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Linear Operators between Nonarchimedean Banach Spaces

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LINEAR OPERATORS BETWEEN NONARCHIMEDEAN BANACH SPACES

by

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Krishnamachari S. Nadathur
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I PRELIMINARIES

The purpose of this dissertation is to extend some classical results on compact and Fredholm operators to the case in which the linear spaces involved are non-archimedean normed linear spaces over a valued field.

In Section I are listed some definitions and basic concepts that are needed for a study of non-archimedean valuation theory. The concepts of tensor products of nonarchimedean Banach spaces over a spherically complete field $F$, and of the tensor product of operators, are also dealt with in this section.

The proofs of most of the results in Section II are based on the known extensions of classical results such as the open mapping theorem and the Hahn-Banach theorem. These results are used repeatedly in the subsequent sections on compact and Fredholm operators.

A few of the significant results proved in Section III and Section IV are briefly explained below:

Let $F$ be a nonarchimedean, nontrivially valued field which is locally compact. Let $X, X_1, Y, Y_1$ be nonarchimedean normed linear spaces over $F$. If $T: X \to X_1$, $S: Y \to Y_1$ are compact, it is shown in Section III that $T \otimes S: X \otimes Y \to X_1 \otimes Y_1$ is compact (where $X \otimes Y$, $X_1 \otimes Y_1$ are the completions of $X \otimes Y$, $X_1 \otimes Y_1$ respectively).
tively and $T^h$ is the unique continuous linear extension of $T^h$). In Section IV, it is shown that if $F$ is spherically complete, then the composition of two Fredholm operators is a Fredholm operator and also that the index of the composition is the sum of the indices of the two operators. If $H(X,Y)$ denotes the metric space consisting of all the Fredholm operators of $X$ into $Y$, then it is proved that the mapping

$\text{Ind}: H(X,Y) \rightarrow Z$ (where $Z$ is the set of all integers and the mapping $\text{Ind}$ maps each Fredholm operator to its index), is continuous.

If $T: X \rightarrow Y$ is a Fredholm operator and $k: X \rightarrow Y$ is a compact operator, it is shown that $T+k$ is Fredholm. Also, a characterization of Fredholm operators of index zero is given.

The corresponding results in the classical functional analysis can be found in [8] and [10].

In Section IV it is also shown that if $X,Y$ are nonarchimedean Banach spaces over a spherically complete field $F$, $T: X \rightarrow Y$ is a Fredholm operator and $M$ is a finite dimensional nonarchimedean Banach space over $F$, then $T^hM$ is Fredholm and further $\text{ind}(T^hM) = \text{ind}(T) \dim(M)$.

**Definition 1.1:** Let $F$ be a field and $R$ the set of all real numbers. A mapping $||: F \rightarrow R$ such that
for every $a,b$ in $F$

(i) $|a| \geq 0; \ |a| = 0$ if and only if $a = 0$,

(ii) $|ab| = |a| |b|$, and

(iii) $|a+b| \leq |a| + |b|,$

is called a valuation of $F$. $F$ is called a field with a valuation or a valued field.

If instead of (iii), the 'ultrametric inequality' is satisfied, that is, for all $a,b$ in $F$

$|a+b| \leq \max(|a|,|b|),$

then the valuation is called a nonarchimedean (N.A.) valuation. Otherwise it is called an archimedean valuation.

Examples 1.2: (i) The field $C$ of complex numbers in which $|a+ib| = \sqrt{a^2+b^2}$, is a valued field.

(ii) Let $Q$ be the field of rational numbers and let $p$ be a prime number. Define $|0| = 0$. If $a$ is in $Q$ and $a = (m/n)p^k$ where $m,n$ are integers each relatively prime to $p$, then define $|a| = p^{-k}$. $Q$ is a field with valuation. This valuation is N.A.

(iii) Let $F$ be a field. If $a \in F$ and $a \neq 0$, let $|a| = 1$. Let $|0| = 0$. The valuation is N.A. and is called the trivial valuation.
The following facts are known:

Remarks 1.3: (i) Let $F$ be a field with a N.A. valuation. Then $V = \{a \in F \mid |a| < 1\}$ is the unique maximal ideal of $V$.

(ii) Let $F$ be a field with a valuation. Then $|F^*| = \{|a| \mid a \in F$ and $a \neq 0\}$ is a multiplicative subgroup of the group of all positive reals, and hence is either cyclic or dense in the set of all positive reals.

Definition 1.4: If $|F^*|$ is cyclic, the valuation is called a discrete valuation.

The valuation in (1.2(ii)) is discrete.

Definition 1.5: Let $F$ be a field with a valuation and $X$ a linear space over $F$. Let $R$ be the set of all real numbers. A mapping $\|\| : X \to R$ such that for all $x, y$ in $X$ and $a$ in $F$

(i) $|x| \geq 0; \quad |x| = 0$ if and only if $x = 0$,

(ii) $|ax| = |a| |x|$, and

(iii) $|x+y| \leq |x| + |y|$, is called a norm on $X$ and $X$ is called a normed space over $F$.

If for all $x, y$ in $X$ we have $|x+y| \leq \max(|x|, |y|)$, then the norm is said to be nonarchimedeian and $X$ is called a nonarchimedean space over $F$. 
Otherwise it is archimedean.

The mapping $||$ defines a norm on $X$. $X$ is said to be a Banach space if $X$ is complete with respect to this norm.

**Examples 1.6:** (i) Any normed space over $\mathbb{R}$ or $\mathbb{C}$ (with valuation the usual absolute value) is an archimedean space.

(ii) Let $F$ be a field with a N.A. valuation. Consider the linear space $F^n$ with the norm defined thus:

$$|| (a_1, a_2, \ldots, a_n) || = \max(|a_1|, |a_2|, \ldots, |a_n|).$$

$F^n$ is a N.A. normed space over $F$.

(iii) Let $F$ be a field with a trivial valuation. Consider the vector space $X$ of all polynomials

$$\sum_{0}^{N} a_n x^n,$$

where $a_1, a_2, \ldots, a_N$ are in $F$. Let $r$ be a fixed positive number less than 1. Let $|0| = 0$.

