Generalized Line Graphs

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Generalized Line Graphs

by

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Mohra Alqahtani
With every nonempty graph, there are associated many graphs. One of the best known and most studied of these is the line graph $L(G)$ of a graph $G$, whose vertices are the edges of $G$ and where two vertices of $L(G)$ are adjacent if the corresponding edges of $G$ are adjacent. This concept was implicitly introduced by Whitney in 1932. Over the years, characterizations of graphs that are line graphs have been given, as well as graphs whose line graphs have some specified property. For example, Beineke characterized graphs that are line graphs by forbidding certain graphs that can be subgraphs. Sedlacek characterized those graphs whose line graph is planar. Harary and Nash-Williams characterized those graphs whose line graph is Hamiltonian. Chartrand and Wall proved that if $G$ is a connected graph all of whose vertices have degree 3 or more, then, although $L(G)$ may not be Hamiltonian, the line graph of $L(G)$ must be Hamiltonian.

Over the years, various generalizations of line graphs have been introduced and studied by many. Among them are Schwenk graphs and $k$-line graphs introduced in 2015 and 2016 here at Western Michigan University. This study introduces a generalization of line graphs and discusses several well-known structural properties of this class of graphs. Furthermore, it establishes a number of characterizations of connected graphs whose generalized line graphs possess some prescribed graph structure.
# TABLE OF CONTENTS

ACKNOWLEDGMENTS ................................................................. ii

LIST OF FIGURES ................................................................. v

CHAPTER

1. Introduction

1.1 Line Graphs ................................................................. 1

1.2 Some Known Results on Line Graphs ................................. 2

1.3 Schwenk Graphs and $\ell$-Line Graphs ............................... 3

1.4 Another Generalization of Line Graphs ............................... 6

2. Planarity of 3-Path Graphs

2.1 Introduction ................................................................. 11

2.2 Trees ................................................................. 14

2.3 On 2-Connected Graphs ................................................ 17

2.4 Non-Acyclic Graphs with Connectivity 1 ............................ 21

2.5 A Characterization of Planar 3-Path Graphs ........................ 27

3. Outerplanarity of 3-Path Graphs

3.1 Introduction ................................................................. 28

3.2 Preliminary Results ...................................................... 30

3.3 Trees and 2-Connected Graphs ........................................ 31

3.4 A Characterization of Outerplanar 3-Path Graphs ................. 33
CHAPTER

4. Hamiltonlicity of 3-Path Graphs

4.1 Introduction ................................................................. 35
4.2 A More General Result ......................................................... 37
4.3 Hamiltonian-Connected 3-Path Graphs ...................................... 44

5. Tree-Connected Graphs

5.1 Introduction ................................................................. 61
5.2 Graphs $C_5 \Box C_5$ and $P \Box K_2$ ........................................ 64
5.3 On $k$-Tree-Connected 3-Path Graphs ..................................... 73

6. The 4-Path Graph of a Graph

6.1 Introduction ................................................................. 84
6.2 The 4-Path Graphs of Some Well-Known Graphs ......................... 89
6.3 Graphs Isomorphic to Their 4-Path Graph ................................ 93
6.4 Graphs Whose 4-Path Graphs are Paths or Cycles ....................... 101
6.5 Graphs Having Connected 4-Path Graphs ............................... 110

7. Topics for Further Study

7.1 Which Graphs are 4-Path Graphs? ......................................... 117
7.2 Distance in 4-Path Graphs ................................................ 124
7.3 Planarity and Hamiltonicity ................................................ 133

Bibliography ................................................................. 135
4.15 Illustrating the proof of Subcase 1.1.................................59
5.1 Showing that $K_{3,3} \lor K_1$ is 3-tree-connected....................63
5.2 The graph $K_{r,r} \lor K_2$..............................................64
5.3 The graph $C_5 \Box C_5$.................................................65
5.4 Illustrating a step of the proof........................................67
5.5 The graph $P \Box K_2$..................................................69
5.6 Illustrating a step of the proof........................................71
5.7 The 3-paths $P$ and $Q$ have the edge $e_2$ in common...............75
5.8 The 3-paths $P$ and $Q$ have no edge in common.....................78
5.9 A step in the proof of Subcase 3.1.................................80
6.1 A graph $G_k$ whose $k$-path graph is disconnected...............86
6.2 A graph $G$ without $k$-path bridges whose $k$-path graph contains cut-vertices...89
6.3 The double star $S_{3,4}$ and the tree $S^*_{3,4}$..........................95
6.4 The trees $T_1$ and $T_2$.................................................97
6.5 Two graphs whose 4-path graph is $P_4$..............................102
6.6 The graphs $F_i$ for $i = 1, 2, \ldots, 6$.................................104
6.7 The graphs $H$ and $P_4(H)$...........................................105
6.8 The subgraphs $F_1, F_2$ and $F_3$ in $G$ in the proof of Lemma 6.4.5...........105
6.9 The graphs $T_1, T_2$ and $T_3$........................................108
6.10 Showing that $P_4(F_5) = C_8$ and $P_4(F_6) = C_9$...............109
6.11 The graph $T_{a,b}$....................................................112
6.12 The subtrees $T_1$ and $T_2$ in Case 1...............................112
6.13 The graphs $T'_i$ for $i = 1, 2, 3, 4$ in Case 2.....................113
6.14 The graphs $H_i$ for $i = 1, 2, ..., 11$ .................................................. 115

7.1 The induced subgraphs not contained in any line graph ....................... 117

7.2 An example of 4-path graph ............................................................... 118

7.3 The graphs $F_1$ and $F_2$ ................................................................. 120

7.4 The graphs $H_1$ and $H_2$ ............................................................... 121

7.5 The graph $C_5 + e$ is not a 4-path graph ........................................ 121

7.6 The graphs $H$ and $\mathcal{P}_4(H)$ ..................................................... 122

7.7 The graphs $S(K_{1,4})$ and its 4-path graph ...................................... 123

7.8 Illustrating the 4-paths in a caterpillar $T$ ........................................ 127

7.9 The caterpillar $T$ and its 4-path graph $\mathcal{P}_4(T)$ ......................... 130

7.10 The graph $K_5$ .............................................................................. 134
Chapter 1

Introduction

1.1 Line Graphs

One of the best known and most studied graphs associated with a graph is its line graph. The line graph $L(G)$ of a nonempty graph $G$ has the set of edges in $G$ as its vertex set and where two vertices of $L(G)$ are adjacent if the corresponding edges of $G$ are adjacent. A graph $F$ of size 6 and its line graph $L(F)$ of order 6 are shown in Figure 1.1, where the six vertices $a, b, c, d, e, f$ of $L(F)$ correspond to the six edges of $F$.

![Figure 1.1: A graph and its line graph](image)

The concept of line graphs was implicitly introduced in 1932 by Whitney [32]. Since then, the study of line graphs has been a classical topic of research in graph theory. Prisner described in [28] what makes line graphs a worthwhile concept to study and how the line graph provides another way of looking at the graph.

Large parts of graph theory could be formulated in terms of line graphs. For instance matching theory deals with independent vertex sets in the line graph, edge coloring is vertex coloring of the corresponding line graph and the edge reconstruction conjecture is the vertex reconstruction conjecture restricted to line graphs.
As Edelsbrunner [16] expressed it (in the context of computational geometry):

“There is some magic to ... transformations which has to do with the way human beings understand ... problems. It is fairly obvious that the transformation of a problem to another problem cannot lead to anything new, in particular if the transformation realizes a one-to-one correspondence. Still, there is an impressive large collection of ... problems where the transformations into other problems plays a crucial role in their solutions. One explanation of this empirical fact is that the transformation of a problem shifts the emphasis to different aspects of the problem which helps the human investigation to study the problem from a new angle, so to speak.

We refer to the book [10] for graph theory notation and terminology not described in this work.

1.2 Some Known Results on Line Graphs

In this section, we present some elementary properties and several of the best known results on line graphs (see [10, 14]). If $G$ is a graph of order $n$ and size $m$ with degree sequence $d_1, d_2, \ldots, d_n$, then the order of its line graph $L(G)$ is $m$ and the size of $L(G)$ is $\sum_{i=1}^{n} \binom{d_i}{2}$ since each edge of $L(G)$ corresponds to a pair of adjacent edges of $G$. Observe that $L(K_3) = L(K_{1,3}) = K_3$. The following result is due to Whitney [32].

**Theorem 1.2.1** [32] Let $G_1$ and $G_2$ be nontrivial connected graphs. If $L(G_1) \cong L(G_2)$, then $G_1 \cong G_2$ unless one of $G_1$ and $G_2$ is $K_3$ and the other is $K_{1,3}$.

A Hamiltonian cycle in a graph $G$ is a cycle containing every vertex of $G$ and a graph having a Hamiltonian cycle is a Hamiltonian graph. A characterization of graphs whose line graph is Hamiltonian was obtained by Harary and Nash-Williams [23]. A circuit $C$ in a graph $G$ is called a dominating circuit if every edge of $G$ is incident with a vertex of $C$.

**Theorem 1.2.2** [23] Let $G$ be a graph without isolated vertices. Then $L(G)$ is Hamiltonian if and only if $G = K_{1,\ell}$ for some $\ell \geq 3$ or $G$ contains a dominating circuit.

For a nonempty graph $G$, we write $L^0(G)$ to denote $G$ and $L^1(G)$ to denote $L(G)$. For an integer $k \geq 2$, the $k$th iterated line graph $L^k(G)$ is defined as $L(L^{k-1}(G))$, where
$L^{k-1}(G)$ is assumed to be nonempty. Iterated line graphs of almost all connected graphs were shown to be Hamiltonian by Chartrand [7].

**Theorem 1.2.3** [7] If $G$ is a connected graph that is not a path, then there exists an integer $k_0$ such that $L^k(G)$ is Hamiltonian for every integer $k \geq k_0$.

A graph $G$ is called a line graph if there exists a graph $H$ such that $G = L(H)$. While several characterizations of line graphs have been obtained, perhaps the best known is a forbidden subgraph characterization due to Beineke [3].

**Theorem 1.2.4** [3] A graph $G$ is a line graph if and only if none of the nine graphs of Figure 7.1 is isomorphic to an induced subgraph of $G$.

![Figure 1.2: The induced subgraphs not contained in any line graph](image)

### 1.3 Schwenk Graphs and $\ell$-Line Graphs

There are many graphs associated with a given graph. Such graphs are referred to as “derived graphs”. For a given graph $G$, a derived graph of $G$ is a graph obtained from $G$ by a graph operation of some type. The study of the structural properties of derived graphs is a popular area of research in graph theory. While one of the most familiar graph operations on a graph is that of the line graph, over the years various generalizations of line graphs have been introduced and studied by many (see [1, 2, 25, 28], for example). Here, we describe two recent generalizations of line graphs (see [1]).
The *girth* of a graph $G$ (with cycles) is the length of a smallest cycle of $G$ and is denoted by $g(G)$. For each pair $r, g$ of integers with $r \geq 2$ and $g \geq 3$, there exists a graph of minimum order that is both $r$-regular graph and has girth $g$ (see [17]). Such a graph is called a *cage*. For $r = 2$ and $g = 5$, the 5-cycle $C_5$ is the unique cage; while for $r = 3$ and $g = 5$, the Petersen graph $P$ is the unique 5-cage. The 5-cycle and the Petersen graph $P$ are shown in Figure 1.3.

![Figure 1.3: The 5-cycle and the Petersen graph $P$](image)

In 2015, Schwenk introduced a new class of derived graphs when he was investigating problems involving cages, and graphs in general having odd girth 5 or more. For a connected graph $G$ having girth $2k + 1 \geq 5$ for some integer $k \geq 2$, the *Schwenk graph* $G^*$ of $G$ has the set of all paths of order $k + 1$ (or $(k + 1)$-paths) of $G$ as its vertex set $V(G^*)$, where two vertices $P$ and $Q$ of $G^*$ are adjacent in $G^*$ if $P = (u_1, u_2, \ldots, u_{k+1})$ and $Q = (v_1, v_2, \ldots, v_{k+1})$ such that $u_{k+1} = v_1$, $V(P) \cap V(Q) = \{u_{k+1}\}$ and $u_1v_{k+1} \in E(G)$. Since the girth of $G$ is $2k + 1$, it follows that the subgraph of $G$ induced by $V(P) \cup V(Q)$ is $G[V(P) \cup V(Q)] \cong C_{2k+1}$. For the special case where $G$ is a connected graph of girth 5, the Schwenk graph $G^*$ of $G$ is defined as that graph whose vertex set is the set of all 3-paths (paths of order 3) of $G$, where two vertices $P$ and $Q$ of $G^*$ (two 3-paths $P$ and $Q$ of $G$) are adjacent in $G^*$ if they have an end-vertex in common but no other vertex in common and the subgraph of $G$ induced by $V(P) \cup V(Q)$ is a 5-cycle. This is illustrated in Figure 1.4 when $P = (u, v, w)$ and $Q = (w, x, y)$.

![Figure 1.4: Illustrating adjacency in a Schwenk graph](image)
In [1], it was shown that the Schwenk graph is triangle-free and for each odd integer $g \geq 5$, there exists a connected graph of girth $g$ whose Schwenk graph contains 4-cycles. Connected graphs of girth 5 whose Schwenk graph contains 4-cycles were characterized in [1]. Structural properties of the Schwenk graphs of the unique 5-cage (the Petersen graph) and the unique 7-cage (the McGee graph) were also investigated in [1]. Other results and open questions were also presented in [1] for the Schwenk graphs of cages.

A more general class of derived graphs was inspired by line graphs and Schwenk graphs. Let $G$ be a connected graph of order at least 3. Two nontrivial paths $P$ and $Q$ in $G$ are said to be adjacent paths in $G$ if $V(P) \cap V(Q) = \{x\}$ where $x$ is an end-vertex of both $P$ and $Q$. For an integer $\ell \geq 2$, the $\ell$-line graph $L_\ell(G)$ of a graph $G$ is the graph whose vertex set is the set of all $\ell$-paths (paths of order $\ell$) of $G$ where two vertices of $L_\ell(G)$ are adjacent if they are adjacent $\ell$-paths in $G$. Since the 2-line graph is the line graph $L(G)$ for every graph $G$, this is a generalization of line graphs. If $G$ is a graph with $g(G) = 2k + 1$ for some integer $k \geq 2$, then the Schwenk graph is a subgraph of $L_{k+1}(G)$. To illustrate this concept, let $T$ be the tree obtained from $P_7 = (u_1, u_2, \ldots, u_7)$ be adding two new edges $xy$ and $vw$ and then joining $x$ to $u_3$ and $v$ to $u_5$ (see Figure 1.5).

The 3-line graph $L_3(T)$ is also shown in that figure.

![Figure 1.5: A tree $T$ and its 3-line graph $L_3(T)$](image)

The concept of $\ell$-line graph was introduced by Chartrand in 2015 and studied extensively in [1], where the emphasis was on the 3-line graphs. Several sufficient conditions have been presented for the 3-line graph of a connected graph to be connected. While the 3-line graph of a connected bipartite graph is disconnected, it has been shown that the
3-line graph of every connected bipartite graph has at most two nontrivial components. Furthermore, planar and outerplanar properties of the 3-line graph of connected graphs have been investigated and characterizations have been established for those trees having a planar or outerplanar 3-line graph by means of forbidden subtrees.

1.4 Another Generalization of Line Graphs

Another more general class of derived graphs was inspired by line graphs. Observe that the vertex set of the line graph $L(G)$ is the set of 2-paths of $G$ (the paths $P_2$ of order 2) and where two vertices of $L(G)$ are adjacent if the corresponding paths of $G$ have a path $P_1$ in common. This observation leads us to a generalization of line graphs.

Let $k \geq 2$ be an integer and let $G$ be a graph containing $k$-paths. The $k$-path graph $P_k(G)$ of $G$ has the set of $k$-paths of $G$ as its vertex set and where two distinct vertices of $P_k(G)$ are adjacent if the corresponding $k$-paths of $G$ have a $(k - 1)$-path in common. Thus, the 2-path graph of a nonempty graph is its line graph. This concept was introduced by Chartrand and first studied in [6].

Let’s consider the case when $k = 3$, that is, the 3-path graph $P_3(G)$ of a connected graph $G$ containing 3-paths, which has the set of 3-paths in $G$ as its vertex set and where two distinct vertices of $P_3(G)$ are adjacent if the corresponding 3-paths of $G$ have a 2-path (an edge) in common. Since every 3-path in a graph $G$ is both a vertex of $P_3(G)$ and an edge of $L(G)$ and every 3-path is obtained from a pair of adjacent edges of $G$, we have the following observation (see [6]).

Proposition 1.4.1  If $G$ is a connected graph of order at least 3, then

$$P_3(G) = L(L(G)).$$

Therefore, the 3-path graph of a graph $G$ provides a different view of $L(L(G))$. We saw that if $G$ is a connected graph of order $n \geq 3$ with degree sequence $d_1, d_2, \ldots, d_n$, then the size of $L(G)$ is $\sum_{i=1}^{n} \binom{d_i}{2}$ and so the order of $P_3(G)$ is $\sum_{i=1}^{n} \binom{d_i}{2}$ by Proposition 1.4.1. For example, consider the graph $F$ of Figure 1.6 along with its 3-path graph $P_3(F)$. Since $\sum_{i=1}^{6} \binom{\deg_F(v_i)}{2} = 8$, it follows that the 3-path graph of $F$ has order 8. The set of 3-paths of $F$ and, consequently, the vertex set of $P_3(F)$ is

$$V(P_3(F)) = \{ (v_2, v_1, v_3), (v_1, v_2, v_3), (v_1, v_3, v_2), (v_1, v_3, v_4), (v_2, v_1, v_4), (v_3, v_4, v_5), (v_3, v_4, v_6), (v_5, v_4, v_6) \}.$$
The vertex set of $\mathcal{P}_3(G)$ of a graph $G$ can be described more simply in terms of the edges of the 3-paths of $G$. For example, if $P$ is a 3-path of $G$ whose two edges are $g$ and $h$, then we represent $P$ by $gh$ (or $hg$). Consequently, two distinct vertices $gh$ and $jk$ of $\mathcal{P}_3(G)$ are adjacent if and only if one of $g$ and $h$ is one of $j$ and $k$. For the graph $F$ of Figure 1.6, the vertex set of $\mathcal{P}_3(F)$ is therefore $V(\mathcal{P}_3(F)) = \{ab, ac, bc, bd, cd, de, df, ef\}$. Let’s now look at how the degrees of the vertices of $\mathcal{P}_3(G)$ are computed for some graph $G$ in terms of the 3-paths of $G$.

**Proposition 1.4.2** Let $G$ be a connected graph of order at least 3. If $x = (u, w, v)$ is a 3-path in $G$, then the degree of $x$ in the 3-path graph $\mathcal{P}_3(G)$ of $G$ is

$$\deg_{\mathcal{P}_3(G)} x = \deg_G u + 2\deg_G w + \deg_G v - 6.$$ 

**Proof.** Let $x = (u, w, v)$ be a 3-path in $G$. Let $N_G(u) - \{w\} = \{u_1, u_2, \ldots, u_a\}$ if $a = \deg_G u - 1 \geq 1$ (where $N_G(u) - \{w\} = \emptyset$ if $a = 0$), let $N_G(w) - \{u, v\} = \{w_1, w_2, \ldots, w_b\}$ if $b = \deg_G w - 2 \geq 1$ (where $N_G(w) - \{u, v\} = \emptyset$ if $b = 0$), and let $N_G(v) - \{w\} = \{v_1, v_2, \ldots, v_c\}$ if $c = \deg_G v - 1 \geq 1$ (where $N_G(v) - \{w\} = \emptyset$ if $c = 0$).

In the 3-path graph $\mathcal{P}_3(G)$ of $G$ then, the vertex $x$ is adjacent only to

(i) each 3-path $(u_i, u, w)$ for $1 \leq i \leq a = \deg_G u - 1$,

(ii) each 3-path $(u, w, w_j)$ and each 3-path $(w_j, w, v)$ for $1 \leq j \leq b = \deg_G w - 2$, and

(iii) each 3-path $(w, v, v_k)$ for $1 \leq k \leq c = \deg_G v - 1$.

Thus, the degree of $x$ in $\mathcal{P}_3(G)$ is

$$\deg_{\mathcal{P}_3(G)} x = a + 2b + c = (\deg_G u - 1) + 2(\deg_G w - 2) + (\deg_G v - 1)$$

$$= \deg_G u + 2\deg_G w + \deg_G v - 6,$$

proving the desired result.

As an illustration, consider the graphs $F$ and $H = \mathcal{P}_3(F)$ of Figure 1.7. Since $\sum_{i=1}^{5} \binom{\deg_F u_i}{2} = 8$, the 3-path graph $\mathcal{P}_3(F)$ has order 8. Therefore,
\[ \deg_H ab = 6, \quad \deg_H ac = \deg_H bc = \deg_H de = 4, \]
\[ \deg_H ad = \deg_H ae = \deg_H bd = \deg_H be = 5. \]

Since the sum of these degrees is 38, the size of \( H = \mathcal{P}_3(F) \) is 19.

\[ F: \quad b \quad d \quad a \quad e \]
\[ a \quad e \quad c \quad v_4 \]
\[ v_1 \quad b \quad d \quad a \quad e \quad c \quad v_4 \]
\[ v_2 \quad v_3 \]

\[ H = \mathcal{P}_3(F): \quad ab \quad ac \quad de \quad ae \quad bd \quad bc \]

\[ \text{Figure 1.7: A graph } F \text{ and its 3-path graph } \mathcal{P}_3(F) \]

Next, we consider a few examples of \( L(L(G)) \) of some well-known graphs \( G \) from the view of the 3-path graph \( \mathcal{P}_3(G) \) of the graph \( G \).

\( * \) It is clear that \( \mathcal{P}_3(K_3) = K_3 \). For an integer \( n \geq 4 \), the order of \( \mathcal{P}_3(K_n) \) is \( n \binom{n-1}{2} \).

Since \( K_n \) is \((n-1)\)-regular, it follows that \( \mathcal{P}_3(K_n) \) is \((4n-10)\)-regular graph. Thus, the size of \( \mathcal{P}_3(K_n) \) is \( \frac{1}{2}(4n-10)n \binom{n-1}{2} = n(2n-5) \binom{n-1}{2} \). In particular, \( \mathcal{P}_3(K_4) \) is a 6-regular graph of order 12 and size 36.

\( \star \) It is clear that \( \mathcal{P}_3(K_2,2) = K_{2,2} \). For each integer \( r \geq 3 \), the order of \( \mathcal{P}_3(K_{r,r}) \) is \( 2r \binom{r}{2} = r^2(r-1) \). Since \( K_{r,r} \) is \( r \)-regular, it follows that \( \mathcal{P}_3(K_{r,r}) \) is \((4r-6)\)-regular and the size of \( \mathcal{P}_3(K_{r,r}) \) is \( \frac{1}{2}(4r-6)r^2(r-1) = r^2(r-1)(2r-3) \).

\( \star \) The 3-path graph \( \mathcal{P}_3(P) \) of the Petersen graph \( P \) is a 6-regular graph of order 30 and size 90.

The following is a useful consequence of Proposition 1.4.1

**Corollary 1.4.3** If \( H \) is a subgraph of \( G \), then \( \mathcal{P}_3(H) \) is a subgraph of \( \mathcal{P}_3(G) \).

It is well known that if \( G \) is a nontrivial connected graph, then \( L(G) \) is connected (see [14], for example). The following is another consequence of Proposition 1.4.1

**Corollary 1.4.4** If \( G \) is a connected graph such that \( L(G) \) is not empty, then \( \mathcal{P}_3(G) \) is also connected.
By Corollary 1.4.4, the 3-path graph of a connected graph of order at least 3 is connected. It is not surprising that the 3-path graph of a connected graph of order at least 3 may contain cut-vertices. Let $G$ be a connected graph that is not complete. A vertex $z$ of $G$ is called a cut-vertex if $G - z$ is disconnected. A 3-path $(u, w, v)$ is called a 3-path bridge of $G$ if $w$ is a cut-vertex of $G$ and $\deg_G w = 2$ and $u$ and $v$ are not end-vertices of $G$. Thus, each of the two edges $uw$ and $uv$ is a bridge of $G$. Therefore, no 3-path bridge lies on a cycle of $G$. Next, we characterize those 3-paths in a connected graph $G$ that are cut-vertices of $P_3(G)$.

**Proposition 1.4.5** Let $G$ be a connected graph of order at least 3 and let $(u, w, v)$ be a 3-path of $G$. Then $(u, w, v)$ is a 3-path bridge of $G$ if and only if $(u, w, v)$ is a cut-vertex of $P_3(G)$.

**Proof.** Let $z = (u, w, v) = ab$ be a 3-path of $G$ where $a = uw$ and $b = vw$. First, suppose that $z$ is a 3-path bridge in $G$. We show that $z$ is a cut-vertex of $P_3(G)$. Since $w$ is a cut-vertex of $G$ and $u$ and $v$ are not end-vertices of $G$, it follows that (i) $G - w$ contains two components $G_1$ and $G_2$ such that $u \in V(G_1)$ and $v \in V(G_2)$ and (ii) there are vertices $u' \in V(G_1) - \{u\}$ and $v' \in V(G_2) - \{v\}$ such that $uw', v'v \in E(G)$. Then $(u', u, w, v, v')$ is a 5-path in $G$. Let $x = (u', u, w)$ and $y = (w, v, v')$ be two 3-paths in $G$.

We show that every $x - y$ path in $P_3(G)$ must contain $z$. Assume, to the contrary, that there is an $x - y$ path $Q = (x = x_0, x_1, \ldots, x_{k-1}, x_k = y)$ in $P_3(G)$ that does not contain $z$. Since $\deg_G w = 2$, it follows that $x_1 = (u, u', u_1) \neq x$ and $x_{k-1} = (v, v', v_1) \neq y$. Thus, $Q$ gives rise to a closed walk $(w, u, u', u_1, \ldots, v_1, v', v, w)$ in $G$. However, this implies that $(u, w, v)$ lies on a cycle in $G$, which is impossible. Since every $x - y$ path in $P_3(G)$ must contain $z$, it follows that $z$ is a cut-vertex of $P_3(G)$.

For the converse, suppose that $z$ is a cut-vertex of $P_3(G)$. We show that $z$ is a 3-path bridge of $G$. Then $P_3(G) - z$ is disconnected and so contains at least two components and $z$ is adjacent to at least one vertex in each component of $P_3(G) - z$. Let $H_1$ and $H_2$ be two components $P_3(G) - z$. Now, let $x \in V(H_1)$ and $y \in V(H_2)$ such that $zx, zy \in E(P_3(G))$. Necessarily, $xy \notin E(P_3(G))$. We may assume, without loss of generality, that $x$ contains $a$ and $y$ contains $b$. First, we show that $\deg_G w = 2$. Assume, to the contrary, that $\deg_G w \geq 3$. Let $e$ be an edge incident with $w$ that is distinct from $a$ and $b$. Let $x' = ae$ and $y' = eb$ be two 3-paths in $P_3(G)$. Then $(x, x', y', y)$ is an $x - y$ path in $P_3(G)$ that avoids $z$, which contradicts the fact that $z$ is a cut-vertex of $P_3(G)$. Thus, $\deg_G w = 2$. This implies that $x = (w, u, u')$ and $y = (w, v, v')$ for some vertices...
\( u', v' \in V(G) - \{w, u, v\} \) and so \( u \) and \( v \) are not end-vertices of \( G \). It remains to show that \( w \) is a cut-vertex of \( G \). If \( w \) is not a cut-vertex of \( G \), then there is a \( u - v \) path \( P = (u = u_0, u_1, \ldots, u_k = v) \) in \( G - w \) that does not contain the 3-path \((u, w, v)\). Since \( x \) is adjacent to the 3-path \((u', u, u_1)\) and \( y \) is adjacent to the 3-path \((u_{k-1}, v, v')\) in \( P_3(G) \), it follows that there is an \( x - y \) path in \( P_3(G) \) (using 3-paths from \( P \)) that avoids \( ab = z \). This is impossible since \( ab \) is a cut-vertex of \( P_3(G) \).

To illustrate Proposition 1.4.5, we consider the graph \( G \) of Figure 1.8. The 3-path \( ab = (u, w, v) \) is a 3-path bridge of \( G \) and \( ab \) is a cut-vertex in \( P_3(G) \).

![Figure 1.8: A graph \( G \) with a 3-path bridge \( ab \) and its 3-path graph](image)

We saw that \( P_2(G) = L(G) = L^1(G) \) and \( P_3(G) = L^2(G) \) for every connected graph \( G \) of order at least 3 by Proposition 1.4.1. However, if \( k \geq 4 \) and \( G \) is a connected graph containing \( k \)-paths, then \( P_k(G) \neq L^{k-1}(G) \) in general. For example, if \( G \) is the double star of order 5, then \( L(G) = K_{1,3} + e \). Thus, \( L^2(G) = K_4 - e \) and so \( L^3(G) = C_4 \lor K_1 \). Since \( P_4(G) = K_2 \), it follows that \( P_4(G) \neq L^3(G) \). In fact, as we will see in Chapter 6, the structure of \( P_k(G) \) can be quite different from that of \( L^{k-1}(G) \) for an arbitrary connected graph \( G \).

In this work, we study some well-known structural properties of the iterated line graph \( L(L(G)) \) of a graph \( G \) by means of its 3-path graph \( P_3(G) \). Because of this new view of 3-path graphs, we are able to apply some different techniques to study this concept. We also study \( k \)-path graphs in general with an emphasis on 4-path graphs. We refer to the book [10] for graph theory notation and terminology not described in this work.
Chapter 2

Planarity of 3-Path Graphs

2.1 Introduction

In the next three chapters, we consider three well-known properties a graph can possess, with an emphasis on 3-path graphs possessing these properties. We begin, in this chapter, by presenting a characterization of those connected graphs having a planar 3-path graph.

A graph $G$ is planar if it can be drawn in the plane without any two of its edges crossing. Such a drawing is also called a planar embedding of $G$. A graph $H$ is a subdivision of a graph $G$ if either $H \cong G$ or $H$ can be obtained from $G$ by inserting vertices of degree 2 into some of the edges of $G$. A subdivision $H$ of a graph $G$ is planar if and only if $G$ is planar. The graphs $K_5$ and $K_{3,3}$ and their subdivisions (see Figure 2.1) play a pivotal role in the study of planar graphs. In fact, Kuratowski [24] obtained the following characterization of planar graphs.

Figure 2.1: Subdivisions of $K_5$ and $K_{3,3}$

**Theorem 2.1.1** [24] A graph $G$ is planar if and only if $G$ contains no subgraph that is a subdivision of $K_5$ or $K_{3,3}$.

In 1962 Sedláček [31] characterized those graphs $G$ for which $L(G)$ is planar.

**Theorem 2.1.2** [31] A nonempty graph $G$ has a planar line graph if and only if (i) $G$ is planar, (ii) $\Delta(G) \leq 4$ and (iii) if $\deg_G v = 4$, then $v$ is a cut-vertex of $G$. 

11
Our goal is to obtain results that correspond to Theorem 2.1.2 for 3-path graphs by determining all those graphs having a planar 3-path graph. Since the 3-path graph is only defined for graphs having 3-paths and the 3-path graph of a disconnected graph $G$ is the union of the 3-path graphs of the components of $G$, it suffices to consider only connected graphs of order 3 or more.

The graphs $K_4 - e$ and $K_{2,3}$ (a subdivision of $K_4 - e$) are shown in Figure 2.2, whose edges are labeled $e_1, e_2, \ldots, e_5$ and $f_1, f_2, f_3, f_4, f_5, f_6$, respectively. We denote a 3-path $e_ie_j$ in $K_4 - e$ and a 3-path $f_if_j$ in $K_{2,3}$ by $ij$. While the 3-path graph $P_3(K_4 - e)$ is nonplanar, as it contains the subdivision $H$ of $K_{3,3}$ shown in Figure 2.2, the 3-path graph $P_3(K_{2,3})$ is planar, where a planar embedding of $P_3(K_{2,3})$ is shown in Figure 2.2. Consequently, it is not true that the 3-path graph of a subdivision $H$ of a graph $G$ is planar if and only if the 3-path graph of $G$ is planar.

![Figure 2.2: Showing that $P_3(K_4 - e)$ is nonplanar and $P_3(K_{2,3})$ is planar.](image)

As another example of this phenomenon, we consider the two planar graphs $F$ and $H$ shown in Figure 2.3, where $F$ is a subdivision of $K_4 - e$ and $H$ is a subdivision of $F$. Since $P_3(F)$ contains a subdivision of $K_{3,3}$ (shown in Figure 2.3), it follows that $P_3(F)$ is nonplanar. Since $P_3(H)$ has a planar embedding (shown in Figure 2.3), it follows that $P_3(H)$ is planar.

We now derive some necessary conditions for a connected graph to have a planar 3-path graph.
Lemma 2.1.3  If $G$ is a connected graph with $\Delta(G) \geq 5$, then $\mathcal{P}_3(G)$ is nonplanar.

Proof.  Let $u$ be a vertex of $G$ of degree 5 or more, where the five edges incident with $u$ are labeled 1, 2, ..., 5. Then $\mathcal{P}_3(G)$ contains a subdivision $F$ of $K_{3,3}$ as a subgraph (shown in Figure 2.4). More precisely, for the graph $K_{3,3}$ with partite sets $\{12, 34, 35\}$ and $\{13, 23, 14\}$, the graph $F$ is obtained from $K_{3,3}$ by inserting the vertex 45 into the edge joining 35 and 14 in $\mathcal{P}_3(G)$. Therefore, $\mathcal{P}_3(G)$ is not planar.  

Lemma 2.1.4  If a connected graph $G$ contains a vertex of degree 4 that is adjacent to a vertex of degree at least 3, then $\mathcal{P}_3(G)$ is nonplanar.

Proof.  The graph $G$ contains the subgraph $G_0$ of Figure 2.5, whose six edges are labeled 1, 2, ..., 6 (where one or two of the edges 1, 2, 3 may be adjacent to one or both...
of the edges 5 and 6). The 3-path graph \( P_3(G_0) \) contains the subdivision of \( K_{3,3} \) shown in Figure 2.5. Since \( P_3(G) \) contains a subdivision of \( K_{3,3} \), the graph \( P_3(G) \) is nonplanar.

**Figure 2.5:** A subgraph \( G_0 \) in \( G \) and a subdivision in \( P_3(G_0) \)

**Lemma 2.1.5** Let \( G \) be a connected graph such that \( P_3(G) \) is nonplanar and let \( e = uv \) be an edge of \( G \) such that at least one of \( u \) and \( v \) has degree at most 2 in \( G \). If the graph \( H \) is obtained from \( G \) by subdividing \( e \) once, then \( P_3(H) \) is also nonplanar.

**Proof.** Let \( x \) be the vertex of degree 2 inserted into \( uv \), obtaining the graph \( H \), where the edge \( uv \) is then replaced by the two edges \( ux \) and \( xv \). We may assume that \( \deg_G u \leq \deg_G v \). First, suppose that \( \deg_G u = 1 \). Then \( H' = H - u \cong G \) and \( H' \subseteq H \). Since \( P_3(H') \cong P_3(G) \) is a subgraph of \( P_3(H) \) and \( P_3(G) \) is nonplanar, it follows that \( P_3(H) \) is also nonplanar. Next, suppose that \( \deg_G u = 2 \), say \( N_G(u) = \{v, y\} \). Let \( (y, u, v, w) \) be a path in \( G \). Then \( (y, u, x, v, w) \) is a path in \( H \). Let \( a = yu, b = uv, c = vw \) in \( G \) and let \( a = yu, b_1 = ux, b_2 = xv, c = vw \) in \( H \). Then the graph \( P_3(H) \) is obtained by replacing the edge joining \( ab \) and \( bc \) in \( P_3(G) \) by the 3-path \( (ab_1, b_1b_2, b_2c) \) in \( P_3(H) \). This implies that \( P_3(H) \) is a subdivision of \( P_3(G) \). Since \( P_3(G) \) is nonplanar, it follows that \( P_3(H) \) is also nonplanar.

**2.2 Trees**

We now proceed to present necessary and sufficient conditions for a graph to have a planar 3-path graph. We first consider the special case where the graph is a tree.

**Theorem 2.2.1** A tree \( T \) of order at least 3 has a planar 3-path graph if and only if \( \Delta(T) \leq 4 \) and every vertex of degree 4 is adjacent only to vertices of degree 1 or 2.

**Proof.** As we have seen in Lemma 2.1.3, no graph with a planar 3-path graph can contain a vertex of degree 5 or more and, by Lemma 2.1.4, no graph with a planar 3-path graph can contain a vertex of degree 4 that is adjacent to a vertex of degree 3 or 4.
Hence, it remains to show that if $T$ is a tree of order 3 or more such that $\Delta(T) \leq 4$ and every vertex of degree 4 is adjacent only to vertices of degree 1 or 2, then $P_3(T)$ is planar.

We proceed by the strong form of induction. The result is clearly true for all trees of order 3, 4 or 5. Assume for an integer $k \geq 5$ that if $T'$ is a tree of order $p$ with $3 \leq p \leq k$ with $\Delta(T') \leq 4$ such that every vertex of degree 4 in $T'$ is only adjacent to vertices of degree 1 or 2 in $T'$, then $P_3(T')$ is planar. Let $T$ be a tree of order $k + 1$ with $\Delta(T) \leq 4$ such that every vertex of degree 4 in $T$ is only adjacent to vertices of degree 1 or 2 in $T$. We show that $P_3(T)$ is planar. Since the 3-path graph of every path of order 3 or more is also a path and therefore planar, we may assume that either $\Delta(T) = 4$ or $\Delta(T) = 3$. We consider these two cases.

Case 1. $\Delta(T) = 4$. Suppose that $\deg_T v = 4$, where

$$N_T(v) = \{v_1, v_2, v_3, v_4\}$$

and $a = vv_1$, $b = vv_2$, $c = vv_3$ and $d = vv_4$. By assumption, each of the vertices $v_i$ ($1 \leq i \leq 4$) has degree 1 or 2. We may assume that $\deg_T v_i = 2$ for $1 \leq i \leq 4$ (for otherwise, the 3-path graph would be the subgraph of this 3-path graph). Let the neighbor of $v_i$ ($1 \leq i \leq 4$) different from $v$ be $u_i$ and let $a_1 = u_1v_1$, $b_1 = u_2v_2$, $c_1 = u_3v_3$ and $d_1 = u_4v_4$. Furthermore, we may assume that $\deg_T u_i = 4$ for $1 \leq i \leq 4$ (for otherwise, the 3-path graph would be the subgraph of this 3-path graph). Denote the edges incident with $u_1$ by $a_1, a_2, a_3, a_4$, those incident with $u_2$ by $b_1, b_2, b_3, b_4$, those incident with $u_3$ by $c_1, c_2, c_3, c_4$ and those incident with $u_4$ by $d_1, d_2, d_3, d_4$. See Figure 2.6.

Let $T_i$ ($1 \leq i \leq 4$) be the branch of $T$ at $v$ containing $v_i$. By the induction hypothesis, $P_3(T_i)$ is planar for $1 \leq i \leq 4$. Thus, there are planar embeddings of the graphs $P_3(T_i)$ such that $aa_1, bb_1, cc_1, dd_1$ lie on the exterior region of

$$P_3(T_1), P_3(T_2), P_3(T_3), P_3(T_4),$$

respectively. Let $T' = K_{1,4}$ be the star of order 5 with the edges $a, b, c, d$. Then $P_3(T')$ is the maximal planar graph $K_{2,2,2}$ and has the planar embedding shown in Figure 2.7.

Four of the eight regions in this embedding are denoted by $R_i$ ($1 \leq i \leq 4$), where each vertex on the boundary of $R_1$ has $a$ in its label, each vertex on the boundary of $R_2$ has $b$ in its label, each vertex on the boundary of $R_3$ has $c$ in its label and each vertex on the boundary of $R_4$ has $d$ in its label.
Now, \( P_3(T_1) \) can be placed in \( R_1 \) and the vertex \( aa_1 \) can be joined to the vertices \( ab, ac \) and \( ad \) without any edges crossing. Similarly, \( P_3(T_2) \) can be placed in \( R_2 \) and the vertex \( bb_1 \) can be joined to the vertices \( ab, bc \) and \( bd \) without any edges crossing, \( P_3(T_3) \) can be placed in \( R_3 \) and the vertex \( cc_1 \) can be joined to the vertices \( ac, bc \) and \( cd \) without any edges crossing and \( P_3(T_4) \) can be placed in \( R_4 \) and the vertex \( dd_1 \) can be joined to the vertices \( ad, bd \) and \( cd \) without any edges crossing. Thus, \( P_3(T) \) is planar.

**Case 2.** \( \Delta(T) = 3 \). Suppose that \( \deg_T v = 3 \), where \( N_T(v) = \{v_1, v_2, v_3\} \) and \( a = vv_1 \), \( b = vv_2 \) and \( c = vv_3 \). We may assume that \( \deg_T v_i = 3 \) for \( 1 \leq i \leq 3 \). Denote the edges incident with \( v_1 \) by \( a, a_1, a_2 \), those incident with \( v_2 \) by \( b, b_1, b_2 \) and those incident with \( v_3 \) by \( c, c_1, c_2 \). Let \( T_i \) \( (1 \leq i \leq 3) \) be the branch of \( T \) at \( v \) containing \( v_i \). By the induction hypothesis, \( P_3(T_i) \) is planar for \( 1 \leq i \leq 3 \). Thus, there are planar embeddings of the graphs \( P_3(T_i) \) such that \( aa_1, bb_1, cc_1 \) lie on the exterior region of \( P_3(T_1), P_3(T_2), P_3(T_3) \),
respectively. Let $F = K_{1,3}$ be the star of order 4 with the edges $a, b, c$. Then $P_3(F)$ is a triangle $(ab, ac, bc, ab)$. Furthermore, each of the vertices $ab, ac, bc$ has degree 6 in $P_3(T)$. In particular, $ab$ is only adjacent to the six vertices $aa_1, aa_2, ac, bb_1, bb_2, bc$ whose labels contain $a$ or $b$, $ac$ is only adjacent to the six vertices $aa_1, aa_2, ab, bc, cc_1, cc_2$ whose labels contain $a$ or $c$ and $cb$ is only adjacent to the six vertices $bb_1, bb_2, ab, bc, cc_1, cc_2$ whose labels contain $b$ or $c$. Then the three vertices $ab, ac, bc$ can be placed in the exterior region and joined to their six neighbors without any edges crossing. This is shown in Figure 2.8. Thus, $P_3(T)$ is planar.

![Figure 2.8: A step in Case 2 in the proof of Theorem 2.2.1](image)

### 2.3 On 2-Connected Graphs

Next, we determine all those 2-connected graphs having a planar 3-path graph. First, if $G$ is a 2-connected graph of order 3 or 4, then $G \in \{C_3, C_4, K_4 - e, K_4\}$. Clearly, $P_3(C_3) = C_3$ and $P_3(C_4) = C_4$ are planar. We have seen that $P_3(K_4 - e)$ is nonplanar and so $P_3(K_4)$ is also nonplanar. We now consider 2-connected graphs of order at least 5.

We begin by presenting a necessary condition for a 2-connected graph to have a planar 3-path graph.

**Lemma 2.3.1** If $G$ is a 2-connected graph containing a vertex of degree 4 or more, then $P_3(G)$ is nonplanar.

**Proof.** Suppose that $G$ contains a vertex $v$ with $\deg_G v \geq 4$. Then $G$ contains the star $K_{1,4}$ of order 5. Let $V(K_{1,4}) = \{v, v_1, v_2, v_3, v_4\}$ where $vv_i$ is an edge of $G$ for
1 \leq i \leq 4. Denote the four edges $vv_1, vv_2, vv_3, vv_4$ of $G$ by $a, b, c, d$, respectively. Then $\mathcal{P}_3(G)$ contains the graph $H = K_{3,3} - e$ of Figure 2.9 as a subgraph. Since $G$ is 2-connected, it follows that there is a $v_1 - v_4$ path $P = (v_1 = x_1, x_2, \ldots, x_t = v_4)$, $t \geq 2$, in $G$ that does not contain $v$. Let $h_i = x_i x_{i+1}$ for $i = 1, 2, \ldots, t - 1$. Since $P$ does not contain $v$, it follows that $h_i \notin \{a, b, c, d\}$ for $1 \leq i \leq t - 1$. Then $\mathcal{P}_3(G)$ contains the path $Q = (ac, ah_1, h_1 h_2, \ldots, h_{t-1} h_i, h_i d)$. However then, $H$ and $Q$ form a subdivision of $K_{3,3}$ in $\mathcal{P}_3(G)$ and so $\mathcal{P}_3(G)$ is nonplanar.

![Figure 2.9](image.png)

Figure 2.9: A subgraph $H$ of $\mathcal{P}_3(G)$ in the proof of Lemma 2.3.1

The proof of Lemma 2.3.1 gives the following more general result.

**Corollary 2.3.2** If a connected graph $G$ contains a vertex $v$ belonging to a block $B$ of $G$ such that $\deg_B v = 4$, then $\mathcal{P}_3(G)$ is nonplanar.

The following well-known result (see [13]) will be useful to us.

**Theorem 2.3.3** A connected graph $G$ of order 3 or more is 2-connected if and only if every two vertices of $G$ lie on a common cycle of $G$.

The following result provides another necessary condition for a 2-connected graph to have a planar 3-path graph.

**Lemma 2.3.4** If a 2-connected graph $G$ of order at least 5 contains two adjacent vertices of degree 3, then $\mathcal{P}_3(G)$ is nonplanar.

**Proof.** By Lemma 2.3.1, we may assume that $\Delta(G) = 3$. Let $u$ and $v$ be two adjacent vertices of degree 3 in $G$. It then follows by Theorem 3.4.1 that $uv$ lies on a cycle $C$ in $G$. Since the order of $G$ is at least 5, it follows that $uv$ lies on a cycle $C$ of order at least 4. If $C$ is a 4-cycle, then $G$ contains a subgraph isomorphic to either $F_1$ or $F_2$ of Figure 2.10, whose edges are labeled $1, 2, \ldots, 6$. In either case, $\mathcal{P}_3(G)$ contains the subdivision $H$ of $K_{3,3}$ shown in Figure 2.10 and so $\mathcal{P}_3(G)$ is nonplanar. If $C$ is a cycle of order 5 or more,
then $P_3(G)$ contains a subgraph that is subdivision of the graph $H$ obtained by replacing the 3-path $(45, 56, 26)$ in $H$ by a longer path. Thus, $P_3(G)$ contains a subdivision of $K_{3,3}$ and so $P_3(G)$ is nonplanar.

Next, we show for 2-connected graphs $G$ that the 3-path graph $P_3(G)$ is planar only if $G$ is planar.

