \ldots, u_{n-1}, u_n = u_1) \) and \( F' = C_{n-1} \lor K_1 \) where \( C_{n-1} = (v_1, v_2, \ldots, v_{n-1}, v_n = v_1) \). The edge coloring \( c_F : E(F) \to [n-1] \) defined by \( c_F(u_iu_{i+1}) = c_F(u_iu) = i \) for \( 1 \leq i \leq n-1 \) is a Hamiltonian-connected rainbow coloring of \( F \) and the edge coloring \( c_{F'} : E(F') \to \{n, n+1, \ldots, 2n-2\} \) defined by \( c_{F'}(v_iv_{i+1}) = c_{F'}(v_iv) = n - 1 + i \) for \( 1 \leq i \leq n-1 \) is a Hamiltonian-connected rainbow coloring of \( F' \). Define the \( (2n-1) \)-edge coloring \( c : E(G) \to [2n-1] \) by

\[
c(e) = \begin{cases} 
  c_F(e) & \text{if } e \in E(F) \\
  c_{F'}(e) & \text{if } e \in E(F') \\
  1 & \text{if } e = u_1v_1 \\
  n + 1 & \text{if } e = u_3v_3 \\
  2n - 1 & \text{if } e = uv \text{ or } e = u_iv_i \text{ for } i = 2 \text{ or } 4 \leq i \leq n - 1.
\end{cases}
\]

For \( n = 6 \), this coloring \( c \) of \( G = W \Box K_2 \) is illustrated in Figure 3.6. Since \( n \geq 5 \), it follows that \( u_1 \) and \( u_3 \) are nonadjacent vertices in \( F \) and \( v_1 \) and \( v_3 \) are nonadjacent vertices in \( F' \). We show that \( c \) is a Hamiltonian-connected rainbow \((2n-1)\)-coloring of \( G \); that is, we show that every two vertices \( x \) and \( y \) of \( G \) are connected by a rainbow Hamiltonian path in \( G \). We consider two cases, according to the locations of \( x \) and \( y \) in \( G \).

**Case 1.** \( x \in V(F) \) and \( y \in V(F') \). Since \( n \geq 5 \), there exists \( z \in V(F) - \{ x \} \) such that (1) the corresponding vertex \( z' \) of \( z \) in \( F' \) is not \( y \) and (2) \( c(zz') = 2n - 1 \) (namely, \( zz' \) is not \( u_1v_1, u_3v_3 \) or the two edges between \( F \) and \( F' \) incident with \( x \) or \( y \)). Let \( P \) be a rainbow Hamiltonian \( x - z \) path in \( F \) and let \( P' \) be a rainbow Hamiltonian \( z' - y \) path in \( F' \). Then the path \( (P, P') \) is a Hamiltonian \( x - y \) path in \( G \).

**Case 2.** \( x, y \in V(F) \) or \( x, y \in V(F') \). We may assume, without loss of generality, that \( x, y \in V(F) \). Let \( Q = (x = x_1, x_2, \ldots, x_n = y) \) be a rainbow Hamiltonian \( x - y \) path in \( F \). Since \( \{c(x_ix_{i+1}) : 1 \leq i \leq n - 1\} = [n - 1] \), there is exactly one integer
Figure 3.6: A Hamiltonian-connected rainbow 11-coloring of $W \Box K_2$ for $n = 6$

$t \in [n - 2]$ such that $c(x_t x_{t+1}) = 1$. Let $x'_t$ and $x'_{t+1}$ be the corresponding vertices of $x_t$ and $x_{t+1}$ in $F'$, respectively. Thus, \{c(x_t x'_t), c(x_{t+1} x'_{t+1})\} = \{1, 2n - 1\}$, say $c(x_t x'_t) = 1$ and $c(x_{t+1} x'_{t+1}) = 2n - 1$. Let $Q'$ be a rainbow Hamiltonian $x'_t - x'_{t+1}$ path in $F'$. Next, let $Q_1$ be the $x_1 - x_t$ subpath of $Q$ and let $Q_2$ be the $x_{t+1} - x_{n-1}$ subpath of $Q$. Then the path $(Q_1, Q', Q_2)$ is a rainbow Hamiltonian $x - y$ in $G$. 

Corollary 3.2.3 For each integer $n \geq 5$, $hrc(K_n \Box K_2) = 2n - 1$.

By Corollary 3.2.3 then, for each integer $n \geq 5$, it follows that

$$hrc(K_n \Box K_2) - 2 hrc(K_n) = 1.$$ 

In fact, for every Hamiltonian-connected graph $H$ of order $n \geq 4$, the number $hrc(H \Box K_2) - 2 hrc(H)$ cannot be much larger than 1.

Theorem 3.2.4 If $H$ is a Hamiltonian-connected graph of order $n \geq 4$, then

$$hrc(H \Box K_2) \leq 2 hrc(H) + 2.$$ 

Proof. Suppose that $hrc(H) = k$. Let $G = H \Box K_2$ be obtained from two copies $F$ and $F'$ of the graph $H$ of order $n \geq 4$, where $V(F) = \{u_1, u_2, \ldots, u_n\}$ and $V(F') = \{v_1, v_2, \ldots, v_n\}$, by adding the $n$ edges $u_i v_i$ for $1 \leq i \leq n$. Since $hrc(H) = k$, it follows that $H$ has a Hamiltonian-connected rainbow $k$-coloring. Let $c_F : V(F) \to \{1, 2, \ldots, k\}$ and $c_{F'} : V(F') \to \{k + 1, k + 2, \ldots, 2k\}$ be a Hamiltonian-connected rainbow $k$-coloring.
of $F$ and $F'$, respectively. Define the $(2k + 2)$-edge coloring $c : E(G) \to [2k + 2]$ by

$$
c(e) = \begin{cases} 
c_F(e) & \text{if } e \in E(F) \\
c_{F'}(e) & \text{if } e \in E(F') \\
2k + 1 & \text{if } e = u_i v_i \text{ and } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
2k + 2 & \text{if } e = u_i v_i \text{ and } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}
$$

We show that $c$ is a Hamiltonian-connected rainbow coloring of $G$; that is, we show that every two vertices $x$ and $y$ of $G$ are connected by a rainbow Hamiltonian path in $G$. We consider two cases, according to the locations of $x$ and $y$ in $G$.

**Case 1.** $x = u_i$ and $y = v_j$ where $1 \leq i, j \leq n$. Let $t \in [n] - \{i, j\}$. Let $P$ be a rainbow Hamiltonian $u_i - u_t$ path in $F$ and let $P'$ be a rainbow Hamiltonian $v_i - v_j$ path in $F'$. Then the path $(P, P')$ is a Hamiltonian $u_i - v_j$ path in $G$.

**Case 2.** $x, y \in V(F)$ or $x, y \in V(F')$, say the former. Suppose that $x = u_i$ and $y = u_j$ where $1 \leq i, j \leq n$ and $i \neq j$. Let $Q$ be a rainbow Hamiltonian $u_i - u_j$ path in $F$, say $Q = (u_i = x_1, x_2, \ldots, x_n = u_j)$. Thus, there is $t \in [n-1]$ such that $c(x_t x'_t) \neq c(x_{t+1} x'_{t+1})$, where $x'_t$ and $x'_{t+1}$ are the corresponding vertices of $x_t$ and $x_{t+1}$ in $F'$, respectively. Let $Q_1$ be the $x_1 - x_t$ subpath of $Q$ and let $Q_2$ be the $x_{t+1} - x_n$ subpath of $Q$. Now, let $Q'$ be a rainbow Hamiltonian $x'_t - x'_{t+1}$ path in $F'$. Then the path $(Q_1, Q', Q_2)$ is a rainbow Hamiltonian $u_i - u_j$ in $G$.

Therefore, $c$ is a Hamiltonian-connected rainbow coloring of $G$ and so $\text{hrc}(G) \leq 2k + 2$.

By Theorem 3.2.1 and Corollary 3.2.3, we then have the following.

**Corollary 3.2.5** $\text{hrc}(K_n \square K_2) = \begin{cases} 
7 & \text{if } n = 3 \\
2n - 1 & \text{if } n \geq 5.
\end{cases}$

Thus, only $\text{hrc}(K_4 \square K_2)$ is unknown. On the other hand, $\text{hrc}(K_4 \square K_2)$ is either 7 or 8 by Theorem 3.2.4.

It is possible to improve the upper bound in Theorem 3.2.4 for some Hamiltonian-connected graphs possessing a Hamiltonian-connected rainbow coloring with a particular property. Before describing this improvement, we first present an additional definition. For a Hamiltonian-connected graph $H$, let $c : E(H) \to [k]$ be an edge coloring of $H$. For each integer $i$ with $1 \leq i \leq k$, let $E_i = \{ e \in E(H) : c(e) = i \}$ be the set of edges of $H$
that are colored \( i \) by \( c \) and let \( H[E_i] \) be the subgraph induced by \( E_i \) in \( H \). If \( T_1 \) and \( T_2 \) are edge-disjoint stars in \( H \) such that their central vertices are not adjacent in \( H \), then \( T_1 \) and \( T_2 \) are referred to as nonadjacent stars in \( H \). The following result is a more general case of Theorem 3.2.2.

**Theorem 3.2.6** Let \( H \) be a Hamiltonian-connected graph of order \( n \geq 5 \) and let \( c : E(H) \rightarrow [k] \) be a Hamiltonian-connected rainbow coloring of \( H \) for some integer \( k \geq n - 1 \). If there are two distinct colors \( \alpha, \beta \in [k] \) such that \( H[E_{\alpha}] \) and \( H[E_{\beta}] \) are nonadjacent stars in \( H \), then \( \text{hrc}(H \square K_2) \leq 2 \text{hrc}(H) + 1 \).

**Proof.** Let \( G = H \square K_2 \) be obtained from two copies \( F \) and \( F' \) of the graph \( H \) of order \( n \geq 5 \), where \( V(F) = \{ u_1, u_2, \ldots, u_n \} \) and \( V(F') = \{ v_1, v_2, \ldots, v_n \} \), by adding the \( n \) edges \( u_i v_i \) for \( 1 \leq i \leq n \). Let \( c_F : E(F) \rightarrow [k] \) be a Hamiltonian-connected rainbow coloring of \( F \) such that there are two distinct colors \( \alpha, \beta \in [k] \) for which \( F[E_{\alpha}] \) and \( F[E_{\beta}] \) are nonadjacent stars in \( F \). We may assume that the central vertex of \( F[E_{\alpha}] \) is \( u_1 \) and the central vertex of \( F[E_{\beta}] \) is \( v_1 \), where then \( u_1 \) and \( v_1 \) are not adjacent in \( F \). For an edge \( e \) of \( F \), let \( e' \) denote the corresponding edge of \( e \) in \( F' \). The coloring \( c_{F'} : E(F') \rightarrow \{ k + 1, k + 2, \ldots, 2k \} \) of \( F' \) defined by \( c_{F'}(e') = c_F(e) + k \) for each \( e' \in E(F') \) is then a Hamiltonian-connected rainbow coloring of \( F' \).

We now define the \((2k + 1)\)-edge coloring \( c : E(G) \rightarrow [2k + 1] \) of \( G \) by

\[
c(e) = \begin{cases} 
c_F(e) & \text{if } e \in E(F) \\
c_{F'}(e) & \text{if } e \in E(F') \\
\alpha & \text{if } e = u_1 v_1 \\
k + \beta & \text{if } e = u_2 v_2 \\
2k + 1 & \text{if } e = u_i v_i \text{ for } 3 \leq i \leq n.
\end{cases}
\]

We now verify that \( c \) is a Hamiltonian-connected rainbow \((2k + 1)\)-coloring of \( G \) by showing that every two vertices \( x \) and \( y \) of \( G \) are connected by a rainbow Hamiltonian path in \( G \). We consider two cases, according to the location of \( x \) and \( y \) in \( G \). In each of these cases, we denote, for a vertex \( w \) of \( F \), the corresponding vertex of \( w \) in \( F' \) by \( w' \).

**Case 1.** \( x \in V(F) \) and \( y \in V(F') \). Since \( n \geq 5 \), there exists \( z \in V(F) - \{ x \} \) such that \( zz' \) is neither \( u_1 v_1 \), \( u_2 v_2 \) nor either of the two edges between \( F \) and \( F' \) incident with \( x \) or \( y \). Let \( P \) be a rainbow Hamiltonian \( x - z \) path in \( F \) and let \( P' \) be a rainbow Hamiltonian \( z' - y \) path in \( F' \). Then the path \(( P, P' )\) is a Hamiltonian \( x - y \) path in \( G \).
Case 2. $x,y \in V(F)$ or $x,y \in V(F')$. We may assume, without loss of generality that $x,y \in V(F)$. Let $Q = (x = x_1, x_2, \ldots, x_n = y)$ be a rainbow Hamiltonian $x - y$ path in $F$ and let $c(Q) = \{c(x_i, x_{i+1}) : 1 \leq i \leq n - 1\}$ be the set of colors of the edges of $Q$.

First, suppose that $\alpha \in c(Q)$. Since $Q$ is a rainbow Hamiltonian path of $F$ and $H[E_\alpha]$ is a star, there is exactly one integer $t \in [n - 1]$ such that $c(x_t, x_{t+1}) = \alpha$. Then either $x_t$ or $x_{t+1}$ is the central vertex of $H[E_\alpha]$. So, $x'_t$ and $x'_{t+1}$ are the corresponding vertices of $x_t$ and $x_{t+1}$ in $F'$, respectively. Since the central vertices of $F[E_\alpha]$ and $F[E_\beta]$ are not adjacent in $F$, it follows that $\{c(x_t, x'_t), c(x_{t+1}, x'_{t+1})\} = \{\alpha, 2k + 1\}$, say $c(x_t, x'_t) = \alpha$ and $c(x_{t+1}, x'_{t+1}) = 2k + 1$. Let $Q'$ be a rainbow Hamiltonian $x'_t - x'_{t+1}$ path in $F'$. Next, let $Q_1$ be the $x_1 - x_t$ subpath of $Q$ and let $Q_2$ be the $x_{t+1} - x_n$ subpath of $Q$. Then the path $(Q_1, Q', Q_2)$ is a rainbow Hamiltonian $x - y$ path in $G$. Next, suppose that $\alpha \notin c(Q)$. Since $Q$ is a rainbow Hamiltonian path of $F$, there is exactly one integer $t \in [n - 1]$ such that $x_t = u_1$ (the central vertex of $F[E_\alpha]$). Let $x'_t$ and $x'_{t+1}$ be the corresponding vertices of $x_t$ and $x_{t+1}$ in $F'$, respectively. Let $Q'$ be a rainbow Hamiltonian $x'_t - x'_{t+1}$ path in $F'$. Next, let $Q_1$ be the $x_1 - x_t$ subpath of $Q$ and let $Q_2$ be the $x_{t+1} - x_n$ subpath of $Q$. Then the path $(Q_1, Q', Q_2)$ is a rainbow Hamiltonian $x - y$ path in $G$. 

A graph is a galaxy if each of its components is a star. If $H_1$ and $H_2$ are edge-disjoint subgraphs of a graph $H$ where $H_1$ and $H_2$ are galaxies, such that no central vertex of any component in $H_1$ is adjacent to a central vertex in a component in $H_2$, then $H_1$ and $H_2$ are referred to as nonadjacent galaxies in $H$. The argument used in the proof of Theorem 3.2.6 yields the following result.