For a nonzero polynomial $\sum_{0}^{N} a_n x^n$, with the last nonzero coefficient $a_N$, let $|\sum_{0}^{N} a_n x^n| = r^{-N}$. $X$ is a nonarchimedean normed space over $F$.

The following remarks are easy to prove:

**Remarks 1.7:** (i) If $X$ is a N.A. normed space over $F$ and if $X$ is not the zero space, then the valuation on $F$ is N.A.

(ii) If $X$ is a N.A. normed space over $F$ and if the sequence $(x_n)$ of elements in $X$ converges to
x and if \( x \neq 0 \), then there exists a positive integer \( N \) such that \( n > N \) implies \( |x_n| = |x| \).

(iii) If \( X \) is a complete N.A. normed space over \( F \), then the series in \( X \), namely \( \sum_{n=0}^{\infty} x_n \), converges if and only if \( \lim x_n = 0 \).

(iv) If \( X,Y \) are N.A. Banach spaces over \( F \), then their direct sum \( X \oplus Y \) with the maximum norm is a N.A. normed space over \( F \).

Definition 1.8: Let \( X \) be a normed space over a non-trivially valued field \( F \). \( X \) is locally compact if every bounded sequence in \( X \) has a convergent subsequence.

Example 1.9: The completion of \( Q \) with the valuation as in 1.2(ii), namely \( Q_p \), is locally compact. (This follows from Theorem 1 in [7], p.23.)

Theorem 1.10: If \( F \) is a locally compact, N.A. nontrivially valued field, then \( F \) is complete and discretely valued.

(For proof see [7], pp. 24,25.)

Definition 1.11: A N.A. normed space \( X \) over the valued field \( F \) is said to be spherically complete if every nest of closed spheres has a nonempty intersection.

Remark 1.12: (i) Since a metric space is complete if
and only if every nested sequence of closed spheres with the sequence of their diameters tending to zero has a nonempty intersection, it is clear that spherical completeness implies completeness.

(ii) There exist nonarchimedean normed spaces which are complete but not spherically complete. (See [7], Example 4, pp. 81-83.)

Definition 1.13: A normed space \( X \) over a valued field \( F \) is said to be discrete if \( 0 \) is the only limit point of the set \( \{|x| \mid x \in X\} \).

Theorem 1.14: Let \( X \) be a N.A. normed space over a valued field. If \( X \) is a complete discrete space, then \( X \) is spherically complete.

(For proof see [7], p. 35.)

Example 1.15: (i) The field \( \mathbb{Q}_p \) of example 1.9 is a normed vector space over itself. It is a complete discrete space. Hence \( \mathbb{Q}_p \) is spherically complete.

(ii) Let \( X = (\mathbb{Q}_p)^n \). If \( x = (a_1, a_2, \ldots, a_n) \) where \( a_1, a_2, \ldots, a_n \) are in \( \mathbb{Q}_p \), let \( |x| = \max(|a_1|, |a_2|, \ldots, |a_n|) \). \( X \) is a N.A. normed space over \( \mathbb{Q}_p \). Since \( \mathbb{Q}_p \) is a complete discrete space, so is \( X \). Hence \( X \) is spherically complete.

Definition 1.16: Let \( X, Y \) be normed spaces over the
valued field $F$. A linear transformation $f$ of $X$ into $Y$ is said to be bounded if there exists a real number $M$ such that $|f(x)| \leq M|x|$ for all $x \in X$.

Remarks 1.17: (i) With the definition $|f| = \inf\{M | f(x)| \leq M|x| \text{ for all } x \in X\}$, the set of all bounded linear transformations of $X$ into $Y$, namely $L(X,Y)$, is a normed linear space over $F$. $X'$ denotes $L(X,F)$.

(ii) If $Y$ is a N.A. normed space, then so is $L(X,Y)$.

(iii) If $f \in L(X,Y)$, then $f'$ denotes the function on $Y'$ into $X'$ defined by $f'(g) = gf$.

Notation 1.18: If $X$ is a normed linear space over the valued field $F$, then $|x|$ denotes the set $\{|x| \mid x \in X\}$.

Proposition 1.19: Let $X,Y$ be N.A. normed spaces over the valued field $F$. Let $f \in L(X,Y)$. If $|X| = |F|$, then $|f| = \sup\{|f(x)| \mid x \in X \text{ and } |x| \leq 1\}$.

Proof: Let $k = \sup\{|f(x)| \mid x \in X \text{ and } |x| \leq 1\}$. By the definition of $|f|$, $|f(x)| \leq |f||x|$. If $|x| \leq 1$, then $|f(x)| \leq |f|$. So, $k \leq |f|$. If $x \in X$, and $x \neq 0$, let $a \in F$ such that $|a| = 1/|x|$. Since $|ax| = 1$, $|f(ax)| \leq k$. Hence, $|a||f(x)| \leq k$. $|f(x)| \leq k/|a| = k|x|$.
Now, for every $x$ in $X$, $|f(x)| \leq k|x|$ and hence $|f| \leq k$. This shows that $|f| = k$.

**Proposition 1.20:** If $X,Y$ are normed linear spaces over a nontrivially valued field $F$ and if $f$ is a linear transformation of $X$ into $Y$, then the following are equivalent:

(i) $f$ is continuous.

(ii) $\{|f(x)| : |x| \leq 1\}$ is a bounded set.

(iii) $f$ is bounded.

**Proof:** See [7], p. 77.

**Definition 1.21:** Let $X,Y$ be normed spaces over the valued field $F$. $Y$ is said to have the extension property if for every subspace $Z$ of $X$, every bounded linear map $f$ of $Z$ into $Y$ can be extended to a bounded linear map $f_1$ of $X$ into $Y$ such that $|f| = |f_1|$.

The following theorem due to Ingleton gives the necessary and sufficient condition for $Y$ to have the extension property:

**Theorem 1.22:** (The Hahn-Banach Theorem for N.A. normed spaces)

A N.A. normed space $Y$ has the extension property if and only if it is spherically complete.
Proof: See [3].

