Western Michigan University
ScholarWorks at WMU

Dissertations Graduate College

6-2018

Induced Graph Colorings

Ian Hart
Western Michigan University, ian.t.hart.01@gmail.com

Follow this and additional works at: https://scholarworks.wmich.edu/dissertations

Part of the Mathematics Commons

Recommended Citation
Hart, Ian, "Induced Graph Colorings" (2018). *Dissertations*. 3309.
https://scholarworks.wmich.edu/dissertations/3309

This Dissertation-Open Access is brought to you for free and open access by the Graduate College at ScholarWorks at WMU. It has been accepted for inclusion in Dissertations by an authorized administrator of ScholarWorks at WMU. For more information, please contact wmu-scholarworks@wmich.edu.

Induced Graph Colorings

by

Ian Hart

A dissertation submitted to the Graduate College
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
Mathematics
Western Michigan University
June 2018

Doctoral Committee:

Ping Zhang, Ph.D., Chair
Gary Chartrand, Ph.D.
Allen Schwenk, Ph.D.
Ebrahim Salehi, Ph.D.

© 2018 Ian Hart

Induced Graph Colorings

Ian Hart, Ph.D.

Western Michigan University, 2018

An edge coloring of a nonempty graph G is an assignment of colors to the edges of G. In an unrestricted edge coloring, adjacent edges of G may be colored the same. If every two adjacent edges of G are colored differently, then this edge coloring is proper and the minimum number of colors in a proper edge coloring of G is the chromatic index $\chi'(G)$ of G. A proper vertex coloring of a nontrivial graph G is an assignment of colors to the vertices of G such that every two adjacent vertices of G are colored differently. The minimum number of colors in a proper vertex coloring of G is the chromatic number $\chi(G)$ of G. In this work, we study a proper vertex coloring of a graph that is induced by an unrestricted edge coloring of the graph.

For a connected graph G of order at least 3, let $c : E(G) \to \{1, 2, \ldots, k\}$ be an unrestricted edge coloring of G where adjacent edges may be colored the same. Then c induces a vertex coloring c' of G obtained by assigning to each vertex v of G the set of colors of the edges incident with v. The edge coloring c is called a majestic k-edge coloring of G if the induced vertex coloring c' is a proper vertex coloring of G. The minimum positive integer k for which a graph G has a majestic k-edge coloring is the majestic chromatic index of G and and denoted by $\text{maj}(G)$. For a graph G with majestic chromatic index k, the minimum number of distinct vertex colors induced by a majestic k-edge coloring is the majestic chromatic number of G and denoted by $\psi(G)$. Thus, $\psi(G)$ is at least as large as the chromatic number $\chi(G)$ of a graph G.

Majestic chromatic indexes and numbers are determined for several well-known classes of graphs, including complete graphs, complete multipartite graphs, paths, cycles and connected bipartite graphs and the Cartesian products $G \,\square\, K_2$ when G is a bipartite graph as well as two classes of nonbipartite graphs G, namely odd cycles and complete multipartite graph. Sharp bounds have been established for these two parameters and relationship between the majestic chromatic number and the chromatic number of a graph has been studied.

The idea of assigning a color a to an edge of a graph can be looking as assigning the color $\{a\}$ to the edge, which gives rise to a natural generalization of majestic colorings. For a positive integer k, let $\mathcal{P}([k])$ be the power set of the set $[k] = \{1, 2, \ldots, k\}$ and let $\mathcal{P}^*([k]) = \mathcal{P}([k]) - \{\emptyset\}$ be the set of nonempty subsets of $[k]$. For each integer t with $1 \leq t < k$, let $\mathcal{P}_t([k])$ be the set of t-element subsets of $\mathcal{P}([k])$. For an edge coloring $c : E(G) \to \mathcal{P}_t([k])$ of a graph G, where adjacent edges may be colored the same, the vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ is defined by $c'(v)$ to be the union of the colors of edges incident with the vertex v in G. If c' is a proper vertex coloring of G, then c is a majestic t-tone k-coloring of G. For a fixed positive integer t, the minimum positive integer k for which a graph G has a majestic t-tone k-coloring is the majestic t-tone index $\mathrm{maj}_t(G)$ of G. In particular, a majestic 1-tone k-coloring is a majestic k-coloring and the majestic 1-tone index of a graph G is the majestic index of G.

First, our emphasis is on the majestic 2-tone colorings in graphs. The values of $\mathrm{maj}_2(G)$ are determined for several well-known classes of graphs G. We study the relationship between $\mathrm{maj}_2(G \vee K_1)$ and $\mathrm{maj}_2(G)$ where $G \vee K_1$ is the join of a graph G and K_1. We then investigate the majestic t-tone indexes of connected graphs for $t \geq 2$ in general with main emphasis on connected bipartite graphs. It is shown that if G is a connected bipartite graph of order at least 3, then $\mathrm{maj}_t(G) = t + 1$ or $\mathrm{maj}_t(G) = t + 2$ for each positive integer t. All trees, unicyclic bipartite graphs and connected bipartite graphs of cycle rank 2 having majestic t-tone index $t + 1$ have been characterized.

We also investigate the majestic t-tone indices of connected bipartite graphs having large cycles. In particular, we consider 2-connected bipartite graphs with large cycles. It is shown that (i) if G is a 2-connected bipartite graph of sufficiently large order n whose longest cycles have length ℓ where $n - 5 \leq \ell \leq n$ and $t \geq 2$ is an integer, then $\mathrm{maj}_t(G) = t + 1$ and (ii) there is a 2-connected bipartite graph F of sufficiently large order n whose longest cycles have length $n - 6$ and $\mathrm{maj}_2(F) = 4$. Furthermore, it is shown for integers $k, t \geq 2$ that there exists a k-connected bipartite graph G such that $\mathrm{maj}_t(G) = t + 2$. Other results and open questions are also presented.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

During the past several decades, there has been increased interest in edge colorings (proper and nonproper) of graphs that give rise to vertex colorings (see [19, 45, 46], for example). Typically, an edge coloring c of a graph G is a function

$$c : E(G) \to [k] = \{1, 2, \ldots, k\}$$

for some positive integer k. Such a coloring c is a k-edge coloring. An edge coloring c is *unrestricted* if no condition is placed on how the edges may be colored. In particular, in an unrestricted edge coloring, adjacent edges may be colored the same. If every two adjacent edges of G are colored differently, then c is a *proper edge coloring* and the minimum positive integer k for which G has a proper k-edge coloring is its *chromatic index*, denoted by $\chi'(G)$. Clearly, for every nonempty graph G that $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G. The most important theorem dealing with chromatic index is one obtained by Vizing [44].

Theorem 1.0.1 *For every nonempty graph G,*

$$\chi'(G) \leq \Delta(G) + 1.$$

Among the vertex colorings c' of a graph G obtained from an edge coloring c of G, the most studied are those for which the color $c'(v)$ of a vertex v of G is either

(1) the set of colors of those edges incident with v,

(2) the multiset of colors of the edges incident with v or

(3) the sum of the colors of the edges incident with v.

Next, we describe some of most-studied induced graph colorings in each of these categories (1)–(3). We refer to the books [19, 45, 46] for more informations on these topics.

1.1 Set-Defined Colorings

Here, we consider edge colorings where the vertex colorings are those in (1). In this case then, for a connected graph G of order 3 or more on which has been defined an edge coloring $c : E(G) \to [k]$, the associated vertex coloring

$$c' : V(G) \to \mathcal{P}^*([k]) = \mathcal{P}([k]) - \{\emptyset\}$$

is defined by

$$c'(v) = \{c(e) : e \in E_v\}, \tag{1.1}$$

where E_v is the set of edges incident with v and $\mathcal{P}([k])$ is the power set of the set $[k]$.

A *neighbor-distinguishing coloring* of a nontrivial graph G is a coloring in which every pair of adjacent vertices of G are colored differently. Such a coloring is more commonly called a *proper coloring*. The minimum number of colors in a proper vertex coloring of a graph G is its chromatic number and is denoted by $\chi(G)$. We state some well-known results about the chromatic number of a graph. The *clique number* $\omega(G)$ of a graph G is the maximum order of a complete subgraph of G.

Theorem 1.1.1 *For every graph G,*

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$$

For each odd integer $n \geq 3$, the connected graphs C_n and K_n have the property that $\chi(C_n) = 3 = \Delta(C_n) + 1$ and $\chi(K_n) = n = \Delta(K_n) + 1$. Brooks [8] showed that these two classes of graphs are the only connected graphs with this property.

Theorem 1.1.2 *If G is a connected graph that is neither an odd cycle nor a complete graph, then*

$$\chi(G) \leq \Delta(G).$$

A *vertex-distinguishing* (or *rainbow*) of a nontrivial graph G if distinct vertices of G are assigned distinct colors. An early example of such an edge coloring was introduced by Harary and Plantholt [28] in 1985. If c is an unrestricted edge coloring of a nontrivial connected graph G and c' is vertex-distinguishing, then c is called a *set irregular edge coloring* of G. The minimum positive integer k for which a graph G has a set irregular edge coloring is the *set irregular chromatic index* of G and is denoted by $\mathrm{si}(G)$. (This parameter was referred to as the *point-distinguishing chromatic index* by Harary and Plantholt.) The set irregular chromatic index does not exist for K_2. Since every two vertices in a connected graph G of order $n \geq 3$ and size $m \geq 2$ are incident with different sets of edges, any edge coloring that assigns distinct colors of $[m]$ to the edges of G is a set irregular edge coloring. Hence, $\mathrm{si}(G)$ exists and $\mathrm{si}(G) \leq m$.

Another edge coloring of a graph G was referred to as a *strong edge coloring* of G. In this case, the edge coloring c is proper and the induced vertex coloring c' is vertex-distinguishing. In this case, the edge coloring has a property which is stronger than that required of a set irregular edge coloring. The minimum positive integer k for which G has a strong k-edge coloring has been called the *strong chromatic index* of G, denoted by $\chi'_s(G)$. Since every strong edge coloring of a nonempty graph G is a proper edge coloring of G, it follows that

$$\Delta(G) \leq \chi'(G) \leq \chi'_s(G).$$

The concept of strong edge colorings of graphs was introduced independently by Aigner, Triesch and Tuza [4], Černý, Horňá and Soták [12, 30, 31] and Burris and Schelp [11]. The terms *strong edge coloring* and *strong chromatic index* were introduced in [11, 23].

In 2002, Zhang, Liu and Wang [47] introduced an edge coloring $c : E(G) \to [k]$ of a connected graph G of order 3 or more for which both c and the induced set vertex coloring c' were required to be proper. They referred to such a coloring c as an *adjacent strong edge coloring*. The minimum positive integer k for which G has an adjacent strong k-edge coloring is called the *adjacent strong chromatic index* $\chi'_{as}(G)$ of G.

In 2008, another edge coloring c of a connected graph G of order 3 or more was introduced by Horňákb, Soták, Palmer and Woźniak [32], namely $c : E(G) \to [k]$ is unrestricted and the induced set vertex coloring c' is proper. They referred to such a coloring c as a *neighbour-distinguishing coloring* and the corresponding parameter was called the *general neighbour-distinguishing index* of G, which was denoted by *gndi*(G). These concepts were studied further by Bi, English, Hart and Zhang [7], under the simplified terminology *majestic edge coloring* and *majestic index*. The majestic index of a connected graph G of order 3 or more was first denoted by $\chi'_m(G)$ and later by $\mathrm{maj}(G)$. In this work, we will use the terminology "majestic edge coloring" and "majestic index" with the notation $\mathrm{maj}(G)$ for this parameter of a graph G. Other concepts related to majestic edge colorings were introduced by Gary Chartrand in 2015 [46] and studied in [7]. We will further study these concepts in this work.

In summary, we have described four set-defined colorings of a graph; namely,

(i) set irregular edge colorings, where the edge coloring c is unrestricted and the induced vertex coloring c' is vertex-distinguishing;

(ii) strong edge colorings, where the edge coloring c is proper and the induced vertex coloring c' is vertex-distinguishing;

(iii) adjacent strong edge colorings, where both the edge coloring c and and the induced vertex coloring c' are proper; and

(iv) unrestricted edge colorings c, where the induced vertex coloring c' is proper.

Because every strong edge coloring of a nonempty graph G is a adjacent strong edge coloring and a adjacent strong edge coloring is a majestic edge coloring, it follows that

$$\mathrm{maj}(G) \leq \chi'_{as}(G) \leq \chi'_s(G). \tag{1.2}$$

An upper bound for the strong chromatic index was obtained by Bazgan, Harkat-Benhamdine, Li and Woźniak [5] which verifies a conjecture by Burris and Schelp [11].

Theorem 1.1.3 [5] *If G is a connected graph of order $n \geq 3$, then $\chi'_s(G) \leq n+1$.*

The bound in Theorem 1.1.3 is sharp. For example, if $n \geq 3$, then

$$\chi'_s(K_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even.} \end{cases}$$

The following is a consequence of (1.2) and Theorem 1.1.3.

Corollary 1.1.4 *If G is a connected graph of order $n \geq 3$, then* $\mathrm{maj}(G) \leq n+1$.

The following conjecture is due to Zhang, Liu and Wang.

Conjecture 1.1.5 [47] *If G is a connected graph of order $n \geq 3$, then*

$$\chi'_{as}(G) \leq \Delta(G) + 2.$$

1.2 Multiset-Defined Colorings

There is also a related edge coloring for which the induced vertex colors are multisets rather than sets. Let $c : E(G) \to [k]$ for some positive integer k be an unrestricted edge coloring of a graph G that associates with each vertex v of G, the multiset $c'(v)$ of colors of the edges incident with v. It is common to write $c'(v)$ as an ordered k-tuple $a_1 a_2 \cdots a_k$, where a_i $(1 \leq i \leq k)$ is the number of edges incident with v that are colored i. Consequently,

$$\sum_{i=1}^{k} a_i = \deg_G v.$$

If $c'(u) \neq c'(v)$ for every two distinct vertices u and v of G, then c is vertex-distinguishing and is called a *multiset irregular edge coloring* of G. If u and v have distinct degrees, then $c'(u) \neq c'(v)$ regardless of how colors are assigned to the edges of G. The minimum number of colors in a multiset irregular edge coloring of G is the *multiset irregular chromatic index* of G, denoted by $\mathrm{mi}(G)$. For every connected graph G of order 3 or more,

$$\mathrm{mi}(G) \leq \mathrm{si}(G).$$

Multiset irregular edge colorings were introduced and studied by Aigner, Triesch and Tuza in [2, 4] and by Burris in [9, 10]. They referred to these colorings as *irregular colorings* or *vertex-distinguishing edge colorings*. This topic was further studied by Chartrand, Escuadro, Okamoto and Zhang (see [13, 21] for example). Multiset irregular edge colorings and the multiset irregular chromatic index of a graph have also been referred to as *detectable labelings* and the *detection number*, respectively, of the graph. See [19, 45] for more information non this topic.

There is a neighbor-distinguishing analogue of the multiset irregular chromatic index. An edge coloring c of a connected graph G of order at least 3 is called *multiset neighbor-distinguishing* if the multisets of colors of the incident edges of every two adjacent vertices are different. The minimum number of colors in such an edge coloring is called the *multiset neighbor-distinguishing chromatic index* of G (also called the *multiset distinguishing index*, denoted by $md(G)$. For every connected graph G of order at least 3,

$$md(G) \leq sd(G).$$

In recent years, there are many papers dealing with this topic (see [1, 22, 35, 40], for example). Addario-Berry, Aldred, Dalal and Reed [1] showed that every connected graph of order 3 or more has multiset distinguishing index at most 4.

Theorem 1.2.1 *If G is a connected graph of order* 3 *or more, then $md(G)$ has one of the values* 1, 2, 3, 4.

1.3 Sum-Defined Colorings

Here, we consider edge colorings where the vertex colorings are those in (1). The unrestricted edge colorings inducing sum-defined vertex colorings that have attracted the most attention are those where the vertex colorings are either vertex-distinguishing or neighbor-distinguishing.

The first of this type of colorings was introduced in 1986. A nontrivial graph has been called *irregular* if its vertices have distinct degrees. It is well known that no graph is irregular. This observation led to a concept introduced by Gary Chartrand at the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne in 1986 (see [45]).

For a connected graph G, a *weighting* w of G is an assignment of numbers (usually positive integers) to the edges of G, where $w(e)$ denotes the weight of an edge e of G. This then converts G into a weighted graph in which the (*weighted*) *degree* of a vertex v is defined as the sum of the weights of the edges incident with v. A weighted graph G is then *irregular* if the vertices of G have distinct degrees. This concept has been viewed in terms of graph colorings.

Let G be a connected graph of order at least 3. An unrestricted edge coloring $c : E(G) \to \mathbb{N}$ induces a vertex coloring $c' : V(G) \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers, defined by

$$c'(v) = \sum_{e \in E_v} c(e) \quad \text{for each vertex } v \text{ of } G. \tag{1.3}$$

Here, the induced vertex coloring c is required to be vertex-distinguishing. In this case, c is called a *vertex-distinguishing edge coloring* of G. The minimum of the largest colors used among the vertex-distinguishing edge colorings of G is called the *irregularity strength* of G and is denoted by $s(G)$.

In 1991, a sum-defined vertex-distinguishing edge coloring was introduced by Jothi [25] where the colors are chosen instead from the sets $\mathbb{Z}_k$ of integers modulo k for integers $k \geq 2$. For a connected graph G of order $n \geq 3$ and an integer $k \geq n$, let $c : E(G) \to \mathbb{Z}_k$ be an unrestricted edge coloring. The edge coloring c induces a vertex coloring $c' : V(G) \to \mathbb{Z}_k$ defined by

$$c'(v) = \sum_{e \in E_v} c(e),$$

where the sum is computed in $\mathbb{Z}_k$, that is vertex-distinguishing. The minimum k for which there exists such a vertex-distinguishing edge coloring of G has been referred to as the *modular edge-gracefulness* $\text{meg}(G)$ of G. Thus, $\text{meg}(G) \geq n$ for every connected graph G of order $n \geq 3$. If $\text{meg}(G) = n$, then G is called a *modular edge-graceful graph* and a vertex-distinguishing edge coloring $c : E(G) \to \mathbb{Z}_n$ is called a *modular edge-graceful labeling* as well as a *modular edge-graceful coloring* of G (see [34, 36]). These concepts were referred to as *line-graceful labelings* and *line-graceful graphs* in [25] (also see [24]).

A number of neighbor-distinguishing vertex colorings different from standard proper colorings have been introduced in the literature (see [19, pp. 379-385], for example). In many situations, vertex or edge colorings (not necessarily proper) of a graph give rise to neighbor-distinguishing colorings of the graph. In 2004, a neighbor-distinguishing edge coloring $c : E(G) \to \{1, 2, \ldots, k\}$ of a connected graph G of order at least 3 was introduced (see [19, p. 385]), where $k \in \mathbb{N}$, in which an induced vertex coloring $c' : V(G) \to \mathbb{N}$ is defined by (1.3). If c' is a proper vertex coloring, then c is called a *proper sum k-edge coloring*. The minimum k for which a graph G has a proper sum k-edge coloring is the *proper sum neighbor-distinguishing chromatic index* and is denoted by $sd(G)$ of G. Since the sum distinguishing index of a connected graph of order at least 3 never exceeds the irregularity strength of the graph, it follows that $sd(G) \leq s(G)$ for each such graph G. Karoński, Łuczak and Thomason [38] conjectured that these are the only three possibilities for every connected graph of order at least 3. This conjecture goes by a rather catchy name.

Conjecture 1.3.1 (The 1-2-3 Conjecture) *If G is a connected graph of order 3 or more, then $sd(G)$ has one of the values 1, 2, 3.*

In 2010 the concept of sigma vertex colorings was introduced in [18]. This is an example of a nonproper vertex coloring of a graph that induces a proper vertex coloring of the graph. More precisely, let G be a nontrivial connected graph, let $\mathbb{N}$ denote the set of positive integers and let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. The *sum color* $\sigma(v)$ of a vertex v in G is the sum of the colors of the vertices in the neighborhood $N(v)$ of v in G. That is,

$$\sigma(v) = \sum_{u \in N(v)} c(u). \tag{1.4}$$

If $\sigma(u) \neq \sigma(v)$ for every two adjacent vertices u and v of G, then c is neighbor-distinguishing and c is a *sigma coloring* of G. The minimum number of colors required of a sigma coloring of G is the *sigma chromatic number* of G and is denoted by $\sigma(G)$. It was shown in [18] that if G is a connected graph of order $n \geq 2$, then $\sigma(G)$ exists and

$$1 \leq \sigma(G) \leq \chi(G) \leq n.$$

Furthermore, $\sigma(K_n) = n$ for every positive integer n and $\sigma(G) = 1$ if and only if every two adjacent vertices of G have different degrees. It was also shown in [18] that for each pair a, b of positive integer with $a \leq b$, there is a connected graph G with $\sigma(G) = a$ and $\chi(G) = b$.

In 2010, a neighbor-distinguishing vertex coloring was introduced in [39] for the purpose of finding solutions to a checkerboard problem. For a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in $N(v)$. The coloring c is called a *modular k-coloring* of G if every pair of adjacent vertices of G have different sum colors in $\mathbb{Z}_k$. The *modular chromatic number* of G is the minimum k for which G has a modular k-coloring. This coloring has been studied extensively, which led to a complete affirmative solution to the checkerboard problem under investigation (see [45]).

In 2012 another neighbor-distinguishing vertex coloring was introduced (in [17]) that is closely related to the modular colorings just described. For a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be a vertex coloring where adjacent vertices may be colored the same. The coloring c induces another vertex coloring $c' : V(G) \to \mathbb{Z}_k$, where $c'(v) = \sum_{u \in N[v]} c(u)$, where $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v and the sum is performed in $\mathbb{Z}_k$. In this case, $c'(v)$ is determined not only by the colors of neighbors of v but by the color of v as well. Such a coloring c of G is called a *closed modular k-coloring* if for every pair x, y of adjacent vertices in G either $c'(x) \neq c'(y)$ or $N[x] = N[y]$, where, in the latter case, we must have $c'(x) = c'(y)$. The minimum k for which G has a closed modular k-coloring is called the *closed modular chromatic number* of G. Closed modular colorings of graphs were introduced in [17] and arose from a domination problem.

We refer to the book [19] for graph theory notation and terminology not described in this paper.

Chapter 2

Majestic Colorings of Graphs

2.1 The Majestic Index of a Graph

Here, we consider unrestricted edge colorings of connected graphs of order 3 or more whose induced vertex coloring is proper. Specifically, let G be a connected graph of order 3 or more on which is defined an unrestricted edge coloring $c : E(G) \to [k]$ for some positive integer k. The coloring c induces a proper vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$, where $c'(v)$ is the set of colors of the edges of G incident with the vertex v of G. Since c' is proper, $c'(x) \neq c'(y)$ whenever $xy \in E(G)$, so there is some color assigned to an edge incident with either x or y that is not assigned to an edge incident with the other vertex. Such a coloring c is called a *majestic k-edge coloring* or *majestic k-coloring* (or simply a *majestic coloring*). The minimum positive integer k for which a graph G has a majestic k-edge coloring is called the *majestic chromatic index* of G or, more simply, the *majestic index* of G and is denoted by $\text{maj}(G)$. The fact that there is no majestic edge coloring of K_2 explains the necessity of considering only connected graphs of order at least 3. Because $\text{si}(G)$ exists for each such graph G, the parameter $\text{maj}(G)$ exists for G as well. If every edge of a connected graph G is assigned the same color, say 1, then $c'(v) = \{1\}$ for every vertex v of G. In this case, $c'(x) = c'(y)$ for every two adjacent vertices x and y of G and so there is no majestic 1-edge coloring of G. For a connected graph G of order 3 or more, no two vertices of G have the same set of incident edges. Therefore, if c assigns distinct colors to the edges of G, then a majestic coloring results. Thus, we have the following observation.

Observation 2.1.1 *If G is a connected graph of size $m \geq 2$, then* $\text{maj}(G)$ *exists and*

$$2 \leq \text{maj}(G) \leq \text{si}(G) \leq m.$$

For complete graphs, an unrestricted edge coloring is majestic if and only if it is vertex-distinguishing. Therefore, every majestic edge coloring of a complete graph is a set irregular coloring. From this observation, the result below follows from a theorem of Harary and Plantholt.

Theorem 2.1.2 [28] *For every integer $n \geq 3$,*

$$\text{maj}(K_n) = \text{si}(K_n) = \lceil \log_2 n \rceil + 1.$$

With the aid of Theorem 2.1.2, we can determine the majestic index of all complete ℓ-partite graphs for $\ell \geq 3$.

Corollary 2.1.3 *If G is a complete ℓ-partite graph where $\ell \geq 3$, then*

$$\text{maj}(G) = \lceil \log_2 \ell \rceil + 1.$$

Proof. Let $G = K_{n_1,n_2,\ldots,n_\ell}$ be a complete ℓ-partite graph with partite sets $V_1, V_2, \ldots, V_\ell$ where $|V_i| = n_i$ for $1 \leq i \leq \ell$ and let K_ℓ be the complete graph of order ℓ with vertex set $\{v_1, v_2, \ldots, v_\ell\}$. Since $\text{maj}(K_\ell) = \lceil \log_2 \ell \rceil + 1$ by Theorem 2.1.2, it follows that K_ℓ has a majestic edge coloring c_0 using colors from the set $\{1, 2, \ldots, \lceil \log_2 \ell \rceil + 1\}$. Define an edge coloring $c : E(G) \to [\lceil \log_2 \ell \rceil + 1]$ by $c(e) = c_0(v_i v_j)$ if $e \in [V_i, V_j]$ for $1 \leq i < j \leq \ell$. Thus, for each integer i with $1 \leq i \leq \ell$, if $v \in V_i$, then $c'(v) = c_0'(v_i)$. Therefore, c is a majestic edge coloring of G and so $\text{maj}(G) \leq \lceil \log_2 \ell \rceil + 1$.

It remains to show that $\text{maj}(G) \geq \lceil \log_2 \ell \rceil + 1$. Let $\text{maj}(G) = k$, where then $k \geq 3$ and let $c : E(G) \to [k]$ be a majestic edge coloring of G. Let $v \in V(G)$. Then $v \in V_i$ for some integer i with $1 \leq i \leq \ell$. For each vertex $u \in V(G) - V_i$, it follows that $c(uv) \in c'(u) \cap c'(v)$ and so $c'(u)$ and $c'(v)$ cannot be disjoint. Thus, for every element $A \in \mathcal{P}^*([k])$, at most one of A and $\overline{A}$ can be a vertex color of G. Consequently, at most 2^{k-1} colors can be used for the vertices of G. Since $c'(u) \neq c'(v)$ if $u \in V_i$ and $v \in V_j$ for $1 \leq i \neq j \leq \ell$, it follows that $\ell \leq 2^{k-1}$ and so $k - 1 \geq \log_2 \ell$. Since $k - 1$ is an integer, $k - 1 \geq \lceil \log_2 \ell \rceil$. Hence, $k \geq \lceil \log_2 \ell \rceil + 1$. Therefore, $k = \text{maj}(G) = \lceil \log_2 \ell \rceil + 1$. ■

There is a lower bound for the majestic index of a graph G in terms of its chromatic number $\chi(G)$.

Proposition 2.1.4 *If G is a connected graph of order at least 3, then*

$$\text{maj}(G) \geq \lceil \log_2(\chi(G) + 1) \rceil.$$

Proof. Let $\text{maj}(G) = k$ and $c : E(G) \to [k]$ a majestic k-edge coloring of G. Since the induced coloring c' is a proper vertex coloring of G using colors from the set of nonempty subsets of $[k]$, it follows that $\chi(G) \le 2^k - 1$ and so $k = \mathrm{maj}(G) \ge \log_2(\chi(G) + 1)$. Since k is an integer, $\mathrm{maj}(G) \ge \lceil \log_2(\chi(G) + 1) \rceil$. ■

The lower bound in Proposition 2.1.4 is sharp, which we now show. If G is a complete ℓ-partite graph for some integer $\ell \ge 3$, then $\chi(G) = \ell$ and $\mathrm{maj}(G) = \lceil \log_2 \ell \rceil + 1$ by Theorem 2.1.3. If $\ell = 2^p$ for some integer $p \ge 2$, then $\lceil \log_2(\ell + 1) \rceil = \lceil \log_2 \ell \rceil + 1$ and so $\mathrm{maj}(G) = \lceil \log_2(\chi(G) + 1) \rceil$. Therefore, there is an infinite class of connected graphs G for which $\mathrm{maj}(G) = \lceil \log_2(\chi(G) + 1) \rceil$.

Since $\chi(G) \ge \omega(G)$ for every graph G, Proposition 2.1.4 leads to a lower bound for the majestic index of G in terms of its clique number.

Corollary 2.1.5 *If G is a nontrivial connected graph, then*

$$\mathrm{maj}(G) \ge \lceil \log_2 \omega(G) \rceil + 1.$$

We now consider the majestic indexes of graphs belonging to certain well-known classes of graphs, beginning with cycles. While the following result appeared in [32], we also present a different proof here.

Proposition 2.1.6 *For an integer $n \ge 3$,*

$$\mathrm{maj}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{if } n \not\equiv 0 \pmod 4. \end{cases}$$

Proof. First, it is immediate that $\mathrm{maj}(C_3) = \mathrm{maj}(C_5) = 3$ and $\mathrm{maj}(C_4) = 2$. So, we may assume that $n \ge 6$. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. In any majestic edge coloring c of C_n, there are two adjacent edges of C_n that are colored differently, say $c(v_n v_1) = 1$ and $c(v_1 v_2) = 2$.

First, suppose that $n \equiv 0 \pmod 4$. If c is a majestic 2-edge coloring of C_n, then we must have $c(v_2 v_3) = 2$, $c(v_3 v_4) = 1$ and $c(v_4 v_5) = 1$. More generally, $c(v_i v_{i+1}) = 2$ when $i \equiv 1, 2 \pmod 4$ and $c(v_i v_{i+1}) = 1$ when $i \equiv 3, 0 \pmod 4$. Since $n \equiv 0 \pmod 4$, it follows that $c'(v_i) = \{1, 2\}$ if i is odd, $c'(v_i) = \{1\}$ if $i \equiv 0 \pmod 4$ and $c'(v_i) = \{2\}$ if $i \equiv 2 \pmod 4$. Hence, c is a majestic 2-edge coloring of C_n and so $\mathrm{maj}(C_n) = 2$.

Next, suppose that $n \equiv 2 \pmod 4$. For the majestic 2-edge coloring of C_n defined above, $c'(v_{n-1}) = c'(v_n) = c'(v_1) = \{1, 2\}$ and so c is not a majestic 2-edge coloring of C_n. Therefore, $\mathrm{maj}(C_n) \ge 3$. In this case, changing the colors of both $v_{n-1}v_n$ and $v_n v_1$ to 3 results in $c'(v_{n-1}) = \{1, 3\}$, $c'(v_n) = \{3\}$ and $c'(v_2) = \{2, 3\}$. Since this is a majestic 3-edge coloring, it follows that $\mathrm{maj}(C_n) = 3$ if $n \equiv 2 \pmod 4$.

Finally, suppose that n is odd. Thus, $c'(v_n) = c'(v_{n-1}) = \{1\}$ if $n \equiv 1 \pmod 4$ and $c'(v_n) = c'(v_1) = \{1, 2\}$ if $n \equiv 3 \pmod 4$ for the majestic 2-edge coloring of C_n defined above. Hence, c is not a majestic 2-edge coloring of C_n when n is odd and so $\text{maj}(C_n) \geq 3$. If $n \equiv 1 \pmod 4$, by changing the color of v_nv_1 from 1 to 3, we have $c'(v_1) = \{2, 3\}$, $c'(v_n) = \{1, 3\}$ and $c'(v_{n-1}) = \{1\}$. If $n \equiv 3 \pmod 4$, by changing the color of v_1v_2 from 2 to 3, we have $c'(v_1) = \{1, 3\}$, $c'(v_2) = \{2, 3\}$ and $c'(v_n) = \{1, 2\}$. This is a majestic 3-edge coloring and so $\text{maj}(C_n) = 3$ if n is odd. ■

We now turn our attention to bipartite graphs, beginning with complete bipartite graphs.

Proposition 2.1.7 *For positive integers r and s with $r \leq s$ and $s \geq 2$,*

$$\text{maj}(K_{r,s}) = 2.$$

Proof. It suffices to show that $K_{r,s}$ has a majestic 2-edge coloring. Let U and W be the partite sets of $K_{r,s}$, where $|U| = r$ and $W = \{w_1, w_2, \ldots, w_s\}$. Assign the color 1 to each edge incident with w_i for $1 \leq i \leq s - 1$ and the color 2 to each edge incident with w_s. Then $c'(w_i) = \{1\}$ for $1 \leq i \leq s - 1$, $c'(w_s) = \{2\}$ and $c'(u) = \{1, 2\}$ for each $u \in U$. Thus, c is a majestic 2-edge coloring of $K_{r,s}$ and so $\text{maj}(K_{r,s}) = 2$. ■

It is well known that if $H \subseteq G$, then $\chi(H) \leq \chi(G)$ and $\chi'(H) \leq \chi'(G)$. This, however, is not the case for the majestic index. For example, $C_6 \subseteq K_{3,3}$; yet, $\text{maj}(C_6) = 3$ and $\text{maj}(K_{3,3}) = 2$ by Propositions 2.1.6 and 2.1.7. (In fact, this is also not true even when H is an induced subgraph of G as we will see soon.) Moreover, for a connected graph G of order at least 3, any of the following could occur:

$$\chi(G) < \text{maj}(G),\ \chi(G) = \text{maj}(G),\ \chi(G) > \text{maj}(G).$$

For example, if $n \equiv 2 \pmod 4$ and $n \geq 6$, then $\chi(C_n) = 2$ and $\text{maj}(C_n) = 3$ by Proposition 2.1.6; while if $n \equiv 0 \pmod 4$ and $n \geq 4$, then $\chi(C_n) = \text{maj}(C_n) = 2$. Furthermore, if $k \geq 4$, then $\chi(K_k) = k$ and $\text{maj}(K_k) = \lceil \log_2 k \rceil + 1$ by Theorem 2.1.2.

We saw in Proposition 2.1.7 that the majestic index of every complete bipartite graph of order at least 3 is 2 and in Proposition 2.1.6 that the majestic chromatic index of every even cycle is 2 or 3. It was shown in [32] that the majestic chromatic index of every connected bipartite graph of order 3 or more is either 2 or 3. Here, we present a proof of this result that employs a concept introduced and studied in [14]. Let u be a vertex in a nontrivial connected graph G. A vertex v distinct from u is called a *boundary vertex* of u if $d(u, v) = k$ for some positive integer k and no $u - w$ geodesic of length greater than k contains v. Equivalently, v is a boundary vertex of u if for every neighbor w of v, it follows that $d(u, w) \leq d(u, v)$. In particular, for every vertex u of G, every end-vertex of G different from u is a boundary vertex of u.

Theorem 2.1.8 *If G is a connected bipartite graph of order 3 or more, then*

$$\operatorname{maj}(G) \leq 3.$$

Proof. Let U and W be the partite sets of G, where U contains at least two vertices. For a vertex u of U, let

$$\begin{aligned} U_1 &= \{v \in V(G) : \ d(u, v) \equiv 0 \pmod 4\} \text{ and} \\ U_2 &= \{v \in V(G) : \ d(u, v) \equiv 2 \pmod 4\}. \end{aligned}$$

Thus, $U = U_1 \cup U_2$ and $W = \{v \in V(G) : \ d(u, v) \text{ is odd}\}$. Assign the color 1 to each edge of G incident with a vertex of U_1 and the color 2 to each edge of G incident with a vertex of U_2. Denote this edge coloring by c and the induced vertex coloring by c'. If no vertex of W is a boundary vertex of u, then every vertex of W has the color $\{1, 2\}$. Since each vertex of U has the color $\{1\}$ or $\{2\}$, the coloring c is a majestic 2-edge coloring of G and so $\operatorname{maj}(G) = 2$. Suppose, on the other hand, that $w \in W$ is a boundary vertex of u. Then $c'(w) = \{1\}$ or $c'(w) = \{2\}$, say the former. For each neighbor x of w on a $u - w$ geodesic, change the color of xw from 1 to 3. Then $c'(w) = \{3\}$ and $c'(x) = \{1, 3\}$. In this case, the color of every vertex of U is $\{1\}$, $\{2\}$, $\{1, 3\}$ or $\{2, 3\}$, while the color of every vertex of W is $\{1, 2\}$ or $\{3\}$. This new edge coloring is a majestic 3-edge coloring of G and so $\operatorname{maj}(G) \leq 3$. ■

The following result describes those bipartite graphs having majestic chromatic index 2.

Theorem 2.1.9 *Let G be a connected bipartite graph of order 3 or more. Then $\operatorname{maj}(G) = 2$ if and only if there exists a partition $\{U_1, U_2, W\}$ of $V(G)$ such that $U = U_1 \cup U_2$ and W are the partite sets of G and each vertex $w \in W$ has a neighbor in both U_1 and U_2.*

Proof. First, suppose that U and W are the partite sets of G such that U can be partitioned into two sets U_1 and U_2 for which every vertex in W has a neighbor in each of U_1 and U_2. Define the edge coloring $c : E(G) \to \{1, 2\}$ by $c(e) = i$ if e is incident with a vertex in U_i for $i = 1, 2$. Then $c'(u) = \{i\}$ if $u \in U_i$ for $i = 1, 2$ and $c'(w) = \{1, 2\}$ for each $w \in W$. Hence, c is a majestic 2-edge coloring of G and so $\operatorname{maj}(G) = 2$.

For the converse, suppose that G is a connected bipartite graph of order 3 or more such that $\operatorname{maj}(G) = 2$. Let U and W be partite sets of G and let $c : E(G) \to \{1, 2\}$ be a majestic 2-coloring of G. Then U is divided into three sets U_1, U_2, and $U_{1,2}$, where U_i is the set of vertices u with $c'(u) = \{i\}$ for $i = 1, 2$ and $U_{1,2}$ is the set of vertices u with $c'(u) = \{1, 2\}$. Similarly, W is divided into three sets W_1, W_2 and $W_{1,2}$. Observe that the vertices in $U_1 \cup U_2$ can only be adjacent to vertices in $W_{1,2}$ and the vertices in $W_1 \cup W_2$ can only be adjacent to vertices in $U_{1,2}$. Since G is connected and no vertex in $U_{1,2}$ can be adjacent to any vertex in $W_{1,2}$, it follows that either $U_1 \cup U_2 \cup W_{1,2} = \emptyset$ or $W_1 \cup W_2 \cup U_{1,2} = \emptyset$, say $W_1 \cup W_2 \cup U_{1,2} = \emptyset$. Then $\{U_1, U_2, W_{1,2}\}$ is the desired partition of the vertex set of G since each vertex $w \in W_{1,2} = W$ must be adjacent to some vertex in U_1 and some vertex in U_2. ■

The following is a consequence of the proof of Theorem 5.1.

Corollary 2.1.10 *Let G be a connected bipartite graph. If G contains a vertex u such that all boundary vertices of u belong to the same partite set of u, then* $\text{maj}(G) = 2$.

The converse of Corollary 2.1.10 is not true, however. For example, Figure 2.1 shows a majestic 2-edge coloring of the 3-cube Q_3, where each solid edge is colored 1 and each dashed edge is colored 2. (In Figure 2.1, $\{a\}$ is denoted by a where $a \in \{1, 2\}$ and $\{1, 2\}$ is denoted by 12.) Thus, so $\text{maj}(Q_3) = 2$. (We'll soon say more about the majestic index of the k-cube Q_k in general.) For each vertex u of Q_3, there is a unique boundary vertex v of u such that $d(u, v) = 3$. Thus, u and v do not belong to the same partite set. Therefore, there is no vertex u in Q_3 all of whose boundary vertices belong to the same partite set as u. In fact, if u and v are boundary vertices of each other, then u and v belong to different partite sets of Q_3. Next, we consider the bipartite graph G of Figure 2.1 that is is not regular. Since G has a majestic 2-edge coloring shown in Figure 2.1 (where a solid edge is colored 1 and a dashed edge is colored 2), it follows that $\text{maj}(G) = 2$. On the other hand, for $i = 1, 2$, the vertex v_i is a boundary vertex of u_i; while $d(u_i, v_i) = 3$ and so v_i and u_i do not belong to the same partite set. Hence, by symmetry, G has no vertex u all of whose boundary vertices belong to the same partite set as u. For trees, the converse of Corollary 2.1.10 is true, however, as we show next.

Theorem 2.1.11 *Let T be a tree of order* 3 *or more. Then* $\text{maj}(T) = 2$ *if and only if the distance between every two end-vertices is even. Equivalently,* $\text{maj}(T) = 2$ *if and only if all end-vertices of T belong to the same partite set of T.*

Proof. Suppose that T is a tree of order 3 or more such that the distance between every two end-vertices is even. Let uv be a pendant edge of T, where u is an end-vertex of T. Assign the color 1 to uv. Let w be any vertex of T such that $d(u, w)$ is

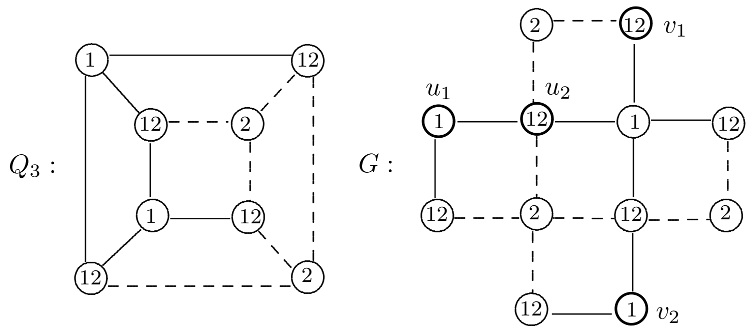


Figure 2.1: A bipartite graph G with $\text{maj}(G) = 2$

even. If $d(u, w) \equiv 2 \pmod 4$, then color all edges incident with w the color 2; while if $d(u, w) \equiv 0 \pmod 4$, then color all edges incident with w the color 1. This is a majestic 2-edge coloring and so $\text{maj}(T) = 2$. Note that for any majestic 2-edge coloring of T, the color of every vertex in one partite set of T is $\{1, 2\}$, while the color of a vertex in the other partite set is $\{1\}$ or $\{2\}$.