**Lemma 2.3.5** If $G$ is a 2-connected nonplanar graph, then $P_3(G)$ is nonplanar.

**Proof.** Since $G$ is nonplanar, it follows that $G$ contains a subdivision of $K_5$ or $K_{3,3}$ by Theorem 2.1.1. If $G$ contains a subdivision of $K_5$, then $G$ contains vertices of degree 4 and, by Corollary 2.3.2, $P_3(G)$ is nonplanar. Thus, we may assume that $G$ contains a subdivision of $K_{3,3}$. If $G$ contains two adjacent vertices of degree 3, then $P_3(G)$ is nonplanar by Lemma 2.3.4. Thus, we may assume that no two vertices of degree 3 of $G$ are adjacent. Hence, if $F$ is a subdivision of $K_{3,3}$ in $G$, then $F$ is obtained by subdividing each edge of $K_{3,3}$ at least once. First, suppose that $G$ contains a subdivision $F$ of $K_{3,3}$ that is obtained by subdividing each edge of $K_{3,3}$ exactly once, namely $F$ is the graph shown in Figure 2.11(a). Since $P_3(F)$ contains the subdivision $H$ of $K_{3,3}$ of Figure 2.11(b), it follows that $P_3(G)$ is nonplanar. Next, suppose that $F$ is obtained by subdividing each edge of $K_{3,3}$ at least once. Then $P_3(G)$ contains a subdivision $H'$ of the graph $H$ in Figure 2.11(b). Since $H'$ is also subdivision of $K_{3,3}$, it follows that $P_3(G)$ is nonplanar.

We are now able to characterize those 2-connected graphs having a planar 3-path graph.

**Theorem 2.3.6** A 2-connected graph $G$ of order at least 3 has a planar 3-path graph if and only if (i) $G$ is a planar graph of maximum degree at most 3 and (ii) no two vertices of degree 3 are adjacent in $G$. 

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Figure 2.10: A step in the proof of Lemma 2.3.4
Proof. First, suppose that $G$ is a 2-connected graph of order at least 3 such that $P_3(G)$ is planar. If $n = 3, 4$, then $G \in \{C_3, C_4, K_4 - e, K_4\}$. Since $P_3(C_n) = C_n$ for each integer $n \geq 3$ and $P_3(K_4 - e)$ and $P_3(K_4)$ are nonplanar, it follows that $G = C_n$ and so $G$ satisfies (i) and (ii). If $n \geq 5$, then $G$ is planar by Lemma 2.3.5. Furthermore, $\Delta(G) \leq 3$ by Lemma 2.3.1 and no two vertices of degree 3 in $G$ are adjacent by Lemma 2.3.4. Thus, $G$ satisfies (i) and (ii).

For the converse, suppose that $G$ is a 2-connected graph of order $n \geq 3$ that satisfies (i) and (ii). If $\Delta(G) = 2$, then $G = C_n$ and $P_3(C_n) = C_n$ is planar. Thus, we may assume that $\Delta(G) = 3$ and no two vertices of degree 3 are adjacent in $G$. We show that there is a planar embedding of $P_3(G)$. Let there be given a planar embedding of $G$ and let $v$ be a vertex of degree 3 in $G$ where $N(v) = \{v_1, v_2, v_3\}$. Therefore, each vertex $v_i$ ($1 \leq i \leq 3$) has degree 2 in $G$. Let $N(v_i) = \{v, u_i\}$ for $1 \leq i \leq 3$. Denote the edge $v v_i$ by $i$ and the edge $v_i u_i$ by $i + 3$. If $u_1, u_2, u_3$ are three distinct vertices, then $G$ contains the subgraph shown in Figure 2.12(a). We can then replace the subgraph $G[N[v]]$ by the planar structure shown in Figure 2.12(a) in $P_3(G)$, thereby showing that $P_3(G)$ is planar.

Should it occur that $u_1 = u_2$, say, and $u_2 \neq u_3$, then $G$ contains the subgraph shown in Figure 2.12(b) and this subgraph is then replaced by the planar structure shown in Figure 2.12(b) in $P_3(G)$. If $u_1 = u_2 = u_3$, then $G = K_{3,3}$, shown in Figure 2.12(c), resulting in a planar graph $P_3(G)$, also shown in Figure 2.12(c).

We saw that $P_3(K_4)$ is nonplanar. If $G$ is the graph obtained by subdividing each edge of $K_4$ exactly once, then $G$ contains four vertices of degree 3 no two of which are adjacent. Since there is a planar embedding of $P_3(G)$ shown in Figure 2.13, it follows
that $\mathcal{P}_3(G)$ is planar.

Figure 2.13: A subdivision of $K_4$ having a planar 3-path graph

2.4 Non-Ayclic Graphs with Connectivity 1

A cyclic block in a graph contains cycles, while an acyclic block does not contain any cycle. The only acyclic block is $K_2$. If $G$ is a graph of order at least 5 with connectivity 1 that are not trees, then $G$ contains at least two blocks, at least one of which is a cyclic
block. There are eight graphs $F_1, F_2, \ldots, F_8$ of order 4 or 5 having connectivity 1 that are not trees, as shown in Figure 2.14. While the graph $\mathcal{P}_3(F_1)$ is planar, $\mathcal{P}_3(F_i)$ is nonplanar for $2 \leq i \leq 8$. In fact, the graph $\mathcal{P}_3(F_2)$ contains the subdivision of $K_{3,3}$ shown in Figure 2.14, $\mathcal{P}_3(F_3)$ contains the subdivision of $K_{3,3}$ shown in Figure 2.10, and $\mathcal{P}_3(F_4)$ contains the subdivision of $K_{3,3}$ shown in Figure 2.14. Since $F_5$ contains $F_4$ as a subgraph, $\mathcal{P}_3(F_5)$ is nonplanar. Each graph $F_i$ ($i = 6, 7, 8$) contains $K_4 - e$ as a subgraph. Since $\mathcal{P}_3(K_4 - e)$ is nonplanar, it follows that $\mathcal{P}_3(F_i)$ is nonplanar for $i = 6, 7, 8$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{eight_graphs.png}
\caption{Eight eight graphs $F_1, F_2, \ldots, F_8$}
\end{figure}

**Lemma 2.4.1** Let $G$ be a graph with connectivity 1 that is not a tree. If $\mathcal{P}_3(G)$ is planar, then no vertex of degree 4 in $G$ belongs to a cyclic block.

**Proof.** Assume, to the contrary, that there is a vertex $v$ of degree 4 in $G$ that belongs to a cyclic block $B$ of $G$. Then $v$ belongs to a cycle $C$ in $B$. Let $N(v) = \{v_1, v_2, v_3, v_4\}$, where $v_1$ and $v_2$ are on $C$. Since no vertex of degree 4 can be adjacent to a vertex of degree 3 by Lemma 2.1.4, it follows that $v_3$ and $v_4$ do not belong to $C$. Thus, $G$ contains a subgraph $F$ obtained by adding the two pendant edges $vv_3$ and $vv_4$ to $C$. If $C$ is a 3-cycle, then $F = F_4$ of Figure 2.14 and we saw that $\mathcal{P}_3(F_4)$ is nonplanar. If $C$ is a larger cycle, then $F$ is obtained by subdividing the edge labeled 1 in $F_4$ (see Figure 2.14). It then follows by Lemma 2.1.5 that $\mathcal{P}_3(F)$ is nonplanar and so $\mathcal{P}_3(G)$ is planar, a contradiction. 

\[\blacksquare\]
As a consequence of Lemma 2.4.1, it now suffices to consider connected graphs with connectivity 1 containing both cyclic blocks and acyclic blocks where no cut-vertex belongs to more than one cyclic block.

**Lemma 2.4.2** Let $G$ be a graph of order at least 4 with connectivity 1 that is not a tree. If $P_3(G)$ is planar, then no two adjacent vertices in a cyclic block have degree 3 in $G$.

**Proof.** Assume, to the contrary, that there are adjacent vertices $u$ and $v$ of degree 3 in $G$ that belong to a cyclic block $B$ of $G$. Thus, $uv$ belongs to a cycle $C$ in $B$. If $\deg_B u = \deg_B v = 3$, then $P_3(B)$ is nonplanar by Lemma 2.3.4. Thus, we may assume that at least one of $u$ and $v$ has degree 2 in $B$, say $\deg_B v = 2$. If $u$ is adjacent to three vertices of $C$, then $B$ contains two adjacent vertices of degree 3 in $B$ and so $P_3(B)$ is nonplanar by Lemma 2.3.4. Hence, $G$ contains a subgraph $F$ that is obtained by adding two edges $uu'$ and $vv'$ to $C$, where $u'$ and $v'$ are not on $C$. Thus, either $F = F_2$ of Figure 2.14 or $F$ is obtained from $F_2$ by subdividing an edge $xy$, where at least one of $x$ and $y$ has degree 2 in $F_2$. It then follows by Lemma 2.1.5 that $P_3(F)$ is nonplanar and so $P_3(G)$ is nonplanar, a contradiction. $\blacksquare$

We are now prepared to present a characterization of those graphs with connectivity 1 that are not trees and having a planar 3-path graph. In order to do this, we first introduce an additional definition. Let $v$ be a cut-vertex of a connected graph $G$. Suppose that the disconnected graph $G - v$ has $k$ components $G_1, G_2, \ldots, G_k$ ($k \geq 2$). The induced subgraphs $B_i = G[V(G_i) \cup \{v\}]$ are connected and referred to as the branches of $G$ at $v$. If a subgraph $G_i$ contains no cut-vertices of $G$, then the branch $B_i$ is a block of $G$, in fact, an end-block of $G$.

**Theorem 2.4.3** Let $G$ be a graph of order at least 4 with connectivity 1 that is not a tree. Then $P_3(G)$ is planar if and only if

1. $\Delta(G) \leq 4$ and every vertex of degree 4 belongs only to acyclic blocks and is adjacent only to a vertex of degree 1 or 2 and

2. no two adjacent vertices in a cyclic block have degree 3 in $G$.

**Proof.** First, suppose that $G$ is a graph with connectivity 1 that is not a tree having a planar 3-path graph. Property (1) is a consequence of Lemmas 2.1.3, 2.1.4 and 2.4.1. Property (2) is a consequence of Lemma 2.4.2.
For the converse, we use the strong form of induction to prove that if $G$ is a graph with connectivity 1 that is not a tree satisfying properties (1) and (2), then $\mathcal{P}_3(G)$ is planar. The result is clearly true for such graphs having order 4, 5 or 6. Assume for an integer $k \geq 6$ that all graphs of order $p$ with $4 \leq p \leq k$ with connectivity 1 that is not a tree and satisfying properties (1) and (2) have a planar 3-path graph. Let $G$ be a graph of order $k + 1$ with connectivity 1 that is not a tree and satisfying properties (1) and (2). We show that $\mathcal{P}_3(G)$ is planar. We consider these two cases.

**Case 1.** $G$ does not contain a cut-vertex belonging only to acyclic blocks. Since no cut-vertex of $G$ can belong to two cyclic blocks by (1), it follows that every cut-vertex of $G$ belongs to a cyclic block and an acyclic block. Let $e = uv$ be the edge in an acyclic block. We may assume that $\deg_G u = \deg_G v = 3$ and so each of $u$ and $v$ belongs to a cyclic block. (If one of $u$ and $v$ is an end-vertex, then the 3-path graph is the subgraph of the 3-path graph when $\deg_G u = \deg_G v = 3$.) Suppose that $u$ belongs to a cyclic block $C$ and $v$ belongs to a cyclic block $C'$. Thus, $u$ is adjacent to vertices $u_1$ and $u_2$ of degree 2 on $C$ and $v$ is adjacent to vertices $v_1$ and $v_2$ of degree 2 on $C'$. Let $a = uu_1, b = uv_2, c = vv_1$ and $d = vv_2$. Also, let $a', b', c'$ and $d'$ be the other edges incident with $u_1, u_2, v_1$ and $v_2$, respectively. (It is possible that $a' = b'$ and $c' = d'$.) Let $H$ and $H'$ be the components of $G - e$, where $H$ contains $C$ and $H'$ contains $C'$. Then each of $H$ and $H'$ is either a cyclic block or a graph of order less than $k$ with connectivity 1 that is not a tree and satisfying properties (1) and (2). In either case, $\mathcal{P}_3(H)$ and $\mathcal{P}_3(H')$ are planar by Theorem 2.3.6 and the induction hypothesis. Hence, there exist planar embeddings of $\mathcal{P}_3(H)$ and $\mathcal{P}_3(H')$ such that the paths $(a' a, ab, bb')$ and $(cc', cd, dd')$ lie on the boundary of the exterior region $R$. We can place a planar embedding of the complete graph $K_4$ with vertex set $\{ea, eb, ec, ed\}$ in $R$ and join $ea$ to $a' a$ and $ab$ in $\mathcal{P}_3(H)$, join $eb$ to $ab$ and $bb'$ in $\mathcal{P}_3(H)$, join $ec$ to $c'$ and $cb$ in $\mathcal{P}_3(H')$ and join $ed$ to $cd$ and $dd'$ in $\mathcal{P}_3(H')$, obtaining a planar embedding of $\mathcal{P}_3(G)$. This is illustrated in Figure 2.15.

**Case 2.** $G$ contains cut-vertices belonging only to acyclic blocks. Among all such cut-vertices of $G$, let $v$ be one of maximum degree. Thus, the degree of $v$ in $G$ is either 2, 3 or 4. We consider these three possibilities.

**Subcase 2.1.** $\deg_G v = 2$. Let $N(v) = \{v_1, v_2\}$. We may assume, without loss of generality, that $\deg_G v_1 = \deg_G v_2 = 3$ and each of $v_1$ and $v_2$ is a cut-vertex of $G$ belonging to a cycle. (If at least one of $v_1$ and $v_2$ has degree at most 2 or at least one of $v_1$ and $v_2$ is a cut-vertex of an acyclic block, then the 3-path graph is a subgraph of
the 3-path graph in the case when both \(v_1\) and \(v_2\) have degree 3 and each of \(v_1\) and \(v_2\) is a cut-vertex belonging to a cycle.) Let \(C\) be the cyclic block containing \(v_1\) and \(C'\) the cyclic block containing \(v_2\). Let \(a\) and \(b\) be the edges incident with \(v_1\) on \(C\) and let \(c\) and \(d\) be the edges incident with \(v_2\) on \(C'\). Let \(e = vv_1\) and \(f = vv_2\). Let \(G_i\) \((i = 1, 2)\) be the branch of \(G\) at \(v\) containing \(v_i\). By the induction hypothesis, \(P_3(G_i)\) is planar for \(i = 1, 2\). Let there be planar embeddings of \(P_3(G_1)\) and \(P_3(G_2)\) such that \(ab\) and \(cd\) lie on the exterior region \(R\). Let \(F = 2K_2 \lor K_1\) be the join of \(2K_2\) and \(K_1\) with vertex set \(\{ae, be, ef, cf, df\}\). We can place a planar embedding of \(F\) in \(R\) and join \(ae, eb\) to \(ab\) in \(P_3(G_1)\) and join \(ef, df\) to \(cd\) in \(P_3(G_2)\) to obtain a planar embedding of \(P_3(G)\). This is illustrated in Figure 2.16.

Subcase 2.2. \(\text{deg}_G v = 3\). Let \(N_T(v) = \{v_1, v_2, v_3\}\). Here too, we may assume for \(i = 1, 2, 3\) that \(\text{deg}_G v_i = 3\) and each vertex \(v_i\) is a cut-vertex of \(G\) belonging to a cycle. Let \(C\) be the cyclic block containing \(v_1\), \(C'\) the cyclic block containing \(v_2\) and \(C''\) the
cyclic block containing \( v_3 \). Let \( a \) and \( b \) be the edges incident with \( v_1 \) on \( C \), let \( c \) and \( d \) be the edges incident with \( v_2 \) on \( C' \) and let \( x \) and \( y \) be the edges incident with \( v_3 \) on \( C'' \). Let \( e = vv_1, f = vv_2 \) and \( g = vv_3 \). Let \( G_i \) (\( 1 \leq i \leq 3 \)) be the branch of \( G \) at \( v \) containing \( v_i \). By the induction hypothesis, \( P_3(G_i) \) is planar for \( 1 \leq i \leq 3 \). Thus, there are planar embeddings of the graphs \( P_3(G_i) \), \( 1 \leq i \leq 3 \), such that \( ab, cd, xy \) lie on the exterior region in \( R \). Let \( F \) be the planar graph with vertex set \( \{ ea, eb, fc, fd, gx, gy, ef, eg, fg \} \). We can place a planar embedding of \( F \) in \( R \) and join \( ea, eb \) to \( ab \) in \( P_3(G_1) \), join \( fc, fd \) to \( cd \) in \( P_3(G_2) \) and join \( gx, gy \) to \( xy \) in \( P_3(G_3) \), obtaining a planar embedding of \( P_3(G) \). This is illustrated in Figure 2.17.

![Figure 2.17: Illustrating a planar embedding of \( P_3(G) \) in Subcase 2.2](image)

**Subcase 2.3.** \( \deg_G v = 4 \). Let \( N_T(v) = \{ v_1, v_2, v_3, v_4 \} \). Thus, \( \deg_G v_i = 1, 2 \) for \( 1 \leq i \leq 4 \) by (2). Again, we may assume that \( \deg_G v_i = 2 \) for \( 1 \leq i \leq 4 \). Let \( a = vv_1, b = vv_2, c = vv_3, d = vv_4 \), where \( a', b', c', d' \) is the other edge incident with \( v_i \) for \( 1 \leq i \leq 4 \), respectively. Let \( G_i \) (\( 1 \leq i \leq 4 \)) be the branch of \( G \) at \( v \) containing \( v_i \). By the induction hypothesis, \( P_3(G_i) \) is planar for \( 1 \leq i \leq 4 \). Thus, there are planar embeddings of the graphs \( P_3(G_i) \) (\( 1 \leq i \leq 4 \)) such that \( aa', bb', cc', dd' \) lie on the exterior region of \( P_3(G_1), P_3(G_2), P_3(G_3), P_3(G_4) \), respectively. Now, for the subgraph \( H = K_{1,4} \) of \( G \) with \( E(H) = \{ a, b, c, d \} \), consider the planar embedding of \( P_3(H) \) shown in Figure 2.18, where four of the eight regions are denoted by \( R_i \) for \( 1 \leq i \leq 4 \). We can place the planar embedding of \( H_i = P_3(G_i) \) into \( R_i \) and join the designated vertex of \( P_3(G_i) \) to the three boundary vertices of \( R_i \) to obtain a planar embedding of \( P_3(G) \). For example, the graph \( H_1 = P_3(G_1) \) is placed in \( R_1 \) and the vertex \( aa' \) of \( H_1 \) is joined to the three boundary vertices \( ab, ac, ad \) of \( R_1 \). This is illustrated in Figure 2.18.

\[\]
Let Theorem 2.5.1 of order at least 3 having a planar 3-path graph, which we summarize below.

As a consequence of Theorems 2.2.1, 2.3.6 and 2.4.3, we now know all connected graphs of order at least 3 having a planar 3-path graph, which we summarize below.

2.5 A Characterization of Planar 3-Path Graphs

As a consequence of Theorems 2.2.1, 2.3.6 and 2.4.3, we now know all connected graphs of order at least 3 having a planar 3-path graph, which we summarize below.

**Theorem 2.5.1** Let $G$ be a connected graph of order at least 3.

1. If $G$ is a tree, then $\mathcal{P}_3(G)$ is planar if and only if $\Delta(G) \leq 4$ and every vertex of degree 4 is adjacent only to vertices of degree 1 or 2.

2. If $G$ is 2-connected, then $\mathcal{P}_3(G)$ is planar if and only if (i) $G$ is a planar graph of maximum degree at most 3 and (ii) no two vertices of degree 3 are adjacent in $G$.

3. If $G$ is neither a tree nor a 2-connected graph, then $\mathcal{P}_3(G)$ is planar if and only if (i) $\Delta(G) \leq 4$ and every vertex of degree 4 belongs only to acyclic blocks and is adjacent only to a vertex of degree 1 or 2 and (ii) no two adjacent vertices in a cyclic block have degree 3 in $G$. 

Figure 2.18: Illustrating a planar embedding of $\mathcal{P}_3(G)$ in Subcase 2.3
Chapter 3

Outerplanarity of 3-Path Graphs

3.1 Introduction

In this chapter, we investigate a related concept dealing with planarity in graphs. A graph $G$ is outerplanar if there exists a planar embedding of $G$ so that every vertex of $G$ lies on the boundary of the exterior region. Equivalently, a graph $G$ is outerplanar if there exists an embedding of $G$ in the plane or on a sphere such that the boundary of some region contains every vertex of $G$. An outerplanar graph $G$ is maximal outerplanar if the addition to $G$ of any edge joining two nonadjacent vertices of $G$ results in a graph that is not outerplanar. Necessarily then, there is a planar embedding of a maximal outerplanar graph $G$ of order at least 3, the boundary of whose exterior region is a Hamiltonian cycle of $G$ and where the boundary of every other region is a triangle. Outerplanar graphs were introduced by Chartrand and Harary [8] in 1967.

There is a characterization of outerplanar graphs (see [13, p. 130]) that is analogous to the characterization of planar graphs in terms of the subdivisions of $K_4$ and $K_{2,3}$. (See Figure 3.1).

![Figure 3.1: Subdivisions of $K_4$ and $K_{2,3}$](image)

**Theorem 3.1.1**  A graph $G$ is outerplanar if and only if $G$ contains no subgraph that is a subdivision of $K_4$ or $K_{2,3}$.

We have seen that a graph $G$ is planar if and only if any subdivision of $G$ is planar.
This is not true, however, for outerplanar graphs. For example, the graph $G = K_4 - e$ is outerplanar. Now, let $uv$ be the edge of $G$ with $\deg u = \deg v = 3$ and let $H$ be the graph obtained from $G$ by subdividing the edge $uv$ at least once. Then $H$ is a subdivision of $K_{2,3}$ and so $H$ is not outerplanar by Theorem 3.1.1.

The following results provides the relationship between planar graphs and outerplanar graphs (see [13]). The join $G_1 \lor G_2$ of $G_1$ and $G_2$ has vertex set $V(G_1 \lor G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \lor G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. In particular, the join $G \lor K_1$ of a graph $G$ and $K_1$ is obtained by joining the vertex of $K_1$ to each vertex of $G$.

**Theorem 3.1.2** A graph $G$ is outerplanar if and only if the join $G \lor K_1$ of $G$ and $K_1$ is planar.

In 1971, Chartrand, Geller and Hedetniemi [9] presented a characterization of graphs whose line graph is outerplanar.

**Theorem 3.1.3** [9] A nonempty graph $G$ has an outerplanar line graph if and only if $\Delta(G) \leq 3$ and every vertex of degree 3 of $G$ is a cut vertex.

Theorem 3.1.3 does not hold for 3-path graphs, however. For example, the graph $G = K_{1,3} + e$ in Figure 3.2 has $\Delta(G) = 3$ and the vertex of degree 3 in $G$ is a cut vertex of $G$. Since $P_3(G)$ contains $K_{2,3}$ as a subgraph (also shown in Figure 3.2), it follows that $P_3(G)$ is not outerplanar.

![Figure 3.2: A graph $G$ and its 3-path graph $P_3(G)$](image)

Our goal is to obtain a result corresponding to Theorem 3.1.3 on 3-path graphs, that is, to determine those graphs having an outerplanar 3-path graph. Since the 3-path graph is only defined for graphs having 3-paths and the 3-path graph of a disconnected graph $G$ is the union of the 3-path graphs of the components of $G$, it suffices to consider only connected graphs of order 3 or more.
3.2 Preliminary Results

First, we derive necessary conditions for a connected graph to have an outerplanar 3-path graph.

**Lemma 3.2.1** If a connected graph $G$ of order at least 3 has an outerplanar 3-path graph, then $\Delta(G) \leq 3$.

**Proof.** Suppose that $G$ contains a vertex of degree 4. Then $G$ contains the star $K_{1,4}$ of order 5. Let $a, b, c, d$ be the four edges of $K_{1,4}$. Since $\mathcal{P}_3(K_{1,4})$ contains the subgraph $K_{2,3}$ (also shown in Figure 3.3), it follows that $\mathcal{P}_3(G)$ is not outerplanar.

![Figure 3.3: A subgraph $K_{2,3}$ of $\mathcal{P}_3(G)$ in the proof of Lemma 3.2.1](image)

For integers $a$ and $b$ with $2 \leq a \leq b$, the double star $S_{a,b}$ is that tree of diameter 3 whose two central vertices have degrees $a$ and $b$. Thus, the order of $S_{a,b}$ is $a + b$.

**Lemma 3.2.2** If a connected graph $G$ contains two adjacent vertices of degree 3, then $\mathcal{P}_3(G)$ is not outerplanar.

**Proof.** If $G$ contains two adjacent vertices of degree 3, then $G$ contains the double star $S_{3,3}$ where the two central vertices have degree 3. Since $\mathcal{P}_3(S_{3,3})$ contains $K_4$ as a subgraph (see Figure 3.4), it follows that $\mathcal{P}_3(G)$ is not outerplanar.

![Figure 3.4: A subgraph $K_4$ of $\mathcal{P}_3(G)$ in the proof of Lemma 3.2.2](image)

**Lemma 3.2.3** Let $G$ be a connected graph such that $\mathcal{P}_3(G)$ is not outerplanar and let $e = uv$ be an edge of $G$ such that at least one of $u$ and $v$ has degree at most 2 in $G$. If $H$ is obtained from $G$ by subdividing $e$ once, then $\mathcal{P}_3(H)$ is not outerplanar.

30
Proof. Let $x$ be the vertex of degree 2 inserted into $uv$, obtaining the graph $H$, where the edge $uv$ is then replaced by the two edges $ux$ and $xv$. We may assume that $\deg_G u \leq \deg_G v$. First, suppose that $\deg_G u = 1$. Then $H' = H - u \cong G$ and $H' \subseteq H$. Since $P_3(H') \cong P_3(G)$ is a subgraph of $P_3(H)$ and $P_3(G)$ is not outerplanar, it follows that $P_3(H)$ is also not outerplanar. Next, suppose that $\deg_G u = 2$, say $\mathcal{N}_G(u) = \{v, y\}$. Let $(y, u, x, v, w)$ be a path in $G$. Then $(y, u, x, v, w)$ is a path in $H$. Let $a = yu, b = uv, c = vw$ in $G$ and let $a = yu, b_1 = ux, b_2 = xv, c = vw$ in $H$. Then the graph $P_3(H)$ is obtained by replacing the edge joining $ab$ and $bc$ in $P_3(G)$ by the 3-path $(ab_1, b_1b_2, b_2c)$ in $P_3(H)$. This implies that $P_3(H)$ is a subdivision of $P_3(G)$. Since $P_3(G)$ is not outerplanar, it follows that $P_3(H)$ is also not outerplanar.

Lemma 3.2.4 If $G$ is a connected graph of order at least 4 containing a triangle, then $P_3(G)$ is not outerplanar. Consequently, if $G$ contains a subdivision of $K_{1,3} + e$ as a subgraph, then $P_3(G)$ is not outerplanar.

Proof. Since $G$ is a connected graph of order at least 4 containing a triangle, it follows that $G$ contains a vertex of degree 3 such that two of its neighbors are adjacent. Let $v$ be a vertex of degree 3 in $G$ where $\mathcal{N}_G(v) = \{v_1, v_2, v_3\}$. By Lemma 3.2.2, it follows that $\deg_G v_i \leq 2$ for $1 \leq i \leq 3$. Assume, without loss of generality, that $v_1$ and $v_2$ are adjacent. Then $K_{1,3} + e$ is a subgraph of $G$. Since $P_3(K_{1,3} + e)$ is not outerplanar (as shown in Figure 3.2) and $P_3(K_{1,3} + e)$ is a subgraph of $P_3(G)$, it follows that $P_3(G)$ is also not outerplanar. If $H$ is a subdivision of $K_{1,3} + e$, then $P_3(H)$ is not outerplanar by Lemma 3.2.3 and so $P_3(G)$ is not outerplanar.

The following corollaries are immediate consequences of Lemma 3.2.4.

Corollary 3.2.5 If $G = K_4$ or $G = K_{2,3}$, then $P_3(G)$ is not outerplanar.

Corollary 3.2.6 If $G$ is a graph that is not outerplanar, then $P_3(G)$ is also not outerplanar.

3.3 Trees and 2-Connected Graphs

Now, we introduce necessary and sufficient conditions for a tree to have an outerplanar 3-path graph.

Theorem 3.3.1 A tree $T$ of order at least 3 has an outerplanar 3-path graph if and only if $\Delta(T) \leq 3$ and every vertex of degree 3 is adjacent to vertices of degree 1 or 2.
Proof. As we have seen in Lemma 3.2.1, no graph with an outerplanar 3-path graph can contain a vertex of degree 4 or more and, by Lemma 3.2.2, no graph with an outerplanar 3-path graph can contain a vertex of degree 3 that is adjacent to a vertex of degree 3. Hence, it remains to show that if $T$ is a tree of order 3 or more such that $\Delta(T) \leq 3$ and every vertex of degree 3 is adjacent only to vertices of degree 1 or 2, then $P_3(T)$ is outerplanar.

We proceed by the strong form of induction. The result is clearly true for all trees of small order. Assume, for some appropriate positive integer $k$, that if $T'$ is a tree of order $p$ with $3 \leq p \leq k$ such that every vertex of degree 3 in $T'$ is only adjacent to vertices of degree 1 or 2, then $P_3(T')$ is outerplanar. Let $T$ be a tree of order $k+1$ with $\Delta(T) \leq 3$ such that every vertex of degree 3 in $T$ is only adjacent to vertices of degree 1 or 2 in $T$. We show that $P_3(T)$ is outerplanar.

Since the 3-path graph of every path of order 3 or more is also a path where in the plane all vertices lie on the boundary of the exterior region and is therefore outerplanar, we may assume that $\Delta(T) = 3$. Suppose that $\deg_T v = 3$ where $N_T(v) = \{v_1, v_2, v_3\}$ and $a = vv_1, b = vv_2$ and $c = vv_3$. By assumption, each of the vertices $v_i$ ($1 \leq i \leq 3$) has degree 1 or 2. We may assume that $\deg_T v_i = 2$ for $1 \leq i \leq 3$. Denote the edges incident with $v_1, v_2$ and $v_3$ by $a', b'$ and $c'$, respectively. Let $T_i$ ($1 \leq i \leq 3$) be the branch of $T$ at $v$ containing $v_i$. By the induction hypothesis, $P_3(T_i)$ is planar for $1 \leq i \leq 3$. Thus, there are planar embeddings of the graphs $P_3(T_1), P_3(T_2), P_3(T_3)$, such that every vertex of $P_3(T_1), P_3(T_2), P_3(T_3)$, lies on the boundary of the exterior region.

Let $T_0 = K_{1,3}$ be the star of order 4 with the edges $a, b, c$. Then $P_3(T_0)$ is a triangle $(ab, ac, bc, ab)$. Furthermore, each of the vertices $ab, ac, bc$ has degree 4 in $P_3(T)$. In particular, the vertex $ab$ is adjacent only to the four vertices $aa', ac, bb', bc$ whose labels contain $a$ or $b$, the vertex $ac$ is adjacent only to the four vertices $aa', ab, bc, cc'$ whose labels contain $a$ or $c$, and the vertex $cb$ is adjacent only to the four vertices $bb', ab, bc, cc'$ whose labels contain $b$ or $c$. Then the three vertices $ab, ac, bc$ can be placed in the exterior region and joined to their four neighbors without any edges crossing, where all vertices lie on the boundary of the exterior region. This is shown in Figure 3.5. Thus, $P_3(T)$ is outerplanar.

\[\square\]
3.4 A Characterization of Outerplanar 3-Path Graphs

We now determine all those connected graphs having an outerplanar 3-path graph, beginning with 2-connected graphs. First, if $G$ is a 2-connected graph of order 3 or 4, then $G \in \{C_3, C_4, K_4 - e, K_4\}$. Since $P_3(C_n) = C_n$, it follows that $P_3(C_3)$ and $P_3(C_4)$ are outerplanar. The graph $K_4 - e$ contains two adjacent vertices of degree 3 and so $P_3(K_4 - e)$ is not outerplanar by Lemma 3.2.2. Furthermore, $P_3(K_4)$ is not outerplanar by Corollary 3.2.6. Thus, we now consider 2-connected graphs of order at least 5. The following well-known result (see [13]) will be useful to us.

**Theorem 3.4.1** A connected graph $G$ of order 3 or more is 2-connected if and only if every two vertices of $G$ lie on a cycle of $G$.

With the aid of Theorem 3.4.1, we show that cycles are the only 2-connected graphs having an outerplanar 3-path graph.

**Theorem 3.4.2** A 2-connected graph $G$ of order at least 3 has an outerplanar 3-path graph if and only if $G$ is a cycle.

**Proof.** If $G$ is a cycle $C_n$ of order $n \geq 3$, then $P_3(G) = C_n$ is outerplanar. For the converse, let $G$ be a 2-connected graph of order at least 3 that is not a cycle. Then $\Delta(G) \geq 3$. Let $v$ be a vertex of degree 3, where $N_G(v) = \{v_1, v_2, v_3\}$. By Lemma 3.2.2, $\deg_G(v_i) \leq 2$ for $i = 1, 2, 3$. Since $G$ is a 2-connected graph, it follows by Theorem 3.4.1 that $G$ has a $v_1 - v_2$ path $P$ that does not contain $v$. Furthermore, since $\deg_G(v_3) \leq 2$ and $v_3$ is adjacent to $v$, the path $P$ cannot contain $v_3$. Hence, $G$ contains a subdivision of $K_{1,3} + e$ as a subgraph. Therefore, $P_3(G)$ is not outerplanar by Lemma 3.2.4.

We are now prepared to characterize all connected graphs having an outerplanar 3-path graph.
Theorem 3.4.3  Let $G$ be a connected graph of order at least 3. Then $G$ has an outer-planar 3-path graph if and only if

(i) $G$ is a cycle or

(ii) $G$ is a tree with $\Delta(G) \leq 3$ such that every vertex of degree 3 of $G$ is adjacent to vertices of degree 1 or 2 in $G$.

Proof. If $G$ satisfies properties (i) or (ii), then $P_3(G)$ is outerplanar by Theorems 3.3.1 and 3.4.2. Thus, it remains to verify the converse. Let $G$ be a connected graph of order $n \geq 3$ having an outerplanar 3-path graph. If $\Delta(G) = 2$, then $G$ is a cycle $C_n$ of order $n$ or a path $P_n$ of order $n$. In either case, $G$ satisfies properties (i) or (ii). Thus, we may assume that $\Delta(G) \geq 3$. By Theorem 3.4.2, the graph $G$ cannot be 2-connected and so $G$ contains cut-vertices. We claim that each block of $G$ is $K_2$. Assume, to the contrary, that there is a block $B$ of $G$ that is not $K_2$. Then $B$ is 2-connected. Let $v$ be a cut-vertex of $G$ belonging to $B$ and let $B' \neq B$ be a block of $G$ that contains $v$. Let $C$ be a cycle of $B$ containing $v$ and let $u$ be a vertex of $B'$ that is adjacent to $v$. Then the cycle $C$ together with $vu$ form a subdivision of $K_{1,3} + e$. However then, $P_3(G)$ is not outerplanar by Lemma 3.2.4, which is a contradiction. Hence, each block of $G$ is $K_2$, as claimed, and so $G$ is a tree. It then follows by Theorem 3.3.1 that $G$ satisfies property (ii).
Chapter 4

Hamiltonicity of 3-Path Graphs

4.1 Introduction

A path or cycle containing every vertex of a graph $G$ is called a Hamiltonian path or a Hamiltonian cycle of $G$, respectively. If a graph $G$ contains a Hamiltonian cycle, then $G$ itself is Hamiltonian. A graph $G$ is Hamiltonian-connected if for every two vertices $u$ and $v$ of $G$, there is a Hamiltonian $u-v$ path in $G$.

Harary and Nash-Williams [23] characterized those graphs $G$ for which the line graph $L(G)$ of $G$ is Hamiltonian. Their characterization primarily involved the existence of a circuit in $G$ called a dominating circuit $C$ in which every edge of $G$ is incident with a vertex of $C$.

**Theorem 4.1.1** (Harary and Nash-Williams) Let $G$ be a graph without isolated vertices. Then $L(G)$ is Hamiltonian if and only if $G$ is a star $K_{1,t}$ for some $t \geq 3$ or $G$ contains a dominating circuit.

While a connected graph $G$ with no vertices of degree 1 or 2 need not have a Hamiltonian line graph (see Figure 4.1), it was verified by Chartrand and Wall in [12] that if $G$ is a connected graph with minimum degree $\delta(G) \geq 3$, then $L(G)$ must have a spanning subgraph containing an Eulerian circuit (a circuit containing every edge of $G$ exactly once), which is a dominating circuit of $L(G)$ and, consequently, gives the following result.

**Theorem 4.1.2** [12] If $G$ is a connected graph with $\delta(G) \geq 3$, then $P_3(G)$ is Hamiltonian.

In [6], an alternative proof of Theorem 4.1.2 was given employing spanning trees and Hamiltonian walks in graphs.
Let $G$ be a nontrivial connected graph. A Hamiltonian walk in $G$ is a closed walk of minimum length that contains every vertex of $G$. This concept was introduced by Goodman and Hedetniemi [19, 20]. They showed that if $G$ is a connected graph of order $n$ and size $m$, then the length of a Hamiltonian walk $W$ in $G$ is at least $n$ and at most $2m$. Furthermore, every edge of $G$ occurs at most twice in $W$. The length of $W$ is $n$ if and only if $G$ is Hamiltonian (in which case $W$ is a Hamiltonian cycle). The length of $W$ is $2m$ if and only if $G$ is a tree in which case each edge of $G$ appears exactly twice in $W$. Hamiltonian walks in graphs have been used to study structural properties of graphs (see [18]).

Every embedding of a tree $T$ in the plane gives rise to a Hamiltonian walk in $T$. For example, suppose that $T$ is a star $K_{1,4}$ whose edges are $a, b, c, d$. Figures 4.2(a) and 4.2(c) show two different embeddings of $T$ in the plane. In (a), the edges $a, b, c, d$ of $T$ appear consecutively in clockwise order about $v$, while in (c), the edges $a, c, b, d$ appear consecutively in clockwise order about $v$.

The embedding of $T$ in Figure 4.2(a) gives rise to the Hamiltonian walk

$$W_1 = (w, v, x, v, y, v, z, v, w),$$

where the edges of $T$ are traced in the manner shown in Figure 4.2(b). The embedding
of $T$ shown in Figure 4.2(c) gives rise to the Hamiltonian walk

$$W_2 = (w, v, y, v, x, z, v, w).$$

In terms of the edges of $T$, these two walks can also be described as

$$W_1 = (a, b, b, c, d, d, a)$$

and

$$W_2 = (a, c, b, b, d, d, a).$$

While every edge of $T$ occurs twice in both $W_1$ and $W_2$, this is not true for all 3-paths in $T$. For example, the 3-path $(w, v, x) = ab$ occurs in $W_1$ but not in $W_2$, while the 3-path $(w, v, y) = ac$ occurs in $W_2$ but not in $W_1$.

### 4.2 A More General Result

With the aid of spanning trees and Hamiltonian walks in graphs, we can not only extend Theorem 4.1.2 but apply this technique to establish sufficient conditions for the 3-path graph of a connected graph to possess stronger Hamiltonian properties. First, we present a few lemmas.

**Lemma 4.2.1** Let $\{f_1, f_2, \ldots, f_k\}$ be the edge set of a star $F$ of size $k \geq 2$ in a connected graph. For $\ell = \binom{k}{2}$, there is a sequence $s : H_1, H_2, \ldots, H_\ell$ of 3-paths in $F$ consisting of $\ell$ distinct ordered pairs $f_if_j$ where $1 \leq i \neq j \leq k$ such that

1. $H_i$ and $H_{i+1}$ have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and
2. $H_1 = f_1f_i$ and $H_\ell = f_jf_1$ for some integers $i, j \geq 2$ where $i \neq j$

**Proof.** We proceed by induction on $k$. For $k = 2$, let $s : H_1 = f_1f_2$. For $k = 3$, let

$$s : H_1 = f_1f_2, H_2 = f_2f_3, H_3 = f_3f_1.$$ 

Thus, the statement is true for $k = 2, 3$. Suppose that the statement is true for some integer $k \geq 3$. Let $\{f_1, f_2, \ldots, f_k, f_{k+1}\}$ be the edge set of a star of size $k + 1 \geq 4$ in a connected graph. Applying the induction hypothesis to the set $\{f_1, f_2, \ldots, f_k\}$, we see that there is a sequence

$$s_0 : H_1, H_2, \ldots, H_\ell,$$

where $\ell = \binom{k}{2}$, consisting of $\ell$ ordered pairs $f_if_j$ where $1 \leq i \neq j \leq k$ such that

1. $H_i$ and $H_{i+1}$ have exactly one edge in common for $i = 1, 2, \ldots, \ell - 1$ and
(ii) \( H_1 = f_1 f_i \) and \( H_\ell = f_j f_1 \) for integers \( i \) and \( j \) where \( 2 \leq i, j \leq k \) and \( i \neq j \).

We may assume, by relabeling \( f_2, f_3, \ldots, f_k \) if necessary, that \( H_1 = f_1 f_2, H_2 = f_2 f_3 \) or \( H_2 = f_1 f_3 \) and \( H_\ell = f_k f_1 \). We now insert the sequence

\[
s' : f_1 f_{k+1}, f_2 f_{k+1}, f_4 f_{k+1}, \ldots, f_k f_{k+1}, f_3 f_{k+1}
\]

between \( H_1 = f_1 f_2 \) and \( H_2 \) to obtain the sequence \( s : H_1, s', H_2, \ldots, H_\ell \), producing

\[
s : H_1 = f_1 f_2, f_1 f_{k+1}, f_2 f_{k+1}, f_4 f_{k+1}, \ldots, f_k f_{k+1}, f_3 f_{k+1}, H_2, H_3, \ldots, H_\ell = f_k f_1
\]

with the desired property.

\[\square\]

**Lemma 4.2.2**  Let \( \{f_1, f_2, \ldots, f_k\} \) be the edge set of a star \( F \) of size \( k \geq 3 \) in a connected graph. For \( \ell = \binom{k}{2} - 1 \), there is a sequence \( H_1, H_2, \ldots, H_\ell \) of 3-paths in \( F \) consisting of \( \ell \) ordered pairs \( f_i f_j \) where \( 1 \leq i \neq j \leq k \) and \( \{i, j\} \neq \{1, k\} \) such that

(i) \( H_i \) and \( H_{i+1} \) have an edge in common for \( i = 1, 2, \ldots, \ell - 1 \) and

(ii) \( H_1 = f_1 f_i \) and \( H_\ell = f_j f_k \) for integers \( i \) and \( j \) with \( i \neq 2 \) and \( j \neq 1 \).

**Proof.** Applying Lemma 4.2.1 to the set \( \{f_1, f_2, \ldots, f_{k-1}\} \), where \( k - 1 \geq 2 \), we see that there is a sequence \( s_0 : F_1, F_2, \ldots, F_\ell \), where \( \ell = \binom{k-1}{2} \), consisting of \( \ell \) distinct ordered pairs \( f_i f_j \) with \( 1 \leq i \neq j \leq k - 1 \) such that

(i) \( F_i \) and \( F_{i+1} \) have exactly one edge in common for \( i = 1, 2, \ldots, \ell - 1 \) and

(ii) \( F_1 = f_1 f_i \) and \( F_\ell = f_j f_1 \) integers \( i \) and \( j \) with \( 2 \leq i, j \leq k - 1 \) and \( i \neq j \).

Let \( s' = f_1 f_k, f_2 f_k, \ldots, f_{k-1} f_k \). The sequence

\[
s : s_0, s' : F_1 = f_1 f_i, F_2, \ldots, F_\ell = f_j f_1, f_1 f_k, f_2 f_k, \ldots, f_{k-1} f_k
\]

has the desired property.

\[\square\]

**Lemma 4.2.3**  Let \( G \) be a connected graph of order at least 4 such that every vertex of degree 2 is adjacent to an end-vertex and a vertex of degree at least 3 and let \( T \) be a spanning tree of \( G \), where \( W \) is a Hamiltonian walk of \( T \). There exists a sequence \( S : A_1, A_2, \ldots, A_p, A_{p+1} = A_1 \) of 3-paths of \( G \) consisting of all 3-paths belonging to \( W \) and all 3-paths of \( G \) whose interior vertex is an end-vertex of \( T \) such that \( A_i \) and \( A_{i+1} \) have an edge in common for \( i = 1, 2, \ldots, p \). Furthermore, all 3-paths of \( S \) are distinct except for those 3-paths belonging to \( W \) whose interior vertex has degree 2 in \( T \) and those 3-paths appear exactly twice in \( S \).
Proof. Let $W = (v_1, v_2, \ldots, v_t, v_{t+1} = v_1)$ be a Hamiltonian walk of $T$. First, consider the sequence

$$S_0 : (v_1, v_2, v_3), (v_2, v_3, v_4), \ldots, (v_t, v_1, v_2)$$

consisting of $t$ walks of length 2 lying on $W$. A walk of length 2 on $W$ is a 3-path on $W$ if its interior vertex is not an end-vertex of $T$. Furthermore, a 3-path on $W$ appears exactly once in $S_0$ if its interior vertex has degree at least 3 and a 3-path on $W$ appears exactly twice in $S_0$ if its interior vertex has degree 2. For each end-vertex $v_j$ of $T$, the walk $(v_{j-1}, v_j, v_{j+1})$ belongs to $S_0$ and so $v_{j-1} = v_{j+1}$. Thus, $(v_{j-2}, v_{j-1}, v_j), (v_{j-1}, v_j, v_{j+1}), (v_j, v_{j+1}, v_{j+2})$ are three consecutive terms in $S_0$, where $v_{j-1} = v_{j+1}$. Let $x = v_{j-2}v_{j-1}$, $a = v_{j-1}v_j = v_jv_{j+1}$ and $y = v_{j+1}v_{j+2}$. Then $xa, aa, ay$ are three consecutive terms in $S_0$ and $aa$ appears exactly once in $S_0$.

We claim that $\deg_G v_j \neq 2$; for otherwise, the two edges incident with $v_j$ in $G$ are bridges of $G$ and hence they must belong to $T$. However then, $\deg_T v_j = 2$, which is a contradiction. Thus, either $\deg_G v_j = 1$ or $\deg_G v_j = k \geq 3$. If $\deg_G v_j = 1$, then we delete $aa$ from $S_0$. If $\deg_G v_j = k \geq 3$, then let $a = e_1, e_2, e_3, \ldots, e_{k-1}, e_k$ be the $k$ distinct edges incident with $v_j$ in $G$ where then only $a = e_1$ belongs to $T$. There are $\binom{k}{2} = \ell$ 3-paths of $G$ with interior vertex $v_j$ that do not belong to $S_0$. Each of these 3-paths has the form $e_re_s$ where $r, s \in \{1, 2, \ldots, k\}$ and $r \neq s$. By Lemma 4.2.1, there is an ordering $H_1, H_2, \ldots, H_\ell$ of these $\ell$ 3-paths such that $H_1 = e_1e_i$ and $H_\ell = e_1e_j$, where $i, j \geq 2$ and $i \neq j$. We replace $aa = e_1e_1$ in $S_0$ by $H_1, H_2, \ldots, H_\ell$ or, equivalently, replace $xe_1, e_1e_1, e_1y$ in $S_0$ by $xe_1, H_1, H_2, \ldots, H_\ell, e_1y$. Applying this procedure to every end-vertex of $T$, we obtain a sequence $S$ with the desired property. 