**Corollary 1.23:** A N.A. normed space \( Y \) which is complete and discrete has the extension property.

**Proof:** By (1.14), \( Y \) is spherically complete and hence has the extension property.

The rest of this section is devoted to tensor products of N.A. Banach spaces over spherically complete \( F \).

Let \( X, Y \) be linear spaces over \( F \). Let \( F(X \times Y) \) be the linear space over \( F \) of all the functions of \( X \times Y \) into \( F \) which have finite support.

If \( x \) is in \( X \) and \( y \) is in \( Y \), let \([x, y]\) denote the function \( f: X \times Y \to F \) defined by \( f(x, y) = 1 \) and \( f(a, b) = 0 \) when \((a, b) \neq (x, y)\). Let \( S \) be the linear subspace of \( F(X \times Y) \) generated by the set of all functions of the following form:

\[
[a, mc + nd] - m[a, c] - n[a, d]
\]

\((m, n \in F, a, b \in X, \text{ and } c, d \in Y)\)

Denote the quotient space \( F(X \times Y) / S \) by \( X \otimes Y \) and the coset \([x, y] + S\) by \( x \otimes y\). It is readily seen that every element of \( X \otimes Y \) can be written as a finite sum

\[\sum (x_i \otimes y_i).\]
The 'tensor map' $T : X \otimes Y \rightarrow X \otimes Y$ defined by
$T(x,y) = x \otimes y$ is bilinear. Also, $T$ is universal
in the following sense:

If $Z$ is a linear space over $F$ and $f : X \otimes Y \rightarrow Z$ is a bilinear map, then there exists a unique linear map
$h : X \otimes Y \rightarrow Z$, such that $f = h_T$. (See [4], p. 321.)

Throughout the rest of this section $F$ is assumed
to be spherically complete. $X,Y$ are N.A. Banach spaces
over $F$.

**Definition 1.24:** If $u$ is in $X \otimes Y$ define $|u| = \inf \{\max_i |x_i| |y_i| \ |u = \Sigma_i (x_i \otimes y_i)\}$.

It will be shown that (1.24) defines a norm
on $X \otimes Y$.

**Lemma 1.25:** If $u$ is in $X \otimes Y$ and $u \neq 0$, then,
there exist $x_1, x_2, \ldots, x_m$ which are linearly indepen-
dent elements of $X$ and $y_1, y_2, \ldots, y_m$ which are linear-
ly independent elements of $Y$, such that
$u = x_1 \otimes y_1 + x_2 \otimes y_2 + \ldots + x_m \otimes y_m$.

**Proof:** Let $u = \sum_{i=1}^{n} a_i \otimes b_i$, where $a_1, a_2, \ldots, a_n$ are in $X$
and $b_1, b_2, \ldots, b_n$ are in $Y$.

If $a_1, a_2, \ldots, a_n$ are not linearly independent,
then, without loss of generality assume $a_1$ is linearly
dependent on \(a_2, a_3, \ldots, a_n\). Let \(a_1 = \sum_{i=2}^{n} k_ia_i\), where \(k_2, k_3, \ldots, k_n\) are in \(F\).

\[
u = \sum_{i=1}^{n} a_i \otimes b_i = (\sum_{i=2}^{n} k_ia_i) \otimes b_1 + (a_2 \otimes b_2) + \ldots + (a_n \otimes b_n)
\]

\[
= a_2 \otimes (k_2b_1 + b_2) + a_3 \otimes (k_3b_1 + b_3) + \ldots + a_n \otimes (k_nb_1 + b_n)
\]

\[
= a_2 \otimes c_2 + a_3 \otimes c_3 + \ldots + a_n \otimes c_n , a \text{ sum of } (n-1) \text{ terms.}
\]

This process can be repeated till finally we obtain

\[
u = \sum_{i=1}^{m} x_i \otimes y_i , \text{ where } x_1, x_2, \ldots, x_m \text{ are linearly independent in } X \text{ and } y_1, y_2, \ldots, y_m \text{ are linearly independent in } Y \text{ or } \nu = z \otimes 0, \text{ or } \nu = 0 \otimes w. \text{ } \nu \neq 0 \text{ implies }
\]

\[
u = \sum_{i=1}^{m} x_i \otimes y_i , \text{ where } x_1, x_2, \ldots, x_m \text{ are linearly independent in } X \text{ and } y_1, y_2, \ldots, y_m \text{ are linearly independent in } Y.
\]

**Lemma 1.26:** Let \(f\) be in \(X'\). If \(u\) is in \(X \otimes Y\) and \(u = \sum_{i=1}^{n} a_i \otimes b_i = \sum_{i=1}^{m} c_i \otimes d_i\), then \(\sum_{i=1}^{n} f(a_i)b_i = \sum_{i=1}^{m} f(c_i)d_i\).

**Proof:** Let \(g: X \times Y \to Y\) be defined by \(g(x, y) = f(x)y\).

It is clear that \(g\) is bilinear. By the universality of the tensor map, there exists a unique linear map \(h: X \otimes Y \to Y\), such that \(h(a \otimes b) = g(a, b) = f(a)b\).

If \(\sum_{i=1}^{n} a_i \otimes b_i = \sum_{i=1}^{m} c_i \otimes d_i\), then \(h(\sum_{i=1}^{n} a_i \otimes b_i) = h(\sum_{i=1}^{m} c_i \otimes d_i)\). This shows that \(\sum_{i=1}^{n} f(a_i)b_i = \sum_{i=1}^{m} f(c_i)d_i\).

**Lemma 1.27:** Let \(u\) be in \(X \otimes Y\) and \(u \neq 0\). If \(u = \sum_{i=1}^{n} a_i \otimes b_i\), then there exists \(f\) in \(X'\) such that \(\sum_{i=1}^{n} f(a_i)b_i \neq 0\).
Proof: Since \( u \neq 0 \), by (1.25), we can find \( c_1, c_2, \ldots, c_m \) linearly independent in \( X \) and \( d_1, d_2, \ldots, d_m \) linearly independent in \( Y \) such that \( u = \sum_{1}^{m} c_i \otimes d_i \).