Next, we verify the converse. Assume to the contrary, that there exists a tree T of order 3 or more such that $\text{maj}(T) = 2$ but T contains a pair u, v of end-vertices for which $d(u, v)$ is odd, say $d(u, v) = 2k + 1$ for some positive integer k. Let $P = (u = u_1, u_2, \ldots, u_{2k+2} = v)$ be the $u - v$ path in T and let c be a majestic 2-edge coloring of T. Since u and v are end-vertices, the colors of u and v are either $\{1\}$ or $\{2\}$. Because u_2 is adjacent to u_1, the color of u_1 must be a proper subset of the color of u_2. So, the color of u_2 is $\{1, 2\}$. Since u_3 is adjacent to u_2, the color of u_3 must be a singleton. Continuing this process, we obtain that $c'(u_{2i}) = \{1, 2\}$ and $c'(u_{2i-1})$ is either $\{1\}$ or $\{2\}$. In particular, $c'(u_{2k+2}) = c'(v) = \{1, 2\}$, which is a contradiction. ∎

By Theorem 5.2.1, for each integer $n \geq 3$,

$$\text{maj}(P_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even.} \end{cases} \tag{2.1}$$

We have seen that there are connected bipartite graphs G for which $\delta(G) = 1$ or $\delta(G) = 2$ such that $\text{maj}(G) = 3$. However, we have seen no such graph G with $\delta(G) \geq 3$. This leads to the following problem [7].

Problem 2.1.12 *If G is a connected bipartite graph with $\delta(G) \geq 3$, is $\text{maj}(G) = 2$?*

This question has a negative answer. As an example, we consider the *Heawood graph* (the unique 6-cage) of Figure 2.2. This graph is a 3-regular bipartite graph of order 14. Next, we show that the majestic index of the Heawood graph is 3.

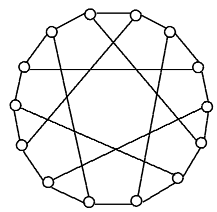


Figure 2.2: The Heawood graph: the unique 6-cage

Proposition 2.1.13 *The majestic index of the Heawood Graph is* 3.

Proof. Assume, to the contrary, that $\text{maj}(H) = 2$. Thus, there exists a majestic 2-edge coloring $c : E(H) \to \{1, 2\}$ of H such that the induced vertex coloring c' of G is proper. Since each edge of H is colored 1 or 2, each vertex of H is colored $\{1\}$, $\{2\}$ or $\{1, 2\}$ and no two adjacent vertices can be colored by a singleton. Thus, every vertex in one partite set of H is colored $\{1, 2\}$ and every vertex in the other partite set of H is colored by a singleton. Let $U = \{u_1, u_2, \ldots, u_7\}$ be the set of vertices colored $\{1, 2\}$ by c', let X be the set of vertices colored $\{1\}$ and let Y be the set of vertices colored $\{2\}$. Thus, U and $X \cup Y$ are the partite sets of H. Observe that the girth of H is 6 and so H has no 4-cycles. Hence,

$$|N(v) \cap N(w)| \le 1 \text{ for every two distinct vertices } v \text{ and } w \text{ in } H. \tag{2.2}$$

Since (i) each vertex in U is adjacent to at least one vertex in X and at least one vertex in Y and (ii) H is 3-regular, it follows by (2.2) that $|X| \ge 3$ and $|Y| \ge 3$. We may assume that $|X| = 3$ and $|Y| = 4$. Let $X = \{x_1, x_2, x_3\}$ and let $Y = \{y_1, y_2, y_3, y_4\}$. By (2.2), $|N(x_1) \cap N(x_2)| \le 1$. Hence, we may assume that $N(x_1) = \{u_1, u_2, u_3\}$ and $\{u_4, u_5\} \subset N(x_2)$. We consider the following two cases.

Case 1. $N(x_1) \cap N(x_2) = \emptyset$. In this case, we may assume $N(x_2) = \{u_4, u_5, u_6\}$ and $N(x_3) = \{u_1, u_4, u_7\}$. Since $N(x_1) = \{u_1, u_2, u_3\}$, no two vertices in $\{u_1, u_2, u_3\}$ can have a common neighbor in Y by (2.2). In addition, exactly one edge incident with u_1 is colored $\{2\}$ and exactly two edges incident with each of u_2 and $_3$ are colored $\{2\}$, say $u_1y_1 \in E(G)$. However then, each of u_2 and u_3 is adjacent to two vertices in $\{y_2, y_3, y_4\}$ and so u_2 and u_3 have a common neighbor in Y, which is impossible.

Case 2. $|N(x_1) \cap N(x_2)| = 1$, *say* $N(x_1) \cap N(x_2) = \{u_1\}$. Thus $N(x_2) = \{u_1, u_4, u_5\}$. Since each of u_6 and u_7 is adjacent to exactly one of x_1 and x_2, it follows that u_6 and u_7 are both adjacent to x_3. Since $\deg x_3 = 3$ and $c'(u_1) = \{1, 2\}$, it follows that x_3 is adjacent to a vertex in $\{u_2, u_3, u_4, u_5\}$. Thus, either $N(x_1) \cap N(x_3) = \emptyset$ or $N(x_2) \cap N(x_3) = \emptyset$. We then apply Case 1 to $N(x_3)$ to produce a contradiction. ■

According to Theorem 5.1, if G is a connected bipartite graph of order 3 or more, then $\mathrm{maj}(G) = 2$ or $\mathrm{maj}(G) = 3$. If G is a graph with $\chi(G) \geq 4$, then it is impossible that $\mathrm{maj}(G) = 2$ by Proposition 2.1.4. If $\chi(G) = 3$, then it is also impossible that $\mathrm{maj}(G) = 2$.

Theorem 2.1.14 *If G is a connected graph with $\chi(G) = 3$, then*

$$\mathrm{maj}(G) \geq 3.$$

Proof. Assume, to the contrary, that there exists a 3-chromatic graph G such that $\mathrm{maj}(G) = 2$. Thus, there exists a majestic 2-edge coloring c of G, where c' is the induced proper vertex coloring of G. Since each edge of G is colored 1 or 2, each vertex of G is colored $\{1\}$, $\{2\}$ or $\{1, 2\}$. Since $\chi(G) = 3$, it follows that G contains an odd cycle C, say $C = (v_1, v_2, \ldots, v_{2k+1}, v_{2k+2} = v_1)$, where k is a positive integer. First, observe that for each integer i $(1 \leq i \leq 2k + 1)$, $c(v_i v_{i+1}) \in c'(v_i)$ and $c(v_i v_{i+1}) \in c'(v_{i+1})$. Since $c'(v_i) \neq c'(v_{i+1})$ and $c(v_i v_{i+1}) \in c'(v_i) \cap c'(v_{i+1})$, it follows that exactly one of $c'(v_i)$ and $c'(v_{i+1})$ is $\{1, 2\}$, say $c'(v_1) = \{1, 2\}$. Then $c'(v_i) = \{1, 2\}$ for every integer $i \in \{1, 3, \ldots, 2k + 1\}$. However then, $c'(v_1) = c'(v_{2k+1}) = \{1, 2\}$, which is a contradiction. ■

We now show that Theorem 4.2.5 cannot be improved further. To do this, we consider four cubic 3-chromatic graphs of order 10 obtained from two 5-cycles. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let α be a permutation of the set $[n] = \{1, 2, \ldots, n\}$. The *permutation graph* $P_\alpha(G)$ of G is the graph of order $2n$ obtained from two copies of G, where the second copy of G is denoted by G' and the vertex v_i in G is denoted by u_i in G' and v_i is joined to the vertex $u_{\alpha(i)}$ in G'. The edges $v_i u_{\alpha(i)}$ are called the *permutation edges* of $P_\alpha(G)$. This concept was first introduced by Chartrand and Harary [15]. Figure 2.3 shows the four permutation graphs of the 5-cycle C_5, where the graph in Figure 2.3(d) is the Petersen graph P. These four graphs appear on the cover of the book *Graph Theory* by Harary [27]. Since each of these graphs has a majestic 3-edge coloring, as shown in Figure 2.3, it follows that $\mathrm{maj}(G) = 3$ for every permutation graph G of C_5.

2.2 The Majestic Number of a Graph

Typically, the graph coloring problems of greatest interest have been those of determining the minimum positive integer k for which it is possible to assign colors from the set $[k]$ to the vertices of a graph G in such a way that adjacent vertices are colored differently. For majestic edge colorings of a graph G, here too the goal is to determine the minimum positive integer k but, in this case, we are assigning colors from the set $[k]$ to the edges

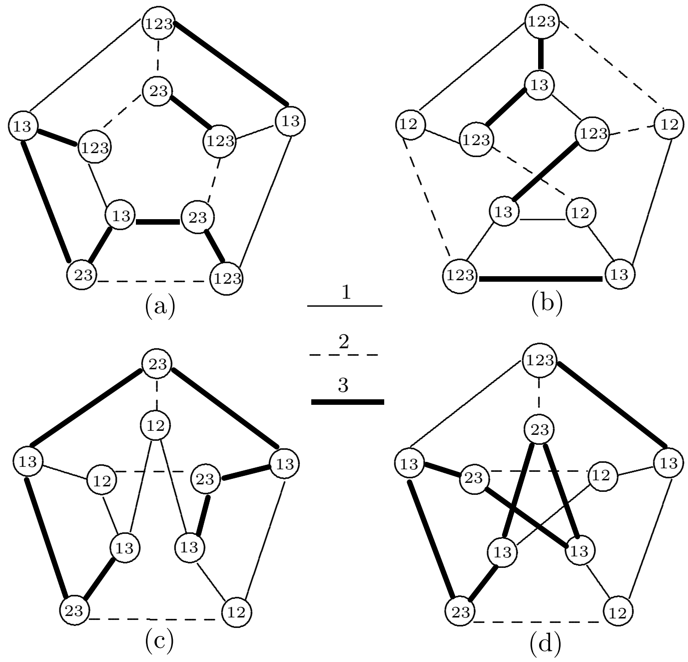


Figure 2.3: The permutation graphs of C_5

of G so that every two adjacent vertices of G receive distinct induced colors. While the vertex colors are selected from the set $\mathcal{P}^*([k])$ of nonempty subsets of $[k]$, it is also of interest to determine the minimum number of vertex colors satisfying these conditions. This leads us to our next topic.

Let G be a connected graph with $\text{maj}(G) = k \geq 2$. Then there exists a majestic k-edge coloring of G where the vertices of G are then colored with the nonempty subsets of $[k]$. Among all majestic k-edge colorings of G, the minimum number of nonempty subsets of $[k]$ needed to color the vertices of G so that two adjacent vertices of G are colored differently is called the *majestic chromatic number* of G or, more simply, the *majestic number* of G and is denoted by $\psi(G)$. This concept was introduced by Chartrand [7] in 2015. First, we present an observation which gives a lower bound for the majestic number of a graph.

Proposition 2.2.1 *If G is a connected graph of order at least 3, then*

$$\psi(G) \geq \max\{3,\ \chi(G)\}.$$

Proof. Since the induced vertex coloring of a majestic edge coloring of G is a proper vertex coloring, it follows that $\psi(G) \geq \chi(G)$. It remains then to show that $\psi(G) \geq 3$. This is certainly the case if $\chi(G) \geq 3$. Hence, we may assume that G is a bipartite graph with $\mathrm{maj}(G) = k \geq 2$. Let c be a majestic k-edge coloring of G for which $\psi(G)$ is minimum. Then the edges of G are colored with at least two colors and there exist two adjacent edges uv and vw that are assigned distinct colors, say 1 and 2, respectively. Thus, $\{1, 2\} \subseteq c'(v)$. Now $c'(u) \neq c'(v) \neq c'(w)$. If $c'(u) \neq c'(w)$, then $\psi(G) \geq 3$. If $c'(u) = c'(w)$, then $\{1, 2\} \subseteq c'(u) \cap c'(w)$. Therefore, there is an edge incident with one of u and w that is assigned a color different from 1 or 2; say ux is colored 3 but no edge incident with v is colored 3. Therefore, $\{1, 2, 3\} \subseteq c'(u)$, $\{1, 2, 3\} \nsubseteq c'(v)$ and $\{3\} \subseteq c'(x)$ but $c'(x) \neq c'(u)$. Hence, $c'(x)$, $c'(u)$ and $c'(v)$ are three distinct colors and $\psi(G) \geq 3$. ■

Since every majestic 2-edge coloring of a graph gives rise to only three distinct vertex colors, we have the following:

Observation 2.2.2 *If G is a connected graph with $\mathrm{maj}(G) = 2$, then*

$$\psi(G) = 3.$$

First, we present a result that gives the majestic number of every complete graph of order 3 or more and the majestic number of every complete ℓ-partite graph where $\ell \geq 3$.

Proposition 2.2.3 *(a) For every integer $n \geq 3$, $\psi(K_n) = n$.*

(b) For every complete ℓ-partite graph G where $\ell \geq 3$, $\psi(G) = \ell$.

Proof. (a) Theorem 2.1.2, $\mathrm{maj}(K_n) = \lceil \log_2 n \rceil + 1$. Let $k = \lceil \log_2 n \rceil + 1$. For every majestic k-coloring of K_n, the vertices of K_n have distinct colors and so $\psi(K_n) = n$.

(b) By Corollary 2.1.3, if G is a complete ℓ-partite graph where $\ell \geq 3$, then $\mathrm{maj}(G) = \lceil \log_2 \ell \rceil + 1$. Let $k = \lceil \log_2 \ell \rceil + 1$. The proof of Corollary 2.1.3 shows that there is a majestic k-coloring of G in which the vertices in each partite set have the same color and the vertices in different partite sets have different colors. Thus, $\psi(G) = \ell$. ■

Next, we determine the majestic index and majestic number of the Petersen graph. The following lemma will be useful for this purpose.

Lemma 2.2.4 *Let G be a nonbipartite connected graph such that*

$$\mathrm{maj}(G) = \psi(G) = 3.$$

If c is a majestic 3-edge coloring whose induced vertex coloring uses exactly three vertex colors, then there is no vertex v of G for which $c'(v)$ is a singleton set.

Proof. Assume, to the contrary, that there is a majestic 3-edge coloring $c : E(G) \to [3]$ of G such that the induced vertex coloring c' of G uses exactly three colors and that $c'(u)$ is a singleton for some vertex u of G. We may assume that $c'(u) = \{1\}$.

Observe that if $e = xy$ is an edge of G, then $c(e) \in c'(x) \cap c'(y)$. Since $\text{maj}(G) = 3$, each of the colors 1, 2, 3 is used to color some edge of G and so each element in $\{1, 2, 3\}$ belongs to at least two vertex colors. Thus, the edge color 1 must also belong to at least one other vertex color, namely either $\{1, 2\}$, $\{1, 3\}$, or $\{1, 2, 3\}$. If $\{1, 2\}$ is the color of a vertex of G, then the edge color 3 must belong to two vertex colors, necessarily distinct from $\{1\}$ and $\{1, 2\}$. However, this contradicts our assumption that there are only three vertex colors. Therefore, $\{1, 2\}$, and similarly $\{1, 3\}$, is not a vertex color of G. It is not possible for $\{3\}$ to be a vertex color because the edge color 2 must belong to at least two vertex colors. Consequently, the only possibilities for the three vertex colors of c' are $\{1\}, \{2, 3\}$ and $\{1, 2, 3\}$. For each i with $0 \le i \le e(u)$, let

$$V_i = \{x \in V(G) :\ d(u, x) = i\}.$$

It therefore follows that

$$c'(v) = \begin{cases} \{1, 2, 3\} & \text{if } v \in V_i \text{ for odd integers } i \text{ with } 1 \le i \le e(u) \\ \{1\} \text{ or } \{2, 3\} & \text{if } v \in V_i \text{ for even integers } i \text{ with } 1 \le i \le e(u). \end{cases}$$

Since c' is a proper vertex coloring and $\{1\} \cap \{2, 3\} = \emptyset$, it follows that each set V_i is independent. Let U be the union of those sets V_i where $0 \le i \le e(u)$ and i is even and let W be the union of those sets V_i where $1 \le i \le e(u)$ and i is odd. Then G is a bipartite graph with partite sets U and W, which is impossible. ■

We saw that the majestic index of the Petersen graph is 3. We now determine the majestic number of this graph.

Proposition 2.2.5 *For the Petersen graph P,*

$$\text{maj}(P) = 3 \text{ and } \psi(P) = 4.$$

Proof. Since $\chi(P) = 3$, it follows by Proposition 2.2.1 that $\psi(P) \ge 3$. The majestic 3-edge coloring of P in Figure 2.4 shows that $\psi(P) \le 4$.

To verify that $\psi(P) = 4$, it is required to show that $\psi(P) = 3$ is impossible. Assume, to the contrary, that there is a majestic 3-edge coloring c of P with an induced proper 3-vertex coloring c'. By Lemma 2.2.4, no vertex color is a singleton. Thus, the vertex colors are necessarily three of $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. There is essentially only

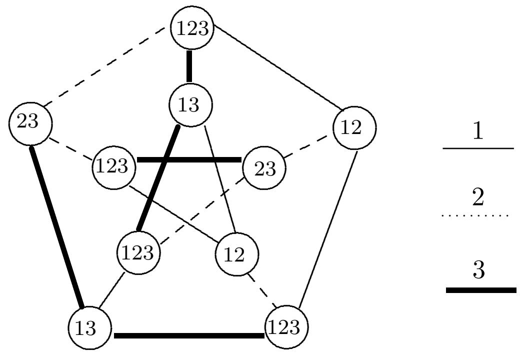


Figure 2.4: A majestic 3-edge colorings of the Petersen graph
with an induced proper 4-vertex coloring

one proper 3-vertex coloring of the Petersen graph, namely that shown in Figure 2.5(a), where the three vertex colors are denoted by p, q, r. Observe that the vertex v_5 is adjacent to three vertices that are assigned the same color r. Suppose that $c(u_5v_5) = 1$. Since $p \neq \{1\}$ by Lemma 2.2.4, either $c(v_2v_5) \neq 1$ or $c(v_3v_5) \neq 1$, say $c(v_2v_5) = 2$. Thus, $\{1,2\} \subseteq p$ and $\{1,2\} \subseteq r$. Hence, we may assume that either p or r is $\{1,2\}$ and the other is $\{1,2,3\}$. Since the edge color 3 belongs to two vetex colors but $\{3\}$ is not a vertex color, we may assume that the third vertex color is $\{1,3\}$. First, observe that $r \neq \{1,2\}$, for otherwise no edge incident with v_5 is colored 3, in which case $p \neq \{1,2,3\}$. Consequently, we may assume that $p = \{1,2\}$, $q = \{1,3\}$ and $r = \{1,2,3\}$. Thus, the color of each edge of P joining vertices colored p and q is 1 (see Figure 2.5(b)). Since $c(u_3u_4) = c(u_4v_4) = 1$, it follows that $c(u_4u_5) = 3$. Since $c(u_1u_2) = c(u_1v_1) = 1$, it follows that $c(u_1u_5) = 2$. Hence, $c(u_5v_5) = 1$. Also, $c(u_3v_3) = 2$, $c(u_2v_2) = 3$ and $c(v_2v_5) = 1$. This implies that that $c'(v_5) = \{1\}$, a contradiction. Therefore, $\psi(P) \neq 3$ and so $\psi(P) = 4$. ■

We now determine the majestic numbers of paths and cycles, beginning with paths.

Theorem 2.2.6 *For each integer $n \geq 3$,*

$$\psi(P_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even and } n \neq 6 \\ 5 & \text{if } n = 6. \end{cases}$$

Proof. If $n \geq 3$ is odd, then $\mathrm{maj}(P_n) = 2$ by (2.1). It then follows by Observation 2.2.2 that $\psi(P_n) = 3$. Thus, we may assume that $n \geq 4$ is even. Let $P_n = (v_1, v_2, \ldots, v_n)$ where $e_i = v_iv_{i+1}$ for $1 \leq i \leq n-1$. First, suppose that $n \neq 6$. To show that $\psi(P_n) \leq 4$,

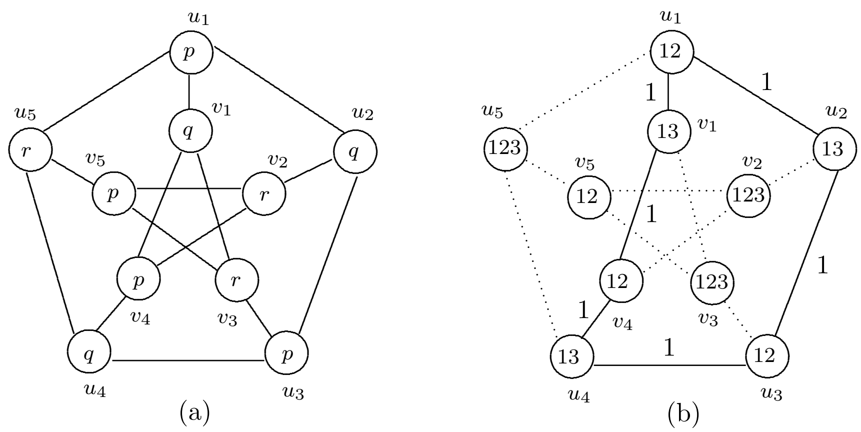


Figure 2.5: A step in the proof of Proposition 2.2.5

we define a majestic 3-edge coloring $c : E(P_n) \to [3]$ of P_n such that its induced vertex coloring uses four colors. Let $c(P_n) = (c(e_1), c(e_2), \ldots, c(e_{n-1}))$.

- If $n \equiv 0 \pmod 6$ and $n \geq 12$, then c is defined such that

$$c(P_n) = (\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \underline{2, 3, 1, 2, 3, 1}, \ldots, \underline{2, 3, 1, 2, 3, 1}).$$

  Then each color $c'(v_i)$, $1 \leq i \leq n$, is one of $\{1\}, \{1,2\}, \{1,3\}$, $\{2,3\}$ and every two adjacent vertices have distinct colors.

- If $n \equiv 2 \pmod 6$ and $n \geq 8$, then c is defined such that

$$c(P_n) = (\underline{1, 2, 3, 1, 2, 3}, \underline{1, 2, 3, 1, 2, 3}, \ldots, \underline{1, 2, 3, 1, 2, 3}, \mathbf{1}).$$

  Here too, each color $c'(v_i)$, $1 \leq i \leq n$, is one of $\{1\}, \{1,2\}, \{1,3\}$, $\{2,3\}$ and every two adjacent vertices have distinct colors.

- If $n \equiv 4 \pmod 6$ and $n \geq 4$, then c is defined such that $c(P_4) = (2, 3, 1)$ and for $n \geq 10$ we have

$$c(P_n) = (\underline{2, 3, 1, 1, 3, 2}, \underline{2, 3, 1, 1, 3, 2}, \ldots, \underline{2, 3, 1, 1, 3, 2}, \mathbf{2}, \mathbf{3}, \mathbf{1}).$$

  In this case, each color $c'(v_i)$, $1 \leq i \leq n$, is one of $\{1\}, \{2\}, \{1,3\}$, $\{2,3\}$ and every two adjacent vertices have distinct colors.

Since in each case, c is a majestic 3-edge coloring of P_n and its induced vertex coloring uses four color, $\psi(P_n) \leq 4$.

Next, we show that $\psi(P_n) \geq 4$. By Proposition 2.2.1, it suffices to show that $\psi(P_n) \neq 3$ when $n \geq 4$ is even. Suppose that c is a majestic 3-edge coloring of P_n. We may assume that $c(v_1v_2) = 1$ and $c(v_2v_3) = 2$. Thus, $c'(v_1) = \{1\}$ and $c'(v_2) = \{1, 2\}$. Since some edge of P_n is colored 3, there are two distinct vertex colors containing 3, which implies that there are at least four distinct vertex colors and so $\psi(P_n) \geq 4$. Therefore, $\psi(P_n) = 4$ where $n \geq 4$ is even and $n \neq 6$.

Finally, we show that $\psi(P_6) = 5$. Since the edge coloring $c_0 : E(P_6) \to [3]$ such that $c_0(P_6) = (1, 2, 3, 1, 2)$ is a majestic 3-edge coloring of P_6 and the induced vertex coloring uses five colors, it follows that $\psi(P_6) \leq 5$. To show that $\psi(P_6) \geq 5$, suppose that $c : E(P_6) \to [3]$ is a majestic 3-edge coloring of P_6. We may assume that $c(v_1v_2) = 1$ and $c(v_2v_3) = 2$. Thus, $c'(v_1) = \{1\}$ and $c'(v_2) = \{1, 2\}$. If $c(v_3v_4) = 1$, then $c'(v_3) = \{1, 2\}$ but this is impossible since $c'(v_2) = \{1, 2\}$ and $v_2v_3 \in E(P_6)$. In addition, if $c(v_3v_4) = 2$, then $c'(v_3) = \{2\}$. However, since some edge of P_6 is colored 3, there are at least two distinct vertex colors containing 3 and so $\psi(P_6) \geq 5$. Therefore, $c(v_3v_4) = 3$ and so $c'(v_3) = \{2, 3\}$. Furthermore, $c(v_4v_5) \neq c(v_5v_6)$. Now, $c(v_5v_6) \neq 3$, for otherwise, $c'(v_4) = c'(v_5)$. Also, $c(v_4v_5) \neq 2$, for otherwise, $c'(v_3) = c'(v_4)$. Hence, $c(v_4v_5)$ is either 1 or 3. If $c(v_4v_5) = 1$, then $c'(v_4) = \{1, 3\}$ and $c(v_5v_6) \neq 1$, so $c'(v_6) \neq \{1\}$. Thus, $\psi(P_6) \geq 5$. If $c(v_4v_5) = 3$, then regardless of the color of $c(v_5v_6)$, there are five distinct vertex colors and so $\psi(P_6) \geq 5$. Therefore, $\psi(P_6) = 5$. ■

We now determine the majestic numbers of cycles.

Theorem 2.2.7 *For each integer $n \geq 4$,*

$$\psi(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 4 \text{ or } n \equiv 0 \pmod 3 \\ 4 & \text{otherwise.} \end{cases}$$

Proof. By Proposition 2.1.6, if $n \equiv 0 \pmod 4$, then $\text{maj}(C_n) = 2$. It then follows by Observation 2.2.2 that $\psi(C_n) = 3$. Thus, we may assume that $n \not\equiv 0 \pmod 4$. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $e_i = v_iv_{i+1}$ for $1 \leq i \leq n$ and let $c(C_n) = (c(e_1), c(e_2), \ldots, c(e_n))$. First, suppose that $n \equiv 0 \pmod 3$. Then $n = 3q$ for some integer $q \geq 2$. Define a majestic 3-edge coloring $c : E(C_n) \to [3]$ of C_n such that

$$c(C_n) = (\underline{1, 2, 3}, \underline{1, 2, 3} \ldots, \underline{1, 2, 3}), \tag{2.3}$$

that is, $c(C_n)$ consists of q subsequences $1, 2, 3$. Here, each color $c'(v_i)$, $1 \leq i \leq n$, is one of $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and every two adjacent vertices have distinct colors. Thus, c is a majestic 3-edge coloring whose induced vertex coloring uses three colors. Therefore, $\psi(C_n) = 3$ when $n \equiv 0 \pmod 3$ by Proposition 2.2.1.

Next, suppose that $n \not\equiv 0 \pmod 4$ and $n \not\equiv 0 \pmod 3$. First, we show that $\psi(C_n) \geq 4$. By Proposition 2.2.1, it suffices to show that $\psi(C_n) \neq 3$. Assume, to the contrary, that there is an integer $n \geq 5$ where $n \not\equiv 0 \pmod 4$ and $n \not\equiv 0 \pmod 3$ such that $\psi(C_n) = 3$. Since $\text{maj}(C_n) = 3$ by Proposition 2.1.6, there is a majestic 3-edge coloring c of C_n. If n is odd, then it follows by Lemma 2.2.4 that $c'(v)$ is $\{1, 2\}$, $\{1, 3\}$ or $\{2, 3\}$. Assume that n is even. Then n is congruent to 2 or 10 modulo 12 and there exists a majestic 3-edge coloring c of C_n. Suppose that some vertex of C_n has a singleton color, say $c'(v_1) = \{1\}$. We may assume that $c(v_2v_3) = 2$ and so $c'(v_2) = \{1, 2\}$. However, some edge of C_n is colored 3, which implies that at least two vertex colors contain 3 and so $\psi(C_n) \geq 4$. Hence, we may assume here as well that $c'(v)$ is $\{1, 2\}$, $\{1, 3\}$ or $\{2, 3\}$ for each vertex v of C_n. Thus, $c(e_i)$, $c(e_{i+1})$ and $c(e_{i+2})$ are distinct for all i with $1 \leq i \leq n$, where each subscript is one of $1, 2, \ldots, n$ modulo n. We may assume, without loss of generality, that $c(e_i) = i$ for $i = 1, 2, 3$. Then c is the edge coloring that satisfies (2.3). This, however, implies that $n \equiv 0 \pmod 3$, which is a contradiction. Therefore, $\psi(C_n) \geq 4$.

To verify that $\psi(C_n) \leq 4$, it remains to show that there exists a majestic 3-edge coloring $c : E(C_n) \to [3]$ of C_n such that its induced vertex coloring uses exactly four colors. We consider two cases, according to whether n is even or n is odd.

Case 1. *n is even.* Since $n \not\equiv 0 \pmod 4$ and $n \not\equiv 0 \pmod 3$, it follows that $n = 12q+2$ or $n = 12q + 10$ for some integer q.

- If $n = 12q + 2$, then define c so that

$$c(C_n) = (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1}, \underline{1, 2, 3}, \underline{1, 2, 3} \ldots, \underline{1, 2, 3}),$$

  that is, $c(C_n)$ starts with the subsequence $1, 1, 2, 3, 1$ and is followed by $4q - 1$ subsequences $1, 2, 3$. Therefore, each color $c'(v_i)$, $1 \leq i \leq n$, is one of $\{1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$. Since every two adjacent vertices have distinct colors, $\psi(C_n) \leq 4$.

- If $n = 12q + 10$, then define c so that

$$c(C_n) = (\mathbf{1}, \underline{1, 2, 3}, \underline{1, 2, 3} \ldots, \underline{1, 2, 3}),$$

  that is, $c(C_n)$ starts with 1 and is followed by $4q + 3$ subsequences $1, 2, 3$. Here as well, each color $c'(v_i)$, $1 \leq i \leq n$, is one of $\{1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$. Furthermore, every two adjacent vertices have distinct colors and $\psi(C_n) \leq 4$.

In either case, c is a majestic 3-edge coloring of C_n and $\psi(C_n) \leq 4$. Therefore, $\psi(C_n) = 4$.

Case 2. *n is odd.* Since $n \not\equiv 0 \pmod 4$ and $n \not\equiv 0 \pmod 3$, it follows that $n = 6q + 1$ or $n = 6q + 5$ for some positive integer q.

- If $n = 6q + 1$, then define c so that

$$c(C_n) = (\mathbf{1}, \underline{1, 2, 3}, \underline{1, 2, 3} \ldots, \underline{1, 2, 3}),$$

that is, $c(C_n)$ starts with 1 and is followed by $2q$ subsequences $1, 2, 3$.

- If $n = 6q + 5$, then define c so that

$$c(C_n) = (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1}, \underline{1, 2, 3}, \underline{1, 2, 3} \ldots, \underline{1, 2, 3}),$$

that is, $c(C_n)$ starts with the subsequence $1, 1, 2, 3, 1$ and is followed by $2q$ subsequences $1, 2, 3$.

In both cases, each color $c'(v_i)$, $1 \leq i \leq n$, is one of $\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$. Since every two adjacent vertices colored differently, c is a majestic 3-edge coloring of C_n and its induced vertex coloring uses four colors. Therefore, $\psi(C_n) \leq 4$ and so $\psi(C_n) = 4$. ■

2.3 Majestic Colorings of Cartesian Products

The *Cartesian product* $G \,\square\, H$ of two graphs G and H has vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and two distinct vertices (u, v) and (x, y) of $G \,\square\, H$ are adjacent if either (1) $u = x$ and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(G)$. The Cartesian product $G \,\square\, K_2$ of a graph G and K_2 is a special case of a more general class of graphs. Note that if α is the identity map on $[n]$, then the permutation graph $P_\alpha(G) = G \,\square\, K_2$. For example, the graph in Figure 2.3(a) is $C_5 \,\square\, K_2$.

We saw that the 3-cube Q_3 has majestic index 2. In fact, every k-cube, $k \geq 2$, has majestic index 2. Since $Q_k = Q_{k-1} \,\square\, K_2$ (the Cartesian product of Q_{k-1} and K_2) for $k \geq 3$, this fact is a consequence of the following result. For two disjoint subsets X and Y of vertices of a graph G, let $[X, Y]$ denote the set of edges joining a vertex of X and a vertex of Y in G.

Theorem 2.3.1 *If G is a nontrivial connected bipartite graph, then*

$$\mathrm{maj}(G \,\square\, K_2) = 2.$$

Proof. Let G_1 and G_2 be two copies of G where G_1 has partite sets U_1 and W_1 and G_2 has corresponding partite sets U_2 and W_2. Then $G \,\square\, K_2$ has partite sets $U_1 \cup W_2$ and $U_2 \cup W_1$. Let $c : E(G) \to \{1, 2\}$ be defined by

$$c(e) = \begin{cases} 1 & \text{if } e \in [U_1, W_1] \cup [U_1, U_2] \\ 2 & \text{if } e \in [U_2, W_2] \cup [W_1, W_2]. \end{cases}$$

Since the induced vertex coloring c' of $G \square K_2$ satisfies that

$$c'(v) = \begin{cases} \{1\} & \text{if } v \in U_1 \\ \{2\} & \text{if } v \in W_2 \\ \{1,2\} & \text{if } v \in U_2 \cup W_1, \end{cases}$$

it follows that c' is a proper coloring of $G \square K_2$ and so $\text{maj}(G \square K_2) = 2$. ■

Corollary 2.3.2 *For each integer $k \geq 2$,* $\text{maj}(Q_k) = 2$.

The following result is a consequence of Theorem 2.3.1 and Observation 2.2.2.

Theorem 2.3.3 *If G is a nontrivial connected bipartite graph, then*

$$\psi(G \square K_2) = 3.$$

Theorems 2.3.1 and 2.3.3 lead us to the study of majestic colorings of graphs $G \square K_2$ where G is not bipartite.

The most common class of nonbipartite graphs is that of the odd cycles. We determine the majestic index of the Cartesian product $G \square K_2$ when G is an odd cycle. For each integer $n \geq 3$, the graph $C_n \square K_2$ is also referred to as a *prism*. We saw that $\text{maj}(C_5 \square K_2) = 3$. In fact, this is the case for $\text{maj}(C_n \square K_2)$ when $n \geq 3$ is an odd integer, as we show next.

Theorem 2.3.4 *For each integer $n \geq 3$,*

$$\text{maj}(C_n \square K_2) = \begin{cases} 2 & \textit{if } n \textit{ is even} \\ 3 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. By Theorem 2.3.1, $\text{maj}(C_n \square K_2) = 2$ for all even integers $n \geq 4$. Thus, we may assume that $n \geq 3$ is odd. By Theorem 4.2.5, $\text{maj}(C_n \square K_2) \geq 3$ for each odd integer $n \geq 3$ and so it remains to show that $C_n \square K_2$ has a majestic 3-edge coloring. Let H and F be two copies of C_n in $C_n \square K_2$, where $H = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$, $F = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ and u_i is adjacent to v_i for $1 \leq i \leq n$. Furthermore, let $h_i = u_i u_{i+1}$, $f_i = v_i v_{i+1}$ and $e_i = u_i v_i$ for $1 \leq i \leq n$. Define a majestic 3-edge coloring $c : E(C_n) \to [3]$ of $C_n \square K_2$ as follows:

- For $n \equiv 0 \pmod 3$, define the edge coloring c by

$$\begin{aligned}
(c(h_1), c(h_2), \ldots, c(h_n)) &= (\underline{1,2,3}, \underline{1,2,3}, \ldots, \underline{1,2,3}) \\
(c(f_1), c(f_2), \ldots, c(f_n)) &= (\underline{3,1,2}, \underline{3,1,2}, \ldots, \underline{3,1,2}) \\
(c(e_1), c(e_2), \ldots, c(e_n)) &= (\underline{3,1,2}, \underline{3,1,2}, \ldots, \underline{3,1,2}).
\end{aligned}$$

  Here, the vertex colors are $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ and adjacent vertices are colored differently.

- For $n \equiv 1 \pmod 6$ or $n \equiv 5 \pmod 6$, define the edge coloring c by

$$\begin{aligned}
(c(h_1), c(h_2), \ldots, c(h_n)) &= (\underline{3,1}, \underline{3,1}, \ldots, \underline{3,1}, \mathbf{2}, \mathbf{3}, \mathbf{1}) \\
(c(f_1), c(f_2), \ldots, c(f_n)) &= (\underline{3,2}, \underline{3,2}, \ldots, \underline{3,2}, \mathbf{3}, \mathbf{1}, \mathbf{2}) \\
(c(e_1), c(e_2), \ldots, c(e_n)) &= (\underline{2,1}, \underline{2,1}, \ldots, \underline{2,1}, \mathbf{3}, \mathbf{3}, \mathbf{3}).
\end{aligned}$$

  Thus, the vertex colors are $\{1,2\}$, $\{2,3\}$ and $\{1,2,3\}$ and adjacent vertices are colored differently.

Since c is a 3-edge majestic coloring in either case,

$$\text{maj}(C_n \,\square\, K_2) = 3$$

for all odd integers $n \geq 3$. ■

By Theorem 2.3.1, if G is a complete bipartite graph, then

$$\text{maj}(G \,\square\, K_2) = \text{maj}(G) = 2.$$

We now extend this result to all complete multipartite graphs.

Theorem 2.3.5 *If G is a complete ℓ-partite graph with $\ell \geq 3$, then*

$$\text{maj}(G \,\square\, K_2) = \text{maj}(G).$$

Proof. Let $k = \lceil \log_2 \ell \rceil + 1$. By Theorem 2.1.3, $\text{maj}(G) = k$. Since $\text{maj}(G \,\square\, K_2) \geq k$ by Theorem 2.1.5, it remains to show that $\text{maj}(G \,\square\, K_2) \leq k$. First, let $G = K_\ell$ be the complete graph of order ℓ with vertex set $\{v_0, v_1, \ldots, v_{\ell-1}\}$. Define a majestic k-edge coloring $c_0 : E(K_\ell) \to [k]$ of K_ℓ as follows. For $\ell = 3, 4$, the coloring c_0 for K_3 and K_4 are shown in Figure 2.6. Hence we may assume that $\ell \geq 5$.

Thus $2^{k-2} + 1 \leq \ell \leq 2^{k-1}$. First, we assign each vertex v_i $(0 \leq i \leq \ell - 1)$ a nonempty subset S_i of $[k]$. For $0 \leq i \leq k$, let

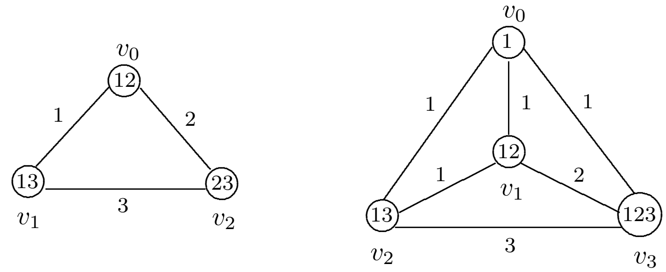


Figure 2.6: Majestic 3-edge colorings of K_3 and K_4

$$S_0 = \{k\},\ S_i = \{i, k\} \text{ for } 1 \le i \le k-1 \text{ and } S_k = [k].$$

For $k+1 \le i \le \ell - 1$, choose the sets S_i in any manner so that $S_0, S_1, \ldots, S_{\ell-1}$ are distinct and $k \in S_i$. We next define an edge coloring c_0 of K_ℓ. For each integer i with $1 \le i \le k-1$, assign the color i to each edge $v_i v_t$ if $i \in S_t$ where $k \le t \le \ell - 1$ and assign the color k to all other edges of K_ℓ. Figure 2.7 shows such a 4-edge coloring of K_8, where edges not drawn are colored 4. Thus $c_0'(v_j) = S_j$ for all j $(0 \le j \le \ell - 1)$ and c_0 is a majestic k-edge coloring of K_ℓ such that $k \in c_0'(v)$ for each vertex v of K_ℓ.

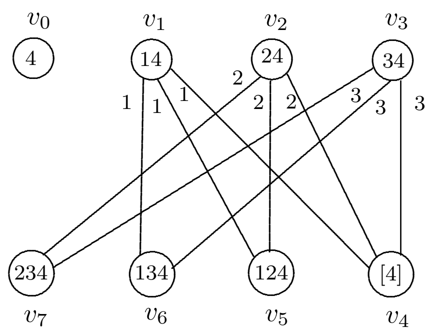


Figure 2.7: A majestic 4-edge coloring of K_8

Next, we construct a majestic k-edge coloring c_ℓ of $K_\ell \,\Box\, K_2$ as follows. For $\ell = 3, 4$, a majestic 3-edge coloring is shown in Figure 2.8 for each of $K_3 \,\Box\, K_2$ and $K_4 \,\Box\, K_2$.