Lemma 4.2.4 Let $G$ be a connected graph of order at least 4 such that every vertex of degree 2 is adjacent to an end-vertex and a vertex of degree at least 3 and let $T$ be a spanning tree of $G$ containing vertices of degree 2, where $W$ is a Hamiltonian walk of $T$. There exists a sequence $S : A_1, A_2, \ldots, A_p, A_{p+1} = A_1$ of distinct 3-paths of $G$ consisting of (i) all 3-paths belonging to $W$, (ii) all 3-paths of $G$ whose interior vertex has degree 1 or 2 in $T$ and (iii) all 3-paths of $G$ whose interior vertex is a neighbor of a vertex of degree 2 in $G$ such that $A_i$ and $A_{i+1}$ have an edge in common for $i = 1, 2, \ldots, p$.

Proof. Let $W = (v_1, v_2, \ldots, v_t, v_{t+1} = v_1)$ be a Hamiltonian walk of $T$. By Lemma 4.2.3, there exists a sequence $S_1 : B_1, B_2, \ldots, B_q, B_{q+1} = B_1$ of 3-paths of $G$ consisting of all 3-paths belonging to $W$ and all 3-paths of $G$ whose interior vertex is an end-vertex.
of $T$ such that $B_i$ and $B_{i+1}$ have an edge in common for $i = 1, 2, \ldots, q$. Furthermore, all 3-paths of $S_1$ are distinct except for those 3-paths belonging to $W$ whose interior vertex has degree 2 in $T$ and those 3-paths appear exactly twice in $S$.

Let $v$ be a vertex of degree 2 in $T$ and let $a$ and $b$ denote the two edges incident with $v$ in $T$. Since $v$ appears exactly twice on $W$, the 3-path $ab$ appears exactly twice in $S_1$. Suppose that $v$ appears first as $v_j$ in $W$ and so the 3-path $ab = (v_{j-1}, v_j, v_{j+1})$ appears first in $S_1$ where $a = v_{j-1}v_j$ and $b = v_jv_{j+1}$. Let $x = v_{j-2}v_{j-1}$ and $y = v_{j+1}v_{j+2}$. Thus, $xa, ab, by$ are three consecutive terms in $S_1$. We consider two cases according to whether $\deg_G v_j = 2$ or $\deg_G v_j = k \geq 3$.

**Case 1.** $\deg_G v_j = 2$. Then $v_j$ is adjacent to an end-vertex and a vertex of degree at least 3 in $G$. We may assume, without loss of generality, that $v_{j+1}$ is an end vertex of $G$ and $v_{j-1}$ is a vertex of degree at least 3 in $G$. There are two subcases, according to whether $\deg_T v_{j-1} = 2$ or $\deg_T v_{j-1} \geq 3$.

**Subcase 1.1.** $\deg_T v_{j-1} = 2$. Since $\deg_G v_{j-1} \geq 3$, there are edges $f_1, f_2, \ldots, f_t$ of $G$ that are incident with $v_{j-1}$ but not in $T$, where $t = \deg_G v_{j-1} - 2 \geq 1$. Then $S_1$ contains

$$zx, xa, ab, ba, ax$$

as consecutive terms for some edge $z$ of $T$. [Note that the walk $bb$ of length 2 in $W$ was deleted from the sequence in the proof of Lemma 4.2.3.] Since $\deg_T v_{j-1} = 2$ and $\deg_G v_{j-1} = t + 2$, there are \( \binom{t+2}{2} - 1 = \ell \) 3-paths of $G$ with interior vertex $v_{j-1}$ that do not belong to $S_1$. Applying Lemma 4.2.2 to the set $\{x, f_1, \ldots, f_t, a\}$, we obtain an ordering

$$s_1 : H_1, H_2, \ldots, H_\ell$$

of distinct 3-paths such that $H_1 = xf_1$ and $H_\ell = af_t$. Now, we replace the consecutive terms $xa, ab$ in (4.1) by the sequence $s_1$, resulting in a new sequence of 3-paths that contains the consecutive terms

$$zx, H_1 = xf_1, H_2, \ldots, H_\ell = af_t, ba, ax.$$  

**Subcase 1.2.** $\deg_T v_{j-1} \geq 3$. Then $S_1$ contains

$$zx, xa, ab, ba, ay$$

as consecutive terms for some edges $y$ and $z$ of $T$, where $x \neq y$. If every edge incident with $v_{j-1}$ belongs to $T$, that is $\deg_G v_{j-1} = \deg_T v_{j-1}$, then every 3-path whose interior
vertex is \( v_{j-1} \) belongs to \( S_1 \). Thus, we may assume that there are edges of \( G \) incident with \( v_{j-1} \) that do not belong to \( T \). Let \( f_1, f_2, \ldots, f_t \) be the edges of \( G \) incident with \( v_{j-1} \) that do not belong to \( T \), where \( t = \text{deg}_G v_{j-1} - \text{deg}_T v_{j-1} \geq 1 \), and let \( x, a, e_1, e_2, \ldots, e_k \) be the edges of \( G \) incident with \( v_{j-1} \) that belong to \( T \), where \( k = \text{deg}_T v_{j-1} - 2 \geq 1 \). We may assume, without loss of generality, that \( T \) is embedded in the plane so that the edges \( x, a, e_1, e_2, \ldots, e_k \) of \( T \) incident with \( v_{j-1} \) appear in this order in \( W \) (not necessarily as consecutive terms). Hence, for every integer \( i \) with \( 1 \leq i \leq d \), each edge \( e_i \) appears in the consecutive terms \( x_i e_i, e_i e_{i+1} \) for some edge \( x_i \) of \( T \) in \( S_1 \). Let \( \ell = \binom{t+2}{2} - 1 \). There are \( \binom{t+2}{2} - 1 + kt = \ell + kt \) 3-paths of \( G \) with interior vertex \( v_{j-1} \) that do not belong to \( S_1 \). Next, we add these 3-paths to \( S_1 \) and delete the 3-path \( ab \) in (4.2) as follows:

\[
\begin{align*}
\star & \text{ Applying Lemma 4.2.2 to the set } \{x, f_1, \ldots, f_t, a\} \text{ gives rise to an ordering } s_0: H_1, H_2, \ldots, H_\ell \text{ of distinct 3-paths of } G \text{ such that } H_1 = xf_1 \text{ and } H_\ell = af_\ell. \text{ We replace } ab \text{ in (4.2) by the sequence } s_0. \\
\star & \text{ For } 1 \leq i \leq k, \text{ let } s_i: e_if_1, e_if_2, \ldots, e_if_\ell. \text{ We insert } s_i \text{ between the two consecutive terms } x_i e_i \text{ and } e_i e_{i+1} \text{ in } S_1. 
\end{align*}
\]

This produces a new sequence consisting of all 3-paths in \( S_1 \) and all distinct 3-paths of \( G \) whose interior vertex is \( v_{j-1} \).

\textbf{Case 2. } \text{deg}_G v_j = k \geq 3. \text{ Let } a = e_1, e_2, e_3, \ldots, e_{k-1}, e_k = b \text{ be the } k \text{ distinct edges incident with } v_j \text{ in } G, \text{ where then only } a = e_1 \text{ and } b = e_k \text{ belong to } T. \text{ There are } \binom{k}{2} - 1 = \ell \text{ 3-paths of } G \text{ with interior vertex } v_j \text{ that do not belong to } S_1. \text{ Each of these 3-paths has the form } e_re_s \text{ where } r, s \in \{1, 2, \ldots, k\} \text{ and } \{r, s\} \not= \{1, k\}. \text{ By Lemma 4.2.2, there is an ordering } H_1, H_2, \ldots, H_\ell \text{ of these } \ell \text{ 3-paths such that } H_1 = e_1e_r \text{ and } H_\ell = e_se_k, \text{ where } r \not= k \text{ and } s \not= 1. \text{ We replace } ab = e_1e_k \text{ in } S_1 \text{ by } H_1, H_2, \ldots, H_\ell, \text{ (or, equivalently, replace } x_1e_1, e_1e_k, e_ky \text{ in } S_1 \text{ by } x_1e_1, H_1, H_2, \ldots, H_\ell, e_ky). \text{ Applying this procedure to every vertex of degree 2 in } T, \text{ we obtain a sequence } S \text{ with the desired property.} 

\textbf{Lemma 4.2.5 } \text{Let } G \text{ be a connected graph of order at least 4 such that every vertex of degree 2 is adjacent to an end-vertex and a vertex of degree at least 3 and let } T \text{ be a spanning tree of } G \text{ containing vertices of degree 3 and let } W \text{ be a Hamiltonian walk of } T. \text{ There exists a sequence } S: A_1, A_2, \ldots, A_p, A_{p+1} = A_1 \text{ of distinct 3-paths of } G \text{ consisting of (i) all 3-paths belonging to } W, \text{ (ii) all 3-paths of } G \text{ whose interior vertex has degree 1, 2 or 3 in } T \text{ and (iii) all 3-paths of } G \text{ whose interior vertex is a neighbor of a vertex of degree 2 in } G \text{ such that } A_i \text{ and } A_{i+1} \text{ have an edge in common for } i = 1, 2, \ldots, p. \]
Proof. Let \( W = (v_1, v_2, \ldots, v_t, v_{t+1} = v_1) \) be a Hamiltonian walk of \( T \). By Lemma 4.2.4, there exists a sequence \( S_1 : B_1, B_2, \ldots, B_q, B_{q+1} = B_1 \) of distinct 3-paths of \( G \) consisting of (i) all 3-paths belonging to \( W \), (ii) all 3-paths of \( G \) whose interior vertex has degree 1 or 2 in \( T \) and (iii) all 3-paths of \( G \) whose interior vertex is a neighbor of a vertex of degree 2 in \( G \) such that such that \( B_i \) and \( B_{i+1} \) have an edge in common for \( i = 1, 2, \ldots, q \).

Let \( v \) be a vertex of degree 3 in \( T \). If \( v \) is a neighbor of a vertex of degree 2 in \( G \), then we needn’t do anything. Thus, we may assume that \( v \) is not a neighbor of a vertex of degree 2 in \( G \). Let \( e_1, e_2 \) and \( e_3 \) denote the three edges incident with \( v \) in \( T \). We may assume that the sequence \( S_1 \) contains

1. three consecutive terms \( ae_1, e_1e_2, e_2b \),
2. three consecutive terms \( ce_2, e_2e_3, e_3d \) and
3. three consecutive terms \( fe_3, e_3e_1, e_1g \).

If every edge incident with \( v \) belongs to \( T \), then we needn’t do anything. On the other hand, suppose that there are \( k \geq 1 \) edges incident with \( v \) that do not belong to \( T \), say \( f_1, f_2, \ldots, f_k \). Applying Lemma 4.2.1 to the set \( \{e_1, f_1, \ldots, f_k\} \), we see that there is a sequence \( s_1 : H_1, H_2, \ldots, H_\ell \) consisting of \( \ell = \binom{k+1}{2} \) distinct ordered pairs \( xy \) where \( x, y \in \{e_1, f_1, \ldots, f_k\} \) such that

(i) \( H_i \) and \( H_{i+1} \) have an edge in common for \( i = 1, 2, \ldots, \ell - 1 \) and

(ii) \( H_1 = e_1f_1 \) and \( H_\ell = f_\ell e_1 \).

Next, we

* insert \( s_1 \) between \( ae_1 \) and \( e_1e_2 \) in \( S_1 \),
* insert \( e_2f_1, e_2f_2, \ldots, e_2f_k \) between \( e_1e_2 \) and \( e_2b \) and
* insert \( e_3f_1, e_3f_2, \ldots, e_3f_k \) between \( e_2e_3 \) and \( e_3d \).

Applying this procedure to every vertex of degree 3 in \( T \), we obtain a sequence \( S \) with the desired property.

We are now prepared to prove the following theorem.

**Theorem 4.2.6** If \( G \) is a connected of order at least 4 such that each vertex of degree 2 is adjacent to an end-vertex and a vertex of degree at least 3, then \( P_3(G) \) is Hamiltonian.
Proof. It suffices to show that there exists an ordering $S$: $A_1, A_2, \ldots, A_p, A_{p+1} = A_1$ of all 3-paths $A_i$ ($1 \leq i \leq p$) of $G$ such that $A_i$ and $A_{i+1}$ have an edge in common for $i = 1, 2, \ldots, p$. We verify this by showing that the following statement is true.

For every spanning tree $T$ of $G$ and a Hamiltonian walk $W$ of $T$, there exists a sequence $S$: $A_1, A_2, \ldots, A_p, A_{p+1} = A_1$ of distinct 3-paths of $G$ consisting of (i) all 3-paths belonging to $W$ (ii) all 3-paths of $G$ whose interior vertex has degree $k$ or less for every integer $k$ with $k = 3, 4, \ldots, \Delta(T)$ in $T$ and (iii) all 3-paths of $G$ whose interior vertex is a neighbor of a vertex of degree 2 in $G$ such that $A_i$ and $A_{i+1}$ have an edge in common for $i = 1, 2, \ldots, p$.

We verify this statement by induction on $k \geq 3$. Let $T$ be a spanning tree of $G$. By Lemma 4.2.5, the result is true for $k = 3$. Therefore, if $\Delta(T) = 2$ or $\Delta(T) = 3$, then the result follows. Hence, we may assume that $\Delta(T) \geq 4$. Assume, for an integer $k$ with $3 \leq k < \Delta(T)$, that there exists a sequence $S_1: B_1, B_2, \ldots, B_q, B_{q+1} = B_1$ of distinct 3-paths of $G$ consisting of (i) all 3-paths belonging to $W$ (ii) all 3-paths of $G$ whose interior vertex has degree $1, 2, \ldots, k$ in $T$ and (iii) all 3-paths of $G$ whose interior vertex is a neighbor of a vertex of degree 2 in $G$ such that $B_i$ and $B_{i+1}$ have an edge in common for $i = 1, 2, \ldots, q$. If $T$ has no vertex of degree $k + 1$ in $T$, then the result follows. Hence, we may assume that $T$ contains one or more vertices of degree $k + 1 \geq 4$ in $T$.

Let $v$ be a vertex of degree $k + 1 \geq 4$ in $T$. If $v$ is a neighbor of a vertex of degree 2 in $G$, then we needn’t do anything. Thus, we may assume that $v$ is not a neighbor of a vertex of degree 2 in $G$. Let $e_1, e_2, \ldots, e_{k+1}$ denote the $k + 1$ edges incident with $v$ in $T$. We may assume, by relabeling $e_1, e_2, \ldots, e_{k+1}$ if necessary, that

(i) $S_1$ contains the 3-paths $e_ie_{i+1}$ for $1 \leq i \leq k$ and $e_{k+1}e_1$ and

(ii) these 3-paths appear in $S_1$ in the order $e_1e_2, e_2e_3, \ldots, e_{k}e_{k+1}, e_{k+1}e_1$

(not necessarily as consecutive terms in $S_1$).

Hence, there are 3-paths having interior vertex $v$ that do not belong to $S_1$. Let $X$ be the set of all such 3-paths and so $|X| = \binom{k+1}{2} - (k + 1) \geq 2$. Divide $X$ into $k - 1$ subsets $X_1, X_2, \ldots, X_{k-1}$ where $X_1 = \{e_1e_s : 3 \leq s \leq k\}$ and $X_r = \{e_re_s \in X : r + 2 \leq s \leq k + 1\}$ for $2 \leq r \leq k - 1$. In particular, $X_1 = \{e_1e_3, e_1e_4, \ldots, e_1e_k\}$, $X_2 = \{e_2e_4, e_2e_5, \ldots, e_2e_k, e_2e_{k+1}\}$ and $X_{k-1} = \{e_{k-1}e_{k+1}\}$. Next, let $s_r$ be an ordering
of vertices of $X_r$ for $1 \leq r \leq k - 1$. The sequence $S_1$ contains the following consecutive terms

$$a_re_r, e_re_{r+1}, e_{r+1}b_r \text{ for } 1 \leq r \leq k \text{ and } a_{k+1}e_{k+1}, e_{k+1}e_1, e_1b_{k+1}$$

for some edges $a_r, b_r, a_{k+1}$ and $b_{k+1}$ of $T$. If every edge incident with $v$ belongs to $T$, then we insert $s_r$ between $a_re_r$ and $e_{r+1}b_r$ for $1 \leq r \leq k$. On the other hand, suppose that there are $d \geq 1$ edges incident with $v$ that do not belong to $T$, say $f_1, f_2, \ldots, f_d$. Applying Lemma 4.2.1 to the set $\{e_1, f_1, \ldots, f_d\}$, we see that there is a sequence $s_0 : H_1, H_2, \ldots, H_\ell$

consisting of $\ell = \binom{d+1}{2}$ distinct ordered pairs $xy$ where $x, y \in \{e_1, f_1, \ldots, f_d\}$ such that

(i) $H_i$ and $H_{i+1}$ have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and

(ii) $H_1 = e_1f_i$ and $H_\ell = f_je_1$ where $1 \leq i \neq j \leq d$ and $H_1 = H_\ell$ if $d = 1$.

Next, we insert $s_1, s_0$ between $a_1e_1$ and $e_1e_2$ in $S_1$ and insert $s_r$ between $a_re_r$ and $e_{r+1}b_r$ for $2 \leq r \leq k - 1$. Applying this procedure to every vertex of degree $k + 1$ in $T$, we obtain a sequence $S$ with the desired property.

Theorem 4.2.6 is best possible in the sense that if a connected graph $G$ contains even one vertex of degree 2 that does not satisfy the conditions in Theorem 4.2.6, then $P_3(G)$ need not be Hamiltonian. For example, the graph $G$ of Figure 4.3 has exactly one vertex of degree 2, each of whose neighbors has degree 3. Since the 3-path $ab$ of $G$ is a cut-vertex of $P_3(G)$, it follows that $P_3(G)$ is not Hamiltonian. If $H$ is the graph obtained from the graph $G$ of Figure 4.3 by subdividing the edge $a$ exactly once, then $H$ contains a vertex of degree 2 that is adjacent to another vertex of degree 2 and a vertex of degree 3 in $H$ and so $H$ contains a path $(u, v, w, x)$ where $v$ and $w$ have degree 2 in $H$ and $u$ and $x$ have degree 3. In this case, $P_3(H)$ contains a bridge joining the two 3-paths $(u, v, w)$ and $(v, w, x)$ and so $P_3(H)$ is not Hamiltonian. Furthermore, if $F$ is a connected graph containing a vertex $v$ of degree 2 such that $v$ is adjacent to an end-vertex $u$ and another vertex $w$ of degree 2, then the 3-path $(u, v, w)$ is an end-vertex in $P_3(F)$ and so $P_3(F)$ is not Hamiltonian.

### 4.3 Hamiltonian-Connected 3-Path Graphs

While $P_3(G)$ is Hamiltonian for every connected graph $G$ with $\delta(G) \geq 3$, the graph $P_3(G)$ need not be Hamiltonian-connected. Figure 4.4 shows a connected 3-regular graph $G$
and $P_3(G) = L(L(G))$. In this graph $G$, the edges of interest are labeled 1, 2, …, 9. Consequently, the corresponding vertices in $L(G)$ are labeled 1, 2, …, 9, producing edges 12, 13, 23, 14, 15, 45, etc. in $L(G)$ and thus those vertices in $P_3(G)$. The graph $P_3(G)$ of Figure 4.4 is not 3-connected and therefore is not Hamiltonian-connected. For example, there is neither a 14-15 nor a 12-13 Hamiltonian path in $P_3(G)$.

The goal of this section is to describe how the concepts of spanning trees, Hamiltonian walks and the technique introduced in [6] can be used to show that the 3-path graph of every connected graph $G$ with $\delta(G) \geq 4$ is Hamiltonian-connected. It is useful to establish the following two lemmas, each of which presents a method in which the edges of a star can be ordered to satisfy certain desirable conditions.

**Lemma 4.3.1** Let $E = \{f_1, f_2, \ldots, f_k\}$ be the edge set of a star of size $k \geq 2$. For $\ell = \binom{k}{2}$, there is a sequence $H_1, H_2, \ldots, H_\ell$ of the $\ell$ distinct pairs $f_if_j$ of edges of $E$ where

(i) $H_i$ and $H_{i+1}$ have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and
\( (ii) \quad H_1 = f_1 f_2 \text{ and } H_\ell = f_1 f_k. \)

**Proof.** We proceed by induction on \( k \). For \( k = 2 \), \( H_1 = f_1 f_2 \) verifies the statement. For \( k = 3 \), the sequence \( H_1 = f_1 f_2, H_2 = f_2 f_3, H_3 = f_1 f_3 \) verifies the statement. Thus, the statement is true for \( k = 2, 3 \). Suppose that the statement is true for some integer \( k \geq 3 \). Let \( \{f_1, f_2, \ldots, f_k, f_{k+1}\} \) be the edge set of a star of size \( k + 1 \geq 4 \). Applying the induction hypothesis to the set \( \{f_1, f_2, \ldots, f_k\} \), we see that there is a sequence 

\[
\begin{align*}
s_0 : H_1, H_2, \ldots, H_\ell,
\end{align*}
\]

where \( \ell = \binom{k}{2} \), consisting of the \( \ell \) distinct ordered pairs \( f_i f_j \) where \( 1 \leq i \neq j \leq k \) such that

\[
(i) \quad H_i \text{ and } H_{i+1} \text{ have an edge in common for } i = 1, 2, \ldots, \ell - 1 \text{ and}
\]

\[
(ii) \quad H_1 = f_1 f_2 \text{ and } H_\ell = f_1 f_k.
\]

Thus, \( H_2 = f_1 f_j \) or \( H_2 = f_2 f_j \) for some integer \( j \) with \( 3 \leq j \leq k \) (necessarily, \( H_2 = f_2 f_3 \) if \( k = 3 \)). We now insert the sequence

\[
\begin{align*}
s' : f_2 f_{k+1}, \ldots, f_{j-1} f_{k+1}, f_j f_{k+1}, \ldots, f_{k} f_{k+1}, f_j f_{k+1}
\end{align*}
\]

between \( H_1 = f_1 f_2 \) and \( H_2 \) and add \( f_1 f_{k+1} \) after \( f_1 f_k \), producing the sequence

\[
\begin{align*}
s : \quad H_1 = f_1 f_2, f_2 f_{k+1}, \ldots, f_{j-1} f_{k+1}, f_j f_{k+1}, \ldots, f_{k} f_{k+1}, f_j f_{k+1},
\quad H_2, H_3, \ldots, H_\ell = f_1 f_k, f_1 f_{k+1},
\end{align*}
\]

which has the desired property. 

\[ \blacksquare \]

**Lemma 4.3.2** Let \( E = \{f_1, f_2, \ldots, f_k\} \) be the edge set of a star of size \( k \geq 2 \). For \( \ell = \binom{k}{2} \), there is a sequence \( H_1, H_2, \ldots, H_\ell \) consisting of the \( \ell \) distinct pairs \( f_i f_j \) of edges of \( E \) where

\[
(i) \quad H_i \text{ and } H_{i+1} \text{ have an edge in common for } i = 1, 2, \ldots, \ell - 1 \text{ and}
\]

\[
(ii) \quad H_1 = f_1 f_2 \text{ and } H_\ell = f_{k-1} f_k.
\]

**Proof.** We proceed by induction on \( k \). First, we observe the following.

* For \( k = 2 \), \( H_1 = f_1 f_2 \) verifies the statement.

* For \( k = 3 \), \( H_1 = f_1 f_2, H_2 = f_1 f_3, H_3 = f_2 f_3 \) verifies the statement.
For $k = 4$, $H_1 = f_1 f_2, f_1 f_3, f_1 f_4, f_2 f_4, f_2 f_3, H_6 = f_3 f_4$ verifies the statement.

Thus, the statement is true for $k = 2, 3, 4$. Suppose that the statement is true for an integer $k \geq 4$. Let $\{f_1, f_2, \ldots, f_k, f_{k+1}\}$ be the edge set of a star of size $k + 1 \geq 5$. Applying the induction hypothesis to the set $\{f_1, f_2, \ldots, f_k\}$, we see that there is a sequence

$$s_0 : H_1, H_2, \ldots, H_\ell,$$

where $\ell = \binom{k}{2}$, consisting of the $\ell$ distinct ordered pairs $f_i f_j$ where $1 \leq i \neq j \leq k$ such that

(i) $H_i$ and $H_{i+1}$ have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and

(ii) $H_1 = f_1 f_2$ and $H_\ell = f_{k-1} f_k$.

Thus, either $H_2 = f_1 f_j$ or $H_2 = f_2 f_j$ for some integer $j$ with $3 \leq j \leq k$. Let

$$s' : f_1 f_{k+1}, f_2 f_{k+1}, \ldots, f_{j-1} f_{k+1}, f_j f_{k+1}, \ldots, f_{k-1} f_{k+1}, f_j f_{k+1}.$$

We now insert $s'$ between $H_1 = f_1 f_2$ and $H_2$ and add $f_k f_{k+1}$ after $f_{k-1} f_k$, producing the sequence

$$s : H_1 = f_1 f_2, f_1 f_{k+1}, f_2 f_{k+1}, \ldots, f_{j-1} f_{k+1}, f_j f_{k+1}, \ldots, f_{k-1} f_{k+1},$$

$$f_j f_{k+1}, H_2, H_3, \ldots, f_{k-1} f_k, f_k f_{k+1},$$

which has the desired property. \hfill \blacksquare

**Theorem 4.3.3** If $G$ is a connected graph with $\delta(G) \geq 4$, then $\mathcal{P}_3(G)$ is Hamiltonian-connected.

**Proof.** It suffices to show that for every two distinct 3-paths $P$ and $Q$ of $G$, there exists a sequence

$$S : P = A_1, A_2, \ldots, A_p = Q \quad (4.3)$$

consisting of the distinct 3-paths $A_i$ ($1 \leq i \leq p$) of $G$ that begins with $P$ and ends with $Q$ such that $A_i$ and $A_{i+1}$ have an edge in common for $i = 1, 2, \ldots, p - 1$. Let there be given two distinct 3-paths $P$ and $Q$ of $G$. We consider five cases, depending on the location of $P$ and $Q$ in the graph $G$. In each case, a spanning tree $T$ of $G$ is constructed based on the location of $P$ and $Q$. The tree $T$ is then appropriately embedded in the
plane from which a Hamiltonian walk $W$ is constructed. The walk $W$ then gives rise to a cyclic sequence $S_1$ consisting of those 3-paths of $T$ that lie on $W$. With the aid of $S_1$, a sequence $S$ as in (4.3) is constructed containing all 3-paths of $G$ possessing the desired property, thereby showing that $P_3(G)$ is Hamiltonian-connected.

**Case 1.** $P$ and $Q$ have an edge in common. There are three possibilities here, namely

1. $P$ and $Q$ have the same interior vertex,
2. $P$ and $Q$ have adjacent interior vertices and form a 4-path or
3. $P$ and $Q$ have adjacent interior vertices and form a 3-cycle.

We consider these three subcases.

**Subcase 1.1.** $P$ and $Q$ have the same interior vertex $v$. Let $P = ab$ and $Q = bc$ and let $d$ be a fourth edge incident with $v$. Let $T$ be a spanning tree of $G$ containing $P$ and $Q$ as well as the edge $d$. The tree $T$ is embedded in the plane so that the edges $c, a, b, d$ appear consecutively in clockwise order about $v$ (see Figure 4.5). Any other edges of $T$ incident with $v$ lie between the edges $d$ and $c$ in this embedding of $T$.

![Figure 4.5: A planar embedding of $T$ in Subcase 1.1](image)

We now consider the Hamiltonian walk $W$ in $T$ obtained from this embedding of $T$ that encounters the edge $a$ before the edge $b$. By proceeding along $W$, we obtain a cyclic sequence $S_1$ of all 3-paths of $T$ lying on $W$. If $T$ contains a 3-path whose interior vertex has degree 2 in $T$, then this 3-path occurs twice in $S_1$. The sequence $S_1$ contains the 3-paths $P = ab$ and $bx$ as two consecutive terms for some edge $x$ (where $x = d$ if $b$ is a pendant edge). The 3-path $Q = bc$ is not a term in $S_1$. We now insert $Q$ between $ab$ and $bx$ to produce a sequence

$$S_2 : P = ab = B_1, B_2, \ldots, B_{\ell-1} = bx, B_\ell = bc = Q$$

consisting of all 3-paths on $W$ as well as the 3-path $bc$ such that $B_i$ and $B_{i+1}$ have an edge in common for $i = 1, 2, \ldots, \ell - 1$. While any 3-path in $T$ having an interior vertex
of degree 2 occurs twice in $S_2$, all other 3-paths on $W$ occur exactly once in $S_2$. We now describe additions that we make to $S_2$ at each vertex $u$ of $T$ depending on the degree of $u$ in $T$.

First, suppose that $u$ is an end-vertex of $T$, where $e$ is the edge in $T$ that is incident with $u$. Let $f_1, f_2, \ldots, f_d$ be the edges of $G$ incident with $u$ that are not in $T$, where $d = \deg_G u - 1 \geq 3$. Applying Lemma 4.3.1 to the set $E = \{e, f_1, f_2, \ldots, f_d\}$ for $\ell = \binom{d+1}{2}$, we see that there is a sequence $s : H_1, H_2, \ldots, H_\ell$ consisting of the $\ell$ distinct pairs of edges of $E$ where (i) $H_i$ and $H_{i+1}$ have one edge in common for $i = 1, 2, \ldots, \ell - 1$ and (ii) $H_1 = ef_1$ and $H_\ell = ef_d$. We insert $s$ between two consecutive terms containing $e$ in $S_2$. This is now done for each end-vertex $u$ of $T$, producing a sequence $S_3$ of 3-paths, consisting of all 3-paths of $T$ lying on $W$ and all 3-paths of $G$ having an interior vertex of degree 1 in $T$ such that every two consecutive terms in $S_3$ have a single edge in common.

Second, suppose that $T$ contains a vertex $u$ of degree 2, incident with edges $e$ and $f$. Then the 3-path $ef$ occurs twice in $S_3$. Since $\deg_G u \geq 4$, there are edges $e_1, e_2, \ldots, e_d$ $(d \geq 2)$ distinct from $e$ and $f$ that are incident with $u$ and belong to $G$ but not to $T$. So, $d = \deg_G u - 2 \geq 2$. Therefore, there are 3-paths in $G$ having the interior vertex $u$ that do not belong to $S_3$, namely, $ee_i$ $(1 \leq i \leq d)$, $fe_i$ $(1 \leq i \leq d)$ and $e_ie_j$ $(1 \leq i < j \leq d)$. In the second occurrence of the 3-path $ef$ in $S_3$, there are three consecutive terms $he, ef, fg$ in $S_3$ for some edges $h$ and $g$. In this case, we replace the 3-path $ef$ here by the sequence $ee_d, eed_{d-1}, \ldots, ee_1, S'', ef_d, f_{e_d-1}, \ldots, fe_1$ of 3-paths, where $S''$ is a sequence of distinct 3-paths $e_ie_j$ $(1 \leq i < j \leq d)$ beginning with $e_1e_2$ and ending with $e_{d-1}e_d$ such that consecutive 3-paths in $S''$ have an edge in common. By Lemma 4.3.2, such a sequence $S''$ exists. We do this for each vertex $u$ of degree 2 in $T$, producing a sequence $S_4$ of distinct 3-paths (having $ab$ and $bc$ are two consecutive terms), consisting of all 3-paths of $T$ lying on $W$ and all 3-paths of $G$ having an interior vertex of degree 1 or 2 in $T$, where every two consecutive terms have an edge in common.

Next, suppose that $T$ contains a vertex $u$ of degree 3 in $T$. Then every 3-path of $T$ having interior vertex $u$ occurs exactly once in both $W$ and $S_4$. Let $e_1, e_2, e_3$ be the three edges of $T$ incident with $u$. We may assume that these three edges appear in clockwise order about $u$ as $e_1, e_2, e_3$ in $T$. Then $xe_1, e_1e_2, e_2y$ are three consecutive terms in $S_4$ for some edges $x$ and $y$. Let $f_1, f_2, \ldots, f_d$ be the edges of $G$ that are incident with $u$ but are not in $T$, where $d = \deg_G u - 3 \geq 1$. Applying Lemma 4.3.1
to the set \( E = \{e_1, f_1, f_2, \ldots, f_d, e_2\}, \) for \( \ell = \binom{d+2}{2} \), we see that there is a sequence \( s : H_1, H_2, \ldots, H_\ell \) consisting of the \( \ell \) distinct pairs of edges of \( E \) where (i) \( H_i \) and \( H_{i+1} \) have exactly one edge in common for \( i = 1, 2, \ldots, \ell - 1 \) and (ii) \( H_1 = e_1 f_1 \) and \( H_\ell = e_1 e_2 \). We now delete \( e_1 e_2 \) from \( S_4 \) and insert \( s \) between \( xe_1 \) and \( e_2 y \). Furthermore, we insert the sequence \( e_3 f_1, e_3 f_2, \ldots, e_3 f_d \) between two consecutive terms in \( S_4 \) containing \( e_3 \). We do this for each vertex \( u \) of degree 3 in \( T \), producing a sequence \( S_5 \) of distinct 3-paths (having \( ab \) and \( bc \) as consecutive terms) consisting of all 3-paths of \( T \) lying on \( W \), the 3-path \( bc \) and all 3-paths of \( G \) having an interior vertex of degree 1, 2 or 3 in \( T \), where every two consecutive terms have an edge in common.

Finally, let \( u \) be a vertex of degree 4 or more in \( T \). First, suppose that every edge incident with \( u \) belongs to \( T \), say \( e_1, e_2, \ldots, e_d \) are the edges incident with \( u \) where \( d = \deg_T u = \deg_G u = d \geq 4 \). We may assume that the edges of \( T \) incident with \( u \) appear consecutively in \( T \) in counter-clockwise order about \( u \) as \( e_1, e_2, \ldots, e_d \). Thus, \( e_1 e_2, e_2 e_3, \ldots, e_{d-1} e_d, e_d e_1 \) are 3-paths in \( W \). Consequently, there are \( \binom{d}{2} - d \geq 2 \) 3-paths of \( G \) with the interior vertex \( u \) that do not lie on \( W \). Let \( X \) be the set of 3-paths whose interior vertex is \( u \) that do not appear in \( S_5 \). For each integer \( i \) with \( 1 \leq i \leq d - 2 \), let \( X_i = \{e_i e_j : i + 1 < j\} \) and let \( s_i \) be any ordering of the 3-paths in \( X_i \). For \( 1 \leq i \leq d - 1 \), insert the 3-paths in \( X_i \) in the order \( s_i \) between two consecutive terms containing \( e_i \) in \( S_5 \). We do this for every vertex \( u \) of degree 4 or more each of whose incident edges belongs to \( T \).

Next, suppose that there are edges of \( G \) incident with \( u \) that do not belong to \( T \). Let \( e_1, e_2, \ldots, e_d \) be the edges incident with \( u \) that belong to \( T \) and let \( f_1, f_2, \ldots, f_d' \) be the edges incident with \( u \) that do not belong to \( T \). Then \( d \geq 4 \) and \( d' \geq 1 \) and \( d + d' = \deg_G u \geq 5 \). We may assume that the edges of \( T \) incident with \( u \) appear consecutively in \( T \) in counter-clockwise order about \( u \) as \( e_1, e_2, \ldots, e_d \). Thus, \( e_1 e_2, e_2 e_3, \ldots, e_{d-1} e_d, e_d e_1 \) are 3-paths in \( W \). Then \( xe_1 e_2, e_2 y \) are three consecutive terms in \( S_5 \) for some edges \( x \) and \( y \). Applying Lemma 4.3.1 to the set \( E = \{e_1, f_1, f_2, \ldots, f_d', e_2\} \), for \( \ell = \binom{d+2}{2} \), we see that there is a sequence \( s : H_1, H_2, \ldots, H_\ell \) consisting of the \( \ell \) distinct pairs of edges of \( E \) where (i) \( H_i \) and \( H_{i+1} \) have exactly one edge in common for \( i = 1, 2, \ldots, \ell - 1 \) and (ii) \( H_1 = e_1 f_1 \) and \( H_\ell = e_1 e_2 \). We now delete \( e_1 e_2 \) from \( S_5 \) and insert \( s \) between \( xe_1 \) and \( e_2 y \). Let \( Y \) be the set of 3-paths whose interior vertex is \( u \), at least one of whose edges is in \( T \) that are not in \( S_5 \). For \( 1 \leq i \leq d \), let \( Y_i = \{e_i e_j : i + 1 < j\} \cup \{e_i f_j : 1 \leq j \leq d'\} \subseteq Y \) for \( 1 \leq i \leq d \), where \( \{e_i e_j : i + 1 < j\} = \emptyset \) if \( i = d - 1, d \) and let \( s_i \) be any ordering of the 3-paths in \( Y_i \). For \( 1 \leq i \leq d \), insert the 3-paths in \( Y_i \) in the order \( s_i \) between two
consecutive terms containing $e_i$ in $S_5$. We do this for every vertex $u$ of degree 4 or more in $G$, producing the sequence $S$ with the desired properties as described in (4.3).

Subcase 1.2. $P$ and $Q$ have adjacent interior vertices and form a 4-path. Let $P = ab$ and $Q = bc$ where $v$ is the interior vertex of $P$, shown in Figure 4.6. Let $T$ be a spanning tree of $G$ containing the 4-path $a, b, c$ as well as an edge $w$ incident with $v$ distinct from $a$ and $b$ (see Figure 4.6).

There exists a Hamiltonian walk $W$ of $T$ and a cyclic sequence $S_1$ of the 3-paths of $T$ on $W$ such that $xa, ab, bc, cy$ are four consecutive terms in $S_1$ for some edges $x$ and $y$ in $T$ (where possibly $x = w$ if $a$ is a pendant edge of $T$), that is,

$$S_2: P = ab = B_1, B_2, \ldots, B_\ell = bc = Q$$

is a sequence of all 3-paths of $T$ on $W$ where $B_i$ and $B_{i+1}$ have a single edge in common for $i = 1, 2, \ldots, \ell - 1$. We now proceed as in Subcase 1.1 to produce a sequence $S$ with the desired properties as described in (4.3).

Subcase 1.3. $P$ and $Q$ have adjacent interior vertices and form a 3-cycle. Let $P = ab$ and $Q = bc$ and let $v$ be the interior vertex of $P$. Let $T$ be a spanning tree of $G$ containing the 3-path $P$ (but not the edge $c$), an edge $w$ incident with $v$ distinct from $a$ and $b$ as well as an edge $z$ incident with the interior vertex of $Q$ different from $b$ and $c$ such that $w$ and $z$ are not adjacent (see Figure 4.7).

There exists a Hamiltonian walk $W$ of $T$ and a cyclic sequence $S_1$ of the 3-paths of $T$ lying on $W$ such that $ab, bz$ are two consecutive terms in $S_1$ but $bc$ is not a term of $S_1$. We insert $bc$ between $ab$ and $bz$ producing the sequence
which consists of $bc$ and all 3-paths of $T$ lying on $W$ where $B_i$ and $B_{i+1}$ have an edge in common for $i = 1, 2, \ldots, \ell - 1$. We now proceed as in Subcase 1.1 to produce a sequence $S$ with the desired properties as described in (4.3).

The remaining four cases deal with situations in which $P$ and $Q$ do not have an edge in common. In these cases, $P = ab$ and $Q = cd$, where then $a, b, c, d$ are four distinct edges of $G$.

Case 2. $P$ and $Q$ do not have an edge in common and there is a path in $G$ containing $P$ and $Q$. Let $R$ be a shortest path in $G$ containing $P$ and $Q$. There are two possibilities here. Either $P$ and $Q$ have a vertex in common or $P$ and $Q$ are vertex-disjoint (see Figure 4.8). We consider these subcases.

Subcase 2.1. $P$ and $Q$ have a vertex in common and so $R = (a, b, c, d)$. Let $T$ be a spanning tree of $G$ containing $R$. Then there is a Hamiltonian walk $W$ of $T$ such that $R$ is a path in $W$, resulting in a cyclic sequence $S_1$ of 3-paths of $T$ occurring in the order they are encountered on $W$. We may assume that $T$ is embedded in the plane so that $ab, bc, cd$ are three consecutive terms in $S_1$. Since each edge of $T$ is encountered twice in $W$, each edge of $T$ that is not a pendant edge of $T$ occurs in two consecutive 3-paths twice in $S_1$. Thus, in addition to $ab, bc$, there is another pair of consecutive 3-paths in $S_1$ containing $b$, say $xb$ and $by$ (where possibly $x = c$ and/or $y = a$). If the 3-path $bc$ occurs twice in $S_1$, then we remove $bc$ from its first occurrence (between $ab$ and $cd$). Otherwise, we remove $bc$ from $S_1$ and insert $bc$ between $xb$ and $by$, producing a sequence

$$S_2 : P = ab = B_1, B_2, \ldots, B_{\ell - 1}, B_\ell = cd = Q$$

consisting of all 3-paths on $W$ where $B_i$ and $B_{i+1}$ have an edge in common for $i =$
1, 2, \ldots, \ell - 1. We now proceed as in Case 1 to place all 3-paths in \( G \) that are not in \( S_2 \) to produce a sequence \( S \) that begins with \( P \) and ends with \( Q \) consisting of all distinct 3-paths of \( G \) such that consecutive 3-paths in \( S \) have an edge in common, as described in (4.3).

**Subcase 2.2.** \( P \) and \( Q \) are vertex-disjoint. Let \( R = (a, b, e_1, e_2, \ldots, e_k, c, d) \), \( k \geq 1 \), be a shortest path in \( G \) containing \( P \) and \( Q \). Let \( T \) be a spanning tree of \( G \) containing \( R \) and embedded in the plane so that there is a Hamiltonian walk \( W \) of \( T \) such that \( R \) is a path in \( W \). This results in a cyclic sequence \( S_1 \) of 3-paths of \( T \) occurring in the order they are encountered on \( W \). Thus,

\[
xa, ab, be_1, e_1e_2, \ldots, e_ke, cd, dy
\]

are consecutive terms in \( S_1 \) for some edges \( x \) and \( y \). Each of the edges \( b, c, e_i \) (\( 1 \leq i \leq k \)) appears between consecutive terms only once in (4.5) and there is another pair of consecutive terms in \( S_1 \) containing each such edge. We can now delete each of the 3-paths \( be_1, e_1e_2, \ldots, e_ke \) in (4.5) whose interior vertex has degree 2 and move every other such 3-path to an appropriate position in \( S_1 \) where the interior vertex of the 3-path is encountered on \( W \). (For example, we can insert \( be_1 \) between consecutive 3-paths in \( S_1 \) containing \( e_1 \), insert \( e_1e_2 \) between consecutive 3-paths in \( S_1 \) containing \( e_1 \) or containing \( e_2 \) and so on.) This creates a new sequence \( S_2 \) as in (4.4). We then proceed as in Case 1 to produce a sequence \( S \) with the desired properties as described in (4.3).

**Case 3.** \( P = ab \) and \( Q = cd \) do not have an edge in common but have two vertices in common. There are two possibilities here, as shown in Figure 4.9(a) and 4.9(b).

![Figure 4.9: The 3-paths P and Q in Case 3](image)

Let \( R = (a, b, d) \) be the 4-path of \( G \) in the graphs shown in both Figures 4.9(a) and 4.9(b). Let \( T \) be a spanning tree of \( G \) containing \( R \) but not the edge \( c \). See Figures 4.9(c) and 4.9(d). There is an embedding of \( T \) in the plane so that the resulting Hamiltonian walk \( W \) of \( T \) contains the 4-path \( R = (a, b, d) \). This, in turn, results in a cyclic sequence \( S_1 \) of those 3-paths in \( T \) occurring in the order they are encountered.
on $W$. Thus, $ab, bd$ are consecutive terms in $S_1$ and the 3-path $cd$ does not occur in $W$ and so not in $S_1$ either. We then insert $cd$ between $ab$ and $bd$ in $S_1$, resulting in a sequence of 3-paths of $G$ that begins with $P = ab$ and ends with $Q = cd$ consisting of all distinct 3-paths of $S_1$, together with $cd$, where consecutive 3-paths have an edge in common. We then proceed as in Case 1 to add all 3-paths in $G$ not in this sequence and produce a sequence $S$ that begins with $P$ and ends with $Q$ consisting of all distinct 3-paths of $G$ such that consecutive 3-paths have an edge in common, as described in (4.3).

Case 4. $P = ab$ and $Q = cd$ do not lie on a common path and have exactly one vertex $v$ in common. Here, $P$ and $Q$ produce one of the two trees of order 5 that is not a path. We consider these two possibilities.

Subcase 4.1. $P = ab$ and $Q = cd$ form the tree of order 5 containing a vertex $v$ of degree 3 in Figure 4.10(a). Let $f$ be an edge incident with $v$ that is distinct from $a, b$ and $c$. The edges $f$ and $d$ may or may not be adjacent.

![Figure 4.10: The 3-paths $P$ and $Q$ in Subcase 4.1](image)

First, suppose that $f$ and $d$ are not adjacent. Let $T$ be a spanning tree of $G$ containing $P, Q$ and $f$, which is embedded in the plane so that $a, f, b, c$ appear consecutively in clockwise order about $v$ as shown in Figure 4.10(b). Then there is a Hamiltonian walk $W$ of $T$ and a cyclic sequence $S_1$ consisting of certain 3-paths of $T$ occurring in the order they are encountered on $W$. Thus, $S_1$ does not contain $ab$ but does contain $bc$ and $cd$ as consecutive terms. We then insert $ab$ between $bc$ and $cd$ in $S_1$, resulting in a sequence of 3-paths of $G$ that begins with $P = ab$ and ends with $Q = cd$ consisting of all distinct 3-paths of $S_1$, together with $ab$, where consecutive 3-paths have an edge in common. We then proceed as in Case 1 to add all 3-paths in $G$ not in this sequence to produce a sequence $S$ that begins with $P$ and ends with $Q$ with the desired properties as described in (4.3).