Let \( Z \) be the subspace of \( X \) generated by \( c_1 \). Define \( f^*: Z \rightarrow F \) by \( f^*(kc_1) = k \) for every \( k \) in \( F \). \( f^* \) is linear and continuous. \( F \) is spherically complete. By the Hahn-Banach theorem (1.22), there exists \( f \) in \( X' \) such that \( f \) is an extension of \( f^* \).

\( d_1, d_2, \ldots, d_m \) are linearly independent elements in \( Y \) and \( f(c_1) = f^*(c_1) = 1 \). Hence \( \sum_{1}^{m} f(c_i)d_i \neq 0 \). By (1.26), \( \sum_{1}^{m} f(a_i)b_i = \sum_{1}^{m} f(c_i)d_i \neq 0 \).

**Proposition 1.28:** (i) The formula \( |u| = \inf \{ \max |x_i||y_i| \} \) defines a N.A. norm on \( X \otimes Y \).

(ii) For every \( x \) in \( X \) and \( y \) in \( Y \),
\[
|x \otimes y| = |x||y|.
\]

**Proof:** (i) For every \( u \) in \( X \otimes Y \), \( |u| \geq 0 \). Also if \( u = 0 \), then \( u = 0 \otimes 0 \) and so \( |u| = 0 \).

Let \( |u| = 0 \). It will be shown that \( u = 0 \). Assume \( u = \sum_{1}^{n} x_i \otimes y_i \neq 0 \). By (1.27), there exists \( f \) in \( X' \) such that \( \sum_{1}^{n} f(x_i)y_i \neq 0 \).

\[
|\sum_{1}^{n} f(x_i)y_i| = \max |f(x_i)||y_i| \leq |f| \max |x_i||y_i|.
\]

If \( \sum_{1}^{n} x_i \otimes y_i = \sum_{1}^{m} a_i \otimes b_i \), then, by (1.26), \( \sum_{1}^{n} f(x_i)y_i = \sum_{1}^{m} f(a_i)b_i \). It follows that \( |\sum_{1}^{n} f(x_i)y_i| \leq |f| \inf \{ \max |a_i||b_i| \} \) |u = \sum_{1}^{m} a_i \otimes b_i \}. This implies
\[ |u| \geq (1/|f|) \sum_{i=1}^{n} f(x_i) y_i > 0. \] This is a contradiction. Hence \( u = 0 \).

Next it will be shown that \( |u+v| < \max(|u|, |v|) \).

Let \( r \) be a positive real number. By definition (2.9),
\[ u = \sum_{k=1}^{p} a_k \otimes b_k, \quad v = \sum_{s=1}^{q} c_s \otimes d_s \]
where \( \max|a_k||b_k| < |u| + r \) and \( \max|c_s||d_s| < |v| + r \).

\[ |u+v| = |\sum_{k=1}^{p} a_k \otimes b_k + \sum_{s=1}^{q} c_s \otimes d_s| < \max(|a_k||b_k|, |c_s||d_s|) \]
\[ < \max(|u|, |v|) + r. \]

For every positive real number \( r \), \( |u+v| < \max(|u|, |v|) + r \). This proves \( |u+v| \leq \max(|u|, |v|) \).

Also it is clear that if \( k \) is in \( F \) and \( u \) is in \( X \otimes Y \), then \( |ku| = |k||u| \).

Thus \( X \otimes Y \) is a N.A. normed space over \( F \).

(ii) If \( x \neq 0 \), it can be shown, as in (1.27), that there exists \( f \) in \( X' \) such that \( f(x) = 1 \) and furthermore \( |f| = 1/|x| \).

If \( x \otimes y = \sum_{i=1}^{n} x_i \otimes y_i \), then by (1.26), \( f(x)y = \sum_{i=1}^{n} f(x_i)y_i \), (i.e.), \( y = \sum_{i=1}^{n} f(x_i)y_i \).

\[ |y| \leq |f| \inf\{\max|x_i||y_i| \mid x \otimes y = \sum_{i=1}^{n} x_i \otimes y_i \} \]

\[ |x \otimes y| \geq (1/|f|) |y| = |x||y| \]

By definition \( |x \otimes y| \leq |x||y| \). So, \( |x \otimes y| = |x||y| \).

Remark 1.29: If \( X, X_1, Y, Y_1 \) are normed spaces over \( F \)
and if \( T : X \to X_1 \) and \( S : Y \to Y_1 \) are linear, then
the formula \((T \& S) (x \& y) = Tx \& Sy\) defines a linear map \(T \& S: X \& Y \to X_1 \& Y_1\).

(For proof see [4], p. 324.)

**Proposition 1.30:** Let \(X,X_1,Y,Y_1\) be N.A. Banach spaces over \(F\). If \(T: X \to X_1, S: Y \to Y_1\) are linear and continuous, then \(T \& S\) is continuous and

\[ |T \& S| \leq |T| |S|. \]

**Proof:** If \(u = \sum_{1}^{n} x_i \& y_i\) is in \(X \& Y\), then \(|T \& S (u)|\)

\[ = \left| \sum_{1}^{n} Tx_i \& Sy_i \right| \leq \max |Tx_i| |Sy_i| \quad \text{(by (1.28))} \]

\[ \leq \max |x_i| |y_i| |T| |S| \]

\[ |(T \& S) u| \leq |T| |S| \inf \{ \max |x_i| |y_i| \mid u = \sum_{1}^{n} x_i \& y_i \} \]

\[ = |T| |S| |u| \]

This implies that \(T \& S\) is continuous and that

\[ |T \& S| \leq |T| |S|. \]

**Notations 1.31:** If \(X,Y\) are normed linear spaces over \(F\), then the completion of \(X\) is denoted by \(\hat{X}\), and the completion of \(X \& Y\) is denoted by \(X \& Y\).

If \(T: X \to Y\) is a continuous linear map, the unique continuous linear map of \(\hat{X}\) into \(\hat{Y}\) which extends \(T\) is denoted by \(\hat{T}\).