**Corollary 3.2.7** Let $H$ be a Hamiltonian-connected graph of order $n \geq 5$ and let $c : E(H) \to [k]$ be a Hamiltonian-connected rainbow coloring of $H$ for some integer $k \geq n - 1$. If there are two distinct colors $\alpha, \beta \in [k]$ such that $H[E_\alpha]$ and $H[E_\beta]$ are nonadjacent galaxies in $H$, then $hrc(H \Box K_2) \leq 2hrc(H) + 1$.

### 3.3 Hamiltonian-Connected Prisms

We have now seen (in Theorem 3.1.6, Corollary 3.1.8, Theorem 3.2.2 and Corollary 3.2.3) Hamiltonian-connected graphs $G$ of order $n$ for which $hrc(G) = n - 1$. We also saw in Theorem 3.2.1 a Hamiltonian-connected graph $G$ of order 6 with $hrc(G) = n + 1$. However, no graph $H$ of order $n \geq 4$ with $hrc(H) = n$ has yet been established. If it is determined that the value of $hrc(K_4 \Box K_2)$ is equal to 7, then this would be the first such graph. Indeed, the examples described thus far may suggest the existence of a constant
There is actually no such constant, as we verify in this section.

For an integer \( n \geq 3 \), the prism \( C_n \diamond K_2 \) is Hamiltonian-connected if and only if \( n \) is odd. Thus, if \( G \) is a Hamiltonian graph of odd order \( n \geq 3 \), then \( G \diamond K_2 \) is Hamiltonian-connected. For an odd integer \( n \geq 3 \), let \( C_n \diamond K_2 \) be obtained from two copies \( C \) and \( C' \) of the \( n \)-cycle \( C_n \), where \( C = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1) \) and \( C' = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1) \), by adding the \( n \) edges \( u_i v_i \) for \( 1 \leq i \leq n \). We will refer to this labeling of the vertices of \( C_n \diamond K_2 \) for all results in this section. By Theorem 3.2.1, \( hrc(C_3 \diamond K_2) = 7 \). Next, we show that \( hrc(C_n \diamond K_2) = 3n \) for each odd integer \( n \geq 5 \). In order to do this, we first present some preliminary results.

**Lemma 3.3.1** Let \( n \geq 5 \) be an odd integer. For each integer \( i \) with \( 1 \leq i \leq n \), there are exactly two \( u_i - v_i \) Hamiltonian paths in \( C_n \diamond K_2 \).

**Proof.** By the symmetry of the graph \( G = C_n \diamond K_2 \), it suffices to show that there are exactly two Hamiltonian \( u_1 - v_1 \) paths in \( G \). First, \( G \) contains the following two Hamiltonian \( u_1 - v_1 \) paths:

\[
P = (u_1, u_n, u_{n-1}, \ldots, u_2, v_2, v_3, v_4, \ldots, v_n, v_1) \quad (3.1)
\]

\[
Q = (u_1, u_2, u_3, \ldots, u_n, v_n, v_{n-1}, \ldots, v_2, v_1) \quad (3.2)
\]

Thus, it remains to show that \( P \) and \( Q \) are the only Hamiltonian \( u_1 - v_1 \) paths in \( G \). Let \( R \) be a Hamiltonian \( u_1 - v_1 \) path in \( G \). Since \( u_1v_1 \notin E(R) \), it follows that exactly one of \( u_1u_n \) and \( u_1u_2 \) belongs to \( R \), say \( u_1u_n \in E(R) \) and \( u_1u_2 \notin E(R) \). Since \( G \) is 3-regular and \( R \) is a Hamiltonian path of \( G \), for each \( x \in V(G) - \{u_1, v_1\} \), exactly one of the three edges incident with \( x \) does not belong to \( R \). Since \( u_1u_2 \notin E(R) \), it follows that \( (v_2, u_2, u_3) \) is a subpath of \( R \) and exactly one of \( v_1v_n \) and \( v_1v_2 \) belongs to \( R \). We consider these two cases.

**Case 1.** \( v_1v_n \in E(R) \) and \( v_1v_2 \notin E(R) \). Since \( v_1v_2 \notin E(R) \) and \( G \) is 3-regular, it follows that \( v_2v_3 \in E(R) \). Hence, \( u_nv_n, u_3v_3 \notin E(R) \). This implies that \( u_nu_{n-1}, v_nv_{n-1}, u_3u_4, v_3v_4 \) are edges of \( R \). Continuing this argument, we see that \( u_iv_i \notin E(R) \) for \( 3 \leq i \leq n \) and so \( R \) is the path \( P \) described in (3.1).

**Case 2.** \( v_1v_n \notin E(R) \) and \( v_1v_2 \in E(R) \). Then \( v_1v_n, v_2v_3 \notin E(R) \) and so \( (u_n, v_n, v_{n-1}) \) and \( (v_3, v_3, v_4) \) are subpaths of \( R \). Thus, \( u_nv_{n-1}, u_3v_4 \notin E(R) \). If \( n = 5 \), then \( v_{n-1} = v_4 \). However then, \( u_4 \) does not belong to \( R \), which is a contradiction. If \( n \geq 7 \), then
(v_4, u_4, u_5) is a subpath of R and so v_4v_5 \notin E(R). Continuing this argument, we obtains that (v_4, u_4, u_5, v_5, v_6, u_6, \ldots, u_{n-2}, v_{n-2}, v_{n-1}) is a subpath of R. However then, u_{n-1} does not belong to R, which is a contradiction. Thus, Case 2 cannot occur.

**Lemma 3.3.2** If c is a Hamiltonian-connected rainbow coloring of \( C_n \square K_2 \) for some odd integer \( n \geq 5 \), then for each integer \( i \) with \( 1 \leq i \leq n \), the coloring c assigns distinct colors to the \( 2(n-2) \) edges in the two paths \( C - u_i \) and \( C' - v_i \) in \( C_n \square K_2 \).

**Proof.** Let c be a Hamiltonian-connected rainbow coloring of \( G = C_n \square K_2 \) for some odd integer \( n \geq 5 \). By the symmetry of the graph \( G \), it suffices to show that c must assign distinct colors to all edges in the two paths \( C - u_1 \) and \( C' - v_1 \) in \( G \). Let \( X = E(C - u_1) \cup E(C' - v_1) \). Then \( |X| = 2(n-2) \). By Lemma 3.3.1, the paths \( P \) in (3.1) and \( Q \) in (3.2) are the only Hamiltonian \( u_1 - v_1 \) paths in \( G \). Thus, at least one of \( P \) and \( Q \) is rainbow. Since \( X \subseteq E(P) \cap E(Q) \) (see Figure 3.7 for \( n = 7 \)), it follows that all edges in \( X \) must be assigned different colors.

![Figure 3.7: The set X of edges in C_7 \square K_2](image)

**Lemma 3.3.3** Let \( n \geq 5 \) be an odd integer. For each integer \( i \) with \( 1 \leq i \leq n \), there are exactly two \( u_i - v_{i+1} \) Hamiltonian paths in \( C_n \square K_2 \).

**Proof.** By the symmetry of the graph \( G = C_n \square K_2 \), it suffices to show that there are exactly two Hamiltonian \( u_1 - v_2 \) paths in \( G \). First, \( G \) contains the following two Hamiltonian \( u_1 - v_2 \) paths:

\[
P = (u_1, v_1, v_3, u_3, u_5, v_5, v_6, \ldots, v_{n-1}, u_{n-1}, u_n, v_n, u_1, v_2)
\]

(3.3)

\[
Q = (u_1, u_2, v_3, v_4, u_4, u_5, v_5, v_6, \ldots, v_{n-1}, u_{n-1}, u_n, v_n, u_1, v_2)
\]

(3.4)

The Hamiltonian \( u_1 - v_2 \) paths \( P \) and \( Q \) are shown in Figure 3.8 for \( n = 7 \), where solid lines indicate edges in \( P \) or \( Q \) while dashed lines are edges not in \( P \) or \( Q \).
Let $R$ be a Hamiltonian $u_1 - v_2$ path in $G$. Then $R$ contains exactly one of $u_1v_1$, $u_1u_2$ or $u_1u_n$. We consider these three cases.

Case 1. $u_1v_1 \in E(R)$ and $u_1u_2, u_1u_n \notin E(R)$. Clearly, $v_1v_2 \notin E(R)$. Hence, $(v_2, u_2, u_3)$ and $(v_1, v_n, u_n, u_{n-1})$ are subpaths of $R$. Since $v_2$ is an end-vertex of $R$, it follows that $v_2v_3 \notin E(R)$. Now $v_2v_3, v_nv_{n-1} \notin E(R)$ and so $(v_3, v_2, v_4)$ is a subpath of $R$. If $n = 5$, then

$$R = (u_1, v_1, v_5, u_4, v_4, v_3, u_3, u_2, v_2),$$

which is the path $P$ in (3.3), while if $n \geq 7$, then, by a similar argument, we also obtain that $R$ is the path $P$ in (3.3).

Case 2. $u_1u_2 \in E(R)$ and $u_1v_1, u_1u_n \notin E(R)$. Then $(v_2, v_1, v_n, u_n, u_{n-1})$ is a subpath of $R$. Since $v_nv_{n-1} \notin E(R)$, it follow that $(u_{n-1}, v_{n-1}, v_n-2)$ is a subpath of $R$. Continuing in this way, we see that $R$ is the path $Q$ in (3.4) in this case.

Case 3. $u_1u_n \in E(R)$ and $u_1v_1, u_1u_2 \notin E(R)$. Since $R$ is a Hamiltonian path and $G$ is 3-regular, it follows that $(v_n, v_1, v_2, u_2, u_3)$ is a subpath of $R$, which is impossible since $v_2$ is an end-vertex of $R$. Thus, Case 3 cannot occur.

**Lemma 3.3.4** Let $n \geq 5$ be an odd integer. For each integer $i$ with $1 \leq i \leq n$, let

$$R_i = (u_{i+1}, u_{i+2}, v_{i+2}, v_{i+3}, u_{i+3}, u_{i+4}, \ldots, v_{i-2}, u_{i-2}, u_{i-1}, v_{i-1}, v_i) \quad (3.5)$$

be the path in $C_n \square K_2$, where the subscript of each vertex is expressed as an integer modulo $n$. If $c$ is a Hamiltonian-connected rainbow coloring of $C_n \square K_2$, then $c$ assigns distinct colors to the edges of $R_i$ for $1 \leq i \leq n$. 

55
**Proof.** Let $c$ be a Hamiltonian-connected rainbow coloring of $G = C_n \square K_2$ for some odd integer $n \geq 5$. It suffices to show that all edges in $R_1$ must be colored differently by $c$. Observe that

$$R_1 = \{u_2, u_3, v_3, v_4, u_4, u_5, v_5, \ldots, u_{n-1}, u_n, v_n, v_{n+1} = v_1\}.$$

By Lemma 3.2, the paths $P$ in (3.3) and $Q$ in (3.4) are the only Hamiltonian $u_1 - v_2$ paths in $G$. Thus, at least one of $P$ and $Q$ is rainbow. Since $E(R_1) \subseteq E(P) \cap E(Q)$ (see Figure 3.9 for $n = 7$), it follows that all edges in $E(R_1)$ must be assigned different colors.

![Figure 3.9](image)

**Figure 3.9:** The path $R_1$ in $C_7 \square K_2$

We now make a useful observation. Let $E[C, C'] = \{u_iv_i : 1 \leq i \leq n\}$ be the set of edges of $C_n \square K_2$ that do not belong to $E(C) \cup E(C')$. For each integer $i$ with $1 \leq i \leq n$, let $R_i$ be the path of $C_n \square K_2$ defined in (3.5). Then $E(C, C') - \{u_iv_i, u_{i+1}v_{i+1}\} \subseteq E(R_i)$ and $u_iv_i, u_{i+1}v_{i+1} \notin E(R_1)$. For example, $R_1$ contains all edges $u_iv_i$ for $3 \leq i \leq n$ and $u_1v_1, u_2v_2 \notin E(R_1)$.

**Theorem 3.3.5** For each odd integer $n \geq 5$, $hrc(C_n \square K_2) = 3n$.

**Proof.** Let $G = C_n \square K_2$ for some odd integer $n \geq 5$. Since the size of $G$ is $3n$, it follows by Observation 3.1.4 that $hrc(C_n \square K_2) \leq 3n$. It remains to show that every Hamiltonian-connected rainbow coloring of $G$ must assign distinct colors to distinct edges of $G$. Let $c$ be a Hamiltonian-connected rainbow coloring and let $e, f \in E(G)$. We show that $c(e) \neq c(f)$.

First, suppose that at least one of $e$ and $f$ belongs to the two $n$-cycles $C$ and $C'$ in $G$, say $e \in E(C) \cup E(C')$. Assume, without loss of generality, that $e = u_1u_2$. We consider two cases, according to whether $f \in E(C) \cup E(C')$ or $f \in E[C, C']$. 

56
Case 1. \( f \in E(C) \cup E(C') \). Thus, either \( f = u_ju_{j+1} \in E(C) \) where \( 2 \leq j \leq n \) or \( f = v_jv_{j+1} \in E(C') \) where \( 1 \leq j \leq n \). Since \( n \geq 5 \), it follows that there exists \( i \in [n] - \{1, 2, j, j+1\} \). Thus, \( e, f \in E(C - u_i) \cup E(C' - v_i) \). It then follows by Lemma 3.3.4 that \( c(e) \neq c(f) \).

Case 2. \( f \in E[C, C'] \). Let \( f = u_jv_j \) where \( 1 \leq j \leq n \). First, suppose that \( f = u_1v_1 \) or \( f = u_2v_2 \), say the former. Then both \( e = u_1u_2 \) and \( f = u_1v_1 \) lie on the path

\[
R_3 = (u_4, u_5, v_5, v_6, u_6, u_7, \ldots, v_{n-1}, u_n, v_n, v_1, u_1, u_2, v_2, v_3),
\]
as described in (3.5). The path \( R_3 \) is shown in Figure 3.10 for \( n = 7 \), where \( e \) and \( f \) are indicated by bold lines, the edges in \( R_3 \) are indicated by solid lines and all other edges (that are not on \( R_3 \)) are indicated by dashed lines.

Figure 3.10: The paths \( R_3 \) and \( R_7 \) in \( C_7 \sqcup K_2 \)

By Lemma 3.3.4, \( c(e) \neq c(f) \). Next, suppose that \( f = u_iv_i \) where \( 3 \leq i \leq n \). By symmetry, we may assume that \( 3 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1 \). Then \( e = u_1u_2 \) and \( f = u_iv_i \) lie on the path

\[
R_n = (u_1, u_2, v_2, v_3, u_3, u_4, \ldots, v_{n-2}, u_{n-2}, u_{n-1}, v_{n-1}, v_n),
\]
as described in (3.5). For \( f = u_4v_4 \), the path \( R_7 \) is shown in Figure 3.10 for \( n = 7 \) which contains both \( e \) and \( f \). Since \( u_iv_i \in E(R_n) \) for \( 3 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + 1 \), it follows that \( f \in E(R_n) \). Thus, both \( e \) and \( f \) belong to \( R_n \) and so \( c(e) \neq c(f) \) by Lemma 3.3.4.

Next, suppose that neither \( e \) nor \( f \) belongs to \( E(C) \cup E(C') \). Assume, without loss of generality, that \( e = u_1v_1 \) and \( f = u_jv_j \) where \( 2 \leq j \leq n \). Since \( n \geq 5 \), there is \( i \in [n] - \{1, j\} \) such that \( e, f \in E(R_i) \). It then follows by Lemma 3.3.4 that \( c(e) \neq c(f) \). \(\blacksquare\)
It then follows from Theorem 3.3.5 that there is no constant $c$ such that $hrc(G) \leq n+c$ for every Hamiltonian-connected graph $G$ of order $n$.