Next, suppose that $f$ and $d$ are adjacent as shown in Figure 4.10(c). Let $T$ be a spanning tree of $G$ containing $a, f, b$ and $c$ (but not $d$), which is embedded in the plane so
that $a, f, b, c$ appear consecutively in clockwise order about $v$ as shown in Figure 4.10(d). Then there is a Hamiltonian walk $W$ of $T$ and a cyclic sequence $S_1$ consisting of certain 3-paths of $T$ occurring in the order they are encountered on $W$. Thus, $S_1$ contains neither $ab$ nor $cd$ but contains $xb, bc$ as consecutive terms for some edge $x$ (where possibly $x = f$).

We now insert $ab, cd$ between $xb$ and $bc$ in $S_1$, resulting in a sequence of 3-paths of $G$ that begins with $P = ab$ and ends with $Q = cd$ consisting of all distinct 3-paths of $S_1$, together with $ab, cd$, where consecutive 3-paths have an edge in common. We then proceed as in Case 1 to produce a sequence $S$ that begins with $P$ and ends with $Q$ with the desired properties as described in (4.3).

Subcase 4.2. $P$ and $Q$ form the star $K_{1,4}$. Let $T$ be a spanning tree of $G$ containing $P$ and $Q$, which is embedded in the plane so that $a, c, b, d$ appear consecutively in clockwise order about $v$. See Figure 4.11.

![Figure 4.11: The 3-paths $P$ and $Q$ in Subcase 4.2](image)

Then there is a Hamiltonian walk $W$ of $T$ and a cyclic sequence $S_1$ consisting of 3-paths of $T$ occurring in the order they are encountered on $W$. Thus, $S_1$ contains neither $ab$ nor $cd$ but contains $xb, bd$ as consecutive terms for some edge $x$, where possibly $x = c$.

Then we insert $ab, cd$ between $xb$ and $bd$ in $S_1$, resulting in a sequence of 3-paths of $G$ that begins with $P = ab$ and ends with $Q = cd$ consisting of all distinct 3-paths of $S_1$, together with $ab, cd$, where consecutive 3-paths have an edge in common. We then proceed as in Case 1 to produce a sequence $S$ with the desired properties as described in (4.3).

Case 5. $P$ and $Q$ do not lie on a common path and are vertex-disjoint. There are two possibilities, shown in Figures 4.12(a) and 4.12(c).

Subcase 5.1. There is no path in $G$ containing one of $P$ and $Q$ and one edge of the other. Necessarily, there is a path in $G$ containing one edge of each of $P$ and $Q$. Let $u$ be the interior vertex of $P$ and $v$ the interior vertex of $Q$ and let $R$ be a shortest $u - v$ path in $G$, say $R = (e_1, e_2, \ldots, e_k)$ for some positive integer $k$. See Figure 4.12(a). We consider two subcases.

Subcase 5.1.1. There is an edge $f$ incident with $u$ distinct from $a, b, e_1$ and an edge
Figure 4.12: The 3-paths $P$ and $Q$ in Case 5

$g$ incident with $v$ distinct from $c,d,e_k$ such that $f$ and $g$ are not adjacent. See Figure 4.12(b). Since $R$ is a shortest $u-v$ path in $G$ and there is no path in $G$ containing one of $P$ and $Q$ and one edge of the other, $f$ is not adjacent to any of the edges $e_2,e_3,\ldots,e_k,c,d,g$ and $g$ is not adjacent to any of $a,b,f,e_1,e_2,\ldots,e_{k-1}$. Let $T$ be a spanning tree of $G$ containing $P$, $Q$, $f$, $g$ and $R$, which is embedded in the plane as shown in Figure 4.12(b). Then there is a Hamiltonian walk $W$ of $T$ and a cyclic sequence $S_1$ consisting of those 3-paths of $T$ occurring in the order they are encountered on $W$. Thus, $S_1$ contains neither $ab$ nor $cd$. If $k = 1$, then $S_1$ contains $be_1,e_1d$ as consecutive terms; while if $k \geq 2$, then $S_1$ contains

$$be_1,e_1e_2,\ldots,e_{k-1}e_k,e_kd$$

(4.6)

as consecutive terms. If $k = 1$, we insert $ab,cd$ between $be_1$ and $e_1d$, creating a new sequence beginning at $P = ab$ and ending at $Q = cd$. If $k \geq 2$, then we insert $ab,cd$ between $be_1$ and $e_1e_2$ and delete each of the 3-paths $e_1e_2,\ldots,e_{k-1}e_k$ in (4.6) whose interior vertex has degree 2 and move every other such 3-path from the sequence in (4.6) to another appropriate position, creating a new sequence that begins at $P = ab$ and ends at $Q = cd$. We then proceed as in Case 1 to produce a sequence $S$ with the desired properties as described in (4.3).

Subcase 5.1.2. Subcase 5.1.1 does not occur. Hence, if $f$ is an edge incident with $u$ distinct from $a,b,e_1$ and $g$ is an edge incident with $v$ distinct from $c,d,e_k$, then $f$ and $g$ are adjacent. Then $\deg_G u = \deg_G v = 4$ and either $k = 1$ or $k = 2$. See Figures 4.13(a) and 4.13(c). Let $T$ be a spanning tree of $G$ containing $P,Q$, $e_1$ (if $k = 1$) or $e_1,e_2$ (if $k = 2$) and the edge $f$ but not $g$, which is embedded in the plane as shown in
Figures 4.13(b) and 4.13(d).

Figure 4.13: The 3-paths $P$ and $Q$ in Subcase 5.1.2

We only consider the case when $k = 2$ since the argument for the case when $k = 1$ is the same. From this planar embedding of $T$, a Hamiltonian walk $W$ of $T$ is produced as well as a cyclic sequence $S_1$ of 3-paths that contains

$$be_1, e_1 e_2, e_2 d, dx, \ldots, yd, dc, cz$$

(4.7)

as consecutive terms for some edges $x, y$ and $z$ of $G$. Note that $S_1$ contains neither $ab$ nor the three 3-paths $dg, e_2 g, cg$ of $G$ whose interior vertex is $v$ but $S_1$ does contain the 3-path $cg$ on $W$. We insert $ab$ between $be_1$ and $e_1 e_2$ and insert $dg, e_2 g, cg$ between $dc$ and $cz$ so that $dc, dg, e_2 g, cg, cz$ are consecutive terms. We move the terms $e_1 e_2, e_2 d, dx, \ldots, yd$ in (4.7) and insert them between $dg$ and $e_2 g$ such that $dg, yd, \ldots, dx, e_2 d, e_1 e_2, e_2 g$ are consecutive terms and then delete each 3-path in the resulting sequence whose interior vertex has degree 2. This produces a sequence

$$P = ab, be_1, \ldots, gd, dc = Q$$

consisting of all 3-paths of $W$ together with the three 3-paths $dg, e_2 g, cg$ of $G$ whose interior vertex is $v$. We then proceed as in Case 1 (but excluding the vertex $v$) to produce a sequence $S$ with the desired properties as described in (4.3).

Subcase 5.2. There is a path in $G$ containing one of $P$ and $Q$ and one edge of the other. See Figure 4.12(c). Let $R$ be a shortest such path, say $R$ contains $b$ and $Q$, where $R = (b, e_1, e_2, \ldots, e_k, c, d)$ for some positive integer $k$. We consider two subcases.

Subcase 5.2.1. There is an edge $f$ incident with $u$ distinct from $a, b, e_1$ that is not adjacent to $d$. Let $T$ be a spanning tree of $G$ containing $P$, $Q$, $f$ and $R$, which is embedded in the plane as shown in Figure 4.12(d). Then there is a Hamiltonian walk
W of T and a cyclic sequence $S_1$ consisting of 3-paths of T occurring in the order they are encountered on W. Thus, $S_1$ does not contain $ab$ but contains

$$be_1, e_1e_2, \ldots, e_k c, cd$$

(4.8)
as consecutive terms. We insert $ab$ between $be_1$ and $e_1e_2$ in $S_1$, delete each of those 3-paths $e_1e_2, \ldots, e_k c$ in the sequence (4.8) having an interior vertex of degree 2 and move every other such 3-path in (4.8) to another appropriate position in the sequence. This creates a new sequence that begins at $P = ab$ and ends at $Q = cd$. We then proceed as in Case 1 to produce a sequence $S$ with the desired properties as described in (4.3).

Subcase 5.2.2. Every edge $f$ incident with $u$ distinct from $a, b, e_1$ is adjacent to $d$. First, suppose that $f$ is incident with $v$. See Figure 4.14(a). Let $T_1$ be the tree as shown in Figure 4.14(b) and let $T$ be a spanning tree of $G$ containing $T_1$ but not the edge $c$. The tree $T$ is embedded as shown in Figure 4.14(b). Then there is a Hamiltonian walk $W$ of $T$ and a cyclic sequence $S_1$ consisting of 3-paths of $T$ occurring in the order they are encountered on $W$ such that $bf, fd$ are consecutive terms in $S_1$ but $ab$ and $cd$ are not in $S_1$. We now insert $ab, cd$ between $bf$ and $fd$, resulting in a sequence of 3-paths of $G$ that begins with $P = ab$ and ends with $Q = cd$. We then proceed as in Case 1 to produce a sequence $S$ with the desired properties as described in (4.3).

Figure 4.14: The 3-paths $P$ and $Q$ in Subcase 5.2.2

Next, suppose that $f$ is incident with $w$. By the defining property of $R$, it follows that $k = 1$ and $R = e_1$. See Figure 4.14(c). Let $T_2$ be the tree as shown in Figure 4.14(d) and let $T$ be a spanning tree of $G$ containing $T_2$ which is embedded in the plane as shown in Figure 4.14(d). Again, there is a Hamiltonian walk $W$ of $T$ and a cyclic sequence $S_1$ consisting of 3-paths of $T$ such that $bf, fd$ are consecutive terms in $S_1$ but $ab$ and $cd$ are not in $S_1$. We then insert $ab, cd$ between $bf$ and $fd$ and proceed as in Subcase 5.2.1. ■
We now present an example to illustrate the proof of Theorem 4.3.3. Consider the
graph $G$ of Figure 4.15 and the two 3-paths $P = ab$ and $Q = bc$ of $G$ having the interior
vertex $v$. Let $T$ be the spanning tree of $G$ containing $P$ and $Q$ that is embedded in the
plane as shown Figure 4.15 and let $W$ be the Hamiltonian walk $W$ of $T$ obtained from
this planar embedding of $T$.

![Graph G and Tree T]

Figure 4.15: Illustrating the proof of Subcase 1.1

A cyclic sequence $S_1$ of all 3-paths lying on $W$ is

$$S_1 : ab, bd, de, cd, dc, ca, ab.$$ 

Since $\text{deg}_T u = 2$, the 3-path $de$ having interior vertex $u$ appears twice in $S_1$. Inserting
$bc$ between $ab$ and $bd$, we obtain the sequence

$$S_2 : ab, bc, bd, de, cd, dc, ca, ab.$$ 

Other 3-paths of $G$ having interior vertex $u$ that are not in $S_2$ are $dh, df, eh, ef, fh$. Replace $ed$ by $eh, ef, fh, dh, df$, resulting in the cyclic sequence

$$ab, bc, bd, de, eh, ef, fh, dh, df, dc, ca, ab.$$ 

Other 3-paths of $G$ not in $S_2$ are

$$ad, ax, ag, ay, xg, xy, gy, cf, cx, cz, fx, fz, xz,$$
$$ek, ey, ez, kx, kz, yz, bg, bh, bk, gh, gk, hk.$$ 

We insert these 3-paths into an appropriate position in $S_2$ to produce the following cyclic
sequence $S$ of the 3-paths of $G$:

$$S : ab, bc, bg, gh, gk, hk, bh, bk, bd, de, ek, ky, yz, kz, ez, ey,$$
$$eh, ef, fh, dh, df, dc, cf, fx, xz, fz, cz, cx,$$
$$ca, ax, xg, xy, gy, ay, ag, ab.$$ 

59
Thus, $S$ is a Hamiltonian cycle in $P_3(G)$. Since $P = ab$ and $Q = bc$ are two consecutive terms in $S$, there is a $P$-$Q$ Hamiltonian path in $P_3(G)$. 
Chapter 5

Tree-Connected Graphs

5.1 Introduction

Recall that a graph \( G \) is Hamiltonian-connected if for every two vertices \( u \) and \( v \) of \( G \), there is a Hamiltonian \( u - v \) path in \( G \). As described in [6] the concept of Hamiltonian-connected graphs can be looked at in a different way. A connected graph \( G \) is Hamiltonian-connected if for every two vertices \( u \) and \( v \), there exists a spanning tree \( T \) of \( G \) whose only end-vertices are \( u \) and \( v \). This observation gives rise to an extension of Hamiltonian-connected graphs.

Let \( G \) be a connected graph of order \( n \geq 3 \) and let \( k \) be an integer with \( 2 \leq k \leq n - 1 \). The graph \( G \) is \( k \)-tree-connected (or \( k \)-leaf-connected) if for every set \( S \) of \( k \) distinct vertices of \( G \), there exists a spanning tree \( T \) of \( G \) whose set of end-vertices is \( S \). Thus, \( G \) is 2-tree-connected if and only if \( G \) is Hamiltonian-connected. A graph \( G \) is therefore 3-tree-connected if for every set \( S \) of three distinct vertices of \( G \), there exists a spanning tree \( T \) of \( G \) in which \( S \) is the set of end-vertices of \( T \). These concepts were studied in [6, 21]. The following elementary result will be useful.

**Proposition 5.1.1** If \( G \) is \( k \)-tree-connected for some integer \( k \geq 2 \), then \( G \) is \((k + 1)\)-connected.

**Proof.** Let \( G \) be a \( k \)-tree-connected graph. Thus, for every set \( S \) of \( k \) vertices, there exists a spanning tree having \( S \) as its set of end-vertices. Assume, to the contrary, that \( G \) is not \((k + 1)\)-connected. Thus, there exists a set \( S \subseteq V(G) \) with \(|S| = k\) such that \( G - S \) is disconnected. Since \( G \) is a \( k \)-tree-connected graph, there exists a spanning tree \( T \) such that \( S \) is the set of end-vertices of \( T \). This implies that \( G - S \) is still connected since \( T - S \) is a spanning tree of \( G - S \), which is a contradiction. Therefore, \( G \) is \((k + 1)\)-connected.
as desired.

We now review some well-known results on Hamiltonian graphs and Hamiltonian-connected graphs. The first theoretical result on Hamiltonian graphs is due to Dirac [15] and occurred in 1952.

**Theorem 5.1.2** [15] *If G is a graph of order n ≥ 3 with δ(G) ≥ n/2, then G is Hamiltonian.*

In 1960 Ore [29] generalized Theorem 5.1.2.

**Theorem 5.1.3** [29] *If G is a graph of order n ≥ 3 such that deg u + deg v ≥ n for every two nonadjacent vertices u and v of G, then G is Hamiltonian.*

In 1963 Ore [30] found a similar sufficient condition for a graph to be Hamiltonian-connected.

**Theorem 5.1.4** [30] *If G is a graph of order n ≥ 3 such that deg u + deg v ≥ n + 1 for every two nonadjacent vertices u and v of G, then G is Hamiltonian-connected.*

This result then gives the following corollary.

**Corollary 5.1.5** *If G is a graph of order n ≥ 3 with δ(G) ≥ (n + 1)/2, then G is Hamiltonian-connected.*

A result analogous to Theorems 5.1.3 and 5.1.4 has been established (see [6, 21]).

**Theorem 5.1.6** *If G is a graph of order n and k is an integer with 2 ≤ k ≤ n − 1 such that deg u + deg v ≥ n + k − 1 for every two nonadjacent vertices u and v of G, then G is k-tree-connected.*

Theorem 5.1.6 then gives the following corollary.

**Corollary 5.1.7** *If G is a connected graph of order n ≥ 5 such that δ(G) ≥ \( \frac{n+k-1}{2} \) for an integer k with 2 ≤ k ≤ n − 1, then G is k-tree-connected.*

If a graph G of order n satisfies the hypothesis in Corollary 5.1.7 for k = 3, that is, if \( δ(G) ≥ \frac{n+2}{2} \), then G is 3-tree-connected. For each vertex v of G, it follows that \( δ(G − v) ≥ \frac{n}{2} = \frac{(n+1)-1}{2} \) and so G − v is Hamiltonian-connected. It was observed in
[6] that it is not true in general, however, that if \( G \) is 3-tree-connected, then \( G - v \) is Hamiltonian-connected for every vertex \( v \) of \( G \). For example, consider the graph \( G = K_{3,3} \lor K_1 \). Figure 5.1 shows, for all possible choices for a set \( S \) of three vertices of \( G \) that there is a spanning tree of \( G \) whose set of end-vertices of \( G \) is \( S \). On the other hand, if one were to remove the vertex of degree 6 from \( G \), then the resulting graph is \( K_{3,3} \) and no bipartite graph of order 3 or more is Hamiltonian-connected (see[6]).

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\[K_{3,3} \lor K_1\]

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only vertex of degree 3 in $T$ and $V(T) = V(P) \cup V(Q)$. If $z' \in U$, then $Q$ has odd order $2\ell + 1$ for some positive integer $\ell$, where $\ell$ vertices of $Q$ belong to $U$ but do not lie on $P$ and $\ell$ vertices of $Q$ belong to $W$. Thus, $|U| = k + \ell + 1$ and $|W| = k + \ell$, a contradiction. Hence, $z' \in W$. However then, $Q$ has even order $2p$, where $p$ vertices of $Q$ belong to $U$ and $p - 1$ vertices of $Q$ belong to $W$. Thus, $|U| = k + p + 1$ and $|W| = k + p - 1$, also a contradiction.

By Proposition 5.1.8, for each integer $r \geq 4$, the graphs $K_{r,r}$ or $K_{r,r} \lor K_2$ are $r$-connected but not 3-tree-connected.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figures/figure52.png}
\caption{The graph $K_{r,r} \lor K_2$}
\end{figure}

### 5.2 The Graphs $C_5 \square C_5$ and $P \square K_2$

The Petersen graph $P$ is a 3-regular 3-connected graph but it is not Hamiltonian-connected. In fact, it is not even Hamiltonian. The graphs $P \square K_2$ and $C_5 \square C_5$ are 4-regular. It can be shown that each of $P \square K_2$ and $C_5 \square C_5$ is

(i) Hamiltonian,

(ii) Hamiltonian-connected and

(iii) 3-tree-connected.

We begin with the graph $C_5 \square C_5$.

**Proposition 5.2.1** The graph $C_5 \square C_5$ is Hamiltonian-connected and 3-tree-connected.

**Proof.** Let $G = C_5 \square C_5$ and label the vertices of $G$ as shown in Figure 5.3.
First, we show that every two vertices $x$ and $y$ of $G$ are connected by a Hamiltonian $x - y$ path; that is, $G$ is Hamiltonian-connected. By the symmetry of the graph $G$, we may assume that $\{x, y\}$ is one of the following six pairs of vertices of $G$:

$$A_1 = \{x_1, v_1\}, A_2 = \{x_1, x_2\}, A_3 = \{x_1, w_3\},$$
$$A_4 = \{x_1, v_3\}, A_5 = \{x_1, v_2\}, A_6 = \{x_1, w_2\}.$$

1. For the two vertices $x_1$ and $v_1$, there is a Hamiltonian $x_1 - v_1$ path $Q_1$
$$\{x_1, w_1, w_2, x_2, y_2, y_1, u_1, u_2, v_2, v_3, v_4, w_4, w_3, x_3, y_3, u_3, u_4, u_5, y_5, y_4, x_4, x_5, w_5, v_5, v_1\}.$$

2. For the two vertices $x_1$ and $x_2$, there is a Hamiltonian $x_1 - x_2$ path $Q_2$
$$\{x_1, y_1, u_1, v_1, w_1, w_2, v_2, u_2, y_2, y_3, u_3, v_3, w_3, v_4, u_4, y_4, y_5, u_5, v_5, w_5, x_5, x_4, x_3, x_2\}.$$

3. For the two vertices $x_1$ and $w_3$, there is a Hamiltonian $x_1 - w_3$ path $Q_3$
$$\{x_1, w_1, v_1, u_1, y_1, y_2, x_2, w_2, v_2, u_2, v_3, v_4, u_4, y_4, y_3, x_3, x_4, x_5, y_5, u_5, v_5, w_5, w_4, w_3\}.$$

4. For the two vertices $x_1$ and $v_3$, there is a Hamiltonian $x_1 - v_3$ path $Q_4$
$$\{x_1, y_1, u_1, v_1, w_1, w_2, v_2, u_2, y_2, x_2, x_3, w_3, w_4, x_4, y_4, y_3, u_3, u_4, u_5, y_5, x_5, w_5, v_4, v_3\}.$$

5. For the two vertices $x_1$ and $v_2$, there is a Hamiltonian $x_1 - v_2$ path $Q_5$
$$\{x_1, y_1, u_1, v_1, w_1, x_2, y_2, u_2, v_3, y_3, x_3, w_3, w_4, x_4, y_4, u_4, u_5, y_5, x_5, w_5, v_5, v_3, v_2\}.$$

Figure 5.3: The graph $C_5 \boxplus C_5$
For the two vertices $x_1$ and $w_2$, there is a Hamiltonian $x_1 - w_2$ path $Q_6$

$$\{x_1, w_1, v_1, u_1, y_1, y_2, x_2, x_3, y_3, u_3, u_2, v_2, v_3, v_4, y_4, x_4, x_5, y_5, u_5, v_5, w_5, w_4, w_3, w_2\}.$$

Hence, $C_5 \boxtimes C_5$ is Hamiltonian-connected. With the aid of the six Hamiltonian paths $Q_1, Q_2, \ldots, Q_6$ of $G$, we now show for any 3-element set $S$ of vertices of $G$, that there exists a spanning tree $T$ such that $S$ is its set of end-vertices. More precisely, we show that for a 3-element set $S = \{x, y, z\}$, there is a Hamiltonian $x - y$ path $P$ and two edges $e$ and $f$ of $G$, where $e \notin E(P)$ and $f \in E(P)$ is incident with $z$, such that $S$ is the set of end-vertices of the tree $T = (P - f) + e$. By the symmetry of the graph $G$, we may assume that $S$ is one of the six 3-element sets $S_i = A_i \cup \{z\}$ for some $z \in V(G) - A_i$ where $1 \leq i \leq 6$. We consider these six cases.

**Case 1.** $S_1 = \{x_1, v_1, z\}$ for some $z \in V(G) - \{x_1, v_1\}$. Denote the Hamiltonian $x_1 - v_1$ path $Q_1$ by

$$Q_1 = (x_1, w_1, x_2, x_3, y_3, u_3, u_2, v_2, v_3, v_4, y_4, x_4, x_5, y_5, u_5, v_5, w_5, w_4, w_3, w_2) = (z_1 = x_1, z_2, \ldots, z_{25} = v_1)$$

and let $z = z_j$ where $2 \leq j \leq 24$. First, let $j = 2$ or $j = 24$.

* For $z = z_2 = w_1$, we consider the $x_1 - w_1$ Hamiltonian path

$$Q'_1 = (x_1, x_2, y_2, y_1, u_1, u_2, u_3, u_4, y_4, y_3, x_3, x_4, x_5, w_5, v_5, v_1, v_2, v_3, v_4, w_4, w_3, w_2, w_1) = (z'_1 = x_1, z_2, \ldots, z'_{25} = w_1).$$

Then the set of end-vertices of the tree $T = (Q'_1 - v_5v_5) + v_4v_5$ is $\{x_1, w_1, v_1\}$.

* For $z = z_{24} = v_5$, we consider the $v_1 - v_5$ Hamiltonian path

$$Q''_1 = (v_1, w_1, y_1, u_1, u_2, y_2, x_2, x_3, w_2, v_2, v_3, w_3, x_3, y_3, u_3, u_4, y_4, x_4, x_5, w_5, v_4, v_5) = (z''_1 = v_1, z_2, \ldots, z''_{25} = v_5).$$

Then the set of end-vertices of the tree $T = (Q''_1 - w_1x_1) + w_1w_2$ is $\{x_1, v_1, v_5\}$.  

66
Next, let $3 \leq j \leq 23$ and $j \neq 10$. We show that either (i) there is an edge $e = z_{j-1}z_t$ for some integer $t$ with $j + 2 \leq t \leq 24$ such that $S_1$ is the set of end-vertices of $T = (Q_1 - z_{j-1}z_j) + e$ or (ii) there is an edge $e = z_tz_{j+1}$ for some integer $t$ with $2 \leq t \leq j - 1$ such that $S_1$ is the set of end-vertices of $T = (Q_1 - z_jz_{j+1}) + e$. See Figure 5.6. By (i) and (ii) then, there are two edges $e$ and $f$ of $G$, where $e \notin E(Q_1)$ and $f \in E(Q_1)$ is incident with $z_j$, such that $S_1$ is the set of end-vertices of the tree $T = (Q_1 - f) + e$.

![Figure 5.4: Illustrating a step of the proof](image)

We list the edges $e$ and $f$ for $z = z_j$, where $3 \leq j \leq 23$ and $j \neq 10$, as follows:

\[
\begin{align*}
z = z_3 &= w_2: \ f = w_1w_2; \ e = w_1w_3, \quad z = z_4 &= x_2: \ f = w_2x_2; \ e = w_2v_2, \\
z = z_5 &= y_2: \ f = x_2y_2; \ e = x_2x_3, \quad z = z_6 &= y_1: \ f = y_1y_2; \ e = y_2u_2, \\
z = z_7 &= u_1: \ f = u_1u_2; \ e = u_2y_2, \quad z = z_8 &= u_2: \ f = u_2v_2; \ e = w_2v_2, \\
z = z_9 &= v_2: \ f = u_2v_2; \ e = u_2u_3, \quad z = z_{11} &= v_4: \ f = v_3v_4; \ e = v_3w_3, \\
z = z_{12} &= w_4: \ f = v_4w_4; \ e = v_4u_4, \quad z = z_{13} &= w_3: \ f = w_3w_4; \ e = w_4x_4, \\
z = z_{14} &= x_3: \ f = x_3v_3; \ e = y_2y_3, \quad z = z_{15} &= y_3: \ f = x_3y_3; \ e = x_3x_4, \\
z = z_{16} &= u_3: \ f = y_3u_3; \ e = y_3y_4, \quad z = z_{17} &= u_4: \ f = u_4u_5; \ e = u_1u_5, \\
z = z_{18} &= u_5: \ f = u_4u_5; \ e = u_4y_4, \quad z = z_{19} &= y_5: \ f = u_5y_5; \ e = u_5v_5, \\
z = z_{20} &= y_4: \ f = y_4y_5; \ e = x_5y_5, \quad z = z_{21} &= x_4: \ f = x_4x_5; \ e = x_5x_5, \\
z = z_{22} &= x_5: \ f = x_5w_5; \ e = w_5w_4, \quad z = z_{23} &= w_5: \ f = w_5v_5; \ e = v_5v_5.
\end{align*}
\]

In each case, the set of end-vertices of $T = (Q_1 - f) + e$ is $\{x_1, v_1, z\}$. Thus, it remains to consider the situation when $z = z_{10} = v_3$. Consider the Hamiltonian $x_1 - v_3$ path

\[
Q_4 = (x_1, y_1, u_1, v_1, w_1, w_2, v_2, y_2, x_2, x_3, w_3, w_4, x_4, y_4, y_3, u_3, u_4, u_5, y_5, x_5, w_5, v_5, v_3) = (z'_1 = x_1, z'_2, \ldots, z'_5 = v_3).
\]

Then $v_1 = z'_4$ in $Q_4$. Let $e = u_1u_2$ and let $T = (Q_4 - u_1v_1) + u_1u_2$. Hence, $S_1$ is the set of end-vertices of $T$.

Case 2. $S_2 = \{x_1, x_2, z\}$ for some $z \in V(G) - \{x_1, x_2\}$. By the same argument used in Case 1, we see that for each $z \in V(G) - \{x_1, x_2\}$, there are edges $f \in E(Q_2)$ incident with $z$ and $e \notin E(Q_2)$ such that $S_2$ is the set of end-vertices of the tree $T = (Q_2 - f) + e$.  

67
Case 3. \( S_3 = \{x_1, w_3, z\} \) for some \( z \in V(G) - \{x_1, w_3\} \). If \( z \neq x_4 \), then we proceed as in Case 1. If \( z = x_4 \), then \( S_3 = \{x_1, x_4, w_3\} \). By the symmetry of the graph \( G \), this can be verified by the argument used in Case 1; that is, we replace \( \{x, y\} \) in the 5-cycle \( (x_1, w_1, v_1, u_1, y_1, x_1) \) in Case 1 by \( \{x, x_4\} \) in the 5-cycle \( (x_1, x_2, x_4, x_5, x_1) \) in this case. More precisely, consider the Hamiltonian \( x_1 - x_4 \) path

\[
P = (x_1, x_2, x_3, y_3, u_3, u_4, y_2, y_1, y_5, y_4, u_2, u_1, u_5, v_1, w_1,
\]

\[
= w_2, v_2, v_3, w_3, w_4, v_4, v_5, w_5, x_5, x_4).
\]

Let \( e = v_3w_4 \) and \( f = w_3v_3 \), where \( e \notin E(P) \) and \( f \in E(P) \). Then the set of end-vertices of the tree \( T = (P - f) + e \) is \( \{x_1, x_4, w_3\} \).

Case 4. \( S_4 = \{x_1, v_3, z\} \) for some \( z \in V(G) - \{x_1, v_3\} \).

The argument is the same as in Case 2.

Case 5. \( S_5 = \{x_1, v_2, z\} \) for some \( z \in V(G) - \{x_1, v_2\} \).

The argument is the same as in Case 2.

Case 6. \( S_6 = \{x_1, w_2, z\} \) for some \( z \in V(G) - \{x_1, w_2\} \). If \( z \neq v_3 \), then we proceed as in Case 1. If \( z = v_3 \), then we consider a Hamiltonian \( x_1 - v_3 \) path \( Q_4 \) in \( G \). By the symmetry of the graph \( G \), this situation can be verified by the argument used in Case 4. For example, for the Hamiltonian path \( Q_4 \), let \( f = w_1w_2 \) and \( e = w_1w_5 \). Thus, the set of end-vertices of \( T = (Q_4 - f) + e \) is \( \{x_1, w_2, v_3\} \).

Therefore, we conclude that \( C_5 \sqcup C_5 \) is 3-tree-connected. \( \blacksquare \)

We now consider the Cartesian product \( P \sqcup K_2 \) of the Petersen graph \( P \) and \( K_2 \).

**Proposition 5.2.2** The graph \( P \sqcup K_2 \) is Hamiltonian-connected and 3-tree-connected.

**Proof.** Let \( G = P \sqcup K_2 \) and label the vertices of \( G \) as shown in Figure 5.5.

First, we show that every two vertices \( x \) and \( y \) of \( G \) are connected by a Hamiltonian \( x - y \) path; that is, \( G \) is Hamiltonian-connected. By the symmetry of the graph \( G \), we may assume that \( \{x, y\} \) is one of the following eleven pairs of vertices of \( G \):

\[
A_1 = \{u_0, u_1\}, \quad A_2 = \{u_0, u_2\}, \quad A_3 = \{u_0, u_5\},
\]

\[
A_4 = \{u_0, u_7\}, \quad A_5 = \{u_0, u_6\}, \quad A_6 = \{u_0, v_0\},
\]

\[
A_7 = \{u_0, v_1\}, \quad A_8 = \{u_0, v_2\}, \quad A_9 = \{u_0, v_5\},
\]

\[
A_{10} = \{u_0, v_6\}, \quad A_{11} = \{u_0, v_7\}.
\]

(1) For the two vertices \( u_0 \) and \( u_1 \), there is a Hamiltonian \( u_0 - u_1 \) path \( Q_1 \)

\[
(u_0, u_5, u_7, u_9, u_6, u_8, u_3, u_4, v_4, v_3, v_8, v_6, v_9, v_7, v_5, v_0, v_1, v_2, u_2, u_1).
\]

68
(2) For the two vertices $u_0$ and $u_2$, there is a Hamiltonian $u_0 - u_2$ path $Q_2$

$$ (u_0, u_5, u_7, u_9, u_6, u_1, v_1, v_6, v_9, v_7, v_5, v_0, u_4, u_3, u_8, v_8, v_3, v_2, u_2). $$

(3) For the two vertices $u_0$ and $u_5$, there is a Hamiltonian $u_0 - u_5$ path $Q_3$

$$ (u_0, u_4, u_3, u_8, u_6, u_9, u_7, u_2, u_1, v_2, v_7, v_9, v_6, v_8, v_3, v_4, v_0, v_5, u_5). $$

(4) For the two vertices $u_0$ and $u_7$, there is a Hamiltonian $u_0 - u_7$ path $Q_4$

$$ (u_0, u_4, u_3, u_2, u_1, u_6, u_8, u_5, v_5, v_8, v_6, v_1, v_0, v_4, v_3, v_2, v_7, v_9, u_9, u_7). $$

(5) For the two vertices $u_0$ and $u_6$, there is a Hamiltonian $u_0 - u_6$ path $Q_5$

$$ (u_0, u_4, u_3, u_2, u_1, v_1, v_6, v_8, v_5, v_0, v_4, v_3, v_2, v_7, v_9, u_9, u_7, u_5, u_8, u_6). $$

(6) For the two vertices $u_0$ and $v_0$, there is a Hamiltonian $u_0 - v_0$ path $Q_6$

$$ (u_0, u_4, u_3, u_2, u_1, u_6, u_8, u_5, u_7, u_9, v_9, v_7, v_5, v_8, v_6v_1, v_2v_3, v_4, v_0). $$

(7) For the two vertices $u_0$ and $v_1$, there is a Hamiltonian $u_0 - v_1$ path $Q_7$

$$ (u_0, u_4, u_3, u_2, u_1, u_6, u_8, u_5, u_7, u_9, v_9, v_6, v_5, v_7, v_2, v_3, v_4, v_0, v_1). $$

(8) For the two vertices $u_0$ and $v_2$, there is a Hamiltonian $u_0 - v_2$ path $Q_8$

$$ (u_0, u_1, u_2, u_3, u_4, u_9, u_7, u_5, u_8, v_6, v_9, v_7, v_5, v_8, v_3, v_4, v_0, v_1). $$
(9) For the two vertices \( u_0 \) and \( v_5 \), there is a Hamiltonian \( u_0 - v_5 \) path \( Q_9 \)

\[(u_0, u_4, u_3, u_8, u_5, u_7, u_9, u_6, u_1, u_2, v_1, v_0, v_4, v_3, v_8, v_6, v_9, v_7, v_5).\]

(10) For the two vertices \( u_0 \) and \( v_6 \), there is a Hamiltonian \( u_0 - v_6 \) path \( Q_{10} \)

\[(u_0, u_4, u_3, u_8, u_5, u_7, u_9, u_6, u_1, v_2, v_1, v_0, v_4, v_3, v_8, v_5, v_7, v_9, v_6).\]

(11) For the two vertices \( u_0 \) and \( v_7 \), there is a Hamiltonian \( u_0 - v_7 \) path \( Q_{11} \)

\[(u_0, u_4, u_9, u_7, u_5, u_8, u_6, u_1, u_2, v_3, v_2, v_1, v_0, v_4, v_9, v_6, v_8, v_5, v_7).\]

Hence, \( P \Box K_2 \) is Hamiltonian-connected. With the aid of the eleven Hamiltonian paths \( Q_1, Q_2, \ldots, Q_{11} \) of \( G \), we now show for any 3-element set \( S \) of vertices of \( G \), that there exists a spanning tree \( T \) such that \( S \) is its set of end-vertices. More precisely, we show that for a 3-element set \( S = \{x, y, z\} \), there is a Hamiltonian \( x - y \) path \( P \) and two edges \( e \) and \( f \) of \( G \), where \( e \notin E(P) \) and \( f \in E(P) \) is incident with \( z \), such that \( S \) is the set of end-vertices of the tree \( T = (P - f) + e \). By the symmetry of the graph \( G \), we may assume that \( S \) is one of the eleven 3-element sets \( S_i = A_i \cup \{z\} \) for some \( z \in V(G) - A_i \) where \( 1 \leq i \leq 11 \). We consider these eleven cases.

**Case 1.** \( S_1 = \{u_0, u_1, z\} \) for some \( z \in V(G) - \{u_0, u_1\} \). Denote the Hamiltonian \( u_0 - u_1 \) path \( Q_1 \) by

\[Q_1 = (u_0, u_5, u_7, u_9, u_6, u_8, u_3, u_4, v_3, v_8, v_6, v_9, v_7, v_5, v_0, v_1, v_2, u_2, u_1)\]

\[= (z_1 = u_0, z_2, \ldots, z_{20} = u_1),\]

and let \( z = z_j \) where \( 2 \leq j \leq 19 \). First, let \( j = 2 \) or \( j = 19 \).

* For \( z = z_2 = u_5 \), we consider the \( u_0 - u_5 \) Hamiltonian path \( Q_3 \).

Then the set of end-vertices of the tree \( T = (Q_3 - u_1 u_2) + u_2 v_2 \) is \( \{u_0, u_1, u_5\} \).

* For \( z = z_{19} = u_2 \), we consider the \( u_0 - u_2 \) Hamiltonian path \( Q_2 \).

Then the set of end-vertices of the tree \( T = (Q_2 - u_6 u_1) + u_6 v_6 \) is \( \{u_0, u_1, u_2\} \).

Next, let \( 3 \leq j \leq 18 \). We show that either (i) there is an edge \( e = z_{j-1} z_t \) for some integer \( t \) with \( j + 2 \leq t \leq 19 \) such that \( S_1 \) is the set of end-vertices of \( T = (Q_1 - z_{j-1} z_j) + e \) or (ii) there is an edge \( e = z_t z_{j+1} \) for some integer \( t \) with \( 2 \leq t \leq j - 1 \) such that \( S_1 \) is the set of end-vertices of \( T = (Q_1 - z_j z_{j+1}) + e \). This is shown in Figure 5.6. By (i)
and (ii) then, there are two edges $e$ and $f$ of $G$, where $e \notin E(Q_1)$ and $f \in E(Q_1)$ is incident with $z_j$, such that $S_1$ is the set of end-vertices of the tree $T = (Q_1 - f) + e$.

We list the edges $f$ and $e$ for $z = z_j$, where $3 \leq j \leq 18$, as follows:

$$z = z_3 = u_7: f = u_5u_7; e = u_5u_8,$$
$$z = z_5 = u_6: f = u_6u_7; e = u_6u_8,$$
$$z = z_7 = u_3: f = u_3u_7; e = u_3u_8,$$
$$z = z_9 = v_4: f = v_3v_4; e = v_3u_3,$$
$$z = z_{11} = v_8: f = v_3v_8; e = v_3v_2,$$
$$z = z_{13} = v_9: f = v_6v_9; e = v_6v_1,$$
$$z = z_{15} = v_5: f = v_7v_5; e = v_7v_2,$$
$$z = z_{17} = v_1: f = v_1v_2; e = v_2v_7,$$
$$z = z_{20} = u_2.$$

Case 2. $S_2 = \{u_0, u_2, z\}$ for some $z \in V(G) - \{u_0, u_2\}$. Denote the Hamiltonian $u_0 - u_2$ path $Q_2$ by

$$Q_2 = (u_0, u_5, u_7, u_9, u_6, u_1, v_1, v_6, v_9, v_7, v_5, v_0, v_4, u_4, u_3, u_8, v_3, v_2, u_2)$$

By the same argument used in Case 1, we see that for each $z \in V(G) - \{u_0, u_2\}$, there are edges $f \in E(Q_2)$ incident with $z$ and $e \notin E(Q_2)$ such that $S_2$ is the set of end-vertices of the tree $T = (Q_2 - f) + e$. More precisely, we have the following.

* For $z = z_2 = u_5$, as in Case 1, we consider the $u_0 - u_5$ Hamiltonian path $Q_3$.

Then the set of end-vertices of the tree $T = (Q_3 - u_1u_2) + u_2v_2$ is $\{u_0, u_1, u_5\}$.

* For $z = z_{19} = v_2$, we consider the $u_0 - v_2$ Hamiltonian path $Q_8$.

Then the set of end-vertices of the tree $T = (Q_8 - u_1u_2) + u_1u_6$ is $\{u_0, u_2, v_2\}$.

We list the edges $f$ and $e$ for $z = z_j$, where $3 \leq j \leq 18$, as follows:
Again, by the same argument used in Case 1, we see that for each end-vertices of the tree, there are edges incident with $z$ such that $z \in V(G) - \{u_0, u_5\}$. More precisely, we have the following.

* For $z = z_2 = u_4$, as in Case 1, we consider a $u_0 - u_4$ Hamiltonian path $P$ (which is equivalent to considering a $u_0 - u_1$ Hamiltonian path). More precisely, consider the Hamiltonian $u_0 - u_4$ path

$$P = (u_0, u_1, u_6, u_9, u_7, u_5, u_8, u_3, u_2, v_2, v_3, v_8, v_6, v_1, v_0, v_5, v_7, v_9, v_4, u_4).$$

Then the set of end-vertices of the tree $T = (P - u_7u_5) + u_7u_2$ is $\{u_0, u_5, u_4\}$.

* For $z = z_{19} = v_5$, we consider the $u_0 - v_5$ Hamiltonian path $Q_9$.

Then the set of end-vertices of the tree $T = (Q_9 - u_8u_5) + u_8u_6$ is $\{u_0, u_5, v_2\}$.

We list the edges $f$ and $e$ for $z = z_j$, where $3 \leq j \leq 18$, as follows:

$$z = z_3 = u_3: \ f = u_4u_3; \ e = u_4u_9, \quad z = z_4 = u_8: \ f = u_3u_8; \ e = u_3u_2,$$

$$z = z_5 = u_6: \ f = u_8u_6; \ e = u_8v_8, \quad z = z_6 = u_9: \ f = u_6u_9; \ e = u_6u_1,$$

$$z = z_7 = u_7: \ f = u_9u_7; \ e = u_9v_9, \quad z = z_8 = u_2: \ f = u_7u_2; \ e = u_7v_7,$$

$$z = z_9 = u_1: \ f = u_2u_1; \ e = u_2v_2, \quad z = z_{10} = v_1: \ f = v_1v_2; \ e = v_1v_6,$$

$$z = z_{11} = v_2: \ f = v_1v_2; \ e = v_1v_6, \quad z = z_{12} = v_7: \ f = v_2v_7; \ e = v_2v_3,$$

$$z = z_{13} = v_9: \ f = v_7v_9; \ e = v_7v_5, \quad z = z_{14} = v_6: \ f = v_9v_6; \ e = v_9v_4,$$

$$z = z_{15} = v_8: \ f = v_9v_8; \ e = v_9v_3, \quad z = z_{16} = v_3: \ f = v_8v_3; \ e = v_8v_5,$$

$$z = z_{17} = v_4: \ f = v_3v_4; \ e = v_3v_4, \quad z = z_{18} = v_0: \ f = v_0v_5; \ e = v_5v_8.$$
The argument is the same as in Case 1.

**Case 5.** $S_5 = \{u_0, u_6, z\}$ for some $z \in V(G) - \{u_0, u_6\}$. If $z \neq v_1$, then we proceed as in Case 1. If $z = v_1$, then we consider a Hamiltonian $u_0 - v_1$ path $Q_7$ in $G$. Let $f = u_6u_8$ and $e = u_8u_3$. Thus, the set of end-vertices of $T = (Q_7 - f) + e$ is $\{u_0, u_6, v_1\}$.

**Case 6.** $S_6 = \{u_0, v_0, z\}$ for some $z \in V(G) - \{u_0, v_0\}$.

The argument is the same as in Case 1.

**Case 7.** $S_7 = \{u_0, v_1, z\}$ for some $z \in V(G) - \{u_0, v_1\}$.

The argument is the same as in Case 1.

**Case 8.** $S_8 = \{u_0, v_2, z\}$ for some $z \in V(G) - \{u_0, v_2\}$.

The argument is the same as in Case 1.

**Case 9.** $S_9 = \{u_0, v_5, z\}$ for some $z \in V(G) - \{u_0, v_5\}$.

The argument is the same as in Case 1.

**Case 10.** $S_{10} = \{u_0, v_6, z\}$ for some $z \in V(G) - \{u_0, v_1\}$.

The argument is the same as in Case 1.

**Case 11.** $S_{11} = \{u_0, v_7, z\}$ for some $z \in V(G) - \{u_0, v_7\}$. If $z \neq v_1$, then we proceed as in Case 1. If $z = v_1$, then we consider a Hamiltonian $u_0 - v_1$ path $Q_7$ in $G$. Let $f = v_5v_7$ and $e = v_5v_0$. Thus, the set of end-vertices of $T = (Q_7 - f) + e$ is $\{u_0, v_7, v_1\}$.

Therefore, we conclude that $P \square K_2$ is 3-tree-connected.

**5.3 On $k$-Tree-Connected 3-Path Graphs**

In this section, we provide a sufficient condition for the 3-path graph of a tree to be 4-tree-connected. First, we present the following two known results that appeared in [6].

**Theorem 5.3.1** [6] If $T$ is a tree of order at least 5 containing no vertices of degree 2 or 3, then $P_3(T)$ is Hamiltonian-connected and, equivalently, 2-tree-connected.

**Theorem 5.3.2** [6] If $T$ is a tree of order at least 6 containing no vertices of degree 2, 3 or 4, then $P_3(T)$ is 3-tree-connected.

Applying the techniques used in the proofs of Theorems 5.3.1 and 5.3.2, we now present an extension of Theorem 5.3.2.

**Theorem 5.3.3** If $T$ is a tree of order at least 6 containing no vertices of degree 2, 3, 4, or 5, then $P_3(T)$ is 4-tree-connected.
Proof. Let \( P, Q, R_1 \) and \( R_2 \) be four 3-paths of \( T \). We show that \( P_3(T) - \{ R_1, R_2 \} \) contains a Hamiltonian \( P-Q \) path. It suffices to show that there exists an ordering

\[
P = A_1, A_2, \ldots, A_p = Q
\]

of those 3-paths \( A_i \) \((1 \leq i \leq p)\) of \( T \) that do not include \( R_1 \) and \( R_2 \), beginning with \( P \) and ending with \( Q \) such that \( A_i \) and \( A_{i+1} \) have an edge in common for \( i = 1, 2, \ldots, p-1 \).

Indeed, since the interior vertices of \( R_1 \) and \( R_2 \) must have degree at least 6, there exist at least five of these 3-paths distinct from \( P, Q \) and \( R_2 \) that contains an edge of \( R_1 \) and at least five of these 3-paths distinct from \( P, Q \) and \( R_1 \) that contains an edge of \( R_2 \). Thus, each of \( R_1 \) and \( R_2 \) shares an edge with \( A_k \) for some \( k \not\in \{1, p\} \). By taking the Hamiltonian \( P-Q \) path together with \( R_1 \) and \( R_2 \) and the edges \( A_i \) and \( A_j \) in \( P_3(T) \) for \( i, j \not\in \{1, p\} \) (we may have \( A_i = A_j \)) joining \( R_1 \) and \( R_2 \), respectively, a spanning tree of \( P_3(T) \) is formed whose set of end-vertices is \( \{P, Q, R_1, R_2\} \).