Hence we may assume that $\ell \ge 5$. Let H and F be two copies of K_ℓ with $V(H) = \{v_0, v_1, \ldots, v_{\ell-1}\}$ and $V(F) = \{u_0, u_1, \ldots, u_{\ell-1}\}$. Define the majestic k-edge coloring $c_H : E(H) \to [k]$ of H by $c_H(e) = c_0(e)$ for each $e \in E(H)$. Next, define the majestic k-edge coloring $c_F : E(F) \to [k]$ of F by $c_F(u_i u_j) = c_H(v_{i+1} v_{j+1})$ for all pairs i, j of integers with $0 \le i < j \le \ell$, where the subscripts of vertices are expressed as integers modulo ℓ. The k-edge coloring

$$c_\ell : E(K_\ell \,\Box\, K_2) \to [k]$$

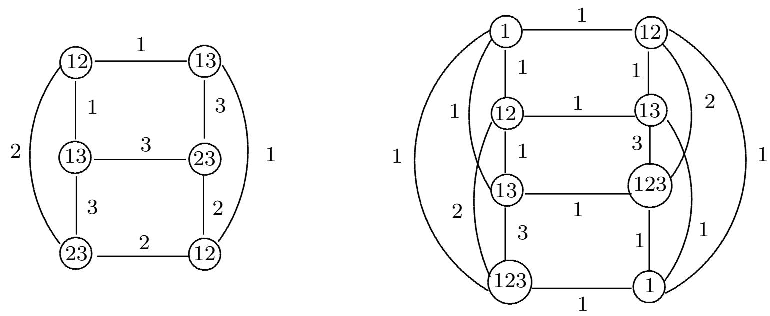


Figure 2.8: Majestic 3-edge colorings of $K_3 \,\square\, K_2$ and $K_4 \,\square\, K_2$

of $K_\ell \,\square\, K_2$ is defined by $c_\ell(e) = c_H(e)$ if $e \in E(H)$, $c_\ell(e) = c_F(e)$ if $e \in E(F)$ and $c_\ell(u_iv_i) = k$ for $0 \le i \le \ell - 1$. This is illustrated in Figure 2.9 for $K_8 \,\square\, K_2$. Since the vertex coloring of $K_\ell \,\square\, K_2$ induced by c_ℓ is proper,

$$\mathrm{maj}(K_\ell \,\square\, K_2) = \mathrm{maj}(K_\ell) = k.$$

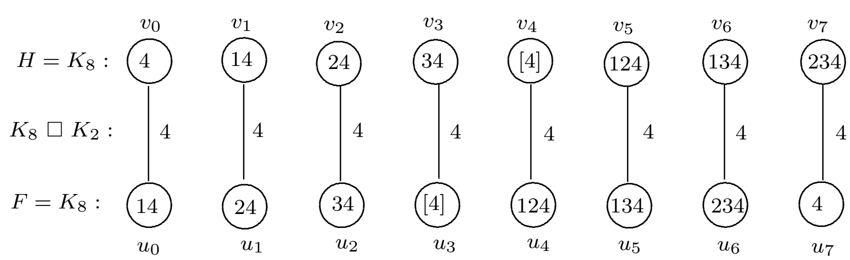


Figure 2.9: A majestic 4-edge coloring of $K_8 \,\square\, K_2$

Next, let $G = K_{n_1,n_2,\ldots,n_\ell}$ be a complete ℓ-partite graph with partite sets V_0, V_1, ..., $V_{\ell-1}$ where $|V_i| = n_i$ for $0 \le i \le \ell - 1$ and $\ell \ge 3$. Let H and F be two copies of G, where the partite sets of H are $V_0, V_1, \ldots, V_{\ell-1}$, the partite sets of F are $U_0, U_1, \ldots, U_{\ell-1}$ and $|V_i| = |U_i| = n_i$ for $0 \le i \le \ell - 1$. Now, let

$$c_\ell : E(K_\ell \,\square\, K_2) \to [k]$$

be the majestic k-edge coloring of $K_\ell \,\square\, K_2$ described above. Define the majestic k-edge coloring $c_H : E(H) \to [k]$ of H by $c_H(e) = c_\ell(e)$ if $e \in [V_i, V_j]$ for $0 \le i < j \le \ell - 1$. Similarly, define the majestic k-edge coloring $c_F : E(F) \to [k]$ of F by $c_F(e) = c_\ell(e)$ if $e \in [U_i, U_j]$ for $0 \le i < j \le \ell - 1$. Now the k-edge coloring $c : E(G \,\square\, K_2) \to [k]$ of $G \,\square\, K_2$ is defined by $c(e) = c_H(e)$ if $e \in E(H)$, $c(e) = c_F(e)$ if $e \in E(F)$ and $c(e) = k$ if $e = xy$ where $x \in V(H)$ and $y \in V(F)$. Then the vertex coloring c' induced by c has the property that $c'(v) = c'_H(v)$ if $v \in V(H)$, $c'(v) = c'_F(v)$ if $v \in V(F)$ and $c'(x) \ne c'(y)$ if x

and y are adjacent where $x \in V(H)$ and $y \in V(F)$. Since c' is a proper vertex coloring of $G \,\square\, K_2$, it follows that $\chi_m(G \,\square\, K_2) = \text{maj}(G) = k$. ■

We include the following question here.

Problem 2.3.6 *For a connected nonbipartite graph G of order at least 3, how are the two numbers $\text{maj}(G)$ and $\text{maj}(G \,\square\, K_2)$ related?*

We now determine the majestic numbers of the Cartesian products $G \,\square\, K_2$ where G is either a cycle or a complete multipartite graph.

Theorem 2.3.7 *For each integer $n \geq 4$, $\psi(C_n \,\square\, K_2) = 3$.*

Proof. By Proposition 2.3.3, if $n \geq 4$ is even, then $\psi(C_n \,\square\, K_2) = 3$. Therefore, it suffices to consider odd integers $n \geq 5$. Let H and F be two copies of C_n in $C_n \,\square\, K_2$, where $H = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$, $F = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ and u_i is adjacent to v_i for $1 \leq i \leq n$. Furthermore, let $h_i = u_i u_{i+1}$, $f_i = v_i v_{i+1}$ and $e_i = u_i v_i$ for $1 \leq i \leq n$. We define a majestic 3-edge coloring $c : E(C_n) \to [3]$ of $C_n \,\square\, K_2$ such that its induced vertex coloring uses three colors.

- For $n \equiv 0 \pmod 3$, define the edge coloring c by

$$\begin{aligned}
(c(h_1), c(h_2), \ldots, c(h_n)) &= (\underline{1,2,3}, \underline{1,2,3}, \ldots, \underline{1,2,3}) \\
(c(f_1), c(f_2), \ldots, c(f_n)) &= (\underline{3,1,2}, \underline{3,1,2}, \ldots, \underline{3,1,2}) \\
(c(e_1), c(e_2), \ldots, c(e_n)) &= (\underline{3,1,2}, \underline{3,1,2}, \ldots, \underline{3,1,2}).
\end{aligned}$$

  Here, the vertex colors are $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ and adjacent vertices are colored differently.

- For $n \equiv 1 \pmod 6$ or $n \equiv 5 \pmod 6$, define the edge coloring c by

$$\begin{aligned}
(c(h_1), c(h_2), \ldots, c(h_n)) &= (\underline{3,1}, \underline{3,1}, \ldots, \underline{3,1}, \mathbf{2}, \mathbf{3}, \mathbf{1}) \\
(c(f_1), c(f_2), \ldots, c(f_n)) &= (\underline{3,2}, \underline{3,2}, \ldots, \underline{3,2}, \mathbf{3}, \mathbf{1}, \mathbf{2}) \\
(c(e_1), c(e_2), \ldots, c(e_n)) &= (\underline{2,1}, \underline{2,1}, \ldots, \underline{2,1}, \mathbf{3}, \mathbf{3}, \mathbf{3}).
\end{aligned}$$

  Thus, the vertex colors are $\{1,2\}$, $\{2,3\}$ and $\{1,2,3\}$.

Since adjacent vertices are colored differently in both instances, it follows that

$$\psi(C_n \,\square\, K_2) = 3$$

for all odd integers $n \geq 5$ by Proposition 2.2.1. ■

We saw that if G is a complete bipartite graph of order at least 3, then

$$\chi(G) = \text{maj}(G) = 2.$$

Thus, $\psi(G) = 3$ by Observation 2.2.2 and so $\psi(G) > \chi(G)$. Such is not the case, however, for complete multipartite graphs that are not bipartite.

Proposition 2.3.8 [7] *If G is a complete ℓ-partite graph with $\ell \geq 3$, then*

$$\psi(G) = \chi(G).$$

In particular, if G is a complete graph of order $n \geq 3$, then $\psi(G) = \chi(G) = n$.

By Proposition 2.3.3, if G is a nontrivial connected bipartite graph, then $\psi(G \,\Box\, K_2) = 3$ and so $\psi(G \,\Box\, K_2) = \chi(G \,\Box\, K_2) + 1$. For a complete ℓ-partite graph with $\ell \geq 3$, this is not the case, as we show next. Although the proof of the following result makes use of the majestic edge colorings described in Theorem 2.3.5, we restate these colorings for the completion of the proof.

Theorem 2.3.9 *If G is a complete ℓ-partite graph with $\ell \geq 3$, then*

$$\psi(G \,\Box\, K_2) = \chi(G \,\Box\, K_2).$$

Proof. Let $G = K_{n_1,n_2,\ldots,n_\ell}$ be a complete ℓ-partite graph with partite sets V_0, V_1, $\ldots$, $V_{\ell-1}$ where $|V_i| = n_i$ for $0 \leq i \leq \ell - 1$ and $\ell \geq 3$. First, we consider the complete graph K_ℓ of order ℓ with vertex set $\{v_0, v_1, \ldots, v_{\ell-1}\}$. By Theorem 2.1.2,

$$\text{maj}(K_\ell) = \lceil \log_2 \ell \rceil + 1.$$

Let $k = \lceil \log_2 \ell \rceil + 1$. First, we define a majestic k-edge coloring $c_0 : E(K_\ell) \to [k]$ of K_ℓ as follows. For $\ell = 3, 4$, the coloring c_0 for K_3 and K_4 are shown in Figure 2.6. Hence we may assume that $\ell \geq 5$.

Thus $2^{k-2} + 1 \leq \ell \leq 2^{k-1}$. First, we assign each vertex v_i $(0 \leq i \leq \ell - 1)$ a nonempty subset S_i of $[k]$. For $0 \leq i \leq k$, let

$$S_0 = \{k\},\ S_i = \{i, k\} \text{ for } 1 \leq i \leq k-1 \text{ and } S_k = [k].$$

For $k + 1 \leq i \leq \ell - 1$, choose the sets S_i in any manner so that $S_0, S_1, \ldots, S_{\ell-1}$ are distinct and $k \in S_i$. We next define an edge coloring c_0 of K_ℓ. For each integer i with $1 \leq i \leq k - 1$, assign the color i to each edge $v_i v_t$ if $i \in S_t$ where $k \leq t \leq \ell - 1$ and assign the color k to all other edges of K_ℓ. Thus $c_0'(v_j) = S_j$ for all j $(0 \le j \le \ell - 1)$ and c_0 is a majestic k-edge coloring of K_ℓ such that $k \in c_0'(v)$ for each vertex v of K_ℓ.

Next, we construct a majestic k-edge coloring c_ℓ of $K_\ell \square K_2$ as follows. For $\ell = 3, 4$, a majestic 3-edge coloring is shown in Figure 2.8 for each of $K_3 \square K_2$ and $K_4 \square K_2$.

Hence we may assume that $\ell \ge 5$. Let H and F be two copies of K_ℓ with $V(H) = \{v_0, v_1, \ldots, v_{\ell-1}\}$ and $V(F) = \{u_0, u_1, \ldots, u_{\ell-1}\}$. Define the majestic k-edge coloring $c_H : E(H) \to [k]$ of H by $c_H(e) = c_0(e)$ for each $e \in E(H)$. Next, define the majestic k-edge coloring $c_F : E(F) \to [k]$ of F by $c_F(u_iu_j) = c_H(v_{i+1}v_{j+1})$ for all pairs i, j of integers with $0 \le i < j \le \ell$, where the subscripts of vertices are expressed as integers modulo ℓ. The k-edge coloring $c_\ell : E(K_\ell \square K_2) \to [k]$ of $K_\ell \square K_2$ is defined by $c_\ell(e) = c_H(e)$ if $e \in E(H)$, $c_\ell(e) = c_F(e)$ if $e \in E(F)$ and $c_\ell(u_iv_i) = k$ for $0 \le i \le \ell - 1$. Since the vertex coloring of $K_\ell \square K_2$ induced by c_ℓ is proper and uses exactly ℓ vertex colors, it follows that $\psi(K_\ell \square K_2) = \chi(K_\ell \square K_2) = \ell$.

We now consider $\psi(G \square K_2)$ where $G = K_{n_1,n_2,\ldots,n_\ell}$. Let H and F be two copies of G, where the partite sets of H are $V_0, V_1, \ldots, V_{\ell-1}$, the partite sets of F are $U_0, U_1, \ldots, U_{\ell-1}$ and $|V_i| = |U_i| = n_i$ for $0 \le i \le \ell - 1$. Now, let $c_\ell : E(K_\ell \square K_2) \to [k]$ be the majestic k-edge coloring of $K_\ell \square K_2$ described above. Define the majestic k-edge coloring $c_H : E(H) \to [k]$ of H by $c_H(e) = c_\ell(e)$ if $e \in [V_i, V_j]$ for $0 \le i < j \le \ell - 1$. Similarly, define the majestic k-edge coloring $c_F : E(F) \to [k]$ of F by $c_F(e) = c_\ell(e)$ if $e \in [U_i, U_j]$ for $0 \le i < j \le \ell - 1$. Now the k-edge coloring $c : E(G \square K_2) \to [k]$ of $G \square K_2$ is defined by $c(e) = c_H(e)$ if $e \in E(H)$, $c(e) = c_F(e)$ if $e \in E(F)$ and $c(e) = k$ if $e = xy$ where $x \in V(H)$ and $y \in V(F)$. Then the vertex coloring c' induced by c has the property that $c'(v) = c_H'(v)$ if $v \in V(H)$, $c'(v) = c_F'(v)$ if $v \in V(F)$ and $c'(x) \ne c'(y)$ if x and y are adjacent where $x \in V(H)$ and $y \in V(F)$. Thus, c' is a proper vertex coloring of $G \square K_2$ using exactly ℓ vertex colors. Therefore $\psi(G \square K_2) = \chi(G \square K_2) = \ell$. ■

By Theorems 2.2.7 and 2.3.7, if $n \ge 4$, $n \not\equiv 0 \pmod 4$ and $n \not\equiv 0 \pmod 3$, then $\psi(C_n \square K_2) = 3$ and $\psi(C_n) = 4$; so $\psi(C_n \square K_2) = \psi(C_n) - 1$. It was shown in [7] that $\psi(P_6) = 5$ and $\psi(P_n) = 4$ for even integers $n \ge 8$. Since $\psi(P_n \square K_2) = 3$ for all $n \ge 6$ by Theorem 2.3.3, it follows that $\psi(P_6 \square K_2) = \psi(P_6) - 2$ and $\psi(P_n \square K_2) = \psi(P_n) - 1$ for even integers $n \ge 8$. By Theorem 2.3.9, if G is a complete ℓ-partite graph with $\ell \ge 3$, then $\psi(G \square K_2) = \psi(G) = \ell$. This leads to the following question.

Problem 2.3.10 *For a connected graph G of order $n \ge 7$, is it true that*

$$\psi(G) - 1 \le \psi(G \square K_2) \le \psi(G)?$$

Chapter 3

Comparing Majestic and Chromatic Numbers

3.1 Introduction

We have seen for every connected graph G of order $n \geq 3$ that $\psi(G) \geq \chi(G)$ and have looked at a number of results involving these two parameters when G has small chromatic number. In the case where G is a complete ℓ-partite graph where $\ell \geq 3$, we saw in Proposition 2.2.3 that $\psi(G) = \chi(G) = \ell$. We now consider additional results dealing with majestic and chromatic numbers of graphs where $\chi(G)$ is large.

3.2 The Composition of a Cycle with K_2

For two graphs G and H, the *composition* $G[H]$ is obtained by replacing each vertex x of G by a copy H_x of H such that every vertex of H_u is adjacent to every vertex of H_v in $G[H]$ where $uv \in E(G)$, that is,

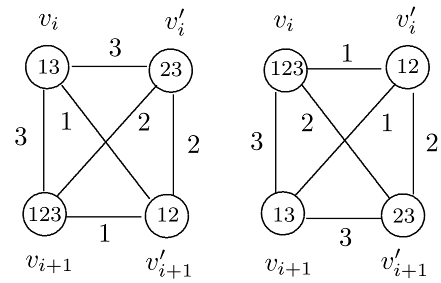
$$E(G[H]) = \{xy : \ x \in V(H_x), y \in V(H_y) \ \text{ and } \ xy \in E(G)\}.$$

First, we consider the composition $G[H]$ where G is a cycle and $H = K_2$. We begin with the case where the cycle has even length.

Theorem 3.2.1 *If $G = C_{2k}[K_2]$ where $k \geq 2$, then* $\text{maj}(G) = 3$ *and* $\psi(G) = 4$.

Proof. Since $\chi(G) = 4$, it follows that $\text{maj}(G) \geq 3$ and $\psi(G) \geq 4$. It suffices to show that there is a majestic 3-edge coloring c of G such that the induced vertex coloring uses exactly four colors.

Let $V_1, V_2, \ldots, V_{2k}, V_{2k+1} = V_1$ be the partite sets of the $2k$ copies of K_2 in G, where $V_i = \{v_i, v_i'\}$ for $1 \leq i \leq 2k$. Thus, $G[V_i \cup V_{i+1}] = K_4$ for $i = 1, 2, \ldots, 2k$. We now define a 3-edge coloring of G. For even integer i with $2 \leq i \leq 2k$ and odd integer i with $1 \leq i \leq 2k-1$, the colors of the edges of the subgraphs $G[V_i \cup V_{i+1}]$ are shown in Figures 3.1(a) and (b).



(a) i even (b) i odd

Figure 3.1: Illustrating a majestic 3-edge coloring
of $C_{2k}[K_2]$ in the proof of Theorem 3.2.1

Since this is a majestic 3-edge coloring, $\mathrm{maj}(G) \leq 3$ and so $\mathrm{maj}(G) = 3$. The vertex colors in this majestic 3-edge coloring are

$$\{1,2\},\ \{1,3\},\ \{2,3\} \text{ and } \{1,2,3\},$$

as shown in Figures 3.1. This is illustrated in Figure 3.2 for the graph G for $k = 2$. Therefore, $\psi(G) \leq 4$ and $\psi(G) = 4$. ■

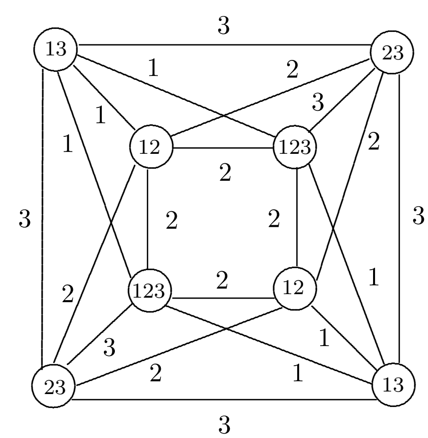


Figure 3.2: A majestic 3-edge coloring of $C_4[K_2]$

Theorem 3.2.2 *If $G = C_{2k+1}[K_2]$ where $k \geq 2$, then* $\mathrm{maj}(G) = 4$ *and* $\psi(G) = 5$.

Proof. The order of G here is $n = 4k+2$, the vertex independence number is k and the clique number of G is $\omega(G) = 4$. Since $\omega(G) = 4$, it follows that $\chi(G) \geq 4$. It is straightforward to show that G has no proper vertex 4-coloring but does have a proper vertex 5-coloring and so $\chi(G) = 5$.

Since $\chi(G) = 5$, it follows that $\mathrm{maj}(G) \geq 3$. First, we show that $\mathrm{maj}(G) \neq 3$. Assume, to the contrary, that $\mathrm{maj}(G) = 3$. Then there is a majestic 3-edge coloring $c : E(G) \to [3]$ of G such that the induced vertex coloring c' is proper. Since $\psi(G) \geq \chi(G) = 5$, it follows that c' uses at least five of the possible seven vertex colors

$$\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}.$$

Each of the three pairs $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, $\{\{3\},\{1,2\}\}$ of possible vertex colors consists of two disjoint sets. The two colors in each pair can be used at most k times, which is also the case for the color $\{1,2,3\}$. Hence, these seven vertex colors can be used at most $4k$ times for the vertices of G. However, since the order of G is $4k+2$, this is impossible. Therefore, $\mathrm{maj}(G) \geq 4$.

To verify that $\mathrm{maj}(G) \leq 4$, we show that G has a majestic 4-edge coloring. Let $V_1, V_2, \ldots, V_{2k}, V_{2k+1} = V_1$ be the partite sets of the $2k+1$ copies of K_2 in G, where $V_i = \{v_i, v_i'\}$ for $1 \leq i \leq 2k+1$. Thus, $G[V_i \cup V_{i+1}] = K_4$ for $i = 1, 2, \ldots, 2k+1$. We now define a 4-edge coloring of G. For even integers i with $2 \leq i \leq 2k-2$ and odd integers i with $1 \leq i \leq 2k-3$, the colors of the edges of the subgraphs $G[V_i \cup V_{i+1}]$ $(1 \leq i \leq 2k-2)$ are shown in Figures 3.3(a) and (b), respectively; while the colors of the edges in the subgraph $G[V_1 \cup V_{2k+1} \cup V_{2k} \cup V_{2k-1}]$ are shown in Figure 3.3(c). This is a majestic 4-edge coloring and so $\mathrm{maj}(G) \leq 4$. Hence, $\mathrm{maj}(G) = 4$.

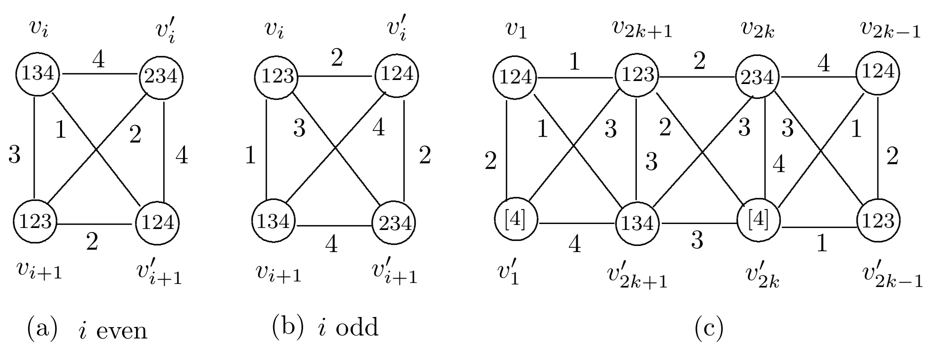


Figure 3.3: Illustrating a majestic 3-edge coloring of $C_{2k+1}[K_2]$ in the proof of Theorem 3.2.2

The vertex colors in this majestic 4-edge coloring are $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$ and $\{1,2,3,4\}$, as shown in Figures 3.3. This is illustrated in Figure 3.4 for the graph G for $k = 2$. Therefore, $\psi(G) \leq 5$ and $\psi(G) = 5$. ■

Combining Theorems 3.2.2 and 3.2.1, we obtain the following.

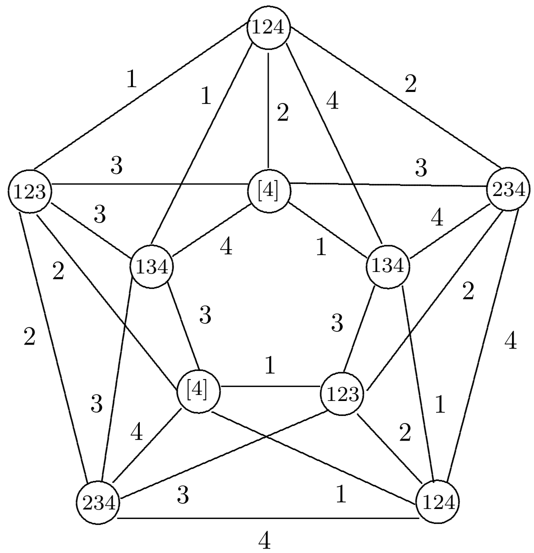


Figure 3.4: A majestic 4-edge coloring of $C_5[K_2]$

Corollary 3.2.3 *For an integer $n \geq 4$, let $G = C_n[K_2]$.*

- $\star$ *If $n \geq 4$ is even, then* $\mathrm{maj}(G) = 3$ *and* $\psi(G) = 4$.
- $\star$ *If $n \geq 5$ is odd, then* $\mathrm{maj}(G) = 4$ *and* $\psi(G) = 5$.

3.3 On Graphs with Two Numbers Differ by at Least 2

We have now seen that there are infinite classes of graphs G for which $\psi(G) = \chi(G)$ and infinite classes of graphs G for which $\psi(G) = \chi(G) + 1$. We next show the existence of an infinite class of graphs G for which $\psi(G) = \chi(G) + 2$.

For a given graph G, the *corona* $\mathrm{cor}(G)$ of G is obtained from G by adding a pendant edge to each vertex of G. For each integer $n \geq 3$, let G_n be the graph obtained from the corona $\mathrm{cor}(K_n)$ of the complete graph K_n by subdividing each pendant edge exactly once. Thus, G_n has order $3n$ and exactly n vertices of degree i for each $i \in \{1, 2, n\}$. Suppose that the vertices of the subgraph K_n in G_n are $w_1, w_2, \ldots, w_n$, the vertices of degree 2 in G_n are $v_1, v_2, \ldots, v_n$ and the end-vertices of G_n are $u_1, u_2, \ldots, u_n$, where (u_i, v_i, w_i) is a path of order 3 in G_n for $1 \leq i \leq n$. The graph G_4 is shown in Figure 3.5.

Theorem 3.3.1 *For each integer $n \geq 3$,*

$$\mathrm{maj}(G_n) = \lceil \log_2 n \rceil + 1 \text{ and } \psi(G_n) = \chi(G_n) + 2.$$

Proof. For a fixed integer $n \geq 3$, let $k = \lceil \log_2 n \rceil + 1$. By Corollary 2.1.5, $\mathrm{maj}(G_n) \geq k$. First, we show that $\psi(G_n) \leq \chi(G_n) + 2$. To do this, we show that there is a majestic

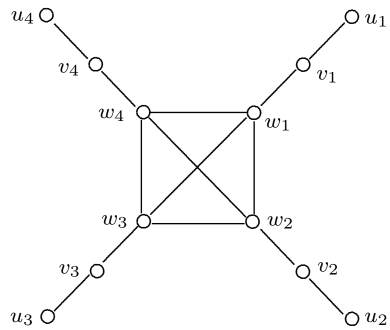


Figure 3.5: The graph G_4

k-edge coloring $c : E(G_n) \to [k]$ of G_n such that the induced vertex coloring c' uses exactly $\chi(G_n) + 2 = n + 2$ colors in $\mathcal{P}^*([k])$. If $n = 3$ or $n = 4$, then $k = 3$. Majestic 3-edge colorings are shown in Figure 3.6 for G_3 and G_4. Hence, we may assume that $n \geq 5$.

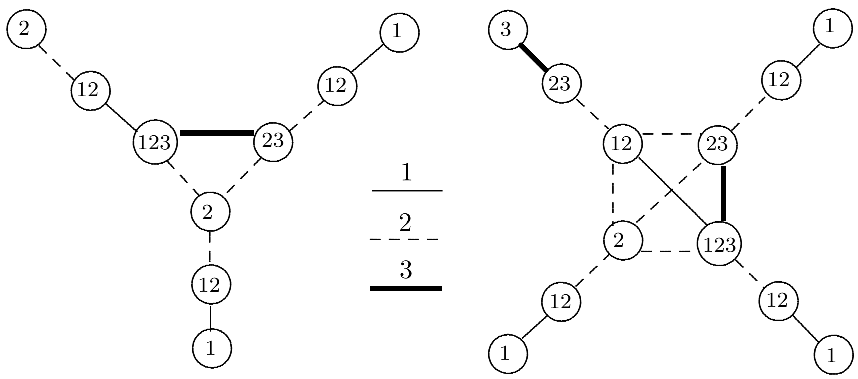


Figure 3.6: Majestic 3-edge colorings of G_3 and G_4

Thus, $2^{k-2} + 1 \leq n \leq 2^{k-1}$. For $0 \leq i \leq k$, let

$$S_0 = \{k\},\ S_i = \{i, k\} \text{ for } 1 \leq i \leq k-1 \text{ and } S_k = [k].$$

For $k+1 \leq i \leq n-1$, choose the sets $S_i \subseteq [k]$ so that $S_0, S_1, \ldots, S_{n-1}$ are distinct and $k \in S_i$. Now, we assign each vertex w_i ($1 \leq i \leq n$) the set S_{i-1}. We next define an edge coloring $c_0 : E(K_n) \to [k]$ of K_n as follows: For each integer i with $1 \leq i \leq k-1$, assign the color i to each edge $w_{i+1}w_{t+1}$ if $i \in S_t$ where $k \leq t \leq n-1$ and assign the color k to all other edges of K_n. Figure 3.7 shows such a 4-edge coloring of K_8, where dashed edges (and edges that are not drawn) are colored 4. Thus $c_0'(w_{j+1}) = S_j$ for all j ($0 \leq j \leq n-1$) and c_0 is a majestic k-edge coloring of K_n such that $k \in c_0'(v)$ for each vertex v of K_n.

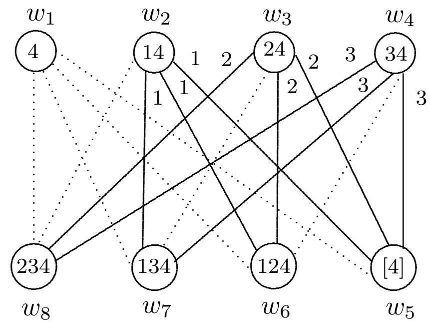


Figure 3.7: A majestic 4-edge coloring of K_8
where dashed edges and undrawn edges are colored 4

Next, we construct a majestic k-edge coloring $c : E(G_n) \to [k]$ of G_n from the coloring c_0 of K_n as follows:

- $\star$ $c(e) = c_0(e)$ if $e \in E(K_n)$;
- $\star$ $c(w_i v_i) = k$ for $1 \le i \le n$;
- $\star$ $c(v_2 u_2) = 2$ and $c(v_i u_i) = 1$ for $1 \le i \le n$ and $i \ne 2$.

Figure 3.8 shows such a 4-edge coloring of G_8, where dashed edges are colored 4. The induced vertex coloring c' then satisfies the following:

- $c'(w) = c_0'(w)$ for each $w \in V(K_n)$,
- $c'(v_2) = \{2, k\} \ne \{1, k\} = c'(w_2)$ and $c'(v_i) = \{1, k\} \ne c'(w_i)$ for $1 \le i \le n$ and $i \ne 2$,
- $c'(u_2) = \{1\}$ and $c'(u_i) = \{2\}$ for $1 \le i \le n$ and $i \ne 2$.

Thus, c' is proper and so c is a majestic k-edge coloring of G_n. Therefore, $\text{maj}(G_n) = k = \lceil \log_2 n \rceil + 1$. Furthermore, c' uses exactly $n+2$ colors (where $\{1\}$ and $\{2\}$ are the only two new colors added to the vertex coloring c_0' of K_n) and so $\psi(G_n) \le n+2$.

It remains to show that $\psi(G_n) \ge \chi(G_n) + 2 = n+2$. Since $\chi(G_n) = n$, it follows that $\psi(G_n) \ge \chi(G_n) = n$. Suppose that $\text{maj}(G_n) = k$. First, we show that $\psi(G_n) \ge n+1$. Assume, to the contrary, that $\psi(G_n) = n$. Let $c : E(G_n) \to [k]$ be a majestic k-edge coloring of G_n such that the induced vertex coloring c' uses exactly n colors in $\mathcal{P}^*([k])$. We may assume that $c(u_1 v_1) = 1$ and $c(v_1 w_1) = 2$. Hence, $c'(u_1) = \{1\}$ and $c'(v_1) = \{1, 2\}$. Since $\psi(G_n) = n$ and no two vertices of K_n can be colored the same, there are two distinct vertices w_i and w_j, $1 \le i < j \le n$, such that $c'(w_i) = \{1\}$ and

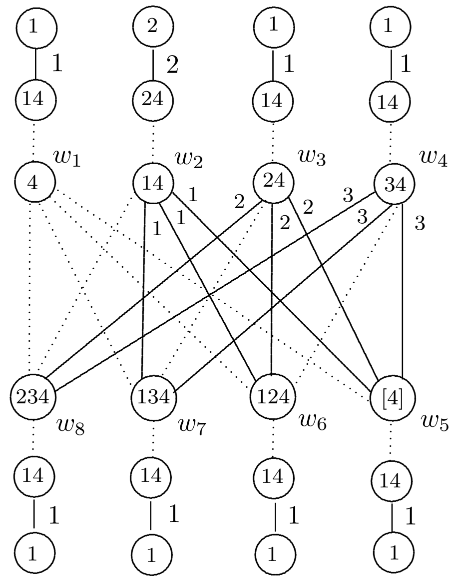


Figure 3.8: A majestic 4-edge coloring of G_8

$c'(w_j) = \{1, 2\}$. This implies that no $c'(w_i)$ $(1 \le i \le n)$ is a singleton different from $\{1\}$; for otherwise, suppose that $c'(w_t) = \{\ell\}$ for some $t, \ell \in [n]$ and $\ell \neq 1$. Since $c'(w_i) = \{1\}$, there is no appropriate color available for $w_i w_t$, a contradiction. Now either $c(v_j w_j) = 1$ or $c(v_j w_j) = 2$. If $c(v_j w_j) = 1$, then $c(u_j v_j) \notin \{1, 2\}$ and so $c(u_j v_j) = 3$. This implies that $c'(u_j) = \{3\}$. However then, the color of some vertex w_t $(1 \le t \le n)$ must be $\{3\}$, which is a contradiction. If $c(v_j w_j) = 2$, then $c(u_j v_j) \notin \{1, 2\}$, which again is a contradiction. Therefore, $\psi(G_n) \ge n + 1$.

Next, we show that $\psi(G_n) \ge n + 2$. Assume, to the contrary, that $\psi(G_n) = n + 1$. Let $c : E(G_n) \to [k]$ be a majestic k-edge coloring of G_n such that the induced vertex coloring c' uses exactly $n + 1$ colors in $\mathcal{P}^*([k])$. Again, we may assume that $c(u_1 v_1) = 1$ and $c(v_1 w_1) = 2$. Hence, $c'(u_1) = \{1\}$ and $c'(v_1) = \{1, 2\}$. Since no two vertices of K_n can be colored the same, there is at least one vertex w_i $(1 \le i \le n)$ such that $c'(w_i) = \{1\}$ or $c'(w_i) = \{1, 2\}$.

First, suppose that $c'(w_i) = \{1\}$ for some i with $1 \le i \le n$. Since $c(v_1 w_1) = 2$, it follows that $i \neq 1$. We may assume that $c'(w_2) = \{1\}$. Thus, $c(v_2 w_2) = 1$ and $c(u_2 v_2) \neq 1$. Assume that $c(u_2 v_2) = a \neq 1$. Then $c'(u_2) = \{a\}$ and $c'(v_2) = \{1, a\}$. Since $c'(w_2) = \{1\}$ and $a \neq 1$, it follows that $c'(w_i) \neq \{a\}$ for all i with $1 \le i \le n$ and so $\{a\}$ is the $(n + 1)$th color. Observe that $c'(v_2) = \{1, a\}$ must be used for some w_i where $4 \le i \le n$, say $c'(w_4) = \{1, a\}$. Thus, $c(v_4 w_4) \in \{1, a\}$, which implies that $c(v_4 u_4) = \ell \neq a$. Hence, $c'(u_4) = \{\ell\}$ and so there is a vertex w_s such that $c'(w_s) = \{\ell\}$. However then, there is no appropriate color available for $w_2 w_s$, which is a contradiction.

Thus, $c'(w_i) \neq \{1\}$ for all i with $1 \le i \le n$ and so $\{1\}$ is the $(n+1)$th color. Therefore, $c'(w_i) = \{1,2\}$ for some i with $1 \le i \le n$. Since $c(v_1w_1) = 2$ and $c'(v_1) = \{1,2\}$, it follows that $i \neq 1$. We may assume that $c'(w_2) = \{1,2\}$. Thus, $c(w_2v_2) \in \{1,2\}$ and so $c(v_2u_2) = a \notin \{1,2\}$. Hence, $c'(u_2) = \{a\}$ and $\{a\}$ must be the color of some vertex w_i. Because $c'(w_2) = \{1,2\}$ and $a \notin \{1,2\}$, it follows that $c'(w_i) \neq \{a\}$ for all i with $1 \le i \le n$, which is impossible. Therefore, $\psi(G_n) \ge n+2$ and so $\psi(G_n) = n+2$. ■

Next, we describe an infinite class of graphs G with arbitrarily large chromatic number for which $\psi(G) = \chi(G) + 3$. A *double corona* of G is obtained from G by adding two pendant edges to each vertex of G. Thus, if the order of G is n, then the order of $\mathrm{cor}(G)$ is $2n$ and the order of the double corona of G is $3n$. For each integer $n \ge 3$, let H_n be the graph constructed from the double corona of the complete graph K_n as follows. For each pair of pendant edges at a vertex of K_n, subdivide one pendant edge exactly once and the other pendant edge exactly twice. Thus, H_n has order $6n$, exactly n vertices of degree i for each $i \in \{1, n\}$ and exactly $3n$ vertices of degree 2. Let $W = \{w_1, w_2, \ldots, w_n\}$ be the vertex set of the subgraph K_n in H_n. For each integer i with $1 \le i \le n$, let (u_i, v_i, w_i) and (x_i, y_i, z_i, w_i) be the two paths of order 3 and 4, respectively, at the vertex w_i. The graph H_4 is shown in Figure 3.9.

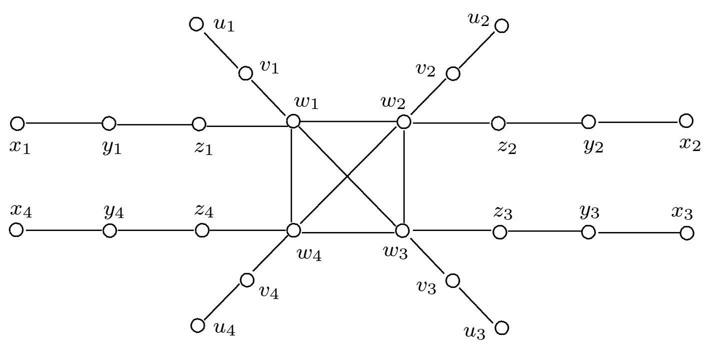


Figure 3.9: The graph H_4

Theorem 3.3.2 *For each integer $n \ge 3$,*

$$\mathrm{maj}(H_n) = \lceil \log_2 n \rceil + 1 \text{ and } \psi(H_n) = \chi(H_n) + 3.$$

Proof. Let $k = \lceil \log_2 n \rceil + 1$. Since $\omega(H_n) = n$, it follows by Corollary 2.1.5 that $\mathrm{maj}(H_n) \ge k$. It remains to construct a majestic k-edge coloring of H_n. Figure 3.10 shows a majestic 3-edge coloring of each of H_3 and H_4. Thus, we may assume that $n \ge 5$.

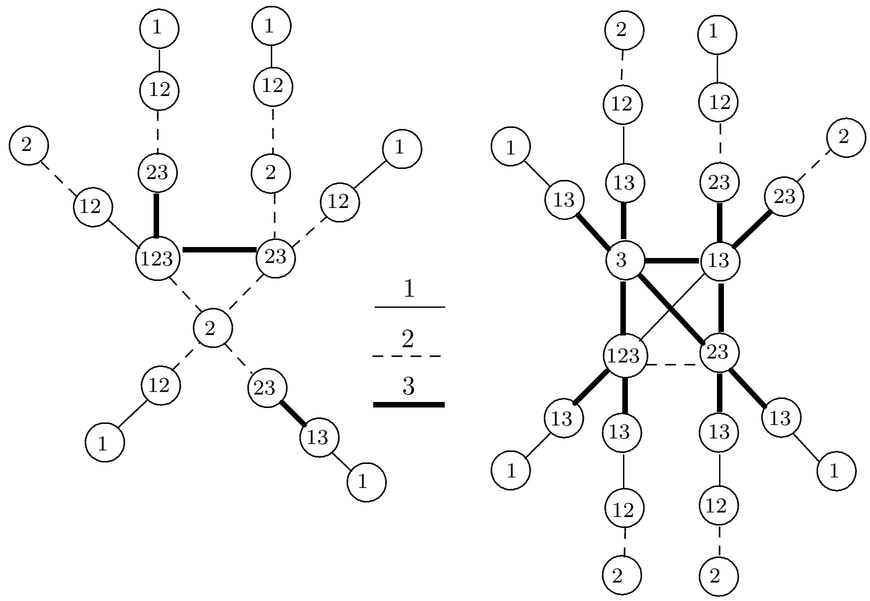


Figure 3.10: Majestic 3-edge colorings of H_3 and H_4

Let c_0 be the majestic k-edge coloring of G_n in the proof of Theorem 3.3.1. We now construct a majestic k-edge coloring $c : E(H_n) \to [k]$ of H_n from the coloring c_0 of G_n as follows:

- ⋆ $c(e) = c_0(e)$ if $e \in E(G_n)$;
- ⋆ $c(w_i z_i) = k$ for $1 \leq i \leq n$;
- ⋆ $c(z_2 y_2) = 2$ and $c(z_i y_i) = 1$ for $1 \leq i \leq n$ and $i \neq 2$;
- ⋆ $c(y_2 x_2) = 1$ and $c(y_i x_i) = 2$ for $1 \leq i \leq n$ and $i \neq 2$.

The induced vertex coloring c' then satisfies the following:

- $c'(w) = c'_0(w)$ for each $w \in V(G_n)$;
- $c'(z_2) = \{2, k\} \neq \{1, k\} = c'(w_2)$ and $c'(z_i) = \{1, k\} \neq c'(w_i)$ for $1 \leq i \leq n$ and $i \neq 2$;
- $c'(y_i) = \{1, 2\}$ for $1 \leq i \leq n$ (Since $k \geq 3$; it follows that $\{1, 2\} \neq \{2, k\}$ and $\{1, 2\} \neq \{1, k\}$ and so $c'(y_i) \neq c'(z_i)$ for $1 \leq i \leq n$);
- $c'(x_2) = \{1\}$ and $c'(x_i) = \{2\}$ for $1 \leq i \leq n$ and $i \neq 2$.

Thus, c' is proper and so c is a majestic k-edge coloring of H_n. Therefore, $\text{maj}(G_n) = k = \lceil \log_2 n \rceil + 1$. This is illustrated in Figure 3.11 for H_8.

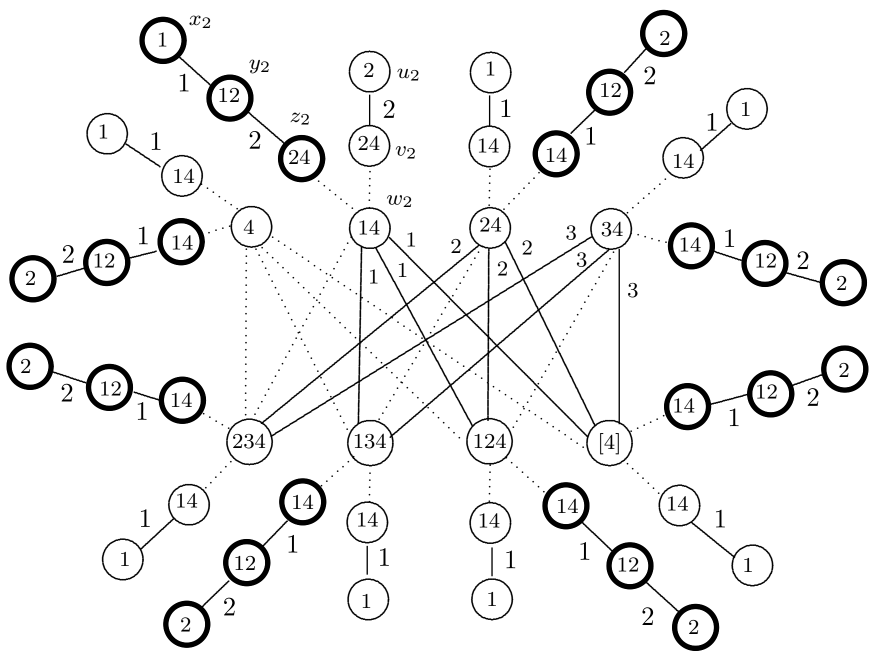


Figure 3.11: A majestic 4-edge coloring of H_8

For $k = \lceil \log_2 n \rceil + 1$, the majestic k-edge coloring of H_n constructed above uses exactly $n+3$ colors for its vertices (where $\{1\}$, $\{2\}$ and $\{1,2\}$ are the only three new colors added to the vertex colors of the vertices of K_n or $\{1,2\}$ is the only new vertex color added to the vertex color of the vertices of G_n). Therefore, $\psi(H_n) \le n+3$. It remains to show that $\psi(H_n) \ge \chi(H_n) + 3$.