If \(X,X_1,Y,Y_1\) are normed linear spaces over \(F\) and if \(T: X \to X_1, S: Y \to Y_1\) are continuous linear maps, then \(\hat{T} \& S\) denotes \((\hat{T} \& S)\).
Definition 1.32: With the notations as in (1.31), $X \hat{\otimes} Y$ is called the Tensor product of the Banach spaces $X,Y$. $T \hat{\otimes} S$ is called the Tensor product of operators $T$ and $S$.

Remark 1.33: With the notations as in (1.31), it can be shown that $|T| = |\hat{T}|$.

The following facts are easy to prove:

Remark 1.34: Let $X,Y,Z$ be N.A. normed spaces over $F$ and $f: X \to Y$, $g: Y \to Z$ be continuous linear maps.

(i) If $h = gf$, then $\hat{h} = \hat{g} \hat{f}$, (ii) $\hat{1}_X = 1_\hat{X}$, and (iii) $(\hat{f})^{-1} = (f^{-1})$.

Proposition 1.35: Let $X,Y,Z$ be N.A. Banach spaces over $F$ and $f: X \to Y$ be an isomorphism of topological vector spaces, then $f \hat{\otimes} 1: X \hat{\otimes} Z \to Y \hat{\otimes} Z$ is an isomorphism of topological vector spaces.

Proof: Let $g = (f^{-1}\otimes 1_Z)(f\otimes 1_Z)$ and $h = (f\otimes 1_Z)(f^{-1}\otimes 1_Z)$.

By (1.34), $\hat{g} = (f^{-1}\otimes 1_Z)(f\otimes 1_Z)$. Also, $g = (f^{-1}f)\otimes 1_Z = 1_X \otimes 1_Z = 1_X \otimes 1_Z$. Therefore $\hat{g} = \hat{1}_X \otimes 1_Z = 1_\hat{X} \otimes 1_Z$; that is, $(f^{-1}\otimes 1_Z)(f\otimes 1_Z) = 1_\hat{X} \otimes 1_Z$.

Likewise, it can be shown that $(f\otimes 1_Z)(f^{-1}\otimes 1_Z) = 1_\hat{Y} \otimes 1_Z$. This shows that $f \hat{\otimes} 1_Z$ is an isomorphism of topological vector spaces.
Definition 1.36: The short exact sequence (S.E.S.) of Banach spaces \( 0 \to X \to Y \to Z \to 0 \) (where the maps are continuous linear) is said to 'split' if there exists a continuous linear map \( h: Y \to X \) such that \( hf = 1_X \).

The following fact (1.37) is known:

Remark 1.37: The S.E.S. of Banach spaces \( 0 \to X \to Y \to Z \to 0 \) splits if and only if there exists an isomorphism of topological vector spaces \( t: Y \to X \oplus Z \), such that the triangles in the following diagram commute:

\[
\begin{array}{ccc}
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
& & \downarrow i & & \downarrow t & & \downarrow p & & \\
& & & \downarrow X \oplus Z & & & & & \\
\end{array}
\]

\( (i \text{ is the insertion and } p \text{ is the natural projection}) \)

Theorem 1.38: Let \( F \) be spherically complete. Let \( X,Y,Z \) be N.A. Banach spaces over \( F \) and let \( 0 \to X \to Y \to Z \to 0 \) be a S.E.S. of Banach spaces which splits. If \( M \) is a N.A. Banach space over \( F \), then the following is a S.E.S. of Banach spaces:

\[
0 \to X \otimes M \to Y \otimes M \to Z \otimes M \to 0 .
\]

Proof: By (1.37) and (1.35), it is enough to show that
the following is a S.E.S. of Banach spaces:

\[(1) \quad 0 \rightarrow X \overset{i}{\rightarrow} X \hat{\otimes} Z \overset{p}{\rightarrow} Z \rightarrow 0\]

(\(i: X \rightarrow X \otimes Z\) is the insertion and \(p: X \otimes Z \rightarrow Z\) is the natural projection.)

Define \(t: (X \otimes Z) \hat{\otimes} M \rightarrow (X \otimes M) \oplus (Z \otimes M)\), by

\[t((x,z) \otimes m) = (x \otimes m, z \otimes m).\]

\(t\) can be shown to be an isomorphism of linear spaces (See [4], p. 323). It will now be shown that \(t\) is an isomorphism of topological vector spaces.

Let \(u\) be in \((X \otimes Z) \hat{\otimes} M\) and let \(u = \sum (x_i, z_i) \otimes m_i\):

\[t(u) = (\sum x_i \otimes m_i, \sum z_i \otimes m_i).\]

Let \(v = \sum x_i \otimes m_i\) and \(w = \sum z_i \otimes m_i\).

\[|v| \leq \max |x_i \otimes m_i| = \max |x_i| |m_i| \quad (\text{by } (1.28)).\]

Similarly, \(|w| \leq \max |z_i| |m_i|\). Therefore, \(|tu| = \max(|v|, |w|) \leq \max(|(x_i, z_i)| |m_i|)\). It follows from the definition of \(|u|\) that \(|tu| \leq |u|\). So \(t\) is continuous.

Let \(a = \sum x_i \otimes m_i\) in \(X \otimes M\) and \(b = \sum z_j \otimes n_j\) in \(Z \otimes M\).

\[|t^{-1}(a,b)| = |\sum (x_i,0) \otimes m_i + \sum (0,z_j) \otimes n_j| \leq \max(|x_i| |m_i|, |z_j| |n_j|).\]

Hence \(|t^{-1}(a,b)| \leq \max(|a|, |b|) = |(a,b)|\). This shows \(t^{-1}\) is continuous. Thus \(t\) is an isomorphism of topological vector spaces. By (1.34), \(\hat{t}\) is an isomorphism of topological vector spaces.
Let \( j = \hat{t}(i \otimes 1) \) and \( q = (p \otimes 1)(\hat{t})^{-1} \).