We consider the following four cases, depending on the location of \( P \) and \( Q \) in \( T \):

1. \( P \) and \( Q \) have an edge in common,
2. \( P \) and \( Q \) do not have an edge in common and there exists a path in \( T \) containing both \( P \) and \( Q \),
3. \( P \) and \( Q \) do not have an edge in common and there exists a path in \( T \) containing one edge of each of \( P \) and \( Q \) but there is no path in \( T \) containing one of these paths and one edge of the other,
4. \( P \) and \( Q \) do not have an edge in common and there is no path in \( T \) containing both \( P \) and \( Q \) but there is a path containing one of \( P \) and \( Q \) and one edge of the other.

Case 1. \( P \) and \( Q \) have an edge in common, say \( P = e_1e_2 \) and \( Q = e_2e_3 \). Thus, either \( P \) and \( Q \) have the same interior vertex or \( P \) and \( Q \) have distinct adjacent interior vertices (see Figure 5.7). We consider these two possibilities.

Subcase 1.1. \( P \) and \( Q \) have the same interior vertex \( v \). See Figure 5.7(a). So, \( v \) is incident with all three edges \( e_1, e_2, e_3 \).

First, suppose that \( v \) is also the interior vertex of \( R_1 \) and \( R_2 \), say \( R_1 = f_1g_1 \) and \( R_2 = f_2g_2 \), where \( f_1, g_1, f_2, g_2 \) are four edges incident with \( v \). Then we consider two situations. In each situation, we will provide an ordering of the edges incident with \( v \)
such that $e_1$ and $e_2$ appear consecutively, and the pairs $e_2, e_3$ and $f_1, g_1, f_2, g_2$ do not appear consecutively.

First, suppose that $R_1$ and $R_2$ have no edge in common. So, we have the following situations.

(i) One of $f_i$ and $g_i$ ($i = 1, 2$) is $e_1$ or $e_2$, say $f_1 = e_1$.

(ii) Neither $f_i$ nor $g_i$ ($i = 1, 2$) is $e_1$ or $e_2$.

* In situation (i), $f_1 = e_1$, $g_1 \neq e_1$ and $g_2 \notin \{e_1, e_2\}$. Let $d \geq 0$ be the number of edges not in $P, Q, R_1, R_2$. For $d \geq 1$, let $e_4, e_5, \ldots, e_{d+3}$ be the distinct edges incident with $v$ that are none of $e_1, e_2, e_3, f_1, g_1, f_2, g_2$.

If $g_1 = e_3$, then $d \geq 2$ and we produce the sequence

$$e_1, e_2, e_{d+3}, \ldots, e_6, g_1 = e_3, f_2, e_5, g_2, e_4.$$  

If $g_1 \neq e_3$, then $d \geq 1$ and we produce the sequence

$$e_1, e_2, e_{d+3}, \ldots, e_5, g_1, g_2, e_4, e_3.$$  

* In situation (ii), let $d \geq 0$ be the number of edges not in $P, Q, R_1, R_2$. For $d \geq 1$, let $e_4, e_5, \ldots, e_{d+3}$ be the distinct edges incident with $v$ that are not $e_1, e_2, e_3, f_1, f_2, g_1, g_2$. If $f_i = e_3$ or $g_i = e_3$ ($i = 1, 2$), then we may assume that $f_1 = e_3$ (since $R_1$ and $R_2$ have no edge in common). So $d \geq 1$. We can then order the edges as follows:

$$e_1, e_2, e_{d+3}, \ldots, e_6, g_1, g_2, e_5, e_4, f_1 = e_3, f_2.$$
If none of $f_i$ and $g_i$ $(i = 1, 2)$ is $e_3$, then $d \geq 0$. Note here that if $d = 0$, then \[ \{e_4, e_5, \ldots, e_{d+3} \} \] is the empty set. In this case, we produce the sequence 

\[ e_1, e_2, e_{d+3}, \ldots, e_5, f_1, e_4, f_2, e_3, g_1, g_2. \]

Next, suppose that $R_1$ and $R_2$ have an edge in common. So, $R_1 = f_1 g_1$ and $R_2 = f_1 g_2$. Then we have the following situations.

(i') One of $f_i$ and $g_i$ $(i = 1, 2)$ is $e_1$ or $e_2$, say $f_1 = e_1$.

(ii') Neither $f_i$ nor $g_i$ $(i = 1, 2)$ is $e_1$ or $e_2$.

* In situation (i'), we may assume that $g_i \notin \{e_1, e_2\}$. Let $d$ be the number of edges not in $P, Q, R_1, R_2$, where then $d \geq 1$. Let $e_4, e_5, \ldots, e_{d+3}$ be the distinct edges incident with $v$ that are not $e_1, e_2, e_3, g_1, g_2$.

If $g_1 = e_3$ or $g_2 = e_3$, say $g_1 = e_3$, then $d \geq 2$ and we produce the sequence 

\[ e_1, e_2, e_{d+3}, e_{d+2}, \ldots, e_5, g_1 = e_3, g_2, e_4. \]

If $g_1 \neq e_3$ and $g_2 \neq e_3$, then $d \geq 1$ and we produce the sequence 

\[ e_1, e_2, e_{d+3}, \ldots, e_5, g_1, g_2, e_4, e_3. \]

* In situation (ii'), Let $d$ be the number of edges not in $P, Q, R_1, R_2$. For $d \geq 1$, let $e_4, e_5, \ldots, e_{d+3}$ be the distinct edges incident with $v$ that are not any of $e_1, e_2, e_3, f_1, f_2, g_1, g_2$.

If $f_i = e_3$ or $g_i = e_3$ $(i = 1, 2)$, then we may assume that $f_1 = e_3$ or $g_1 = e_3$, say $f_1 = e_3$, and so $d \geq 1$. We can then order the edges as follows:

\[ e_1, e_2, e_{d+3}, \ldots, e_6, g_1, g_2, e_5, e_4, f_1 = e_3. \]

If none of $f_1, g_1, g_2$ is $e_3$, then $d \geq 0$. Note here that if $d = 0$, then \{e_4, e_5, \cdots, e_{d+3}\} is the empty set. In this case, we produce the sequence 

\[ e_1, e_2, e_{d+3}, \ldots, e_5, f_1 = f_2, e_4, e_3, g_1, g_2. \]

Therefore, in any situation, we can embed $T$ so that the edges incident with $v$ appear counterclockwise in the order given in one of the situations above. Then there exists a Hamiltonian walk $W$ of $T$ and a resulting ordering $S_1$ of those distinct 3-paths of $T$ belonging to $W$ with the following properties.
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with the vertex $u$, as shown in Figure 5.7(b). If $R_1$ and/or $R_2$ contains an edge that is incident with either $u$ or $v$, then we may choose an ordering of the edges incident with one of these vertices such that the edges of $R_1$ and $R_2$ do not appear consecutively. There exists a Hamiltonian walk $W$ of $T$ and a resulting ordering $S_1$ of those 3-paths of $T$ on $W$ such that $e_1e_2$, $e_2e_3$ are consecutive terms in $S_1$. There are edges $a$ and $b$ in $T$ such that $ae_1, e_1e_2, e_2e_3, e_3b$ are consecutive terms in $S_1$. Again, we may embed $T$ so that neither $ae_1$ nor $e_3b$ is $R_1$ or $R_2$. For each vertex $w$ having degree 6 or more, we insert all 3-paths containing an edge $e$ and with interior vertex $w$ not already in $S_1$ into the sequence $S_1$, except for the 3-paths $R_1$ and $R_2$, between two consecutive terms containing $e$ except $e_1e_2$ and $e_2e_3$. For the 3-paths containing $e_2$ that are not $R_1$ or $R_2$, we insert such 3-paths between two other consecutive terms containing $e_2$. Since the edge $e_2$ occurs elsewhere in $W$ and in $S_1$, this can be done. This produces a sequence of all 3-paths of $T$ with the desired properties.

**Case 2.** $P$ and $Q$ do not have an edge in common and there exists a path $\Gamma$ in $T$ containing both $P$ and $Q$. Let $P = ab$ and $Q = cd$. See Figure 5.8(a).

![Figure 5.8: The 3-paths P and Q have no edge in common](image)

By assumption, there exists a path $\Gamma$ in $T$ containing both $P$ and $Q$. If none of the edges of $R_1$ and $R_2$ appear as 3-paths of $\Gamma$, then there is a Hamiltonian walk $W$ of $T$ such that $\Gamma$ is a path in $W$ such that $R_1$ and $R_2$ do not appear in $W$. Thus, either

$$\Gamma : a, b, c, d \quad \text{or} \quad \Gamma : a, b, e_1, e_2, \ldots, e_k, c, d$$

for some positive integer $k$.

Let $S_1$ be a cyclic sequence consisting of those 3-paths of $T$ appearing in the order as they are encountered on $W$. Thus, either

$$ab, bc, cd \quad \text{or} \quad ab, be_1, e_1e_2, \ldots, e_kc, cd$$
are consecutive terms in $S_1$.

* If $\Gamma : a, b, c, d$, then we delete $bc$ from $S_1$.

* if $\Gamma : a, b, e_1, e_2, \ldots, e_k, c, d$, then we delete the terms $be_1, e_1e_2, \ldots, e_kc$ from $S_1$.

In either situation, a new sequence $S_2$ is created. Since each edge of $T$ is encountered twice in $W$, each edge of $T$ occurs in two consecutive terms of $S_2$. Each 3-path deleted from $S_1$ and each 3-path in $T$ not in $S_1$ may now be added in an appropriate position in $S_2$, with the exception of not adding $R_1$ and $R_2$, creating a new sequence $S : A_1, A_2, \ldots, A_p$ of all 3-paths of $T$ such that (1) $P = A_1$ and $Q = A_p$, (2) $A_i$ and $A_{i+1}$ have a single edge in common for $i = 1, 2, \ldots, p-1$ and (3) $R_1$ and $R_2$ are not terms of this sequence.

Now, we may assume that $R_1$ or $R_2$ (or both) appear in $\Gamma$. Thus again, either $\Gamma : a, b, c, d$ where $R_1 = bc$ or $R_2 = bc$, say $R_1 = bc$ or

\[ \Gamma : a, b, e_1, e_2, \ldots, e_k, c, d \]

for some positive integer $k$,

where each of $R_1$ and $R_2$ (or perhaps one of them) appears as two consecutive terms in this sequence not including $a, b$ or $c, d$. Let $S_1$ be a cyclic sequence consisting of those 3-paths of $T$ appearing in the order as they are encountered on $W$. Thus, either

$ab, bc, cd$ or $ab, be_1, e_1e_2, \ldots, e_kc, cd$ are consecutive terms in $S_1$.

* If $\Gamma : a, b, c, d$, then we delete $bc$ from $S_1$.

* If $\Gamma : a, b, e_1, e_2, \ldots, e_k, c, d$, then we delete the terms $be_1, e_1e_2, \ldots, e_kc$ from $S_1$.

In either situation, a new sequence $S_2$ is created that has the property that neither of $R_1$ and $R_2$ appears in $S_2$. Since each edge of $T$ is encountered twice in $W$, each edge of $T$ occurs in two consecutive terms of $S_2$. Each 3-path deleted from $S_1$ and each 3-path in $T$ not in $S_1$, except for the two 3-paths $R_1$ and $R_2$, may now be added in an appropriate position in $S_2$, creating a new sequence $S : A_1, A_2, \ldots, A_p$ of all 3-paths of $T$ such that (1) $P = A_1$ and $Q = A_p$, (2) $A_i$ and $A_{i+1}$ have a single edge in common for $i = 1, 2, \ldots, p-1$ and (3) $R_1$ and $R_2$ are not terms of this sequence. The existence of the sequence $S$ shows that $P_3(T)$ contains a Hamiltonian $P$-$Q$ path that avoids $R_1$ and $R_2$.

Case 3. $P$ and $Q$ do not have an edge in common and there is no path in $T$ containing one of these paths and one edge of the other. Let $P = ab$ and $Q = cd$. See Figure 5.8(b).
Necessarily, there exists a path in $T$ containing one edge of $P$ and $Q$. Let $\Gamma$ be the path in $T$ connecting the interior vertices of $P$ and $Q$. We consider two possibilities, depending on whether $\Gamma$ is a trivial path.

**Subcase 3.1. $\Gamma$ is a trivial path.** Thus, $P$ and $Q$ have the same interior vertex $v$. See Figure 5.8(b). Suppose first that $\deg v = 6$. The tree $T$ is embedded in the plane so that the six edges $a, b, c, d, e, f$ incident with $v$ appear as in Figure 5.9(a).

If $R_1$ and $R_2$ also have $v$ as their interior vertex, we may select an ordering of these edges so that none the edges of $P$, $Q$, $R_1$ and $R_2$ appear consecutively about $v$. Possibly as many as three neighbors of $v$ are end-vertices in $T$. This embedding gives rise to a Hamiltonian walk $W$ of $T$ and a cyclic sequence $S_1$ of distinct 3-paths of $T$ lying on $W$. So, $S_1$ has the appearance

$$S_1 : cf, \ldots, fb, \ldots, bd, \ldots, de, \ldots, ea, \ldots, xa, ac, cy, \ldots, zc, cf$$

for edges $x, y, z$ in $T$, where, for example, possibly $x = e$ and/or $y = f$. Among the 3-paths in $T$ not in $S_1$ are $P = ab$ and $Q = cd$. We now insert the pair $P = ab, Q = cd$ between $ac$ and $cy$, arriving at

$$S_2 : P = ab, ac, xa, \ldots, ea, \ldots, de, \ldots, bd, \ldots, cf, zc, \ldots, cy, cd = Q.$$ 

Thus, this noncyclic sequence $S_2$ of distinct 3-paths begins at $P$, ends at $Q$ and contains all 3-paths of $T$ on $W$ in addition to $P$ and $Q$. Furthermore, every two consecutive 3-paths on $S_2$ have an edge in common. Each 3-path $rs$ in $T$ not in $S_2$, except for the 3-paths $R_1$ and $R_2$, can then be added to $S_2$, either between two 3-paths containing $r$ or between two 3-paths containing $s$, to produce a new sequence $S : A_1, A_2, \ldots, A_p$ of all 3-paths of $T$ such that (1) $P = A_1$ and $Q = A_p$, (2) $A_i$ and $A_{i+1}$ have a single edge in common for $i = 1, 2, \ldots, p - 1$ and (3) $R_1$ and $R_2$ are not terms of this sequence.
Next, suppose that $\text{deg} \, v = k \geq 7$, say $f_1, f_2, \ldots, f_{k-6}$ are the remaining $k - 6$ edges incident with $v$. Let $T$ be embedded as in Figure 5.9(b). We then proceed as above to produce a sequence $\mathcal{S}$ with the desired properties.

**Subcase 3.2.** $\Gamma$ is not a trivial path. Let $u$ be the interior vertex of $P = ab$ and $v$ the interior vertex of $Q = cd$. Since the interior vertices of $P$ and $Q$ have degree at least 6, it follows that if $R_1$ and/or $R_2$ contains an edge that is incident with one of these interior vertices, then we may select an ordering of these edges such that none of the edges $P, Q, R_1$ and $R_2$ appear consecutively about $v$. Let

$$\Gamma : e_1, e_2, \ldots, e_k, k \geq 1,$$

be the $u - v$ path in $T$.

Since $\text{deg} \, u \geq 6$ and $\text{deg} \, v \geq 6$, there exists an edge $f$ incident with $u$ different from $a, b$ and $e_1$ and an edge $g$ incident with $v$ different from $c, d$ and $e_k$. Let $T$ be embedded in the plane, as shown in Figure 5.8(b). Let $W$ be the resulting Hamiltonian walk of $T$ which contains none of $P, Q, R_1$ and $R_2$ but contains the path $\Gamma$ and let $\mathcal{S}_1$ be the resulting cyclic sequence of 3-paths lying on $W$. Hence, $\mathcal{S}_1$ has the appearance

$$\mathcal{S}_1 : af, \ldots, fb, bx, \ldots, yb, be_1, e_1e_2, \ldots, e_{k-1}e_k, e_kd, dz, \ldots, wa, af$$

for edges $x, y, z$ and $w$ in $T$, where possibly $y = f$ and $z = g$. Among the 3-paths in $T$ not belonging to $\mathcal{S}_1$ are $P = ab$ and $Q = cd$ (see Figure 5.8(b)). Hence, we may insert $P = ab$ between $yb$ and $be_1$ and $Q = cd$ between $e_kd$ and $dz$. We then delete the 3-paths $be_1, e_1e_2, \ldots, e_{k-1}d$ from $\mathcal{S}_1$, arriving at the sequence

$$\mathcal{S}_2 : P = ab, yb, \ldots, bx, fb, \ldots, af, wa, \ldots, dz, cd = Q,$$

consisting of $P, Q$ and all 3-paths of $T$ lying on $W$, except $be_1, e_1e_2, \ldots, e_{k-1}d$. These 3-paths along with all 3-paths of $T$ not in $\mathcal{S}_2$, except the 3 paths $R_1$ and $R_2$, can be appropriately inserted into $\mathcal{S}_2$ to produce a sequence $\mathcal{S} : A_1, A_2, \ldots, A_p$ consisting of the distinct 3-paths such that (1) $P = A_1$ and $Q = A_p$, (2) $A_i$ and $A_{i+1}$ have a single edge in common for $i = 1, 2, \ldots, p - 1$ and (3) the 3-paths $R_1$ and $R_2$ are not terms of this sequence. The existence of the sequence $\mathcal{S}$ shows that $\mathcal{P}_3(T)$ contains a Hamiltonian $P$-$Q$ path that avoids the 3-paths $R_1$ and $R_2$.

**Case 4.** $P$ and $Q$ do not have an edge in common and there is no path in $T$ containing both $P$ and $Q$ but there is a path containing one of $P$ and $Q$ and one edge of the other. Let $P = ab$ and $Q = cd$. See Figure 5.8(c). We may assume that there exists a path $\Gamma$ in $T$ containing the 3-path $Q$ and the edge $b$ but not $a$. Then there is a Hamiltonian walk $W$ of $T$ such that $\Gamma$ is a path in $W$. Thus, either

81
Γ : b, c, d or Γ : b, e₁, e₂, ..., eₖ, c, d for some positive integer k.

Let T be embedded in the plane, as shown in Figure 5.8(c). Since no vertex of T has degree 3, there is an edge f adjacent to a and b but not belonging to Γ that lies between a and b. Consider the following three cases:

1. If one of R₁ or R₂ is af or fb, say R₁ = af or R₁ = fb, then since there are no vertices of degree 5 in T, there is some other edge g that lies either between a and f or between f and b so that the edges of R₁ do not appear consecutively.

2. Let R₁ = af and R₂ = fb. Then since there are no vertices of degree 5 in T, there exist some other edges g₁ and g₂ such that (a) g₁ lies between a and f and (b) g₂ lies between f and b so that the edges of R₁ and R₂ do not appear consecutively.

3. If R₁ = e₁a or R₂ = e₁a, then we place g between these edges instead.

Let S₁ be the cyclic sequence consisting of those 3-paths of T appearing in the order as they are encountered on W. The 3-path ab therefore does not lie on W. Thus, either

(i) xb, bc, cd are three consecutive terms in S₁ for some edge x of T or

(ii) xb, be₁, e₁e₂, ..., eₖc, cd are consecutive terms in S₁ for some edge x of T.

If (i) occurs, then we insert the 3-path ab between xb and bc and delete bc; while if (ii) occurs, we insert ab between xb and be₁ and delete the terms be₁, e₁e₂, ..., eₖc. In either situation, a new sequence S₂ is created. Since each edge of T is encountered twice in W, each edge of T occurs twice in two consecutive terms of S₁. Each 3-path deleted from S₁ and each 3-path in T not in S₁ may now be added in an appropriate position in S₂ except for the 3-paths R₁ and R₂.

Specifically, if (i) occurs and R₁, R₂ ≠ bc, then the 3-path bc can now be inserted between two consecutive terms containing c. If (ii) occurs and R₁ and R₂ are not 3-paths of Γ, then the 3-path be₁ can be inserted between two consecutive terms containing e₁, the 3-path e₁e₂ can be inserted between two consecutive terms containing e₂, and so on. This creates a new sequence S : A₁, A₂, ..., Aₚ of all 3-paths of T that begins at P and ends at Q such that every two consecutive terms of S have an edge in common and R₁ and R₂ are not terms of S. The existence of the sequence S shows that P₃(T) contains a Hamiltonian P-Q path that avoids R₁ and R₂.

In each case, there is a Hamiltonian P-Q path in P₃(T) that avoids R₁ and R₂ and so P₃(T) is 4-tree connected.

82
Theorems 5.3.1–5.3.3 give rise to the following conjecture.

**Conjecture 5.3.4**  If $T$ is a tree of sufficiently large order containing no vertices of any of the degrees $2, 3, \ldots, k + 1$ for each integer $k \geq 2$, then $P_3(T)$ is $k$-tree-connected.
Chapter 6

The 4-Path Graph of a Graph

6.1 Introduction

In this chapter, we study the $k$-path graph of a graph in general with emphasis on the 4-path graph. First, we review some terminology and notation that we introduced earlier. For a nonempty graph $G$, we write $L^0(G)$ to denote $G$ and $L^1(G)$ to denote $L(G)$. For an integer $k \geq 2$, the $k$th iterated line graph $L^k(G)$ is defined as $L(L^{k-1}(G))$, where $L^{k-1}(G)$ is assumed to be nonempty. In particular, $L(L(G)) = L^2(G)$. For an integer $k \geq 2$ and a graph $G$ containing $k$-paths, recall that the $k$-path graph $P_k(G)$ of a connected graph $G$ has the set of $k$-paths of $G$ as its vertex set where two distinct vertices of $P_k(G)$ are adjacent if the corresponding $k$-paths of $G$ have a $(k-1)$-path in common. As we indicated earlier, $P_2(G) = L(G)$ and $P_3(G) = L(L(G)) = L^2(G)$. However, if $k \geq 4$ and $G$ is a connected graph, then $P_k(G) \neq L^{k-1}(G)$ in general. For example, $L(K_{1,3}) = K_3$ and so $L^2(K_{1,3}) = L(K_3) = K_3$; thus, $L^3(K_{1,3}) = K_3$. Since $K_{1,3}$ contains no 4-paths, it follows that $P_4(K_{1,3})$ does not exist. Thus, $P_4(K_{1,3}) \neq L^3(K_{1,3})$. As another example, if $G$ is the double star $S_{2,3}$, then $L(G) = K_{1,3} + e$. Thus, $L^2(G) = K_4 - e$ and so $L^3(G) = C_4 \vee K_1$. Since $P_4(G) = K_2$, it follows that $P_4(G) \neq L^3(G)$. On the other hand, for each integer $k \geq 4$, there are also connected graphs $G$ for which $P_k(G) = L^{k-1}(G)$.

Proposition 6.1.1 If $k$ and $n$ are integers with $n \geq k \geq 4$, then $P_k(P_n) = P_{n-k+1}$.

Proof. For given integers $k$ and $n$ with $n \geq k \geq 4$, let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n$. For each integer $i$ with $1 \leq i \leq n - k + 1$, let $Q_i = (v_i, v_{i+1}, \ldots, v_{i-k+1})$ be the subpath of order $k$ in $P_n$. Then $V(P_k(P_n)) = \{Q_1, Q_2, \ldots, Q_{n-k+1}\}$. In $P_k(P_n)$, the $k$-path $Q_1$ is adjacent only to $Q_2$, the $k$-path $Q_{n-k+1}$ is adjacent only to $Q_{n-k}$, and for $2 \leq i \leq n - k$ and the $k$-path $Q_i$ is adjacent only to both $Q_{i-1}$ and $Q_{i+1}$. Therefore,
\( \mathcal{P}_k(P_n) = P_{n-k+1} \) for all integers \( n \) and \( k \) with \( n \geq k \geq 4 \).

**Proposition 6.1.2** If \( k \) and \( n \) are integers with \( n \geq k \geq 4 \), then \( \mathcal{P}_k(C_n) = C_n \).

**Proof.** For given integers \( k \) and \( n \) with \( n \geq k \geq 4 \), let \( C_n = (v_1, v_2, \ldots, v_n, v_1) \) be a cycle of order \( n \). For each integer \( i \) with \( 1 \leq i \leq n-k+1 \), let \( Q_i = (v_{i_1}, v_{i_2}, \ldots, v_{i_k}) \) be the subpath of order \( k \) in \( C_n \), where the subscripts are expressed as positive integers modulo \( n \). Then \( V(\mathcal{P}_k(C_n)) = \{Q_1, Q_2, \ldots, Q_{n-k+1}\} \). For each integer \( i \) with \( 1 \leq i \leq n \), the \( k \)-path is adjacent only to both \( Q_{i-1} \) and \( Q_{i+1} \) in \( \mathcal{P}_k(C_n) \) (where the subscripts are expressed as positive integers modulo \( n \)). Therefore, \( \mathcal{P}_k(C_n) = C_n \).

Since \( L(P_n) = P_{n-1} \) and \( L(C_n) = C_n \) for each integer \( n \geq 4 \), the following corollary is a consequence of Propositions 6.1.1 and 6.1.2.

**Corollary 6.1.3** Let \( k \) and \( n \) be integers with \( n \geq k \geq 4 \).

If \( G \in \{P_n, C_n\} \), then \( \mathcal{P}_k(G) = L^{k-1}(G) \).

From the examples we have investigated, Corollary 6.1.3 gives rise to the following conjecture.

**Conjecture 6.1.4** Let \( k \) and \( n \) be integers with \( n \geq k \geq 4 \) and let \( G \) be a connected graph of order \( n \) containing \( k \)-paths. Then

\[ \mathcal{P}_k(G) = L^{k-1}(G) \] if and only if \( G \in \{P_n, C_n\} \).

We have seen that if \( G \) is a nontrivial connected graph, then \( L(G) \) is connected. Thus, for an integer \( k \geq 2 \) such that \( L^{k-1}(G) \) is nonempty, the \( k \)th iterated line graph \( L^k(G) = L(L^{k-1}(G)) \) of \( G \) is also connected. In particular, \( \mathcal{P}_3(G) = L^2(G) \) is connected. However, this is not true for \( \mathcal{P}_k(G) \) in general when \( k \geq 4 \). In fact, more can be said.

**Proposition 6.1.5** For each integer \( k \geq 4 \), there is a connected graph \( G_k \) such that \( \mathcal{P}_k(G_k) \) is disconnected.

**Proof.** For an integer \( k \geq 4 \), let \( G_k \) be the connected graph obtained from the path

\[ P = (u_{k-2}, u_{k-3}, \ldots, u_1, w, v_1, v_2, \ldots, v_{k-3}, v_{k-2}) \]
of order $2k-3 \geq 5$ by adding the edge $u_1v_1$ (see Figure 6.1). Observe that the graph $G_k$ contains exactly $2k-3$ distinct $k$-paths. We show that $P_k(G_k) = P_{k-1} + P_{k-2}$, which is the union of two vertex-disjoint paths $P_{k-1}$ and $P_{k-2}$.

First, we consider the following $k-1$ distinct $k$-paths in $G_k$:

\[
Q_1 = (w, v_1, u_1, u_2, \ldots, u_{k-3}, u_{k-2}) \\
Q_2 = (v_2, v_1, u_1, u_2, \ldots, u_{k-3}, u_{k-2}) \\
Q_3 = (v_3, v_2, v_1, u_1, u_2, \ldots, u_{k-4}, u_{k-3}) \\
Q_4 = (v_4, v_3, v_2, v_1, u_1, u_2, \ldots, u_{k-5}, u_{k-4}) \\
\vdots \\
Q_{k-3} = (v_{k-3}, v_{k-4}, \ldots, v_2, v_1, u_1, u_2, u_3) \\
Q_{k-2} = (v_{k-2}, v_{k-3}, \ldots, v_2, v_1, u_1, u_2) \\
Q_{k-1} = (w, u_1, v_1, v_2, \ldots, v_{k-3}, v_{k-2}).
\]

We make some observations concerning these $k$-paths in $G_k$. For each integer $i$ with $1 \leq i \leq k-2$, the paths $Q_i$ and $Q_{i+1}$ have a $(k-1)$-path in common and so $Q_i$ is adjacent to $Q_{i+1}$ in $P_4(G_k)$. For each pair $i, j$ with $1 \leq i < j \leq k-1$ and $j - i \geq 2$, the paths $Q_i$ and $Q_j$ do not have any $(k-1)$-path in common and so $Q_i$ and $Q_j$ are not adjacent in $P_4(G_k)$. Thus, $Q = (Q_1, Q_2, \ldots, Q_{k-1})$ is a path of order $k-1$ in $P_k(G_k)$. 

86
Next, we consider the remaining \( k-2 \) distinct \( k \)-paths in \( G_k \):

\[
Q_1' = (v_1, w, u_1, u_2, \ldots, u_{k-3}, u_{k-2}) \\
Q_2' = (v_2, v_1, w, u_1, u_2, \ldots, u_{k-3}) \\
Q_3' = (v_3, v_2, v_1, w, u_1, u_2, \ldots, u_{k-4}) \\
Q_4' = (v_4, v_3, v_2, v_1, w, u_1, u_2, \ldots, u_{k-5}) \\
\vdots \\
Q_{k-3}' = (v_{k-3}, v_{k-4}, \ldots, v_2, v_1, w, u_1, u_2) \\
Q_{k-2}' = (v_{k-2}, v_{k-3}, \ldots, v_2, v_1, w, u_1)
\]

Similarly, for each integer \( i \) with \( 1 \leq i \leq k-3 \), the paths \( Q_i' \) and \( Q_{i+1}' \) have a \((k-1)\)-path in common and so \( Q_i' \) is adjacent to \( Q_{i+1}' \) in \( P_4(G_k) \). For each pair \( i, j \) with \( 1 \leq i < j \leq k-2 \) and \( j - i \geq 2 \), the paths \( Q_i' \) and \( Q_j' \) do not have any \((k-1)\)-path in common and so \( Q_i' \) and \( Q_j' \) are not adjacent in \( P_4(G_k) \). Thus, \( Q' = (Q_1', Q_2', \ldots, Q_{k-2}') \) is a path of order \( k-2 \) in \( P_k(G_k) \).

It remains to show that no vertex in \( Q \) is adjacent to any vertex in \( Q' \) in \( P_k(G_k) \). Observe that each path \( Q_j' \) contains \((v_1, w, u_1)\) as a subpath for \( 1 \leq j \leq k-2 \). Since

(i) \( Q_1 \) contains \((w, v_1, u_1)\) as a subpath,

(ii) \( Q_i \) does not contain the two edges \( v_1w \) and \( wu_1 \) for \( 2 \leq i \leq k-2 \) and

(iii) \( Q_{k-1} \) contains \((w, u_1, v_1)\) as a subpath, it follows that \( Q_i \) and \( Q_j' \) have no \((k-1)\)-path in common for every pair \( i, j \) of integers with \( 1 \leq i \leq k-1 \) and \( 1 \leq j \leq k-2 \).

Hence, no vertex in \( Q \) is adjacent to any vertex in \( Q' \). Therefore, \( P_k(G_k) = P_{k-1} + P_{k-2} \).

The following is an immediate consequence of Proposition 6.1.5.

**Corollary 6.1.6** For each integer \( k \geq 4 \), there is a connected graph \( G_k \) such that

\[
\mathcal{P}_k(G_k) \neq L^{k-1}(G_k).
\]

Proposition 6.1.5 suggests the following question.

**Problem 6.1.7** For each integer \( k \geq 4 \), is there a connected graph \( G \) such that \( \mathcal{P}_\ell(G) \) is connected for every integer \( \ell \) with \( 2 \leq \ell < k \) but \( \mathcal{P}_k(G) \) is disconnected?
A path \( P = (u_1, u_2, \ldots, u_k) \) of order \( k \geq 3 \) in a connected graph \( G \) is called a \textit{k-path bridge} of \( G \) if (i) \( \deg_G u_i \geq 2 \) for \( i = 1, k \), (ii) \( \deg_G u_i = 2 \) for \( 2 \leq i \leq k - 1 \), and (iii) each edge of \( P \) is a bridge of \( G \). First, we make two useful observations about a \( k \)-path bridge \( P \) in a connected graph \( G \).

* Since each edge of \( P \) is a bridge, it follows that no edge of \( P \) lies on a cycle of \( G \).

* Each interior vertex of \( P \) is a cut-vertex of \( G \) as it is incident with a bridge in \( G \).

Thus, a \( k \)-path bridge contains \( k - 1 \) bridges of \( G \).

\textbf{Proposition 6.1.8} \textit{If} \( G \) \textit{is a connected graph of order at least} \( k \geq 3 \), \textit{then every} \( k \)-\textit{path bridge of} \( G \) \textit{is a cut-vertex of the} \( k \)-\textit{path graph} \( \mathcal{P}_k(G) \) \textit{of} \( G \).

\textbf{Proof.} Let \( G \) be a connected graph of order at least \( k \) and let \( z = (u_1, u_2, \ldots, u_k) \) be a \( k \)-path bridge of \( G \). To verify that \( z \) is a cut-vertex of \( \mathcal{P}_k(G) \), we show that there exist two vertices \( x \) and \( y \) in \( \mathcal{P}_k(G) \) such that every \( x - y \) path contains \( z \). Since \( \deg_G u_i = 2 \) for \( i = 2, 3, \ldots, k - 1 \) and \( u_1 \) and \( u_k \) are not end-vertices of \( G \), it follows that \( G - \{u_2, u_3, \ldots, u_{k-1}\} \) contains two nontrivial components \( G_1 \) and \( G_2 \) such that \( u_1 \in V(G_1) \) and \( u_k \in V(G_2) \). Let \( v \in V(G_1) - \{u_1\} \) and \( w \in V(G_2) - \{u_k\} \) such that \( u_1v, u_kw \in E(G) \). Then \((v, u_1, u_2, \ldots, u_{k-1}, u_k, w)\) is a \((k + 2)\)-path in \( G \). Let \( x = (v, u_1, u_2, u_3, \ldots, u_{k-1}) \) and \( y = (u_2, u_3, u_4, \ldots, u_{k-1}, u_k, w) \) be two \( k \)-paths in \( G \). We claim that every \( x - y \) path in \( \mathcal{P}_k(G) \) must contain \( z \). Assume, to the contrary, that there is an \( x - y \) path \( Q = (x = x_0, x_1, \ldots, x_{\ell-1}, x_{\ell} = y) \) in \( \mathcal{P}_k(G) \) that does not contain \( z \). Since \( \deg_G u_i = 2 \) for \( i = 2, 3, \ldots, k - 1 \), it follows that \( x_1 = (v, u_1, u_2, \ldots, u_{k-2}) \neq x \) and \( x_{\ell-1} = (u_2, u_3, \ldots, u_{k-1}, u_k, w, w_1) \neq y \). Thus, the path \( Q \) in \( \mathcal{P}_k(G) \) gives rise to a closed walk \((u_{k-1}, u_{k-2}, \ldots, u_2, u_1, v, v_1, \ldots, w_1, w, u_k, u_{k-1})\) in \( G \). However then, this implies that \((u_1, u_2, \ldots, u_k)\) lies on a cycle in \( G \), which is impossible. Thus, as claimed, every \( x - y \) path in \( \mathcal{P}_k(G) \) contains \( z \) and so \( z \) is a cut-vertex of \( \mathcal{P}_k(G) \).

The converse of Proposition 6.1.8 is not true in general. For example, let \( G \) be the graph constructed from the path \((u_1, u_2, \ldots, u_{k-1})\) of order \( k - 1 \geq 3 \) by adding an edge \( vw \) and joining both \( v \) and \( w \) to the vertex \( u_1 \) (see Figure 6.2). Then \( G \) contains no \( k \)-path bridges but the \( k \)-path graph \( \mathcal{P}_k(G) \cong P_4 \) contains two cut-vertices.
Chapter 6
Graphs

6.2 The 4-Path Graphs of Some Well-Known Graphs

For the purpose of giving examples of 4-path graphs of some well-known connected graphs and providing information about such graphs, our emphasis here will be on such graphs. The following is a consequence of Propositions 6.1.1 and 6.1.2.

Proposition 6.2.1 If \( n \geq 4 \), then \( P_4(P_n) = P_{n-3} \) and \( P_4(C_n) = C_n \).

For integers \( a \) and \( b \) with \( 1 \leq a \leq b \), the double star \( S_{a+1,b+1} \) is the tree of diameter 3, whose two central vertices have degrees \( a+1 \) and \( b+1 \), respectively. Thus, the order of \( S_{a+1,b+1} \) is \( a+b+2 \). The Cartesian product \( G_1 \square G_2 \) of two graphs \( G_1 \) and \( G_2 \) has vertex set \( V(G_1 \square G_2) = V(G_1) \times V(G_2) \), where two distinct vertices \( (u,v) \) and \( (x,y) \) of \( G_1 \square G_2 \) are adjacent if either (i) \( u = x \) and \( vy \in E(G_2) \) or (ii) \( v = y \) and \( ux \in E(G_1) \).

For a 4-path \( P = (w,x,y,z) \) in a graph \( G \), the edge \( xy \) is referred to as the interior edge of \( P \) and each of the two edges \( wx \) and \( yz \) is referred to as an end-edge of \( P \).

Proposition 6.2.2 For positive integers \( a \) and \( b \), \( P_4(S_{a+1,b+1}) = K_a \square K_b \). In particular, \( P_4(S_{2,b+1}) = K_b \) for each positive integer \( b \).

Proof. First, we consider the 4-path graph \( P_4(S_{a+1,b+1}) \) of \( S_{a+1,b+1} \). Let \( u \) and \( v \) be the central vertices of \( S_{a+1,b+1} \), where \( u_1, u_2, \ldots, u_a \) are the leaves adjacent to \( u \) and \( v_1, v_2, \ldots, v_b \) are the leaves adjacent to \( v \). Since any 4-path in \( S_{a+1,b+1} \) must have \( uv \) as its interior edge, it follows that a 4-path in \( S_{a+1,b+1} \) can be denoted by \( P_{i,j} = (u_i, u, v, v_j) \) for some integers \( i \) and \( j \) with \( 1 \leq i \leq a \) and \( 1 \leq j \leq b \). Hence, the 4-path \( P_{i,j} \) in \( P_4(S_{a+1,b+1}) \) is uniquely determined by the 2 digit code \( ij \) and so

\[
V(P_4(S_{a+1,b+1})) = \{P_{i,j} : 1 \leq i \leq a \text{ and } 1 \leq j \leq b\}.
\]

Furthermore, if \( P_{i,j} \) and \( P_{i',j'} \), where \( 1 \leq i, i' \leq a \) and \( 1 \leq j, j' \leq b \), are two distinct vertices of \( P_4(S_{a+1,b+1}) \) (or two distinct 4-paths of \( S_{a+1,b+1} \)), then \( P_{i,j} \) and \( P_{i',j'} \) are adjacent in \( P_4(S_{a+1,b+1}) \) if and only if either \( i = i' \) and \( j \neq j' \) or \( i \neq i' \) and \( j = j' \).

Next, consider the Cartesian product \( K_a \square K_b \) of \( K_a \) and \( K_b \). If
V(K_a) = \{x_1, x_2, \ldots, x_a\} and V(K_b) = \{y_1, y_2, \ldots, y_b\},

then

V(K_a \Box K_b) = \{(x_i, y_j) : 1 \leq i \leq a and 1 \leq j \leq b\}.

Furthermore, if \((x_i, y_j)\) and \((x_{i'}, y_{j'})\) are two distinct vertices in \(K_a \Box K_b\), then \((x_i, y_j)\) and \((x_{i'}, y_{j'})\) are adjacent in \(K_a \Box K_b\) if and only if either \(i = i'\) and \(j \neq j'\) or \(i \neq i'\) and \(j = j'\).

Now, define the bijection \(\phi : V(\mathcal{P}_4(S_{a+1,b+1})) \to V(K_a \Box K_b)\) by \(\phi(P_{i,j}) = x_iy_j\) for each pair \(i, j\) of integers with \(1 \leq i \leq a\) and \(1 \leq j \leq b\). It then follows by the definitions of \(\mathcal{P}_4(S_{a+1,b+1})\) and \(K_a \Box K_b\) that \(\phi\) preserves the adjacency and nonadjacency of the vertices of \(\mathcal{P}_4(S_{a+1,b+1})\). Therefore, \(\phi\) is an isomorphism from \(\mathcal{P}_4(S_{a+1,b+1})\) to \(K_a \Box K_b\) and so \(\mathcal{P}_4(S_{a+1,b+1}) \cong K_a \Box K_b\). 

**Proposition 6.2.3** For every integer \(r \geq 2\), the 4-path graph \(\mathcal{P}_4(K_{r,r})\) of the complete \(r\)-regular bipartite graph \(K_{r,r}\) is a \((4r - 6)\)-regular graph of order \(r^2(r - 1)^2\).

**Proof.** Let \(U = \{u_1, u_2, \ldots, u_r\}\) and \(V = \{v_1, v_2, \ldots, v_r\}\) be the partite sets of \(K_{r,r}\). Since each 4-path in \(K_{r,r}\) must contain the edge \((u_i, v_j)\) for some pair \(i, j\) of integers with \(1 \leq i, j \leq r\), we partition the vertex set of \(\mathcal{P}_4(K_{r,r})\) into the \(r^2\) subsets \(B_{i,j}\) \((1 \leq i, j \leq r)\) that consists of all 4-paths of \(K_{r,r}\) whose interior edge is \(u_iv_j\). For each pair \(i, j\) of integers with \(1 \leq i, j \leq r\), let

\[E_{i,j} = \{u_iv_j \cup \{u_iv : v \in V - \{v_j\} \cup \{uv_j : u \in U - \{u_i\}\}\}

be the set of all edges that belong to the 4-paths in \(B_{i,j}\) and let \(T_{i,j} = K_{r,r}[E_{i,j}]\) be the subgraph induced by \(E_{i,j}\). Then \(T_{i,j} \cong S_{r,r}\) for each such pair \(i, j\) of integers. By Proposition 6.2.2, \(\mathcal{P}_4(T_{i,j}) = K_{r-1,r-1}\) for each pair \(i, j\) of integers with \(1 \leq i, j \leq r\) and so \(|B_{i,j}| = (r - 1)^2\). Therefore, the order of \(\mathcal{P}_4(K_{r,r})\) is

\[|V(\mathcal{P}_4(K_{r,r}))| = \sum_{1 \leq i,j \leq r} |B_{i,j}| = r^2(r - 1)^2.\]

Next, we show that each vertex of \(\mathcal{P}_4(K_{r,r})\) has degree \(4r - 6\). By the symmetry of the graph \(K_{r,r}\), it suffices to determine the degree of the 4-path \(P = (v_2, u_1, v_1, u_2)\) in \(\mathcal{P}_4(K_{r,r})\). Since \(P\) is a 4-path in \(T_{1,1}\), it follows that \(P\) is adjacent to exactly \(2(r - 2)\) vertices in \(\mathcal{P}_4(T_{1,1}) = K_{r-1,r-1}\). Furthermore, \(P\) is adjacent to a 4-path \(Q\) that is not in \(T_{1,1}\) if and only if \(Q = (u_i, v_1, u_2, v_j)\) for \(2 \leq j \leq r\) or \(Q = (v_1, u_i, v_2, u_j)\) for \(2 \leq i \leq r\).
Thus, \( P \) is adjacent to exactly \( 2(r-1) \) vertices that are not in \( \mathcal{P}_4(T_{1,1}) \). Therefore, the degree of the 4-path \( P = (v_2, u_1, v_1, u_2) \) in \( \mathcal{P}_4(K_{r,p}) \) is \( 2(r-2) + 2(r-1) = 4r - 6 \). 

**Proposition 6.2.4** For each integer \( n \geq 4 \), the 4-path graph \( \mathcal{P}_4(K_n) \) of the complete graph \( K_n \) is a \( (4n-14) \)-regular graph of order \( 12\binom{n}{4} \).

**Proof.** First, we determine the order of \( \mathcal{P}_4(K_n) \). Let \( e = uv \) be an edge of \( K_n \). We show that the number of distinct 4-paths in \( K_n \) having interior edge \( e \) is \( (n-2)(n-3) \). Let \( (x,u,v,y) \) be such a 4-path in \( K_n \). Then there are \( n - 2 \) choices for \( x \) and once \( x \) has been chosen, there are \( n - 3 \) choices for \( y \). Thus, there are exactly \( (n-2)(n-3) \) distinct 4-paths in \( K_n \) having interior edge \( e \). Since there are \( \binom{n}{2} \) choices for an edge \( e \), the total number of distinct 4-paths in \( K_n \) is

\[
\binom{n}{2}(n-2)(n-3) = \frac{n(n-1)}{2}(n-2)(n-3) = \frac{n!}{2(n-4)!} = 12\binom{n}{4}
\]

and so the order of \( \mathcal{P}_4(K_n) \) is \( 12\binom{n}{4} \).

Next, we determine the regularity of \( \mathcal{P}_4(K_n) \). Let \( P = (x,u,v,y) \) be a 4-path in \( K_n \). Suppose that \( P \) is adjacent to a 4-path \( Q \) of \( K_n \) in \( \mathcal{P}_4(K_n) \). Thus, \( P \) and \( Q \) either (i) have the 3-path \( (x,u,v) \) in common or (ii) have the 3-path \( (u,v,y) \) in common. First, suppose that (i) occurs. Then either \( Q = (x,u,v,z_1) \) for some \( z_1 \in V(K_n) - \{x,u,v,y\} \) or \( Q = (v,u,x,z_2) \) for some \( z_2 \in V(K_n) - \{x,u,v\} \). There are \( n - 4 \) choices for \( z_1 \) and \( n - 3 \) choices for \( z_2 \). Thus, there are exactly \( (n-4) + (n-3) = 2n - 7 \) distinct 4-paths that contain the 3-path \( (x,u,v) \) and are adjacent to \( P \). Similarly, there are exactly \( 2n - 7 \) distinct 4-paths that contain the 3-path \( (u,v,y) \) and are adjacent to \( P \). Therefore, \( P \) is adjacent to exactly \( 2(2n-7) = 4n - 14 \) distinct 4-paths of \( K_n \) and so the degree of \( P \) in \( \mathcal{P}_4(K_n) \) is \( 4n - 14 \). 

Examples presented thus far illustrate the following result.