Since $\chi(H_n) = n$, it follows that $\psi(H_n) \ge \chi(H_n) = n$. First, we show that $\psi(H_n) > n$. Assume, to the contrary, that $\psi(H_n) = n$. Let $c : E(H_n) \to [k]$ be a majestic k-edge coloring of H_n such that the induced vertex coloring c' uses exactly n colors in $\mathcal{P}^*([k])$. We claim that all end-vertices of H_n must be colored the same, for otherwise, we may assume that $\{1\}$ and $\{2\}$ are the colors of two end-vertices of H_n. However then, both $\{1\}$ and $\{2\}$ are used for the colors of vertices of W, which is impossible. Thus, as claimed, all end-vertices are colored the same, say $\{1\}$ is the color of each end-vertex of H_n. Thus, $c(u_1v_1) = 1$ and we may assume that $c(v_1w_1) = 2$. Hence, $c'(v_1) = \{1,2\}$ is used for some vertex of W, say $c'(w_2) = \{1,2\}$. Since $c'(u_2) = \{1\}$, it follows that $c(v_2w_2) \in \{1,2\}$, which is impossible. Therefore, $\psi(H_n) > n$.

Next, we show that $\psi(H_n) > n+1$. Assume, to the contrary, that $\psi(H_n) = n+1$. Let $c : E(H_n) \to [k]$ be a majestic k-edge coloring of H_n such that the induced vertex coloring c' uses exactly $n+1$ colors in $\mathcal{P}^*([k])$. We consider the following two situations.

- *All end-vertices of H_n are colored the same, say $\{1\}$ is the color of each end-*

vertex of H_n. We claim that $\{1\}$ cannot be used for the color of any vertex of W; for otherwise, suppose that $c'(w_1) = \{1\}$.Thus $c(v_1w_1) = 1 = c(v_1u_1)$, which is impossible. Thus, $\{1\}$ is the $(n+1)$th color. Suppose, without loss of generality, that $c(w_1v_1) = 2$ and so $c'(v_1) = \{1, 2\}$ is used for some vertex of W, say $c'(w_2) = \{1, 2\}$. Since $c(v_2w_2) \in \{1, 2\}$ and $c(v_2u_2) = 1$, either $c'(w_2) = c'(v_2)$ or $c'(v_2) = c'(u_2)$, a contradiction.

- *Two end-vertices of H_n are colored differently, say $\{1\}$ and $\{2\}$ are the colors of two end-vertices of H_n.* Since $\{1\}$ and $\{2\}$ cannot both be the colors of the vertices of W and $\psi(H_n) = n+1$, it follows that exactly one of $\{1\}$ and $\{2\}$ is the color of a vertex of W, say $c'(w_1) = \{2\}$. Thus, $c(v_1w_1) = 2$. Since $c(v_1u_1) \in \{1, 2\}$, it follows that $c'(v_1) = \{1, 2\}$ and $\{1, 2\}$ is the color of a vertex of W, say $c'(w_2) = \{1, 2\}$. Hence, $c(v_2w_2) \in \{1, 2\}$ and $c'(v_2u_2) \in \{1, 2\}$, which is impossible.

Therefore, $\psi(H_n) > n+1$. Finally, we show that $\psi(H_n) > n+2$. Assume, to the contrary, that $\psi(H_n) = n + 2$. Then there exists a majestic k-edge coloring $c : E(H_n) \to [k]$ of H_n such that the induced vertex coloring c' uses exactly $n + 2$ colors in $\mathcal{P}^*([k])$. Since all vertices of W must be colored differently by c', there are n distinct colors used for the vertices of W and there are two additional colors that are not used for the colors of vertices in W. We consider two cases, according to whether a singleton in $\mathcal{P}^*([k])$ is used for the color of a vertex in W or not.

Case 1. *A singleton in $\mathcal{P}^*([k])$ is used for the color of a vertex in W, say $c'(w_1) = \{1\}$.* Since $c(v_1w_1) = 1$, it follows that $c(u_1v_1) \neq 1$, say $c(u_1v_1) = 2$. Therefore, $c'(v_1) = \{1, 2\}$ and $c'(u_1) = \{2\}$. Necessary, no vertex in W is colored $\{2\}$ and so $\{2\}$ is the $(n + 1)$th color used by c'.

Claim 1: *The color $\{1, 2\}$ must be used for some vertex in W.*

If no vertex in W is colored $\{1, 2\}$, then $\{1, 2\}$ is the $(n+2)$th color used by c'. Hence, each end-vertex of H_n is colored by $\{1\}$ or $\{2\}$. Since $c(w_1z_1) = 1$, it follows that $c(x_1y_1) \neq 1$ and so $c(x_1y_1) = 2$. Thus, $c(y_1z_1) = \ell \notin \{1, 2\}$. However then, $c'(y_1) = \{2, \ell\}$ must be used for some vertex of W. Since $c'(w_1) = \{1\}$ and $\{1\} \cap \{2, \ell\} = \emptyset$, this is impossible. Hence, Claim 1 holds.

We may assume that $c'(w_2) = \{1, 2\}$. We now consider the colors of the edges in the path (w_1, z_1, y_1, x_1). Since $c(w_1z_1) = 1$ and $c'(w_1) = \{1\}$, it follows that $c(x_1y_1) \neq 1$.

- $\star$ If $c(x_1y_1) = 2$, then $c'(x_1) = \{2\}$. If $c(y_1z_1) = 1$, then $c'(w_1) = c'(z_1)$; while if $c(y_1z_1) = 2$, then $c'(y_1) = c'(x_1) = \{2\}$. Hence, $c(y_1z_1) = \ell \notin \{1, 2\}$. Now $c'(y_1) = \{2, \ell\}$. Since $c'(w_1) = \{1\}$ and $\{1\} \cap \{2\} = \emptyset$, it follows that $\{2\}$ cannot be used for any vertex in W. Similarly, $\{2, \ell\}$ cannot be used for any vertex of W as $\{1\} \cap \{2, \ell\} = \emptyset$. This implies that each end-vertex of G is colored by $\{2\}$. Since $c'(w_2) = \{1, 2\}$ and $c'(u_2) = \{2\}$, it follows that $c(v_2w_2) \in \{1, 2\}$ and $c(u_2v_2) = 2$. However then, either $c'(w_2) = c'(v_2)$ or $c'(v_2) = c'(u_2)$, which is a contradiction.

⋆ If $c(x_1y_1) = t \notin \{1, 2\}$, then $c'(x_1) = \{t\}$ is not used for a vertex of W and so $\{t\}$ is the $(n + 2)$th color used by c'. Let $c(z_1, y_1) = s \neq 1$. Then $c'(y_1) = \{s, t\}$ must be used for some vertex of W. Since $\{1\} \cap \{s, t\} = \emptyset$, this is impossible.

Case 2. *No singleton in* $\mathcal{P}^*([k])$ *is used for the color of any vertex in* W. We consider two subcases, according whether all end-vertices of H_n are colored the same or not.

Subcase 2.1. *All end-vertices of* H_n *are colored the same, say* $\{1\}$ *is the color of each end-vertex of* H_n. We may assume that $c(v_1w_1) = 2$ and so $c'(v_1) = \{1, 2\}$.

Claim 2: *No vertex in* W *is colored by* $\{1, i\}$ *for all* $1 \leq i \leq k$.

Assume, to the contrary, that some vertex in W is colored $\{1, i\}$ for some $i \neq 1$, say $c'(w_2) = \{1, i\}$. Thus, $c(v_2w_2) \in \{1, i\}$. Since $c'(u_2) = \{1\}$, it follows that $c(v_2w_2) = i$ and so $c'(v_2) = \{1, i\}$, a contradiction. Hence, Claim 2 holds. In particular, $\{1, 2\}$ is not used for any vertex in W and so $\{1, 2\}$ is the $(n + 2)$th color used by c'. This implies that $c'(v_i) = c'(y_i) = \{1, 2\}$ and $c(v_iw_i) = c(y_iz_i) = 2$ for $1 \leq i \leq n$. Suppose that $c(w_1z_1) = \ell \neq 1$. Since $\{1\}$ and $\{1, 2\}$ are not used for any vertex of W, it follows that $c'(z_1) \neq \{2\}$ (for otherwise, $\{2\}$ must be used for some vertex in W) and so $\ell \neq 2$. However then, $c'(z_1) = \{2, \ell\}$ must be used for some vertex of W, say $c'(w_2) = \{2, \ell\}$. Again, since $c'(z_2) \neq \{2\}$, it follows that $c(w_2z_2) = \ell$ and so $c'(w_2) = c'(z_2) = \{2, \ell\}$, a contradiction.

Subcase 2.2. *Two end-vertices of* H_n *are colored differently, say* $\{1\}$ *and* $\{2\}$ *are the colors of two end-vertices of* H_n. Since no singleton is the color of a vertex of W, it follows that $\{1\}$ and $\{2\}$ cannot be used as the colors of W.

Claim 3: *No vertex in* W *is colored by* $\{1, 2\}$.

Assume, to the contrary, that some vertex in W is colored $\{1, 2\}$, say $c'(w_1) = \{1, 2\}$. Thus, $c(v_1w_1) \in \{1, 2\}$ and $c(u_1v_1) \in \{1, 2\}$. Thus, either $c'(w_1) = c'(v_1)$ or $c'(v_1) = c'(u_1)$, a contradiction. Hence, Claim 3 holds. We may assume that $c(u_1v_1) = 1$ and $c(v_1w_1) = 3$. Hence, $c'(v_1) = \{1, 3\}$ and some vertex of W is colored $\{1, 3\}$, say $c'(w_2) = \{1, 3\}$. Since $c'(v_2)$ is not any of $\{1\}$, $\{2\}$, $\{1, 2\}$ or $\{1, 3\}$, it follows that $c'(v_2) = \{2, 3\}$

(or $c(v_2w_2) = 3$ and $c(u_2v_2) = 2$).Furthermore, $\{2, 3\}$ must be used for the color of a vertex of W. Therefore,

$$\text{both } \{1, 3\} \text{ and } \{2, 3\} \text{ must be used for the color of a vertex of } W. \tag{3.1}$$

We now consider the colors of the edges in $P = (w_2, z_2, y_2, x_2)$, where $c(w_2z_2) \in \{1, 3\}$, $c(x_2y_2) \in \{1, 2\}$ and $c(z_2y_2) = \ell \in [k]$.

- $\star$ If $\ell = 1, 2$, then $c'(y_2) = \{1, 2\}$ and so some vertex of W must be colored by $\{1, 2\}$. This is impossible by Claim 3.
- $\star$ If $\ell = 3$, then since $c'(z_2y_2) \neq \{1, 3\}$, it follows that $c'(z_2) = \{3\}$. This implies that some vertex of W must be colored by $\{3\}$, which is impossible.
- $\star$ If $\ell \geq 4$, then there are three possibilities for $(c(w_2z_2), c(z_2y_2), c(y_2x_2))$, namely

$$\text{(i) } (1, \ell, 2), \quad \text{(ii) } (3, \ell, 1) \quad \text{and} \quad \text{(iii) } (3, \ell, 2).$$

If (i) or (ii) occurs, then $\{1, \ell\}$ must be used for the color of a vertex in W. Since $\{2, 3\} \cap \{1, \ell\} = \emptyset$, this is impossible by (3.1). If (iii) occurs, then $c'(y_2) = \{2, \ell\}$ for some $\ell \geq 4$ and so $\{2, \ell\}$ must be used for some vertex in W. However, $\{1, 3\} \cap \{2, \ell\} = \emptyset$, which is impossible by (3.1). ■

3.4 Open Questions

We have now seen that there infinitely many connected graphs G satisfying each of the following:

(i) $\psi(G) = \chi(G)$;

(ii) $\psi(G) = \chi(G) + 1$;

(iii) $\psi(G) = \chi(G) + 2$;

(iv) $\psi(G) = \chi(G) + 3$.

This brings up the following related questions:

Problem 3.4.1 *Does there exist a graph G such that $\psi(G) = \chi(G) + 4$?*

Problem 3.4.2 *For a given positive integer k, does there exist a connected graph F_k such that $\psi(F_k) = \chi(F_k) + k$?*

Problem 3.4.3 *Does there exist a positive integer K such that $\psi(F) \leq \chi(F) + K$ for every connected graph F?*

We conclude with an additional question.

Problem 3.4.4 *Does there exist a connected graph G having a majestic k-edge coloring with $k > \mathrm{maj}(G)$ such that the number of vertex colors is p where $p < \psi(G)$?*

Chapter 4

Majestic 2-Tone Colorings

4.1 Introduction

For a positive integer t, a *t-tone coloring* c of a graph G is an assignment of t-element subsets of $[k]$ for some integer $k > t$ to the vertices of G such that for every two vertices u and v of G if the distance $d(u, v)$ between u and v is d, then $|c(u) \cap c(v)| < d$. Since a 1-tone coloring is a proper coloring, t-tone colorings provide a generalization of proper colorings of graphs. The *t-tone chromatic number* of G is the minimum number of colors in a t-tone coloring of G. The concept of t-tone colorings was introduced by Chartrand in 2009 and first studied as a research project by graduate students at Western Michigan University [19]. Since then, this topic has attracted the attention of several researchers and has become a popular topic of study and a t-tone coloring of a graph G has been used to describe any assignment of t-element subsets of some set $[k]$ with $k > t$ to the vertices of G so that certain conditions are met. Applying this observation to majestic colorings suggests the concept that we introduce here.

4.2 Majestic t-Tone Indices

Here, we are considering set-defined colorings. The idea of assigning a color a to an edge can be looking as assigning the color $\{a\}$ to the edge. Looking at an edge coloring in this way gives rise to a natural generalization of majestic colorings.

Recall, for a positive integer k, that $\mathcal{P}([k])$ denotes the power set of the set $[k]$ and $\mathcal{P}^*([k]) = \mathcal{P}([k]) - \{\emptyset\}$ denotes the set of nonempty subsets of $[k]$. For each integer t with $1 \leq t < k$, let $\mathcal{P}_t([k])$ denote the set of t-element subsets of $\mathcal{P}([k])$; consequently, $|\mathcal{P}_t([k])| = \binom{k}{t}$.

For an unrestricted edge coloring $c : E(G) \to \mathcal{P}_t([k])$ of a graph G, the vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ is defined by

$$c'(v) = \bigcup_{e \in E_v} c(e),$$

where again E_v denotes the set of edges of G incident with the vertex v. So, instead of assigning a 1-element set to each edge of G, a t-element set is assigned each edge of G. The color of a vertex v is obtained by taking the union of the t-element sets assigned to the edges incident with v.

If c' is a proper vertex coloring of G, then c is called a *majestic t-tone k-coloring* of G. An edge coloring of G is a *majestic t-tone coloring* if it is a majestic t-tone k-coloring for some integer $k > t$. For a fixed positive integer t, the minimum positive integer k for which a graph G has a majestic t-tone k-coloring is called the *majestic t-tone index* $\text{maj}_t(G)$ of G. In particular, a majestic 1-tone k-coloring is a majestic k-coloring and the majestic 1-tone index of a graph G is the majestic index of G.

As an illustration, we determine $\text{maj}_2(H)$ for the graph $H = K_4 - e$. We claim that $\text{maj}_2(H) = 4$. First, observe that the 2-tone 4-coloring of H shown in Figure 4.1 (where we write $\{a, b, c, \ldots\}$ as $abc\ldots$, for simplicity) is majestic and so $\text{maj}_2(H) \le 4$. (In fact, this is a 2-tone rainbow 4-coloring.)

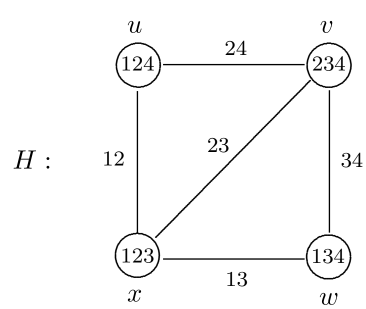


Figure 4.1: A 2-tone majestic edge coloring of a graph H

It remains to show that $\text{maj}_2(H) \ne 3$. Assume, to the contrary, that $\text{maj}_2(H) = 3$. Then there is a majestic 2-tone 3-coloring $c : E(H) \to \mathcal{P}_2([3])$ of H, where the color assigned to each edge of H is one of $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$. Thus, the color of each vertex of H is either one of these three sets or the set $[3]$. Since v and x are adjacent, at least one of $c'(v)$ and $c'(x)$ is a 2-element set, say $c'(v) = \{1, 2\}$. Thus, $c(uv) = c(xv) = c(wv) = \{1, 2\}$. Since $\{1, 2\} \subseteq c'(u)$ and $\{1, 2\} \subseteq c'(x)$ but $c'(u) \ne c'(v)$ and $c'(x) \ne c'(v)$, it follows that $c'(u) = c'(x) = [3]$, which is impossible since $ux \in E(H)$. Hence, $\text{maj}_2(H) = 4$.

We now present bounds for the majestic t-tone index of a graph in terms of t and its majestic index.

Proposition 4.2.1 *For a connected graph G of order at least 3 and an integer $t \geq 2$,*

$$t+1 \leq \mathrm{maj}_t(G) \leq \mathrm{maj}(G) + (t-1).$$

Proof. Since no majestic t-tone coloring can assign the same color (t-element subset) to each edge of G, it follows that $\mathrm{maj}_t(G) \geq t+1$. It remains to show that $\mathrm{maj}_t(G) \leq \mathrm{maj}(G)+(t-1)$. Suppose that $\mathrm{maj}(G) = k$. Let c_0 be a majestic k-edge coloring of G and let $T = \{k+1, k+2, \ldots, k+t-1\}$. Define an edge coloring $c : E(G) \to \mathcal{P}_t([k+t-1])$ of G by $c(e) = \{c_0(e)\} \cup T$ for each $e \in E(G)$. Thus, $|c(e)| = t$ for all $e \in E(G)$. Furthermore, for each $v \in V(G)$, the induced vertex coloring $c' : V(G) \to \mathcal{P}^*([k+t-1])$ satisfies $c'(v) = c'_0(v) \cup T$. Since $c'_0(u) \neq c'_0(v)$ for every two adjacent vertices u and v of G and $c'_0(x) \cap T = \emptyset$ for each $x \in V(G)$, it follows that $c'(u) \neq c'(v)$. Thus, c' is a proper vertex coloring and so c is a majestic t-tone $(k+t-1)$-coloring of G. Therefore, $\mathrm{maj}_t(G) \leq \mathrm{maj}(G) + (t-1)$. ■

We have seen that if G is a connected graph of order $n \geq 3$, then $\mathrm{maj}(G) \leq n+1$. Thus, the following is a consequence of Proposition 4.2.1.

Corollary 4.2.2 *If G is a connected graph of $n \geq 3$, then*

$$\mathrm{maj}_t(G) \leq n+t$$

for all integers $t \geq 2$.

The upper bound in Proposition 4.2.1 states that

$$\mathrm{maj}_t(G) \leq \mathrm{maj}(G) + (t-1).$$

This observation gives rise to the following result.

Proposition 4.2.3 *If G is a connected graph of order at least 3 and s and t are positive integers with $s \leq t$, then*

$$\mathrm{maj}_t(G) \leq \mathrm{maj}_s(G) + (t-s)$$

Proof. Suppose that $\mathrm{maj}_s(G) = k$. Let c_s be a majestic s-tone k-edge coloring of G, let c'_s be the resulting induced vertex coloring and let $T = \{k+1, k+2, \ldots, k+t-s\}$. Define an edge coloring $c : E(G) \to \mathcal{P}_t([k+t-s])$ of G by $c(e) = \{c_s(e)\} \cup T$ for each $e \in E(G)$. Thus, $|c(e)| = s+t-s = t$ for all $e \in E(G)$. Furthermore, for each $v \in V(G)$, the induced vertex coloring $c' : V(G) \to \mathcal{P}^*([k+t-s])$ satisfies $c'(v) = c'_s(v) \cup T$. Since $c'_s(u) \neq c'_s(v)$ for every two adjacent vertices u and v of G and $c'_s(x) \cap T = \emptyset$ for each $x \in V(G)$, it follows that $c'(u) \neq c'(v)$. Thus, c' is a proper vertex coloring and so c is a majestic t-tone $(k+t-s)$-coloring of G. Therefore,

$$\mathrm{maj}_t(G) \leq k+t-s = \mathrm{maj}_s(G) + (t-s),$$

as desired. ■

In this chapter, our emphasis is on majestic 2-tone colorings of graphs. The following results on majestic 2-tone 3-colorings of graphs will be useful to us.

Lemma 4.2.4 *Let G be a connected graph with $\mathrm{maj}_2(G) = 3$ and let $c : E(G) \to \mathcal{P}_2([3])$ be a majestic 2-tone 3-coloring of G. If u and v are two distinct vertices of G such that $|c'(u)| = |c'(v)| = 2$, then there is no $u-v$ path of odd length in G. In particular, if u and v are end-vertices of G, then there is no $u-v$ path of odd length in G.*

Proof. Assume, to the contrary, that there are paths of odd length in G connecting two vertices whose colors are 2-element subsets of [3]. Among all such paths, let $P = (v_0, v_1, \ldots, v_s)$ be one of minimum length, where then $|c'(v_0)| = |c'(v_s)| = 2$ and $s \geq 3$ is odd. Since $|c'(v_0)| = |c'(v_s)| = 2$, each edge incident with v_0 is colored by $c'(v_0)$ and each edge incident with v_s is colored $c'(v_s)$. In particular, $c(v_0v_1) = c'(v_0)$ and $c(v_{s-1}v_s) = c'(v_s)$. Since c' is proper, it follows that $c'(v_0) \neq c'(v_1)$ and $c'(v_{s-1}) \neq c'(v_s)$. On the other hand, $c'(v_0) = c(v_0v_1) \subseteq c'(v_1)$ and $c'(v_s) = c(v_{s-1}v_s) \subseteq c'(v_{s-1})$, implying that $c'(v_1) = c'(v_{s-1}) = [3]$. Hence, $s \geq 5$ and $|c'(v_2)| = |c'(v_{s-2})| = 2$. However then, v_2 and v_{s-2} are connected by a path of odd length $s-4$, contradicting the defining property of v_0 and v_s. In particular, if u and v are end-vertices of G, then $|c'(u)| = |c'(v)| = 2$ and so there is no $u-v$ path of odd length in G. ■

Proposition 4.2.5 *If G is a connected graph with $\mathrm{maj}_2(G) = 3$, then G is bipartite.*

Proof. Assume, to the contrary, that there exists a connected graph G with $\mathrm{maj}_2(G) = 3$ that is not bipartite. Then G contains a cycle C_a of odd order $a \geq 3$. Let $c : E(G) \to \mathcal{P}_2([3])$ be a majestic 2-tone 3-coloring of G and let uv be an edge of C_a. Since $|c(uv)| = 2$, either $c'(u) = [3]$ or $c'(v) = [3]$, say the latter. Thus, $|c'(u)| = 2$. Let w be a neighbor of v on C_a distinct from u. Then $|c'(w)| = 2$. Hence, there is a $u-w$ path on C_a of length $a-2$. Since $a-2$ is odd and $|c'(u)| = |c'(w)| = 2$, this contradicts Lemma 4.2.4. ■

4.3 Majestic 2-Tone Indices of Paths and Cycles

In this section, we determine the majestic 2-tone indices of two well-known classes of graphs, namely paths and cycles. In order to do this, we first provide some preliminary information. The following corollary is an immediate consequence of Proposition 4.2.1.

Corollary 4.3.1 *If G is a connected graph with $\mathrm{maj}(G) = 2$, then*

$$\mathrm{maj}_t(G) = t + 1$$

for every positive integer t.

The converse of Corollary 4.3.1 is not true in general as we will soon see.

Majestic indices of paths and cycles were determined in Chapter 3, which we review below. For each integer $n \geq 3$,

$$\mathrm{maj}(P_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even} \end{cases} \tag{4.1}$$

$$\mathrm{maj}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{if } n \not\equiv 0 \pmod 4. \end{cases} \tag{4.2}$$

We are now prepared to determine $\mathrm{maj}_2(P_n)$ and $\mathrm{maj}_2(C_n)$ for each integer $n \geq 3$. The following result also shows that the converse of Proposition 4.2.5 is false.

Proposition 4.3.2 *For an integer $n \geq 3$,*

$$\mathrm{maj}_2(P_n) = \begin{cases} 3 & \textit{if } n \textit{ is odd} \\ 4 & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 3$. If n is odd, then $\mathrm{maj}(P_n) = 2$ by (4.1) and so $\mathrm{maj}_2(P_n) = 3$ by Corollary 4.3.1. It remains to show that if $n \geq 4$ is even, then $\mathrm{maj}_2(P_n) = 4$.

First, we show that $\mathrm{maj}_2(P_n) \geq 4$. Assume, to the contrary, that $\mathrm{maj}_2(P_n) < 4$ for some even integer $n \geq 4$. Thus, $\mathrm{maj}_2(P_n) = 3$ by Proposition 4.2.1. Hence, there is a majestic 2-tone 3-coloring $c : E(P_n) \to \mathcal{P}_2([3])$ of P_n and the possible colors of the induced vertex coloring c' are $\{1, 2\}, \{1, 3\}, \{2, 3\}$, $\{1, 2, 3\}$. Necessarily, $|c'(v_1)| = |c'(v_n)| = 2$. Since n is even, P_n is a $v_1 - v_n$ path of odd length, which is impossible by Lemma 4.2.4. Thus, $\mathrm{maj}_2(P_n) \geq 4$.

It remains only to show that P_n has a majestic 2-tone 4-coloring. Define the edge coloring $c : E(P_n) \to \mathcal{P}_2([4])$ of P_n by

$$c(v_i v_{i+1}) = \{j, j+1\} \text{ if } j \equiv i \pmod 4 \text{ and } j \in [4].$$

Thus, the induced vertex coloring c' is given by

$$c'(v_i) = \begin{cases} \{1,2\} & \text{if } i = 1 \\ \{j-1, j, j+1\} & \text{if } 2 \le i \le n-1,\ j \equiv i \ (\text{mod } 4) \\ & \ \text{where } j \in [4] \\ \{a, a+1\} & \text{if } i = n,\ a \equiv n-1 \ (\text{mod } 4),\ a \in [4]. \end{cases}$$

Since c' is proper, it follows that c is a majestic 2-tone 4-coloring. Therefore, $\text{maj}_2(P_n) = 4$ when $n \ge 4$ is even. ■

Figure 4.2 illustrates the colorings of P_n described in the proof of Proposition 4.3.2 for $n = 4, 6, 8, 10$, where again we write $\{a, b, c, \ldots\} = abc\ldots$ for simplification.

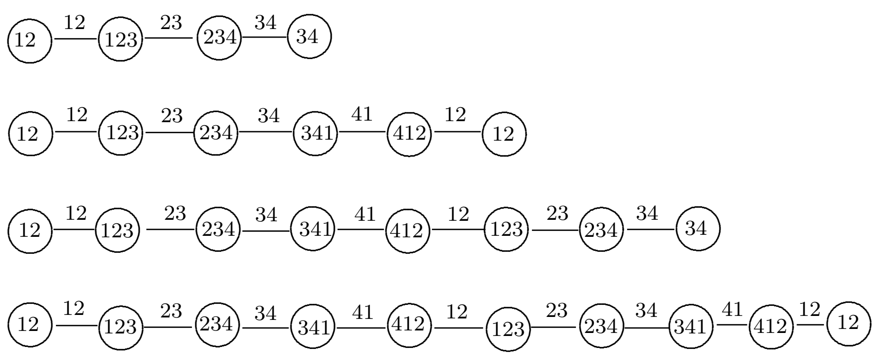


Figure 4.2: Majestic 2-tone 4-colorings of P_n for $n = 4, 6, 8, 10$

Next, we determine the majestic 2-tone indices of cycles.

Proposition 4.3.3 *For an integer $n \ge 3$,*

$$\text{maj}_2(C_n) = \begin{cases} 3 & \textit{if } n \textit{ is even} \\ 4 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a cycle of order $n \ge 3$. We consider two cases, according to whether n is even or n is odd.

Case 1. *$n \geq 4$ is even.* Thus, either $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. If $n \equiv 0 \pmod 4$, then $\mathrm{maj}(C_n) = 2$ by (4.2) and so $\mathrm{maj}_2(C_n) = 3$ by Corollary 4.3.1. Hence, we may assume that $n \equiv 2 \pmod 4$. Define an edge coloring $c : E(C_n) \to \mathcal{P}_2[3])$ of C_n by

$$c(e) = \begin{cases} \{1,2\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 1\,(\text{mod } 4),\ 1 \leq i \leq n-3 \\ \{1,3\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 3\,(\text{mod } 4),\ 1 \leq i \leq n-3 \\ \{2,3\} & \text{if } e = v_{n-2}v_{n-1} \text{ or } e = v_{n-1}v_n. \end{cases}$$

Then the induced vertex coloring $c'(v_i) = \cup_{e \in E_{v_i}} c(e)$, $1 \leq i \leq n$, is given by

$$c'(v_i) = \begin{cases} \{1,2\} & \text{if } i \equiv 1 \pmod 4 \text{ and } 1 \leq i \leq n-3 \\ \{1,3\} & \text{if } i \equiv 3 \pmod 4 \text{ and } 1 \leq i \leq n-3 \\ \{2,3\} & \text{if } i = n-1 \\ \{1,2,3\} & \text{if } i \text{ is even and } 2 \leq i \leq n. \end{cases}$$

Since c' is proper, c is a majestic 2-tone coloring of C_n and so $\mathrm{maj}_2(C_n) \leq 3$. It then follows by Proposition 4.2.1 that $\mathrm{maj}_2(C_n) = 3$ if $n \equiv 2 \pmod 4$. Therefore, $\mathrm{maj}_2(C_n) = 3$ when n is even.

Case 2. *$n \geq 3$ is odd.* Since C_n is an odd cycle, it follows by Proposition 4.2.5 that $\mathrm{maj}_2(C_n) \geq 4$. It remains to show that C_n has a majestic 2-tone 4-coloring. Define the edge coloring $c : E(C_n) \to \mathcal{P}_2([4])$ of C_n as follows. For $1 \leq i \leq n$, let

$$c(v_i v_{i+1}) = \{j, j+1\} \text{ if } j \equiv i \pmod 4 \text{ and } j \in [4].$$

For $2 \leq i \leq n$, it follows that

$$c'(v_i) = \{j-1, j, j+1\} \text{ where } j \equiv i \pmod 4 \text{ and } j \in [4].$$

Here, $|c'(v_1)|$ is either 2 or 4 and $|c'(v_i)| = 3$ for $2 \leq i \leq n$. Since $c'(v_i) \neq c'(v_{i+1})$ for $1 \leq i \leq n$, it follows that c' is proper and so c is a majestic 2-tone 4-coloring. Therefore, $\mathrm{maj}_2(C_n) = 4$ when $n \geq 4$ is odd. ■

Figure 4.3 illustrates the colorings of C_n described in the proof of Proposition 4.3.3 for $n = 5, 6, 7$, where we write $\{a, b, c, \ldots\} = abc\ldots$, for simplification.

By (4.2), if $n \geq 4$ and $n \equiv 2 \pmod 4$, then $\mathrm{maj}(C_n) = 3$. Since $\mathrm{maj}_2(C_n) = 3$ if n is even by Proposition 4.3.3, it follows that $\mathrm{maj}_t(C_n) = t+1$ where $t = 2$ but $\mathrm{maj}(C_n) \neq 2$. Thus, there is an infinite class of graphs G for which $\mathrm{maj}(G) \neq 2$ but $\mathrm{maj}_t(G) = t+1$

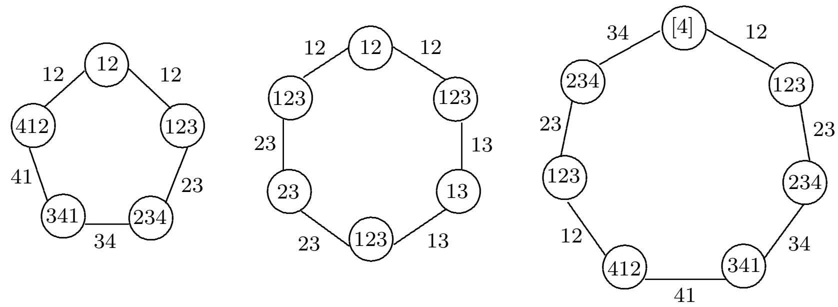


Figure 4.3: Majestic 2-tone colorings of C_n for $n = 5, 6, 7$

for some integer $t \geq 2$. Therefore, the converse of Corollary 4.3.1 is not true as we mentioned earlier.

We have now seen that $\mathrm{maj}_2(P_n) = 3$ when $n \geq 3$ is odd by Proposition 4.3.2 and $\mathrm{maj}_2(C_n) = 3$ when $n \geq 4$ and $n \equiv 0 \pmod 4$ by Proposition 4.3.3. There two classes of bipartite graphs therefore have the property that the distance between every two antipodal vertices is even. Consequently, if v is a peripheral vertex of each such graph G and u and v are antipodal vertices of G, then $d(u, v)$ is even and $\mathrm{maj}_2(G) = 3$. If "antipodal" is replaced by "boundary", then we have the following more general result.

Proposition 4.3.4 *If a connected bipartite graph G of order 3 or more has a peripheral vertex v such that the distance from v to every boundary vertex of v is even, then* $\mathrm{maj}(G) = 3$.

Proof. For each integer i with $0 \leq i \leq e(v)$, let $V_i = \{w \in V(T) : d(v, w) = i\}$. Furthermore, let $[V_i, V_{i+1}]$ denote the set of edges of G joining a vertex of V_i and a vertex in V_{i+1} for $0 \leq i \leq e(v) - 1$. For every vertex $u \in V_i$ such that i is odd and $1 \leq i \leq e(v) - 1$, there is a vertex $w \in V_{i+1}$ with $uw \in E(G)$ since u is not a boundary vertex of v. Define the edge coloring $c : E(G) \to \mathcal{P}_2([3])$ by

$$c(e) = \begin{cases} \{1, 2\} & \text{if } e \in [V_i, V_{i+1}] \text{ where } i \equiv 0, 3 \pmod 4 \text{ and } 0 \leq i \leq e(v) - 1 \\ \{1, 3\} & \text{if } e \in [V_i, V_{i+1}] \text{ where } i \equiv 1, 2 \pmod 4 \text{ and } 1 \leq i \leq e(v) - 1. \end{cases}$$

Then the induced vertex coloring c' is given by

$$c'(v_i) = \begin{cases} \{1, 2\} & \text{if } i \equiv 0 \pmod 4 \text{ and } 0 \leq i \leq e(v) \\ \{1, 3\} & \text{if } i \equiv 2 \pmod 4 \text{ and } 2 \leq i \leq e(v) \\ \{1, 2, 3\} & \text{if } i \text{ is odd and } 1 \leq i \leq e(v) \end{cases}$$

Since c' is proper, c is a majestic 2-tone 3-coloring. Therefore, $\text{maj}_2(G) = 3$. ■

The converse of Proposition 4.3.4 is not true in general, however. For example, $\text{maj}_2(C_n) = 3$ when $n \geq 4$ and $n \equiv 2 \pmod 4$. Every vertex v of C_n is a peripheral vertex of C_n and the only boundary vertex u of v is the antipodal vertex of v. Since $d(u, v)$ is odd, there is no peripheral vertex of C_n that satisfies the conditions in Proposition 4.3.4.

4.4 Majestic 2-Tone Indices of the Graphs $G \vee K_1$

For two vertex-disjoint graphs F and H, the graph $F \vee H$ denotes the *join* of F and H. In particular, the graph $G \vee K_1$ is obtained by adding a new vertex to G and joining it to each vertex of G. In this section, we study majestic 2-tone colorings and the majestic 2-tone indices of $G \vee K_1$ for various connected graphs G. The graph $W_n = C_n \vee K_1$ is the *wheel* of order $n + 1$ and $F_n = P_n \vee K_1$ is the *fan* of order $n + 1$.

Proposition 4.4.1 *For each integer $n \geq 3$, $\text{maj}_2(W_n) = 4$.*

Proof. Let $W_n = C_n \vee K_1$, where $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $V(K_1) = \{v_0\}$. Since W_n is a not bipartite graph, $\text{maj}_2(W_n) \geq 4$ by Proposition 4.2.5. To verify that $\text{maj}_2(W_n) \leq 4$, we need only show that W_n has a majestic 2-tone 4-edge coloring. For each even integer $n \geq 4$, define the edge coloring $c : E(W_n) \to \mathcal{P}_2([4])$ by

$$c(e) = \begin{cases} \{1,2\} & \text{if } e = v_0v_i \text{ where } i \text{ is odd and } 1 \leq i \leq n-1 \\ \{1,3\} & \text{if } e = v_0v_i \text{ where } i \text{ is even and } 1 \leq i \leq n \\ \{1,4\} & \text{otherwise.} \end{cases}$$

Then the induced vertex coloring c' is given by

$$c'(v_i) = \begin{cases} \{1,2,3\} & \text{if } i = 0 \\ \{1,2,4\} & \text{if } i \text{ is odd and } 1 \leq i \leq n-1 \\ \{1,3,4\} & \text{if } i \text{ is even and } 2 \leq i \leq n. \end{cases}$$

For each odd integer $n \geq 3$, define the edge coloring $c : E(W_n) \to \mathcal{P}_2([4])$ by

$$c(e) = \begin{cases} \{1,2\} & \text{if } e = v_0v_i \text{ where } i \text{ is odd and } 1 \leq i \leq n-2 \\ \{1,3\} & \text{if } e = v_0v_i \text{ where } i \text{ is even and } 2 \leq i \leq n-1 \\ \{2,3\} & \text{if } e = v_0v_n \\ \{1,4\} & \text{otherwise.} \end{cases}$$

Then the induced vertex coloring c' is given by

$$c'(v_i) = \begin{cases} \{1,2,3\} & \text{if } i = 0 \\ \{1,2,4\} & \text{if } i \text{ is odd and } 1 \le i \le n-2 \\ \{1,3,4\} & \text{if } i \text{ is even and } 2 \le i \le n-1 \\ [4] & \text{if } i = n. \end{cases}$$

In each case, the induced vertex coloring c' is proper and so the edge coloring c is majestic. Therefore, $\text{maj}_2(W_n) = 4$. ■

Majestic 2-tone 4-edge colorings of W_n described in the proof of Proposition 4.4.1 are illustrated in Figure 4.4 for $n = 3, 4, 5$.

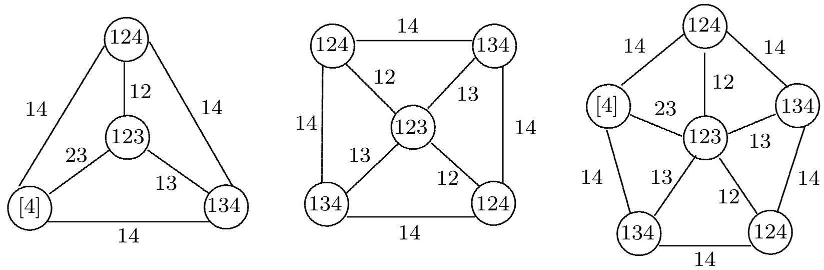


Figure 4.4: Majestic 2-tone 4-edge colorings of W_n for $n = 3, 4, 5$

Proposition 4.4.2 *For each integer $n \ge 2$, $\text{maj}_2(F_n) = 4$.*

Proof. If $n = 2$, then $F_2 = P_2 \vee K_1 = C_3$ and so $\text{maj}_2(F_2) = \text{maj}_2(C_3) = 4$. Thus, we may assume that $n \ge 3$. Let $F_n = P_n \vee K_1$, where $P_n = (v_1, v_2, \ldots, v_n)$ and $V(K_1) = \{v_0\}$. Since F_n is not bipartite, $\text{maj}_2(W_n) \ge 4$ by Proposition 4.2.5. To see that $\text{maj}_2(F_n) = 4$, we show that F_n has a majestic 2-tone 4-edge coloring. Define the edge coloring $c : E(F_n) \to \mathcal{P}_2([4])$ by

$$c(e) = \begin{cases} \{1,2\} & \text{if } e = v_0v_i \text{ where } i \text{ is odd and } 1 \le i \le n \\ \{1,3\} & \text{if } e = v_0v_i \text{ where } i \text{ is even and } 2 \le i \le n \\ \{1,4\} & \text{otherwise.} \end{cases}$$

Then the induced vertex coloring c' is given by

$$c'(v_i) = \begin{cases} \{1,2,3\} & \text{if } i = 0 \\ \{1,2,4\} & \text{if } i \text{ is odd and } 1 \le i \le n \\ \{1,3,4\} & \text{if } i \text{ is even and } 2 \le i \le n. \end{cases}$$

Since c' is proper, the edge coloring c is majestic. Therefore, $\text{maj}_2(F_n) = 4$. ■

The majestic 2-tone 4-edge coloring of F_n described in the proof of Proposition 4.4.2 is illustrated in Figure 4.5 for $n = 5$.

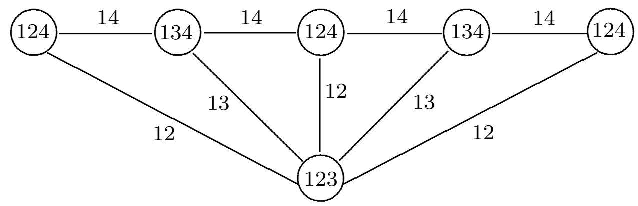


Figure 4.5: A majestic 2-tone 4-edge colorings of F_5

By Propositions 4.3.2, 4.3.3, 4.4.1 and 4.4.2, if $n \geq 3$ is odd, then $\text{maj}_2(C_n \vee K_1) = \text{maj}_2(C_n)$ and $\text{maj}_2(P_n \vee K_1) = \text{maj}_2(P_n)+1$; while if $n \geq 4$ is even, then $\text{maj}_2(P_n \vee K_1) = \text{maj}_2(P_n)$ and $\text{maj}_2(C_n \vee K_1) = \text{maj}_2(C_n) + 1$. Therefore, if G is a path or a cycle, then $\text{maj}_2(G) \leq \text{maj}_2(G \vee K_1) \leq \text{maj}_2(G) + 1$. This observation gives rise to the following question:

For which connected graphs G of order at least 3*, is*

$$\text{maj}_2(G \vee K_1) \leq \text{maj}_2(G) + 1?$$

We now answer this question by showing that this inequality holds for *all* connected graphs G of order at least 3.