\((\hat{X} \otimes M) \oplus (\hat{Z} \otimes M)\) is the completion of \((X \otimes M) \oplus (Z \otimes M)\). By (1.35), it follows that in order to prove the exactness of (1), it is enough to prove the exactness of the following S.E.S.:

\[
\begin{align*}
(2) \quad 0 \to \hat{X} \otimes M & \xrightarrow{i} (\hat{X} \otimes M) \oplus (\hat{Z} \otimes M) \to \hat{Z} \otimes M \to 0.
\end{align*}
\]

Let \( u \) be in \( \hat{X} \otimes M \) and \( u = \lim u_n \) where \( u_n \) is in \( X \otimes M \). \( j(u) = \lim t(i \otimes 1)(u_n) = \lim (u_n, 0) = (\lim u_n, 0) = (u, 0) \). This shows that \( j \) is the injection map.

Let \( v \) be in \( \hat{Z} \otimes M \) and \( v = \lim v_n \) where

\[
\begin{align*}
v_n &= \Sigma(z_n, j \otimes m_n, j).
\end{align*}
\]

\( u_n = \Sigma(x_n, i \otimes m_n, i) \).

\[
\begin{align*}
q(u, v) &= \lim_{n \to \infty} (p \otimes 1)t^{-1}(\Sigma x_n, i \otimes m_n, i, j, \Sigma z_n, j \otimes m_n, j),
\end{align*}
\]

\[
\begin{align*}
&= \lim_{n \to \infty} (p \otimes 1)(\Sigma(x_n, i, 0) \otimes m_n, i + \Sigma(0, z_n, j) \otimes m_n, j)
\end{align*}
\]

\[
\begin{align*}
&= \lim v_n
\end{align*}
\]

\[
\begin{align*}
&= v
\end{align*}
\]

This shows that \( q \) is the natural projection. It follows that (2) is a S.E.S. of Banach spaces.

This completes the proof.
II LINEAR OPERATORS

In this section and the subsequent sections the following notations are used:

Notations 2.1: (i) $F$ denotes a field with a complete nontrivial valuation.

(ii) If $X$ is a N.A. Banach space over $F$ and $S$ is a closed subspace of $X$, then $X/S$ is the N.A. Banach space formed by the quotient space of $X$ by $S$ with the norm defined thus:

$$|x + S| = \inf \{|x + s| : s \in S\}.$$

Theorem 2.2: (The open mapping theorem) Let $X, Y$ be N.A. Banach spaces over $F$. If $T$ is in $L(X, Y)$ and is onto, then $T$ is open.

Proof: See [2].

Proposition 2.3: If $X, Y$ are N.A. Banach spaces over $F$ and if $T: X \to Y$ is linear, continuous and surjective, then the induced bijection $T_1: X/\ker(T) \to Y$ is an isomorphism of topological vector spaces.

Proof: If $T = 0$, then $X/\ker(T)$ and $Y$ are each zero and hence $T_1$ is clearly an isomorphism of topological vector spaces.

Let $T \neq 0$. $|Tx| = |T(x + z)|$ for all $z$ in
\[ \ker(T). \quad |T(x+z)| \leq |T||x+z|. \text{ So, } |x+z| \geq |Tx|/|T| \]
and \[ |x+\ker(T)| = \inf\{|x+z| \mid z \in \ker(T)\} \geq |Tx|/|T| . \]

Now \[ |T_1(x+\ker(T))| = |Tx| \leq |T||x+\ker(T)| . \]

Hence \( T_1 \) is continuous.

\( T_1 \) is a continuous linear bijection. So, by the open mapping theorem, \( T_1 \) is an isomorphism of topological vector spaces.

**Proposition 2.4:** Let \( F \) be spherically complete and let \( 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \) be a S.E.S. of N.A. Banach spaces over \( F \). Then \( 0 \to Z' \xrightarrow{g'} Y' \xrightarrow{f'} X' \to 0 \) is a S.E.S. of Banach spaces.

**Proof:** For every \( h \in Y' \), \( |f'(h)| = |hf| \leq |h||f| \). \( f' \) is continuous and so is \( g' \). \( g' \) is one to one since \( g \) is onto.

Let \( h \) be in \( X' \). \( \text{Im}(f) = \ker(g) \) is closed in \( Y \) and hence is a Banach subspace of \( Y \). Also \( f \) is one to one. Hence by the open mapping theorem, \( f_1: X \to \text{Im}(f) \) defined by \( f_1(x) = f(x) \), is an isomorphism of topological vector spaces. \( F \) being spherically complete \( hf^{-1}_1: \text{Im}(f) \to F \) can be extended to \( h_1: Y \to F \). Clearly, \( h_1f = h \); that is, \( f'(h_1) = h \). So \( f' \) is onto.

Since \( gf = 0 \), \( f'g' = 0 \). Therefore \( \text{Im}(g') \subset \ker(f') \). It will be shown that \( \ker(f') \subset \text{Im}(g') \).

Let \( p \) be in \( \ker(f') \). So \( f'(p) = 0 \); that is,
pf = 0. This implies \( \ker(p) \supset \text{Im}(f) = \ker(g) \). There exists a unique map \( k : Z \to F \) which is linear and is such that \( kg = p \).

If \( S \) is an open subset of \( F \), \( g^{-1}k^{-1}(S) = p^{-1}(S) \). Since \( g \) is onto, this means \( k^{-1}(S) = gp^{-1}(S) \). Now \( p \) is continuous and by the open mapping theorem, \( g \) is open. Therefore \( gp^{-1}(S) \) is open. Hence \( k^{-1}(S) \) is open. This shows that \( k \) is continuous.

Now \( k \) is in \( Z' \) and \( g'(k) = p \); that is, \( p \) is in \( \text{Im}(g') \). Thus \( \ker(f') \subset \text{Im}(g') \) and hence \( \ker(f') = \text{Im}(g') \).

This completes the proof.

\textbf{Remark 2.5:} Let \( F \) be spherically complete. Let \( X, Y, Z \) be N.A. Banach spaces over \( F \). It follows that if \( X \overset f \to Y \overset q \to Z \) is exact at \( Y \) and \( g(y) \) is closed in \( Z \), then \( Z' \overset{g'} \to Y' \overset{f'} \to X' \) is exact at \( Y' \).