### 3.4 The Square of Hamiltonian Graphs

During 1960-1980, there was a great deal of activity in researching Hamiltonian properties of powers of graphs. For a connected graph $G$ and a positive integer $k$, the $k$th power $G^k$ of $G$ is the graph whose vertex set is $V(G)$ such that $uv$ is an edge of $G^k$ if $1 \leq d_G(u,v) \leq k$, where $d_G(u,v)$ is the distance between two vertices $u$ and $v$ in $G$ (the length of a shortest $u-v$ path in $G$). The graph $G^2$ is called the square of $G$ and $G^3$ is the cube of $G$. In 1960, Sekanina [73] proved the following result.

**Theorem 3.4.1** If $G$ is a nontrivial connected graph, then the cube of $G$ is Hamiltonian-connected.

In the 1960s, it was conjectured independently by Nash-Williams [62] and Plummer (see [25, p.147]) that the square of every 2-connected graph is Hamiltonian. In 1974, Fleischner [39] verified this conjecture. Also, in 1974, using Fleischner’s result, Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [22] proved the following.

**Theorem 3.4.2** If $G$ is a 2-connected graph, then the square of the graph $G$ is Hamiltonian-connected. In particular, the square of every Hamiltonian graph is Hamiltonian-connected.

Since the square of a Hamiltonian graph $G$ of order $n \geq 3$ contains the square $C_n^2$ of an $n$-cycle $C_n$ as a spanning subgraph, it then follows by Observation 3.1.3 that $hrc(G^2) \leq hrc(C_n^2)$. Thus, it is of interest to investigate the rainbow Hamiltonian-connection numbers of the squares of cycles. Since $hrc(K_n) = n - 1$ for $n \geq 4$, it follows that $hrc(C_n^2) = n - 1$ for $n = 4, 5$. For $n \geq 6$, $hrc(C_n^2) \geq n - 1$ by Observation 3.1.4. The following theorem says that for every integer $n \geq 6$, either $hrc(C_n^2) = n - 1$ or $hrc(C_n^2) = n$.

**Theorem 3.4.3** For each integer $n \geq 6$, $hrc(C_n^2) \leq n$.

**Proof.** Let $G = C_n^2$, where $C_n = (v_1, v_2, \ldots, v_n, v_1)$. We show that $G$ has a Hamiltonian-connected rainbow coloring using $n$ colors. Define the $n$-edge coloring $c : E(G) \to [n]$ by $c(v_i v_{i+1}) = i$ and $c(v_i v_{i+2}) = i + 1$ for $1 \leq i \leq n$, where the subscript of each vertex is expressed as a positive integer modulo $n$. 

58
Corollary 3.4.4 If $G$ is a Hamiltonian graph of order at least $n \geq 6$, then $hrc(G^2) \leq n$. 

Figure 3.11: A Hamiltonian-connected rainbow coloring of $C_n^2$ using $n$ colors

We show that every two vertices $x$ and $y$ in $G$ are connected by a rainbow Hamiltonian path in $G$. Since $C_n$ is a rainbow Hamiltonian cycle in $G$, we may assume that $x$ and $y$ are not consecutive vertices of $C_n$. By symmetry of the graph $G$ and the coloring $c$, we may further assume that $x = v_1$ and $y = v_i$ for $3 \leq i \leq n - 1$.

* If $i$ is odd, then
  
  $$(v_1, v_n, v_{n-1}, \ldots, v_{i+1}, v_{i-1}, v_{i-3}, \ldots, v_2, v_3, v_5, v_7, \ldots, v_1)$$

  is a rainbow Hamiltonian $v_1 - v_i$ path.

* If $i$ is even, then

  $$(v_1, v_n, v_{n-1}, \ldots, v_{i+1}, v_{i-1}, v_{i-3}, \ldots, v_3, v_2, v_4, v_6, \ldots, v_1)$$

  is a rainbow Hamiltonian $v_1 - v_i$ path.

For example, if $n = 10$, then $(v_1, v_{10}, v_9, v_8, v_7, v_6, v_4, v_2, v_3, v_5)$ is a rainbow Hamiltonian $v_1 - v_5$ path and $(v_1, v_{10}, v_9, v_8, v_7, v_5, v_3, v_2, v_4, v_6)$ is a rainbow Hamiltonian $v_1 - v_6$ path. Thus, $c$ is a Hamiltonian-connected rainbow coloring of $G$ and so $hrc(G) \leq n$. 

If $G$ is a Hamiltonian graph of of order $n \geq 4$ having diameter at most 2, then $G^2 = K_n$ and so $hrc(G^2) = n - 1$. In general, the following is an immediate consequence of Theorem 3.4.3 and Observation 3.1.3.

Corollary 3.4.4 If $G$ is a Hamiltonian graph of order at least $n \geq 6$, then $hrc(G^2) \leq n$. 

59
In fact, the rainbow Hamiltonian-connection number of $C_6^2$ is 5. To show this, consider the 5-edge coloring of the graph $G = C_6^2$ shown in Figure 3.12.

![Graph of $C_6^2$ with edge coloring](image)

**Figure 3.12: A Hamiltonian-connected rainbow coloring of $C_6^2$ using 5 colors**

This edge coloring is a Hamiltonian-connected rainbow 5-coloring since each of the following $\binom{5}{2} = 15$ Hamiltonian paths are rainbow paths. Therefore, $\text{hrc}(G) = 5$ by Observation 3.1.4.

1. rainbow Hamiltonian $v_1 - v_2$ path: $(v_1, v_6, v_5, v_4, v_3, v_2)$
2. rainbow Hamiltonian $v_1 - v_3$ path: $(v_1, v_6, v_5, v_4, v_2, v_3)$
3. rainbow Hamiltonian $v_1 - v_4$ path: $(v_1, v_6, v_5, v_3, v_2, v_4)$
4. rainbow Hamiltonian $v_1 - v_5$ path: $(v_1, v_6, v_4, v_2, v_3, v_5)$
5. rainbow Hamiltonian $v_1 - v_6$ path: $(v_1, v_2, v_3, v_4, v_5, v_6)$
6. rainbow Hamiltonian $v_2 - v_3$ path: $(v_2, v_4, v_5, v_6, v_1, v_3)$
7. rainbow Hamiltonian $v_2 - v_4$ path: $(v_2, v_3, v_5, v_1, v_6, v_4)$
8. rainbow Hamiltonian $v_2 - v_5$ path: $(v_2, v_6, v_1, v_3, v_4, v_5)$
9. rainbow Hamiltonian $v_2 - v_6$ path: $(v_2, v_1, v_3, v_4, v_5, v_6)$
10. rainbow Hamiltonian $v_3 - v_4$ path: $(v_3, v_2, v_6, v_1, v_5, v_4)$
11. rainbow Hamiltonian $v_3 - v_5$ path: $(v_3, v_1, v_6, v_2, v_4, v_5)$
12. rainbow Hamiltonian $v_3 - v_6$ path: $(v_3, v_2, v_1, v_5, v_4, v_6)$
13. rainbow Hamiltonian $v_4 - v_5$ path: $(v_4, v_2, v_6, v_1, v_3, v_5)$
14. rainbow Hamiltonian $v_4 - v_6$ path: $(v_4, v_2, v_1, v_3, v_5, v_6)$
15. rainbow Hamiltonian $v_5 - v_6$ path: $(v_5, v_4, v_3, v_2, v_1, v_6)$

By Theorem 3.4.3, for each integer $n \geq 7$, the rainbow Hamiltonian-connection number of $C_n^2$ is either $n - 1$ or $n$. By Theorem 3.4.1, if $G$ is a nontrivial connected graph of
order \( n \), then \( G^3 \) is Hamiltonian-connected and so \( G^k \) is Hamiltonian-connected for each \( k \geq 3 \). Therefore, if \( G \) is a Hamiltonian graph of order \( n \geq 3 \), then \( hrc(G^k) \in \{n - 1, n\} \) for each integer \( k \geq 3 \).

### 3.5 Minimum Hamiltonian-Connected Graphs

If \( G \) is a Hamiltonian-connected graph that is not complete and \( u \) and \( v \) are nonadjacent vertices of \( G \), then \( G + uv \) is also Hamiltonian-connected and \( hrc(G + uv) \leq hrc(G) \) by Observation 3.1.3. This suggests that Hamiltonian-connected graphs having the greatest rainbow Hamiltonian-connection numbers are minimal Hamiltonian-connected graphs. This leads us to consider Hamiltonian-connected graphs of order \( n \) and minimum size. Every Hamiltonian-connected graph of order at least 4 is 3-connected. Therefore, if \( G \) is a Hamiltonian-connected graph of \( n \geq 4 \), then \( \delta(G) \geq 3 \), which implies that the minimum size of a Hamiltonian-connected graph of order \( n \) is \( \lfloor \frac{3n+1}{2} \rfloor \). The following result is due to Moon [61].

**Theorem 3.5.1** For each integer \( n \geq 4 \), there exists Hamiltonian-connected graph of order \( n \) and size \( \lfloor \frac{3n+1}{2} \rfloor \).

For each integer \( k \geq 2 \), let \( P_k \square K_2 \) be the grid of order \( 2k \) in which we have two paths of order \( k \), \( P_k = (x_1, x_2, \ldots, x_k) \) and \( P'_k = (y_1, y_2, \ldots, y_k) \), with \( x_iy_i \in E(P_k \square K_2) \) for \( 1 \leq i \leq k \). Now, let \( H_k \) be the cubic graph of order \( 2k + 2 \) obtained by adding two adjacent vertices \( u \) and \( v \) to the grid \( P_k \square K_2 \) and joining (1) the vertex \( u \) to \( x_1 \) and \( y_1 \) and (2) the vertex \( v \) to \( x_k \) and \( y_k \) in \( P_k \square K_2 \) (see Figure 3.13).

![Graphs H2, H3 and Hk](image)

Figure 3.13: Graphs \( H_2, H_3 \) and \( H_k \)
The graph $H_3$ has order 8 and rainbow Hamiltonian-connection number 10, as we show next.

**Theorem 3.5.2** \(\text{hrc}(H_3) = 10.\)

**Proof.** Let $G = H_3$. First, we make an observation. For each pair $w, z$ of distinct vertices of $G$, there are exactly two Hamiltonian $w - z$ paths except for $\{w, z\} = \{x_1, y_3\}$ and $\{w, z\} = \{x_3, y_1\}$, in which case, there are exactly four Hamiltonian $w - z$ paths. Since there are $\binom{8}{2} = 28$ pairs of vertices of $G$, there are $2\binom{8}{2} + 4 = 60$ Hamiltonian paths in $G$. For each pair $w, z$ of distinct vertices of $G$, we list the Hamiltonian $w - z$ paths in $G$ as well as the edges commonly belonging to each of these $w - z$ paths. We label the vertices and the edges of $G$ as indicated in Figure 3.14.

![Figure 3.14: The graph $H_3$](image-url)

1. $u - v$: $(u, x_1, y_1, y_2, x_2, x_3, y_3, v), (u, y_1, x_1, x_2, y_2, y_3, x_3, v), e_3, e_6, e_9$
2. $u - x_1$: $(u, v, y_3, x_3, x_2, y_2, y_1, x_1), (u, y_1, y_2, y_3, y_1, x_3, x_2, x_1), e_5, e_7, e_11$
3. $u - y_1$: $(u, v, x_3, y_3, y_2, x_2, x_1, y_1), (u, x_1, x_2, x_3, y_3, y_2, y_1), e_4, e_8, e_10$
4. $u - x_2$: $(u, v, x_3, y_3, y_2, x_1, y_1, x_2), (u, x_1, y_1, y_2, y_3, x_3, x_2, x_1), e_3, e_5, e_8, e_{10}$
5. $u - y_2$: $(u, v, y_3, x_3, x_2, x_1, y_1, y_2), (u, y_1, x_1, x_2, x_3, y_3, y_2, y_1), e_3, e_4, e_7, e_{11}$
6. $u - y_3$: $(u, v, y_3, y_2, y_1, x_1, x_2, x_3), (u, y_1, x_1, x_2, y_2, y_3, y_1, y_2), e_3, e_4, e_8, e_{11}$
7. $x_1 - v$: $(x_1, u, y_1, y_2, x_2, x_3, y_1, v), (x_1, x_2, x_3, y_1, y_2, y_1, u, v), e_2, e_5, e_7, e_9$
8. $x_1 - y_1$: $(x_1, u, v, y_3, x_3, x_2, y_2, y_1), (x_1, x_2, y_2, y_3, x_3, x_2, y_1, u, v), e_6, e_9, e_{12}$
9. $x_1 - x_2$: $(x_1, u, y_1, y_2, y_3, v, x_3, x_2), (x_1, y_1, u, v, x_3, y_2, y_2, x_2), e_2, e_8, e_{10}$
10. $x_1 - y_2$: $(x_1, y_1, u, v, y_3, x_3, x_2, y_2), (x_1, x_2, x_3, y_1, y_2, y_1, u, y_1), e_2, e_7, e_9, e_{11}, e_{12}$
11. $x_1 - x_3$: $(x_1, x_2, y_2, y_1, u, v, y_3, x_3, y_3), (x_1, y_1, u, v, y_3, y_2, x_2, x_3), e_2, e_6, e_{11}, e_{12}$
12. $x_1 - y_3$: $(x_1, x_2, y_2, y_1, u, v, x_3, x_3, y_3), (x_1, u, y_1, y_2, x_2, x_3, v, y_3), (x_1, x_2, x_3, v, u, y_1, y_2, y_3), (x_1, y_1, u, v, x_3, x_2, y_2, y_1), e_2, e_{10}$
13. $y_1 - v$: $(y_1, u, x_1, x_2, y_2, y_3, x_3, x_1, x_2, y_2, y_1), e_1, e_4, e_8, e_9$
14. $y_1 - x_2$: $(y_1, x_1, u, v, x_3, y_3, y_2, x_2), (y_1, y_2, y_3, y_1, x_3, x_2, x_1, u, v), e_1, e_4, e_8, e_9$
15. $y_1 - y_2$: $(y_1, x_1, u, v, x_3, y_3, x_2, y_2), (y_1, y_2, y_3, x_3, x_2, x_1, u, v), e_1, e_8, e_9, e_{10}, e_{12}$
16. $y_1 - y_3$: $(y_1, x_1, u, v, y_3, x_3, x_2, y_2), (y_1, x_1, x_2, x_3, v, y_3, y_2), e_1, e_7, e_{11}$
17. $y_1 - x_3$: $(y_1, y_2, x_2, x_1, u, v, y_3, x_3), (y_1, y_2, y_3, v, u, x_1, x_2, x_3), (y_1, x_1, u, v, y_3, y_2, x_2, x_3), (y_1, u, x_1, x_2, y_2, y_3, v, x_3), e_1, e_{11}$
18. $y_1 - y_3$: $(y_1, y_2, x_2, x_1, u, v, x_3, y_3), (y_1, x_1, u, v, x_3, x_2, y_2, y_3), e_1, e_6, e_{10}, e_{12}$
For each Hamiltonian-connected rainbow coloring of \( G \) and each pair \( w, z \) of distinct vertices of \( G \), at least one of these \( w - z \) paths is necessarily a rainbow path. First, we consider the 10-edge coloring of the graph \( G \) shown in Figure 3.15. A Hamiltonian path \( P \) is a rainbow path if \( P \) contains at most one edge in \( \{ux_1, uy_1\} \) and at most one edge from \( \{vx_3, vy_3\} \). For each pair \( w, z \) of distinct vertices of \( G \), at least one of these \( w - z \) paths contains at most one edge in \( \{ux_1, uy_1\} \) and at most one edge from \( \{vx_3, vy_3\} \). Thus, this edge coloring is a Hamiltonian-connected rainbow 10-coloring and so \( \text{hrc}(G) \leq 10 \).