Theorem 4.4.3 *For a connected graph G of order* 3 *or more,*

$$\text{maj}_2(G \vee K_1) \leq \text{maj}_2(G) + 1.$$

Proof. Let $H = G \vee K_1$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_1) = \{v\}$. Suppose that $\text{maj}_2(G) = k$ and let $c : E(G) \to \mathcal{P}_2([k])$ be a majestic 2-tone k-edge coloring of G.

First, we claim that for each integer $i \in [k]$, there is an edge e of G such that $i \in c(e)$. If this were not the case, then there is $i_0 \in [k]$ such that i_0 is not assigned to any edge of H. However then, c is a majestic 2-tone $(k-1)$-edge coloring of G using colors in $[k] - \{i_0\}$, which is a contradiction. Therefore, $c'(v_1) \cup c'(v_2) \cup \cdots \cup c'(v_n) = [k]$. We may assume, without loss of generality, that the vertices $v_1, v_2, \ldots, v_n$ are labeled so that

$$|c'(v_1)| = \min\{|c'(v_i)| : 1 \leq i \leq n\},$$

where then $|c'(v_1)| < k$ and where $k \notin c'(v_1)$ and $k \in c'(v_2)$. Let $a_1 \in c'(v_1)$ and $a_i, b_i \in c'(v_i)$ for $2 \leq i \leq n$ where $a_2 = k$ and $a_i \neq b_i$. Now, let

$$S = \{a_1, a_2, \ldots, a_n\} \cup \{b_2, b_3, \ldots, b_n\},$$

where then $k \in S$. Define the edge coloring $c_H : E(H) \to \mathcal{P}_2([k+1])$ of H by

$$c_H(e) = \begin{cases} c(e) & \text{if } e \in E(G) \\ \{a_1, k+1\} & \text{if } e = vv_1 \\ \{a_i, b_i\} & \text{if } e = vv_i \text{ where } 2 \le i \le n. \end{cases}$$

Then the colors of vertices of G induced by c_H are given by

$$c'_H(v_i) = \begin{cases} c'(v_1) \cup \{k+1\} & \text{if } i = 1 \\ c'(v_i) & \text{if } 2 \le i \le n \\ S \cup \{k+1\} & \text{if } v_i = v. \end{cases}$$

Since $k, k+1 \in c'_H(v)$, $k \notin c'(v_1)$ and $k+1 \notin c'_H(v_i)$ for $2 \le i \le n$, it follows that $c'_H(v) \ne c'_H(v_i)$ for $1 \le i \le n$. Because $k+1 \in c'_H(v_1)$ and $k+1 \notin c'_H(v_i)$ for $2 \le i \le n$, we have $c'_H(v_1) \ne c'_H(v_i)$. Since $c'_H(v_i) = c'(v_i)$ for $2 \le i \le n$ and c is a majestic 2-tone k-edge coloring of G, it follows that if v_i and v_j are adjacent vertices of G where $2 \le i \le j \le n$ and $i \ne j$, then $c'_H(v_i) = c'(v_i) \ne c'(v_j) = c'_H(v_j)$. Hence, c'_H is proper and so c_H is a majestic 2-tone $(k+1)$-edge coloring of H. Therefore, $\mathrm{maj}_2(H) \le \mathrm{maj}_2(G)+1$. ■

The cases where Proposition 4.4.1 concerns wheels constructed from even cycles and Proposition 4.4.2 concerns fans constructed from paths of odd order are special cases of a more general class of connected graphs G for which $\mathrm{maj}_2(G \vee K_1) = \mathrm{maj}_2(G) + 1$.

Proposition 4.4.4 *If G is a connected bipartite graph, then* $\mathrm{maj}_2(G \vee K_1) = 4$. *In particular,* $\mathrm{maj}_2(T \vee K_1) = 4$ *for every nontrivial tree T.*

Proof. Let G be a connected bipartite graph with partite sets U and W and let $G \vee K_1$ be constructed from G by adding the new vertex v and joining v to each vertex of G. Since $G \vee K_1$ contains a triangle, $G \vee K_1$ is not bipartite and so $\mathrm{maj}_2(G \vee K_1) \ge 4$ by Proposition 4.2.5. Next, define the 2-tone edge-coloring $c : E(G \vee K_1) \to [4]$ of $G \vee K_1$ by

$$c(e) = \begin{cases} \{1,2\} & \text{if } e \in E(G) \\ \{1,3\} & \text{if } e = uv \text{ where } u \in U \\ \{1,4\} & \text{if } e = wv \text{ where } w \in W. \end{cases}$$

Thus, the induced vertex coloring c' is given by

$$c(x) = \begin{cases} \{1, 2, 3\} & \text{if } x \in U \\ \{1, 2, 4\} & \text{if } x \in W \\ \{1, 3, 4\} & \text{if } x = v. \end{cases}$$

Since c' is proper, it follows that c is a majestic 2-tone 4-edge coloring of G. Therefore, $\text{maj}_2(G \vee K_1) = 4$. ■

Next, we investigate the following question:

If G is a connected graph of order at least 3, then under what conditions does

$$\text{maj}_2(G) = \text{maj}_2(G \vee K_1)?$$

Since $G \vee K_1$ is not bipartite for each nontrivial connected graph G, it follows by Proposition 4.2.5 that $\text{maj}_2(G \vee K_1) \geq 4$. Thus, if $\text{maj}_2(G) = 3$, then $\text{maj}_2(G) \neq \text{maj}_2(G \vee K_1)$. Therefore, we consider connected graphs G for which $\text{maj}_2(G) \geq 4$. In order to answer this question, we first provide information about majestic 2-tone colorings of graphs in general.

Proposition 4.4.5 *Let G be a connected graph with $\text{maj}_2(G) = k$. If c is a majestic 2-tone k-coloring with the induced vertex coloring c', then*

$$\bigcup_{v \in V(G)} c'(v) = [k].$$

Proof. First, we claim that for each integer $i \in [k]$, there is $e \in E(G)$ such that $i \in c(e)$. If this were not the case, then there is $i_0 \in [k]$ such that i_0 is not assigned to any edge of H. However then, c is a majestic 2-tone $(k-1)$-edge coloring of G using colors in $[k] - \{i_0\}$, which is a contradiction. Therefore, for each color $i \in [k]$, there are at least two vertices x_i and y_i of G such that $i \in c'(x_i) \cap c'(y_i)$ and so $\bigcup_{v \in V(G)} c'(v) = [k]$. ■

Theorem 4.4.6 *Let G be a connected graph with $\text{maj}_2(G) = k \geq 3$. For every majestic 2-tone k-coloring c with the induced vertex coloring c', it is possible to select k vertices $v_1, v_2, \ldots, v_k$ of G that satisfies the following two properties:*

Property 1: $i \in c'(v_i)$ for $1 \leq i \leq k$ and

Property 2: $|\{v_{i_1}, v_{i_2}, v_{i_3}\}| \geq 2$ for every three distinct integers $i_1, i_2, i_3 \in \{1, 2, \ldots, k\}$.

Proof. Assume, to the contrary, that there is a connected graph G with $\mathrm{maj}_2(G) = k$ for which there is a majestic 2-tone k-coloring c such that it is impossible to select any k vertices of G that satisfy (1) and (2). Since $|c(e)| = 2$ for each edge e of G and c' is proper, there is a maximum integer $\ell \geq 3$ for which we can select ℓ vertices, say $v_1, v_2, \ldots, v_\ell$, such that $i \in c'(v_i)$ for $1 \leq i \leq \ell$ and $|\{v_{i_1}, v_{i_2}, v_{i_3}\}| \geq 2$ for every three distinct integers $i_1, i_2, i_3 \in \{1, 2, \ldots, \ell\}$. Let s be the sequence of the ℓ selected vertices (not necessarily distinct); that is,

$$s:\ v_1, v_2, \ldots, v_\ell. \tag{4.3}$$

By the assumption of this coloring c, it follows that $3 \leq \ell < k$ and so $\ell + 1 \in [k]$. By Lemma 4.4.5, there is $v \in V(G)$ such that $\ell + 1 \in c'(v)$.

> **Claim 1:** *If $\ell + 1 \in c'(v)$, then there are integers i and j with $1 \leq i < j < \ell$ such that $v = v_i = v_j$ or v occurs exactly twice in the sequence* (4.3)*.*

If Claim 1 is false, then we can select a vertex $v_{\ell+1} = v$ such that v_1, v_2, $\ldots$, $v_{\ell+1}$ have the desired property, contradicting the maximality of ℓ. Also, no vertices can appears in s more than twice by Property 2. Thus, Claim 1 holds.

Let $E_{\ell+1} = \{e \in E(G) : c(e) = \ell + 1\}$ be the set of edges colored $\ell + 1$ by c. We now perform the following procedure to each edge in $E_{\ell+1}$.

- ⋆ Let $e_0 = vu_0$ be an edge such that $\ell + 1 \in c(e_0)$. Thus, $\ell + 1 \in c'(u_0)$. By Claim 1, there are integers a_0, b_0 with $1 \leq a_0 < b_0 \leq \ell$ such that $u_0 = v_{a_0} = v_{b_0}$. Since $\ell + 1 \in c(e_0)$, $1 \leq a_0 < b_0 \leq \ell$ and $|c(e_0)| = 2$, it follows that at least one of a_0 and b_0 does not belong to the 2-element set $c(e_0)$, say $a_0 \notin c(e_0)$. This implies that there is some edge $e_1 = u_0u_1$ such that $a_0 \in c(e_1)$.

  > **Claim 2:** *The vertex u_1 occurs exactly twice in the sequence* (4.3)*.*

  If Claim 2 is false, then u_1 occurs at most once in (4.3). However then, we can replace $v_{a_0} = u_0$ in (4.3) by u_1 and then select $v_{\ell+1} = u_0$. Since $a_0 \in c'(u_1)$ and $\ell + 1 \in c'(u_0)$, it follows that $\ell + 1$ vertices $v_1, v_2, \ldots, v_\ell, v_{\ell+1}$ can be selected such that $i \in c'(v_i)$ for $1 \leq i \leq \ell$ and $|\{v_{i_1}, v_{i_2}, v_{i_3}\}| \geq 2$ for every three distinct integers $i_1, i_2, i_3 \in \{1, 2, \ldots, \ell + 1\}$. This contradicts the maximality of ℓ. Hence, Claim 2 holds.

  Suppose that $u_1 = v_{a_1} = v_{b_1}$ where $1 \leq a_1 < b_1 \leq \ell$. Since $a_0 \in c(e_1)$ where $e_1 = u_0u_1$ and $|\{a_0, a_1, b_1\}| = 3$, at least one of a_1 and b_1 does not belong to $c(e_1)$, say $a_1 \notin c(e_1)$.

* ⋆ Let $e_2 = u_1u_2$ such that $a_1 \in c(e_2)$. An argument similar to the one used in verifying Claim 2 shows that u_2 must occur at least twice in the sequence (4.3). Suppose that $u_2 = v_{a_2} = v_{b_2}$ where $1 \leq a_2 < b_2 \leq \ell$. Again, we may assume that $a_2 \notin c(e_2)$ and so there is an edge $e_3 = u_2u_3$ such that $a_2 \in c(e_3)$ and u_3 must appear at least twice in the sequence (4.3).

* ⋆ Continuing this procedure, we arrive at a sequence of vertices $u_1, u_2, \ldots,$ for which $e_{i+1} = u_iu_{i+1} \in E(G)$ and $u_i = v_{a_i} = v_{b_i}$ where $1 \leq a_i < b_i \leq \ell$ such that $a_i \notin c(e_i)$ but $a_i \in c(e_{i+1})$. This procedure terminates when we reach a vertex u_t such that $u_t = u_j$ for some integer j with $0 \leq j \leq t-3$ (note that $u_t \neq u_{t-1}$ and $u_t \neq u_{t-2}$). Necessarily, $t \geq 3$. Since there are only finitely many vertices in $v_1, v_2, \ldots, v_\ell$, such a vertex u_t exists and this process terminates at u_t. Thus, $u_0, u_1, \ldots, u_{t-1}$ are t distinct vertices in $\{v_1, v_2, \ldots, v_\ell\}$.

In summary, we have the walk $(u_0, u_1, \ldots, u_{t-1}, u_t)$ of length $t+1 \geq 4$ in G, where $u_0, u_1, \ldots, u_{t-1}$ are distinct and $u_t = u_j$ for some integer j with $0 \leq j \leq t-3$, such that

(1) $\ell + 1 \in c(vu_0)$ and $u_i = v_{a_i} = v_{b_i}$ where $1 \leq a_i < b_i \leq \ell$ for $0 \leq i \leq t-1$

(2) $a_0 \notin c(vu_0)$ and $a_i \notin c(u_{i-1}u_i)$ but $a_i \in c(u_iu_{i+1})$ for $1 \leq i \leq t-1$.

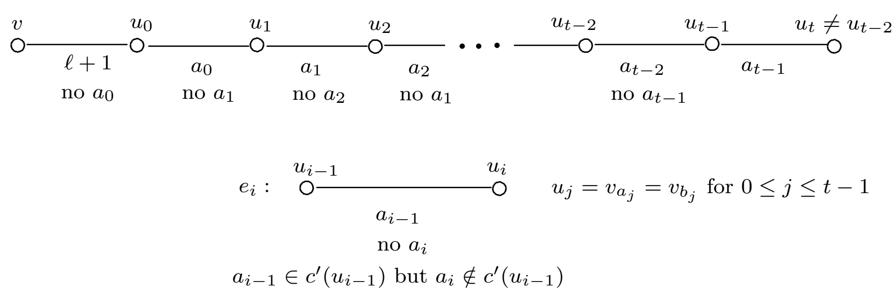


Figure 4.6: Illustrating the proof of Theorem 4.4.6

Next, we verify two claims on the subscripts (or colors) $a_i, b_i \in [\ell]$ for $0 \leq i \leq t-1$.

Claim 3: *For each a_i, where $1 \leq i \leq t-1$, if $a_i \in c'(x)$ for some vertex x of G, then x occurs exactly twice in the sequence* (4.3).

If Claim 3 is false, then there is a_i where $1 \leq i \leq t-1$ such that $a_i \in c'(x)$ for some vertex x of G that occurs at most once in the sequence (4.3). Then we may replace $v_{a_i} = u_i$ by x, replace $v_{a_{j-1}} = u_{j-1}$ by u_j for $1 \leq j \leq i-1$ and then select $v_{\ell+1} = u_0$. Since $a_i \in c'(v)$ and $\ell + 1 \in c'(v_{\ell+1})$, we obtain a sequence of $\ell + 1$ vertices with the two desired properties, contradicting the maximality of ℓ. Thus, Claim 3 holds.

Claim 4: *For each b_i, where $1 \leq i \leq t-1$, if $b_i \in c'(y)$ for some vertex y of G, then y occurs exactly twice in the sequence* (4.3)*.*

If Claim 4 is false, then there is b_i where $1 \leq i \leq t-1$ such that $b_i \in c'(y)$ for some vertex y of G that occurs at most once in the sequence (4.3). Then we may replace $v_{b_i} = u_i$ by y, replace $v_{a_{j-1}} = u_{j-1}$ by u_j for $1 \leq j \leq i-1$ and then select $v_{\ell+1} = u_0$. Since $b_i \in c'(v)$ and $\ell+1 \in c'(v_{\ell+1})$, we obtain a sequence of $\ell+1$ vertices with the two desired properties, contradicting the maximality of ℓ. Thus, Claim 4 holds.

Thus, starting with the edge $e_0 = vu_0 \in E_{\ell+1}$, we produce the following two sets

$$\begin{aligned} S_{e_0} &= \{u_0, u_1, \ldots, u_{t-1}\} \subseteq \{v_1, v_2, \ldots, v_\ell\} \\ T_{e_0} &= \{a_0, b_0, a_1, b_1, \ldots, a_{t-1}, b_{t-1}\} \subseteq [\ell], \end{aligned}$$

where $|S_{e_0}| = t \geq 3$ and $|T_{e_0}| = 2|S_{e_0}|$.

We repeat this procedure to each of the remaining edges of $E_{\ell+1}$. Hence, each edge $e \in E_{\ell+1}$ gives rise to two sets S_e and T_e such that $|S_e| \geq 3$ and $2 \cdot |S_e| = |T_e|$. (Note that for an edge e, the sets S_e and T_e may not be unique.) Let

$$S_0 = \bigcup_{e \in E_{\ell+1}} S_e \quad \text{and} \quad T_0 = \bigcup_{e \in E_{\ell+1}} T_e.$$

Necessarily, $|S_0| \geq 3$ and $2 \cdot |S_0| = |T_0|$. There are two possible situations.

Case 1. *For every color $a_0 \in T_0$, if $a_0 \in c(e)$ where $e = xy$, then $x, y \in S_0$.*

Let $H = G[S_0]$ be the subgraph of G induced by S_0. By Claim 1, we have the following.

The coloring c assigns each of the $2 \cdot |S_0| + 1$ colors in $T_0 \cup \{\ell+1\}$ to the edges in $H = G[S_0]$ only; that is, if $c(e) \in T_0 \cup \{\ell+1\}$, then $e \in E(H)$.

Since $\text{maj}_2(H) \leq |S_0| + 2 < 2|S_0| + 1$ by Corollary 4.2.2, it follows that H has a majestic 2-tone $(|S_0|+2)$-coloring

$$c_H : E(H) \to \{k+1, k+2, \ldots, k+|S_0|+2\}.$$

Let

$$\mathcal{C} = [[k] - (T_0 \cup \{\ell+1\})] \cup \{k+1, k+2, \ldots, k+|S_0|+2\}.$$

Then $|\mathcal{C}| = k - (2|S_0|+1) + |S_0| + 2 < k$. Next, we define a new edge coloring

$$c_G : E(G) \to \mathcal{C}$$

of G by

$$c_G(e) = \begin{cases} c(e) & e \notin E(H) \\ c_H(e) & e \in E(H). \end{cases}$$

We claim that c_G is a majestic 2-tone coloring using few than k colors, which produces a contradiction. Let x and y be adjacent vertices in G. There are three possible situations.

- ⋆ First, suppose that $x, y \in S_0$; that is, x and y are vertices of $H = G[S_0]$. Since c_H is a majestic 2-tone coloring of H, it follows that $c'_H(x) \neq c'_H(y)$. We may assume, without loss of generality, there is $\alpha \in c'_H(x)$ and $\alpha \notin c'_H(y)$. Thus,

$$\alpha \in \{k+1, k+2, \ldots, k+|S_0|+2\} \tag{4.4}$$

  Assume, to the contrary, that $c'_G(x) = c'_G(y)$. This implies that there is an edge e incident with y such that $e \notin E(H)$ and $\alpha \in c_G(e)$. However then, since $\alpha \in c_G(e)$ and $e \notin E(H)$, it follows that $\alpha \in \{\ell+1, \ell+2, \ldots, k\}$, which contradicts (4.4). Thus, $c'_G(x) \neq c'_G(y)$.

- ⋆ Next, suppose that exactly one of x and y belongs to S_0, say $x \in S_0$ and $y \notin S_0$. Then there is $\beta \in \{k+1, k+2, \ldots, k+|S_0|+2\}$ such that $\beta \in c'_G(x)$; while $\beta \notin c'_G(y)$ since no edge incident with y belongs to $E(H)$. Thus $c'_G(x) \neq c'_G(y)$.

- ⋆ Finally, suppose that $x, y \notin S_0$. Hence, no edge incident with x or y belong $E(H)$ and so $c'_G(x) = c'(x) \neq c'(y) = c'_G(y)$.

Therefore, c'_G is a proper vertex coloring of G and so c_G is a majestic 2-tone coloring using few than k colors. This contradicts the hypothesis that $\mathrm{maj}_2(G) = k$.

Case 2. There exists a color $a_0 \in T_0$ such that $a_0 \in c(e)$ for some edge $e = xy$ but one of x and y does not belong to S_0, say $y \notin S_0$

Since $a_0 \in c'(y)$, it follows by the proof of Claim 3 (or Claim 4) that $y \in \{v_1, v_2, \ldots, v_\ell\}$. Furthermore, by the proof of Claim 1, the vertex y must occur at least twice in the sequence $v_1, v_2, \ldots, v_\ell$. Hence, $y = v_i = v_j$ for $i, j \notin T_0$. Such a vertex y is referred to as a non-S_0 vertex. Let S_1 be the set of all non-S_0 vertices of G and let T_1 be the set of all subscripts i and j such that $v_i = v_j = y \in S_1$. That is,

$$\begin{aligned} S_1 &= \{y : \ y \text{ is a non-}S_0 \text{ vertex of } G\} \\ T_1 &= \{i, j : \ v_i = v_j = y \in S_1 \text{ where } i, j \in [\ell] \ \}. \end{aligned}$$

Hence, $2 \cdot |S_1| = |T_1|$. Furthermore, $S_1 \cap S_0 = \emptyset$ and $T_1 \cap T_0 = \emptyset$. There are two subcases.

Subcase 2.1. *For every $a \in T_0 \cup T_1$, if $a \in c(e)$ for an edge $e = uv$, then $u, v \in S_0 \cup S_1$.* Replacing S_0 by $S_0 \cup S_1$ and T_0 by $T_0 \cup T_1$, we can apply the argument in Case 1 to obtain a majestic 2-tone coloring of G using fewer than k colors, a contradiction.

Subcase 2.2. *Subcase* 2.1 *does no occur.* We now repeat the procedure above to the sets $S_0 \cup S_1$ and $T_0 \cup T_1$, arriving at two sets S_2 and T_2 such that

$$\begin{aligned} S_2 &= \{y : \ y \text{ is a non-}(S_0 \cup S_1) \text{ vertex of } G\} \\ T_2 &= \{i, j : \ v_i = v_j = y \in S_0 \cup S_1 \text{ where } i, j \in [\ell] \ \}. \end{aligned}$$

Hence, $2 \cdot |S_2| = |T_2|$. Furthermore, $S_2 \cap (S_1 \cup S_0) = \emptyset$ and $T_2 \cap (T_1 \cup T_0) = \emptyset$. If $S_0 \cup S_1 \cup S_2$ and $T_0 \cup T_1 \cup T_2$ have the property described in Subcase 2.1, then we apply the argument in Case 1 to obtain a majestic 2-tone coloring of G using fewer than k colors, a contradiction. Otherwise, we repeat this procedure to $S_0 \cup S_1 \cup S_2$ and $T_0 \cup T_1 \cup T_2$, arriving at two sets S_3 and T_3. Since there are only finitely many vertices $v_1, v_2, \ldots, v_\ell$, this procedure will eventually terminate at two sets S_p and T_p for some integer $p \geq 2$ such that the following two sets

$$S^* = \bigcup_{i=1}^{p} S_i \ \text{ and } \ T^* = \bigcup_{i=1}^{p} T_i.$$

have the property described in Subcase 2.1. Applying the argument in Case 1 to the sets S^* and T^*, we obtain a majestic 2-tone coloring of G using fewer than k colors, a contradiction. This completes the proof. ■

With the aid of Theorem 4.4.6, we are now able to provide a sufficient condition for certain connected graphs G such that $\text{maj}_2(G) = \text{maj}_2(G \vee K_1)$.

Theorem 4.4.7 *Let G be a nontrivial connected graph with* $\text{maj}_2(G) = k \geq 4$. *If G has a majestic 2-tone k-edge coloring $c : E(G) \to \mathcal{P}_2([k])$ such that $c'(v) \neq [k]$ for each $v \in V(G)$, then*

$$\text{maj}_2(G) = \text{maj}_2(G \vee K_1).$$

Proof. Suppose that G is a nontrivial connected graph of order n with $\text{maj}_2(G) = k \geq 3$. Let $c : E(G) \to \mathcal{P}_2([k])$ be a majestic 2-tone k-edge coloring of G such that $c'(v) \neq [k]$ for each $v \in V(G)$. By Theorem 4.4.6, it is possible to order the n vertices of G as $v_1, v_2, \ldots v_n$ such that $c'(v_i)$ contains two distinct colors a_i and b_i for $1 \leq i \leq n$ and

$$\{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, b_3, \ldots, b_n\} = [k].$$

Let $H = G \vee K_1$. Define the edge coloring $c_H : E(H) \to \mathcal{P}_2([k])$ of H by

$$c_H(e) = \begin{cases} c(e) & \text{if } e \in E(G) \\ \{a_i, b_i\} & \text{if } e = vv_i \text{ for } 1 \leq i \leq n. \end{cases}$$

Then the colors of vertices of H are given by $c'_H(v_i) = c'(v_i) \neq [k]$ and $c'_H(v) = [k]$. Thus, $c'_H(v) \neq c'_H(v_i)$ for $1 \leq i \leq n$. Furthermore, since c is a majestic 2-tone k-edge coloring of G, it follows that if v_i and v_j are adjacent vertices of G where $2 \leq i \leq j \leq n$ and $i \neq j$, then $c'_H(v_i) = c'(v_i) \neq c'(v_j) = c'_H(v_j)$. Therefore, c_H is a majestic 2-tone k-coloring of H and so $\mathrm{maj}_2(H) = \mathrm{maj}_2(G)$. ■

Chapter 5

Majestic t-Tone Colorings

5.1 Introduction

In this chapter we turn our attention to majestic t-tone colorings of connected graphs for all integers $t \geq 2$. Our primary emphasis will be on majestic t-tone colorings of connected bipartite graphs. Specifically, we will extend some results on majestic 2-tone colorings of bipartite graphs that were presented in Chapter 4. By Corollary 4.3.1, if G is a connected graph with $\text{maj}(G) = 2$, then $\text{maj}_t(G) = t+1$ for every positive integer t. First, we show that the only connected graphs G for which $\text{maj}_t(G) = t+1$ for some integer $t \geq 2$ are bipartite.

Theorem 5.1.1 *If G is a connected graph such that*

$$\text{maj}_t(G) = t + 1$$

for some integer $t \geq 2$, then G is bipartite.

Proof. Let G be a connected graph such that $\text{maj}_t(G) = t+1$ for some integer $t \geq 2$. Then there exists a majestic t-tone $(t+1)$-coloring $c : E(G) \to \mathcal{P}_t([t+1])$ of G. Thus, for each vertex v of G, either $c'(v) = [t+1]$ or $c'(v) = [t+1] - \{i\}$ for some $i \in [t+1]$. Consequently, the vertex set of G can be partitioned into the two sets

$$U = \{v \in V(G) : \ c'(v) = [t+1]\} \text{ and } W = \{v \in V(G) : \ |c'(v)| = t\}.$$

We show that G is a bipartite graph with partite sets U and W. Since $c'(u) = [t+1]$ for each $u \in U$ and c' is proper, it follows that U is an independent set of vertices of G. It remains only to show that W is also an independent set of vertices of G. Assume, to the contrary, that there are distinct vertices $x, y \in W$ such that $xy \in E(G)$. Since $c'(x) \neq c'(y)$ and $|c'(x)| = |c'(y)| = t$, it follows that

$$c'(x) = [t+1] - \{i\} \text{ and } c'(y) = [t+1] - \{j\}$$

for some $i, j \in [t+1]$ where $i \neq j$. However then,

$$c(xy) \subseteq c'(x) \cap c'(y) = [t+1] - \{i, j\}$$

and so $|c(xy)| \leq t-1$, which is a contradiction. Therefore, W is an independent set of vertices of G, as desired. ■

We have seen that $\text{maj}_2(P_n) = 4$ when $n \geq 4$ is even. Thus, the converse of Theorem 5.1.1 is false for $t = 2$. As we will soon see, the converse of Theorem 5.1.1 is also false for $t \geq 3$.

Next, we recall the following result that appeared in Chapter 2 (Theorem 2.1.8).

If G is a connected bipartite graph of order 3 or more, then $\text{maj}(G) \leq 3$. (5.1)

Thus, if G is a connected bipartite graph of order at least 3. then either $\text{maj}(G) = 2$ or $\text{maj}(G) = 3$. The following is a consequence of (5.1), Proposition 4.2.1 and Corollary 4.3.1. In fact, it is an extension of (5.1).

Corollary 5.1.2 *If G is a connected bipartite graph of order at least 3 and $t \geq 2$ is an integer, then*

$$t+1 \leq \text{maj}_t(G) \leq t+2.$$

Proof. Let G be a connected bipartite graph of order at least 3. By (5.1), either $\text{maj}(G) = 2$ or $\text{maj}(G) = 3$. If $\text{maj}(G) = 2$, then $\text{maj}_t(G) = t+1$ by Corollary 4.3.1; while if $\text{maj}(G) = 3$, then $\text{maj}_t(G) \leq 3 + (t-1) = t+2$ by Proposition 4.2.1. ■

The statement in (5.1) and Corollary 5.1.2 give rise to a problem concerning the majestic t-tone index of connected bipartite graphs.

Problem 5.1.3 *Let t be a positive integer. Characterize those connected bipartite graphs G for which* $\text{maj}_t(G) = t+1$.

Before presenting some results on majestic t-tone colorings of bipartite graphs, we obtain one additional result on majestic colorings of bipartite graphs

5.2 Majestic Colorings of Unicyclic Bipartite Graphs

A well-known class of connected bipartite graphs is that of trees. By (5.1), if T is a tree order at least 3, then $\text{maj}(T) = 2$ or $\text{maj}(G) = 3$. All trees T of order at least 3 having $\text{maj}(T) = 2$ were characterized in [7].

Theorem 5.2.1 [7] *Let T be a tree of order 3 or more. Then* $\text{maj}(T) = 2$ *if and only if all end-vertices of T belong to the same partite set of T.*

If T is a tree of order at least 3, then the graph obtained by adding an edge between two nonadjacent vertices of G is unicyclic. Specifically, a *unicyclic graph* is a connected graph containing exactly one cycle. Thus, if G is a unicyclic graph of order $n \geq 3$, then the size of G is also n. In particular, the graph C_n is a unicyclic graph. A unicyclic graph G is bipartite if and only if the unique cycle of G is an even cycle. Recall for each integer $n \geq 3$ that

$$\text{maj}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{if } n \not\equiv 0 \pmod 4. \end{cases} \tag{5.2}$$

Thus, $\text{maj}(C_n) = 2$ if and only if $n \equiv 0 \pmod 4$. In order to characterize all unicyclic bipartite graphs having $\text{maj}(G) = 2$ in general, we first present a lemma.

Lemma 5.2.2 *If G is a connected bipartite graph and $c : E(G) \to [2]$ is a majestic coloring of G, then all vertices colored $\{1, 2\}$ by the induced vertex coloring c' belong to one partite set of G and all vertices colored $\{1\}$ or $\{2\}$ belong to the other partite set of G.*

Proof. Suppose that G is a connected bipartite graph and $c : E(G) \to [2]$ is a majestic coloring of G. Since the induced vertex coloring c' is proper, no two vertices colored the same color are adjacent. Since $\{1\}$ and $\{2\}$ are disjoint, no vertex colored $\{1\}$ is adjacent to a vertex colored $\{2\}$. Therefore, two vertices are adjacent if and only if one is colored $\{1, 2\}$ and the other is colored $\{1\}$ or $\{2\}$. ■

We begin with those connected bipartite unicyclic graphs whose cycle has order ℓ where $\ell \equiv 0 \pmod 4$.

Theorem 5.2.3 *Let G be a connected bipartite unicyclic graph whose cycle has length ℓ where $\ell \equiv 0 \pmod 4$. Then* $\text{maj}(G) = 2$ *if and only if all end-vertices of G belong to the same partite set of G.*

Proof. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be the cycle of G, where $\ell \equiv 0 \pmod 4$. If G has no end-vertices, then the result holds by (5.1). Hence, we may assume that $G \neq C$ and so G has end-vertices. First, suppose that $\text{maj}(G) = 2$. Let $c : E(G) \to [2]$ be a majestic coloring of G. If v is an end-vertex of G and v is incident with the edge e, then $c'(v) = \{c(e)\}$ and so $|c'(v)| = 1$. It then follows by Lemma 5.2.2 that all end-vertices of G belong to a same partite set of G.

For the converse, suppose that all the end-vertices of G belong to the same partite set of G. We show that $\mathrm{maj}(G) = 2$. By relabeling the vertices of C if necessary, we may assume, without loss of generality, that if v is an end-vertex of T_i for some i with $1 \le i \le \ell$, then $d(v, v_i)$ and i are of the same parity. Define the majestic coloring $c_0 : E(C) \to [2]$ of C by

$$c_0(e) = \begin{cases} 1 & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 0 \pmod 4 \text{ and } 4 \le i \le \ell \\ 2 & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 2 \pmod 4 \text{ and } 2 \le i \le \ell - 2. \end{cases}$$

Let $F = G - E(C)$ be the forest whose ℓ components are the trees T_i rooted at v_i $(1 \le i \le \ell)$. Define the coloring $c_i : E(T_i) \to [2]$ as follows:

⋆ If i is odd, then

$$c_i(e) = \begin{cases} 1 & \text{if } e \text{ is incident with a vertex } v \text{ with } d(v, v_i) \equiv 1 \pmod 4 \\ 2 & \text{if } e \text{ is incident with a vertex } v \text{ with } d(v, v_i) \equiv 3 \pmod 4 \end{cases}$$

⋆ If i is even and $i \equiv 0 \pmod 4$, then

$$c_i(e) = \begin{cases} 1 & \text{if } e \text{ is incident with a vertex } v \text{ with } d(v, v_i) \equiv 0 \pmod 4 \\ 2 & \text{if } e \text{ is incident with a vertex } v \text{ with } d(v, v_i) \equiv 2 \pmod 4 \end{cases}$$

⋆ If i is even and $i \equiv 2 \pmod 4$, then

$$c_i(e) = \begin{cases} 2 & \text{if } e \text{ is incident with a vertex } v \text{ with } d(v, v_i) \equiv 0 \pmod 4 \\ 1 & \text{if } e \text{ is incident with a vertex } v \text{ with } d(v, v_i) \equiv 2 \pmod 4 \end{cases}$$

The edge coloring $c : E(G) \to [2]$ is then defined by

$$c(e) = \begin{cases} c_0(e) & \text{if } e \in E(C) \\ c_i(e) & \text{if } e \in E(T_i) \text{ where } 1 \le i \le \ell. \end{cases}$$

This is illustrated in Figure 5.1 for $\ell = 8$.

It remains to show that c is a majestic coloring of G. Let c' be the induced vertex coloring of G. Note that if $v_i \in V(C)$ where $1 \le i \le \ell$, then $c'(v_i) = c'_0(v_i)$. Let u and v

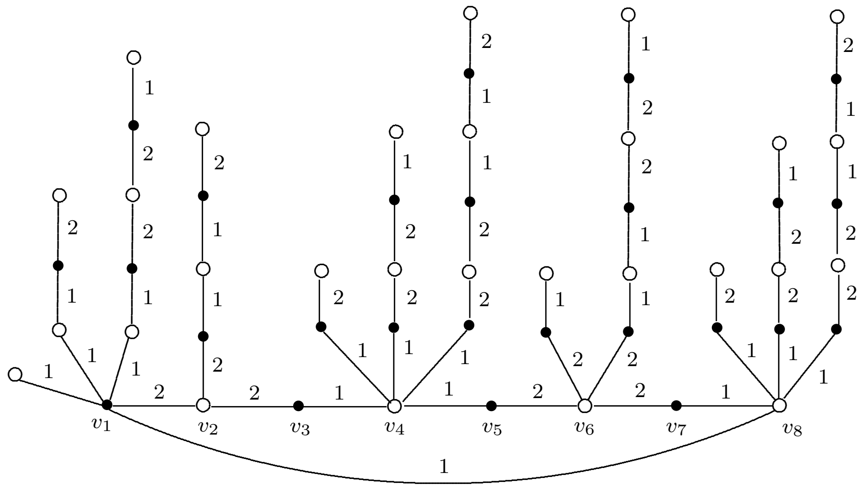


Figure 5.1: A majestic 2-edge coloring in Theorem 5.2.3

be two adjacent vertices of G. Thus, either $u, v \in V(C)$ or $u, v \in V(T_i)$ for some integer i with $1 \le i \le \ell$. If $u, v \in V(C)$, then $c'(u) = c'_0(u) \neq c'_0(v) = c'(v)$. If $u, v \in V(T_i)$, then we may assume, without loss of generality, that $d(v_i, u)$ is odd while $d(v_i, v)$ is even. If i is odd, then $c'(u) = 1$ or $c'(u) = 2$ while $c'(v) = \{1, 2\}$. If i is even then $c'(v) = 1$ or 2 while $c'(u) = \{1, 2\}$. In either situation, $c'(u) \neq c'(v)$ and so c is a majestic coloring of G. ■

Next, we consider those connected bipartite unicyclic graphs whose cycle has length ℓ where $\ell \equiv 2 \pmod 4$. First, we introduce an additional definition. For a vertex v of a connected graph G and a subgraph H of G, the *distance* from v to H is defined by

$$d(v, H) = \min\{d(v, w) : \; w \in V(H)\}.$$

Thus, $d(v, H) = 0$ if and only if $v \in V(H)$.

Theorem 5.2.4 *Let G be a bipartite unicyclic graph whose cycle C has length ℓ where $\ell \equiv 2 \pmod 4$. Then $\mathrm{maj}(G) = 2$ if and only if*

(i) all end-vertices of G belong to the same partite set of G and

(ii) there is an end-vertex v of G such that $d(v, C)$ is odd.

Proof. Let $C = (v_1, v_2, \ldots, v_\ell)$ be the cycle in G, where $\ell \equiv 2 \pmod 4$. If G has no end-vertex, then $G = C_\ell$ and the result follows by (5.1). Hence, we may assume that $G \neq C$ and so G contains and-vertices. Let $F = G - E(C)$ be the forest whose ℓ components are the trees $T_1, T_2, \ldots, T_\ell$, where $v_i \in V(T_i)$ for $1 \le i \le \ell$.

First, suppose that $\text{maj}(G) = 2$. Let $c : E(G) \to [2]$ be a majestic coloring of G. If v is an end-vertex of G and v is incident with the edge e, then $c'(v) = \{c(e)\}$ and so $|c'(v)| = 1$. It then follows by Lemma 5.2.2 that all end-vertices of G belong to a same partite set of G. Next, we show that there is an end-vertex v of G such that $d(v, C)$ is odd. Assume, to the contrary, that no such end-vertex exists. Since all end-vertices of G belong to a same partite set of G, it follows that we can relabel the vertices of C if necessary such that for each odd integer i with $1 \le i \le \ell - 1$, the tree T_i at v_i is a trivial tree. Since G contains end-vertices, we may further assume that T_2 contains an end-vertex u. Thus, $d(v_2, u)$ is even and so v_2 and u belong to the same partite set of G. By Lemma 5.2.2 then, $|c'(v_2)| = |c'(u)| = 1$. Thus, we may assume that $c'(v_2) = \{1\}$. This then implies that

$$c'(v_i) = \begin{cases} \{1,2\} & \text{if } i \text{ is odd} \\ \{1\} & \text{if } i \equiv 2 \pmod 4 \\ \{2\} & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

Since $c'(v_2) = c'(v_\ell) = \{1\}$, it follows that $c(v_1 v_\ell) = c(v_1 v_2) = 1$ and so $c'(v_1) = \{1\}$, a contradiction.

For the converse, suppose that all end-vertices of G belong to the same partite set and there is an end-vertex v such that $d(v, C)$ is odd. We show that $\text{maj}(G) = 2$. By relabeling the vertices of C if necessary, we may assume that

(1) for each integer i with $1 \le i \le \ell$, if u is an end-vertex T_i, then $d(u, v_i)$ and i have the same parity and

(2) the tree T_1 at v_1 contains an end-vertex v and so $d(v, v_1)$ is odd.

We now define the $\ell + 1$ edge colorings c_i $(0 \le i \le \ell)$, where

$$c_0 : E(C) \to [2] \text{ and } c_i : E(T_i) \to [2] \ (1 \le i \le \ell),$$

as follows:

$\star$ If $e \in E(C)$, then

$$c_0(e) = \begin{cases} 1 & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 0 \ (\text{mod } 4) \\ & \text{and } 4 \le i \le \ell - 2 \\ 2 & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 2 \ (\text{mod } 4) \\ & \text{and } 2 \le i \le \ell. \end{cases}$$

⋆ If $e \in E(T_i)$ for some odd integer i, then

$$c_i(e) = \begin{cases} 1 & \text{if } e \text{ is incident with a vertex } w \text{ with } d(w, v_i) \equiv 1 \pmod 4 \\ 2 & \text{if } e \text{ is incident with a vertex } w \text{ with } d(w, v_i) \equiv 3 \pmod 4 \end{cases}$$

⋆ If $e \in E(T_i)$ for some integer i with $i \equiv 0 \pmod 4$, then

$$c_i(e) = \begin{cases} 1 & \text{if } e \text{ is incident with a vertex } w \text{ with } d(w, v_i) \equiv 0 \pmod 4 \\ 2 & \text{if } e \text{ is incident with a vertex } w \text{ with } d(w, v_i) \equiv 2 \pmod 4 \end{cases}$$

⋆ If $e \in E(T_i)$ for some integer i with $i \equiv 2 \pmod 4$, then

$$c_i(e) = \begin{cases} 2 & \text{if } e \text{ is incident with a vertex } w \text{ with } d(w, v_i) \equiv 0 \pmod 4 \\ 1 & \text{if } e \text{ is incident with a vertex } w \text{ with } d(w, v_i) \equiv 2 \pmod 4 \end{cases}$$

Next, we define the edge coloring $c : E(G) \to [2]$ by

$$c(e) = \begin{cases} c_0(e) & \text{if } e \in E(C) \\ c_i(e) & \text{if } e \in E(T_i) \text{ where } 1 \leq i \leq \ell. \end{cases}$$

This is illustrated in Figure 5.2 for $\ell = 6$.