**Proposition 6.2.5** Let \( G \) be a connected graph containing 4-paths. If \( x = (u_1, u_2, u_3, u_4) \) is a 4-path in \( G \), where \( s(x) = \sum_{i=1}^{4} \deg_G u_i \), then the degree of \( x \) in \( \mathcal{P}_4(G) \) is

\[
\deg_{\mathcal{P}_4(G)} x = \begin{cases} 
  s(x) - 6 & \text{if } u_1u_3, u_2u_4 \notin E(G), \\
  s(x) - 8 & \text{if } u_1u_3 \in E(G) \text{ or } u_2u_4 \in E(G) \text{ but not both}, \\
  s(x) - 10 & \text{if } u_1u_3, u_2u_4 \in E(G).
\end{cases}
\]

**Proof.** First, suppose that \( u_1u_3, u_2u_4 \notin E(G) \). Let \( N_G(u_1) - \{u_2\} = \{u_{1,1}, u_{1,2}, \ldots, u_{1,a}\} \) if \( a = \deg_G u_1 - 1 \geq 1 \) (where \( N_G(u_1) - \{u_2\} = \emptyset \) if \( a = 0 \) and it is possible that
Next, suppose that exactly one of $u_4 \in N_G(u_1) - \{u_2\}$, let $N_G(u_2) - \{u_1, u_3\} = \{u_{2,1}, u_{2,2}, \ldots, u_{2,b}\}$ if $b = \deg_G u_2 - 2 \geq 1$
(\text{where } N_G(u_2) - \{u_1, u_3\} = \emptyset \text{ if } b = 0), let $N_G(u_3) - \{u_2, u_4\} = \{u_{3,1}, u_{3,2}, \ldots, u_{3,c}\}$ if $c = \deg_G u_3 - 2 \geq 1$ (\text{where } N_G(u_3) - \{u_2, u_4\} = \emptyset \text{ if } c = 0) and let $N_G(u_4) - \{u_3\} = \{u_{4,1}, u_{4,2}, \ldots, u_{4,d}\}$ if $d = \deg_G u_4 - 1 \geq 1$ (\text{where } N_G(u_4) - \{u_3\} = \emptyset \text{ if } d = 0 and it is possible that $u_1 \in N_G(u_4) - \{u_3\}$). In $P_4(G)$ then, the vertex $x$ is adjacent only to

1. each 4-path $(u_{1,i}, u_1, u_2, u_3)$ for $1 \leq i \leq a = \deg_G u_1 - 1,$

2. each 4-path $(u_{2,j}, u_2, u_3, u_4)$ for $1 \leq j \leq b = \deg_G u_2 - 2,$

3. each 4-path $(u_1, u_2, u_3, u_{3,k})$ for $1 \leq k \leq c = \deg_G u_3 - 2,$

4. each 4-path $(u_2, u_3, u_4, u_{4,\ell})$ for $1 \leq \ell \leq d = \deg_G u_4 - 1.$

Thus, the degree of $x$ in $P_4(G)$ is

$$\deg_{P_4(G)} x = a + b + c + d$$

$$= (\deg_G u_1 - 1) + (\deg_G u_2 - 2) + (\deg_G u_3 - 2) + (\deg_G u_4 - 1)$$

$$= \deg_G u_1 + \deg_G u_2 + \deg_G u_3 + \deg_G u_4 - 6 = s(x) - 6.$$

Next, suppose that exactly one of $u_1 u_3$ and $u_2 u_4$ is an edge of $G$, say $u_1 u_3 \in E(G)$ and $u_2 u_4 \notin E(G)$. Since $u_1 u_3$ is an edge in $G$, it follows that $u_1 u_3$ cannot belong to any 4-path of $G$ that is adjacent to $x$. Let $N_G(u_1) - \{u_2, u_3\} = \{u_{1,1}, u_{1,2}, \ldots, u_{1,a'}\}$ if $a' = \deg_G u_1 - 2 \geq 1$ (where $N_G(u_1) - \{u_2, u_3\} = \emptyset$ if $a' = 0$), let $N_G(u_2) - \{u_1, u_3\} = \{u_{2,1}, u_{2,2}, \ldots, u_{2,b}\}$ if $b = \deg_G u_2 - 2 \geq 1$ (where $N_G(u_2) - \{u_1, u_3\} = \emptyset$ if $b = 0$), let $N_G(u_3) - \{u_1, u_2, u_4\} = \{u_{3,1}, u_{3,2}, \ldots, u_{3,c'}\}$ if $c' = \deg_G u_3 - 3 \geq 1$ (where $N_G(u_3) - \{u_1, u_2, u_4\} = \emptyset$ if $c' = 0$) and let $N_G(u_4) - \{u_3\} = \{u_{4,1}, u_{4,2}, \ldots, u_{4,d}\}$ if $d = \deg_G u_4 - 1 \geq 1$ (where $N_G(u_4) - \{u_3\} = \emptyset$ if $d = 0$). Thus, the vertex $x$ is adjacent in $P_4(G)$ only to

1. each 4-path $(u_{1,i}, u_1, u_2, u_3)$ for $1 \leq i \leq a' = \deg_G u_1 - 2,$

2. each 4-path $(u_{2,j}, u_2, u_3, u_4)$ for $1 \leq j \leq b = \deg_G u_2 - 2,$

3. each 4-path $(u_1, u_2, u_3, u_{3,k})$ for $1 \leq k \leq c' = \deg_G u_3 - 3,$

4. each 4-path $(u_2, u_3, u_4, u_{4,\ell})$ for $1 \leq \ell \leq d = \deg_G u_4 - 1.$

Therefore, the degree of $x$ in $P_4(G)$ is

$$\deg_{P_4(G)} x = a' + b + c' + d$$

$$= (\deg_G u_1 - 2) + (\deg_G u_2 - 2) + (\deg_G u_3 - 3) + (\deg_G u_4 - 1)$$

$$= \deg_G u_1 + \deg_G u_2 + \deg_G u_3 + \deg_G u_4 - 8 = s(x) - 8.$$
Finally, suppose that \( u_1u_3, u_2u_4 \in E(G) \). Then neither \( u_1u_3 \) nor \( u_2u_4 \) belong to any 4-path of \( G \) that is adjacent to \( x \). Let \( N_G(u_1) - \{u_2, u_3\} = \{u_{1,1}, u_{1,2}, \ldots, u_{1,a'}\} \) if \( a' = \deg_G u_1 - 2 \geq 1 \) (where \( N_G(u_1) - \{u_2, u_3\} = \emptyset \) if \( a' = 0 \)), let \( N_G(u_2) - \{u_1, u_3, u_4\} = \{u_{2,1}, u_{2,2}, \ldots, u_{2,b'}\} \) if \( b' = \deg_G u_2 - 3 \geq 1 \) (where \( N_G(u_2) - \{u_1, u_3, u_4\} = \emptyset \) if \( b' = 0 \)), let \( N_G(u_3) - \{u_1, u_2, u_4\} = \{u_{3,1}, u_{3,2}, \ldots, u_{3,c'}\} \) if \( c' = \deg_G u_3 - 3 \geq 1 \) (where \( N_G(u_3) - \{u_1, u_2, u_4\} = \emptyset \) if \( c' = 0 \)) and let \( N_G(u_4) - \{u_2, u_3\} = \{u_{4,1}, u_{4,2}, \ldots, u_{4,d'}\} \) if \( d' = \deg_G u_4 - 2 \geq 1 \) (where \( N_G(u_4) - \{u_2, u_3\} = \emptyset \) if \( d' = 0 \)). Thus, the vertex \( x \) is adjacent in \( \mathcal{P}_4(G) \) only to

(1) each 4-path \((u_{1,i}, u_1, u_2, u_3)\) for \(1 \leq i \leq a' = \deg_G u_1 - 2\),

(2) each 4-path \((u_{2,j}, u_2, u_3, u_4)\) for \(1 \leq j \leq b' = \deg_G u_2 - 3\),

(3) each 4-path \((u_1, u_2, u_3, u_{3,k})\) for \(1 \leq k \leq c' = \deg_G u_3 - 3\),

(4) each 4-path \((u_2, u_3, u_4, u_{4,\ell})\) for \(1 \leq \ell \leq d' = \deg_G u_4 - 2\).

Hence, the degree of \( x \) in \( \mathcal{P}_4(G) \) is

\[
\deg_{\mathcal{P}_4(G)} x = a' + b' + c' + d' = (\deg_G u_1 - 2) + (\deg_G u_2 - 3) + (\deg_G u_3 - 3) + (\deg_G u_4 - 2) = \deg_G u_1 + \deg_G u_2 + \deg_G u_3 + \deg_G u_4 - 10 = s(x) - 10.
\]

This completes the proof.

By Proposition 6.2.5, if \( G \) is an \( r \)-regular and triangle-free graph, then \( \deg_{\mathcal{P}_4(G)} x = s(x) - 6 = 4r - 6 \) for each vertex \( x \) of \( \mathcal{P}_4(G) \). Thus, the following is a consequence of Proposition 6.2.5.

**Corollary 6.2.6** If \( G \) is an \( r \)-regular triangle-free graph for some integer \( r \geq 2 \), then \( \mathcal{P}_4(G) \) is \( (4r - 6) \)-regular.

### 6.3 Graphs Isomorphic to Their 4-Path Graph

In this section, we determine all graphs that are isomorphic to their 4-path graphs. First, we recall the following two known results (see [14], for example).

**Proposition 6.3.1** The line graph of every nontrivial connected graph is connected.

**Proposition 6.3.2** Let \( G \) be a connected graph of order \( n \geq 3 \). Then
(a) \( G \cong L(G) \) if and only if \( G = C_n \) and

(b) \( L(G) \cong L^2(G) \) if and only if \( L(G) = C_n \).

With the aid of Propositions 6.3.1 and 6.3.2, we now show that cycles are the only graphs that are isomorphic to its 3-path graph.

**Theorem 6.3.3** If \( G \) is a connected graph of order \( n \geq 3 \), then

\[ \mathcal{P}_3(G) \cong G \] if and only if \( G = C_n \).

**Proof.** If \( G = C_n \) for some integer \( n \geq 3 \), then, by Proposition 6.3.2, \( \mathcal{P}_3(C_n) = L(L(C_n)) = L(C_n) = C_n \). For the converse, suppose that \( G \) is a connected graph of order \( n \geq 3 \) and size \( m \) such that \( \mathcal{P}_3(G) \cong G \). Then \( m \geq n - 1 \). If \( m = n - 1 \), then \( G \) is a tree. Let \( L(G) = H \) which is connected by Proposition 6.3.1. Moreover, \( |V(H)| = |E(G)| = n - 1 \) and \( |E(H)| = n \). Thus, \( L(G) = H \) is a connected graph containing exactly two cycles. This implies that \( L(H) = L(L(G)) \) must contain a cycle. Hence, \( L(L(G)) \not\cong G \) which is a contradiction. Thus \( m \geq n \). Assume, to the contrary, that \( G \not\cong C_n \) for \( n \geq 4 \). Since \( m \geq n \), it follows that \( \Delta(G) = \Delta \geq 3 \). If \( \delta(G) \geq 2 \), then

\[
|V(\mathcal{P}_3(G))| = |E(L(G))| = \sum_{i=1}^{n} \left( \frac{d_i}{2} \right) > n
\]

where \( d_1, d_2, \ldots, d_n \) is the degree sequence of \( G \) which means that the order of \( \mathcal{P}_3(G) \) exceeds \( n \), a contradiction. Thus, \( G \) contains end-vertices. For each integer \( i \) with \( 1 \leq i \leq \Delta \), let \( n_i \) be the number of vertices of degree \( i \) in \( G \). Since \( 2m \geq 2n \), it follows that

\[
2m = \sum_{i=1}^{\Delta} in_i \geq 2n = 2(n_1 + n_2 + \cdots + n_\Delta)
\]

and so \( n_1 \leq n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta \). Hence,

\[
n = n_1 + n_2 + \cdots + n_\Delta \\
\leq [n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta] + (n_2 + n_3 + \cdots + n_\Delta) \\
= n_2 + 2n_3 + 3n_4 + \cdots + (\Delta - 1)n_\Delta = \sum_{i=2}^{\Delta} (i - 1)n_i.
\]
On the other hand, since \( L(L(G)) \cong G \) and \( \binom{i}{2} > i - 1 \) for each integer \( i \) with \( 3 \leq i \leq \Delta \), it follows that

\[
n = \sum_{i=2}^{\Delta} \binom{i}{2} n_i > \sum_{i=2}^{\Delta} (i - 1) n_i \geq n,
\]

which is impossible.

Theorem 6.3.3 can be extended to 4-path graphs; that is, the \( n \)-cycle \( C_n \) for \( n \geq 4 \) is also the only connected graph of order \( n \) for which \( P_4(G) \cong G \). In order to show this, we first introduce some additional definitions and notation. The clique number \( \omega(G) \) of a graph \( G \) is the order of a largest clique (complete subgraph) of \( G \). Let \( G \) be a connected graph with \( \omega(G) = \omega \geq 3 \) and let \( K_\omega \) be a clique of order \( \omega \) in \( G \). An edge \( e = uv \) of \( G \) is called a pendant edge of \( K_\omega \) at \( u \) if \( u \) is a vertex of \( K_\omega \) and \( v \) is not a vertex of \( K_\omega \).

Let \( p(u, K_\omega) \) denote the number of pendant edges of \( K_\omega \) at \( u \) and let \( p(K_\omega) \) denote the total number of pendant edges of \( K_\omega \). Thus,

\[
p(K_\omega) = \sum_{u \in V(K_\omega)} p(u, K_\omega).
\]

Next, let \( p_{\text{max}}(G, K_\omega) \) be the maximum number of pendant edges of a clique \( K_\omega \) in \( G \), that is,

\[
p_{\text{max}}(G, K_\omega) = \max \{ p(K_\omega) : K_\omega \subseteq G \}.
\]

For positive integers \( a \) and \( b \), let \( S_{a+1,b+1} \) be the double star of order \( a + b + 2 \) whose central vertices \( u \) and \( v \) have degree \( a + 1 \) and \( b + 1 \), respectively, and let \( S^*_a \) be the tree obtained from \( S_{a+1,b+1} \) by subdividing the non-pendant edge \( uv \) exactly once. This is illustrated in Figure 6.3.

![Figure 6.3: The double star \( S_{3,4} \) and the tree \( S^*_{3,4} \)](image)

**Lemma 6.3.4** For positive integers \( a \) and \( b \), \( P_4(S^*_{a+1,b+1}) = K_{a+b} \).
Proof. Suppose that $u$ and $v$ are the central vertices of $F = S_{a+1,b+1}$, where $u$ is adjacent to the $a$ end-vertices $u_1, u_2, \ldots, u_a$ and $v$ is adjacent to the $a$ end-vertices $v_1, v_2, \ldots, v_b$. Let $G = S_{a+1,b+1}^*$ be obtained from $F$ by inserting the vertex $w$ of degree 2 to the edge $uv$ of $F$. Then $G$ has exactly $a + b$ distinct 4-paths, namely $(u_i, u, w, v)$ and $(u, w, v, v_j)$ where $1 \leq i \leq a$ and $1 \leq j \leq b$. Since every pair of 4-paths of $G$ are adjacent in $P_4(G)$, it follows that $P_4(G) = K_{a+b}$.

An interesting fact derived from Propositions 6.2.2 and 6.3.4 is that there are nontrivial nonisomorphic trees $T_1$ and $T_2$ for which $P_4(T_1) \cong P_4(T_2)$. For example, if $a$ and $b$ are positive integers, then $S_{a+1,b+1}^* \not\cong S_{2,a+b+1}^*$, while $P_4(S_{a+1,b+1}^*) \cong P_4(S_{2,a+b+1}^*) \cong K_{a+b}$.

We are now prepared to present the following characterization of those graphs that are isomorphic to their 4-path graph.

**Theorem 6.3.5** Let $G$ be a connected graph of order $n \geq 4$. Then

$$P_4(G) \cong G \text{ if and only if } G = C_n.$$ 

**Proof.** By Proposition 6.1.2, if $G = C_n$, then $P_4(G) \cong G$. For the converse, assume that $G$ is a connected graph of order $n \geq 4$ and $G \not\cong C_n$. We show that $P_4(G) \not\cong G$. First, suppose that $G$ is a tree. If $\text{diam}(G) = 2$ or $\text{diam}(G) = 3$, then $G$ is a star or a double star. If $G$ is a star, then $G$ has no 4-paths; while if $G$ is a double star, then $P_4(G) \not\cong G$ by Proposition 6.2.2. Thus, we may assume that $\text{diam}(G) \geq 4$. Let $\Delta = \Delta(G)$. If $\Delta = 2$, then $G = P_n$ and $P_4(P_n) = P_{n-3} \not\cong P_n$ by Proposition 6.1.1. Thus, we may assume that $\Delta \geq 3$. Let $v \in V(G)$ with $\text{deg}_G(v) = \Delta$.

* First, suppose that $\Delta \geq 4$. Then $G$ contains $S_{\Delta,2}$ as a subgraph. It then follows by Proposition 6.2.2 that $P_4(G)$ contains $K_{\Delta-1}$ as a subgraph. Thus, $P_4(G)$ contains $K_3$ as a subgraph and so $P_4(G) \not\cong G$.

* Next, suppose that $\Delta = 3$. Let $T_1$ be the tree of order 6 obtained from the 5-path $(v_1, v_2, v_3, v_4, v_5)$ by adding a new vertex $v$ and the edge $vv_4$ and let $T_2$ be the tree obtained from the 5-path $(v_1, v_2, v_3, v_4, v_5)$ by adding a new vertex $v$ and the edge $vv_3$. Thus, $G$ either contains $T_1$ or $T_2$ as a subgraph. First, suppose that $G$ contains $T_1$ as a subgraph. Since $P_4(T_1)$ contains $K_3$ as a subgraph, it follows that $P_4(G)$ contains $K_3$ as a subgraph and so $P_4(G) \not\cong G$. Next, suppose that $G$ does not contain $T_1$ as a subgraph and so $G$ contains $T_2$ as a subgraph. If $G = T_2$, then $P_4(T_2) = P_4 \not\cong T_2$. If $G \neq T_2$, then, since $G$ does not contain $T_1$ as a subgraph, $G$
is the tree obtained from $T_2$ by adding a new vertex $w$ and the edge $vw$. That is, $G = S(K_{1,3})$, which is obtained from $K_{1,3}$ by subdividing each edge of $K_{1,3}$ exactly once. However then, $P_4(G) = C_6$ and so $P_4(G) \neq G$.

![Figure 6.4: The trees $T_1$ and $T_2$](image)

Next, suppose that $G$ contains cycles but is triangle-free. Let $g = g(G)$ be the girth of $G$. Since $G$ is triangle-free and $G \neq C_g$, it follows that $4 \leq g \leq n - 1$. Let $F$ be the unicyclic graph of order $g + 1$ obtained from $C_g$ by adding a pendant edge at a vertex of $C_g$. Then $P_4(F)$ contains a triangle and so $P_4(G)$ contains a triangle. Hence, $P_4(G) \neq G$.

Finally, suppose that $G$ contains triangles. By Proposition 6.2.4, we may assume that $G$ is a connected graph of order $n \geq 4$ and $G \neq K_n$. Let $\omega(G) = \omega$ and let $p_{\text{max}}(G, K_\omega) = p_{\text{max}}$. Thus, $3 \leq \omega \leq n - 1$ and $1 \leq p_{\text{max}} \leq n - \omega$. Suppose that $H = K_\omega$ is a maximum clique of $G$ with $p(H) = p_{\text{max}}$. Let $\omega' = \omega(P_4(G))$ and let $p'_{\text{max}} = p_{\text{max}}(P(G), \omega')$. In what follows, we show that

(i) $P_4(G)$ contains $K_{\omega+1}$ as a subgraph or

(ii) $P_4(G)$ contains $H' = K_\omega$ having $p(H') \geq p_{\text{max}} + 1$ or

(iii) $P_4(G)$ contains a subgraph that does not appear in $G$.

If (i) occurs, then $\omega' \geq \omega + 1$; while if (ii) occurs, then $p'_{\text{max}} > p_{\text{max}} + 1$. In either case, $P_4(G) \not\equiv G$. Clearly, if (iii) occurs, then $P_4(G) \not\equiv G$. We consider four cases, depending on the values of $\omega$. In each case, we assume that $H = K_\omega$ is a maximum clique of $G$ with $p(H) = p_{\text{max}}$ and $V(H) = \{u_1, u_2, \ldots, u_\omega\}$ where $\deg u_i = \deg u_{i+1}$ for $1 \leq i \leq \omega - 1$.

**Case 1.** $\omega(G) = 3$. There are four subcases, depending on the degrees of vertices of $H$.

**Subcase 1.1.** Exactly one of $u_1$, $u_2$ and $u_3$ has degree at least 3 in $G$. Then $\deg_{G^c} u_1 \geq 3$ and $\deg_{G^c} u_2 = \deg_{G^c} u_3 = 2$. We now consider four situations, according to the values of $p(H)$. 

97
Subcase 1.1.1. \( p(H) = 1 \). Then \( p(u_1, H) = 1 \). Suppose that \( u_1v \) is the pendant edge at \( u_1 \). If \( \deg_G v = 1 \), then \( \mathcal{P}_4(G) = K_2 \neq G \). Thus, we may assume that \( \deg_G v \geq 2 \). First, suppose that \( \deg_G v = 2 \) and that \( v \) is adjacent to a vertex \( w \) not on \( H \). If \( \deg_G w = 1 \), then \( \mathcal{P}_4(G) = P_4 \neq G \). If \( \deg_G w = 2 \), then \( \mathcal{P}_4(G) \) contains a clique \( H' = K_3 \) such that \( p(H') \geq 2 \). If \( \deg_G w \geq 3 \), then \( G \) contains \( S^*_3,3 \) as a subgraph and so \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph by Lemma 6.3.4. Next, suppose that \( \deg_G v \geq 3 \). Then \( \mathcal{P}_4(G) \) contains a clique \( H' = K_3 \) such that \( p(H') \geq 2 \).

Subcase 1.1.2. \( p(H) = 2 \). Then \( p(u_1, H) = 2 \). Suppose that \( u_1v_1 \) and \( u_1v_2 \) are the pendant edges of \( H \) at \( u_1 \). If \( \deg_G v_1 = \deg_G v_2 = 1 \), then \( \mathcal{P}_4(G) = K_2 + K_2 \neq G \). If both \( v_1 \) and \( v_2 \) have degree at least 2, then \( \mathcal{P}_4(G) \) contains a clique \( H' = K_3 \) with \( p(H') \geq 3 \). Hence, we may assume that \( \deg_G v_1 \geq 2 \). If \( \deg_G v_1 \geq 3 \), then \( \mathcal{P}_4(G) \) contains a clique \( H' = K_3 \) with \( p(H') \geq 3 \). So, we now assume that \( \deg_G v_1 = 2 \), where say \( v_1 \) is adjacent to some vertex \( w \in V(G) - V(H) \). If \( \deg_G w = 1 \), then \( G \) is a graph of order 6, where \( V(G) = \{u_1, u_2, u_3, v_1, v_2, w\} \), and \( \mathcal{P}_4(G) \) is a graph of order 7 and so \( \mathcal{P}_4(G) \neq G \). If \( \deg_G w \geq 2 \), then \( G \) contains \( S^*_2,4 \) as a subgraph and so \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph by Lemma 6.3.4.

Subcase 1.1.3. \( p(H) = 3 \). Then \( p(u_1, H) = 3 \). Suppose that \( u_1v_1, u_1v_2, u_1v_3 \) are the pendant edges at \( u_1 \). If \( \deg_G v_1 = \deg_G v_2 = \deg_G v_3 = 1 \), then \( \mathcal{P}_4(G) = K_3 + K_3 \neq G \). If at least one vertex of \( \{v_1, v_2, v_3\} \) has degree 2, then \( G \) contains the double star \( S^*_2,5 \) as a subgraph and so \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph by Proposition 6.2.2.

Subcase 1.1.4. \( p(H) \geq 4 \). Then \( p(u_1, H) = 4 \) and so there are four pendant edges at \( u_1 \). Thus, \( G \) contains \( S^*_2,5 \) as a subgraph and so \( \mathcal{P}(G) \) contains \( K_4 \) as a subgraph by Proposition 6.2.2.

Subcase 1.2. Exactly two of \( u_1, u_2 \) and \( u_3 \) have degree 3 or more. Then \( \deg_G u_1 \geq \deg_G u_2 \geq 3 \) and \( \deg_G u_3 = 2 \). We now consider three situations, according to the values of \( p(H) \).

Subcase 1.2.1. \( p(H) = 2 \). Thus, \( p(u_1, H) = p(u_2, H) = 1 \). Suppose that \( u_1v_1 \) and \( u_2v_2 \) are the pendant edges of \( H \). If \( \deg_G v_1 \geq 2 \) and \( \deg_G v_2 \geq 2 \), then \( \mathcal{P}_4(G) \) contains a clique \( H' = K_3 \) with \( p(H') = 3 \). If \( \deg_G v_1 = \deg_G v_2 = 1 \), then \( \mathcal{P}_4(G) = K_2 + P_3 \neq G \). Thus, we may assume that \( \deg_G v_1 \geq 2 \) and \( \deg_G v_2 = 1 \). First, suppose that \( \deg_G v_1 = 2 \), where say \( v_1 \) is adjacent to some vertex \( w \in V(G) - V(H) \). If \( \deg_G w = 1 \), then \( G \) has order 6, where \( V(G) = \{u_1, u_2, u_3, v_1, v_2, w\} \) and \( \mathcal{P}_4(G) \) has order 8; so \( \mathcal{P}_4(G) \neq G \). If \( \deg_G w \geq 2 \), then \( \mathcal{P}_4(G) \) contains a clique \( H' = K_3 \) with \( p(H') \geq 3 \). Next, suppose
that \( \deg_G v_1 \geq 3 \). Then \( G \) contains \( S_{3,3}^* \) as a subgraph and so \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph by Lemma 6.3.4.

**Subcase 1.2.2.** \( p(H) = 3 \). Then \( p(u_1, H) = 2 \) and \( p(u_2, H) = 1 \). Suppose that \( u_1v_1, u_1v_2 \) and \( u_2v_3 \) are the pendant edges of \( H \). If \( \deg_G v_1 = \deg_G v_2 = \deg_G v_3 = 1 \), then the order of \( G \) is 6 while the order of \( \mathcal{P}_4(G) \) is 9; so \( \mathcal{P}_4(G) \not\cong G \). Thus we may assume that at least one of \( v_1, v_2 \) and \( v_3 \) has degree 2 or more. First, suppose that \( \deg_G v_3 \geq 2 \).

Since \( G \) contains \( S_{2,4}^* \) as a subgraph, it follows by Lemma 6.3.4 that \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph. Thus, we may assume that \( v_1 \) or \( v_2 \) has degree 2, say \( \deg_G v_1 = 2 \) and \( v_1 \) adjacent to \( v \). If \( \deg_G w = 1 \), the order of \( \mathcal{P}_4(G) \) is greater than the order of \( G \). If \( \deg_G w \geq 2 \), then \( G \) contains \( S_{2,4}^* \) as a subgraph and so \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph by Lemma 6.3.4. Moreover, if at least one of \( v_1 \) and \( v_2 \) has degree at least 3, then \( G \) contains \( S_{3,3}^* \) as a subgraph and so \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph by Lemma 6.3.4.

**Subcase 1.2.3.** \( p(H) \geq 4 \). If \( p(u_1, H) = 2 \) and \( p(u_2, H) = 2 \), then \( G \) contains \( S_{3,3}^* \); while if \( p(u_1, H) = 3 \) and \( p(u_2, H) = 1 \), then \( G \) contains \( S_{2,4}^* \) as a subgraph. In either case, \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph by Lemma 6.3.4.

**Subcase 1.3.** All three vertices \( u_1, u_2 \) and \( u_3 \) have degree at least 3. We now consider three situations, according to the values of \( p(H) \).

**Subcase 1.3.1.** \( p(H) = 3 \). Thus, \( p(u_1, H) = p(u_2, H) = p(u_3, H) = 1 \). Suppose that \( u_1v_1, u_2v_2 \) and \( u_3v_3 \) are the pendant edges of \( H \) where \( v_1, v_2, v_3 \in V(G) - V(H) \). If \( \deg_G v_1 = \deg_G v_2 = \deg_G v_3 = 1 \), then \( \mathcal{P}_4(G) = C_9 \not\cong G \). Thus, we may assume, without loss of generality, that \( \deg_G v_1 \geq 2 \). If \( \deg_G v_1 \geq 3 \), then \( G \) contains \( S_{3,3}^* \) and so \( \mathcal{P}_4(G) \) contains a clique \( K_4 \). Thus, assume that \( \deg_G v_1 = 2 \) and \( v_1 \) is adjacent to some \( w \in V(G) - V(H) \). If \( \deg_G w \geq 3 \), then \( G \) contains \( S_{3,3}^* \) and so \( \mathcal{P}_4(G) \) contains \( K_4 \). If \( \deg_G w = 2 \), then \( \mathcal{P}_4(G) \) contains a clique \( H' = K_3 \) with \( p(H') \geq 4 \). If \( \deg_G w = 1 \), then the order of \( \mathcal{P}_4(G) \) is greater than that of \( G \) and so \( \mathcal{P}_4(G) \not\cong G \).

**Subcase 1.3.2.** \( p(H) = 4 \). Then \( p(u_1, H) = 2 \) and \( p(u_2, H) = p(u_3, H) = 1 \). Let \( u_1v_1, u_1v_2, u_2v_3, u_3v_4 \) be the pendant edges of \( H \). If each of \( v_1, v_2, v_3 \) and \( v_4 \) is an end-vertex, then the order of \( G \) is 7 and the order of \( \mathcal{P}_4(G) \) is at least 9; so \( \mathcal{P}_4(G) \not\cong G \). Thus, we may assume that at least one of \( v_1, v_2, v_3 \) and \( v_4 \) has degree at least 2. First, suppose that either \( v_1 \) or \( v_2 \) has degree at least 2, say \( \deg_G v_1 \geq 2 \). If \( \deg_G v_1 = 2 \), then \( \mathcal{P}_4(G) \) contains a clique \( H' = K_3 \) with \( p(H') = 5 \); while if \( \deg_G v_1 \geq 3 \), then \( G \) contains \( S_{3,3}^* \) and so \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph. Next, suppose that either \( v_3 \) or \( v_4 \) has degree at least 2. Then \( G \) contains \( S_{2,4}^* \) and so \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph by Lemma 6.3.4.
Subcase 1.3.3. \( p(H) \geq 5 \). If \( p(u_1, H) \geq 3 \), then \( G \) contains \( S_{2,4}^* \) as a subgraph; while if \( p(u_1, H) = 2 \), then \( p(u_2, H) = 2 \) and so \( G \) contains \( S_{3,3}^* \) as a subgraph. In either case, \( \mathcal{P}_4(G) \) contains \( K_4 \) as a subgraph by Lemma 6.3.4.

Case 2. \( \omega(G) = 4 \). We now consider three subcases, according to the values of \( p(H) \).

Subcase 2.1. \( p(H) = 1 \). Then \( p(u_1, H) = 1 \). Suppose that \( u_1v \) is the pendant edge of \( H \).

* If \( \deg_G v = 1 \), then the order of \( G \) is 5 and the order of \( \mathcal{P}_4(G) \) is 12. Thus, \( \mathcal{P}_4(G) \not\cong G \).

* If \( \deg_G v = 2 \), then let \( w \neq u_1 \) be the vertex adjacent to \( v \). If \( \deg_G w = 1 \), then \( \mathcal{P}_4(G) \not\cong G \) since the order of \( \mathcal{P}_4(G) \) is greater than that of \( G \). If \( \deg_G w = 2 \), then \( G \) contains \( S_{2,4}^* \), which creates a clique \( H' = K_4 \) in \( \mathcal{P}_4(G) \) with \( p(H') = 6 \). If \( \deg_G w \geq 3 \), then \( G \) contains \( S_{3,4}^* \) and so \( \mathcal{P}_4(G) \) contains \( K_5 \) as a subgraph.

* If \( \deg_G v = 3 \), then \( G \) contain \( S_{2,4}^* \), which creates a clique \( H' = K_4 \) in \( \mathcal{P}_4(G) \) with \( p(H') = 6 \).

* If \( \deg_G v \geq 4 \), then \( G \) contain \( S_{3,4}^* \) and so \( \mathcal{P}_4(G) \) contains \( K_5 \) as a subgraph.

Subcase 2.2. \( p(H) = 2 \). First, suppose that \( p(u_1, H) = 2 \) and let \( u_1v_1 \) and \( u_1v_2 \) be the two pendant edges of \( H \) at \( u_1 \), where \( v_1, v_2 \in V(G) - V(H) \). Let \( H' = K_4 \) in \( \mathcal{P}_4(G) \) whose four vertices are \( x_1 = (u_3, u_2, u_1, v_1), x_2 = (u_3, u_2, u_1, v_2), x_3 = (u_3, u_2, u_1, u_4) \) and \( x_4 = (u_1, u_2, u_3, u_4) \). Since \( p(x_1, H') = p(x_2, H') = p(x_3, H') = p(x_4, H') = 1 \), it follows that \( p(H') \geq 4 \). Next, suppose that \( p(u_1, H) = p(u_2, H) = 1 \). Suppose that \( u_1v_1 \) and \( u_2v_2 \) are the two pendant edges of \( H \), where \( v_1, v_2 \in V(G) - V(H) \). Let \( H' = K_4 \) in \( \mathcal{P}_4(G) \) whose four vertices are \( x_1 = (u_1, u_3, u_2), x_2 = (u_4, u_1, u_3, u_2), x_3 = (u_1, u_3, u_2, v_2) \) and \( x_4 = (u_1, u_3, u_2, u_4) \). Since \( p(x_1, H') = p(x_2, H') = p(x_3, H') = p(x_4, H') = 1 \), it follows that \( p(H') \geq 4 \).

Subcase 2.3. \( p(H) \geq 3 \). First, suppose that \( p(u_1, H) = 2 \) and \( p(u_2, H) = 1 \). Let \( u_1v_1, u_1v_2 \) and \( u_2v_3 \) be the pendant edges at \( u_1 \) and \( u_2 \). Then \( G \) contains \( S_{3,4}^* \) as a subgraph and so \( \mathcal{P}_4(G) \) contains \( K_5 \) as a subgraph by Lemma 6.3.4. Next, suppose that \( p(u_1, H) = p(u_2, H) = p(u_3, H) = 1 \). Let \( u_1v_1, u_2v_2 \) and \( u_3v_3 \) be the pendant edges at \( u_1, u_2 \) and \( u_3 \). If \( \deg_G v_i = 1 \) for all \( i = 1, 2, 3 \), then \( \mathcal{P}_4(G) \) contains a clique \( H' = K_4 \) with \( p(H') \geq 4 \). So we may assume that one of \( v_1, v_2 \) and \( v_3 \) has degree at least 2, say, \( \deg_G v_1 \geq 2 \). If \( \deg_G v_1 = 2 \), then \( \mathcal{P}_4(G) \not\cong G \) since the order of \( \mathcal{P}_4(G) \) is greater that
the order of $G$. If $\deg_v v_1 \geq 3$, then $G$ contains $S_{3,4}$ as a subgraph so $\mathcal{P}_4(G)$ contains $K_5$ as a subgraph by Lemma 6.3.4.

Case 3. $\omega(G) = 5$. We now consider two subcases, according to the values of $p(H)$.

Subcase 3.1. $p(H) = 1$. Then $p(u_1, H) = 1$. Suppose that $u_1 v$ is the pendant edge of $H$. Let $H' = K_5$ in $\mathcal{P}_4(G)$ whose five vertices are $x_1 = (v, u_1, u_2, u_3), x_2 = (u_4, u_1, u_2, u_3), x_3 = (u_5, u_1, u_2, u_3), x_4 = (u_1, u_2, u_3, u_4)$ and $x_5 = (u_1, u_2, u_3, u_5)$. Since $p(x_1, H') = p(x_2, H') = p(x_3, H') = p(x_4, H') = p(x_5, H') = 2$, it follows that $p(H') \geq 10$.

Subcase 3.2. $p(H) \geq 2$. First, suppose that $p(u_1, H) = 2$ and let $u_1 v_1$ and $u_1 v_2$ be the two pendant edges of $H$ at $u_1$. Then $G$ contains $S_{3,5}^*$ which creates a clique $H' = K_6$ in $\mathcal{P}_4(G)$. Next, suppose that $p(u_1, H) = 1$ and $p(u_2, H) = 1$. Let $u_1 v_1$ and $u_2 v_2$ be the two pendant edges of $H$ at $u_1$ and $u_2$, respectively. Then $G$ contains $S_{4,4}^*$ as a subgraph and so $\mathcal{P}_4(G)$ contains $K_6$ as a subgraph by Lemma 6.3.4.

Case 4. $\omega(G) = \omega \geq 6$. Let $p(H) \geq 1$. Let $u_1 v$ be a pendant edge at $u_1$ where $v \in V(G) - V(H)$. Observe that the 4-paths $(v, u_1, u_2, u_3), (u_i, u_1, u_2, u_3), 4 \leq i \leq \omega,$ and $(u_1, u_2, u_3, u_1), 4 \leq j \leq \omega,$ create a clique of order $1 + (\omega - 3) + (\omega - 3) = 2\omega - 5$ in $\mathcal{P}_4(G)$. Since $2\omega - 5 = \omega + (\omega - 5) \geq \omega + 1$ for $\omega \geq 6$, it follows that $\mathcal{P}_4(G)$ contains a clique $K_{\omega + 1}$ as a subgraph. Therefore, $\mathcal{P}_4(G) \not\cong G$.

### 6.4 Graphs Whose 4-Path Graphs are Paths or Cycles

By Proposition 6.2.1, if $G$ is a path or a cycle, then $\mathcal{P}_4(G)$ is also a path or a cycle. This gives rise to the following question:

Are there graphs other than a path or a cycle whose 4-path graph is a path or a cycle?

First, we show that there are graphs that are not paths but whose 4-path graph is a path.

**Proposition 6.4.1** Let $G$ be a connected graph of order at least 4. Then

(a) $\mathcal{P}_4(G) = P_2$ if and only if $G = P_5$ or $G = S_{2,3}$.

(b) $\mathcal{P}_4(G) = P_4$ if and only if (i) $G = P_7$ or (ii) $G$ is one of the two graphs in Figure 6.5.
Proof. It is straightforward to see that (a) holds. For (b), we have seen that if $G = P_7$ or $G$ is one of the graphs in Figure 6.5, then $\mathcal{P}_4(G) = P_4$. It remains to verify the converse. Suppose that $\mathcal{P}_4(G) = P_4 = (x_1, x_2, x_3, x_4)$. Assume, to the contrary, that $G \neq P_7$ and $G$ is neither of the two graphs in Figure 6.5. We consider two cases.

Case 1. $G$ is not a tree. Since $\mathcal{P}_4(C_k) = C_k$ for each integer $k \geq 4$, it follows that $G$ cannot contain a $k$-cycle for any integer $k \geq 4$. Thus, $G$ contains a triangle, say $C_3 = (u, v, w, u)$, as a subgraph. If two or more vertices of $C_3$ have degree at least 3, then the order of $\mathcal{P}_4(G)$ is at least 5. Thus, exactly one vertex of $C_3$ has degree at least 3, say $\deg_G u \geq 3$. If $\deg_G u \geq 5$, then the order of $\mathcal{P}_4(G)$ is at least 5, a contradiction. If $\deg_G u = 4$ and $G$ does not contain the second graph of Figure 6.5 as a subgraph, then $\mathcal{P}_4(G) = 2K_2$, a contradiction. Thus, $\deg_G u = 3$ and so $G$ must contain the second graph of Figure 6.5 as a subgraph, a contradiction.

Case 2. $G$ is a tree. Let $x_1 = (u_1, u_2, u_3, u_4)$ be a 4-path in $G$. Let $U = \{u_1, u_2, u_3, u_4\}$. We may assume, without loss of generality, that $x_2 = (u_1, u_2, u_3, v)$ or $x_2 = (v, u_1, u_2, u_3)$ for some $v \notin U$. There are two subcases.

Subcase 2.1. $x_2 = (u_1, u_2, u_3, v)$. Since $x_3$ is not adjacent to $x_1$, it follows that (i) $x_3 = (w, u_2, u_3, v)$ or (ii) $x_3 = (w, u_3, v, w)$ for some $w \notin U \cup \{v\}$. If (i) occurs, then $G$ contains $S_{3,3}$ and so $\mathcal{P}_4(G)$ contains a 4-cycle, a contradiction. If (ii) occurs, then $G$ contains the first graph of Figure 6.5 as a subgraph, a contradiction.

Subcase 2.2. $x_2 = (v, u_1, u_2, u_3)$. Since $x_3$ is not adjacent to $x_1$, it follows that $x_3 = (v, u_1, u_2, w)$ or $x_3 = (v, w, u_1, u_2)$ for some $w \notin U \cup \{v\}$. If $x_3 = (v, u_1, u_2, w)$, then $G$ contains the first graph of Figure 6.5 as a subgraph, a contradiction. Thus, $x_3 = (w, v, u_1, u_2)$. Thus, $x_4$ contains either $(w, v, u_1)$ or $(v, u_1, u_2)$. Since $x_4$ is not adjacent to $x_2$, it follows that $x_4$ does not contain $(v, u_1, u_2)$ and so $x_4$ contains $(w, v, u_1)$. However then, either $G$ contains $P_7$ as a subgraph or $G$ contains the first graph of Figure 6.5 as a subgraph, a contradiction.

Next, we show that paths $P_2$ and $P_4$ are the only two exceptions.
**Proposition 6.4.2** Let $G$ be a connected graph of order at least 4. If $n$ is a positive integer and $n \not\in \{2, 4\}$, then $\mathcal{P}_4(G) = P_n$ if and only if $G = P_{n+3}$.

**Proof.** Let $n \geq 2$ and $n \not\in \{2, 4\}$. By Proposition 6.2.1, we have $\mathcal{P}_4(P_{n+3}) = P_n$. It remains to verify the converse. Suppose that $G \not= P_{n+3}$ and we show that $\mathcal{P}_4(G) \not= P_n$.

Assume, to the contrary, that $\mathcal{P}_4(G) = P_n$. We consider two cases, according to whether $G$ is a tree.

**Case 1. $G$ is a tree.** Since $G$ is not a path, $\Delta(G) = \Delta \geq 3$. Let $v \in V(G)$ such that $\deg_G v = \Delta$. Since $G$ is not a star, some neighbor of $v$ has degree at least 2 and so $G$ contains $S_{2, \Delta}$ as a subgraph. If $\Delta \geq 4$, then $\mathcal{P}_4(S_{2, \Delta}) = K_{\Delta - 1}$ by Proposition 6.2.2 and so $\mathcal{P}_4(G)$ contains a triangle. Hence, $\mathcal{P}_4(G) \not= P_n$ for $n \geq 4$, a contradiction. Thus, $\Delta = 3$. If $G$ contains either $S_{2, 3}^*$ or $S_{3, 3}$ as a subgraph, then $\mathcal{P}_4(G)$ contains $C_3$ or $C_4$ by Propositions 6.4.6 and 6.4.7, respectively. In either case, $\mathcal{P}_4(G) \not= P_n$ for any integer $n \geq 2$ and $n \not\in \{2, 4\}$, a contradiction. Hence, $G$ contains neither $S_{2, 3}^*$ nor $S_{3, 3}$ as a subgraph. Thus, each neighbor of $v$ has degree at most 2. For $i = 1, 2, 3$, let $T_i$ be the tree obtained by subdividing exactly $i$ edges of $K_{1, 3}$ exactly once. Since $\mathcal{P}_4(T_1) = K_2$, $\mathcal{P}_4(T_2) = P_4$ and $\mathcal{P}_4(T_3) = C_6$, it follows that $\mathcal{P}_4(T_i) \not= P_n$ ($i = 1, 3$) for any integer $n \geq 2$ and $n \not\in \{2, 4\}$, which is a contradiction.

**Case 2. $G$ is not a tree.** Thus, $G$ contains a $k$-cycle $C$ for some integer $k \geq 3$. Since $G \not= C$, there is $v \in V(G) - V(C)$ such that $v$ is adjacent to a vertex of $C$. If $k \geq 4$, then $G$ contains $S_{2, 3}^*$ as a subgraph and so $\mathcal{P}_4(G)$ contains a triangle by Lemma 6.3.4. Hence, $k = 3$ and $C = (v_1, v_2, v_3, v_1)$ is a triangle in $G$. Since $G$ cannot contain $S_{2, 3}^*$, $S_{2, 4}$, $S_{3, 3}$ or $S(K_{1, 3})$ as a subgraph, it follows that $\deg_G v_i \leq 4$ for $i = 1, 2, 3$ and, in fact, $G = F_i$ of Figure 6.6 for some $i$ with $1 \leq i \leq 6$. Since $\mathcal{P}_4(F_1) = 2K_1$, $\mathcal{P}_4(F_2) = 2K_2$, $\mathcal{P}_4(F_3) = P_4$, $\mathcal{P}_4(F_4) = P_2 + P_3$, $\mathcal{P}_4(F_5) = C_8$ and $\mathcal{P}_4(F_6) = C_9$, it follows that $\mathcal{P}_4(G) \not= P_n$ for $n \geq 2$ and $n \not\in \{2, 4\}$, a contradiction. Thus, if $\mathcal{P}_4(G) = P_n$ where $n \geq 2$ and $n \not\in \{2, 4\}$, then $G = P_{n+3}$. \hfill $\blacksquare$

Combining Propositions 6.4.1 and 6.4.2, we have the following characterization of all graphs whose 4-path graph is a path.

**Theorem 6.4.3** If $G$ is a connected graph of order at least 4, then

* $\mathcal{P}_4(G) = P_2$ if and only if $G = P_5$ or $G = S_{2, 3}$;

* $\mathcal{P}_4(G) = P_4$ if and only if $G = P_7$ or $G$ is one of the graphs in Figure 6.5;
We have seen that $P_4(C_n) = C_n$ for each integer $n \geq 4$. However, there are graphs $G$ of order $n$ such that $G \neq C_n$ but $P_4(G) = C_n$ for some integer $n \geq 3$. In order to show this, we first present two lemmas.

**Lemma 6.4.4** Let $G$ be a connected graph that is not a cycle. If $G$ contains an $\ell$-cycle for some integer $\ell \geq 4$, then the order of $P_4(G)$ is at least $\ell + 2$ and $P_4(G)$ contains two triangles.

**Proof.** Let $C = (u_1, u_2, \ldots, u_\ell, u_1)$ be an $\ell$-cycle in $G$ where $\ell \geq 4$. Since $G$ is a connected graph that is not a cycle, there is a vertex $v \in V(G) - V(C)$ such that $v$ is adjacent to at least one vertex of $C$, say $vu_1 \in E(G)$. Thus, $G$ contains the subgraph $H$ obtained from $C$ by adding the pendant edge $vu_1$ (see Figure 6.7). Since $H$ contains both the $\ell$ distinct 4-paths $Q_i = (u_i, u_{i+1}, u_{i+2}, u_{i+3})$, where $i \in [\ell]$ and the subscripts are expressed as positive integers modulo $\ell$, and the two 4-paths $Q_{\ell+1} = (v, u_1, u_2, u_3)$ and $Q_{\ell+2} = (v, u_1, u_\ell, u_{\ell-1})$, it follows that the order of $P_4(H)$ is $\ell + 2$ and so the order of $P_4(G)$ is at least $\ell + 2$. In fact, $P_4(H)$ is the graph obtained from the cycle $(Q_1, Q_2, \ldots, Q_\ell, Q_1)$ by adding the two vertices $Q_{\ell+1}$ and $Q_{\ell+2}$ and (i) joining $Q_{\ell+1}$ to the two vertices $Q_1$ and $Q_\ell$ and (ii) joining $Q_{\ell+2}$ to the two vertices $Q_{\ell-1}$ and $Q_{\ell-2}$ (see Figure 6.7). Furthermore, $P_4(G)$ contains two triangles. 