\textbf{Theorem 2.6:} Let \( F \) be spherically complete and let \( X, Y \) be N.A. Banach spaces over \( F \). If \( T \) is in \( L(X,Y) \) and has closed range, then there are linear topological isomorphisms:

\[ \ker(T') = (\text{coker}(T))' \]
\[ \text{coker}(T') = (\ker(T))' \]
Proof: Since $T$ has closed range, $\text{coker}(T)$ is a Banach space. Consider the exact sequence of Banach spaces:

$$0 \to \ker(T) \overset{i}{\to} X \overset{T}{\to} Y \overset{j}{\to} \text{coker}(T) \to 0$$

By (2.5), this sequence gives the exact sequence of Banach spaces:

$$0 \to (\text{coker}(T))' \overset{i'}{\to} Y' \overset{T'}{\to} X' \overset{j'}{\to} (\ker(T))' \to 0.$$

So, by the open mapping theorem, $(\text{coker}(T))' \sim \text{Im}(j') = \ker(T')$. Now $\text{Im}(T') = \ker(i')$ is closed in $X'$. Hence $\text{coker}(T')$ is a Banach space. From the exact sequence of Banach spaces:

$$0 \to \ker(T') \overset{i'}{\to} Y' \overset{T'}{\to} X' \overset{j'}{\to} \text{coker}(T') \to 0,$$

it follows that $\ker(v) = \text{Im}(T') = \ker(i')$. Also $v$ is onto. Therefore there exists a unique $h$:

$$(\ker(T))' \to \text{coker}(T')$$

which is linear and a bijection such that $hi' = v$. $i'$ being open (by the open mapping theorem) and $v$ being continuous, it is easy to see that $h$ is continuous. Hence, by the open mapping theorem, $h$ is an isomorphism of topological vector spaces.

**Proposition 2.7**: Let $F$ be spherically complete and let $X$ be a N.A. Banach space over $F$. If $S$ is a
closed subspace of $X$ and $S^\perp = \{ f \in X' | f(x) = 0 \}
for all $x$ in $S$}, then $S^\perp$ and $(X/S)'$ are isomorphic
as topological vector spaces.

**Proof:** $X$ is a Banach space and $S$ is a closed sub-

space of $X$. $X/S$ with the norm defined by $\|x+S\|
= \inf\{|x+s| : s \in S\}$, is a Banach space. Consider the

S.E.S. Banach spaces: $0 \rightarrow S \rightarrowtail X \twoheadrightarrow X/S \rightarrow 0$

where $i$ is the insertion map and $\pi$ is the natural
projection. Since $F$ is spherically complete, we get
by (2.4), the S.E.S. of Banach spaces

$$0 \rightarrow (X/S)' \twoheadrightarrow X' \rightarrowtail S' \rightarrow 0.$$ 

Hence $\text{Im}(\pi') = \text{ker}(i')$. So $\text{Im}(\pi')$ is closed in $X'$.

Since $X'$ is a Banach space, $\text{Im}(\pi')$ is also a Banach
space. $\pi'$ induces a linear, continuous, one to one
map of the Banach space $(X/S)'$ onto the Banach space

$\text{Im}(\pi')$. By the open mapping theorem $(X/S)'$ and

$\text{Im}(\pi')$ are isomorphic as topological vector spaces.

But $\text{Im}(\pi') = \text{ker}(i')$. If $f$ is in $X'$, then

$i'(f) = fi = f|S$. So, $\text{ker}(i') = S^\perp$. Hence $(X/S)'$

and $S^\perp$ are isomorphic as topological vector spaces.

**Proposition 2.8:** If $S$ and $T$ are complementary

closed subspaces of a Banach space $X$, then

$f: S \oplus T \rightarrow X$ defined by $f(s,t) = s+t$, is an isomor-

phism of topological vector spaces.
Proof: \( S, T \) being closed subspaces of the Banach space \( X \), are complete and hence are Banach spaces. So, \( S \oplus T \) is a Banach space. \( f \) is linear, one to one and onto. Also, \(|f(s,t)| = |s+t| \leq \max(|s|,|t|) = |(s,t)|\). So \( f \) is continuous. Hence, by the open mapping theorem, \( f \) is an isomorphism of topological vector spaces.

**Proposition 2.9:** Let \( X \) be a N.A. Banach space over \( F \) and \( S \) a closed subspace of \( X \).

(i) If \( F \) is spherically complete and if \( S \) is finite dimensional then \( S \) is closed and has a complementary closed subspace \( T \).

(ii) If \( S \) is closed and \( X/S \) is finite dimensional, then \( S \) has a complementary closed subspace \( T \).

(So, in either case, \( X \) and \( S \oplus T \) are isomorphic as topological vector spaces.)

**Proof:** (i) Let \( \dim(S) = n \). So \( \dim(S') = n \). Let \( B = \{x_1, x_2, \ldots, x_n\} \) be a basis of \( S \). If \( i: S \to X \) is the insertion map, then by (2.4), \( i': X' \to S' \) is onto. (Note that \( F \) is spherically complete.) There exist \( k_1, k_2, \ldots, k_n \) in \( X' \) such that \( \{k_1|S, \ldots, k_n|S\} \) is a basis of \( S' \) and also such that \( k_i(x_j) = 0 \) if \( i \neq j \) and \( k_i(x_i) = 1 \).

Define \( P: X \to X \) by \( P(x) = k_1(x)x_1 + k_2(x)x_2 + \)
Clearly $P$ is linear, continuous and $P(X) = S$. Also $P^2 = P$. If $x$ is in $X$, $x = Px + (x-Px)$ where $Px \in P(X)$ and $x-Px \in \ker(P)$. Also, $P(X) \cap \ker(P) = \{0\}$. $PX$ and $\ker(P)$ are complementary subspaces of $X$. $\ker(P)$ is closed since it is finite dimensional. (See [7], p. 71.) $S$ and $\ker(P)=T$ are complementary closed subspaces of $X$. By (2.8), $X$ and $S \oplus T$ are isomorphic as linear topological spaces.