![Figure 3.15: A rainbow Hamiltonian coloring of \( G \)](image)

It remains to show that \( \text{hrc}(G) \geq 10 \). By Observation 3.1.5, it follows that \( \text{hrc}(G) \geq 8 \). In fact, every Hamiltonian-connected rainbow coloring of \( G \) must use at least 10 colors, as we show next.

Let \( c \) be a Hamiltonian-connected rainbow coloring of \( G \). If \( e \) and \( f \) are two distinct edges of \( G \) such that \( e \) and \( f \) belong to every Hamiltonian \( w - z \) path of \( G \) for some pair \( w, z \) of distinct vertices of \( G \), then \( e \) and \( f \) cannot be assigned the same color by \( c \). For example, since \( e_3 \) and \( e_9 \) belong to both two Hamiltonian \( u - v \) paths of \( G \), it follows that \( c(e_3) \neq c(e_9) \). We now construct a graph \( G^* \) with \( V(G^*) = E(G) \) such that two vertices \( x \) and \( y \) of \( G^* \), that is, two edges \( x \) and \( y \) of \( G \), are adjacent in \( G^* \) if the edges \( x \) and \( y \) of \( G \) are adjacent in \( G \).
and $y$ of $G$ belong to every Hamiltonian $w - z$ path of $G$ for some pair $w, z$ of distinct vertices of $G$. The graph $G^*$ is shown in Figure 3.16. The degrees of the vertices in $G^*$ are

$$\text{deg}_{G^*} e_i = \begin{cases} 7 & i = 6 \\ 9 & i = 4, 5, 7, 8 \\ 10 & i = 1, 2, 10, 11 \\ 11 & i = 3, 9, 12. \end{cases}$$

![Figure 3.16: The graph $G^*$](image-url)

The minimum number of independent sets into which $V(G^*)$ can be partitioned is therefore 7. Since $\text{deg}_{G^*} e_i = 11$ for $i = 3, 9, 12$, it follows that $\{e_3\}, \{e_9\}, \{e_{12}\}$ are maximal independent sets. Since $\text{deg}_{G^*} e_i = 10$ for $i = 1, 2, 10, 11$ and $\{e_1, e_2\}$ and $\{e_{10}, e_{11}\}$ are independent sets, they are both maximal independent sets. This leaves the five vertices $e_4, e_5, e_6, e_7, e_8$. The two sets $\{e_4, e_5\}$ and $\{e_7, e_8\}$ are independent, while each vertex of $\{e_4, e_5\}$ is adjacent to each vertex of $\{e_7, e_8\}$. The vertex $e_6$ of $G^*$ is adjacent to none of $e_4, e_5, e_7, e_8$. Hence, a Hamiltonian-connected rainbow coloring $c$ of $G$ must assign distinct colors to the vertices of $G^*$ in the seven sets $\{e_3\}, \{e_9\}, \{e_{12}\}, \{e_1, e_2\}, \{e_{10}, e_{11}\}, \{e_4, e_5\}, \{e_7, e_8\}$, while $e_6$ may be assigned the same color that is assigned to $e_4$ and $e_5$ or to $e_7$ and $e_8$. Next, let $A = \{\{e_1, e_2\}, \{e_{10}, e_{11}\}, \{e_4, e_5\}, \{e_7, e_8\}\}$. The digraph $D$ of Figure 3.17 has vertex set $A$. For $S, T \in A$, $(S, T)$ is an arc of $D$ if
the vertices in $S$ are colored the same by $c$, then the vertices of $T$ must be colored differently. By the two Hamiltonian $u - y_2$ paths, $(S, T)$ and $(T, S)$ are both arcs in $D$ for $S = \{e_4, e_5\}$ and $T = \{e_7, e_8\}$. While with the aid of the Hamiltonian $u - y_3$ paths and Hamiltonian $x_1 - v$ paths, the remaining arcs of $D$ in Figure 3.17 are determined. Consequently, the minimum number of colors can be obtained only when $\{e_1, e_2\}$ and $\{e_{10}, e_{11}\}$ are color classes and the remaining eight vertices of $G^*$ are in individual color classes. Therefore, $\text{hrc}(G) = 10$.

Figure 3.17: The digraph $D$ in the proof of Theorem 3.5.2

We saw that there are infinitely many Hamiltonian-connected graphs $G$ of order $n$ for which $\text{hrc}(G) = n - 1$. If $G = K_3 \square K_2$, then $\text{hrc}(G) = 7$, while if $G = H_3$, then $\text{hrc}(G) = 10$. Thus, there exist Hamiltonian-connected graphs $G$ of order $n$ for which $\text{hrc}(G) \in \{n + 1, n + 2\}$. However, it is not known whether there exists a Hamiltonian-connected graph $F$ of order $n$ for which $\text{hrc}(F) = n$. A more challenging question is to determine an infinite class of Hamiltonian-connected graphs $G$ of order $n$ such that $\text{hrc}(G) = k$ for each integer $k \in \{n, n + 1, n + 2\}$.
Chapter 4

Chromatic Connection in Graphs

4.1 Introduction

Recall that a proper edge coloring $c$ of a nonempty graph $G$ is a function $c$ on $E(G)$ with the property that $c(e) \neq c(f)$ for every two adjacent edges $e$ and $f$ of $G$. If the colors are chosen from a set of $k$ colors, then $c$ is called a $k$-edge coloring of $G$. The minimum positive integer $k$ for which $G$ has a proper $k$-edge coloring is called the chromatic index of $G$ and is denoted by $\chi'(G)$. It is immediate for every nonempty graph $G$ that $\chi'(G) \geq \Delta(G)$. Also, recall the most important theorem concerning chromatic index, obtained by Vizing [78], that for every nonempty graph $G$,

$$\chi'(G) \leq \Delta(G) + 1.$$ 

Let $G$ be an edge-colored connected graph, where adjacent edges may be colored the same. A path $P$ in $G$ is properly colored or, more simply, $P$ is a proper path in $G$ if no two adjacent edges of $P$ are colored the same. An edge coloring $c$ is a proper-path coloring of a connected graph $G$ if every pair $u, v$ of distinct vertices of $G$ are connected by a proper $u - v$ path in $G$. If $k$ colors are used, then $c$ is referred to as a proper-path $k$-coloring. The minimum $k$ for which $G$ has a proper-path $k$-coloring is called the proper connection number of $G$. Much research has been done in this area recently (see [5, 6, 16], for example). In fact, there is a dynamic survey of this topic due to Li and Magnant [75].

As we have observed in the case of rainbow colorings of graphs, in a proper-path coloring of a connected graph $G$, every two vertices $u$ and $v$ of $G$ are connected by a proper $u - v$ path and there is no condition on what the length of such a path must be. For certain graphs $G$, however, it is natural to ask whether there may exist an edge coloring of $G$ using a certain number of colors such that every two vertices of $G$ are
connected by a proper path of a prescribed length. This gives rise to one of the main concepts in this chapter.

4.2 Proper Hamiltonian-Path Colorings

If \( G \) is a Hamiltonian-connected graph with a proper edge coloring, then for every two vertices \( u \) and \( v \) of \( G \), there is a proper Hamiltonian \( u-v \) path in \( G \). However, it may be possible to find an edge coloring of \( G \) such that there exists a proper Hamiltonian \( u-v \) path in \( G \) between every two vertices \( u \) and \( v \) of \( G \) using fewer than \( \chi'(G) \) colors. Of course, graphs possessing such edge colorings are necessarily Hamiltonian-connected. For a Hamiltonian-connected graph \( G \), an edge coloring \( c : E(G) \rightarrow [k] \) is a \textit{proper Hamiltonian-path \( k \)-coloring} if every two vertices of \( G \) are connected by a proper Hamiltonian path in \( G \). An edge coloring \( c \) is a \textit{proper Hamiltonian-path coloring} if \( c \) is a proper Hamiltonian-path \( k \)-coloring for some positive integer \( k \). The minimum number of colors in a proper Hamiltonian-path coloring of \( G \) is the \textit{proper Hamiltonian-connection number} of \( G \), denoted by \( \text{hpc}(G) \). This concept was introduced by Gary Chartrand and first studied in [14].

Since every proper edge coloring of a Hamiltonian-connected graph \( G \) is a proper Hamiltonian-path coloring of \( G \) and there is no proper Hamiltonian-path 1-coloring of \( G \), it follows that

\[
2 \leq \text{hpc}(G) \leq \chi'(G). 
\] (4.1)

To illustrate these concepts, consider the graph \( G = C^2_6 \). Since \( \Delta(G) = 4 \) and the edge coloring of \( G \) in Figure 4.1(a) is a proper 4-edge coloring, it follows that \( \chi'(G) = \Delta(G) = 4 \). Next, consider the 2-edge coloring \( c \) of \( G \) shown in Figure 4.1(b).

![Figure 4.1: A proper 4-edge coloring and a proper Hamiltonian-path 2-coloring of \( C^2_6 \)](image-url)
We show that \( c \) is a proper Hamiltonian-path coloring of \( G \); that is, every two vertices \( u \) and \( v \) of \( G \) are connected by a proper Hamiltonian \( u-v \) path \( P \) in \( G \). If \( \{u, v\} = \{v_1, v_2\} \) or \( \{u, v\} = \{v_1, v_6\} \), say the former, let \( P = (v_1, v_6, v_5, v_4, v_3, v_2) \). And, if \( \{u, v\} = \{v_1, v_3\} \) or \( \{u, v\} = \{v_1, v_5\} \), say the former, then let \( P = (v_1, v_2, v_6, v_5, v_4, v_3) \). While if \( \{u, v\} = \{v_1, v_4\} \), let \( P = (v_1, v_2, v_6, v_5, v_3, v_4) \). By the symmetry of this edge coloring, \( c \) is proper a Hamiltonian-path 2-coloring and so \( hpc(G) = 3 \). Thus, \( hpc(G) < \chi(G) \).

We now consider some well-known Hamiltonian-connected graphs, beginning with

[Diagram of a graph labeled K3 \( \square \) K2]

Figure 4.2: A proper 3-edge coloring of \( K_3 \square K_2 \)

Next, we give an example of a graph \( G \) with \( hpc(G) = \chi'(G) \). Let \( G = K_3 \square K_2 \), where the two triangles \( K_3 \) in \( G \) are \((u, x, w, u)\) and \((v, y, z, v)\) and \( uw, xy, wz \in E(G) \).

Since there is a proper 3-edge coloring of \( G \) shown in Figure 4.2 and \( \Delta(G) = 3 \), it follows that \( \chi'(G) = 3 \). Hence, \( hpc(G) \leq 3 \).

We now show that \( hpc(G) \geq 3 \). Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring \( c \) of \( G \) using the colors red (color 1) and blue (color 2).

There are only two Hamiltonian \( u-v \) paths, namely \((u, w, x, y, z, v)\) and \((u, x, w, z, y, v)\). Because of the symmetry of these paths, we may assume that the first path is a proper Hamiltonian \( u-v \) path and whose edges are colored as \( c(uw) = c(xy) = c(zv) = 1 \) and \( c(wx) = c(yz) = 2 \). Next, we consider a proper Hamiltonian \( x-z \) path. There are only two Hamiltonian \( x-z \) paths in \( G \), namely, \( Q_1 = (x, w, u, v, y, z) \) and \( Q_2 = (x, y, v, u, w, z) \). Since the path \( Q = (w, u, v, y) \) lies on both \( Q_1 \) and \( Q_2 \), it follows that \( Q \) must be proper. This implies that \( c(uv) = 2 \) and \( c(vy) = 1 \). Similarly, there are only two Hamiltonian \( w-y \) paths in \( G \), each of which contains the path \((x, u, v, z)\), and so this path must be proper. This implies that \( c(ux) = 1 \). We now consider a proper Hamiltonian \( x-v \) path. There are only two Hamiltonian \( x-v \) paths in \( G \), namely, \( R_1 = (x, u, w, z, y, v) \) and \( R_2 = (x, y, z, w, u, v) \). Since the path \( R = (y, z, w, u) \) lies on both \( R_1 \) and \( R_2 \), it follows that \( R \) must be properly colored by the colors 1 and 2. Since \( c(yz) = 2 \) and \( c(wu) = 1 \), this is impossible. Thus, there is no proper Hamiltonian \( x-v \) path in \( G \), which is a contradiction. Therefore, \( hpc(G) \geq 3 \) and so \( hpc(G) = 3 \).

We now consider some well-known Hamiltonian-connected graphs, beginning with
complete graphs, which are clearly supergraphs of all graphs, and in particular, all Hamiltonian-connected graphs. It is easy to see that \( hpc(K_3) = 3 \). However, when \( n \geq 4 \), then \( hpc(K_n) = 2 \), which we verify next.

**Theorem 4.2.1** For every integer \( n \geq 4 \), \( hpc(K_n) = 2 \).

**Proof.** We consider two cases, according to whether \( n \) is even or \( n \) is odd.

*Case 1. \( n \) is even.* The complete graph \( G = K_n \) contains a 1-factor \( F \). Define an edge coloring \( c \) of \( G \) by assigning the color red to each edge of \( F \) and the color blue to the remaining edges of \( G \). We show that \( c \) is a proper Hamiltonian-path 2-coloring of \( G \); that is, for every two vertices \( u \) and \( v \) of \( G \), there is a proper Hamiltonian \( u - v \) path in \( G \). Let \( n = 2k \) and let \( V(G) = \{v_1, v_2, \ldots, v_{2k}\} \). Suppose that \( E(F) = \{v_{2i-1}v_{2i} : 1 \leq i \leq k\} \). There are two possibilities, depending on whether \( uv \) is a blue edge or \( uv \) is a red edge. Thus, we may assume that either (1) \( u = v_1 \) and \( v = v_{2k} \) or (2) \( u = v_2 \) and \( v = v_1 \). Consider the properly colored Hamiltonian cycle \( C = (v_1, v_2, \ldots, v_{2k}, v_1) \) of \( G \). If (1) occurs, then \( (u = v_1, v_2, \ldots, v_{2k} = v) \) is a proper Hamiltonian \( u - v \) path in \( G \); while if (2) occurs, then \( (u = v_2, v_3, \ldots, v_{2k}, v_1 = v) \) is a proper Hamiltonian \( u - v \) path in \( G \). Therefore, \( hpc(K_n) = 2 \).

*Case 2. \( n \geq 5 \) is odd.* Let \( C = (v_1, v_2, \ldots, v_n, v_1) \) be a Hamiltonian cycle in \( G = K_n \). Define a coloring \( c \) of \( G \) by assigning the color red to each edge of \( C \) and the color blue to the remaining edges of \( G \). We show that \( c \) is a proper Hamiltonian-path 2-coloring of \( G \); that is, for every two vertices \( u \) and \( v \) of \( G \), there is a proper Hamiltonian \( u - v \) path in \( G \). We may assume that \( v = v_n \) and \( u = v_i \) for some integer \( i \) with \( 1 \leq i \leq (n-1)/2 \).

First, suppose that \( u = v_1 \). If \( n \equiv 1 \) (mod 4), then

\[
(u = v_1, v_2, v_4, v_5, v_6, v_8, v_7, v_9, \ldots, v_{n-3}, v_{n-1}, v_{n-2}, v_n = v)
\]

is a proper Hamiltonian \( u - v \) path in \( G \); while if \( n \equiv 3 \) (mod 4), then

\[
(u = v_1, v_2, v_4, v_3, v_5, v_6, v_8, v_7, v_9, \ldots, v_{n-3}, v_{n-4}, v_{n-5}, v_{n-3}, v_{n-2}, v_n = v)
\]

is a proper Hamiltonian \( u - v \) path in \( G \).