**Lemma 6.4.5** Let $G$ be a connected graph of order at least 5. If $G$ contains a triangle but no $\ell$-cycle for any integer $\ell \geq 4$, then the order of $P_4(G)$ is at least 4.

**Proof.** Let $C = (u, v, w, v)$ be a triangle in $G$ and let $x$ and $y$ be two vertices of $G$ that are not on $C$. Since $G$ has no $\ell$-cycle for any integer $\ell \geq 4$, it follows that $G$ contains
at least one of the following graphs $F_1, F_2$ and $F_3$ of Figure 6.8 as a subgraph. Since $P_4(F_1) = 2K_2$, $P_4(F_2) = P_4$ and $P_4(F_3) = P_2 + P_3$, the order of $P_4(G)$ is at least 4. 

First, we determine all graphs whose 4-path graph is a cycle of small order, beginning with the 3-cycle.

**Proposition 6.4.6** If $G$ is a connected graph of order at least 4, then

$$P_4(G) = C_3$$ if and only if $G = S_{2,4}$ or $G = S^{*}_{2,3}$.

**Proof.** If $G = S_{2,4}$ or $G = S^{*}_{2,3}$, then $P_4(G) = K_3$ by Proposition 6.2.2 and Lemma 6.3.4. For the converse, suppose that $G$ is a connected graph such that $P_4(G) = C_3$. We show that $G = S_{2,4}$ or $G = S^{*}_{2,3}$.

First, we claim that $G$ is a tree. If this is not the case, then $G$ contains a cycle $C$. If $C$ is an $\ell$-cycle for some integer $\ell \geq 4$, then $P_4(G)$ contains $C_\ell$ as a subgraph and so $P_4(G) \neq K_3$. Hence, $C = K_3$ is a triangle and $G \neq C$. If the order of $G$ is 4, then since $G$ contains no 4-cycle, it follows that $G$ is the graph obtained from $C$ by adding a pendant edge. However then, $P_4(G) = K_2$, a contradiction. If the order of $G$ is at least 5, then $P_4(G)$ has at least four vertices by Lemma 6.4.5, a contradiction again. Thus, as claimed, $G$ is a tree. Let $V(P_4(G)) = K_3 = (x_1, x_2, x_3, x_4)$ where $x_1 = (u_1, u_2, u_3, u_4)$ is a 4-path in $G$. Suppose, without loss of generality, that $x_1, x_2$ and $x_3$ share the 3-path $(u_1, u_2, u_3)$ and that $x_2 = (u_1, u_2, u_3, a)$ for some $a \in V(G) - \{u_4\}$. Then $x_3 = (u_1, u_2, u_3, b)$ or $x_3 = (b, u_1, u_2, u_3)$ for some $b \in V(G) - \{u_4\}$ (in the latter case, $b \neq u_4$ since $G$ is a tree).
If \( x_3 = (u_1, u_2, u_2, b) \), then \( G = S_{2,4} \); while if \( x_3 = (b, u_1, u_2, u_3) \), then \( G = S_{2,3}^* \), giving the desired result.

**Proposition 6.4.7** If \( G \) is a connected graph of order at least 4, then

\[ \mathcal{P}_4(G) = C_4 \text{ if and only if } G = C_4 \text{ or } G = S_{3,3}. \]

**Proof.** If \( G = C_4 \) or \( G = S_{3,3} \), then by Proposition 6.2.2 and Theorem 6.3.5, it follows that \( \mathcal{P}_4(G) = C_4 \). For the converse, suppose that \( \mathcal{P}_4(G) = C_4 \). We show that \( G \in \{C_4, S_{3,3}\} \). We consider two cases.

**Case 1. The graph \( G \) is not a tree.** We show that \( G = C_4 \) in this case. Assume, to the contrary, that \( G \neq C_4 \). Since \( G \) is not a tree, it follows that \( G \) contains a cycle \( C \). If \( C = C_\ell \) for some integer \( \ell \geq 5 \), then \( \mathcal{P}_4(G) \) contains \( C_\ell \) as a subgraph and \( \mathcal{P}_4(G) \neq C_4 \), a contradiction. If \( C = C_4 \), then, since \( G \neq C_4 \), it follows by Lemma 6.4.4 that \( \mathcal{P}_4(G) \) has at least six vertices and so \( \mathcal{P}_4(G) \neq C_4 \), a contradiction. Hence, \( C = C_3 \). If the order of \( G \) is 4, then \( \mathcal{P}_4(G) = \overline{K}_2 \), a contradiction. If the order of \( G \) is 5, then \( G = F_i \) of Figure 6.8 for some integer \( i = 1, 2, 3 \). Since \( \mathcal{P}_4(F_i) \neq C_4 \) for \( i = 1, 2, 3 \), it follows that \( \mathcal{P}_4(G) \neq C_4 \), which is a contradiction. Thus, the order of \( G \) is at least 6 and so \( G \) contains a subgraph \( H \) that is isomorphic to some graph \( F_i \) (\( i = 1, 2, 3 \)) of Figure 6.8. Furthermore, there is a vertex \( v \) that is not in \( H \) but adjacent to some vertex of \( H \). However then, there are at least two distinct 4-paths in \( G \) containing \( v \). This implies that the order of \( \mathcal{P}_4(G) \) is at least 6 and so \( \mathcal{P}_4(G) \neq C_4 \), which is a contradiction.

**Case 2. The graph \( G \) is a tree.** We show that \( G = S_{3,3} \) in this case. Suppose that \( \mathcal{P}_4(G) = C_4 = (x_1, x_2, x_3, x_4, x_1) \), where, say, \( x_1 = (u_1, u_2, u_3, u_4) \) is a 4-path in \( G \). We may assume, without loss of generality, that \( x_2 \) contains the 3-path \( (u_1, u_2, u_3) \) and that \( x_4 \) contains the 3-path \( (u_2, u_3, u_4) \). Thus, either \( x_2 = (u_1, u_2, u_3, a) \) or \( x_2 = (a, u_1, u_2, u_3) \) for some vertex \( a \). Since \( G \) is a tree and \( x_2 \neq x_1 \), it follows that \( a \notin U = \{u_1, u_2, u_3, u_4\} \).

We consider these two subcases.

**Subcase 2.1.** \( x_2 = (u_1, u_2, u_3, a) \), where \( a \notin U \). We now consider \( x_3 \). Since \( x_1 x_3 \notin E(\mathcal{P}_4(G)) \), it follows that \( x_3 \) contains neither \( (u_1, u_2, u_3) \) nor \( (u_2, u_3, u_4) \). On the other hand, because \( x_2 x_3 \in E(\mathcal{P}_4(G)) \), we conclude that \( x_3 \) must contain \( (u_2, u_3, a) \). Hence, either \( x_3 = (u_2, u_3, a, b) \) or \( x_3 = (b, u_2, u_3, a) \) for some vertex \( b \). Since \( G \) is not a tree and \( x_3 \) does not contain \( (u_1, u_2, u_3) \), it follows that \( b \notin U \). Therefore, \( a, b \notin U \).

* First, suppose that \( x_3 = (u_2, u_3, a, b) \) where \( a, b \notin U \). We now consider \( x_4 \). Since \( x_4 \) is adjacent to \( x_3 \), it follows that \( x_4 \) contains either \( (u_2, u_3, a) \) or \( (u_3, a, b) \). However,
$x_4$ also contains $(u_2, u_3, u_4)$ and $\{a, b\} \cap U = \emptyset$, which is impossible.

* Next, suppose that $x_3 = (b, u_2, u_3, a)$. This implies that $x_4 = (b, u_2, u_3, a)$. Hence, $G = S_{3,3}$ whose central vertices are $u_2$ and $u_3$.

Subcase 2.2. $x_2 = (a, u_1, u_2, u_3)$ where $a \notin U$. Since $x_3$ is adjacent to $x_2$ but not adjacent to $x_1$, it follows that $x_3 = (a, u_1, u_2, b)$ or $x_3 = (b, a, u_1, u_2)$ for some $b \notin U$. In either case, this is impossible since $x_4$ contains $(u_2, u_3, u_4)$ and $x_4$ is adjacent to $x_3$. ■

We have now seen that there are two connected graphs $G$ for which $\mathcal{P}_4(G) = C_3$ or for which $\mathcal{P}_4(G) = C_4$. We now show that this is also the case for $C_6, C_8$ and $C_9$. Later we will see that such is not the case for all other cycles. First, we present an additional definition. The corona $\text{cor}(H)$ of a graph $H$ is the graph obtained from $H$ by attaching a pendant edge to each vertex of $H$. Thus, if $H$ has order $n$, then the corona $\text{cor}(H)$ has order $2n$ and has precisely $n$ leaves.

**Proposition 6.4.8** If $G$ is a connected graph of order at least 4, then

1. $\mathcal{P}_4(G) = C_6$ if and only if $G \in \{C_6, S(K_{1,3})\}$;
2. $\mathcal{P}_4(G) = C_8$ if and only if $G \in \{C_8, 2K_2 \lor K_1\}$;
3. $\mathcal{P}_4(G) = C_9$ if and only if $G \in \{C_9, \text{cor}(K_3)\}$.

**Proof.** By Proposition 6.1.2, $\mathcal{P}_4(C_n) = C_n$ for each integer $n \geq 4$. Moreover,

$$\mathcal{P}_4(S(K_{1,3})) = C_6, \mathcal{P}_4(2K_2 \lor K_1) = C_8 \text{ and } \mathcal{P}_4(\text{cor}(K_3)) = C_9.$$ 

Therefore, it remains to verify the converse. Let $G$ be a graph such that $G \neq C_n$ for $n \in \{6, 8, 9\}$ and $G \notin \{S(K_{1,3}), 2K_2 \lor K_1, \text{cor}(K_3)\}$. We show that $\mathcal{P}_4(G) \neq C_n$ for $n \in \{6, 8, 9\}$. Assume, to the contrary, that $\mathcal{P}_4(G) = C_n$ for some $n \in \{6, 8, 9\}$. We consider two cases.

Case 1. The graph $G$ is not a tree. Then $G$ contains a $k$-cycle $C$ for some integer $k \geq 3$. Since $G \neq C$, there is a vertex $v \in V(G) - V(C)$ such that $v$ is adjacent to a vertex of $C$. If $k \geq 4$, then $G$ contains $S_{2,3}^{k}$ as a subgraph and so $\mathcal{P}_4(G)$ contains a triangle by Lemma 6.4.6, a contradiction. Hence, $k = 3$ and $C$ is a triangle in $G$. Since $G$ contains at least six 4-paths and does not contain any of $S_{2,3}^{k}, S_{2,3}^{k}$ and $S_{3,3}$ as a subgraph, it follows that $G \not\in \{S(K_{1,3}), 2K_2 \lor K_1, \text{cor}(K_3)\}$, which contradicts the assumption. Therefore, $\mathcal{P}_4(G) \neq C_n$ for $n = 6, 8, 9$. 

107
Case 2. The graph $G$ is a tree. Since $P_4(G) = C_n$ where $n \in \{6, 8, 9\}$, it follows that $P_4(G)$ contains the path $P_{n-1}$ as an induced subgraph. By Theorem 6.4.9, the graph $G$ contains $P_{n+2}$ as a subgraph. Since $n + 2 \geq 8$ and $G$ is not a path, $G$ contains a vertex of degree at least 3. However then, $G$ contains $S^*_2$ as a subgraph and so $P_4(G)$ contains a triangle, a contradiction. 

Proof. Let $n \geq 3$ and $n \notin \{3, 4, 6, 8, 9\}$. Since $P_4(C_n) = C_n$, it remains to verify the converse. Suppose that $G \neq C_n$. We show that $P_4(G) \neq C_n$. Assume, to the contrary, that $P_4(G) = C_n$. We consider two cases, according to whether $G$ is a tree.

Case 1. $G$ is a tree. By Proposition 6.2.1, we may assume that $G$ is not a path and so $\Delta(G) = \Delta \geq 3$. Let $v \in V(G)$ such that $\deg_G v = \Delta$. Since $G$ is not a star, some neighbor of $v$ has degree at least 2 and so $G$ contains $S_{2,\Delta}$ as a subgraph. If $\Delta \geq 4$, then $P_4(S_{2,\Delta}) = K_{\Delta-1}$ by Proposition 6.2.2 and so $P_4(G)$ contains a triangle. Hence, $P_4(G) \neq C_n$ for $n \geq 4$, a contradiction. Thus, $\Delta = 3$. If $G$ contains either $S^*_2$ or $S_{3,3}$ as a subgraph, then $P_4(G)$ contains $C_3$ or $C_4$ by Propositions 6.2.2 and Lemma 6.3.4, respectively. In either case, $P_4(G) \neq C_n$ for any integer $n \geq 5$ and $n \notin \{3, 4, 6, 8, 9\}$, a contradiction. Hence, $G$ contains neither $S^*_2$ nor $S_{3,3}$ as a subgraph. Thus, each neighbor of $v$ has degree at most 2. For $i = 1, 2, 3$, let $T_i$ be the tree obtained by subdividing exactly $i$ edges of $K_{1,3}$ exactly once. In particular, $T_1 = S_{2,3}$ and $T_3 = S(K_{1,3})$. Since $G$ cannot contain $S^*_2$ or $S_{3,3}$ as a subgraph, it follows that $G$ must be one of the trees $T_1, T_2, T_3$. Since $P_4(T_1) = K_2$, $P_4(T_2) = P_4$ and $P_4(T_3) = C_6$, it follows that $P_4(T_i) \neq C_n$ ($i = 1, 2, 3$) for any integer $n \geq 5$ and $n \notin \{3, 4, 6, 8, 9\}$, which is a contradiction.

![Figure 6.9: The graphs $T_1, T_2$ and $T_3$](image)

Case 2. $G$ is not a tree. Thus, $G$ contains a $k$-cycle $C$ for some integer $k \geq 3$. Since $G \neq C$, there is $v \in V(G) - V(C)$ such that $v$ is adjacent to a vertex of $C$. If $k \geq 4$, then $G$ contains $S^*_2$ as a subgraph and so $P_4(G)$ contains a triangle by Lemma 6.3.4.
Hence, \( k = 3 \) and \( C = (v_1, v_2, v_3, v_1) \) is a triangle in \( G \). Since \( G \) cannot contain \( S_{2,3}^*, S_{2,4} \) or \( S_{3,3} \) as a subgraph, it follows that \( \deg_G v_i \leq 4 \) for \( i = 1, 2, 3 \) and, in fact, \( G = F_i \) of Figure 6.6 for some \( i \) with \( 1 \leq i \leq 6 \).

Since \( \mathcal{P}_4(F_1) = 2K_1, \mathcal{P}_4(F_2) = 2K_2, \mathcal{P}_4(F_3) = P_4, \mathcal{P}_4(F_4) = P_2 + P_3, \mathcal{P}_4(F_5) = C_8 \) and \( \mathcal{P}_4(F_6) = C_9 \) (see Figure 6.10), it follows that \( \mathcal{P}_4(G) \neq C_n \) for \( n \geq 3 \) and \( n \notin \{3, 4, 6, 8, 9\} \), which is a contradiction.

Figure 6.10: Showing that \( \mathcal{P}_4(F_5) = C_8 \) and \( \mathcal{P}_4(F_6) = C_9 \).

Combining Propositions 6.4.6, 6.4.7, 6.4.8 and 6.4.9, we have the following characterization of all graphs whose 4-path graph is a cycle.

**Theorem 6.4.10** If \( G \) is a connected graph of order at least 4 and \( n \geq 3 \), then

* \( \mathcal{P}_4(G) = C_3 \) if and only if \( G \in \{S_{2,4}, S_{2,3}^*\} \);
* \( \mathcal{P}_4(G) = C_4 \) if and only if \( G \in \{C_4, S_{3,3}\} \);
* \( \mathcal{P}_4(G) = C_6 \) if and only if \( G \in \{C_6, S(K_{1,3})\} \);
* \( \mathcal{P}_4(G) = C_8 \) if and only if \( G \in \{C_8, 2K_2 \vee K_1\} \);
* \( \mathcal{P}_4(G) = C_9 \) if and only if \( G \in \{C_9, \text{cor}(K_3)\} \);
* for \( n \notin \{3, 4, 6, 8, 9\} \), then \( \mathcal{P}_4(G) = C_n \) if and only if \( G = C_n \).

Theorem 6.4.10 suggests the following question.

**Problem 6.4.11** Does there exist some connected graph \( H \) and three non-isomorphic connected graphs \( H_1, H_2 \) and \( H_3 \) for which \( \mathcal{P}_4(H_i) \cong H \) for \( i = 1, 2, 3 \)?
6.5 Graphs Having Connected 4-Path Graphs

Our primary goal in this section is to characterize those connected graphs \( G \) for which \( \mathcal{P}_4(G) \) is connected. First, we establish some preliminary results.

**Proposition 6.5.1** If \( T \) is a tree of order \( n \geq 4 \) with \( \text{diam}(T) \geq 3 \), then \( \mathcal{P}_4(T) \) is connected.

**Proof.** We proceed by induction on the order \( n \geq 4 \) of a tree. Since \( \mathcal{P}_4(P_4) = K_1 \) is connected, the statement is true for \( n = 4 \). Assume that if \( T' \) is a tree of order \( n \geq 4 \) and \( \text{diam}(T') \geq 3 \), then \( \mathcal{P}_4(T') \) is connected. Let \( T \) be a tree of order \( n + 1 \) and \( \text{diam}(T) = d \geq 3 \). We show that \( \mathcal{P}_4(T) \) is connected. If \( d = 3 \), then \( T \) is a double star and so \( \mathcal{P}_4(T) \) is connected. Thus, we may assume that \( d \geq 4 \). Let \( v \) be a peripheral vertex of \( T \). Then \( v \) is an end-vertex of \( T \) and so \( T' = T - v \) is a tree of order \( n \). Since \( \text{diam}(T') \geq d - 1 \geq 3 \), it follows by the induction hypothesis that \( \mathcal{P}_4(T') \) is connected. Suppose that \( v \) is adjacent to the vertex \( u \) of \( T \). Let \( P = (v, u, u_1, u_2) \) be a 4-path in \( T \). Thus, \( P \) is not in \( T' \). We show that \( P \) is connected to a vertex of \( \mathcal{P}_4(T') \) and possibly \( P \) is adjacent to a vertex of \( \mathcal{P}_4(T') \). Since \( T \neq P \) and \( v \) is an end-vertex of \( T \), there is a vertex \( w \in V(T) - V(P) \) that is adjacent to one of the vertices \( u, u_1 \) and \( u_2 \). If \( w \) is adjacent to \( u \), then \( P \) is adjacent to the vertex \( (w, u, u_1, u_2) \) of \( \mathcal{P}_4(T') \). If \( w \) is adjacent to \( u_2 \), then \( P \) is adjacent to the vertex \( (u, u_1, u_2, w) \) of \( \mathcal{P}_4(T') \). Thus, we may assume that \( w \) is adjacent to \( u_1 \). Furthermore, we may also assume that \( \deg_T u = 2 \) and \( \deg_T u_2 = 1 \). Since \( T \) is not a double star, it follows that \( w \) is adjacent to \( u' \) in \( T' \). Let \( Q_1 = (v, u, u_1, w) \) and \( Q_2 = (u, u_1, w, w') \). Then \( Q_2 \) is a vertex of \( \mathcal{P}_4(T') \). Since \( (P, Q_1, Q_2) \) is a path in \( \mathcal{P}_4(T) \), it follows that \( P \) is connected to a vertex of \( \mathcal{P}_4(T') \). Therefore, \( \mathcal{P}_4(T) \) is connected. ■

**Proposition 6.5.2** Let \( G \) be a connected graph containing 4-paths. If \( G \) is triangle-free, then \( \mathcal{P}_4(G) \) is connected.

**Proof.** Let \( Q = (u_1, u_2, u_3, u_4) \) and \( Q' = (v_1, v_2, v_3, v_4) \) be two distinct 4-paths of \( G \). We show that \( Q \) and \( Q' \) are connected in \( \mathcal{P}_4(G) \). Let \( H = G[E(Q) \cup E(Q')] \) be the subgraph induced by the edge sets of \( Q \) and \( Q' \). Thus, \( H \) is a graph of order at most 8 and size at most 6. We consider two cases, according to whether \( H \) is acyclic.

Case 1. \( H \) is acyclic. If \( H \) is connected, then \( H \) is a tree. Since \( \mathcal{P}_4(H) \) is connected by Proposition 6.5.1, it follows that \( Q \) and \( Q' \) are connected in \( \mathcal{P}_4(H) \) and so in \( \mathcal{P}_4(G) \). If \( H \) is disconnected, then \( H = Q + Q' \). Let \( Q'' \) be the shortest path joining a vertex
of $Q$ and a vertex of $Q'$ in $G$. Let $F = G[E(Q) \cup E(Q') \cup E(Q'')]$ be the subgraph induced by the edge sets of $Q, Q'$ and $Q''$. Then $F$ is a tree. Again, $\mathcal{P}_4(F)$ is connected by Proposition 6.5.1. Therefore, $Q$ and $Q'$ are connected in $\mathcal{P}_4(F)$ as well as in $\mathcal{P}_4(G)$.

Case 2. $H$ contains cycles. Thus, $H$ is connected. This implies that $Q$ and $Q'$ have at least two vertices in common. Let $H$ have order $n$ and size $m$ where $4 \leq n, m \leq 6$. We show that $\mathcal{P}_4(H)$ is connected. Thus, $Q$ and $Q'$ are connected in $\mathcal{P}_4(H)$ and so in $\mathcal{P}_4(G)$. Let $C_\ell$ be a largest cycle in $H$, where then $\ell \in \{4, 5, 6\}$.

* If $\ell = 4$, then there are seven possible non-isomorphic graphs $H_1, H_2, \ldots, H_7$ for $H$. To see this, let $C_4 = (x_1, x_2, x_3, x_4, x_1)$. Then $H_1 = C_4$, $H_2$ is the graph of order 5 obtained from $C_4$ by adding a pendant edge at $x_1$, $H_3$ is the graph of order 6 obtained from $C_4$ by adding two pendant edges at $x_1$, $H_4$ is the graph of order 6 obtained from $C_4$ by adding a vertex adjacent to both $x_1$ and $x_3$, that is $H_4 = K_{2, 3}$, $H_5$ is the graph of order 6 obtained from $C_4$ by adding a pendant edge at $x_1$ and a pendant edge at $x_2$, $H_6$ is the graph of order 6 obtained from $C_4$ by adding a pendant edge at $x_1$ and a pendant edge at $x_3$ and $H_7$ is the graph of order 6 obtained from $C_4$ and $P_3$ by identifying an end-vertex of $P_3$ with $x_1$. In each case, $\mathcal{P}_4(H)$ is connected.

* If $\ell = 5$, then $H = C_5$ and or $H$ is the graph obtained from $C_5$ by adding a pendant edge. Hence, $\mathcal{P}_4(H)$ is connected.

* If $\ell = 6$, then $H = C_6$ and $\mathcal{P}_4(H)$ is connected.

We will see that the main result of this section concerns a particular class of unicyclic graphs. For integers $a$ and $b$ with $0 \leq a \leq b$ and $b \geq 1$, let $T_{a, b}$ be the graph obtained from the triangle $(u, v, w, u)$ by adding $a$ pendant edges at $u$ and $b$ pendant edges at $v$.

**Lemma 6.5.3** For integers $a$ and $b$ with $0 \leq a \leq b$ and $b \geq 1$, the 4-path graph $\mathcal{P}_4(T_{a, b})$ of the graph $T_{a, b}$ is disconnected with two nontrivial components.

**Proof.** In the graph $T_{a, b}$, let $u_1, u_2, \ldots, u_a$ be the end-vertices adjacent to $u$ (if $a \geq 1$) and let $v_1, v_2, \ldots, v_b$ be the end-vertices adjacent to $v$. See Figure 6.11.

We consider two cases, according to whether $a = 0$ or $a \geq 1$.

Case 1. $a = 0$. Then $\text{deg}_{T_{a, b}} u = 2$. Let $T_1$ be the spanning tree obtained by removing the edge $uv$ from $T_{0, b}$. Thus, $T_1 = S_{2, b+1}$ is the subtree of $T_{0, b}$ whose central vertices are $w$ and $v$ and whose end-vertices are $u, v_1, v_2, \ldots, v_b$. Let $T_2$ be the spanning
tree obtained by removing the edge $wv$ from $T_{0,b}$. Thus, $T_2 = S_{2,b+1}$ is the sub-tree of $T_{0,b}$ whose central vertices are $u$ and $v$ and whose end-vertices are $w, v_1, v_2, \ldots, v_b$. See Figure 6.12. Hence, $T_1 \cong T_2$. Then $\mathcal{P}_4(T_1) \cong \mathcal{P}_4(T_2) \cong K_b$. Every 4-path in $T_{0,b}$ is either a 4-path in $T_1$ or a 4-path in $T_2$. If $Q$ is a 4-path of $T_1$ and $Q'$ is a 4-path of $T_2$, then $Q = (u, w, v, v_i)$ and $Q' = (w, u, v, v_j)$ where $1 \leq i, j \leq b$. Since $Q$ and $Q'$ have no 3-path in common, it follows that $Q$ and $Q'$ are not adjacent in $\mathcal{P}_4(T_{0,b})$. Therefore, $\mathcal{P}_4(T_{0,b}) = \mathcal{P}_4(T_1) + \mathcal{P}_4(T_2) = 2K_b$.

\begin{figure}[h]
\centering
\includegraphics{image.png}
\caption{The subtrees $T_1$ and $T_2$ in Case 1.}
\end{figure}

**Case 2.** $a \geq 1$. First, we consider four sub-trees $T'_1, T'_2, T'_3, T'_4$ of $T_{a,b}$. See Figure 6.13.

* Let $T'_1 \cong S^*_{a+1,b+1}$, where $(u, w, v)$ is a 3-path in $T'_1$ and $u_1, u_2, \ldots, u_a, v_1, v_2, \ldots, v_b$ are end-vertices of $T'_1$. Then $\mathcal{P}_4(T'_1) = \mathcal{P}_4(S^*_{a+1,b+1}) = K_{a+b}$.

* Let $T'_2 \cong S_{a+1,b+1}$ whose central vertices are $u$ and $v$ and whose end-vertices are $u_1, u_2, \ldots, u_a, v_1, v_2, \ldots, v_b$. Then $\mathcal{P}_4(T'_2) = \mathcal{P}_4(S_{a+1,b+1}) = K_a \square K_b$.

* Let $T'_3 \cong S_{2,a+1}$ whose central vertices are $u$ and $v$ and whose end-vertices are $w, u_1, u_2, \ldots, u_a$. Then $\mathcal{P}_4(T'_3) = K_a$.

* Let $T'_4 \cong S_{2,b+1}$ whose central vertices are $u$ and $v$ and whose end-vertices are $w, v_1, v_2, \ldots, v_b$. Then $\mathcal{P}_4(T'_4) = K_b$.

In $T_{a,b}$, every 4-path belongs to exactly one of the sub-trees $T'_i$ ($1 \leq i \leq 4$). Furthermore, if $Q$ is a 4-path of $T'_i$ and $Q'$ is a 4-path of $T'_i$ where $i = 2, 3, 4$, then $Q$ and $Q'$ do not have any 3-path in common and so $Q$ and $Q'$ are not adjacent in $\mathcal{P}_4(T_{a,b})$. Hence, $\mathcal{F}_1 = \mathcal{P}_4(T'_1) = K_{a+b}$ is a non-trivial component of $\mathcal{P}_4(T_{a,b})$. Next, let $H_i = \mathcal{P}_4(T'_i)$ for $i = 1, 2, 3$, where then $H_1 = K_a \square K_b$, $H_2 = K_a$ and $H_3 = K_b$. Notice that (i) every
vertex of $H_1$ belonging to the subgraph $K_a$ of $H_1$ is adjacent to every vertex of $H_2 = K_a$ and (ii) every vertex of $H_1$ belonging to the subgraph $K_b$ of $H_1$ is adjacent to every vertex of $H_3 = K_b$. Thus, $H_1, H_2$ and $H_3$ give rise to another nontrivial component $F_2$ of $P_4(T_{a,b})$. Therefore, $P_4(T_{a,b}) = F_1 + F_2$ consisting of two nontrivial components $F_1$ and $F_2$, as desired.

Lemma 6.5.4 For integers $a$ and $b$ with $0 \leq a \leq b$ and $b \geq 1$, let $T_{a,b}^*$ be the graph obtained from $T_{a,b}$ by subdividing exactly one pendant edge of $T_{a,b}$ exactly once. Then $P_4(T_{a,b}^*)$ is connected.

Proof. Label the vertices of $T_{a,b}$ as described in the proof of Lemma 6.5.3 (see Figure 6.11). Since $T_{a,b}^*$ is the graph obtained from $T_{a,b}$ by subdividing exactly one pendant edge of $T_{a,b}$ exactly once, we can also assume that $T_{a,b}^*$ is obtained from $T_{a,b}$ by adding a new vertex $z$ and joining $z$ to an end-vertex of $T_{a,b}$. We consider two cases, according to whether $a = 0$ or $a \geq 1$.

Case 1. $a = 0$. We may assume that $z$ is adjacent to $v_1$. Here, we use the notation of the proof in Case 1 of Lemma 6.5.3. The graph $P_4(T_{0,b})$ consists of two nontrivial components $F_1$ and $F_2$. Certainly, each 4-path of $T_{0,b}$ is a 4-path of $T_{0,b}^*$. Moreover, $T_{0,b}^*$ contains the 4-paths $(z, v_1, v, w)$, $(z, v, v_1, w)$, $(z, v_1, v, v_i)$ for $i = 2, 3, \ldots, b$ (if $b \geq 2$), which are all mutually adjacent and so form a complete subgraph in $P_4(T_{0,b}^*)$. Furthermore, the 4-path $(z, v_1, v, w)$ in $P_4(T_{0,b}^*)$ is adjacent to $(v_1, v, u, w)$ in the component $P_4(T_1)$ of $P_4(T_{0,b})$ and $(z, v_1, v, w)$ in $P_4(T_{0,b}^*)$ is adjacent to $(v_1, v, u, w)$ in the component $P_4(T_2)$ of $P_4(T_{0,b})$. Therefore, $P_4(T_{0,b}^*)$ is connected.

Case 2. $a \geq 1$. We may assume that $z$ is adjacent to $u_1$ or $v_1$. Here, we use the notation of the proof in Case 2 of Lemma 6.5.3. The graph $P_4(T_{a,b})$ consists two nontrivial

Figure 6.13: The graphs $T_i'$ for $i = 1, 2, 3, 4$ in Case 2.
components \( F_1 \) and \( F_2 \). First, suppose that \( z \) is adjacent to \( u_1 \). By Lemma 6.5.3, in addition to the 4-paths of \( T_{a,b} \), the graph \( T_{a,b}^* \) contains the 4-paths \((z, u_1, u, v), (z, u_1, u, w)\) and \((z, u_1, u, u_j)\) for \( j = 2, 3, \ldots, a \) (if \( a \geq 2 \)) which are all mutually adjacent. Furthermore, the 4-path \((z, u_1, u, w)\) is adjacent to \((u_1, u, w, v)\) in \( F_1 \) and \((z, u_1, u, v)\) is adjacent to \((u_1, u, v, v_1)\) in \( F_2 \). Hence, \( P_4(T_{a,b}^*) \) is connected. Next, suppose that \( z \) is adjacent to \( v_1 \). In addition to the 4-paths of \( T_{a,b} \), the graph \( T_{a,b}^* \) contains the 4-paths \((z, v_1, v, u), (z, v_1, v, w)\) and \((z, v_1, v, v_1)\) for \( i = 2, 3, \ldots, b \) which are all adjacent. Observe that the 4-path \((z, v_1, v, w)\) is adjacent to \((v_1, v, w, u)\) in \( F_1 \) and \((z, v_1, v, u)\) is adjacent to \((v_1, v, u, u_1)\) in \( F_2 \). Hence, \( P_4(T_{a,b}^*) \) is connected. 

We are now prepared to present the following characterization of those connected graphs whose 4-path graph is disconnected.

**Theorem 6.5.5** Let \( G \) be a connected graph containing 4-paths. Then \( P_4(G) \) is disconnected if and only if \( G = T_{a,b} \) for some integers \( a \) and \( b \) with \( 0 \leq a \leq b \) and \( b \geq 1 \).

**Proof.** If \( G = T_{a,b} \) for \( 0 \leq a \leq b \) and \( b \geq 1 \), then \( P_4(T_{a,b}) \) is disconnected by Lemma 6.5.3. For the converse, let \( G \) be a connected graph containing 4-paths such that \( G \neq T_{a,b} \) for all integers \( a \) and \( b \) with \( 0 \leq a \leq b \) and \( b \geq 1 \). We show that \( P_4(G) \) is connected. Let \( Q = (u_1, u_2, u_3, u_4) \) and \( Q' = (v_1, v_2, v_3, v_4) \) be two distinct 4-paths of \( G \). We show that \( Q \) and \( Q' \) are connected in \( P_4(G) \).

Let \( H = G[E(Q) \cup E(Q')] \) be the subgraph induced by the edge sets of \( Q \) and \( Q' \). Thus, \( H \) is a graph of order at most 8 and size at most 6. If \( H \) is acyclic, then the proof in Case 1 of Proposition 6.5.2 shows that \( Q \) and \( Q' \) are connected in \( P_4(G) \). Thus, we may assume that \( H \) contains cycles. This implies that \( Q \) and \( Q' \) have at least two vertices in common. Hence, \( H \) is a connected graph of order \( n \) and size \( m \) where \( 4 \leq n, m \leq 6 \). If \( H \) is triangle-free, then \( P_4(H) \) is connected by Proposition 6.5.2. Hence, \( Q \) and \( Q' \) are connected in \( P_4(H) \), as well as in \( P_4(G) \). Thus, we may assume that \( H \) contains a triangle.

Since \( H \) is a connected graph of order \( n \) and size \( m \) where \( 4 \leq n, m \leq 6 \) containing at least one triangle, there are eleven possible non-isomorphic graphs \( H_1, H_2, \ldots, H_{11} \) for \( H \). See Figure 6.14. To see this, let \( C_3 = (x_1, x_2, x_3, x_1) \). Then \( H_1 \) is the graph of order 4 obtained from \( C_3 \) by adding a pendant edge at \( x_1 \), \( H_2 = K_4 - e \), \( H_3 \) is the graph of order 5 obtained from \( C_3 \) by adding two pendant edges at \( x_1 \), \( H_4 \) is the graph of order 5 obtained from \( K_4 - e \) by adding a pendant edge at a vertex of \( K_4 - e \) of degree 2, \( H_5 \) is the graph of order 5 obtained from \( K_4 - e \) by adding a pendant edge at a vertex
of $K_4 - e$ of degree 3, $H_6$ is the graph of order 5 obtained from $C_3$ by adding a pendant edge at $x_1$ and a pendant edge at $x_2$, $H_7$ is the graph of order 5 obtained from $C_4$ by adding a vertex to be adjacent to two adjacent vertices of $C_4$, $H_8$ is the graph of order 6 obtained from $C_3$ by adding a pendant edge at $x_1$ and a pendant edge at $x_2$, $H_9$ is the graph of order 6 obtained from $C_3$ by adding two pendant edges at $x_1$ and a pendant edge at $x_2$, $H_{10}$ is the graph of order 6 obtained from $C_3$ by adding a pendant edge at $x_1$ and joining a vertex of a $K_2$ to $x_2$ (or $H_{10}$ is obtained by adding a pendant edge at an end-vertex of $H_6$) and $H_{11}$ is the bowtie graph.

It can be verified that if $H \neq T_{a,b}$, then $P_4(H)$ is connected. Thus, we may assume $H = T_{a,b}$ for some integers $a$ and $b$ where $0 \leq a \leq b$ and $b \geq 1$. Since $G \neq T_{a,b}$ and $G$ is connected, it follows that there is a vertex $z \in V(G) - V(H)$ that is adjacent to some vertex of $H$. If $z$ is adjacent to an end-vertex of $H = T_{a,b}$, then $H^* = T_{a,b}^*$. It then follows by Lemma 6.5.4 that $P_4(H^*)$ is connected. Thus, $Q$ and $Q'$ are connected in $P_4(H^*)$ and in $P_4(G)$ as well. If $z$ is not adjacent to an end-vertex of $H$, then $H$ is the corona $\text{cor}(C_3)$ of $C_3$ and so $P_4(F)$ is connected. Hence, $Q$ and $Q'$ are connected in $P_4(F)$ as well as in $P_4(G)$. Therefore, $P_4(G)$ is connected.

Stating Theorem 6.5.5 in its contrapositive, we have the following characterization of those connected graphs whose 4-path graph is connected, which is equivalent to Theorem 6.5.5.

**Corollary 6.5.6** Let $G$ be a connected graph containing 4-paths. Then $P_4(G)$ is connected if and only if $G \neq T_{a,b}$ for any pair $a, b$ of integers with $0 \leq a \leq b$ and $b \geq 1$. 

115
Next, we characterize all connected graphs whose 4-path graph is a connected bipartite graph. The star $K_{1,3}$ is also referred to as a claw. A graph is said to be claw-free if it does not contain $K_{1,3}$ as an induced subgraph.

**Proposition 6.5.7** For every connected graph $G$ having 4-paths, the 4-path graph $P_4(G)$ of $G$ is claw-free. Furthermore, if $P_4(G)$ is a connected bipartite graph, then $P_4(G)$ is either a path or an even cycle.

**Proof.** Assume, to the contrary, that there is a connected graph $G$ whose 4-path graph $H = P_4(G)$ contains $K_{1,3}$ as an induced subgraph. Let $V(K_{1,3}) = \{x, x_1, x_2, x_3\}$ in $H$, where $x$ is the central vertex of $K_{1,3}$. Suppose that $x = (u_1, u_2, u_3, u_4)$ is a 4-path in $G$. Since $x$ is adjacent to $x_1, x_2, x_3$ in $H$, it follows that each of $x_1, x_2, x_3$ must contain the 3-path $(u_1, u_2, u_3)$ or the 3-path $(u_2, u_3, u_4)$. Hence, at least two of $x_1, x_2$ and $x_3$, say $x_1$ and $x_2$, must contain the same 3-path in $x$. However then, $x_1x_2 \in E(H)$, which is a contradiction. Hence, $P_4(G)$ is claw-free. Furthermore, if $P_4(G)$ is a connected bipartite graph that is neither a path nor a cycle, then $\Delta(P_4(G)) \geq 3$. However then, $P_4(G)$ contains $K_{1,3}$ as an induced subgraph, which is impossible.

Proposition 6.2.1, Theorems 6.4.3, 6.4.10 and 6.5.5 and Proposition 6.5.7 give rise to the following characterization of those connected graphs whose 4-path graph is a connected bipartite graph.

**Corollary 6.5.8** Let $G$ be a connected graph of order $n \geq 4$ containing 4-paths. Then $P_4(G)$ is a connected bipartite graph if and only if $P_4(G)$ is either a path or an even cycle. Furthermore, $P_4(G)$ is a connected bipartite graph if and only if

(i) $G = P_n$ for some integer $n \geq 4$ or $G \in \{S_{2,3}, T_0\}$ where $T_0$ is the tree obtained from a 5-path $P_5$ by adding a pendant edge at the central vertex of $P_5$, in which case $P_4(G)$ is a path, or

(ii) $G = C_n$ for some even integer $n \geq 4$ or

$$G \in \{S_{2,4}, S_{2,3}^*, S_{3,3}, S(K_{1,3}), 2K_2 \vee K_1, \text{cor}(K_3)\},$$

in which case $P_4(G)$ is an even cycle.
Chapter 7

Topics for Further Study

In this chapter, we present concepts and topics for further study on $k$-path graphs for integers $k \geq 4$. Again, our emphasis will be on 4-path graphs. We present some preliminary results on these topics as well as some open questions and conjectures.

7.1 Which Graphs are 4-Path Graphs?

Recall that a graph $G$ is called a line graph if there exists a graph $H$ such that $G = L(H)$ and the following characterization due to Beineke [3].

**Theorem 7.1.1** [3] A graph $G$ is a line graph if and only if none of the nine graphs of Figure 7.1 is isomorphic to an induced subgraph of $G$.

![Figure 7.1: The induced subgraphs not contained in any line graph](image)
A graph $G$ is called a 4-path graph if there exists a graph $H$ such that $G = \mathcal{P}_4(H)$. By Proposition 6.2.1, every nontrivial path and every cycle of order at least 3 is a 4-path graph. A natural question to ask is whether a given graph is a 4-line graph. By Proposition 6.2.2, every complete graph is a 4-path graph.

**Proposition 7.1.2** If $G$ is a complete graph, a path or a cycle of order at least 3, then $G$ is a 4-path graph.

There are other connected graphs that are 4-path graphs. For example, for each integer $n \geq 4$, let $G_n$ be the graph of order $n + 2$ that is obtained from the $n$-cycle $(x_1, x_2, \ldots, x_n, x_1)$ by adding two new vertices $y$ and $z$ and joining (i) $y$ to $x_1$ and $x_2$ and (ii) $z$ to $x_3$ and $x_4$. Then $G_n$ is a 4-path graph. To see this, let $L_n$ be the unicycle graph obtained from the $n$-cycle by adding a pendant edge. Then $\mathcal{P}_4(L_n) = G_n$.

![Figure 7.2: An example of 4-path graph](image)

Next, we present a class of connected graphs that are 4-path graphs. For positive integers $a$ and $b$, let $S_{a+1,b+1}$ be the double star of order $a + b + 2$ whose central vertices $u$ and $v$ have degree $a + 1$ and $b + 1$ respectively and let $S_{a+1,b+1}^{**}$ be the tree obtained from $S_{a+1,b+1}$ by subdividing the non-pendant edge $uv$ exactly twice.

**Proposition 7.1.3** For positive integers $a$ and $b$, $\mathcal{P}_4(S_{a+1,b+1}^{**}) = (K_a + K_b) \lor K_1$.

**Proof.** Suppose that $u$ and $v$ are the central vertices of $F = S_{a+1,b+1}$, where $u$ is adjacent to the $a$ end-vertices $u_1, u_2, \ldots, u_a$ and $v$ is adjacent to the $a$ end-vertices $v_1, v_2, \ldots, v_b$. Let $G = S_{a+1,b+1}^{**}$ be obtained from $F$ by inserting the two vertices $x$ and $y$ of degree 2 to the edge $uv$ of $F$. Then $G$ has exactly $a + b + 1$ distinct 4-paths, namely $(u, x, y, v), (u_i, u, x, y)$ and $(x, y, v, v_j)$ where $1 \leq i \leq a$ and $1 \leq j \leq b$. Observe that

(i) the 4-path $(u, x, y, v)$ is adjacent to every other 4-path in $G$,

(ii) every pair of 4-paths $(u_i, u, x, y)$ (1 \(\leq i \leq a\)) of $G$ are adjacent in $\mathcal{P}_4(G)$ and every pair of 4-paths $(x, y, v, v_j)$ (1 \(\leq j \leq b\)) of $G$ are adjacent in $\mathcal{P}_4(G)$ and
(iii) no 4-path \((u_i, u, x, y)\) is adjacent to any 4-path \((x, y, v, v_j)\) for each pair \(i, j\) of integers with \(1 \leq i \leq a\) and \(1 \leq j \leq b\).

Hence, \(P_4(G) = (K_a + K_b) \lor K_1\) and the vertex \((u, x, y, v)\) in \(K_1\) is the cut-vertex of \(P_4(G)\). In particular, if \(a = b = 1\), then \((K_1 + K_1) \lor K_1 = P_3\) is a 4-path graph; while if \(a = b = 2\), then the bowtie graph \((2K_2) \lor K_1\) is a 4-path graph. 

We now determine some forbidden subgraphs of 4-path graphs. The following is a consequence of Proposition 6.5.7 and Corollary 6.5.8, which completely characterizes all connected bipartite graphs that are 4-path graphs.

**Corollary 7.1.4** No 4-path graph contains \(K_{1,3}\) as an induced subgraph. Consequently, a connected bipartite graph \(G\) is a 4-path graph if and only if \(G\) is a path or an even cycle.

By Corollary 7.1.4, we now consider connected non-bipartite graphs only.

**Proposition 7.1.5** No 4-path graph contains \(K_4 - e\) as an induced subgraph. Consequently, any graph containing \(K_4 - e\) as an induced subgraph is not a 4-path graph.

**Proof.** Suppose, to the contrary, that there is a 4-path graph \(H\) that contains \(K_4 - e\) as an induced subgraph. Let \(H = P_4(G)\) for some graph \(G\) and let \(K_4 - e\) be obtained from the 4-cycle \((x_1, x_2, x_3, x_4, x_1)\) by adding the edge \(x_1x_3\). Suppose that \(x_1 = (u_1, u_2, u_3, u_4)\) is a 4-path in \(G\). Assume, without loss of generality, that \(x_2 = (u_1, u_2, u_3, a)\) where \(a \in V(G)\) and \(a \neq u_4\). Since \(x_2x_3 \in E(H)\), we may assume that

\[
(i) \quad x_3 = (u_1, u_2, u_3, b) \quad \text{for some vertex } b \neq u_4, \text{ or}
\]

\[
(ii) \quad x_3 = (b', u_1, u_2, u_3) \quad \text{for some vertex } b' \quad \text{(where it is possible that } b' = u_4).\]

Since \(x_1x_4 \in E(H)\) and \(x_2x_4 \notin E(H)\), we may assume either \(x_4 = (u_2, u_3, u_4, c)\), where it is possible that \(c = u_1\), or \(x_4 = (c', u_2, u_3, u_4)\) where \(c' \neq u_1\).

\* First, suppose that \((i)\) occurs. If \(x_4 = (u_2, u_3, u_4, c)\), then \(x_3x_4 \notin E(H)\) regardless of \(c\). If \(x_4 = (c', u_2, u_3, u_4)\), where \(c' \neq u_1\), then since \(b \neq u_4\), it follows that \(x_3x_4 \notin E(H)\), a contradiction.

\* Next, suppose that \((ii)\) occurs. If \(x_4 = (u_2, u_3, u_4, c)\), then \(x_3x_4 \notin E(H)\) regardless of \(c\). If \(x_4 = (c', u_2, u_3, u_4)\) where \(c' \neq u_1\), then \(x_3x_4 \notin E(H)\).
In either case, a contradiction is produced. □

Next, we show that every 4-path graph of maximum degree at least 3 must contain a triangle.