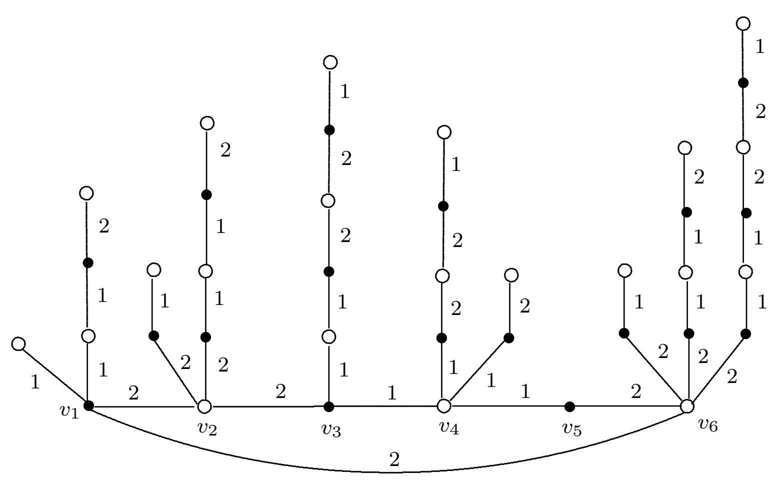


Figure 5.2: A majestic 2-edge coloring in Theorem 5.2.4

It remains to show that c is a majestic coloring of G. Let c' be the induced vertex coloring of G. Since T_1 is a nontrivial tree, it follows that $c'(v_1) = \{1, 2\}$. Let x and y be two adjacent vertices of G. Then either $x, y \in V(C)$ or $x, y \in V(T_i)$ for some integer i with $1 \le i \le \ell$. If $x, y \in V(C)$, say $x = v_i$ where i even and $y = v_{i+1}$, then $c'(y) = \{1, 2\}$ while $c'(x) = \{1\}$ or $c'(x) = \{2\}$. If $x, y \in V(T_i)$, then we may assume, without loss of generality, that $d(v_i, x)$ is odd while $d(v_i, y)$ is even. If i is odd, then $c'(x) = \{1\}$ or $c'(x) = \{2\}$ while $c'(y) = \{1, 2\}$. If i is even, then $c'(y) = \{1\}$ or $c'(y) = \{2\}$ while $c'(x) = \{1, 2\}$. In each case, $c'(x) \neq c'(y)$ and so c is a majestic 2-coloring of G. ■

Combining Theorems 5.2.3 and 5.2.4, we have the following result which characterizes all bipartite unicyclic graphs having majestic index 2.

Corollary 5.2.5 *Let G be a bipartite unicyclic graph and let C_ℓ be the unique cycle of even order $\ell \ge 4$ in G.*

- ⋆ *If $\ell \equiv 0 \pmod 4$, then $\mathrm{maj}(G) = 2$ if and only if all end-vertices of G belong to the same partite set of G.*
- ⋆ *If $\ell \equiv 2 \pmod 4$, then $\mathrm{maj}(G) = 2$ if and only if*
  - *(i) all end-vertices of G belong to the same partite set of G and*
  - *(ii) there is an end-vertex v of G such that $d(v, C)$ is odd.*

5.3 Majestic t-Tone Indices of Bipartite Graphs

We have seen that if G is a connected bipartite graph of order at least 3 with $\mathrm{maj}(G) = 2$, then $\mathrm{maj}_t(G) = t + 1$ for each integer $t \ge 2$. It was shown in [7] that if $r + s \ge 3$, then $\mathrm{maj}(K_{r,s}) = 2$. Hence, we have a class of connected bipartite graphs G for which $\mathrm{maj}_t(G) = t + 1$ for every integer $t \ge 2$, as we state next.

Proposition 5.3.1 *For integers r, s, t where $r \le s$ and $s, t \ge 2$.*

$$\mathrm{maj}_t(K_{r,s}) = t + 1.$$

As an illustration of Proposition 5.3.1, let $G = K_{r,s}$ for $1 \le r \le s$ and $s \ge 2$ where U and W are the partite sets of G with $|U| = r$ and $|W| = s$. Select a vertex v in W and define the edge coloring $c : E(G) \to [t + 1]$ of G by

$$c(e) = \begin{cases} [t+1] - \{1\} & \text{if } e \text{ is adjacent to } v \\ [t+1] - \{2\} & \text{otherwise.} \end{cases}$$

The induced vertex coloring c' is given by

$$c'(u) = \begin{cases} [t+1] - \{1\} & \text{if } u = v \\ [t+1] - \{2\} & \text{if } u \in W \text{ and } u \neq v \\ [t+1] & \text{otherwise.} \end{cases}$$

Since c' is proper, c is a majestic t-tone $(t+1)$-coloring of G and so $\text{maj}_t(G) = t+1$.

In order to describe a class of connected bipartite graphs G for which $\text{maj}_t(G) = t+2$ for every integer $t \geq 2$, we first present a lemma.

Lemma 5.3.2 *For an integer $t \geq 2$, let G be a connected graph with $\text{maj}_t(G) = t+1$ and let $c : E(G) \to \mathcal{P}_t([t+1])$ be a majestic t-tone $(t+1)$-coloring of G. If u and v are two distinct vertices of G such that $|c'(u)| = |c'(v)| = t$, then there is no $u-v$ path of odd length in G. In particular, if u and v are end-vertices of G, then there is no $u-v$ path of odd length in G.*

Proof. Assume, to the contrary, that there are paths of odd length in G that join two vertices whose colors are t-element subsets of $[t+1]$. Among all such paths, let $P = (v_0, v_1, \ldots, v_s)$ be a shortest one, where then $|c'(v_0)| = |c'(v_s)| = t$ and $s \geq 3$ is odd. Since $|c'(v_0)| = |c'(v_s)| = t$, all edges incident with v_0 are colored by the t-element set $c'(v_0)$ and all edges incident with v_s are colored with the t-element set $c'(v_s)$. In particular, $c(v_0v_1) = c'(v_0)$ and $c(v_{s-1}v_s) = c'(v_s)$. Since c' is proper, it follows that $c'(v_0) \neq c'(v_1)$ and $c'(v_{s-1}) \neq c'(v_s)$. On the other hand, $c'(v_0) = c(v_0v_1) \subseteq c'(v_1)$ and $c'(v_s) = c(v_{s-1}v_s) \subseteq c'(v_{s-1})$, implying that $c'(v_1) = c'(v_{s-1}) = [t+1]$. Hence, $s \geq 5$ and $|c'(v_2)| = |c'(v_{s-2})| = t$. However then, v_2 and v_{s-2} are connected by a path of odd length $s-4$, which contradicts the defining property of v_0 and v_s. In particular, if u and v are end-vertices of G, then $|c'(u)| = |c'(v)| = t$ and so there is no $u-v$ path of odd length in G. ■

Theorem 5.3.3 *Let G be a connected bipartite graph of order at least 3. If each partite set of G contains an end-vertex, then $\text{maj}_t(G) = t+2$ for every integer $t \geq 2$.*

Proof. Suppose that u and v are two end-vertices of G that belong to different partite sets of G. Assume, to the contrary, that $\text{maj}_t(G) \neq t+2$. It then follows by Corollary 5.1.2 that $\text{maj}_t(G) = t+1$. Let $c : E(G) \to \mathcal{P}_t([t+1])$ be a majestic t-tone $(t+1)$-coloring of G. Since u and v are end-vertices that belong to different partite sets of G, $|c'(u)| = |c'(w)| = t$ and there is a $u-v$ path of odd length in G, which is impossible by Lemma 5.3.2. Therefore, $\text{maj}_t(G) = t+2$. ■

With the aid of Theorem 5.2.1, Corollary 4.3.1 and Theorem 5.3.3, we now present a characterization of all trees of order 3 having the majestic t-tone index $t+1$ for each integer $t \geq 2$.

Theorem 5.3.4 *Let T be a tree of order 3 or more and let $t \geq 2$ be an integer. Then $\mathrm{maj}_t(T) = t+1$ if and only if all end-vertices of T belong to the same partite set of T and $\mathrm{maj}_t(T) = t+2$ otherwise.*

Proof. First, suppose that all end-vertices of T belong to the same partite set of T. By Theorem 5.2.1, $\mathrm{maj}(T) = 2$ and so $\mathrm{maj}_t(T) = t+1$ by Corollary 4.3.1. For the converse, suppose that each partite set of T contains an end-vertex of T. Then $\mathrm{maj}_t(G) = t+2$ by Theorem 5.3.3 and so $\mathrm{maj}_t(T) \neq t+1$. ■

By Theorem 5.3.4, for each integer $t \geq 2$, if $n \geq 3$ is odd, then $\mathrm{maj}_t(P_n) = t+1$; while if $n \geq 4$ is even, $\mathrm{maj}_t(P_n) = t+2$. Furthermore, if T is a double star (a tree of diameter 3), then $\mathrm{maj}_t(T) = t+2$ for each integer $t \geq 2$.

Next, we characterize all unicyclic bipartite graphs having the majestic t-tone index $t+1$ for each integer $t \geq 2$, beginning with even cycles. First, we determine the majestic t-tone index of cycles for each integer $t \geq 2$.

Proposition 5.3.5 *For integers $n \geq 3$ and $t \geq 2$,*

$$\mathrm{maj}_t(C_n) = \begin{cases} t+1 & \text{if } n \text{ is even} \\ t+2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a cycle of order $n \geq 3$. We consider two cases, according to whether n is even or n is odd.

Case 1. *$n \geq 4$ is even.* Thus, either $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$. If $n \equiv 0 \pmod 4$, then $\mathrm{maj}(C_n) = 2$ by (5.2) and so $\mathrm{maj}_t(C_n) = t+1$ by Corollary 4.2.1. Hence, we may assume that $n \equiv 2 \pmod 4$. Define an edge coloring $c : E(C_n) \to \mathcal{P}_t[t+1])$ by

$$c(e) = \begin{cases} [t+1] - \{t+1\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 2 \pmod 4 \\ & \text{and } 2 \leq i \leq n-2 \\ [t+1] - \{t\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 0 \pmod 4 \\ & \text{and } 2 \leq i \leq n-2 \\ [t+1] - \{1\} & \text{if } e \text{ is incident with } v_n. \end{cases}$$

Then the induced vertex coloring $c'(v_i) = \cup_{e \in E_{v_i}} c(e)$, $1 \le i \le n$, is given by

$$c'(v_i) = \begin{cases} [t+1] - \{t+1\} & \text{if } i \equiv 2 \pmod 4 \text{ and } 2 \le i \le n-2 \\ [t+1] - \{t\} & \text{if } i \equiv 0 \pmod 4 \text{ and } 2 \le i \le n-2 \\ [t+1] - \{1\} & \text{if } i = n \\ [t+1] & \text{if } i \text{ is odd} \end{cases}$$

Since c' is proper, c is a majestic t-tone coloring of C_n and so $\mathrm{maj}_t(C_n) \le t+1$. It then follows by Proposition 4.2.1 that $\mathrm{maj}_t(C_n) = t+1$ if $n \equiv 2 \pmod 4$. Therefore, $\mathrm{maj}_t(C_n) = t+1$ when n is even.

Case 2. $n \ge 3$ is odd. Since C_n is an odd cycle, it follows by Theorem 5.1.1 that $\mathrm{maj}_t(C_n) \ge t+2$. By Proposition 4.2.1 and (5.2) , $\mathrm{maj}_t(C_n) \le (t-1) + \mathrm{maj}(C_n) = t+2$. Thus $\mathrm{maj}_t(C_n) = t+2$ if n is odd. ■

By (5.2) and Proposition 5.3.5, if $n \ge 6$ is an integer such that $n \equiv 2 \pmod 4$, then $\mathrm{maj}(C_n) = 3$ and $\mathrm{maj}_t(C_n) = t+1$. Thus, there are connected bipartite graphs G for which $\mathrm{maj}(G) = 3$ and $\mathrm{maj}_t(G) = t+1$. Next, we consider unicyclic bipartite graphs in general and present a result which extends Theorems 5.2.3 and 5.2.4 to integers $t \ge 3$.

Theorem 5.3.6 *Let G be a unicyclic bipartite connected graph and let $t \ge 2$ be an integer. Then $\mathrm{maj}_t(G) = t+1$ if and only if all end-vertices of G belong to the same partite set of G and $\mathrm{maj}_t(G) = t+2$ otherwise.*

Proof. Let G be a bipartite unicyclic graph with partite sets U and W and let C be the cycle of G. By Proposition 5.3.5, we may assume that $G \ne C$ and so G contains end-vertices. Let X be the set of all end-vertices of G. If $X \cap U \ne \emptyset$ and $X \cap W \ne \emptyset$, then $\mathrm{maj}_t(G) = t+2$ by Theorem 5.3.3. Assume that X is a subset of one partite set of G, say $X \subseteq U$. We show that $\mathrm{maj}_t(G) = t+1$. By Proposition 4.2.1, it suffices to show that G has a t-tone majestic $(t+1)$-edge coloring. Let $C = (v_1, v_2, \ldots, v_\ell, v_{\ell+1} = v_1)$ where $v_1 \in W$ and $\ell \ge 4$ is an even integer. Let $F = G - E(C)$ be the forest whose ℓ components are the trees T_i $(1 \le i \le \ell)$, where T_i is rooted at v_i. Since $X \subseteq U$ and $v_1 \in W$, it follows that if $x \in X$ and $1 \le i \le \ell$, then $d(v_i, x)$ and i are of the same parity; that is, $d(v_i, x)$ is odd if i is odd and $d(v_i, x)$ is even if i is even for all $x \in X$. First, we define a t-tone edge coloring $c_0 : E(C) \to \mathcal{P}_t([t+1])$ of C and a t-tone edge coloring $c_i : E(T_i) \to \mathcal{P}_t([t+1])$ of T_i for $1 \le i \le \ell$ when T_i is a nontrivial tree.

⋆ For the cycle C in G, let c_0 be defined by

$$c_0(e) = \begin{cases} [t+1]-\{t+1\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 2 \pmod 4 \\ & \text{ and } 2 \le i \le \ell-2 \\ [t+1]-\{t\} & \text{if } e \text{ is incident with } v_i \text{ for } i \equiv 0 \pmod 4 \\ & \text{and } 2 \le i \le \ell-2 \\ [t+1]-\{1\} & \text{if } e \text{ is incident with } v_\ell. \end{cases}$$

Note that c_0 is in fact a majestic t-tone $(t+1)$-edge coloring of C.

⋆ If i is odd, then let c_i be defined by

$$c_i(e) = \begin{cases} [t+1]-\{t+1\} & \text{if } e \text{ is incident with a vertex } v \\ & \text{with } d(v,v_i) \equiv 1 \pmod 4 \\ [t+1]-\{t\} & \text{if } e \text{ is incident with a vertex } v \\ & \text{with } d(v,v_i) \equiv 3 \pmod 4). \end{cases}$$

⋆ If $i \equiv 2 \pmod 4$ and $2 \le i \le \ell-2$, then let c_i be defined by

$$c_i(e) = \begin{cases} [t+1]-\{t+1\} & \text{if } e \text{ is incident with a vertex } v \\ & \text{with } d(v,v_i) \equiv 0 \pmod 4 \\ [t+1]-\{t\} & \text{if } e \text{ is incident with a vertex } v \\ & \text{with } d(v,v_i) \equiv 2 \pmod 4). \end{cases}$$

⋆ If $i \equiv 0 \pmod 4$ and $2 \le i \le \ell-2$, then

$$c_i(e) = \begin{cases} [t+1]-\{t\} & \text{if } e \text{ is incident with a vertex } v \\ & \text{with } d(v,v_i) \equiv 0 \pmod 4). \\ [t+1]-\{t+1\} & \text{if } e \text{ is incident with a vertex } v \\ & \text{with } d(v,v_i) \equiv 2 \pmod 4 \end{cases}$$

⋆ If $i = \ell$, then

$$c_\ell(e) = \begin{cases} [t+1]-\{t+1\} & \text{if } e \text{ is incident with a vertex } v \\ & \text{with } d(v,v_i) \equiv 2 \pmod 4 \\ [t+1]-\{t\} & \text{if } e \text{ is not incident with } v_i \neq v_\ell \\ & \text{with } d(v,v_i) \equiv 0 \pmod 4 \\ [t+1]-\{1\} & \text{if } e \text{ is incident with } v_\ell. \end{cases}$$

The t-tone edge coloring $c : E(G) \to \mathcal{P}_t([t+1])$ of G is then defined by

$$c(e) = \begin{cases} c_0(e) & \text{if } e \in E(C) \\ c_i(e) & \text{if } e \in E(T_i) \text{ when } T_i \neq K_1 \text{ for } 1 \leq i \leq \ell. \end{cases}$$

This coloring c is illustrated in Figure 5.3 for $\ell = 8$ and $t = 2$.

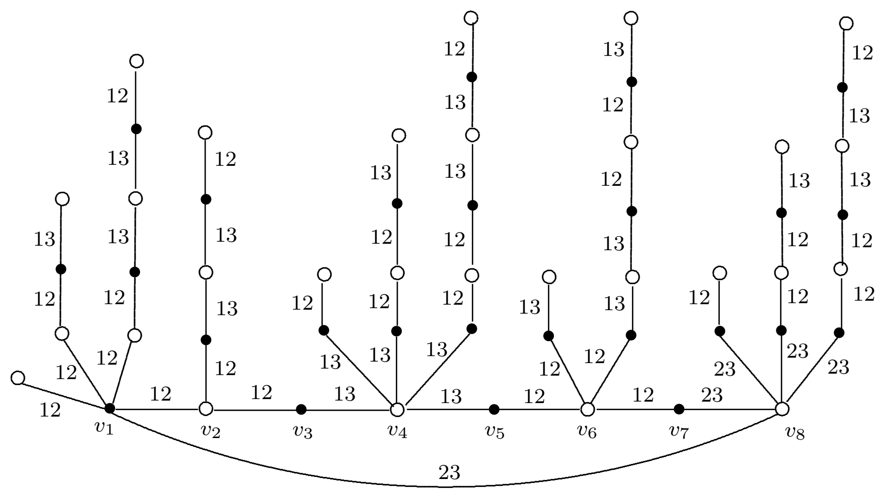


Figure 5.3: A majestic 2-tone 3-edge coloring in Theorem 5.3.6

Let c' be the vertex coloring of G induced by c. We show that c' is proper. Let x and y be two adjacent vertices of G. Since $c'(v_i) = c_0'(v_i)$ for $1 \leq i \leq \ell$ and c_0 is a majestic t-tone $(t+1)$-edge coloring of C, we may assume that x and y are adjacent vertices in some nontrivial T_i for $1 \leq i \leq \ell$. We may assume, without loss of generality, that $d(x, v_i)$ is even and $d(y, v_i)$ is odd. If i is odd, then $c'(x) = [t+1]$ and $c'(y) = [t+1] - \{t+1\}$ or $c'(y) = [t+1] - \{t\}$; while if i is even, then $c'(y) = [t+1]$ and $c'(x)$ is one of the t-element subsets $[t+1] - \{t+1\}, [t+1] - \{t\}, [t+1] - \{1\}$ of the set $[t+1]$. In either case, $c'(x) \neq c'(y)$ and so c' is proper. Therefore, c is a majestic t-tone $(t+1)$-edge coloring of G and so $\mathrm{maj}_t(G) = t+1$. ■

If G is the corona of an even cycle C_n for some even integer $n \geq 4$ (that is, G is obtained from C_n by adding a pendant edge to each vertex of C_n), then by Theorem 5.3.6, $\mathrm{maj}_t(G) = t+2$ for all even integers $n \geq 4$ and each integer $t \geq 2$.

With the aid of the proof of Theorem 5.3.6, we are able to extend this result to a larger class of connected bipartite graphs. First, we introduce an additional definition. Let H and F be two vertex-disjoint graphs. Then F is *attached* to a vertex u of H if a vertex v of F is identified with the vertex u. In particular, if $F = K_2$, then attaching F to a vertex u of G is adding a pendant edge at u.

Theorem 5.3.7 *Let H be a Hamiltonian bipartite graph of order $\ell \geq 3$, let C=(v_1, v_2, …, v_ℓ) be a Hamiltonian cycle of H and let $t \geq 2$ be an integer. If F is the graph obtained from C by attaching a tree T_i at the vertex v_i for each integer i with $1 \leq i \leq \ell$ and G is the graph obtained from H by attaching the same tree T_i at the vertex v_i for $1 \leq i \leq \ell$, then* $\mathrm{maj}_t(F) = \mathrm{maj}_t(G)$.

Proof. First, suppose that $\mathrm{maj}_t(F) = t + 1$. We show that $\mathrm{maj}_t(G) = t + 1$. Let $c_F : E(F) \to \mathcal{P}_t([t+1])$ be the majestic t-tone $(t+1)$-coloring of F given in the proof of Theorem 5.3.6 and let C=(v_1, v_2, …, v_ℓ) be a Hamiltonian cycle of H. If $G \neq F$, then G contains chords of C. Each chord e of C joins vertices v_i and v_j of C such that either (1) $|c'_C(v_i)| = t$ and $c'_C(v_j) = [t + 1]$ or (2) $c'_C(v_i) = [t + 1]$ and $|c'_C(v_j)| = t$. We may assume, without loss of generality, that $|c'_C(v_i)| = t$. In this case, define $c(v_iv_j) = c'_C(v_i)$. Coloring all chords of C in this way produces an edge coloring $c : E(G) \to \mathcal{P}_t([t+1])$ such that $c'_C(v_i) = c'(v_i)$ for all i with $1 \leq i \leq n$. Thus, c is a majestic t-tone $(t + 1)$-coloring of G and so $\mathrm{maj}_t(G) = t + 1$.

Next, suppose that $\mathrm{maj}_t(F) = t + 2$. Since F is a unicyclic bipartite connected graph, it follows by Theorem 5.3.6 that each partite set of F contains an end-vertex of F. Hence, each partite set of G contains an end-vertex. It then follows by Theorem 5.3.3 that $\mathrm{maj}_t(G) = t + 2$ for every integer $t \geq 2$. ■

Next, we extend Theorem 5.3.6 further to another well-known class of connected graphs. Let G be a connected graph of order n and size m. The number of edges that must be deleted from G to obtain a spanning tree of G is $m-n+1$. The number $m-n+1$ is called the *cycle rank* (or *Betti number*) of G. Thus, the cycle rank of a tree is 0 and the cycle rank of a unicyclic graph (a connected graph with exactly one cycle) is 1. The cycle rank of a connected graph of order n and size $m = n + 1$ is therefore 2. A graph H is called a *subdivision* of a graph G if H is obtained from G by inserting vertices of degree 2 into one or more edges of G. For this purpose, we also say that a graph is vacuously a subdivision of itself. If H is a subdivision of a graph G, then H and G have the same cycle rank. If G is a connected graph of order $n \geq 4$ with cycle rank 2, then G contains at least two cycles and so G has a subgraph F that is isomorphic to one of three types of graphs in Figure 5.4.

- ⋆ a graph obtained from two cycles C and C', by identifying a vertex in C and a vertex in C', as shown in Figure 5.4(a),
- ⋆ a graph obtained from two disjoint cycles C and C' and a path P of length 1 or more by identifying an end-vertex u of P with a vertex of C and identifying the other end-vertex v of P with a vertex of C', as shown in Figure 5.4(b),

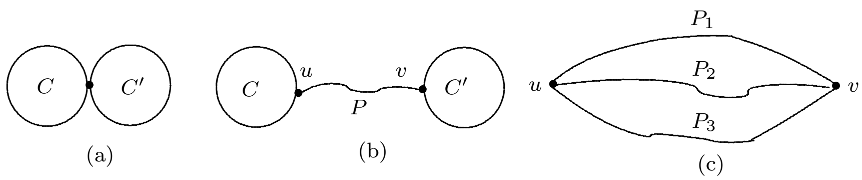


Figure 5.4: Three possible types of subgraphs

- ⋆ a subdivision of $K_4 - e$, that is, a graph consisting of three internally disjoint $u - v$ paths P_i $(1 \leq i \leq 3)$, as shown in Figure 5.4(c), where at least two paths P_i $(1 \leq i \leq 3)$ have length 2 or more.

Therefore, if G is a connected graph of cycle rank 2, then G contains an edge $e = uv$ lying on a cycle of G such that $G - e$ is a unicyclic graph. Clearly, neither u nor v can be an end-vertex of G; while u and v may be end-vertices of $G - e$. Furthermore, if u (or v) is an end-vertex of $G - e$, then $\deg_G u = 2$ and so $G - u$ is a connected graph having with a unique cycle, that is, $G - u$ is a unicyclic graph. We are now prepared to present the following result.

Theorem 5.3.8 *Let G be a bipartite graph of cycle rank* 2 *and let $t \geq 2$ be an integer. Then* $\text{maj}_t(G) = t + 1$ *if and only if all end-vertices of G belong to the same partite set and* $\text{maj}_t(G) = t + 2$ *otherwise.*

Proof. Let G be a bipartite graph of cycle order 2 with partite sets U and W and let X be the set of all end-vertices of G. If $X \cap U \neq \emptyset$ and $X \cap W \neq \emptyset$, then $\text{maj}_t(G) = t+2$ by Theorem 5.3.3. Thus, it remains to verify the converse. Suppose that X is a subset of one partite set of G, say $X \subseteq U$. We show that $\text{maj}_t(G) = t+1$. By Proposition 4.2.1, it suffices to show that G has a majestic t-tone $(t+1)$-edge coloring. Since G be a bipartite graph of cycle order 2, there is an edge uw of G, where $u \in U$ and $w \in W$, such that $H = G - uw$ is a unicyclic bipartite graph with the partite sets U and W. Hence, uw lies on a cycle of G and so neither u nor w is an end-vertex of G. There are two possibilities.

Case 1. *w is not an end-vertex of H.* Hence, the set of the end-vertices of H is either X or $X \cup \{u\}$, which is a subset of U. By Theorem 5.3.6, there exists a majestic t-tone (t+1) coloring c_H of H such that $|c'_H(u)| = t$ and $c'_H(w) = [t+1]$. Define a t-tone $(t+1)$-coloring c of G by $c(e) = c_H(e)$ if $e \neq uw$ and $c(uw) = c'_H(u)$. Since $c'(v) = c'_H(v)$ for each vertex of G, it follows that c is a majestic t-tone $(t+1)$-coloring of G.

Case 2. *w is an end-vertex of H.* Thus, W contains only one end-vertex of H, namely w, and U contains all other end-vertices of H. Let x be the vertex adjacent to

w in H, where then $x \in U$, and let $F = H - w$. Then F is a unicyclic bipartite graph with partite sets U and $W - \{w\}$, all of whose end-vertices belong to U. Now, let c_F be the majestic t-tone $(t+1)$-coloring of F (as described in the proof of Theorem 5.3.6) such that $|c'_F(u)| = |c'_F(x)| = t$. We consider two subcases.

Subcase 2.1. *u and x are colored differently by c'_F.* We now extend the coloring c_F to a t-tone $(t+1)$-coloring c of G as follows: For each edge $e \in E(G) - \{uw, ux\}$, define $c(e) = c_F(e)$, define $c(uw) = c'_F(u)$ and $c(wx) = c'_F(x)$. Since (i) $c'(v) = c'_F(v)$ for each $v \in V(G) - \{w\}$ and (ii) $|c'_F(u)| = |c'_F((x)| = t$, and $c'(w) = [t+1]$, it follows that c is a majestic t-tone $(t+1)$-coloring of G.

Subcase 2.2. *u and x are colored the same by c'_F.* First, we make a useful observation.

> *In the proof of Theorem* 5.3.6*, the induced vertex coloring of the majestic t-tone* $(t+1)$*-coloring of a unicyclic bipartite graph assigns the color* $[t+1]-\{1\}$ *to only one vertex of the graph and this vertex belongs to the cycle of the graph.*

Now, let C be the cycle of F and let c_F be a majestic t-tone $(t+1)$-coloring of F such that c'_F assigns the color $[t+1] - \{1\}$ to only one vertex of F and this vertex belongs to C. Hence, neither x nor u is colored $[t+1] - \{1\}$ by c'_F. We consider two possibilities:

- $\star$ At least one of u and x belongs to C, say u belongs to C. By relabeling the vertices of C if necessary, we may assume that $c'_F(u) = [t+1] - \{1\}$ and so $c'_F(u) \neq c'_F(x)$. We then proceed as in Subcase 2.1

- $\star$ Neither u nor x belongs to C. By relabeling the vertices of C if necessary, we may assume that u is at least distance 4 from the vertex on C that is colored $[t+1]-\{1\}$ by c'_F. So, we adjust the coloring c_F of F by reassigning the color $[t+1] - \{1\}$ to each edge incident with u and leave all other colors of edges unchanged. We still denote the resulting edge coloring by c_F. This preserves c_F as a majestic t-tone coloring of F but u and x are colored differently by c_F. We then proceed as in Subcase 2.1. ■

We have seen in Theorems 5.3.4, 5.3.6 and 5.3.8 that if G is a tree, a unicyclic bipartite connected graph or a bipartite graphs of cycle rank 2 and $t \geq 2$ is an integer, then $\text{maj}_t(G) = t+1$ if and only if all end-vertices of G belong to the same partite set of G. However, this is not true for bipartite graphs in general. As we will see later, there are connected bipartite graphs G, all whose end-vertices belonging to the same partite set of G, but $\text{maj}_t(G) = t+2$ for some integers $t \geq 2$.

Chapter 6

Bipartite Graphs with Large Cycles

We have seen in Proposition 5.3.5 that if C_n is an even graph of order $n \geq 4$, then $\text{maj}_t(C_n) = t + 1$. We now investigate the majestic t-tone indices of connected bipartite graphs having large cycles. In particular, we consider 2-connected bipartite graphs with large cycles.

6.1 Hamiltonian and Near-Hamiltonian Bipartite Graphs

Since the largest cycle that a graph can have is a Hamiltonian cycle, we begin with Hamiltonian bipartite graphs. The result below follows immediately from the proof of Proposition 5.3.5.

Corollary 6.1.1 *If G is a Hamiltonian bipartite graph of even order at least 4 and $t \geq 2$ is an integer, then* $\text{maj}_t(G) = t + 1$.

Proof. Let $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ be a Hamiltonian cycle of G, where $n \geq 4$ is even, and let $c_C : E(C) \to \mathcal{P}_t([t+1])$ be the majestic t-tone $(t+1)$-coloring of C given in the proof of Proposition 5.3.5. If $G \neq C$, then G contains chords. Each chord e of C joins vertices v_i and v_j of C such that either (1) $|c'_C(v_i)| = t$ and $c'_C(v_j) = [t+1]$ or (2) $c'_C(v_i) = [t+1]$ and $|c'_C(v_j)| = t$. We may assume, without loss of generality, that $|c'_C(v_i)| = t$. In this case, define $c(v_iv_j) = c'_C(v_i)$. Coloring all chords of C in this way produces an edge coloring $c : E(G) \to \mathcal{P}_t([t+1])$ such that $c'_C(v_i) = c'(v_i)$ for all i with $1 \leq i \leq n$. Thus, c is a majestic t-tone $(t+1)$-coloring of G and so $\text{maj}_t(G) = t + 1$. ■

In the proof of Corollary 6.1.1, the restriction of the majestic t-tone $(t+1)$-coloring c of the graph G to the Hamiltonian cycle C of G is the majestic t-tone $(t+1)$-coloring

c_C of the even cycle C, as described in Proposition 5.3.5. We now show that even if G is not Hamiltonian but $G - v$ is Hamiltonian for some vertex v of G (that is, G is *near-Hamiltonian*, then we have the same conclusion as in Corollary 6.1.1.

Theorem 6.1.2 *If G is a 2-connected bipartite graph of odd order $n \geq 5$ whose longest cycles have order $n-1$ and $t \geq 2$ is an integer, then* $\mathrm{maj}_t(G) = t+1$.

Proof. Let $C = (v_1, v_2, \ldots, v_{n-1}, v_n = v_1)$ be a longest cycle of G and let v be the vertex of G that is not on C. Then the subgraph $H = G - v$ is Hamiltonian. By Corollary 6.1.1, $\mathrm{maj}_t(H) = t+1$. Let $c_H : E(H) \to \mathcal{P}_t([t+1])$ be a majestic t-tone $(t+1)$-coloring of H. We now construct a majestic t-tone $(t+1)$-coloring $c : E(G) \to \mathcal{P}_t([t+1])$ of G.

Necessarily, the d neighbors of v in G all have an odd subscript or all have an even subscript. Hence, either each neighbor of v has vertex color $[t+1]$ or each neighbor of v has a t-element subset of $[t+1]$ as its vertex color. If each neighbor of v has vertex color $[t+1]$, then assign each edge incident with v the color $[t]$. Then $c'(v) = [t]$ and the resulting edge coloring is a majestic t-tone $(t+1)$-coloring of G. If each neighbor of v has a t-element subset of $[t+1]$ as its vertex color, then relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \leq j \leq n-1$. Thus, each neighbor of v has vertex color $[t+1]$ and we proceed as above. ■

Recall that for each integer $t \geq 2$, the majestic t-tone $(t+1)$-coloring c of the even cycle C_n of order $n \geq 4$ in the proof of Proposition 5.3.5 only uses three distinct t-element subsets of $[t+1]$. Furthermore, if $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$, then either

(1) $c'(v_i) = [t+1]$ for all odd integers i and $|c'(v_i)| = t$ for all even integers i or

(2) $|c'(v_i)| = t$ for all odd integers i and $c'(v_i) = [t+1]$ for all even integers i, where $1 \leq i \leq n$.

This is illustrated in Figure 6.1 for $t = 2$ and $n = 10, 12$.

6.2 More Non-Hamiltonian Bipartite Graphs

We now show that if G is a 2-connected bipartite graph of sufficiently large order n which fails to contain either an n-cycle or an $(n-1)$-cycle but does contain an $(n-2)$-cycle, then we still have the same conclusion as in Theorem 6.1.2.

Theorem 6.2.1 *If G is a 2-connected bipartite graph of even order $n \geq 12$ whose longest cycles have order $n-2$ and $t \geq 2$ is an integer, then* $\mathrm{maj}_t(G) = t+1$.

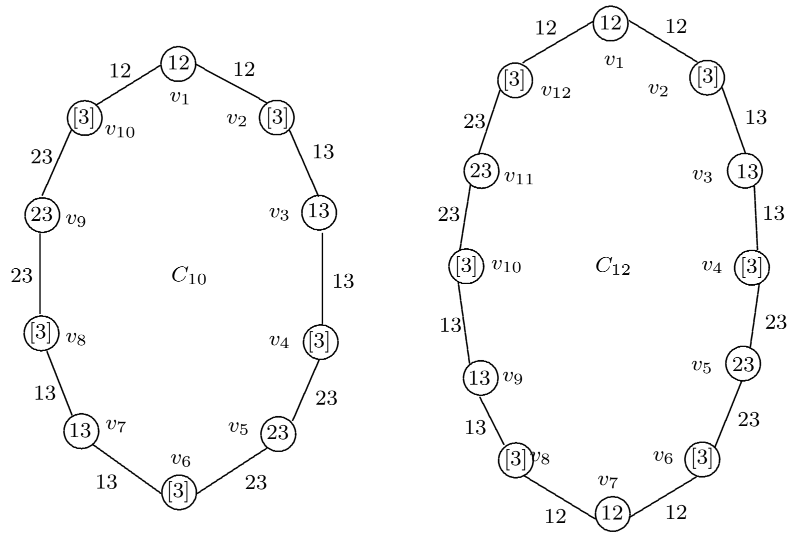


Figure 6.1: Majestic 2-tone 3-colorings of C_{10} and C_{12}

Proof. Let $C = (v_1, v_2, \ldots, v_{n-2}, v_{n-1} = v_1)$ be a longest cycle of G and let $H = G[V(C)]$ be the subgraph of G induced by $V(C)$. Let $u, v \in V(G)$ such that $H = G - u - v$. Thus, H is a Hamiltonian bipartite graph of order $n-2$. By Corollary 6.1.1, $\text{maj}_t(H) = t+1$. Let $c_H : E(H) \to \mathcal{P}_t([t+1])$ be a majestic t-tone $(t+1)$-coloring of H. We show that there is a majestic t-tone $(t+1)$-coloring c of G.

Since G is bipartite, the vertex u is adjacent to vertices v_i $(1 \le i \le n-2)$ of C such that all integers i are odd or all integers i are even. The same is true of the vertex v. Consequently, either (1) each neighbor of u has a t-element subset of $[t+1]$ as its vertex color or (2) each neighbor of u has $[t+1]$ as its vertex color. The same is true of the vertex v. There are three possibilities.

Case 1. *u and v are not adjacent.*

Subcase 1.1. *Each neighbor of both u and v is colored* $[t+1]$. For each edge e incident with u or v, define $c(e) = [t]$ and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$ and $c'(u) = c'(v) = [t]$. Thus, c is a majestic t-tone $(t+1)$-coloring of G and so $\text{maj}_t(G) = t+1$.

Subcase 1.2. *Without loss of generality, each neighbor of u is colored* $[t+1]$ *and each neighbor of v is colored by a t-element subset of* $[t+1]$. There are two possibilities here.

Subcase 1.2.1. *Two neighbors of v are colored differently.* For each edge e incident with u, define $c(e) = [t]$, for each edge vv_i $(1 \le i \le n-2)$, define $c(vv_i) = c_H(v_i)$ and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-2$, $c'(u) = [t]$ and $c'(v) = [t+1]$. Thus, c is a majestic t-tone $(t+1)$-coloring of G and so

$\mathrm{maj}_t(G) = t + 1$.

Subcase 1.2.2. *All neighbors of* v *are colored with the same* t*-element subset of* $[t+1]$. Let $S_1, S_2, \ldots, S_{t+1}$ denote the $t + 1 \geq 3$ distinct t-element subsets of $[t + 1]$. We may assume that every neighbor of v is colored S_1.

Since G is 2-connected, it follows that v is adjacent to two or more vertices of the cycle C in G. We may assume that v_5 is one of the neighbors of v and $c'_H(v_5) = S_1$. Thus, $c_H(v_4v_5) = c_H(v_5v_6) = S_1$. Furthermore, $c'_H(v_3) = c_H(v_2v_3) = c_H(v_3v_4) \neq S_1$ and $c'_H(v_7) = c_H(v_6v_7) = c_H(v_7v_8) \neq S_1$.

- ⋆ If $c'_H(v_3) = c'_H(v_7)$, say $c'_H(v_3) = c'_H(v_7) = S_2$, then we recolor the edges v_4v_5 and v_5v_6 with the color S_3 as well as recolor any edge of H not on C that is incident with v_5 with the color S_3. We then proceed as in Subcase 3.1.

- ⋆ If $c'_H(v_3) \neq c'_H(v_7)$, then we may assume that $c'_H(v_3) = c_H(v_2v_3) = c_H(v_3v_4) = S_2$ and $c'_H(v_7) = c_H(v_6v_7) = c_H(v_7v_8) = S_3$. If $c'_H(v_9) = c_H(v_8v_9) = c_H(v_9v_{10}) = S_1$, then we recolor the edges v_4v_5 and v_5v_6 (and any other edges with incident with v_5) the color S_3 and recolor the edges v_6v_7 and v_7v_8 (and any other edges with incident with v_7) the color S_2. We then proceed as in Subcase 3.1. If $c'_H(v_9) = c_H(v_8v_9) = c_H(v_9v_{10}) = S_2$, then we recolor the edges v_4v_5 and v_5v_6 (and any other edges with incident with v_5) the color S_3 and recolor the edges v_6v_7 and v_7v_8 (and any other edges with incident with v_7) the color S_1. We then proceed as in Subcase 3.1.

Subcase 1.3. *Each neighbor of both* u *and* v *is colored by a* t*-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \leq j \leq n - 2$. We then proceed as in Case 1.

Case 2. u *and* v *are adjacent.* Without loss of generality, we may assume some neighbor of v is colored S_1 For each edge e incident with u define $c(e) = S_2$, for each edge $e = vv_i$ where $1 \leq i \leq n-3$, define $c(e) = c'(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$, $c'(u) = S_2$ and $c'(v) = [t + 1]$. Thus, c is a majestic t-tone $(t + 1)$-coloring of G and so $\mathrm{maj}_t(G) = t + 1$.

In each case, G has a majestic t-tone $(t + 1)$-coloring and so $\mathrm{maj}_t(G) = t + 1$. ■

Next, we show that Theorem 6.2.1 can be extended even further in the next three results.

Theorem 6.2.2 *If* G *is a* 2*-connected bipartite graph of odd order* $n \geq 13$ *whose longest cycles have order* $n - 3$ *and* $t \geq 2$ *is an integer, then* $\mathrm{maj}_t(G) = t + 1$.

Proof. Let $C = (v_1, v_2, \ldots, v_{n-3}, v_{n-2} = v_1)$ be a longest cycle of G and let $H = G[V(C)]$ be the subgraph of G induced by $V(C)$. Let $u, v, w \in V(G)$ such that $H = G - \{u, v, w\}$. Thus, H is a Hamiltonian bipartite graph of order $n-3$. By Corollary 6.1.1, $\mathrm{maj}_t(H) = t+1$. Let $c_H : E(H) \to \mathcal{P}_t([t+1])$ be a majestic t-tone $(t+1)$-coloring of H. Let $S_1, S_2, \ldots, S_{t+1}$ denote the $t+1 \geq 3$ distinct t-element subsets of $[t+1]$. We show that there is a majestic t-tone $(t+1)$-coloring c of G. We now consider three cases.

Case 1. *No two of u, v, w are adjacent.* There are four subcases.

Subcase 1.1. *Each neighbor of u, v and w is colored* $[t+1]$. For each edge e incident with u, v and w, define $c(e) = [t]$ and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$ and $c'(u) = c'(v) = c'(w) = [t]$. Thus, c is a majestic t-tone $(t+1)$-coloring of G and so $\mathrm{maj}_t(G) = t+1$.

Subcase 1.2. *Each neighbor of u, v and w is colored by a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \leq j \leq n-3$. We then proceed as in Subcase 1.1.

Subcase 1.3. *Each neighbor of two of u, v and w is colored* $[t+1]$ *and each neighbor of the other, say w, is colored by a t-element subset of* $[t+1]$.

Subcase 1.3.1. *Two neighbors of w are colored differently.* For each edge e incident with u or v, define $c(e) = [t]$, for each edge $e = v_i w$ where $1 \leq i \leq n-3$, define $c(e) = c'(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$, $c'(u) = c'(v) = [t]$ and $c'(w) = [t+1]$. Thus, c is a majestic t-tone $(t+1)$-coloring of G and so $\mathrm{maj}_t(G) = t+1$.

Subcase 1.3.2. *All neighbors of w are colored with the same t-element subset of* $[t+1]$. We may assume that v_4 is one of the neighbors of w and $c'_H(v_4) = S_1$. Thus, $c_H(v_3v_4) = c_H(v_4v_5) = S_1$. Hence $c_H(v_1v_2) = c_H(v_2v_3) \neq S_1$ and $c_H(v_5v_6) = c_H(v_6v_7) \neq S_1$. We may assume, without loss of generality, that $c_H(v_1v_2) = c_H(v_2v_3) = S_2$.

- ⋆ First, suppose that $c_H(v_5v_6) = c_H(v_6v_7) = S_2$. we recolor the edges v_3v_4 and v_4v_5 (and any other edges with incident with v_4) with the color S_3. We then proceed as in Subcase 1.3.1.