Next, suppose that \( u = v_j \) where \( 2 \leq j \leq (n-1)/2 \). If \( n = 5 \), then \( u = v_2 \) and \( (v_5, v_3, v_1, v_2) \) is a proper Hamiltonian \( u - v \) path in \( G \). Thus, we may assume that \( n \geq 7 \) is odd. Let \( A = \{v_1, v_2, \ldots, v_{j-1}\} \) and \( B = \{v_{j+1}, v_{j+2}, \ldots, v_{n-1}\} \). Let \( |A| = a \) and \( |B| = b \). Since \( n \geq 7 \) is odd, it follows that (1) \( b \geq 3 \) and (2) \( a + b = n - 2 \) is odd.
and so \(a\) and \(b\) are of opposite parity. We consider two subcases, according to whether \(a\) is even or \(a\) is odd.

**Subcase 2.1. \(a\) is even.** Then

\[
Q = (u = v_j, v_{j-2}, v_{j-1}, v_{j-4}, v_{j-3}, v_{j-6}, v_{j-5}, \ldots, v_1, v_2, v_{j+2})
\]

is a proper \(u - v_{j+2}\) path in \(G\) with \(V(Q) = \{v_1, \ldots, v_j\} \cup \{v_{j+2}\}\) and

\[
Q' = (u = v_{j+2}, v_{j+1}, v_{j+4}, v_{j+3}, u_{j+6}, v_{j+5}, v_{j+8}, u_{j+7}, \ldots,
v_{n-2}, v_{n-3}, v_{n-1}, v_n = v)
\]

is a proper \(v_{j+2} - v\) path in \(G\) with \(V(Q') = \{v_{j+1}, v_{j+2}, \ldots, v_n\}\). Thus, \(V(Q) \cup V(Q') = V(G)\), \(V(Q) \cap V(Q') = \{v_{j+1}\}\) and \(v_2v_{j+2}\) and \(v_{j+1}v_{j+2}\) have distinct colors (namely, \(v_2v_{j+2}\) is blue and \(v_{j+1}v_{j+2}\) is red). Therefore, the path \(Q\) followed by \(Q'\) produces a proper Hamiltonian \(u - v\) path in \(G\).

**Subcase 2.2. \(a\) is odd.** If \(a \equiv 3 \pmod{4}\), then

\[
Q = (u = v_j, v_{j-1}, v_{j-3}, v_{j-2}, u_{j-4}, v_{j-5}, v_{j-7}, u_{j-6}, \ldots, v_1, v_2, v_{j+1})
\]

is a proper \(u - v_{j+1}\) path in \(G\); while if \(a \equiv 1 \pmod{4}\), then

\[
Q = (u = v_j, v_{j-1}, v_{j-3}, v_{j-2}, u_{j-4}, v_{j-5}, v_{j-7}, u_{j-6}, \ldots, v_3, v_4, v_1, v_2, v_{j+1})
\]

is a proper \(u - v_{j+1}\) path in \(G\). We now show that \(Q\) can be extended to a proper Hamiltonian \(u - v\) path in \(G\). If \(b \equiv 0 \pmod{4}\), then

\[
Q' = (v_{j+1}, v_{j+2}, v_{j+4}, v_{j+3}, u_{j+5}, v_{j+6}, v_{j+8}, u_{j+7}, \ldots,
v_{n-3}, v_{n-1}, v_{n-2}, v_n = v)
\]

is a proper \(v_{j+1} - v\) path in \(G\); while if \(b \equiv 2 \pmod{4}\), then \(b \geq 6\) (since \(b \geq 3\)) and

\[
Q' = (v_{j+1}, v_{j+2}, v_{j+4}, v_{j+3}, u_{j+5}, v_{j+6}, v_{j+8}, u_{j+7}, \ldots,
v_{n-4}, v_{n-1}, v_{n-2}, v_n = v)
\]

is a proper \(v_{j+1} - v\) path in \(G\). Thus, as in Case 1, the path \(Q\) followed by \(Q'\) produces a proper Hamiltonian \(u - v\) path in \(G\).

In the proof of Theorem 4.2.1, the two vertices incident with an edge on each proper Hamiltonian path in \(K_n\) have distance at most 3 on the Hamiltonian cycle \(C\) of \(K_n\). Thus, the following is a consequence of the proof of Theorem 4.2.1.
Corollary 4.2.2 For integers \( n, k \) where \( n \geq 6 \) and \( 3 \leq k \leq \lfloor n/2 \rfloor \), \( hpc(C^k_n) = 2 \).

While we have seen that \( hpc(C^2_6) = 2 \), the following problem remains.

Problem 4.2.3 For each integer \( n \geq 7 \), determine \( hpc(C^2_n) \).

4.3 Hamiltonian-Connected Prisms

Recall that if \( G \) is a Hamiltonian-connected graph of order at least 4, then \( \delta(G) \geq 3 \). Actually, there are infinitely many Hamiltonian-connected cubic graphs. For example, the prism \( C_n \square K_2 \) is cubic and Hamiltonian-connected for each odd integer \( n \geq 3 \) (see [47]). We saw that \( hpc(C^3_3 \square K_2) = 3 \). In fact, \( hpc(C_n \square K_2) = 3 \) for all odd integers \( n \geq 3 \).

Theorem 4.3.1 For each odd integer \( n \geq 3 \), \( hpc(C_n \square K_2) = 3 \).

Proof. For an odd integer \( n \geq 3 \), let \( G = C_n \square K_2 \), which is constructed from the two \( n \)-cycles \((u_1, u_2, \ldots, u_n, u_1)\) and \((v_1, v_2, \ldots, v_n, v_1)\) by adding the \( n \) edges \( u_iv_i \) for \( 1 \leq i \leq n \). Since \( \chi'(G) = 3 \), it follows by (4.1) that \( hpc(G) \leq 3 \). It remains to show that \( hpc(G) \geq 3 \). Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring \( c \) of \( G \) using the colors 1 and 2.

First, consider a proper Hamiltonian \( u_1 - u_3 \) path \( P \) in \( G \). Observe that either \( P \) begins with \( u_1, u_2 \) or \( P \) ends with \( u_2, u_3 \). Suppose first that \( P \) begins with \( u_1, u_2 \). Hence, \( P \) must begin with \( u_1, u_2, v_2 \) and so \( u_1u_nv_i \notin E(P) \). Since each vertex in \( V(G) - \{u_1, u_3\} \) has degree 2 in \( P \), it follows that \( v_1v_n, v_1v_2 \in E(P) \) and so \( P \) begins with the subpath \((u_1, u_2, v_2, v_1, v_n)\). Since \( u_nv_1 \notin E(P) \) and \( u_n \) has degree 2 in \( P \), it follows that \( u_nv_n, u_nv_{n-1} \in E(P) \) and so \( P \) contains the subpath \((u_1, u_2, v_2, v_1, v_n, u_n, u_{n-1})\). Similarly, \( v_nv_{n-1} \notin E(P) \) and \( u_{n-1}v_{n-1}, v_{n-1}v_{n-2} \in E(P) \). Continuing in this way, we see that \( P \) is the following path

\[
P_1 = (u_1, u_2, v_2, v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_4, v_4, v_3, u_3).
\]

Next, suppose that \( P \) ends with \( u_2, u_3 \). This implies that \( u_1u_2, u_3v_3, u_3u_4 \notin E(P) \) and so \( u_2v_2, v_2v_3, v_3v_4 \in E(P) \). Hence, \( P \) ends at the subpath \((v_4, v_3, v_2, u_2, u_3)\). An argument similar to the one above shows that \( P \) is the following path

\[
P_2 = (u_1, v_1, v_n, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_4, v_4, v_3, u_2, u_3).
\]
In either case, $P$ must contain the subpath

$$P' = (v_1, v_n, u_n, u_{n-1}, v_{n-1}, u_{n-2}, \ldots, u_4, v_4, v_3).$$

The paths $P_1$ and $P_2$ are illustrated in Figure 4.3 for $C_9 \square K_2$.

Figure 4.3: Two Hamiltonian $u_1 - u_3$ paths in $C_9 \square K_2$

By the symmetry of the graph $G$, we may assume, without loss of generality, that $P = P_1$, described in (4.2). Since $c$ is a proper Hamiltonian-path 2-coloring of $G$ using the colors 1 and 2, we may assume, without loss of generality, that $c(u_1u_2) = 1$. Since $P_1$ is a proper path and $c(u_1u_2) = 1$, it follows that $c(u_2v_2) = 2$ and $c(v_1v_2) = 1$. For the remaining edges $e$ of $P_1$, it follows that $c(e) = 1$ if $e = uiv_i$ and $c(e) = 2$ if $e$ belongs to one of the two $u$-cycles. In particular, $c(v_1v_n) = 2$. Next, consider a proper Hamiltonian $u_3 - u_5$ path $Q$ in $G$. An argument above shows that there are two possibilities for $Q$. This is illustrated in Figure 4.4 for $C_9 \square K_2$. Furthermore, $Q$ must contain the subpath

$$Q' = (v_3, v_2, u_2, u_1, v_1, u_n, u_{n-1}, v_{n-1}, v_{n-2}, u_{n-2}, \ldots, u_6, v_6, v_5).$$

Since $Q'$ is proper and $c(u_2v_2) = 2$, it follows that $c(v_3v_2) = 1$ and so the colors of $Q'$ are alternately colored by 1 and 2, beginning with 1. In particular, $c(v_1v_n) = 1$, which contradicts the fact that $c(v_1v_n) = 2$.

### 4.4 Minimum Hamiltonian-Connected Graphs

We start this section with a useful observation.

**Observation 4.4.1** If $H$ is a Hamiltonian-connected spanning subgraph of a graph $G$, then \(hpc(G) \leq hpc(H)\).
If $G$ is a Hamiltonian-connected graph that is not complete and $u$ and $v$ are nonadjacent vertices of $G$, then $G + uv$ is also Hamiltonian-connected and $\text{hpc}(G + uv) \leq \text{hpc}(G)$ by Observation 4.4.1. This suggests that Hamiltonian-connected graphs having the greatest proper Hamiltonian connection numbers are minimal Hamiltonian-connected graphs. This leads us to consider Hamiltonian-connected graphs of order $n$ with minimum size. Every Hamiltonian-connected graph of order at least 4 is 3-connected. Therefore, if $G$ is a Hamiltonian-connected graph of order $n \geq 4$, then $\delta(G) \geq 3$, which implies that the minimum possible size of a Hamiltonian-connected graph of order $n$ is $\lceil \frac{3n+1}{2} \rceil$. The following result is due to Moon [61].

\textbf{Theorem 4.4.2} For each integer $n \geq 4$, there exists a Hamiltonian-connected graph of order $n$ and size $\lceil \frac{3n+1}{2} \rceil$.

We now determine the proper Hamiltonian connection numbers of graphs belonging to two classes of Hamiltonian-connected graphs of order $n$ and size $\lceil \frac{3n+1}{2} \rceil$, one class for when $n$ is even and the other class for when $n$ is odd. Let’s begin with the case when $n$ is even.

For each integer $k \geq 2$, let $P_k \square K_2$ be the grid of order $2k$ in which $P_k = (x_1, x_2, \ldots, x_k)$ and $P'_k = (y_1, y_2, \ldots, y_k)$ are two paths of order $k$ such that $x_iy_i \in E(P_k \square K_2)$ for $1 \leq i \leq k$. Now, let $H_k$ be the cubic graph of order $2k + 2$ obtained by adding two adjacent vertices $u$ and $v$ to the grid $P_k \square K_2$ and joining (1) the vertex $u$ to $x_1$ and $y_1$ and (2) the vertex $v$ to $x_k$ and $y_k$ in $P_k \square K_2$. Each graph $H_k$ has the property that it is Hamiltonian-connected (see [61]) and

$$\text{hpc}(H_k) = \chi'(H_k) = \Delta(H_k) = 3.$$
We verify this now.

![Graphs H2, H3 and Hk](image)

**Theorem 4.4.3**  
For each integer $k \geq 2$, $hpc(H_k) = 3$.

**Proof.** Let $C = (u, x_1, x_2, \ldots, x_k, v, y_k, y_{k-1}, \ldots, y_3, y_2, y_1, u)$ be a Hamiltonian cycle of $H_k$. Define a proper 3-edge coloring of $H_k$ by alternately assigning the colors 1 and 3 to the edges of $C$ and assigning the color 2 to the remaining edges of $H_k$. Thus, $hpc(H_k) \leq \chi'(H_k) = 3$. Figure 4.6(a) shows this edge coloring for the case when $k$ is odd and Figure 4.6(b) shows this edge coloring for the case when $k$ is even.

![Edge colorings of Hk](image)

It therefore remains to show that $hpc(H_k) \geq 3$. Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring $c$ of $H_k$ using the colors 1 and 2. First, consider a proper Hamiltonian $u-v$ path. There are only two Hamiltonian $u-v$ paths in $G$. Because
of the symmetry of these paths, we consider the path \((u, x_1, y_1, y_2, x_2, x_3, y_3, \ldots, x_k, y_k, v)\) if \(k\) is odd and \((u, x_1, y_1, y_2, x_2, x_3, \ldots, y_k, x_k, v)\) if \(k\) is even. Choosing \(c(ux_1) = 1\), the colors of the remaining edges on the path are determined as shown in Figure 4.7 when \(k\) is odd.

![Figure 4.7: A step in the proof of Proposition 4.4.3 when \(k\) is odd](image)

Next, consider a proper Hamiltonian \(u - x_2\) path \(P\) in \(H_k\). If \(P\) begins with \(u, y_1\), then \(P\) cannot contain \(x_1\), which is impossible. Suppose that \(P\) begins with \(u, v\). Then \(P\) must end as \(x_3, y_3, y_1, x_1, x_2\). Since \(c(x_3y_3) = 2\), it follows that \(c(y_2y_3) = 1\), which is impossible as \(c(y_1y_2) = 1\). Hence, \(P\) must begin with \(u, x_1\) and so

\[
P = (u, x_1, y_1, y_2, \ldots, y_k, v, x_k, x_{k-1}, \ldots, x_2).
\]

Furthermore,

the edges of \(P\) are alternately colored 1 and 2. \hfill (4.3)

We now consider the Hamiltonian \(x_1 - x_2\) paths in \(G\). There are only two Hamiltonian \(x_1 - x_2\) paths \(Q\) and \(Q'\) in \(G\), where

\[
Q = (x_1, u, y_1, y_2, \ldots, y_k, v, x_k, x_{k-1}, \ldots, x_2)
\]

and

\[
Q' = \begin{cases} 
(x_1, y_1, u, v, y_k, x_k, x_{k-1}, y_{k-1}, \ldots, y_2, x_2) & \text{if } k \text{ is even} \\
(x_1, y_1, u, v, x_k, y_k, y_{k-1}, x_{k-1}, \ldots, y_2, x_2) & \text{if } k \text{ is odd}
\end{cases}
\]

If \(c(uy_1) = 1\), then \(Q\) is not proper and so \(Q'\) must be proper. However then, \(c(x_ix_{i+1}) = 1\) for each integer \(i\) with \(1 \leq i \leq k - 1\), which contradicts (4.3). Hence, the edges of the Hamiltonian \(x_1 - x_2\) path \(Q\) are alternately colored 1 and 2, beginning and ending with 1.