**Proposition 7.1.6** If $G$ is a connected triangle-free graph with $\Delta(G) \geq 3$, then $G$ is not a 4-path graph.

**Proof.** Let $v \in V(G)$ with $\deg_G v = \Delta(G) \geq 3$ and let $N(v) = \{v_1, v_2, \ldots, v_{\Delta(G)}\}$. Since $G$ is triangle-free, it follows that $G$ contains $K_{1,3}$ as an induced subgraph where $V(K_{1,3}) = \{v, v_1, v_2, v_3\}$. Therefore, by Corollary 7.1.4, $G$ is not a 4-path graph. □

We have seen that the triangle $K_3$ and the bowtie graph $2K_2 \vee K_1$ having two triangles are both 4-path graphs. By Proposition 7.1.5, the wheel $W_n = C_{n-1} \vee K_1$ of order $n \geq 5$ is not a 4-path graph. Thus, Proposition 7.1.6 cannot be improved. There are other graphs containing one or two triangles that are 4-path graphs. For example, let $F_1$ be the graph of order $k + 2 \geq 4$ obtained from the $k$-path $Q = (x_1, x_2, x_3, \ldots, x_k)$ by adding the edge $y_1y_2$ and joining $y_1$ and $y_2$ to $x_1$ and let $F_2$ be a graph obtained from the $k$-path $Q = (x_1, x_2, x_3, \ldots, x_k)$ by (i) adding the edge $y_1y_2$ and joining $y_1$ and $y_2$ to $x_1$ and (ii) adding the edge $z_1z_2$ and joining $z_1$ and $z_2$ to $x_k$. The graphs $F_1$ and $F_2$ are shown in Figure 7.3. Thus, $F_i$ contains exactly $i$ triangles for $i = 1, 2$.

![Figure 7.3: The graphs $F_1$ and $F_2$](http://example.com/figure7.3)

Now, let $H_1$ be the tree of order $k + 5$ obtained from the $(k + 3)$-path $P = (u_1, u_2, \ldots, u_{k+3})$ by adding two pendant edges at $u_{k+3}$ and let $H_2$ be the the tree of order $k + 7$ obtained from the $(k + 3)$-path $P = (u_1, u_2, \ldots, u_{k+3})$ by adding two pendant edges at $u_1$ and at $u_{k+3}$. The graphs $H_1$ and $H_2$ are shown in Figure 7.4. Since $P_4(H_i) = F_i$ for $i = 1, 2$, it follows that $F_1$ and $F_2$ are 4-path graphs.

For each integer $n \geq 5$, let $C_{n,\Delta}$ denote the graph $C_n + e$ that contains a triangle. Thus, $C_5 + e = C_{5,\Delta}$ while if $n = 6, 7$, then there are two nonisomorphic graphs obtained
from $C_n$ by adding an edge $e$ joining two nonadjacent vertices of $C_n$, one of which contains a triangle, namely $C_{n,\Delta}$, and the other one is triangle-free.

**Proposition 7.1.7** For each integer $n \geq 5$, the graph $C_n + e$ is a 4-path graph if and only if $C_n + e = C_{7,\Delta}$.

**Proof.** First, let $H$ be the tree obtained from $S(K_{1,3})$ by subdividing exactly one pendant edge exactly once. Then $P_4(H) = C_{7,\Delta}$ and so $C_{7,\Delta}$ is a 4-path graph.

For the converse, we show that if $C_n + e \neq C_{7,\Delta}$, then $C_n + e$ is not a 4-path graph. First, suppose that $n = 5$. Let $C_5 + e$, where say $C_5 = (x_1, x_2, x_3, x_4, x_5, x_1)$ and $e = x_1x_3$.

Assume, to the contrary, that there is a graph $G$ such that $P_4(G) = C_5 + e$. By Lemma 6.4.4, the graph $G$ cannot contain a 4-cycle $C_4$ with a vertex of degree at least 3 in $G$. However, since $C_5 + e$ contains the 4-cycle $(x_1, x_3, x_4, x_5, x_1)$, it follows by Proposition 6.4.7 that $G$ contains a subgraph $F$ isomorphic to the double star $S_{3,3}$. Let $V(F) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ whose central vertices are $v_1$ and $v_2$. Thus, $\deg_G v_1 = \deg_G v_2 = 3$. Since $G$ is connected and $G \neq S_{3,3}$, there is $v \in V(G) - V(F)$ that is adjacent to a vertex of $F$. First, suppose that $v$ is adjacent to $v_i$ for $i = 1, 2$, say $vv_1 \in E(G)$. Then $G$ contains $S_{3,4}$ as a subgraph. Since $P_4(S_{3,4}) = K_2 \boxtimes K_3$ by Proposition 6.2.2, this is impossible. Next, suppose that $v$ is adjacent to $v_i$ for $i = 3, 4, 5, 6$, say $vv_3 \in E(G)$. Then $G$ contains a subgraph $H$ of Figure 7.6. Since $P_4(H)$ contains $C_5 + e$ as a proper subgraph, as shown in Figure 7.6, this is a contradiction. Therefore, $C_5 + e$ is not a 4-path graph. Observe that $C_5 + e$ is an induced subgraph of the 4-path graph of the graph $H$ of Figure 7.6.
Thus, we may assume that \( n \geq 6 \). Let \( C_n = (x_1, x_2, \ldots, x_n, x_1) \), where \( n \geq 6 \). We may assume that \( e = x_1x_i \) for some integer \( i \) with \( 3 \leq i \leq n - 1 \). If \( i \neq 3, n - 1 \), then \( C_n + e \) contains \( K_{1,3} \) as an induced subgraph and so \( C_n + e \) is not a 4-path graph (or by Proposition 7.1.6). Hence, we may assume that \( e = v_1v_3 \) and \( n \neq 7 \). Thus, \( C_n + e \) contains the \((n - 1)\)-cycle \((x_1, x_3, x_4, \ldots, x_n, x_1)\), where \( n - 1 \neq 6 \). If \( n - 1 \neq 8, 9 \) or \( n \neq 9, 10 \), then \( C_n + e \) is not a 4-path graph by Lemma 6.4.4 and Theorem 6.4.9. Thus, we may assume that \( n = 9, 10 \) and that \( C_n = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_1) \) where \( e = x_1x_3 \). Assume, to the contrary, that there is a graph \( G \) such that \( P_4(G) = C_n + e \) for \( n = 9, 10 \). We consider these two cases.

**Case 1.** \( n = 9 \). Since (i) \( C_9 + e \) contains the 8-cycle and (ii) \( G \) cannot contain an 8-cycle with a vertex of degree at least 3 in \( G \) by Lemma 6.4.4, it follows by Theorem 6.4.10 that \( G \) contains a subgraph \( F \) isomorphic to \( 2K_2 \vee K_1 \). Let \( V(F) = \{v_1, v_2, v_3, v_4, v_5\} \) where \( \deg_F v_1 = 4 \), \( \deg_F v_i = 2 \) for \( i \in \{2, 3, 4, 5\} \). Since \( G \) is connected and \( G \neq 2K_2 \vee K_1 \), there is \( v \in V(G) - V(F) \) that is adjacent to a vertex of \( F \). First, suppose that \( v \) is adjacent to \( v_1 \), say \( vv_1 \in E(G) \). Then \( G \) contains at least twelve 4-paths, so \( C_9 + e \neq P_4(G) \). Next, suppose that \( v \) is adjacent to \( v_i \) for \( i \in \{2, 3, 4, 5\} \), say \( vv_2 \in E(G) \). Then \( G \) contains at least eleven 4-paths, a contradiction.

**Case 2.** \( n = 10 \). Since (i) \( C_{10} + e \) contains the 9-cycle and (ii) \( G \) cannot contain a 9-cycle with a vertex of degree at least 3 in \( G \) by Lemma 6.4.4, it follows by Theorem 6.4.10 that \( G \) contains a subgraph \( F \) isomorphic to \( \text{cor}(K_3) \). Let \( V(F) = \{v_1, v_2, v_3, v_4, v_5, v_6\} \) where \( \deg_F v_i = 3 \) for \( i \in \{1, 2, 3\} \) and \( \deg_F v_j = 1 \) for \( j \in \{4, 5, 6\} \). Since \( G \) is connected and \( G \neq \text{cor}(K_3) \), there is \( v \in V(G) - V(F) \) that is adjacent to a vertex of \( F \). First, suppose that \( v \) is adjacent to \( v_i \), for \( i \in \{1, 2, 3\} \), say \( vv_1 \in E(G) \). Then \( G \) contains at least thirteen 4-paths, so \( C_{10} + e \neq P_4(G) \). Next, suppose that \( v \) is adjacent to \( v_j \) for \( j \in \{4, 5, 6\} \), say \( vv_4 \in E(G) \). Then \( G \) contains at least eleven 4-paths, a contradiction. ■
For an integer $n \geq 5$, let $G$ be the unicycle graph obtained from an $(n-1)$-cycle $(v_1, v_2, \ldots, v_{n-1}, v_1)$ by adding the pendant edge $vv_1$ at the vertex $v_1$. Then $\mathcal{P}_4(G)$ is the graph obtained from an $(n-1)$-cycle $(x_1, x_2, \ldots, x_{n-1}, x_1)$ by adding two new vertices $y$ and $z$ and (i) joining $y$ to $x_1$ and $x_2$ and (ii) joining $z$ to $x_3$ and $x_4$. Thus, $\mathcal{P}_4(G)$ contains $C_{n,\Delta}$ as an induced subgraph. Therefore, if $n \geq 5$, then $C_{n,\Delta}$ is not a 4-path graph but it can be an induced subgraph of some 4-path graph.

Suppose that $G$ is an $s$-regular graph of order $n$ and size $m$. Recall that the subdivision graph $S(G)$ of $G$ is obtained from $G$ by subdividing each edge $G$ exactly once. Then $S(G)$ is a graph of order $n + m$ having $n$ vertices of degree $s$ and $m$ vertices of degree 2. If $T$ is the tree obtained from the star $K_{1,3}$ by subdividing two edges of $K_{1,3}$ exactly once, then $T \neq P_7$ but $\mathcal{P}_4(T) = \mathcal{P}_4(P_7) = P_4$. Let $S(K_{1,3})$ be the subdivision of $K_{1,3}$ obtained from $K_{1,3}$ by subdividing each edge $K_{1,3}$ exactly once. Note that $S(K_{1,3}) \neq C_6$ but $\mathcal{P}_4(S(K_{1,3})) = \mathcal{P}_4(C_6) = C_6$. For the subdivision graph $S(K_{1,4})$ of the star $K_{1,4}$, its 4-path graph is the truncated tetrahedral graph shown in Figure 7.7.

![Figure 7.7: The graphs $S(K_{1,4})$ and its 4-path graph](image)

**Proposition 7.1.8** For each positive integer $r$, there exists an $r$-regular 4-path graph.

**Proof.** For a positive integer $r \geq 3$, let $G = S(K_{1,r+1})$ be the subdivision of the star $K_{1,r+1}$. For a 4-path $P = (u, v, w, x)$ in $G$, where $u$ is an end-vertex of $G$. Then $\deg_G u = 1$, $\deg_G v = 2$, $\deg_G w = r + 1$ and $\deg_G x = 2$. Since $G$ is triangle-free, it follows by Proposition 6.2.5 that $\deg_{\mathcal{P}_4(G)} P = 1 + 2 + (r + 1) + 2 - 6 = r$. Therefore, $\mathcal{P}_4(G)$ is $r$-regular.

There are other regular graphs whose 4-path graph is also regular. For example, we saw that $\mathcal{P}_4(C_n) = C_n$ which is 2-regular for each integer $n \geq 4$. For an even integer $r \geq 4$, let $r = 2(s - 1)$ for some integer $s \geq 3$ and let $G$ be an $s$-regular graph. Now,
let $H = S(G)$ be the subdivision graph of $G$ and let $P = (u, v, w, x)$ be any 4-path in $G$. We may assume, without loss of generality, that $\deg_H u = \deg_H w = s$ and $\deg_H v = \deg_H x = 2$. Suppose that $u$ is adjacent to the vertices $u_1, u_2, \ldots, u_{s-1}$ of $G$ that are not on $P$, $w$ is adjacent to the vertices $w_1, w_2, \ldots, w_{s-2}$ of $G$ that are not on $P$ and $x$ is adjacent to $x' \neq w$. In $P_4(H)$, the vertex $P$ is adjacent only to the 4-paths $(u_i, u, v, w)$ where $1 \leq i \leq s-1$, the 4-paths $(u, v, w_j)$ where $1 \leq j \leq s-2$ and the 4-path $(v, w, x, x')$. Thus, $P$ is adjacent exactly to $(s-1) + (s-2) + 1 = 2(s-1) = r$ vertices in $P_4(H)$ and so the degree of $P$ is $r$. Therefore, $P_4(H)$ is an $r$-regular graph.

7.2 Distance in 4-Path Graphs

First, we review some definitions and notation in distance in graphs. Let $G$ be a nontrivial connected graph. The distance $d(u, v)$ between vertices $u$ and $v$ in $G$ is the minimum number of edges in a $u - v$ path in $G$. The eccentricity $e(v)$ of a vertex $v$ of $G$ is the distance between $v$ and a vertex farthest from $v$ in $G$; that is,

$$e(v) = \max\{d(v, w) : w \in V(G)\}.$$ 

The following result is well known (see [14]).

**Theorem 7.2.1** If $u$ and $v$ are adjacent vertices in a connected graph $G$, then

$$|e(u) - e(v)| \leq 1.$$ 

The diameter of $G$ is the largest eccentricity among the vertices of $G$, denoted by $\text{diam}(G)$, and the radius is the smallest eccentricity among the vertices of $G$, denoted by $\text{rad}(G)$. Therefore, the diameter of $G$ is the greatest distance between any two vertices of $G$. The following result is well known (see [14]).

**Theorem 7.2.2** For every nontrivial connected graph $G$,

$$\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G).$$

A vertex $v$ with $e(v) = \text{rad}(G)$ is called a central vertex of $G$ and a vertex $v$ with $e(v) = \text{diam}(G)$ is called a peripheral vertex of $G$. Two vertices $u$ and $v$ of $G$ with $d(u, v) = \text{diam}(G)$ are antipodal vertices of $G$. Necessarily, if $u$ and $v$ are antipodal vertices in $G$, then both $u$ and $v$ are peripheral vertices. If $G$ is a tree, then every peripheral vertex of $G$ is an end-vertex of $G$. The subgraph induced by the central
vertices of a connected graph $G$ is the center of $G$ and the subgraph induced by the peripheral vertices of a connected graph $G$ is the periphery of $G$. We now present two best known results on the center and the periphery of a connected graph, the first of which is due to Hedetniemi (see [5]) while the send one is due to Bielak and Sysło (see [4]).

**Theorem 7.2.3** [5] Every graph is the center of some graph.

While every graph is the center of some graph, this is not true for the periphery.

**Theorem 7.2.4** [4] A nontrivial graph $G$ is the periphery of some graph if and only if every vertex of $G$ has eccentricity 1 or no vertex of $G$ has eccentricity 1.

For a nontrivial tree $T$, the center of $T$ is either $K_1$ or $K_2$. If the center of $T$ is $K_1$, then $T$ is said to be central, while if the center of $T$ is $K_2$, then $T$ is bicentral. It is well known that, for a nontrivial tree $T$, either diam$(T) = 2\text{ rad}(T)$ or diam$(T) = 2\text{ rad}(T) - 1$. Furthermore, if diam$(T) = 2\text{ rad}(T)$, then $T$ is central; while if diam$(T) = 2\text{ rad}(T) - 1$, then $T$ is bicentral.

We have seen that if $a$ and $b$ are positive integers, then $P_4(S_{a+1,b+1}) = K_a \Box K_b$. In particular, $P_4(S_{2,b+1}) = K_b$ for each positive integer $b$. Thus, if $a, b \geq 2$, then
\[
\text{diam}(P_4(S_{a+1,b+1})) = \text{diam}(S_{a+1,b+1}) - 1.
\]
Moreover, if $b \geq 2$, then
\[
\text{diam}(P_4(S_{2,b+1})) = \text{diam}(S_{2,b+1}) - 2.
\]
We have seen that $P_4(P_n) = P_{n-3}$ for each integer $n \geq 4$ and so
\[
\text{diam}(P_4(P_n)) = \text{diam}(P_n) - 3.
\]
Thus, if $T$ is a path of order at least 4 or if $T$ is a double star, then
\[
\text{diam}(P_4(T)) = \text{diam}(T) - i \text{ for an integer } i \in \{1, 2, 3\}.
\]
In general, we have the following conjecture.

**Conjecture 7.2.5** If $T$ be a tree with diam$(T) \geq 3$, then
\[
\text{diam}(T) - 3 \leq \text{diam}(P_4(T)) \leq \text{diam}(T) - 1.
\]
We now verify Conjecture 7.2.5 for a well-known class of trees. A caterpillar is a tree $T$ of order 3 or more, such that the removal of all vertices of degree one (leaves) from $T$ results in a path (called the spine of $T$). Thus, every path, every star (of order at least 3) and every double star (a tree of diameter 3) is a caterpillar.

**Theorem 7.2.6** If $T$ is a caterpillar of order $n \geq 4$ with $\text{diam}(T) \geq 3$, then

$$\text{diam}(T) - 3 \leq \text{diam}(\mathcal{P}_4(T)) \leq \text{diam}(T) - 1.$$  

**Proof.** Since the statement is true for double stars, we may assume that $\text{diam}(T) = d \geq 4$. Let $P = (u_0, u_1, u_2, \ldots, u_{d-1}, u_d)$, $d \geq 4$, be a longest path in $T$. Any other vertex of $T$ is an end-vertex of $T$ that is adjacent to some vertex $u_i$, $1 \leq i \leq d - 1$.

First, we show that $\text{diam}(\mathcal{P}_4(T)) \leq \text{diam}(T) - 1$, that is, we show that the distance between every two vertices of $\mathcal{P}_4(T)$ (every two 4-paths of $T$) is at most $d - 1$. For $0 \leq i \leq d - 3$, let $P_i = (u_i, u_{i+1}, u_{i+2}, u_{i+3})$. These paths $P_i$ are the only 4-paths all of whose edges lie on $P$. Thus $(P_0, P_1, P_2, \ldots, P_{d-3})$ is a path in $\mathcal{P}_4(T)$.

First, we show that it suffices to assume that each interior vertex of $P$ is adjacent to at most one end-vertex not on $P$, for suppose, to the contrary, that $v_i$ and $w_i$ are end-vertices adjacent to $u_i$ and $v_i+1$ and $w_i+1$ are end-vertices adjacent to $u_i+1$. We now determine the distance between two 4-paths of $T$, where each 4-path contains a pendant edge of $T$ incident with the same vertex of $P$. Let $A_1 = (v_i, u_i, u_{i+1}, v_{i+1})$ and let $A_2 = (w_i, u_i, u_{i+1}, w_{i+1})$. Then let $A = (v_i, u_i, u_{i+1}, w_{i+1})$. Here, $(A_1, A, A_2)$ is an $A_1-A_2$ path of length 2 in $\mathcal{P}_4(T)$. Next, let $B_1 = (v_i, u_i, u_{i+1}, u_{i+2})$ and $B_2 = (w_i, u_i, u_{i+1}, u_{i+2})$. Then $B_1$ and $B_2$ are adjacent vertices in $\mathcal{P}_4(T)$.

Next, let $F_1 = (u_{i-2}, u_{i-1}, u_i, v_i)$ and $F_2 = (w_i, u_i, u_{i+1}, u_{i+2})$, here $d \geq 4$. Then let $P_{i-2} = (u_{i-2}, u_{i-1}, u_i, u_{i+1})$ and $P_{i-1} = (u_{i-1}, u_i, u_{i+1}, u_{i+2})$. Here $(F_1, P_{i-2}, P_{i-1}, F_2)$ is an $F_1 - F_2$ path of length 3. We may therefore assume that each interior vertex of $P$ is adjacent to exactly one end-vertex of $T$ not on $P$, say $v_i$ is an end-vertex of $T$ not on $P$ that is adjacent to $u_i$, where $1 \leq i \leq d - 1$. In addition to the paths $P_i$, $0 \leq i \leq d - 3$, the only 4-paths in $T$ are as follows:

1. $Q_i^- = (u_{i-2}, u_{i-1}, u_i, v_i)$, where $2 \leq i \leq d - 1$,
2. $Q_i^+ = (v_i, u_i, u_{i+1}, u_{i+2})$, where $1 \leq i \leq d - 2$,
3. $R_{i,i+1} = (v_i, u_i, u_{i+1}, v_{i+1})$, where $1 \leq i \leq d - 2$. 

126
In $T$:

In $P_4(T)$:

Figure 7.8: Illustrating the 4-paths in a caterpillar $T$

We now show that the distance between any two vertices $s_1$ and $s_2$ is at most $d - 1$. Certainly the distance between any two paths $P_i$ and $P_j$, where $0 \leq i, j \leq d - 3$, $i \neq j$ is less than $d - 1$. Consider the following cases.

Case 1. $s_1 = Q_i^−$ and $s_2 = P_j$. Since $s_1$ is adjacent to $P_{i−2}$ and $d(P_{i−2}, P_j) \leq d − 3$, it follows that $d(s_1, s_2) \leq d − 2$.

Case 2. $s_1 = Q_i^−$ and $s_2 = Q_j^−$. Since $s_1$ is adjacent to $P_{i−2}$ and $s_2$ is adjacent to $P_{j−2}$, it follows that $d(s_1, s_2) \leq d − 1$.

Case 3. $s_1 = Q_i^+$ and $s_2 = P_j$. Since $s_1$ is adjacent to $P_i$ and $d(P_i, P_j) \leq d − 3$, it follows that $d(s_1, s_2) \leq d − 2$.

Case 4. $s_1 = Q_i^+$ and $s_2 = Q_j^+$. Since $s_1$ is adjacent to $P_i$ and $s_2$ is adjacent to $P_j$ and $d(P_i, P_j) \leq d − 3$, it follows that $d(s_1, s_2) \leq d − 1$.

Case 5. $s_1 = Q_i^+$ and $s_2 = Q_j^-$. Since $s_1$ is adjacent to $P_i$ and $s_2$ is adjacent to $P_{j−2}$, it follows that $d(s_1, s_2) \leq d − 1$.

Case 6. $s_1 = R_{i,j+1}$ and $s_2 = P_j$. Let $s' = (v_i, u_i, u_{i+1}, u_{i+2})$. Then $(s_1, s', P_i)$ is a path of length 2 in $P_4(T)$. Since $d(P_i, P_j) \leq d − 3$, it follows that $d(s_1, s_2) \leq d − 1$.

We may let $s'' = (u_{i-1}, u_i, u_{i+1}, v_{i+1})$, then $(s, s'', P_{i-1})$ is a path of length 2 in $P_4(T)$. Again, $d(P_{i-1}, P_j) \leq d − 3$, it follows that $d(s_1, s_2) \leq d − 1$.

Case 7. $s_1 = R_{i,j+1}$ and $s_2 = Q_j^-$. Here, $1 \leq i \leq d − 2$ and $0 \leq j \leq d − 3$. Let $s'' = Q_{i+1}^- = (u_{i-1}, u_i, u_{i+1}, v_{i+1})$. Then $(s_1, s'', P_{i-1})$ is a path of length 2 in $P_4(T)$. Thus in this case, $d(s_1, P_{i−1}) = 2$ and $d(s_2, P_{j−2}) = 1$. The maximum distance between $s_1$ and $s_2$ occurs where $s_1 = R_{d−2,d−1}$ and $s_2 = Q_2^-$. Since $(s_1, Q_{d−1}^−, P_{d−4}, P_{d−5}, \ldots, P_2, P_1, P_0, s_2)$ is an $s_1 − s_2$ path of length $d − 1$ in $P_4(T)$, it follows that $d(s_1, s_2) \leq d − 1$.
Case 8. \( s_1 = R_{i,i+1} \) and \( s_2 = Q_j^+ \). Here, \( 1 \leq i \leq d - 2 \) and \( 1 \leq j \leq d - 2 \). Let \( s'' = Q_{i+1}^- \). Then \((s_1, s'', P_{i-1})\) is a path of length 2 in \( P_4(T) \). Thus, in this case, \( d(s_1, P_{i-1}) = 2 \) and \( d(s_2, P_j) = 1 \). The maximum distance between \( s_1 \) and \( s_2 \) occurs where \( s_1 = R_{d-2,d-1} \) and \( s_2 = Q_1^+ \). Since \((s_1, Q_{d-1}^-, P_{d-4}, P_{d-5}, \ldots, P_2, P_1, P_0, s_2)\) is an \( s_1 \) \(-\) \( s_2 \) path of length \( d - 1 \) in \( P_4(T) \), it follows that \( d(s_1, s_2) \leq d - 1 \).

Case 9. \( s_1 = R_{i,i+1} \) and \( s_2 = R_{j,j+1} \). In this case, \( d(s_1, P_i) = d(s_1, P_{i-1}) = 2 \) and \( d(s_2, P_j) = d(s_2, P_{j-1}) = 2 \), since \( 1 \leq i, j \leq d - 2 \) with \( i \neq j \), we may assume that \( s_1 = R_{1,2} \) and \( s_2 = R_{d-2,d-1} \). Since \((s_1, P_0, P_1, P_2, \ldots, P_{d-4}, Q_{d-1}^-, s_2)\) is an \( s_1 \) \(-\) \( s_2 \) path of length \( d - 1 \), it follows that \( d(s_1, s_2) \leq d - 1 \).

Next, we establish the lower bound. As discussed earlier, it suffices to assume that each interior vertex of \( P \) is adjacent to at most one end-vertex not on \( P \). We consider two cases.

Case I. Each interior vertex of \( P \) is adjacent to exactly one end-vertex not on \( P \). As we saw, there are four types of 4-paths in \( T \), namely

* \( P_i = (u_i, u_{i+1}, u_{i+1}, u_{i+3}) \), where \( 0 \leq i \leq d - 3 \),

* \( Q_i^- = (u_{i-2}, u_{i-1}, u_i, v_i) \), where \( 2 \leq i \leq d - 1 \),

* \( Q_i^+ = (v_i, u_i, u_{i+1}, u_{i+2}) \), where \( 1 \leq i \leq d - 2 \),

* \( R_{i,i+1} = (v_i, u_i, u_{i+1}, v_{i+1}) \), where \( 1 \leq i \leq d - 2 \).

We show that the eccentricity of every vertex of \( P_4(T) \) is at least \( d - 3 \). Let

\[ S = \{ P_0, Q_1^+, Q_2^-, R_{1,2}, P_{d-3}, Q_{d-1}^-, Q_{d-2}^+, R_{d-2,d-1} \} \]

First, we consider the eccentricities of vertices in \( S \). Observe that

\[
e(P_0) = d(P_0, Q_{d-2}^+) = d(P_0, R_{d-2,d-1}) = d - 2
e(Q_1^+) = d(Q_1^+, Q_{d-2}^+) = d(Q_1^+, R_{d-2,d-1}) = d - 2
e(Q_2^-) = d(Q_2^-, Q_{d-2}^+) = d(Q_2^-, R_{d-2,d-1}) = d - 1
e(R_{1,2}) = d(R_{1,2}, Q_{d-2}^+) = d(R_{1,2}, R_{d-2,d-1}) = d - 1.
\]

The \( Q_2^- - Q_{d-2}^+ \) geodesic, the \( Q_2^- - R_{d-2,d-1} \) geodesic, the \( R_{1,2} - Q_{d-2}^+ \) geodesic and the
Thus, if $d$ is the caterpillar obtained from the path $(u_0, u_1, u_2, \ldots, u_{d-1}, u_d)$ of length $d \geq 4$ by adding two pendant edges $u_2v_2$ and $u_{d-2}v_{d-2}$, where $u_2v_2 = u_{d-2}v_{d-2}$ when $d = 4$, then $\text{diam}(P_4(T_0)) = \text{diam}(P_4(T)) = d - 1$.

Case II. Some interior vertices of $P$ are adjacent to no end-vertex of $T$. By observations above, it suffices to consider only the degrees of $u_2$ and $u_{d-2}$ in $T$. First, let $d = 4$. In this case, $u_2 = u_{d-2}$.

* If $\deg_T u_2 = 3$ then $\text{diam}(P_4(T)) = d - 1 = 3$. In this case, $e(Q_2) = d(Q_2, Q_2^+) = d - 1 = 3$, where the $Q_2 - Q_2^+$ geodesic is $(Q_2, P_0, P_1, Q_2^+)$ and $e(F) \leq 3$ for each $F$ in $P_4(T)$. 

\[ R_{1,2} - R_{d-2,d-1} \text{ geodesic in } P_4(T) \text{ are listed as follows:} \]

\[ Q_2 - Q_{d-2}^- : (Q_2, P_0, P_1, P_2, \ldots, P_{d-4}, P_{d-3}, Q_{d-2}^+) \] (7.5)

\[ Q_2 - R_{d-2,d-1}^- : (Q_2, P_0, P_1, P_2, \ldots, P_{d-4}, Q_{d-1}^-, R_{d-2,d-1}) \] (7.6)

\[ R_{1,2} - Q_{d-2}^+ : (R_{1,2}, Q_1^+, P_1, P_2, \ldots, P_{d-4}, P_{d-3}, Q_{d-2}^+) \] (7.7)

\[ R_{1,2} - R_{d-2,d-1}^- : (R_{1,2}, Q_1^+, P_1, P_2, \ldots, P_{d-4}, Q_{d-1}^-, R_{d-2,d-1}) \] (7.8)

For each pair of two vertices involved in (7.5)-(7.8), there is a unique geodesic connecting them in $P_4(T)$. By symmetry, $e(P_{d-3}) = e(Q_{d-1}^-) = d - 2$ and $e(Q_{d-2}^+) = e(R_{d-2,d-1}) = d - 1$. Furthermore, if $F$ is a 4-path of $T$ such that $F \notin S$, then $e(F) = d(F, H)$ for some $H \in S$. Thus, $e(F) = d(F, H) \leq e(H)$. This implies that

\[ \text{diam}(P_4(T)) = \max\{e(H) : H \in S\} = d - 1. \]

We now make some observations from the argument in Case I. By the four equations in (7.1)-(7.4) and the four geodesics described in (7.5)-(7.8), (or the structure of the graph $P_4(T)$), it would be useful to define

\[ X = \{Q_2, R_{1,2}\} \text{ and } Y = \{Q_{d-2}^+, R_{d-2,d-1}\}. \]

- The set of peripheral vertices of $P_4(T)$ is $X \cup Y$. Furthermore, an antipodal vertex of a vertex in $X$ belongs to a vertex in $Y$ and vice versa. Hence, the diameter of $P_4(T)$ is completely determined by the eccentricities of the vertices in $X \cup Y$, regardless of the degrees of the remaining interior vertices $u_j$ ($3 \leq j \leq d - 3$) in $P$.

- All vertices $P_0, Q_1^+, R_{2,3}, P_{d-3}, Q_{d-1}^-, R_{d-3,d-2}$ have the same eccentricity in $P_4(T)$ regardless of the degrees of the other interior vertices in $P$. 

Thus, if $T_0$ is the caterpillar obtained from the path $(u_0, u_1, u_2, \ldots, u_{d-1}, u_d)$ of length $d \geq 4$ by adding two pendant edges $u_2v_2$ and $u_{d-2}v_{d-2}$, where $u_2v_2 = u_{d-2}v_{d-2}$ when $d = 4$, then $\text{diam}(P_4(T_0)) = \text{diam}(P_4(T)) = d - 1$.

Case II. Some interior vertices of $P$ are adjacent to no end-vertex of $T$. By observations above, it suffices to consider only the degrees of $u_2$ and $u_{d-2}$ in $T$. First, let $d = 4$. In this case, $u_2 = u_{d-2}$.

* If $\deg_T u_2 = 3$ then $\text{diam}(P_4(T)) = d - 1 = 3$. In this case, $e(Q_2) = d(Q_2, Q_2^+) = d - 1 = 3$, where the $Q_2 - Q_2^+$ geodesic is $(Q_2, P_0, P_1, Q_2^+)$ and $e(F) \leq 3$ for each $F$ in $P_4(T)$. 

129
If \( \deg_T u_2 = 2 \), then \( X \cup Y = \emptyset \) and \( \text{diam}(\mathcal{P}_4(T)) = d - 3 = 1 \). In this case, if \( \deg_T(u_1) = \deg_T(u_3) = 3 \), then \( e(P_0) = e(P_1) = e(Q_1^+) = e(Q_3^-) = 1 \). If \( \deg_T(u_i) = 2 \) for \( i \in \{1, 2, 3\} \), then \( e(P_0) = d(P_0, P_1) = 1 \).

Next, let \( d \geq 5 \). In this case, \( u_2 \neq u_{d-2} \).

- If \( \deg_T u_2 = \deg_T u_{d-2} = 3 \), then \( \text{diam}(\mathcal{P}_4(T)) = d - 1 \). In this case, \( e(Q_2^-) = d(Q_2^-, Q_{d-2}^+) = d - 1 \), where the \( Q_2^- - Q_{d-2}^+ \) geodesic is \( (Q_2^-, P_0, P_1, \ldots, P_{d-3}, Q_{d-2}^+) \) and \( e(F) \leq d - 1 \) for each 4-path \( F \) in \( \mathcal{P}_4(T) \). Note that if \( d = 4 \), then \( Q_{d-2}^+ = Q_2^+ \).

- If exactly one of \( u_2 \) and \( u_{d-2} \) has degree 2, say \( \deg_T u_2 = 2 \) and \( \deg_T u_{d-2} = 3 \), then \( X = \emptyset \) and \( \text{diam}(\mathcal{P}_4(T)) = d - 2 \). In this case, \( e(P_0) = d(P_0, Q_{d-2}^+) = d - 2 \) and \( e(F) \leq d - 2 \) for each 4-path \( F \) in \( \mathcal{P}_4(T) \).

- If \( \deg_T u_2 = \deg_T u_{d-2} = 2 \), then \( X \cup Y = \emptyset \) and \( \text{diam}(\mathcal{P}_4(T)) = d - 3 \). In this case, \( e(P_0) = d(P_0, P_{d-3}) = d - 3 \) and \( e(F) \leq d - 3 \) for each 4-path \( F \) in \( \mathcal{P}_4(T) \).

The bounds in Theorem 7.2.6 are sharp. In fact, for each integer \( d \geq 4 \), there is (i) an infinite class \( \mathcal{T}_1 \) of caterpillars of diameter \( d \) such that \( \text{diam}(\mathcal{P}_4(T)) = \text{diam}(T) - 1 \) for each \( T \in \mathcal{T}_1 \) where \( \mathcal{T}_1 \) contains caterpillars with spine \( (u_0, u_1, \ldots, u_d) \) such that each interior vertex is adjacent to at least one end-vertex and (ii) an infinite class \( \mathcal{T}_2 \) of caterpillars of diameter \( d \) such that \( \text{diam}(\mathcal{P}_4(T)) = \text{diam}(T) - 3 \) for each \( T \in \mathcal{T}_2 \) where \( \mathcal{T}_2 \) contains caterpillars with spine \( (u_0, u_1, \ldots, u_d) \) such that \( \deg u_i = 2 \) for \( i = 1, 2, d - 2, d - 1 \).

Figure 7.9 shows the caterpillar \( T \) of diameter \( d \geq 5 \) in which each of interior vertices on its spine is adjacent to exactly one end vertex and the 4-path graph \( \mathcal{P}_4(T) \) of \( T \).

![Figure 7.9: The caterpillar T and its 4-path graph P4(T)](image)

The following corollary is a consequence of Theorems 7.2.2 and 7.2.5.

130
Corollary 7.2.7  Let $T$ be a caterpillar with $\text{diam}(T) \geq 3$.

(i)  If $T$ is central, then $\text{rad}(T) \leq \text{rad}(P_4(T)) + 1 \leq 2 \text{rad}(T)$.

(ii) If $T$ is bicentral, then $\text{rad}(T) \leq \text{rad}(P_4(T)) + 2 \leq 2 \text{rad}(T)$.

Proof.  First, suppose that $T$ is central, that is, $\text{diam}(T) = 2 \text{rad}(T)$. By Theorems 7.2.2 and 7.2.5 then,

$$\text{rad}(P_4(T)) \leq \text{diam}(P_4(T)) \leq \text{diam}(T) - 1 = 2 \text{rad}(T) - 1.$$  

Moreover,

$$\text{rad}(P_4(T)) \geq \frac{1}{2} \text{diam}(P_4(T)) \geq \frac{1}{2}(\text{diam}(T) - 3) = \frac{1}{2}(2 \text{rad}(T) - 3) = \text{rad}(T) - \frac{3}{2}.$$  

Since the radius of a graph is an integer, it follows that $\text{rad}(P_4(T)) \geq \text{rad}(T) - 1$. Therefore, $\text{rad}(T) \leq \text{rad}(P_4(T)) + 1 \leq 2 \text{rad}(T)$.

Next, suppose that $T$ is bicentral, that is, $\text{diam}(T) = 2 \text{rad}(T) - 1$. By Theorems 7.2.2 and 7.2.5 again,

$$\text{rad}(P_4(T)) \leq \text{diam}(P_4(T)) \leq \text{diam}(T) - 1 = 2 \text{rad}(T) - 2.$$  

Moreover,

$$\text{rad}(P_4(T)) \geq \frac{1}{2} \text{diam}(P_4(T)) \geq \frac{1}{2}(\text{diam}(T) - 3) = \frac{1}{2}(2 \text{rad}(T) - 1 - 3) = \frac{1}{2}(2 \text{rad}(T) - 4) = \text{rad}(T) - 2.$$  

Therefore, $\text{rad}(T) \leq \text{rad}(P_4(T)) + 2 \leq 2 \text{rad}(T)$.

The lower bounds described in Corollary 7.2.8 are both sharp. For example, the path $P$ of order $d + 1 \geq 4$ has diameter $d$ and radius

$$\text{rad}(P) = \begin{cases} 
\frac{1}{2}d & \text{if } d \text{ is even,} \\
\frac{1}{2}(d+1) & \text{if } d \text{ is odd.}
\end{cases}$$  

Now, $\text{diam}(P_4(P)) = d - 3$, and

$$\text{rad}(P_4(P)) = \begin{cases} 
\frac{1}{2}(d-2) & \text{if } d \text{ is even,} \\
\frac{1}{2}(d-3) & \text{if } d \text{ is odd.}
\end{cases}$$
If \(d\) is even, then \(\text{rad}(\mathcal{P}_4(P)) + 1 = \text{rad}(P)\). Indeed,
\[
\text{rad}(\mathcal{P}_4(P)) + 1 = \frac{1}{2}(d - 2) + 1 = \frac{1}{2}d - 1 + 1 = \frac{1}{2}d = \text{rad}(P).
\]
If \(d\) is odd, then \(\text{rad}(\mathcal{P}_4(P)) + 2 = \text{rad}(P)\). Indeed,
\[
\text{rad}(\mathcal{P}_4(P)) + 2 = \frac{1}{2}(d - 3) + 2 = \frac{1}{2}d + \frac{1}{2} = \frac{1}{2}(d + 1) = \text{rad}(P).
\]
Therefore, the lower bounds described in Corollary 7.2.8 are both sharp.

Next, we show that the upper bounds described in Corollary 7.2.8 can be improved. If \(T\) is a double star, then \(\text{diam}(T) = 3\) and \(\text{rad}(\mathcal{P}_4(T)) = \text{rad}(T) = 2\). If \(T\) is not a star or a double star, then \(\text{rad}(\mathcal{P}_4(T)) < \text{rad}(T)\). In fact, more can be said.

**Corollary 7.2.8**  Let \(T\) be a caterpillar with \(\text{diam}(T) \geq 4\).

(i) If \(T\) is central, then \(\text{rad}(T) - 1 \leq \text{rad}(\mathcal{P}_4(T)) \leq \text{rad}(T)\).

(ii) If \(T\) is bicentral, then \(\text{rad}(T) - 2 \leq \text{rad}(\mathcal{P}_4(T)) \leq \text{rad}(T) - 1\).

**Proof.** Let \(\text{diam}(T) = d \geq 4\) and \(P = (u_0, u_1, u_2, \ldots, u_{d-1}, u_d)\) be a longest path in \(T\). Any other vertex of \(T\) is an end-vertex of \(T\) that is adjacent to some vertex \(u_i\), \(1 \leq i \leq d - 1\). As discussed in the proof of Theorem 7.2.6, we may assume that each interior vertex of \(P\) is adjacent to at most one end-vertex not on \(P\). We consider two cases, according to whether \(T\) is central or \(T\) is bicentral.

**Case 1.** \(T\) is central. Then \(\text{diam}(T) = 2\text{rad}(T)\). Assume that \(d \geq 4\) let \(d = 2a\) for some integer \(a \geq 2\), where then \(\text{rad}(T) = a\). Using the same notation in the proof of Theorem 7.2.6, let
\[
S_1 = \{P_{a-2}, P_{a-1}, Q_{a-1}^+, Q_{a+1}^-\}.
\]
First, suppose that \(d = 4\). As shown in the proof of Theorem 7.2.6, \(\text{diam}(\mathcal{P}_4(T)) \in \{d - 1, d - 3\}\).

\[\star\text{ If } (\mathcal{P}_4(T)) = d - 1, \text{ then } \deg_T u_2 = 3 \text{ and } \text{rad}(\mathcal{P}_4(T)) = e(H) = a = 2 \text{ for each } H \in S_1.\]

\[\star\text{ If } (\mathcal{P}_4(T)) = d - 3, \text{ then } \deg_T u_2 = 2 \text{ and } \text{rad}(\mathcal{P}_4(T)) = e(H) = a - 1 = 1 \text{ for each } H \in S_1.\]

We now assume that \(d \geq 6\). By Theorem 7.2.6, \(\text{diam}(\mathcal{P}_4(T)) \in \{d - 1, d - 2, d - 3\}\).
* If \( \text{diam}(P_4(T)) = d - 1 \), then \( \text{deg}_T u_{d-2} = \text{deg}_T u_2 = 3 \) and \( \text{rad}(P_4(T)) = e(H) = a = \text{rad}(T) \) for each \( H \in S_1 \).

* If \( \text{diam}(P_4(T)) = d - 2 \), then, by the proof of Theorem 7.2.6, exactly one of \( u_2 \) and \( u_{d-2} \) has degree 2. First, suppose that \( \text{deg}_T u_{d-2} = 2 \). Then \( \text{rad}(P_4(T)) = e(P_a-2) = a - 1 = \text{rad}(T) - 1 \). Next, suppose that \( \text{deg}_T u_2 = 2 \), then \( \text{rad}(P_4(T)) = e(P_a-1) = a - 1 = \text{rad}(T) - 1 \).

* If \( \text{diam}(P_4(T)) = d - 3 \), then \( \text{deg}_T u_{d-2} = \text{deg}_T u_2 = 2 \) and \( \text{rad}(P_4(T)) = e(H) = a - 1 = \text{rad}(T) - 1 \) for each \( H \in S_1 \).

Thus, if \( T \) is central, then \( \text{rad}(P_4(T)) \in \{ \text{rad}(T), \text{rad}(T) - 1 \} \) and (i) holds.

**Case 2.** \( T \) is bicentral. Then \( \text{diam}(T) = 2 \text{rad}(T) - 1 \). Let \( d = 2b + 1 \) for some integer \( b \geq 2 \), where then \( \text{rad}(T) = b \). By Theorem 7.2.6 again, \( \text{diam}(P_4(T)) \in \{ d-1, d-2, d-3 \} \).

* If \( \text{diam}(P_4(T)) = d - 1 \), then \( \text{deg}_T u_{d-2} = \text{deg}_T u_2 = 3 \) and \( \text{rad}(P_4(T)) = e(P_{b-1}) = b - 1 = \text{rad}(T) - 1 \).

* If \( \text{diam}(P_4(T)) = d - 2 \), then by the proof of Theorem 7.2.6, exactly one of \( u_2 \) and \( u_{d-2} \) has degree 2. First, let \( \text{deg}_T u_{d-2} = 2 \). Then \( \text{rad}(P_4(T)) = e(F) \) for each \( F \in \{ P_{b-2}, P_{b-1}, Q_{b-1}^+, Q_{b+1}^- \} \). In particular, \( e(P_{b-2}) = \text{rad}(T) - 1 \). Next, let \( \text{deg}_T u_2 = 2 \). Then \( \text{rad}(P_4(T)) = e(F) \) for each \( F \in \{ P_{b-1}, P_b, Q_b^+, Q_{b+2}^- \} \). In particular, \( e(P_{b-1}) = b - 1 = \text{rad}(T) - 1 \).

* If \( \text{diam}(P_4(T)) = d - 3 \), then \( \text{deg}_T u_{d-2} = \text{deg}_T u_2 = 2 \) and \( \text{rad}(P_4(T)) = e(P_{b-1}) = b - 2 = \text{rad}(T) - 2 \).

Thus, if \( T \) is bicentral, then \( \text{rad}(P_4(T)) \in \{ \text{rad}(T) - 1, \text{rad}(T) - 2 \} \) and (ii) holds. ■

### 7.3 Planarity and Hamiltonicity

We also plan to investigate two well-known graphical properties of graphs, namely, planar and Hamiltonian properties of the 4-path graphs of graphs. In particular, we plan to investigate the following two problems.

**Problem 7.3.1** Which connected graphs have a planar 4-path graph?

Observe that if \( H \) is a subgraph of \( G \), then \( P_4(H) \) is a subgraph of \( P_4(G) \). Furthermore, it is well known that if \( F \) contains a nonplanar subgraph, then \( F \) cannot be planar.
Since every connected graph contains a spanning tree, we plan first to investigate the planarity of the 4-path graph of a tree.

**Problem 7.3.2** Which graphs have a Hamiltonian 4-path graph?

In this case, we plan to start with the 4-path graph of a complete graph of order at least 5. The 6-regular 4-path graph $P_4(\mathcal{K}_5)$ of $\mathcal{K}_5$ of order 60 is Hamiltonian. To see this, we label the edges of $\mathcal{K}_5$ as shown in Figure 7.10. The following cyclic sequence of all sixty 4-paths in $\mathcal{K}_5$ gives rise to a Hamiltonian cycle in $P_4(\mathcal{K}_5)$.

$$
547, 478, 789, 894, 492, 290, 093, 395, 198, 981, 815, 517, 174, 749,
946, 640, 401, 216, 163, 365, 659, 951, 154, 543, 340, 408, 082, 287,
876, 670, 705, 056, 567, 762, 629, 921, 217, 173, 732, 325, 528, 826,
264, 461, 618, 183, 836, 639, 937, 370, 709, 901, 012, 123, 234, 438,
380, 805, 052, 254, | 547
$$

![Figure 7.10: The graph $\mathcal{K}_5$](image)

**Conjecture 7.3.3** For each integer $n \geq 5$, the graph $P_4(\mathcal{K}_n)$ is Hamiltonian.
Bibliography