- ⋆ Next, suppose that $c_H(v_5v_6) = c_H(v_6v_7) \neq S_2$, say $c_H(v_5v_6) = c_H(v_6v_7) = S_3$. Thus, $c_H(v_7v_8) = c_H(v_8v_9) \neq S_3$. If $c_H(v_7v_8) = c_H(v_8v_9) = S_1$, then we (1) recolor the edges v_3v_4 and v_4v_5 (and any other edges with incident with v_4) with the color S_3 and (2) recolor the edges v_5v_6 and v_6v_7 (and any other edges with incident with v_6) with the color S_2. We then proceed as in Subcase 1.3.1. If $c_H(v_7v_8) = c_H(v_8v_9) \neq S_1$, then $c_H(v_7v_8) = c_H(v_8v_9) = S_2$ and we (1) recolor the edges v_3v_4 and v_4v_5 (and any other edges with incident with v_4) with the color S_3 and (2) recolor the edges v_5v_6 and v_6v_7 (and any other edges with incident with v_6) with the color S_1. We then proceed as in Subcase 1.3.1.

Subcase 1.4. *Each neighbor of two of u, v and w is colored by a t-element subset of* $[t+1]$ *and each neighbor of the other, say w, is colored* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \le j \le n-3$. We then proceed as in Subcase 1.3.

Case 2. $G[\{u,v,w\}] = K_2 + K_1$. We may assume that $uv \in E(G)$.

Subcase 2.1. *Each neighbor of w is colored* $[t+1]$. Since $uv \in E(G)$, we may assume that each neighbor of u is colored $[t+1]$ and each neighbor of v is colored by a t-element subset of $[t+1]$, at least one of which is colored S_1. For each edge e incident with u or w, define $c(e) = S_2$, for each edge $e = v_iv$ where $1 \le i \le n-3$, define $c(e) = c'(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$, $c'(u) = c'(w) = S_2$ and $c'(v) = [t+1]$. Thus, c is a majestic t-tone $(t+1)$-coloring of G and so $\mathrm{maj}_t(G) = t+1$.

Subcase 2.2. *Each neighbor of w is colored by a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \le j \le n-3$. We then proceed as in Subcase 2.1.

Case 3. $G[\{u,v,w\}] = P_3$. We may assume that (u,v,w) is a path in G.

Subcase 3.1. *Each neighbor of v is colored by a t-element subset of* $[t+1]$. Thus, every neighbor of u and w is colored $[t+1]$. We define a coloring c of G as follows: For each edge e incident with u, define $c(e) = S_1$, for each edge e incident with w, define $c(e) = S_2$ for each edge $e = v_iv$ where $1 \le i \le n-3$, define $c(e) = c'(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then $c'(x) = c'_H(x)$ for each $x \in V(H)$, $c'(u) = S_1$, $c'(w) = S_2$, and $c'(v) = [t+1]$. Thus, c is a majestic t-tone $(t+1)$-coloring of G and so $\mathrm{maj}_t(G) = t+1$.

Subcase 3.2. *Each neighbor of v is colored* $[t+1]$. Thus, every neighbor of u and w is colored by a t-element subset of $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for all integers j with $1 \le j \le n-3$. We then proceed as in Subcase 3.1. ■

Theorem 6.2.3 *If G is a 2-connected bipartite graph of even order $n \ge 14$ whose longest cycles have order $n-4$ and $t \ge 2$ is an integer, then* $\mathrm{maj}_t(G) = t+1$.

Proof. Let $C = (v_1, v_2, \ldots, v_{n-4}, v_{n-4} = v_1)$ be a longest cycle of G and let $H = G[V(C)]$ be the subgraph of G induced by $V(C)$. Let $w, x, y, z, \in V(G)$ such that $H = G - \{w, x, y, z\}$. Thus, H is a Hamiltonian bipartite graph of order $n-4$. By Corollary 6.1.1, $\text{maj}_t(H) = t+1$. Let $c_H : E(H) \to \mathcal{P}_t([t+1])$ be a majestic t-tone $(t+1)$-coloring of H. Let $S_1, S_2, \ldots, S_{t+1}$ be the $t+1$ distinct t-element subsets of $[t+1]$. We show that there is a majestic t-tone $(t+1)$-coloring c of G. Since there are seven bipartite graphs of order 4, namely $\overline{K}_4$, $K_2 + 2K_1$, $2K_2$, $P_3 + K_1$, P_4, $K_{1,3}$ and C_4, we consider these seven cases.

Case 1. *$G[\{w, x, y, z\}] = \overline{K_4}$, that is, no two of w, x, y, z are adjacent.*

Subcase 1.1. *Each neighbor of w, x, y and z is colored $[t+1]$.*

For each edge incident with w, x, y, or z, define $c(e) = [t]$ and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$ -coloring of G with $c'(w) = c'(x) = c'(y) = c'(z) = [t]$ and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-4$.

Subcase 1.2. *Without loss of generality, assume that each neighbor of w is colored by a t-element subset of $[t+1]$ and each neighbor of x, y, z is colored $[t+1]$.*

Subcase 1.2.1. *Two neighbors of w are colored differently.*

For each edge incident with x, y or z, define $c(e) = [t]$, for each edge wv_i where $v_i \in V(H)$, define $c(wv_i) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$ -coloring of G with $c'(w) = [t+1], c'(x) = c'(y) = c'(z) = [t]$ and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-4$.

Subcase 1.2.2. *Each neighbor of w is colored by the same t-element subset of $[t+1]$.*

We may assume, without loss of generality, that v_4 is one of the neighbors of w and $c'_H(v_4) = S_1$. Thus, $c_H(v_3v_4) = c_H(v_4v_5) = S_1$. Furthermore, $c'_H(v_2) = c_H(v_1v_2) = c_H(v_2v_3) \ne S_1$ and $c'_H(v_6) = c_H(v_5v_6) = c_H(v_6v_7) \ne S_1$. We consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 1.2.1.
- ⋆ If $c'_H(v_2) \ne c'_H(v_6)$, then we may assume that $c'_H(v_2) = c_H(v_1v_2) = c_H(v_2v_3) = S_2$ and $c'_H(v_6) = c_H(v_5v_6) = c_H(v_6v_7) = S_3$. If $c'_H(v_8) = c_H(v_7v_8) = c_H(v_8v_9) = S_1$, then we recolor the edges v_3v_4 and v_4v_5 (and any other edges with incident with v_4) the color S_3 and recolor the edges v_5v_6 and v_6v_7 (and any other edges with incident with v_6) the color S_2. We then proceed as in Subcase 1.2.1. If $c'_H(v_8) = c_H(v_7v_8) = c_H(v_8v_9) \ne S_1$, then we recolor the edges v_3v_4 and v_4v_5 (and any other edges with

incident with v_5) the color S_3 and recolor the edges v_5v_6 and v_5v_6 (and any other edges with incident with v_6) the color S_1. We then proceed as in Subcase 1.2.1.

Subcase 1.3. *Without loss of generality, assume that each neighbor of w and x is colored by a t-element subset of $[t+1]$ and each neighbor of y and z is colored $[t+1]$.* Then the neighbors of w and x belong to a partite set of G and the neighbors of y and z belong to the other partite set of G.

Subcase 1.3.1. *Two neighbors of w are colored differently and two neighbors of x are colored differently.*

For each edge incident with y or z, define $c(e) = [t]$, for each edge wv_i where $v_i \in V(H)$, define $c(wv_i) = c'_H(v_i)$, for each edge xv_i where $v_i \in V(H)$, define $c(xv_i) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$ -coloring of G with $c'(w) = c'(x) = [t+1], c'(y) = c'(z) = [t]$ and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-4$.

Subcase 1.3.2. *Without loss of generality, assume that the neighbors of w are colored with the same t-element subset of $[t+1]$ and two neighbors of x are colored differently.*

Without loss of generality, we may assume that each neighbor of w is colored S_1.

Subcase 1.3.2.1. *For every neighbor v_i of w, there exists a neighbor v_j of x such that $d_C(v_i, v_j) \leq 2$.* Since the neighbors of w and x belong to the same partite set of G, it follows that there are two possible situations as shown in Figure 6.2.

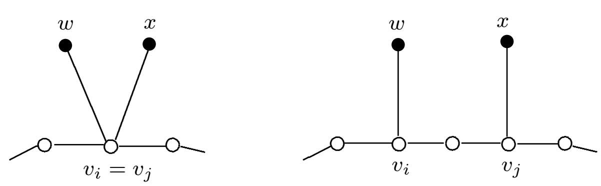


Figure 6.2: Two possible situations in Subcase 1.3.2.1

Subcase 1.3.2.1.1. *There exists a vertex v_i that is a neighbor of both w and x.*

Without loss of generality, we may assume that v_6 is a common neighbor of w and x. Furthermore, we may assume that S_2 is the color of another neighbor v_j, $j \geq 8$, of x and that the path $(v_6, v_7, \ldots, v_j)$ does not contain every neighbor of w in C. We consider the following two possibilities:

- ⋆ Suppose that $c'_H(v_4) = S_2$. We adjust the coloring along the edges in the path $(v_5, v_6, v_7, \ldots, v_{j-1})$ as follows: For an edge e incident with v_{2k} for some even integer k, let $c_H(e) = S_1$. For an edge e incident with v_{2k} for some odd integer k, let

$c_H(e) = S_3$. This preserves c_H as a majestic coloring of H such that $c'_H(v_6) = S_3$, while $c'_H(v_j) = S_2$, and some other neighbor of w is colored S_1. We then proceed as in Subcase 1.3.1.

⋆ Suppose that $c'_H(v_4) \neq S_2$. If $c'_H(v_2) \neq S_2$, then we may recolor the edges incident with v_4 by S_2 and proceed as above. If $c'_H(v_2) = S_2$, then we adjust the coloring along the path $(v_3, \dots, v_{j-1})$ as follows: For an edge e incident with v_{2k} for some even integer k, let $c_H(e) = S_1$. For an edge e incident with v_{2k} for some odd integer k, let $c_H(e) = S_3$. This preserves c_H as a majestic coloring of H such that $c'_H(v_6) = S_3$ while $c'_H(v_j) = S_2$, and some other neighbor of w is colored S_1. We then proceed as in Subcase 1.3.1.

Subcase 1.3.2.1.2. *There exist neighbors v_{i_1}, v_{i_2} of w and neighbors v_{j_1}, v_{j_2} of x such that $i_1 + 2 = j_1, i_2 + 2 = j_2$, and $j_1 < i_2$.* This is illustrated in Figure 6.3.

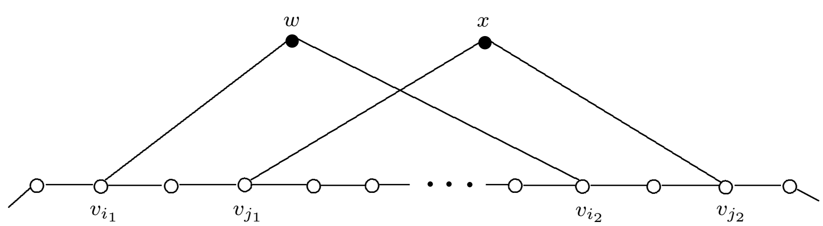


Figure 6.3: A step in the proof of Subcase 1.3.2.1.2

Without loss of generality, we may assume that $v_{i_1} = v_4$ so $v_{j_1} = v_6$. We consider the following four possibilities:

⋆ Suppose that $v_{i_2} = v_8$ and $C = C_{10}$. We may redefine the coloring c_H as follows: For each edge e incident to v_2 or v_8, define $c_H(e) = S_1$. For each edge e incident to v_4 or v_{10}, define $c_H(e) = S_2$. For each edge e incident to v_6, define $c_H(e) = S_3$. This defines a majestic t-tone coloring on H with $c'_H(v_4) = c'_H(v_{10}) = S_2$, $c'_H(v_6) = S_3$ and $c'_H(v_8) = S_1$. We then proceed as in Subcase 1.3.1.

⋆ Suppose that $v_{i_2} = v_8$ and $n \geq 16$. If $c'_H(v_{12}) \neq S_1$. Then we may adjust the coloring c_H as follows: For each edge e incident to v_6, define $c_H(e) = S_3$. For each edge e incident to v_8, define $c_H(e) = S_2$. For each edge e incident to v_{10}, define $c_H(e) = S_1$. This preserves c_H as a majestic coloring with $c'_H(v_4) = c'_H(v_{10}) = S_1$, $c'_H(v_6) = S_3$, and $c'_H(v_8) = S_2$. If $c'_H(v_{12}) = S_1$, then we may assume that $c'_H(v_2) = S_2$ and adjust the coloring c_H as follows: For each edge e incident to v_4 or v_{10}, define $c_H(e) = S_3$. For each edge e incident to v_6, define $c_H(e) = S_1$. For each edge e incident to v_8, define $c_H(e) = S_2$. This preserves c_H as a majestic coloring with $c'_H(v_4) = c'_H(v_{10}) = S_3$, $c'_H(v_6) = S_1$ and $c'_H(v_8) = S_2$. In either case, We then proceed as in Subcase 1.3.1.

- ⋆ Suppose that $v_{i_2} \neq v_8$, that is, $i_2 \geq 10$ and $c'_H(v_{j_2+2}) = S_3$. Then we adjust the coloring c_H as follows: For each edge e incident to v_{2k} with k even and $6 \leq 2k \leq i_2 - 4$, define $c_H(e) = S_2$. For each edge e incident to v_{2k} with k odd and $6 \leq 2k \leq i_2 - 4$, define $c_H(e) = S_3$. For each edge e incident to v_{i_2-2}, define $c_H(e) = S_1$. For each edge e incident to v_{i_2}, define $c_H(e) = S_3$. For each edge e incident to v_{j_2}, define $c_H(e) = S_2$. This preserves c_H as a majestic coloring with $c'_H(v_4) = S_1$, $c'_H(v_6) = c'_H(v_{i_2}) = S_3$ and $c'_H(v_{j_2}) = S_2$. We then proceed as in Subcase 1.3.1.

- ⋆ Suppose that $v_{i_2} \neq v_8$, that is, $i_2 \geq 10$ and $c'_H(v_{j_2+2}) \neq S_3$, then we adjust the coloring c_H as follows: For each edge e incident to v_{2k} with k even and $6 \leq 2k \leq i_2 - 4$, define $c_H(e) = S_3$. For each edge e incident to v_{2k} with k odd and $6 \leq 2k \leq i_2 - 4$, define $c_H(e) = S_2$. For each edge e incident to v_{i_2-2}, define $c_H(e) = S_1$. For each edge e incident to v_{i_2}, define $c_H(e) = S_2$. For each edge e incident to v_{j_2}, define $c_H(e) = S_3$. This preserves c_H as a majestic coloring with $c'_H(v_4) = S_1$, $c'_H(v_6) = c'_H(v_{i_2}) = S_2$ and $c'_H(v_{j_2}) = S_3$. We then proceed as in Subcase 1.3.1.

Subcase 1.3.2.1.3. *Neither Subcase* 1.3.2.1.1 *nor Subcase* 1.3.2.1.2 *is satisfied.* This implies that there exist neighbors v_{i_1}, v_{i_2} of w and neighbors v_{j_1}, v_{j_2} of x such that $i_1 < j_1 < j_2 < i_2$, $j_1 = i_1 + 2$, $j_2 = i_2 - 2$. This is illustrated in Figure 6.4.

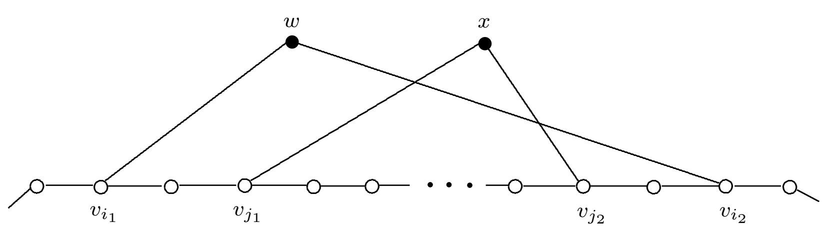


Figure 6.4: A step in the proof of Subcase 1.3.2.1.3

Without loss of generality, we may assume that $v_{i_1} = v_4$, $v_{j_1} = v_6$, and $c'_H(v_6) = S_2$. If $c'_H(v_2) = S_2$, then for each edge e incident with $v_{i_1} = v_4$, we may define $c_H(e) = S_3$ and proceed as in Subcase 1.3.1. Similarly, if $c'_H(v_{i_2+2}) = c'_H(v_{j_2})$, say this color is S_2, then for each edge e incident with v_{i_2}, we may define $c_H(e) = S_3$ and proceed as in

Subcase 1.3.1. So we may assume that $c'_H(v_2) = S_3$. Since $i_1 = 4$ and i_1 and i_2 are of the same parity, i_2 is even. We consider two possibilities:

- $\star$ If $i_2 \equiv 0 \pmod 4$, then we may change the coloring of c_H by recoloring edges as follows: If e is incident with v_{2k} with k even and $4 \leq 2k \leq j_2 - 2$, then define $c_H(e) = S_2$. If e is incident with v_{2k} with k odd and $4 \leq 2k \leq j_2 - 2$, then define $c_H(e) = S_1$. If e is incident with v_{j_2}, then define $c_H(e) = S_3$. So, c_H is still majestic, $c'_H(v_4) = S_2$, $c'_H(v_6) = c'_H(v_{i_2}) = S_1$, and $c'_H(v_{j_2}) = S_3$. We then proceed as in Subcase 1.3.1.

- $\star$ If $i_2 \equiv 2 \pmod 4$, then we may change the coloring of c_H by recoloring edges as follows: If e is incident with v_{2k} with k even and $4 \leq 2k \leq i_2$, then define $c_H(e) = S_2$. If e is incident with v_{2k} with k odd and $4 \leq 2k \leq i_2$, then define $c_H(e) = S_1$. Thus, c_H is still majestic, $c'_H(v_4) = c'_H(v_{j_2}) = S_2$, and $c'_H(v_6) = c'_H(v_{i_2}) = S_1$. We then proceed as in Subcase 1.3.1.

Subcase 1.3.2.2. *There exists a neighbor v_i of w such that $d_C(v_i, v_j) \geq 4$ for every neighbor v_j of x.*

We may assume that v_4 is a neighbor of w satisfying the condition above and $c'_H(v_4) = S_1$. We consider the following two possibilities:

- $\star$ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 1.3.1.

- $\star$ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_2. We then proceed as in Subcase 1.3.1. If $c'_H(v_8) \neq S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_1. We then proceed as in Subcase 1.3.1.

Subcase 1.3.3. *The neighbors of w and x are all colored with the same t-element subset of* $[t+1]$.

Without loss of generality, we may assume that the neighbors of w and x are all colored S_1.

Subcase 1.3.3.1. *w and x share a neighbor v_i.*

Without loss of generality, we may assume that this neighbor is v_4. We consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor each edge incident with v_4 with the color S_3. We then proceed as in Subcase 1.3.1.

- ⋆ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_2. We then proceed as in Subcase 1.3.1. If $c'_H(v_8) \neq S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_1. We then proceed as in Subcase 1.3.1.

Subcase 1.3.3.2. *w and x share no neighbors.*

This implies that for any two vertices v_i and v_j of H that are the neighbors of w and x, respectively, $d_c(v_i, v_j) \geq 4$. We may assume that v_4 is a neighbor of x and consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 1.3.2.2.

- ⋆ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_2. We then proceed as in Subcase 1.3.2.2. If $c'_H(v_8) \neq S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_1. We then proceed as in Subcase 1.3.2.2.

Subcase 1.4. *Without loss of generality, assume that each neighbor of w, x and y is colored by a t-element subset of $[t+1]$ and each neighbor of z is colored $[t+1]$.*

We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-4$ and apply c_H to the new labeling. We then proceed as in Subcase 1.2.

Subcase 1.5. *Each neighbor of w, x, y and z is colored by a t-element subset of $[t+1]$.*

We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-4$ and apply c_H to the new labeling. We then proceed as in Subcase 1.1.

Case 2. $G[\{w, x, y, z\}] = K_2 + \overline{K_2}$. Without loss of generality, we may assume that $wx \in E(G)$.

Subcase 2.1. *Without loss of generality, we may assume that each neighbor of w is colored by a t-element subset of $[t+1]$.*

Subcase 2.1.1. *Each neighbor of y and z is colored $[t+1]$.*

We may assume that v_2 is a neighbor of w and $c'_H(v_2) = S_1$. For each edge incident with x, define $c(e) = S_2$, for each edge e incident with y or z, define $c(e) = [t]$, for each edge wv_i where $v_i \in V(H)$, define $c(wv_i) = c'_H(v_i)$, and for $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(w) = [t+1], c'(x) = S_2, c'(y) = c'(z) = [t]$ and $c'(v_i) = c'_H(v_i)$.

Subcase 2.1.2. *Without loss of generality, assume that each neighbor y is colored by a t-element subset of $[t+1]$ and each neighbor of z is colored $[t+1]$.*

Subcase 2.1.2.1. *There are two neighbors of y colored differently in c_H.*

We may assume that v_2 is a neighbor of w and $c'_H(v_2) = S_1$. For each edge incident with x, define $c(e) = S_2$, for each edge e incident with z, define $c(e) = [t]$, for each edge wv_i where $v_i \in V(H)$, define $c(wv_i) = c'_H(v_i)$, for each edge yv_i where $v_i \in V(H)$, define $c(yv_i) = c'_H(v_i)$, and for $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(w) = c'(y) = [t+1], c'(x) = S_2, c'(z) = [t]$ and $c'(v_i) = c'_H(v_i)$.

Subcase 2.1.2.2. *All neighbors of y are colored with the same t-element subset of $[t+1]$ in c_H.*

We may assume that v_8 is a neighbor of y and $c'_H(v_8) = S_1$. Consider the following two possibilities:

- ⋆ If $c'_H(v_6) = c'_H(v_{10})$, say $c'_H(v_6) = c'_H(v_{10}) = S_2$, then we recolor the edges incident with v_8 with the color S_3. We then proceed as in Subcase 2.1.2.1.
- ⋆ If $c'_H(v_6) \neq c'_H(v_{10})$, then we may assume that $c'_H(v_6) = S_2$ and $c'_H(v_{10}) = S_3$. If $c'_H(v_4) = S_1$, then we recolor the edges incident with v_8 with the color S_2 and recolor the edges incident with v_6 with the color S_3. We then proceed as in Subcase 2.1.2.1. If $c'_H(v_4) \neq S_1$, then we recolor the edges incident with v_8 with the color S_2 and recolor the edges incident with v_6 with the color S_1. We then proceed as in Subcase 2.1.2.1.

Subcase 2.1.3. *Each neighbor of y and z is colored by a t-element subset of $[t+1]$.*

We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-4$ and apply the coloring c_H to the new labeling. We then proceed as in Subcase 2.1.1.

Case 3. $G[\{w, x, y, z\}] = 2K_2$. Without loss of generality, we may assume that $wx, yz \in E(G)$. Additionally, we may assume that the neighbors of w in C are colored with t-element subsets of $[t+1]$ in c'_H, the neighbors of y in C are colored with t-element subsets of $[t+1]$ in c'_H, and that there are neighbors v_i and v_j of w and y respectively such that $c'(v_i) \neq S_3$ and $c'(v_j) \neq S_3$. For each edge e incident with x or z, define $c(e) = S_3$, for each edge e incident with either w or y and v_i for some integer $1 \leq i \leq n-4$, define $c(e) = c'_H(v_i)$, and for each edge e in $E(H)$, define $c(e) = c_h(e)$. Then c is a majestic t-tone $(t+1)$ -coloring of G with $c'(w) = c'(y) = [t+1]$, $c'(x) = c'(z) = S_3$ and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-4$.

Case 4. $G[\{w,x,y,z\}] = P_3 + K_1$. Without loss of generality, we may assume that $(w,x,y) = P_3$ in G.

Subcase 4.1. *The neighbors of x are colored with t-element subsets of $[t+1]$ in c'_H.*

Subcase 4.1.1. *All neighbors of z are colored $[t+1]$ in c'_H.*

For each edge e incident with w, define $c(e) = S_1$, for each edge e incident with y or z, define $c(e) = S_2$, for each edge xv_i where $v_i \in V(H)$, define $c(xv_i) = c'_H(v_i)$, and for each edge e in $E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(w) = S_1, c'(x) = [t+1], c'(y) = c'(z) = S_2$ and $c'(v_i) = c'_H(v_i)$.

Subcase 4.1.2. *All neighbors of z are colored with t-element subsets of $[t+1]$ in c'_H, two of which are different.*

For each edge e incident with w, define $c(e) = S_1$, for each edge e incident with y, define $c(e) = S_2$, for each edge xv_i where $v_i \in V(H)$, define $c(xv_i) = c'_H(v_i)$, for each edge zv_i where $v_i \in V(H)$, define $c(zv_i) = c'_H(v_i)$, and for each edge e in $E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(w) = S_1, c'(x) = c'(z) = [t+1], c'(y) = S_2$ and $c'(v_i) = c'_H(v_i)$.

Subcase 4.1.3. *All neighbors of z are colored with the same t-element subset of $[t+1]$ in c'_H.*

We may assume that v_4 is a neighbor of z such that $c'_H(v_4) = S_1$. Consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 4.1.2.
- ⋆ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_2. We then proceed as in Subcase 4.1.2. If $c'_H(v_8) \neq S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_1. We then proceed as in Subcase 4.1.2.

Subcase 4.2. *The neighbors of x are colored with $[t+1]$ in c'_H.* We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-4$. We then proceed as in Subcase 4.1.

Case 5. $G[\{w,x,y,z\}] = P_4$. Without loss of generality, we may assume that $(w,x,y,z) = P_4$ in G.

Subcase 5.1. *The neighbors of w are colored with t-element subsets of* $[t+1]$.

Without loss of generality, we may assume that v_4 is a neighbor of w and $c'_H(v_4) = S_1$. For each edge e incident with x, define $c(e) = S_2$, for each edge incident with z, define $e = S_1$, for each edge wv_i where $v_i \in V(H)$, define $c(wv_i) = c'_H(v_i)$, for each edge yv_i where $v_i \in V(H)$, define $c(yv_i) = c'_H(v_i)$, and for each edge e in $E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(w) = c'(y) = [t+1], c'(x) = S_2, c'(z) = S_1$ and $c'(v_i) = c'_H(v_i)$.

Subcase 5.2. *The neighbors of w are colored* $[t+1]$.

We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \le j \le n-4$. We then proceed as in Subcase 5.1.

Case 6. $G[\{w,x,y,z\}] = K_{1,3}$. We may assume that x is the central vertex of this $K_{1,3}$ in G.

Subcase 6.1. *The neighbors of x are colored with t-element subsets of* $[t+1]$.

We may assume that v_4 is a neighbor of x and $c'_H(v_4) = S_1$. For each edge e incident with w, y, or z, define $c(e) = S_2$, for each edge xv_i where $v_i \in V(H)$, define $c(xv_i) = c'_H(v_i)$, and for each edge e in $E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(w) = c'(y) = c'(z) = S_2, c'(x) = [t+1]$ and $c'(v_i) = c'_H(v_i)$.

Subcase 6.2. *The neighbors of x are colored* $[t+1]$.

We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \le j \le n-3$. We then proceed as in Subcase 6.1.

Case 7. $G[\{w,x,y,z\}] = C_4$. We may assume that (w,x,y,z,w) is C_4 in G.

Subcase 7.1. *The neighbors of w are colored with t-element subsets of* $[t+1]$.

For each edge e incident with x, define $c(e) = S_1$, for each edge e incident with z, define $c(e) = S_2$, for each edge wv_i where $v_i \in V(H)$, define $c(wv_i) = c'_H(v_i)$, for each edge yv_i where $v_i \in V(H)$, define $c(yv_i) = c'_H(v_i)$, and for each edge e in $E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(w) = c'(y) = [t+1], c'(x) = S_1, c'(z) = S_2$ and $c'(v_i) = c'_H(v_i)$.

Subcase 7.2. *The neighbors of w are colored* $[t+1]$.

We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \le j \le n-3$. We then proceed as in Subcase 7.1. ■

Again, recall that if G is a Hamiltonian bipartite graph of order $n \ge 4$ and C_n is an Hamiltonian cycle of G, then a majestic t-tone $(t+1)$-coloring c of G (in the proof of Corollary 6.1.1) only uses three distinct t-element subsets of $[t+1]$ such that for $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ and $1 \le i \le n$, either

(1) $c'(v_i) = [t+1]$ for all odd (even) integers i and $|c'(v_i)| = t$ for all even (odd) integers i or

(2) $|c'(v_i)| = t$ for all odd (even) integers i and $c'(v_i) = [t+1]$ for all even (odd) integers i.

Theorem 6.2.4 *If G is a 2-connected bipartite graph of odd order $n \geq 15$ whose longest cycles have order $n-5$ and $t \geq 2$ is an integer, then* $\mathrm{maj}_t(G) = t+1$.

Proof. Let $C = (v_1, v_2, \ldots, v_{n-5}, v_{n-4} = v_1)$ be a longest cycle of G and let $H = G[V(C)]$ be the subgraph of G induced by $V(C)$. Let $u, w, x, y, z, \in V(G)$ such that $H = G - \{u, w, x, y, z\}$. Thus, H is a Hamiltonian bipartite graph of order $n-5$. By Corollary 6.1.1, $\mathrm{maj}_t(H) = t+1$. Let $c_H : E(H) \to \mathcal{P}_t([t+1])$ be the majestic t-tone $(t+1)$-coloring of H defined in the proof of Corollary 6.1.1. Let $S_1, S_2, \ldots, S_{t+1}$ be the $t+1$ distinct t-element subsets of $[t+1]$. We show that there is a majestic t-tone $(t+1)$-coloring c of G. Since there is one trivial bipartite graph of order 5 (namely $\overline{K}_5$) and twelves nontrivial bipartite graphs of order 5 (shown in Figure 6.5), we consider these thirteen cases.

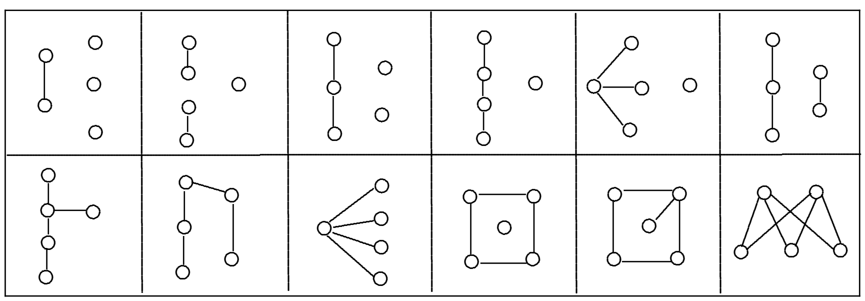


Figure 6.5: Twelve nontrivial bipartite graphs of order 5

Case 1. *$G[\{u, w, x, y, z\}] = \overline{K}_5$, that is, no two of u, w, x, y, z are adjacent.*

Subcase 1.1. *Each neighbor of u, w, x, y, z is colored $[t+1]$.* We define the coloring c as follows: For each edge incident with u, w, x, y, or z, define $c(e) = [t]$ and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(w) = c'(x) = c'(y) = c'(z) = [t]$ and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 1.2. *Without loss of generality, we may assume that each neighbor of u is colored by a t-element subset of $[t+1]$ and each neighbor of w, x, y, z is colored $[t+1]$.*

Subcase 1.2.1. *Two neighbors of u are colored differently.* We define the coloring c as follows: For each edge $e = uv_i$, define $c(e) = c'_H(v_i)$, for each edge e incident with

w, x, y, or z, define $c(e) = [t]$ and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = [t+1], c'(w) = c'(x) = c'(y) = c'(z) = [t]$ and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 1.2.2. *Without loss of generality, we may assume that the neighbors of u are all colored S_1.* We may assume, without loss of generality, that v_4 is one of the neighbors of u. Thus, $c_H(v_3v_4) = c_H(v_4v_5) = S_1$. Furthermore, $c'_H(v_2) = c_H(v_1v_2) = c_H(v_2v_3) \neq S_1$ and $c'_H(v_6) = c_H(v_5v_6) = c_H(v_6v_7) \neq S_1$. We consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 1.2.1.

- ⋆ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = c_H(v_1v_2) = c_H(v_2v_3) = S_2$ and $c'_H(v_6) = c_H(v_5v_6) = c_H(v_6v_7) = S_3$. If $c'_H(v_8) = c_H(v_7v_8) = c_H(v_8v_9) = S_1$, then we recolor the edges v_3v_4 and v_4v_5 (and any other edges with incident with v_4) the color S_3 and recolor the edges v_5v_6 and v_6v_7 (and any other edges with incident with v_6) the color S_2. We then proceed as in Subcase 1.2.1. If $c'_H(v_8) = c_H(v_7v_8) = c_H(v_8v_9) \neq S_1$, then we recolor the edges v_3v_4 and v_4v_5 (and any other edges with incident with v_4) the color S_3 and recolor the edges v_5v_6 and v_5v_6 (and any other edges with incident with v_6) the color S_1. We then proceed as in Subcase 1.2.1.

Subcase 1.3. *Without loss of generality, we may assume that each neighbor of u and w is colored by a t-element subset of $[t+1]$ and each neighbor of x, y, z is colored $[t+1]$.* Thus, the neighbors of u and w belong to a partite set of G and the neighbors of x, y, z belong to the other partite set of G.

Subcase 1.3.1. *Two neighbors of u are colored differently and two neighbors of w are colored differently.* For each edge $e = uv_i$, define $c(e) = c'_H(v_i)$, for each edge $e = wv_i$, define $c(e) = c'_H(v_i)$, for each edge e incident with x, y, or z, define $c(e) = [t]$ and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(w) = [t+1], c'(x) = c'(y) = c'(z) = [t]$ and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 1.3.2. *Without loss of generality, we may assume that each neighbor of u is colored S_1 and two neighbors of w are colored differently.*

Subcase 1.3.2.1. *For every neighbor v_i of u, there exists a neighbor v_j of w such that $d_C(v_i, v_j) \leq 2$.* Since the neighbors of u and w belong to a partite set of G, there are two possible situations as shown in Figure 6.6.

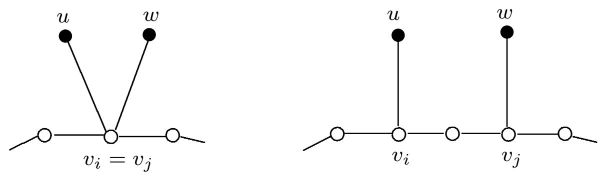


Figure 6.6: Two possible situations in Subcase 1.3.2.1

Subcase 1.3.2.1.1. *There exists a vertex v_i that is a neighbor of both u and w.* Without loss of generality, we may assume that v_6 is a common neighbor of u and w. Furthermore, we may assume that S_2 is the color of another neighbor v_j, $j \geq 8$, of w and that the path $(v_6, v_7, \ldots, v_j)$ does not contain every neighbor of u in C. We consider the following two possibilities:

- ⋆ Suppose that $c'_H(v_4) = S_2$. We adjust the coloring along the path $(v_5, v_6, v_7, \ldots, v_{j-1})$ as follows: For an edge e incident with v_{2k} for some even integer k, let $c_H(e) = S_1$. For an edge e incident with v_{2k} for some odd integer k, let $c_H(e) = S_3$. This preserves c_H as a majestic coloring of H such that $c'_H(v_6) = S_3$, while $c'_H(v_j) = S_2$, and some other neighbor of u is colored S_1. We then proceed as in Subcase 1.3.1.

- ⋆ Suppose that $c'_H(v_4) \neq S_2$. If $c'_H(v_2) \neq S_2$, then we may recolor the edges incident with v_4 by S_2 and proceed as above. If $c'_H(v_2) = S_2$, then we adjust the coloring along the path $(v_3, v_4, \ldots, v_{j-1})$ as follows: For an edge e incident with v_{2k} for some even integer k, let $c_H(e) = S_1$. For an edge e incident with v_{2k} for some odd integer k, let $c_H(e) = S_3$. This preserves c_H as a majestic coloring of H such that $c'_H(v_6) = S_3$ while $c'_H(v_j) = S_2$, and some other neighbor of u is colored S_1. We then proceed as in Subcase 1.3.1.

Subcase 1.3.2.1.2. *There exist neighbors v_{i_1}, v_{i_2} of u and neighbors v_{j_1}, v_{j_2} of w such that $i_1 + 2 = j_1, i_2 + 2 = j_2$, and $j_1 < i_2$.* Since G is a bipartite graph, i_1, i_2, j_1, j_2 are of the same parity. This is illustrated in Figure 6.7.

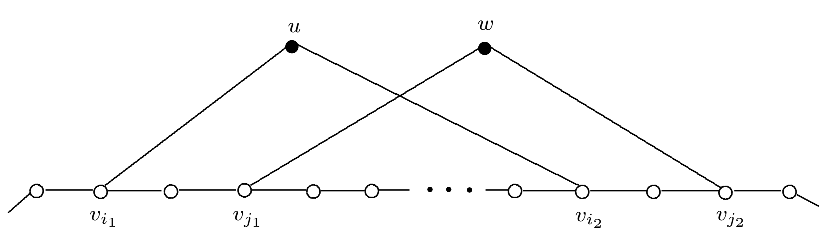


Figure 6.7: A step in the proof of Subcase 1.3.2.1.2

Without loss of generality, we may assume that $v_{i_1} = v_4$ so $v_{j_1} = v_6$ (and so i_2 and j_2 are even). We consider the following four possibilities:

- $\star$ Suppose that $v_{i_2} = v_8$ and $n = 15$ (or $C = C_{10}$). We may redefine the coloring c_H as follows: For each edge e incident with v_2 or v_8, define $c_H(e) = S_1$. For each edge e incident with v_4 or v_{10}, define $c_H(e) = S_2$. For each edge e incident with v_6, define $c_H(e) = S_3$. This defines a majestic t-tone coloring on H with $c'_H(v_4) = c'_H(v_{10}) = S_2$, $c'_H(v_6) = S_3$ and $c'_H(v_8) = S_1$. We then proceed as in Subcase 1.3.1.

- $\star$ Suppose that $v_{i_2} = v_8$ and $n \geq 17$. If $c'_H(v_{12}) \neq S_1$, then we may adjust the coloring c_H as follows: For each edge e incident with v_6, define $c_H(e) = S_3$. For each edge e incident with v_8, define $c_H(e) = S_2$. For each edge e incident with v_{10}, define $c_H(e) = S_1$. This preserves c_H as a majestic coloring with $c'_H(v_4) = c'_H(v_{10}) = S_1$, $c'_H(v_6) = S_3$, and $c'_H(v_8) = S_2$. If $c'_H(v_{12}) = S_1$, then we may assume that $c'_H(v_2) = S_2$ and adjust the coloring c_H as follows: For each edge e incident with v_4 or v_{10}, define $c_H(e) = S_3$. For each edge e incident with v_6, define $c_H(e) = S_1$. For each edge e incident with v_8, define $c_H(e) = S_2$. This preserves c_H as a majestic coloring with $c'_H(v_4) = c'_H(v_{10}) = S_3$, $c'_H(v_6) = S_1$ and $c'_H(v_8) = S_2$. In either case, We then proceed as in Subcase 1.3.1.

- $\star$ Suppose that $v_{i_2} \neq v_8$, that is, $i_2 \geq 10$ and $c'_H(v_{j_2+2}) = S_3$. Then we adjust the coloring c_H as follows: For each edge e incident with v_{2k} with k even and $6 \leq 2k \leq i_2 - 4$, define $c_H(e) = S_2$. For each edge e incident with v_{2k} with k odd and $6 \leq 2k \leq i_2 - 4$, define $c_H(e) = S_3$. For each edge e incident with v_{i_2-2}, define $c_H(e) = S_1$. For each edge e incident with v_{i_2}, define $c_H(e) = S_3$. For each edge e incident with v_{j_2}, define $c_H(e) = S_2$. This preserves c_H as a majestic coloring with $c'_H(v_4) = S_1$, $c'_H(v_6) = c'_H(v_{i_2}) = S_3$ and $c'_H(v_{j_2}) = S_2$. We then proceed as in Subcase 1.3.1.

- $\star$ Suppose that $v_{i_2} \neq v_8$, that is, $i_2 \geq 10$ and $c'_H(v_{j_2+2}) \neq S_3$, then we adjust the coloring c_H as follows: For each edge e incident with v_{2k} with k even and $6 \leq 2k \leq i_2 - 4$, define $c_H(e) = S_3$. For each edge e incident with v_{2k} with k odd and $6 \leq 2k \leq i_2 - 4$, define $c_H(e) = S_2$. For each edge e incident with v_{i_2-2}, define $c_H(e) = S_1$. For each edge e incident with v_{i_2}, define $c_H(e) = S_2$. For each edge e incident with v_{j_2}, define $c_H(e) = S_3$. This preserves c_H as a majestic coloring with $c'_H(v_4) = S_1$, $c'_H(v_6) = c'_H(v_{i_2}) = S_2$ and $c'_H(v_{j_2}) = S_3$. We then proceed as in Subcase 1.3.1.

Subcase 1.3.2.1.3. *Neither Subcase* 1.3.2.1.1 *nor Subcase* 1.3.2.1.2 *is satisfied.* This implies that there exist neighbors v_{i_1}, v_{i_2} of u and neighbors v_{j_1}, v_{j_2} of w such that $i_1 < j_1 < j_2 < i_2$, $j_1 = i_1 + 2$, $j_2 = i_2 - 2$. This is illustrated in Figure 6.8.

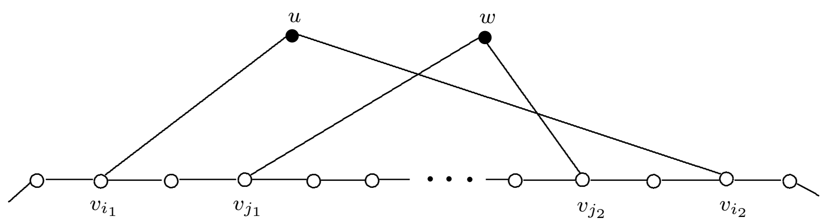


Figure 6.8: A step in the proof of Subcase 1.3.2.1.3

Without loss of generality, we may assume that $v_{i_1} = v_4$, $v_{j_1} = v_6$, and $c'_H(v_6) = S_2$. If $c'_H(v_2) = S_2$, then for each edge e incident with $v_{i_1} = v_4$, we may define $c_H(e) = S_3$ and proceed as in Subcase 1.3.1. Similarly, if $c'_H(v_{i_2+2}) = c'_H(v_{j_2})$, say this color is S_2, then for each edge e incident with v_{i_2}, we may define $c_H(e) = S_3$ and proceed as in Subcase 1.3.1. So we may assume that $c'_H(v_2) = S_3$. Since $i_1 = 4$ and i_1 and i_2 are of the same parity, i_2 is even. We consider two possibilities:

- ⋆ If $i_2 \equiv 0 \pmod 4$, then we may change the coloring of c_H by recoloring edges as follows: If e is incident with v_{2k} with k even and $4 \le 2k \le j_2 - 2$, then define $c_H(e) = S_2$. If e is incident with v_{2k} with k odd and $4 \le 2k \le j_2 - 2$, then define $c_H(e) = S_1$. If e is incident with v_{j_2}, then define $c_H(e) = S_3$. So, c_H is still majestic, $c'_H(v_4) = S_2$, $c'_H(v_6) = c'_H(v_{i_2}) = S_1$, and $c'_H(v_{j_2}) = S_3$. We then proceed as in Subcase 1.3.1.