Now, consider a Hamiltonian \(u - y_2\) path \(Q\). Proceeding as above with the path \(P\), we see that \(Q\) must contain \(x_1y_1, x_1x_2, x_2x_3\) as consecutive edges on \(Q\). Since \(c(x_1y_1) = 2\), it follows that \(c(x_1x_2) = 1\). However, \(c(x_2x_3) = 1\), which is impossible. Thus, no such proper Hamiltonian \(u - y_2\) path exists. Therefore, \(hpc(H_k) \geq 3\) and so \(hpc(H_k) = 3\). \hfill \(\blacksquare\)
For each integer \( k \geq 3 \), recall that \( P_k \Box K_2 \) is the grid of order \( 2k \) in which \( P_k = (x_1, x_2, \ldots, x_k) \) and \( P'_k = (y_1, y_2, \ldots, y_k) \) are two paths of order \( k \) such that \( x_iy_i \in E(P_k \Box K_2) \) for \( 1 \leq i \leq k \). The graph \( F_k \) of order \( 2k + 1 \) is constructed from \( P_k \Box K_2 \) by adding a new vertex \( u \) and joining \( u \) to each vertex in \( \{x_1, x_k, y_1, y_k\} \). Thus, \( F_k \) has \( 2k \) vertices of degree 3 and one vertex of degree 4. It is known [61] that \( F_k \) is a Hamiltonian-connected graph of odd order \( 2k + 1 \) and has the minimum size of a Hamiltonian-connected graph of that order for each integer \( k \geq 3 \). Furthermore, \( \chi'(F_k) = \Delta(F_k) = 4 \). We show that \( \text{hpc}(F_k) = 3 \).

![Figure 4.8: Graphs \( F_3, F_4 \) and \( F_k \)](image)

**Theorem 4.4.4** For each integer \( k \geq 3 \), \( \text{hpc}(F_k) = 3 \).

**Proof.** For each integer \( k \geq 3 \), let \( P_k \Box K_2 \) be the grid of order \( 2k \) in which two paths of order \( k \) are \( P_k = (x_1, x_2, \ldots, x_k) \) and \( P'_k = (y_1, y_2, \ldots, y_k) \) such that \( x_iy_i \in E(P_k \Box K_2) \) for \( 1 \leq i \leq k \). The graph \( F_k \) of order \( 2k + 1 \) is constructed from \( P_k \Box K_2 \) by adding a new vertex \( u \) and joining \( u \) to each vertex in \( \{x_1, x_k, y_1, y_k\} \). Define an edge coloring \( c : E(F_k) \to \{1, 2, 3\} \) of \( F_k \) by alternately assigning the colors 1 and 3 to the edges of \( P_k \) and \( P'_k \) beginning with 1 and assigning the color 2 to the remaining edges of \( P_k \Box K_2 \). Furthermore, if \( k \geq 3 \) is odd, then let \( c(ux_1) = c(uy_1) = 3 \) and \( c(ux_k) = c(uy_k) = 1 \) and if \( k \geq 4 \) is even, then let \( c(ux_1) = c(uy_1) = 3 \) and \( c(ux_k) = c(uy_k) = 2 \). Figure 4.8(a) shows this edge coloring for the case when \( k \) is odd and Figure 4.8(b) shows this edge coloring for the case when \( k \) is even.

Next, we show that the 3-edge coloring of \( F_k \) described in Figure 4.8 is a proper Hamiltonian-path 3-coloring of \( F_k \); that is, we show that \( F_k \) contains a proper Hamilto-
nian $w - z$ path for each pair $w, z$ of distinct vertices of $F_k$. First, observe that every Hamiltonian path $P$ of $F_k$ is proper unless $P$ contains both $ux_1$ and $uy_1$ or contains both $ux_k$ and $uy_k$. Hence, if either $w$ or $z$ is $u$, then $F_k$ contains a proper Hamiltonian $w - z$ path with initial vertex $u$. Therefore, we may assume that neither $w$ nor $z$ is $u$. We consider the following cases.

Case 1. $\{w, z\} = \{x_i, x_j\}$ or $\{w, z\} = \{y_i, y_j\}$, where $i < j$, say the former. If $i$ is even, then consider the $x_i - u$ path

$$P' = (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, y_i, y_{i-1}, x_i-1, x_i-2, \ldots, x_1, x_1, u);$$

while if $i$ is odd, then consider the $x_i - u$ path

$$P' = (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, y_i, y_{i-1}, x_i-1, x_i-2, \ldots, x_1, y_1, u).$$

Next, if $k - j$ is even, then consider the $u - x_j$ path

$$P'' = (u, y_k, x_k, x_{k-1}, y_{k-1}, y_{k-2}, \ldots, y_j, x_j);$$

while if $k - j$ is odd, then consider the $u - x_j$ path

$$P'' = (u, x_k, y_k, x_{k-1}, x_{k-2}, y_{k-2}, \ldots, y_j, x_j).$$

Then, $P'$ followed by $P''$ is a proper Hamiltonian $x_i - x_j$ path.
Case 2. \{w, z\} = \{x_i, y_j\}. We may assume that \(i \leq j\). There are two subcases.  

Subcase 2.1. \(i = j\). If \(i\) is even, then consider the \(x_i - u\) path  
\[
P' = (x_i, x_{i-1}, y_{i-1}, y_{i-2}, x_{i-3}, \ldots, x_1, y_1, u);  
\]
while if \(i\) is odd, then consider the \(x_i - u\) path  
\[
P' = (x_i, x_{i-1}, y_{i-1}, y_{i-2}, x_{i-3}, \ldots, y_1, x_1, u).  
\]
Next, if \(k - i\) is even, then consider the \(u - y_i\) path  
\[
P'' = (u, y_k, x_k, x_{k-1}, y_{k-1}, y_{k-2}, \ldots, x_{i+1}, y_{i+1}, y_i);  
\]
while if \(k - i\) is odd, then consider the \(u - y_i\) path  
\[
P'' = (u, x_k, y_k, y_{k-1}, x_{k-1}, x_{k-2}, \ldots, x_{i+1}, y_{i+1}, y_i).  
\]
Then, \(P'\) followed by \(P''\) is a proper Hamiltonian \(x_i - y_i\) path.  

Subcase 2.2. \(i < j\). If \(i\) is even, then consider the \(x_i - u\) path  
\[
P' = (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, y_1, x_1, u);  
\]
while if \(i\) is odd, then consider the \(x_i - u\) path  
\[
P' = (x_i, x_{i+1}, \ldots, x_{j-1}, y_{j-1}, y_{j-2}, \ldots, y_i, y_{i-1}, x_{i-1}, x_{i-2}, y_{i-2}, \ldots, x_1, y_1, u).  
\]
If \(k - j\) is even, then consider the \(u - y_j\) path  
\[
P'' = (u, x_k, y_k, x_{k-1}, x_{k-2}, y_{k-2}, \ldots, x_j, y_j);  
\]
while if \(k - j\) is odd, then consider the \(u - y_j\) path  
\[
P'' = (u, y_k, x_k, x_{k-1}, y_{k-1}, y_{k-2}, x_{k-2}, \ldots, x_j, y_j).  
\]
Then, \(P'\) followed by \(P''\) is a proper Hamiltonian \(x_i - y_j\) path.  

It therefore remains to show that \(\text{hpc}(F_k) \geq 3\). Assume, to the contrary, that there is a proper Hamiltonian-path 2-coloring \(c\) of \(F_k\) using the colors 1 and 2. First, consider a proper Hamiltonian \(u - v\) path. We consider two cases, according to whether \(k\) is odd or \(k\) is even.
Case 1. \( k \geq 3 \) is odd. Let \( k = 2t + 1 \) for some positive integer \( t \). First, consider the vertices \( x_{t+1} \) and \( u \). Let \( P \) be a proper Hamiltonian \( x_{t+1} - u \) path in \( F_k \). First, observe that \( P \) cannot start with \( x_{t+1}, y_{t+1} \). Thus, either \( P \) starts with \( x_{t+1}, x_t \) or starts with \( x_{t+1}, x_{t+2} \). Suppose, without loss of generality, that \( P \) starts with \( x_{t+1}, x_t \). Since \( x_{t+1}x_{t+2}, x_{t+1}y_{t+1} \notin E(P) \) and \( y_{t+1} \) and \( x_{t+2} \) have degree 2 on \( P \), it follows that

\[
(y_t, y_{t+1}, y_{t+2}, x_{t+2}, x_{t+3}) \text{ is a subpath of } P.
\] (4.4)

If \( t \geq 2 \), then \( x_t y_t \notin E(P) \) (for otherwise, \( y_{t-1} \) cannot belong to \( P \)). Similarly, \( x_i y_i \notin E(P) \) for \( 2 \leq i \leq t \). Hence, \( P \) contains the subpath \( (x_{t+1}, x_t, \ldots, x_1, y_1, y_2, \ldots, y_{t+1}, y_{t+2}) \).

By (4.4), if \( t \) is odd, then

\[
P = (x_{t+1}, x_t, \ldots, x_1, y_1, y_2, \ldots, y_{t+1}, y_{t+2}, x_{t+2}, x_{t+3}, y_{t+3}, \ldots, y_k, x_k, u);
\]

while if \( t \) is even, then

\[
P = (x_{t+1}, x_t, \ldots, x_1, y_1, y_2, \ldots, y_{t+1}, y_{t+2}, x_{t+2}, x_{t+3}, y_{t+3}, \ldots, x_k, y_k, u).
\]

Since \( c \) is a proper Hamiltonian-path 2-coloring of \( F_k \) using the colors 1 and 2, we may assume that \( P \) is alternately colored 1 and 2, beginning with 1 and ending with 2. Thus, the colors of some edges of \( P_k \boxtimes K_2 \) are determined. This is shown for \( k \in \{5, 7\} \) in Figure 4.9 where each bold edge belongs to the path \( P \). In particular, \( \{c(y_1 y_2), c(x_2 x_3)\} = \{c(x_{t+1} x_t), c(x_{t+2} x_{t+3})\} = \{1, 2\} \).

Figure 4.10: The colors of some edges of \( P_k \boxtimes K_2 \) in Case 1 for \( k \in \{5, 7\} \)

Next, consider the vertices \( x_1 \) and \( u \). Let \( Q \) be a proper Hamiltonian \( x_1 - u \) path in \( F_k \). Since \( Q \) cannot begin with \( x_1, u \), exactly one of \( x_1 x_2 \) and \( x_1 y_1 \) is an edge of \( Q \). We consider these two subcases.

Subcase 1.1. \( x_1 x_2 \in E(Q) \) and \( x_1 y_1 \notin E(Q) \). Then
\[ Q = (x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_1, u). \]

Since \( c(x_t x_{t+1}) = 1 \), it follows that \( c(x_{t+1} x_{t+2}) = 2 \) and \( c(x_{t+2} x_{t+3}) = 1 \), which is a contradiction.

**Subcase 1.2.** \( x_1 x_2 \notin E(Q) \) and \( x_1 y_1 \in E(Q) \). Here,

\[ Q = (x_1, y_1, y_2, x_2, x_3, \ldots, y_{k-2}, y_{k-1}, x_k, x_k, y_k, u). \]

Since \( \{c(y_1 y_2), c(x_2 x_3)\} = \{1, 2\} \), there is no color for \( y_2 x_2 \) and so \( Q \) is not proper.

**Case 2.** \( k \geq 4 \) is even. Let \( k = 2t \) for some integer \( t \geq 2 \). First, consider the vertices \( x_t \) and \( u \). Let \( P \) be a proper Hamiltonian \( x_t - u \) path in \( F_k \). As in Case 1, the path \( P \) cannot start with \( x_t, y_t \). Thus, either \( P \) starts with \( x_t, x_{t-1} \) or \( x_t, x_{t+1} \). We consider these two subcases.

**Subcase 2.1.** \( P \) starts with \( x_t, x_{t-1} \). Since \( y_t \) and \( x_{t+1} \) have degree 2 in \( P \), it follows that

\[ (y_{t-1}, y_t, y_{t+1}, x_{t+1}, x_{t+2}) \] is a subpath of \( P \). \hspace{1cm} (4.5)

If \( t \geq 3 \), then \( x_{t-1} y_{t-1} \notin E(P) \) (for otherwise, \( y_{t-2} \) cannot belong to \( P \)). Hence, \( P \) begins with the subpath \( (x_t, x_{t-1}, \ldots, x_1, y_1, y_2, \ldots, y_t) \). Because of (4.5), if \( t \geq 3 \) is odd, then

\[ P = (x_t, x_{t-1}, \ldots, x_1, y_1, y_2, \ldots, y_t, y_{t+1}, x_{t+1}, x_{t+2}, \ldots, y_{k-1}, y_k, x_k, u); \]

while if \( t \geq 2 \) is even, then

\[ P = (x_t, x_{t-1}, \ldots, x_1, y_1, y_2, \ldots, y_t, y_{t+1}, x_{t+1}, x_{t+2}, \ldots, x_{k-1}, x_k, y_k, u). \]

Since \( c \) is a proper Hamiltonian-path 2-coloring of \( F_k \) using the colors 1 and 2, we may assume that \( P \) is alternately colored 1 and 2, beginning with 1 which is shown in Figure 4.10. In particular, \( c(x_{t-1} x_t) = 1 \) and \( c(x_{t+1} x_{t+2}) = 2 \) whether \( t \) is odd or even.

Next, consider the vertices \( x_1 \) and \( u \). Let \( Q \) be a proper Hamiltonian \( x_1 - u \) path in \( F_k \). Since \( Q \) cannot begin with \( x_1, u \), exactly one of \( x_1 x_2 \) and \( x_1 y_1 \) is an edge of \( Q \).

\* First, suppose that \( x_1 x_2 \) is an edge of \( Q \) and \( x_1 y_1 \) is not an edge of \( Q \). Since each of \( x_2 \) and \( y_1 \) has degree 2 in \( Q \), it follows that \( Q \) starts with \( (x_1, x_2, x_3) \) and ends at \( (y_2, y_1, u) \). This forces that \( Q \) is the following path

\[ Q = (x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_2, y_1, u). \]
Figure 4.11: The colors of some edges of \( P_k \square K_2 \) in Subcase 2.1 for \( k \in \{6, 8\} \)

Since \( c(x_{t-1}x_t) = 1 \) and \( c(x_{t+1}x_{t+2}) = 2 \), regardless of the color of \( x_t x_{t+1} \), it follows that \( Q \) is not proper.

* Next, suppose that \( x_1y_1 \) is an edge of \( Q \) and \( x_1x_2 \) is not an edge of \( Q \). Since each of \( x_2 \) and \( y_1 \) has degree 2 in \( Q \), it follows that \( Q \) must start with \((x_1, y_1, y_2, x_2, x_3)\). This forces that \( Q \) is the following path

\[
Q = (x_1, y_1, y_2, x_2, x_3, y_3, y_4, \ldots, x_{k-2}, x_{k-1}, y_{k-1}, y_k, x_k, u).
\]

Since \( \{c(y_1y_2), c(x_2x_3)\} = \{1, 2\} \) (see Figure 4.10), regardless of the color of \( x_2y_2 \), it follows that \( Q \) is not proper.