- ⋆ If $i_2 \equiv 2 \pmod 4$, then we may change the coloring of c_H by recoloring edges as follows: If e is incident with v_{2k} with k even and $4 \le 2k \le i_2$, then define $c_H(e) = S_2$. If e is incident with v_{2k} with k odd and $4 \le 2k \le i_2$, then define $c_H(e) = S_1$. Thus, c_H is still majestic, $c'_H(v_4) = c'_H(v_{j_2}) = S_2$, and $c'_H(v_6) = c'_H(v_{i_2}) = S_1$. We then proceed as in Subcase 1.3.1.

Subcase 1.3.2.2. *There exists a neighbor* v_i *of* u *such that for every neighbor* v_j *of* w, $d_C(v_i, v_j) \ge 4$. We may assume that v_4 is a neighbor of u satisfying the condition above. We consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 1.3.1.

- ⋆ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_2. We then proceed as in Subcase 1.3.1. If $c'_H(v_8) \neq S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_1. We then proceed as in Subcase 1.3.1.

Subcase 1.3.3. *Without loss of generality, we may assume that each neighbor of u and w is colored S_1.*

Subcase 1.3.3.1. *u and w share a neighbor v_i.* Without loss of generality, we may assume that this neighbor is v_4. We consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor each edge incident with v_4 with the color S_3. We then proceed as in Subcase 1.3.1.

- ⋆ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_2. We then proceed as in Subcase 1.3.1. If $c'_H(v_8) \neq S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_1. We then proceed as in Subcase 1.3.1.

Subcase 1.3.3.2. *u and w share no neighbors.* This implies that for any two vertices v_i and v_j of H that are the neighbors of u and w, respectively, $d_c(v_i, v_j) \geq 4$. We may assume that v_4 is a neighbor of w and consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 1.3.2.2.

- ⋆ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_2. We then proceed as in Subcase 1.3.2.2. If $c'_H(v_8) \neq S_1$, then we recolor the edges incident with v_4 by the color S_3 and recolor the edges incident with v_6 by the color S_1. We then proceed as in Subcase 1.3.2.2.

Subcase 1.4. *Without loss of generality, we may assume that each neighbor of u, w, x is colored by a t-element subset of $[t+1]$ and each neighbor of y, z is colored $[t+1]$.* We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 1.3.

Subcase 1.5. *Without loss of generality, we may assume that each neighbor of* u, w, x, y *is colored by a* t*-element subset of* $[t+1]$ *and each neighbor of* z *is colored* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 1.2.

Subcase 1.6. *Each neighbor of* u, w, x, y, z *is colored by a* t*-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 1.1.

Case 2. $G[\{u, w, x, y, z\}] = 3K_1 + K_2$*, where say* $uw \in E(G)$.

Subcase 2.1. *Each neighbor of* x, y, z *is colored* $[t+1]$. Assume, without loss of generality, that the neighbors of u are colored by t-element subsets of $[t+1]$ and that u has a neighbor colored S_1. Since G is bipartite, each neighbor of w is colored $[t+1]$. We define the coloring c as follows: For each edge incident with w, x, y, or z, define $c(e) = S_2$, for each edge $e = uv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = [t+1]$, $c'(w) = c'(x) = c'(y) = c'(z) = S_2$ and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 2.2. *Assume, without loss of generality, that each neighbor of* x *is colored by a* t*-element subset of* $[t+1]$ *and each neighbor of* y, z *is colored* $[t+1]$.

Subcase 2.2.1. *Two neighbors of* x *are colored differently.* Assume, without loss of generality, that the neighbors of u are colored by t-element subsets of $[t+1]$ and that u has a neighbor colored S_1. We define the coloring c as follows: For each edge incident with w, y, or z, define $c(e) = S_2$, for each edge $e = uv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = xv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$ and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(x) = [t+1]$, $c'(w) = c'(y) = c'(z) = S_2$ and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 2.2.2. *All neighbors of* x *are colored the same.* Assume, without loss of generality, that v_4 is one of the neighbors of x and is colored S_1. We consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 2.2.1.

- ⋆ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 the color S_3 and recolor the edges incident with v_6 the color S_2. We then proceed as in Subcase 2.2.1. If

$c'_H(v_8) \neq S_1$, then we recolor the edges incident with v_4 the color S_3 and recolor the edges incident with v_6 the color S_1. We then proceed as in Subcase 2.2.1.

Subcase 2.3. *Assume, without loss of generality, that each neighbor of x and y is colored by a t-element subset of $[t+1]$ each neighbor of z is colored $[t+1]$.* We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 2.2.

Subcase 2.4. *Assume, without loss of generality, that each neighbor of x, y and z is colored by a t-element subset of $[t+1]$.* We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 2.1.

Case 3. *$G[\{u, w, x, y, z\}] = K_1 + 2K_2$, where say $uw, xy \in E(G)$.*

Subcase 3.1. *Each neighbor of z is colored $[t+1]$.* Assume, without loss of generality, that the neighbors of u and x are colored by t-element subsets of $[t+1]$, that a neighbor of u is colored S_1, and that a neighbor of x is colored either S_1 or S_2. Since G is bipartite, the neighbors of w and y are colored $[t+1]$. We define the coloring c as follows: For each edge incident with w, y, or z, define $c(e) = S_3$, for each edge $e = uv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = xv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(x) = [t+1]$, $c'(w) = c'(y) = c'(z) = S_3$ and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 3.2. *Each neighbor of z is colored by a t-element subset of $[t+1]$.* We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 3.1.

Case 4. *$G[\{u, w, x, y, z\}] = 2K_1 + P_3$, where say $(x, y, z) = P_3$ in G.*

Subcase 4.1. *Each neighbor of u and w is colored $[t+1]$.*

Subcase 4.1.1. *Each neighbor of x is colored with a t-element subset of $[t+1]$.* We may assume that a neighbor of x is colored S_1 and a neighbor of z is colored either S_1 or S_2. We define the coloring c as follows: For each edge incident with u, w, or y, define $c(e) = S_3$, for each edge $e = xv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = zv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(x) = c'(z) = [t+1]$, $c'(u) = c'(w) = c'(y) = S_3$ and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 4.1.2. *Each neighbor of x is colored $[t+1]$.* Thus, each neighbor of z in H is colored $[t+1]$. We define the coloring c as follows: For each edge incident with

u, w, x define $c(e) = S_1$, For each edge incident with z define $c(e) = S_2$, for each edge $e = yv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(y) = [t+1]$, $c'(u) = c'(w) = c'(x) = S_1$, $c'(z) = S_2$ and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 4.2. *Without loss of generality, assume that each neighbor of u is colored by a t-element subset of* $[t+1]$ *and each neighbor of w is colored* $[t+1]$.

Subcase 4.2.1. *Two neighbors of u are colored differently.*

Subcase 4.2.1.1. *Each neighbor of x is colored with a t-element subset of* $[t+1]$. Assume, without loss of generality, that a neighbor of x is colored S_1 and a neighbor of z is colored either S_1 or S_2. We define the coloring c as follows: For each edge incident with w or y, define $c(e) = S_3$, for each edge $e = uv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = xv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = zv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(x) = c'(z) = [t+1]$, $c'(w) = c'(y) = S_3$ and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 4.2.1.2. *Each neighbor of x is colored with* $[t+1]$. We define the coloring c as follows: For each edge incident with w or x, define $c(e) = S_1$, for each edge incident with z, define $c(e) = S_2$ for each edge $e = uv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = yv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(y) = [t+1]$, $c'(w) = c'(x) = S_1$, $c'(z) = S_2$ and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 4.2.2. *All neighbors of u are colored the same.* Assume, without loss of generality, that v_4 is one of the neighbors of u and is colored S_1. We consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 4.2.1.

- ⋆ If $c'_H(v_2) \ne c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 the color S_3 and recolor the edges incident with v_6 the color S_2. We then proceed as in Subcase 4.2.1. If $c'_H(v_8) \ne S_1$, then we recolor the edges incident with v_4 the color S_3 and recolor the edges incident with v_6 the color S_1. We then proceed as in Subcase 4.2.1.

Subcase 4.3. *Without loss of generality, assume that each neighbor of u and w is colored by a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \le j \le n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 4.1.

Case 5. *$G[\{u, w, x, y, z\}] = K_1 + P_4$, where say $(w, x, y, z) = P_4$ in G.*

Subcase 5.1. *Each neighbor of u is colored* $[t+1]$. Assume, without loss of generality, that the neighbors of w are colored with t-element subsets of $[t+1]$ and that w has a neighbor colored S_1. Thus, each neighbor of y in H is colored with a t-element subset of $[t+1]$ and each neighbor of x and z is colored $[t+1]$ in H.

We define the coloring c as follows: For each edge incident with u or x, define $c(e) = S_2$, f each edge incident with z, define $c(e) = S_3$, for each edge $e = wv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = yv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(x) = S_2$, $c'(w) = c'(y) = [t+1]$, $c'(z) = S_3$, and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 5.2. *Each neighbor of u is colored with a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \le j \le n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 5.1.

Case 6. *$G[\{u, w, x, y, z\}] = K_1 + K_{1,3}$, where say $wx, wy, wz \in E(G)$.*

Subcase 6.1. *Each neighbor of x is colored* $[t+1]$. Thus, each neighbor of y and z is also colored $[t+1]$.

Subcase 6.1.1. *Each neighbor of u is colored* $[t+1]$. We define the coloring c as follows: For each edge incident with u, x, or y, define $c(e) = S_1$, for each edge incident with z, define $c(e) = S_2$, for each edge $e = wv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(x) = c'(y) = S_1$, $c'(w) = [t+1]$, $c'(z) = S_2$, and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 6.1.2. *Each neighbor of u is colored with a t-element subset of* $[t+1]$.

Subcase 6.1.2.1. *Two neighbors of u are colored differently.* We define the coloring c as follows: For each edge incident with x or y, define $c(e) = S_1$, for each edge incident with z, define $c(e) = S_2$, for each edge $e = uv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = wv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(w) = [t+1]$, $c'(x) = c'(y) = S_1$, $c'(z) = S_2$, and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 6.1.2.2. All neighbors of u are colored the same. Assume, without loss of generality, that v_4 is one of the neighbors of u and is colored S_1. We consider the following two possibilities:

- ⋆ If $c'_H(v_2) = c'_H(v_6)$, say $c'_H(v_2) = c'_H(v_6) = S_2$, then we recolor the edges v_3v_4 and v_4v_5 with the color S_3 as well as recolor any edge of H not on C that is incident with v_4 with the color S_3. We then proceed as in Subcase 6.1.2.1.
- ⋆ If $c'_H(v_2) \neq c'_H(v_6)$, then we may assume that $c'_H(v_2) = S_2$ and $c'_H(v_6) = S_3$. If $c'_H(v_8) = S_1$, then we recolor the edges incident with v_4 the color S_3 and recolor the edges incident with v_6 the color S_2. We then proceed as in Subcase 6.1.2.1. If $c'_H(v_8) \neq S_1$, then we recolor the edges incident with v_4 the color S_3 and recolor the edges incident with v_6 the color S_1. We then proceed as in Subcase 6.1.2.1.

Subcase 6.2. *Each neighbor of x is colored with a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 6.1.

Case 7. *$G[\{u, w, x, y, z\}] = K_2 + P_3$, where say $uw \in E(G)$ and $(x, y, z) = P_3$.*

Subcase 7.1. *Each neighbor of x is colored* $[t+1]$. Thus, each neighbor of z is also colored $[t+1]$. Assume, without loss of generality, that there is a neighbor of u colored S_1. Thus, each neighbor of w in H is colored $[t+1]$. We define the coloring c as follows: For each edge incident with w or x, define $c(e) = S_2$, for each edge incident with z, define $c(e) = S_1$, for each edge $e = uv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = yv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(y) = [t+1]$, $c'(w) = c'(x) = S_2$, $c'(z) = S_1$, and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 7.2. *Each neighbor of x is colored with a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 7.1.

Case 8. *$G[\{u, w, x, y, z\}] = S_{2,3}$, a double star whose central vertices have degrees* 2 *and* 3. Assume, without loss of generality, that $uw, wx, xy, xz \in E(G)$.

Subcase 8.1. *Each neighbor of u is colored with a t-element subset of* $[t+1]$. Thus, each neighbor of x in H is also colored with a t-element subset of $[t+1]$; while each neighbor of w, y, z in H is colored $[t+1]$. Assume, without loss of generality, there is a neighbor of u colored S_1. We define the coloring c as follows: For each edge incident with w, define $c(e) = S_2$, for each edge incident with y or z, define $c(e) = S_1$, for each edge $e = uv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = xv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(x) = [t+1]$, $c'(w) = S_2$, $c'(y) = c'(z) = S_1$, and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 8.2. *Each neighbor of u is colored* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 8.1.

Case 9. *$G[\{u,w,x,y,z\}] = P_5$, where say $(u,w,x,y,z) = P_5$ in G.*

Subcase 9.1. *Each neighbor of u is colored* $[t+1]$. Thus, each neighbor of x and z in H is also colored $[t+1]$; while each neighbor of w and y in H is colored by a t-element subset of $[t+1]$. We define the coloring c as follows: For each edge incident with u or z, define $c(e) = S_1$, for each edge incident with x, define $c(e) = S_2$, for each edge $e = wv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = yv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(z) = S_1$, $c'(w) = c'(y) = [t+1]$, $c'(x) = S_2$, and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 9.2. *Each neighbor of u is colored with a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 9.1.

Case 10. *$G[\{u,w,x,y,z\}] = K_{1,4}$, where say $uw, ux, uy, uz \in E(G)$.*

Subcase 10.1. *Each neighbor of w is colored* $[t+1]$. Thus, each neighbor of x, y and z is also colored $[t+1]$. We define the coloring c as follows: For each edge incident with w, define $c(e) = S_1$, for each edge incident with x, y, or z, define $c(e) = S_2$, for each edge $e = uv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = [t+1]$, $c'(w) = S_1$, $c'(x) = c'(y) = c'(z) = S_2$, and $c'(v_i) = c'_H(v_i)$ for $1 \leq i \leq n-5$.

Subcase 10.2. *Each neighbor of w is colored by a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \leq j \leq n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 10.1.

Case 11. *$G[\{u,w,x,y,z\}] = K_1 + C_4$, where say $(w,x,y,z,w) = C_4$ in G.*

Subcase 11.1. *Each neighbor of u is colored* $[t+1]$. Assume, without loss of generality, that w has a neighbor in $V(H)$.

Subcase 11.1.1. *Each neighbor of w is colored* $[t+1]$. Thus, each neighbor of y in H is also colored $[t+1]$ and each neighbor of x and z in H is colored by a t-element subset of $[t+1]$. We define the coloring c as follows: For each edge incident with u or w, define $c(e) = S_1$, for each edge incident with y, define $c(e) = S_2$, for each edge $e = xv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = zv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic

t-tone $(t+1)$-coloring of G with $c'(u) = c'(w) = S_1$, $c'(x) = c'(z) = [t+1]$, $c'(y) = S_2$, and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 11.1.2. *Assume, without loss of generality, that w has a neighbor colored S_1.* Thus, each neighbor of y in H is colored a t-element subset of $[t+1]$ and each neighbor of x and z in H is colored by $[t+1]$. We define the coloring c as follows: For each edge incident with u or x, define $c(e) = S_1$, for each edge incident with z, define $c(e) = S_2$, for each edge $e = wv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = yv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(x) = S_1$, $c'(w) = c'(y) = [t+1]$, $c'(z) = S_2$, and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 11.2. *Each neighbor of u is colored by a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \le j \le n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 11.1.

Case 12. *$G[\{u, w, x, y, z\}]$ is the graph obtained from the* 4*-cycle C_4 by adding a pendant edge at a vertex of C_4.* Assume, without loss of generality, that $(w, x, y, z, w) = C_4$ in G and $uw \in E(G)$

Subcase 12.1. *Each neighbor of u is colored* $[t+1]$. Thus, each neighbor of w and y in H is colored by a t-element subset of $[t+1]$ and each neighbor of x and z in H is colored $[t+1]$. We define the coloring c as follows: For each edge incident with u or x, define $c(e) = S_1$, for each edge incident with z, define $c(e) = S_2$, for each edge $e = wv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = yv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = c'(x) = S_1$, $c'(w) = c'(y) = [t+1]$, $c'(z) = S_2$, and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 12.2. *Each neighbor of u is colored by a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \le j \le n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 12.1.

Case 13. *$G[\{u, w, x, y, z\}] = K_{2,3}$, where say the partite sets are $\{u, w\}$ and $\{x, y, z\}$.*

Subcase 13.1. *Either u or w has a neighbor in $V(H)$, say the former.*

Subcase 13.1.1. *Each neighbor of u is colored* $[t+1]$. Thus, each neighbor of w in H is also colored $[t+1]$ and each neighbor of x, y and z is colored by a t-element subset of $[t+1]$. We define the coloring c as follows: For each edge incident with u, define $c(e) = S_1$, for each edge incident with w, define $c(e) = S_2$, for each edge $e = xv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = yv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = zv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$ define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = S_1$, $c'(w) = S_2$, $c'(x) = c'(y) = c'(z) = [t+1]$, and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 13.2.1. *Each neighbor of u is colored by a t-element subset of* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \le j \le n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 13.1.1.

Subcase 13.2. *Neither u nor w has a neighbor in $V(H)$.* Thus, one of x, y, z has a neighbor in $V(H)$, say x has a neighbor in $V(H)$.

Subcase 13.2.1. *Each neighbor of x is colored by a t-element subset of* $[t+1]$. We define the coloring c as follows: For each edge incident with u, define $c(e) = S_1$, for each edge incident with w, define $c(e) = S_2$, for each edge $e = xv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = yv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, for each edge $e = zv_i$ where $v_i \in V(H)$, define $c(e) = c'_H(v_i)$, and for each edge $e \in E(H)$, define $c(e) = c_H(e)$. Then c is a majestic t-tone $(t+1)$-coloring of G with $c'(u) = S_1$, $c'(w) = S_2$, $c'(x) = c'(y) = c'(z) = [t+1]$, and $c'(v_i) = c'_H(v_i)$ for $1 \le i \le n-5$.

Subcase 13.2.2. *Each neighbor of x is colored* $[t+1]$. We relabel the vertices of H such that v_j is relabeled as v_{j+1} for $1 \le j \le n-5$ and apply c_H to the new labeling. We then proceed as in Subcase 13.2.1. ∎

In summary, it follows by Corollary 6.1.1 and Theorems 6.1.2, 6.2.1 and 6.2.3 that if G is a 2-connected bipartite graph of sufficiently large order n whose longest cycles have length $\ell \in \{n, n-1, n-2, n-3, n-4, n-5\}$ and $t \ge 2$ is an integer, then $\mathrm{maj}_t(G) = t+1$. There is a limit, however, on how small the length of a longest cycle can be in terms of the order of a 2-connected bipartite graph G to guarantee that G has a majestic 2-tone 3-coloring, as we will see in the following section.

6.3 Bipartite Graphs with Majestic t-Tone Index $t+2$

First, we present a class of 2-connected bipartite graph G having $\text{maj}_t(G) = t + 2$.

Theorem 6.3.1 *For each integer $t \geq 2$, there exists a 2-connected bipartite graph G such that* $\text{maj}_t(G) = t + 2$.

Proof. Let $s = \binom{t+2}{2}$. First, we construct a bipartite graph G of order $2s + 2(t+2) = 2(s+t+2)$ as follows. The partite sets of G are

$$\begin{aligned} U &= \{u_1, u_2, \ldots, u_s\} \cup \{y_1, y_2, \ldots, y_{t+2}\} \\ W &= \{w_1, w_2, \ldots, w_s\} \cup \{x_1, x_2, \ldots, x_{t+2}\}. \end{aligned}$$

Let $X = \{x_1, x_2, \ldots, x_{t+2}\}$ and $Y = \{y_1, y_2, \ldots, y_{t+2}\}$. The subgraph $G[X \cup Y]$ of G induced by $X \cup Y$ is $K_{t+2,t+2}$. Let $X_1, X_2, \ldots, X_s$ be the s distinct 2-element subsets of X and $Y_1, Y_2, \ldots, Y_s$ the s distinct 2-element subsets of Y. For each integer i with $1 \leq i \leq s$, the vertex u_i is joined to the two vertices in X_i and the vertex w_i is joined to the two vertices in Y_i. This completes the construction of G. This is illustrated in Figure 6.9 for $t = 2$. Thus, $\deg u_i = \deg w_i = 2$ for $1 \leq i \leq s$. The graph G is 2-connected but not Hamiltonian (as a vertex in $X \cup Y$ is adjacent to three vertices of degree 2 in G).

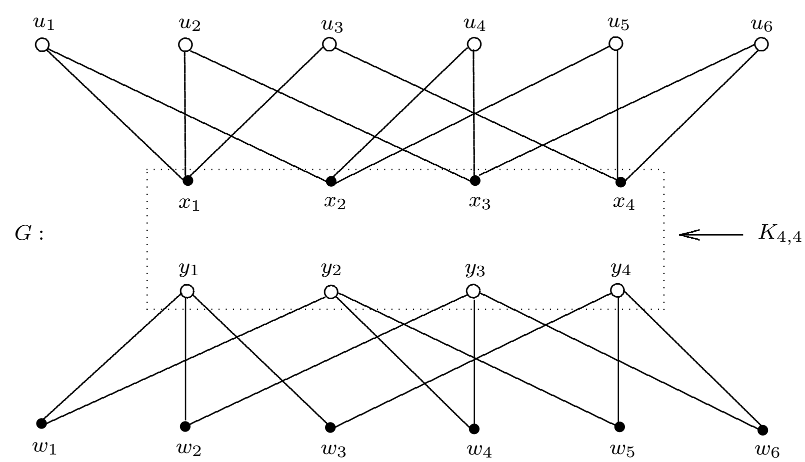


Figure 6.9: A 2-connected bipartite graph G with $\text{maj}_2(G) = 4$

Next, we show that $\text{maj}_t(G) = t + 2$. Assume, to the contrary, that $\text{maj}_t(G) = t+1$. Then there exists a majestic t-tone $(t+1)$-coloring $c : E(G) \to [t+1]$ of G. Thus, either $|c'(u)| = t$ for each $u \in U$ or $|c'(w)| = t$ for each $w \in W$. Because of the symmetry of G, we may assume that $|c'(u)| = t$ for each $u \in U$. Therefore, $c'(w) = [t+1]$ for each $w \in W$. In particular, $c'(w_i) = [t+1]$ for $i = 1, 2, \ldots, s$. Since $\deg_G w_i = 2$ and $c'(w_i) = [t+1]$ for each integer i with $1 \le i \le s$, the two neighbors of w_i have distinct vertex colors. Since $|c'(u)| = t$ for each $u \in U$ and there are exactly $t+1$ distinct t-element sets of $[t+1]$, there are two vertices in Y that have the same vertex color, say $c'(y_p) = c'(y_q)$, where $p, q \in \{1, 2, \ldots, t+2\}$ and $p \ne q$. Let $w \in W$ such that w is only adjacent to y_p and y_q in G. However then, $|c'(w)| = t$, which is a contradiction. Therefore, $\mathrm{maj}_t(G) = t+2$. ■

As an example, we provide a majestic 2-tone 4-coloring c of the graph in Figure 6.9 as follows: Color each edge incident with u_1 and u_6 by the 2-element set $\{1, 4\}$, color each edge incident with w_1 and w_6 by the 2-element set $\{3, 4\}$ and color all remaining edges by the 2-element set $\{2, 4\}$. Observe that

⋆ $c'(u_1) = c'(u_6) = \{1, 4\}$, $c'(u_i) = \{2, 4\}$ for $2 \le i \le 5$ and $c'(y_j) = \{2, 3, 4\}$ for $1 \le j \le 4$ and

⋆ $c'(w_1) = c'(w_6) = \{3, 4\}$, $c'(w_i) = \{2, 4\}$ for $2 \le i \le 5$ and $c'(x_j) = \{1, 2, 4\}$ for $1 \le j \le 4$.

Thus, c' is a proper vertex coloring of G and so c is a majestic 2-tone 4-coloring of G.

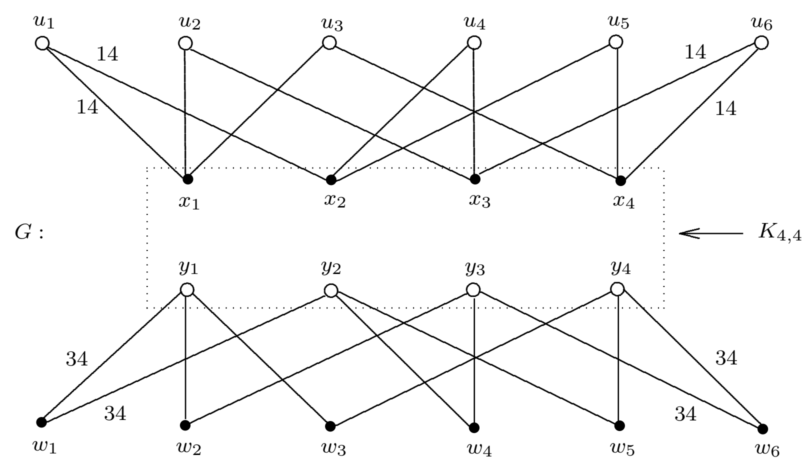


Figure 6.10: A majestic 2-tone 4-coloring of the graph G of Figure 6.9

In general, if $t \ge 2$, then we start with a majestic 3-coloring c_0 of the bipartite graph G and define $c(e) = c_0(e) \cup \{4, 5, \ldots, t+2\}$ for each edge e of G to obtain a majestic t-tone $(t+2)$-coloring c of G in which each edge color is a t-element subset of the set $[t+2]$.

Theorem 6.3.1 can, in fact, be extended to k-connected bipartite graphs for all integers $k \ge 2$.

Theorem 6.3.2 *Let k and t be integers such that $k, t \ge 2$. Then there exists a k-connected bipartite graph G such that* $\mathrm{maj}_t(G) = t+2$.

Proof. Let $s = \binom{tk-t+k}{k}$. First, we construct a bipartite graph G of order $2s + 2(tk - t + k) = 2(s + tk - t + k)$ as follows. The partite sets of G are

$$\begin{aligned} U &= \{u_1, u_2, \ldots, u_s\} \cup \{y_1, y_2, \ldots, y_{tk-t+k}\} \\ W &= \{w_1, w_2, \ldots, w_s\} \cup \{x_1, x_2, \ldots, x_{tk-t+k}\}. \end{aligned}$$

Let $X = \{x_1, x_2, \ldots, x_{tk-t+k}\}$ and $Y = \{y_1, y_2, \ldots, y_{tk-t+k}\}$. The subgraph $G[X \cup Y]$ of G induced by $X \cup Y$ is $K_{tk-t+k,tk-t+k}$. There are s distinct k-element subsets X_i $(1 \le i \le s)$ of X and s distinct k-element subsets Y_i $(1 \le i \le s)$ of Y. For each integer i with $1 \le i \le s$, the vertex u_i is joined to the k vertices in X_i and the vertex w_i is joined to the k vertices in Y_i. This completes the construction of G. This is illustrated in Figure 6.9 for $t = 2$. Thus, $\deg u_i = \deg w_i = k$ for $1 \le i \le s$. The graph G is k-connected.

Next, we show that $\text{maj}_t(G) = t + 2$. Assume, to the contrary, that $\text{maj}_t(G) = t + 1$. Then there exists a majestic t-tone $(t+1)$-coloring $c : E(G) \to [t+1]$ of G. Thus, either $|c'(u)| = t$ for each $u \in U$ or $|c'(w)| = t$ for each $w \in W$. Because of the symmetry of G, we may assume that $|c'(u)| = t$ for each $u \in U$. Therefore, $c'(w) = [t+1]$ for each $w \in W$. In particular, $c'(w_i) = [t+1]$ for $i = 1, 2, \ldots, s$. Since $\deg_G w_i = k$ and $c'(w_i) = [t+1]$ for each integer i with $1 \le i \le s$, the two neighbors of w_i have distinct vertex colors. Since (1) $|Y| = tk - t + k$, (2) $|c'(u)| = k$ for each $u \in U$, (3) there are exactly $t + 1$ distinct t-element sets of $[t + 1]$ and (4) $tk - t + k > tk - t + k - 1 = (k - 1)(t + 1)$, it follows that there are k vertices in Y that have the same vertex color, say Y_j consists of these k vertices of Y for some integer j with $1 \le j \le s$. However, w_j is only adjacent to the k vertices of Y_j in G, which implies that $|c'(w_i)| = t$, which is a contradiction. Therefore, $\text{maj}_t(G) = t + 2$. ■

As we mentioned before, by Theorems 5.3.4 and 5.3.6, if G is a tree or a unicyclic bipartite connected graph and let $t \ge 2$ be an integer, then $\text{maj}_t(G) = t + 1$ if and only if all end-vertices of G belong to the same partite set of G. However, this is not true for bipartite graphs in general. We are able to show this now. Let $t \ge 2$ be an integer and let H be graph given in the proof of Theorem 6.3.1. Now, let G be the graph obtained from the graph H by adding an end-vertex z that is adjacent to x_1. This is illustrated in Figure 6.11 for $t = 2$.

Then G is a bipartite graph having only one end-vertex z and so only one partite set of G contain this end-vertex of G. We show that $\text{maj}_t(G) = t + 2$. Assume, to the contrary, that there is a majestic t-tone $(t+1)$-coloring c of G. Since z is an end-vertex, it follows that $|c'(z)| = t$. Since z is adjacent to x_1, this implies that z and x_1 belong to different partite sets of G. Now, x_1 is adjacent to some vertices in Y and so z and Y

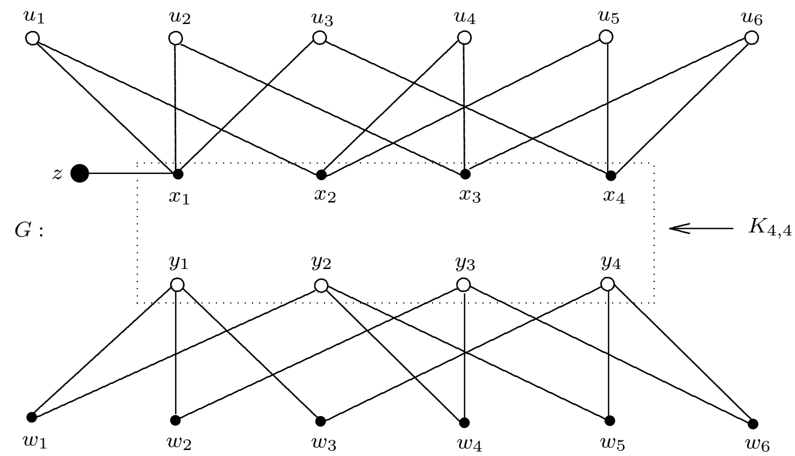


Figure 6.11: A bipartite graph G with $\mathrm{maj}_2(G) = 4$

belong to the same partite set of G. Hence, $|c'(y_i)| = t$ for each vertex $y_i \in Y$. Since there are exactly $t+1$ distinct t-element sets of $[t+1]$, there are two vertices in Y that have the same vertex color, say $c'(y_p) = c'(y_q)$, where $p, q \in \{1, 2, \ldots, t+2\}$ and $p \neq q$. Let $w \in W$ such that w is only adjacent to y_p and y_q in G. However then, $|c'(w)| = t$, which is a contradiction. Therefore, $\mathrm{maj}_t(G) = t+2$ and so, there exists connected bipartite graphs G whose end-vertices belong to the same partite set but $\mathrm{maj}_t(G) \neq t+1$.

We finish this section with a short discussion of the $n-6$ case of bipartite graphs with large cycles. For $t = 2$, the length of a longest cycle in the 2-connected bipartite graph G of order n constructed in the proof of Theorem 6.3.1 is $n-6$ and $\mathrm{maj}_2(G) = 4$. This graph is shown in Figure 6.12.

If we wished to continue in the same vein as Theorems 6.1.2–6.2.4 to classify the 2-connected bipartite graphs having maximum cycle length $n-6$, it would be necessary to require n to be at least 21. However, the graph G of Figure 6.3 also provides us with a means of constructing infinitely many 2-connected bipartite graphs having maximum cycle length $n-6$ and 2-tone majestic index 4. Let $P = (v_1, v_2, \ldots, v_{2k})$ be a path of length $2k \geq 2$ and define the graph H by $V(H) = V(G) \cup V(P)$ and $E(H) = E(G) \cup E(P) \cup \{x_1v_1, y_1v_{2k}\}$. Then H is a 2-connected bipartite graph of order $n = 20+2k$ with maximum cycle length $n-6$. An argument similar to the one given in the proof of Theorem 6.3.1 shows that $\mathrm{maj}_2(H) = 4$. Thus, there are infinitely many 2-connected bipartite graphs of sufficiently large order n having maximum cycle length $n-6$ and 2-tone majestic index 4.

On the other hand, the proofs of Theorems 6.2.4 and 6.3.1 suggest the following conjecture.

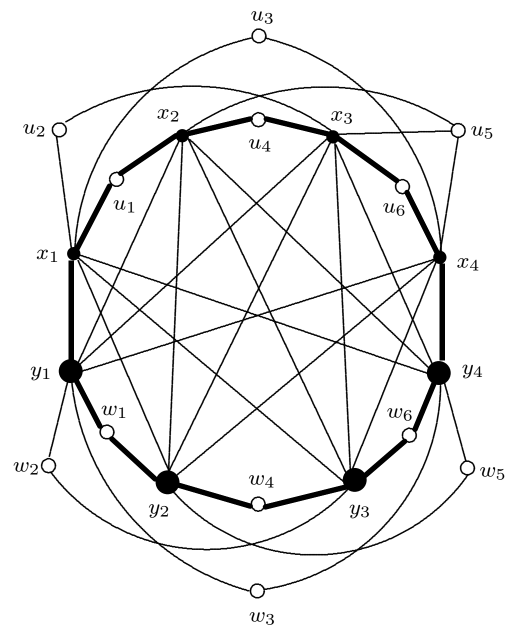


Figure 6.12: A 2-connected bipartite graph G of order 20
whose longest cycle has length 14 with $\mathrm{maj}_2(G) = 4$

Conjecture 6.3.3 *Let G be a connected bipartite graph of sufficiently large order n whose longest cycles have order $n-6$. If C is a cycle of length $n-6$ in G and the subgraph induced by the set $V(G) - V(C)$ of six vertices is not an empty graph, then* $\mathrm{maj}_t(G) = t+1$ *for each integer $t \geq 2$.*

We are not aware of any 2-connected bipartite graph of sufficiently large order n having maximum cycle length $n-6$ and t-tone majestic index $t+2$ for all integers $t \geq 2$. We conclude with the following more general question.

Problem 6.3.4 *For given integers t and r with $t \geq 2$ and $r \geq 6$, does there exist a 2-connected bipartite graph G of sufficiently large order n having maximum cycle length $n-r$ such that* $\mathrm{maj}_t(G) = t+2$*?*

Bibliography

[1] L. Addario-Berry, R. E. L. Aldred, K. Dalal and B. A. Reed, Vertex colouring edge partitions. *J. Combin. Theory Ser. B* **94** (2005) 237-244.

[2] M. Aigner and E. Triesch, Irregular assignments and two problems á la Ringel. *Topics in Combinatorics and Graph Theory.* (R. Bodendiek and R. Henn, eds.). Physica, Heidelberg (1990) 29–36.

[3] M. Aigner and E. Triesch, Irregular assignments of trees and forests, *SIAM J. Discrete Math.* **3** (1990) 439–449.

[4] M. Aigner, E. Triesch and Z. Tuza, Irregular assignments and vertex-distinguishing edge-colorings of graphs. *Combinatorics' 90* Elsevier Science Pub., New York (1992) 1-9.

[5] C. Bazgan, A. Harkat-Benhamdine, H. Li and M. Woźniak, On the vertex-distinguishing proper edge-colorings of graphs. *J. Combin. Theory Ser. B.* **75** (1999) 288-301.

[6] Jean-Luc Baril and O. Togni, Neighbor-distinguishing $k-$tuple edge-colorings of graphs *Discrete Math.* **309** (2009), 5147-5157

[7] Z. Bi, S. English, I. Hart and P. Zhang, Majestic colorings of graphs. *J. Combin. Math. Combin. Comput.* To appear.

[8] R. L. Brooks, On coloring the nodes of a network. *Proc. Cambridge Philos. Soc.* **37** (1941), 194–197.

[9] A. C. Burris, On graphs with irregular coloring number 2. *Congr. Numer.* **100** (1994), 129–140.

[10] A. C. Burris, The irregular coloring number of a tree. *Discrete Math.* **141** (1995), 279–283.

[11] A. C. Burris and R. H. Schelp, Vertex-distinguishing proper edge colorings. *J. Graph Theory.* **26** (1997) 73-82.

[12] J. Černý, M. Horňák and R. Soták, Observability of a graph. *Math. Slovaca* **46** (1996) 21-31.

[13] G. Chartrand, H. Escuadro, F. Okamoto, and P. Zhang, Detectable colorings of graphs. *Util. Math.* **69** (2006), 13–32.

[14] G. Chartrand, D. Erwin, G. L. Johns and P. Zhang, Boundary vertices in graphs. *Discrete Math.* **263** (2003), 25–34.

[15] G. Chartrand and F. Harary, Planar permutation graphs. *Ann. Inst. H. Poincaré* (*Sect. B*) **3** (1967), 433–438.

[16] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs: 6th Edition*, Chapman & Hall/CRC, Boca Raton, FL (2015).

[17] G. Chartrand, B. Phinezy and P. Zhang, On closed modular colorings of regular graphs *Bull. Inst. Combin. Appl.* **66** (2012) 7–32.

[18] G. Chartrand, F. Okamoto and P. Zhang, The sigma chromatic number of a graph. *Graphs Combin.* **26** (2010) 755-773.

[19] G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Chapman & Hall/CRC Press, Boca Raton, FL (2009).

[20] G. Chartrand and P. Zhang, *A First Course in Graph Theory*, Dover, New York (2012).

[21] H. Escuadro, *Detectable Colorings of Graphs*, Ph.D. Dissertation, Western Michigan University (2006).

[22] H. Escuadro, F. Okamoto, and P. Zhang, A three-color problem in graph theory. *Bull. Inst. Combin. Appl.* **52** (2008), 65–82.

[23] O. Favaron, H. Li and R. H. Schelp, Strong edge colorings of graphs. *Discrete Math.* **159** (1996) 103-109.

[24] J. A. Gallian, A dynamic survey of graph labeling. *Electron. J. Combin.* **16** (2009) #DS6.

[25] R. B. Gnana Jothi, *Topics in Graph Theory*, Ph.D. Thesis, Madurai Kamaraj University (1991).

[26] E. Györi, M. Horňák, C. Palmer and M. Woźniak, General neighbour-distinguishing index of a graph. *Discrete Math.* **308** (2008) 827-831.

[27] F. Harary, *Graph Theory.* Addison-Wesley, Reading, MA (1969).

[28] F. Harary and M. Plantholt, The point-distinguishing chromatic index. *Graphs and Applications.* Wiley, New York (1985) 147-162.

[29] J. E. Hopcroft and M. S. Krishnamoorthy, On the harmonious coloring of graphs. *SIAM J. Algebraic Discrete Methods* **4** (1983), 306–311.

[30] M. Horňák and R. Soták, Observability of complete multipartite graphs with equipotent parts. *Ars Combin.* **41** (1995) 289-301.

[31] M. Horňák and R. Soták, Asymptotic behaviour of the observability of Q_n. *Discrete Math.* 176 (1997) 139-148.

[32] M. Horňák, R. Soták, C. Palmer and M. Woźniak, General neighbour-distinguishing index of a graph. *Discrete Math.* **308** (2008) 827-831.

[33] M. Horňák and R. Soták, General neighbour-distinguishing index via chromatic number. *Discrete Math.* **310** (2010) 1733-1736.

[34] R. Jones, *Modular and Graceful Edge Colorings of Graphs.* Ph D Dissertation, Western Michigan University (2011).

[35] M. Kalkowski, M. Karoński, and F. Pfender, Vertex-coloring edge-weightings: Towards the 1-2-3 Conjecture. *J. Combin. Theory Ser. B.* **100** (2010), 347–349.

[36] R. Jones, K. Kolasinski, F. Okamoto and P. Zhang, On modular edge-graceful graphs, *Graphs Combin.* **29** (2013), 901–912.

[37] T. Meagher, Multi-coloring and Mycielskis construction (2010)

[38] M. Karoński, T. Łuczak, and A. Thomason, Edge weights and vertex colours. *J. Combin. Theory Ser. B* **91** (2004), 151–157.

[39] F. Okamoto, E. Salehi and P. Zhang, A checkerboard problem and modular colorings of graphs. *Bull. Inst. Combin Appl.* **58** (2010) 29-47.

[40] J. Przybylo and M. Woźniak, On a 1, 2 Conjecture, *Discrete Math. Theor. Comput. Sci.* **12** (2010) 101–108.

[41] S. Poljak and F. S. Roberts, An application of Stahl's conjecture about the k-tuple chromatic numbers of Kneser graphs. *The mathematics of preference, choice and order*, Stud. Choice Welf., Springer, Berlin, (2009) 345-352.

[42] J. Rieder On k-Tuple-Colorings of Graphs, Technical Report , 16 p. (1986)

[43] S. Stahl n-Tuple colorings and associated graphs. *J. Combin. Theory, Series B* **20** (1976), 185-203

[44] V. G. Vizing, On an estimate of the chromatic class of a p-graph. (Russian) *Diskret. Analiz.* **3** (1964) 25-30.

[45] P. Zhang, *Color-Induced Graph Colorings.* Springer, New York (2015).

[46] P. Zhang, *A Kaleidoscopic View of Graph Colorings.* Springer, New York (2016).

[47] Z. Zhang, L. Liu and J. Wang, Adjacent strong edge coloring of graphs. *Appl. Math. Lett.* **15** (2002) 623-626.