Subcase 2.2. \( P \) starts with \( x_t, x_{t+1} \). Since \( x_t x_{t-1}, x_t y_t \notin E(P) \), it follows that \((x_{t-2}, x_{t-1}, y_{t-1}, y_t, y_{t+1})\) is a subpath of \( P \). Thus, if \( t \geq 3 \) is odd, then

\[
P = (x_t, x_{t+1}, \ldots, x_k, y_k, y_{k-1}, \ldots, y_t, y_{t-1}, x_{t-1}, x_{t-2}, \ldots, x_2, x_1, y_1, u)
\]

and if \( t \geq 2 \) is even, then

\[
P = (x_t, x_{t+1}, \ldots, x_k, y_k, y_{k-1}, \ldots, y_t, y_{t-1}, x_{t-1}, x_{t-2}, \ldots, y_2, y_1, x_1, u).
\]

Since \( c \) is a proper Hamiltonian-path 2-coloring of \( F_k \) using the colors 1 and 2, we may assume that \( P \) is alternately colored 1 and 2, beginning with 1 which is shown in Figure 4.11.

Next, consider the vertices \( x_1 \) and \( u \). Let \( Q \) be a proper Hamiltonian \( x_1 - u \) path in \( F_k \). Since \( Q \) cannot begin with \( x_1, u \), exactly one of \( x_1x_2 \) and \( x_1y_1 \) is an edge of \( Q \).

* First, suppose that \( x_1x_2 \) is an edge of \( Q \) and \( x_1y_1 \) is not an edge of \( Q \). Since \( y_1 \) has degree 2 in \( Q \), it follows that \( Q \) ends at \((y_2, y_1, u)\). Furthermore, \( x_2y_2 \notin E(Q) \) and so \( x_2x_3, y_2y_3 \in E(Q) \). This forces that \( Q \) is the following path
$Q = (x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_2, y_1, u)$.

Since $\{c(x_t, x_{t+1}), c(x_{t-2}, x_{t-1})\} = \{1, 2\}$, there is no color for $x_{t-1}x_t$ and so $Q$ is not proper.

* Next, suppose that $x_1y_1$ is an edge of $Q$ and $x_1x_2$ is not an edge of $Q$. Since each of $x_2$ and $y_1$ has degree 2 in $Q$ and $y_1u \notin E(Q)$, it follows that

$Q = (x_1, y_1, y_2, x_2, x_3, y_3, \ldots, y_{t-1}, y_t, x_t, x_{t+1}, \ldots, y_k, x_k, u)$.

Since $\{c(y_{t-1}y_t), c(x_t, x_{t+1})\} = \{1, 2\}$, there is no color for $c(x_t y_t)$ and so $Q$ is not proper.

It has been shown in [16] that if $G$ is a 2-connected graph, then the proper connection number of $G$ is at most 3. Since every Hamiltonian-connected graph $G$ of order at least 4 is 2-connected (in fact, 3-connected), $pc(G) \leq 3$. We have seen no Hamiltonian-connected graph $G$ where $hpc(G) > 3$, which leads to the following conjecture.

**Conjecture 4.4.5** If $G$ is a Hamiltonian-connected graph, then $hpc(G) \leq 3$.

### 4.5 $k$-Rainbow Connection in Graphs

Let $G$ be an edge-colored connected graph, where adjacent edges may be colored the same. Recall that an edge coloring $c$ of $G$ is a *rainbow coloring* of $G$ if every pair of distinct vertices of $G$ are connected by a rainbow path in $G$ and an edge coloring $c$ of a connected graph $G$ is a *proper-path coloring* of $G$ if every pair of distinct vertices of $G$ are connected by a proper path in $G$. As described in [17], these two edge colorings of
a graph $G$ can be looked at in another way. If $G$ is an edge-colored graph such that for every two vertices $u$ and $v$, there exists a $u - v$ path $P$ having the property that every subpath of $P$ is a rainbow path, then this edge coloring is clearly a rainbow coloring. On the other hand, if for every two vertices $u$ and $v$, there exists a $u - v$ path $Q$ having the property that every subpath of $Q$ of length (at most) 2 is a rainbow path, then this edge coloring is a proper-path coloring. However, what if we require this property to hold for subpaths of length greater than 2 as well? Looking at rainbow colorings and proper-path colorings in this way brings up, quite naturally, edge colorings that are intermediate to rainbow and proper-path colorings (see [17]).

For an integer $k \geq 2$, a path $P$ in an edge-colored graph $G$ is a $k$-rainbow path if every subpath of $P$ having length at most $k$ is a rainbow path. In particular, every $k$-rainbow path of length at most $k$ is a rainbow path. An edge coloring $c$ of a connected graph $G$ is a $k$-rainbow coloring if every pair of distinct vertices of $G$ are connected by a $k$-rainbow path in $G$. In this case, the graph $G$ is $k$-rainbow connected (with respect to $c$). If $j$ colors are used in a $k$-rainbow coloring of $G$, then $c$ is referred to as a $k$-rainbow $j$-edge coloring (or simply a $k$-rainbow $j$-coloring). The minimum $j$ for which $G$ has a $k$-rainbow $j$-coloring is called the $k$-rainbow connection number $rc_k(G)$ of $G$. Hence, $rc_2(G) = pc(G)$ and if $\ell$ is the length of a longest path in $G$, then $rc_\ell(G) = rc(G)$. Moreover, if $G$ is a nontrivial connected graph of size $m$ for which the length of a longest path in $G$ is $\ell$, then

$$1 \leq pc(G) = rc_2(G) \leq rc_3(G) \leq \cdots \leq rc_\ell(G) = rc(G) \leq m.$$  \hfill (4.6)

These concepts were introduced in [17], where 3-rainbow colorings were studied, and studied further in [34]. To illustrate these concepts, Figure 4.12 shows a proper-path 2-coloring, a 3-rainbow 3-coloring and a rainbow 4-coloring of a graph $G$. In fact, $pc(G) = 2$, $rc_3(G) = 3$ and $rc(G) = 4$ for this graph $G$.

![Figure 4.13: Three edge colorings of a graph $G$](image)
As we mentioned before, in a rainbow coloring or a proper-path coloring of a connected graph \( G \), every two vertices \( u \) and \( v \) of \( G \) are connected by a rainbow \( u - v \) path or a proper \( u - v \) path and there is no condition on what the length of such a path must be. For certain graphs \( G \), however, it is natural to ask whether there may exist an edge coloring of \( G \) using a certain number of colors such that every two vertices of \( G \) are connected by a rainbow path or proper path of a prescribed length. Inspired by proper Hamiltonian-path colorings, \( k \)-rainbow colorings and Hamiltonian-connected rainbow colorings, we study the concept of \( k \)-rainbow Hamiltonian-path colorings of Hamiltonian-connected graphs.

Let \( G \) be an edge colored Hamiltonian-connected graph, where adjacent edges may be colored the same. For an integer \( k \geq 2 \), a Hamiltonian path \( P \) in \( G \) is a \( k \)-rainbow Hamiltonian path if every subpath of \( P \) having length at most \( k \) is a rainbow path. An edge coloring \( c \) of \( G \) is a \( k \)-rainbow Hamiltonian-path coloring if every two vertices of \( G \) are connected by a \( k \)-rainbow Hamiltonian path in \( G \). If \( j \) colors are used in a \( k \)-rainbow Hamiltonian-path coloring \( c \) of \( G \), then \( c \) is referred to as a \( k \)-rainbow Hamiltonian-path \( j \)-edge coloring (or simply a \( k \)-rainbow Hamiltonian-path \( j \)-coloring). The minimum number of colors required of a \( k \)-rainbow Hamiltonian-path coloring of \( G \) is the \( k \)-rainbow Hamiltonian-connection number of \( G \) and is denoted by \( hrc_k(G) \). As expected, \( k \)-rainbow Hamiltonian-path colorings are intermediate to Hamiltonian-connected rainbow colorings and proper Hamiltonian-path colorings. In particular, if \( G \) is a Hamiltonian-connected graph of order \( n \geq 4 \) and size \( m \), then

\[
2 \leq hpc(G) = hrc_2(G) \leq hrc_3(G) \leq \cdots \leq hrc_{n-1}(G) = hrc(G) \leq m. \tag{4.7}
\]

If \( H \) is a Hamiltonian-connected spanning subgraph of a graph \( G \) and \( c \) is a \( k \)-rainbow Hamiltonian-path coloring of \( H \) for some integer \( k \geq 2 \), then the coloring \( c \) can be extended to a \( k \)-rainbow Hamiltonian-path coloring of \( G \) by assigning any color used by \( c \) to each edge in \( E(G) - E(H) \). Thus, we have the following observation.

**Observation 4.5.1** [17] If \( H \) is a Hamiltonian-connected spanning subgraph of a graph \( G \), then \( hrc_k(G) \leq hrc_k(H) \) for every integer \( k \geq 2 \).

### 4.6 On \( k \)-Rainbow Colorings of Wheels

For two vertex-disjoint graphs \( F \) and \( H \), let \( F \lor H \) denote the join of \( F \) and \( H \). In particular, the join \( G \lor K_1 \) of a graph \( G \) and the trivial graph \( K_1 \) is obtained by joining
the vertex of $K_1$ to each vertex of $G$. If $G$ is a Hamiltonian graph of order $n \geq 3$, then $G \vee K_1$ is a Hamiltonian-connected graph of order $n + 1$ and so the length of a longest path in $G \vee K_1$ is $n$. Since the wheel $W_n = C_n \vee K_1$ is a spanning Hamiltonian-connected subgraph of $G \vee K_1$, it follows by Observation 4.5.1 that $\text{hrc}_k(G \vee K_1) \leq \text{hrc}_k(W_n)$. This suggests investigating $k$-rainbow Hamiltonian-path colorings of the wheels $W_n$ since $\text{hrc}_k(W_n)$ is an upper bound for $\text{hrc}_k(G \vee K_1)$ for every Hamiltonian graph $G$ of order $n$.

It is known [17] that $\text{rc}_2(W_3) = \text{rc}(W_3) = 1$ and for integers $k$ and $n$ with $3 \leq k \leq n$ and $n \geq 4$,

\[
\text{rc}_2(W_n) = \text{pc}(W_n) = 2,
\]

\[
\text{rc}_k(W_n) = \begin{cases} 
2 & \text{if } 4 \leq n \leq 6 \\
3 & \text{if } n \geq 7.
\end{cases}
\]

The rainbow Hamiltonian-connection number $\text{hrc}(W_n)$ has been determined for each integer $n \geq 3$. Since $\text{hrc}(W_n) = n$ for each integer $n \geq 3$, it follows that $\text{hrc}_n(W_n) = n$.

We now investigate the following problem:

**What can be said about the value of $\text{hrc}_k(W_n)$ for integers $k$ and $n$ with $2 \leq k \leq n - 1$?**

It is known that

(i) $\text{lpc}(K_3) = 3$ and $\text{lpc}(K_n) = 2$ for $n \geq 4$ and

(ii) $\text{hrc}(K_n) = n - 1$ for $n \geq 4$.

Since $W_3 = K_4$, it follows that $\text{lpc}(K_4) = \text{hrc}_2(W_3) = 2$ and $\text{hrc}(K_4) = \text{hrc}_3(W_3) = 3$.

Thus, we now assume that $n \geq 4$. In order to present a lower bound for $\text{hrc}_k(W_n)$, we first present a lemma.

**Lemma 4.6.1** Let $k$ and $n$ be integers with $2 \leq k \leq n - 2$. If $c$ is a $k$-rainbow Hamiltonian-path coloring of $W_n = C_n \vee K_1$, then every path of length $k$ in $C_n$ is a rainbow path. In particular, the restriction of $c$ to $C_n$ is a proper edge coloring of $C_n$.

**Proof.** For an integer $n \geq 3$, let $W_n = C_n \vee K_1$ where $C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ and $V(K_1) = \{u_0\}$. For each integer $i$ with $1 \leq i \leq n$, there are exactly two $u_i - u_0$ Hamiltonian paths in $W_n$, namely

\[
P_{i,0} = (u_i, u_{i+1}, \ldots, u_n, u_1, u_2, \ldots, u_{i-1}, u_0)
\]

\[
Q_{i,0} = (u_i, u_{i-1}, \ldots, u_n, u_{n-1}, u_{n-2}, \ldots, u_{i+1}, u_0)
\]

85
where all subscripts are expressed as integers modulo $n$. Assume, to the contrary, there is a path of length $k$ in $C_n$ that is not a rainbow path, say $P = (u_1, u_2, \ldots, u_{k+1})$ is not a rainbow path. Since $P$ is a subpath of the two $u_{k+2} - u_0$ Hamiltonian paths $P_{k+2,0}$ and $Q_{k+2,0}$ in $W_n$, there is no $k$-rainbow Hamiltonian $u_{k+2} - u_0$ path in $W_n$, which is a contradiction. Consequently, the restriction of $c$ to $C_n$ is a proper edge coloring of $C_n$. 

**Theorem 4.6.2** If $k$ and $n$ are integers with $2 \leq k \leq n - 1$ and $n \geq 4$, then 

$$\text{hrc}_k(W_n) \geq k + 1.$$ 

In particular, $\text{hpc}(W_n) \geq 3$.

**Proof.** For an integer $n \geq 4$, let $W_n = C_n \lor K_1$ where $C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ and $V(K_1) = \{u_0\}$. By Lemma 4.6.1, it follows that $\text{hrc}_k(W_n) \geq k$. We now show that $\text{hrc}_k(W_n) \neq k$. Assume, to the contrary, that there is a $k$-rainbow Hamiltonian-path $k$-coloring $c : E(W_n) \to [k]$ of $W_n$. As we saw in the proof of Lemma 4.6.1, for each integer $i$ with $1 \leq i \leq n$, there are exactly two $u_i - u_0$ Hamiltonian paths in $W_n$, namely 

$$P_{i,0} = (u_i, u_{i+1}, \ldots, u_n, u_1, u_2, \ldots, u_j, u_0)$$

$$Q_{i,0} = (u_i, u_{i-1}, \ldots, u_n, u_{n-1}, u_{n-2}, \ldots, u_{i+1}, u_0)$$

where all subscripts are expressed as the integers $1, 2, \ldots, n$ modulo $n$. Since $c$ is a $k$-rainbow Hamiltonian-path coloring of $W_n$, at least one of $P_{1,0}$ and $Q_{1,0}$ is a $k$-rainbow Hamiltonian path. By the symmetry of $W_n$, we may assume that $P_{1,0} = (u_1, u_2, \ldots, u_n, u_0)$ is a $k$-rainbow Hamiltonian path and $c(u_n u_0) = k$. Let $n = kq + r$ for some integers $q$ and $r$ where $q \geq 1$ and $0 \leq r \leq k - 1$ and let $C_{P_{1,0}} = (c(u_1 u_2), c(u_2 u_3), \ldots, c(u_{n} u_0))$. By Lemma 4.6.1, we may assume, without loss of generality, that 

$$C_{P_{1,0}} = \begin{cases} (1, 2, \ldots, k, ***, 1, 2, \ldots, k) & \text{if } r = 0 \\ (1, 2, \ldots, k, ***, 1, 2, \ldots, k, 1, 2, \ldots, r) & \text{if } 1 \leq r \leq k - 1. \end{cases} \tag{4.8}$$

where *** represents repeating the next $k$-tuple as the preceding $k$-tuple. We consider two cases, according to whether $r = 0$ or $1 \leq r \leq k - 1$.

**Case 1.** $r = 0$. There are exactly two Hamiltonian $u_1 - u_{n-1}$ paths in $W_n$, namely, 

$$P_{1,n-1} = (u_1, u_n, u_0, u_2, u_3, \ldots, u_{n-1})$$

$$Q_{1,n-1} = (u_1, u_2, \ldots, u_{n-2}, u_0, u_n, u_{n-1}).$$
Since \( c(u_1u_n) = c(u_nu_0) = k \), it follows that \( P_{1,n-1} \) is not a \( k \)-rainbow Hamiltonian path. Because \( c(u_{n-k}u_{n-k+1}) = c(u_0u_n) \), the path \( (u_{n-k}, u_{n-k+1}, \ldots, u_{n-2}, u_0, u_n) \) of length \( k \) in \( Q_{1,n-1} \) is not a rainbow path and so \( Q_{1,n-1} \) is not a \( k \)-rainbow Hamiltonian path. Thus, \( u_1 \) and \( u_{n-1} \) are not connected by any \( k \)-rainbow Hamiltonian path in \( W_n \), a contradiction.

**Case 2.** \( 1 \leq r \leq k-1 \). Consider the path \( P = (u_n, u_1, u_2, \ldots, u_k) \) of length \( k \) on the cycle \( C_n \). By (4.8) then, \( c(P - u_n) = \{1, 2, \ldots, k-1\} \) and \( c(u_nu_1) = r \). Since \( 1 \leq r \leq k-1 \), it follows that \( u_nu_1 \) has the same color as one of the edges on \( P - u_n \) and so \( P \) is not a rainbow path, which contradicts Lemma 4.6.1.

Next, we present an upper bound for \( hrc_k(W_n) \) where \( 2 \leq k \leq n-1 \) and \( n \geq 4 \).

**Theorem 4.6.3** For integers \( k \) and \( n \) with \( 2 \leq k \leq n-1 \) and \( n \geq 4 \), let \( n = kq + r \) where \( 0 \leq r < k \). If \( n \) is even and \( n > 2k + 2r \) or \( n \) is odd and \( n > 2k + 2r + 1 \), then

\[
hrc_k(W_n) \leq k + r + \left\lceil \frac{n}{2} \right\rceil.
\]

**Proof.** Let \( W_n = C_n \lor K_1 \) where \( C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1) \) and \( V(K_1) = \{u_0\} \). It suffices to show that there is a \( k \)-rainbow Hamiltonian-path coloring \( c \) of \( W_n \) using \( k + r + \left\lceil \frac{n}{2} \right\rceil \) colors. First, we assign the \( k + r \) colors \( 1, 2, \ldots, k + r \) to the edges of \( C_n \), where

\[
c(C_n) = (1, 2, \ldots, k, *, *, *, 1, 2, \ldots, k, k+1, k+2, \ldots, k+r),
\]

where * * * represents repeating the next \( k \)-tuple as the preceding \( k \)-tuple. Next, we assign \( \left\lceil \frac{n}{2} \right\rceil \) new colors to the spokes of \( W_n \). We consider two cases, according to whether \( n \) is even or \( n \) is odd.

**Case 1.** \( n \) is even and \( n > 2k + 2r \). Then we define

\[
c(u_0u_{2t-1}) = c(u_0u_{2t}) = k + r + t \text{ for } 1 \leq t \leq \frac{n}{2}.
\]

Thus, we obtain an edge coloring of \( W_n \) with \( k + r + \frac{n}{2} \) colors. Since \( n > 2k + 2r \), it follows that \( k + r + \frac{n}{2} < n \).

**Case 2.** \( n \) is odd and \( n > 2k + 2r + 1 \). Then we define

\[
c(u_0u_{2t-1}) = c(u_0u_{2t}) = k + r + t \text{ for } 1 \leq t \leq \frac{n-1}{2}
\]

\[
c(u_0u_n) = k + r + \frac{n+1}{2}.
\]

87
Thus, we obtain an edge coloring of $W_n$ with $k + r + \frac{n+1}{2}$ colors. Since $n > 2k + 2r + 1$, it follows that $k + r + \frac{n+1}{2} < n$.

It remains to show that $c$ is a $k$-rainbow Hamiltonian-path coloring of $W_n$. In what follows, all subscripts are expressed as the integers $1, 2, \ldots, n$ modulo $n$. For every two distinct vertices $u_i$ and $u_j$ of $W_n$, we show that there is a $k$-rainbow Hamiltonian $u_i - u_j$ path $P_{i,j}$ in $W_n$ as follows:

$\star$ For $1 \leq i \leq n$, $P_{i,0} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_n, u_1, u_2, \ldots, u_{i-1}, u_0)$.

$\star$ For $i \neq n$, $P_{i,i+1} = (u_i, u_{i-1}, u_{i-2}, \ldots, u_1, u_0, u_n, u_{n-1}, u_{n-2}, \ldots, u_{i+1})$.

$\star$ For $i = n$, $P_{n,1} = (u_n, u_{n-1}, u_0, u_{n-2}, u_{n-3}, \ldots, u_2, u_1)$ if $n$ is even and

$P_{n,1} = (u_n, u_0, u_{n-1}, u_{n-2}, u_{n-3}, \ldots, u_2, u_1)$ if $n$ is odd.

$\star$ For $1 \leq i, j \leq n$ and $j \geq i + 2$,

$P_{i,j} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_{j-1}, u_0, u_{i-1}, u_{i-2}, \ldots, u_n, u_{n-1}, \ldots, u_j)$.

Therefore, $c$ is a $k$-rainbow Hamiltonian-path $(k + r + \lceil \frac{n}{2} \rceil)$-coloring of $W_n$ and so $\text{hrc}_k(W_n) \leq k + r + \lceil \frac{n}{2} \rceil$.

The upper bound for $\text{hrc}_k(W_n)$ in Theorem 4.6.3 can be improved. As an example, we consider the case when $k = 2$. By Theorem 4.6.3, $\text{hpc}(W_n) \leq 2 + \lceil \frac{n}{2} \rceil$. For each integer $n \in \{4, 5, 6\}$, a proper Hamiltonian-path coloring of $W_n$ using the colors $1, 2, 3$ is shown in Figure 4.13 and so $\text{hpc}(W_n) = 3$ by Theorem 4.6.2. Therefore, $\text{hpc}(W_n) = \lceil \frac{n}{2} \rceil$ for $n = 5, 6$. In general, $\lceil \frac{n}{2} \rceil$ is an upper bound for $\text{hpc}(W_n)$ for all $n \geq 5$, as we show next.

![Proper Hamiltonian-path colorings of $W_n$ for $n = 4, 5, 6$](image)

**Figure 4.14:** Proper Hamiltonian-path colorings of $W_n$ for $n = 4, 5, 6$

**Theorem 4.6.4** For each integer $n \geq 5$, $\text{hpc}(W_n) \leq \lceil \frac{n}{2} \rceil$.

**Proof.** For an integer $n \geq 5$, let $W_n = C_n \cup K_1$ where $C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ and $V(K_1) = \{u_0\}$. We consider two cases, according to whether $n$ is even or $n$ is odd.
Case 1. \( n \geq 6 \) is even. Let \( n = 2t \) for some integer \( t \geq 2 \). Define an edge coloring \( c : E(W_n) \rightarrow [t] \) of \( W_n \) by

\[
c(u_iu_{i+1}) = \begin{cases} 
  i & \text{if } 1 \leq i \leq t \\
  i - t & \text{if } t + 1 \leq i \leq n 
\end{cases}
\]

\[
c(u_iu_0) = \begin{cases} 
  2 & \text{if } i = 1, 2 \\
  1 & \text{if } i = 3, 4 \\
  p & \text{if } i = 2p - 1, 2p \text{ and } 3 \leq p \leq t \\
  t + 1 & \text{if } i = n.
\end{cases}
\]

It remains to show that \( c \) is a proper Hamiltonian-path coloring of \( W_n \). In what follows, all subscripts are expressed as the integers 1, 2, ..., \( n \) modulo \( n \). For every two distinct vertices \( u_i \) and \( u_j \) of \( W_n \), we show that there is a proper Hamiltonian \( u_i - u_j \) path \( P_{i,j} \) in \( W_n \) as follows:

\[
\star \text{ For } 1 \leq i \leq n, \ P_{i,0} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_n, u_1, u_2, \ldots, u_{i-1}, u_0).
\]

\[
\star \text{ For } i \neq n, \ P_{i,i+1} = (u_i, u_{i-1}, u_{i-2}, \ldots, u_1, u_0, u_n, u_{n-1}, u_{n-2}, \ldots, u_{i+1}).
\]

\[
\star \text{ For } i = n, \ P_{n,1} = (u_n, u_{n-1}, u_0, u_{n-2}, u_{n-3}, \ldots, u_2, u_1).
\]

\[
\star \text{ For } 1 \leq i, j \leq n \text{ and } j \geq i + 2,
\]

\[
P_{i,j} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_{j-1}, u_0, u_{i-1}, u_{i-2}, \ldots, u_n, u_{n-1}, \ldots, u_j).
\]

Therefore, \( c \) is a proper Hamiltonian \( t \)-coloring and so \( \text{hpc}(W_n) \leq t = \frac{n}{2} \).

Case 2. \( n \geq 5 \) is odd. Let \( n = 2t + 1 \) for some integer \( t \geq 2 \). Define an edge coloring \( c : E(W_n) \rightarrow [t + 1] \) of \( W_n \) by

\[
c(u_iu_{i+1}) = \begin{cases} 
  i & \text{if } 1 \leq i \leq t + 1 \\
  i - (t + 1) & \text{if } t + 2 \leq i \leq n 
\end{cases}
\]

\[
c(u_iu_0) = \begin{cases} 
  2 & \text{if } i = 1, 2 \\
  1 & \text{if } i = 3, 4 \\
  p & \text{if } i = 2p - 1, 2p \text{ and } 3 \leq p \leq t \\
  t + 1 & \text{if } i = n.
\end{cases}
\]

It remains to show that \( c \) is a proper Hamiltonian-path coloring of \( W_n \). As before, all subscripts are expressed as the integers 1, 2, ..., \( n \) modulo \( n \). Since it is routine to show
that \( c \) is a proper Hamiltonian-path coloring of \( W_n \) when \( n = 5 \), we may assume that \( n \geq 7 \). For every two distinct vertices \( u_i \) and \( u_j \) of \( W_n \), we show that there is a proper Hamiltonian \( u_i - u_j \) path \( P_{i,j} \) in \( W_n \) as follows:

- For \( 1 \leq i \leq n \), \( P_{i,0} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_n, u_1, u_2, \ldots, u_{i-1}, u_0) \).
- For \( i \neq n \), \( P_{i,i+1} = (u_i, u_{i-1}, u_{i-2}, \ldots, u_1, u_0, u_n, u_{n-1}, u_{n-2} \ldots, u_{i+1}) \).
- For \( i = n \), \( P_{n,1} = (u_n, u_0, u_{n-1}, u_{n-2}, u_{n-3}, \ldots, u_2, u_1) \).
- For \( 1 \leq i, j \leq n \) and \( j \geq i + 2 \),
  \[ P_{i,j} = (u_i, u_{i+1}, u_{i+2}, \ldots, u_{j-1}, u_0, u_{i-1}, u_{i-2}, \ldots, u_n, u_{n-1}, \ldots, u_j) \].

Therefore, this coloring \( c \) is a proper Hamiltonian \((t + 1)\)-coloring and so \( \text{hpc}(W_n) \leq t + 1 = \lceil \frac{n}{2} \rceil \).

The following is a consequence of Theorems 4.6.2 and 4.6.4

**Corollary 4.6.5** If \( n \geq 5 \), then \( 3 \leq \text{hpc}(W_n) \leq \lceil \frac{n}{2} \rceil \).

### 4.7 On \( k \)-Rainbow Colorings in Prisms

For a graph \( G \), let \( G \square K_2 \) denote the *Cartesian product* of \( G \) and \( K_2 \). It is known that if \( G \) is a Hamiltonian graph of odd order \( n \geq 3 \), then \( G \square K_2 \) is a Hamiltonian-connected graph of order \( 2n \) and so the length of a longest path in \( G \square K_2 \) is \( 2n - 1 \). Since the prism \( C_n \square K_2 \) is a spanning Hamiltonian-connected subgraph of \( G \square K_2 \), it follows by Observation 4.5.1 that \( \text{hrc}_k(G \square K_2) \leq \text{hrc}_k(C_n \square K_2) \) for each integer \( k \) with \( 2 \leq k \leq 2n - 1 \). This leads us to study the \( k \)-rainbow Hamiltonian-path colorings of the prisms \( C_n \square K_2 \) for odd integers \( n \geq 3 \). The \( k \)-rainbow connection numbers \( \text{rc}_k(C_n \square K_2) \) have been determined for all integers \( k \) and \( n \) with \( 2 \leq k \leq 2n - 1 \) (see [8]).

**Theorem 4.7.1** For integers \( k \) and \( n \) with \( 2 \leq k \leq 2n - 1 \) and \( n \geq 3 \),

\[
\text{rc}_k(C_n \square K_2) = \min \left\{ k, \left\lfloor \frac{n}{2} \right\rfloor + 1 \right\} .
\]

Recall that \( C_n \square K_2 \) is Hamiltonian-connected if and only if \( n \geq 3 \) is odd. The proper and rainbow Hamiltonian-connection numbers \( \text{hpc}(C_n \square K_2) \) and \( \text{hrc}(C_n \square K_2) \) have been determined for each integer \( n \geq 3 \) (see Theorem 4.3.1 and Theorem 3.3.5). We have seen for each odd integer \( n \geq 3 \) that
\* \( h_{pc}(C_n \square K_2) = 3 \),
\* \( h_{rc}(C_3 \square K_2) = 7 \) and \( h_{rc}(C_n \square K_2) = 3n \) for \( n \geq 5 \).

This suggests the following problem:

Investigate \( h_{rc_k}(C_n \square K_2) \) for integers \( k \) and \( n \) where \( 3 \leq k \leq 2n - 2 \) and \( n \geq 3 \) is odd.

First, we determine the value of \( h_{rc_k}(C_3 \square K_2) \) for \( 2 \leq k \leq 5 \).

**Theorem 4.7.2** If \( G = C_3 \square K_2 \), then \( h_{rc_2}(G) = 3 \) and \( h_{rc_k}(G) = 7 \) for \( k = 3, 4, 5 \).

**Proof.** By Theorem 4.3.1 and the fact that \( h_{rc}(C_3 \square K_2) = 7 \), it follows that \( h_{rc_2}(C_3 \square K_2) = 3 \) and \( h_{rc_3}(C_3 \square K_2) = 7 \). It remains therefore to determine \( h_{rc_k}(C_3 \square K_2) \) for \( k = 3, 4 \).

Let \( G = C_3 \square K_2 \) where the vertices and the edges of \( G \) are labeled as shown in Figure 4.14. Since \( h_{rc}(G) = 7 \), it follows that \( h_{rc_k}(G) \leq 7 \) for \( k = 3, 4 \). By (4.7), it suffices to show that there is no 3-rainbow Hamiltonian-path coloring of \( C_3 \square K_2 \) using at most six colors.

![Figure 4.15: The graph \( C_3 \square K_2 \)](image)

Assume, to the contrary, that there is a 3-rainbow Hamiltonian-path coloring \( c \) of \( C_3 \square K_2 \) using at most six colors. For each of the \( \binom{6}{2} = 15 \) pairs \( x, y \) of vertices of \( G \), there are exactly two Hamiltonian \( x - y \) paths. At least one of these two paths is necessarily a 3-rainbow Hamiltonian path. These \( 2 \binom{6}{2} = 30 \) Hamiltonian paths are shown below.

1. \( u_1 - u_2 \) paths: \((h_1, f_1, f_2, h_3, e_2), (e_3, h_3, f_3, f_1, h_2)\)
2. \( u_1 - u_3 \) paths: \((e_1, h_2, f_1, f_3, h_3), (h_1, f_3, f_2, h_2, e_2)\)
3. \( u_1 - v_1 \) paths: \((e_1, e_2, h_3, f_2, f_1), (e_3, e_2, h_2, f_2, f_3)\)