(ii) Let $\dim(\operatorname{coker}(S)) = m$. Let $x_1, x_2, \ldots, x_m$ be elements in $X$ such that $\{x_i+S \mid i = 1, 2, \ldots, m\}$ is a basis for $X/S$. Every element $x$ in $X$ can be written in the form $x = a_1x_1 + a_2x_2 + \ldots + a_mx_m + s$ where $a_1, a_2, \ldots, a_m$ are in $F$ and $s$ is in $S$. Further, $a_1x_1 + a_2x_2 + \ldots + a_mx_m$ is in $S$ implies $\sum_{1}^{m} a_i(x_i+S) = S$. This means $a_i = 0$ $(1 \leq i \leq m)$ since $\{x_i+S \mid i = 1, 2, \ldots, m\}$ is a basis for $X/S$. If $T$ is the subspace of $X$ generated by $x_1, x_2, \ldots, x_m$ then it is clear that $X = S + T$ where $S \oplus T = \{0\}$.

Now $S$ is closed in $X$. $T$ is finite dimensional and so is closed in $X$. (See [6], p. 71.) Hence $S$ and $T$ are complementary closed subspaces of $X$. By (2.8), it follows that $X$ and $S \oplus T$ are isomorphic as topological vector spaces.
Proposition 2.10: If $X, Y$ are N.A. Banach spaces over $F$ and $X \neq \{0\}$, $Y \neq \{0\}$, then the set of all linear topological isomorphisms of $X$ onto $Y$ is open in $L(X, Y)$.

Proof: If the set $U$ of all topological linear isomorphisms of $X$ onto $Y$ is empty, $U$ is open in $L(X, Y)$.

If $U$ is nonempty, and $T$ is in $U$, let $S$ be the continuous inverse of $T$. Let $T_1$ be in $L(X, Y)$ such that $|T - T_1| < |S|^{-1}$ and let $A = T - T_1$. Since $|SA| \leq |S| |T - T_1| < 1$ and $|(SA)^n| \leq |SA|^n$, the series $1_X + SA + (SA)^2 + \ldots + (SA)^n + \ldots$ converges (say) to $B$ in $L(X, X)$.

\[ \lim_{n \to \infty} (1_X - SA) (1_X + SA + (SA)^2 + \ldots + (SA)^n - 1) = (1_X - SA)B \]

\[ 1_X - \lim_{n \to \infty} (SA)^n = (1_X - SA)B \]

\[ 1_X = (1_X - SA)B \]

Similarly, it can be shown that $B(1_X - SA) = 1_X$. So $1_X - SA$ is a linear topological automorphism of $X$. Therefore $T(1_X - SA) = T - A = T_1$ is a linear topological isomorphism of $X$ onto $Y$. Hence $U$ is open.
III COMPACT OPERATORS

In this section the concept of a compact operator is defined and some known facts on compact operators are given. In the case when $F$ is locally compact, it is shown that the tensor product of two compact operators is compact.

Definition 3.1: If $X,Y$ are N.A. Banach spaces over $F$, then $k$ in $L(X,Y)$ is said to be compact if for every bounded subset $S$ of $X$, $kS$ is compact. (Equivalently, for every bounded sequence $(x_n)$ in $X$, the sequence $(kx_n)$ in $Y$ has a convergent subsequence.)

The following facts (3.2) are known:

Remarks 3.2: Let $X,Y$ be N.A. Banach spaces over $F$.

(i) If $k$ in $L(X,Y)$ is compact and $k \neq 0$, then $F$ is locally compact.

(For proof see [7], p. 86.)

(ii) The subset $C(X,Y)$ of all compact operators in $L(X,Y)$ is a closed subspace of $L(X,Y)$.

(For proof see [9], p. 84.)

(iii) If $k$ is in $C(X,Y)$, $f$ is in $L(Y,Z)$ and $g$ is in $L(Z,X)$, then $fk$ is in $C(X,Z)$ and
Proposition 3.3: Let $F$ be locally compact. If $X,Y$ are N.A. Banach spaces over $F$ and if $k$ is in $L(X,Y)$ and has finite dimensional image, then $k$ is compact.

Proof: If $k = 0$, then $k$ is compact.

If $k \neq 0$, then $\text{Im}(k)$ has dimension $n$. $\text{Im}(k)$ and $F^n$ are isomorphic as topological vector spaces. (See [7], p.71) $F$ being locally compact, $F^n$ is locally compact and hence $\text{Im}(k)$ is locally compact.

If $(x_n)$ is a bounded sequence in $X$, $(kx_n)$ is a bounded sequence in $\text{Im}(k)$ and hence has a convergent subsequence. This proves $k$ is compact.

Theorem 3.4: Let $X,Y$ be N.A. Banach spaces over $F$.

If $k$ is in $L(X,Y)$ and is compact, then $k$ is a limit of a sequence of operators of finite dimensional image.

Proof: In [11], Serre proves this result in the case when $|X| \subseteq |F|$ and $|Y| \subseteq |F|$. It will be shown that the result is true in general.

If $k = 0$, the theorem is true. If $k \neq 0$, then by (3.2(i)), $F$ is locally compact. Hence $F$ is complete and discretely valued. (See [7], pp. 24,25.) Also, $X,Y$ are complete. So there are linear topo-
logical isomorphisms $f: X \rightarrow X^*$, $g: Y \rightarrow Y^*$ where $X^*, Y^*$ are N.A. Banach spaces over $F$ and $|X^*| \subseteq |F|$, $|Y^*| \subseteq |F|$. (For proof see [5], p.44.)

$k \in C(X,Y)$, $f^{-1} \in L(X^*,X)$ and $g \in L(Y,Y^*)$. By (3.2(iii)), $gkf^{-1} \in C(X^*,Y^*)$. Since $|X^*| \subseteq |F|$ and $|Y^*| \subseteq |F|$, there exists a sequence $(k_n)$ of elements in $L(X^*,Y^*)$ which have finite dimensional image such that $gkf^{-1} = \lim k_n$. So, $k = \lim (g^{-1}k_nf)$ and for each $n$, $g^{-1}k_nf$ has finite dimensional image.

This completes the proof.

**Theorem 3.5:** Let $X,Y$ be N.A. Banach spaces over $F$. If $k \in L(X,Y)$ is compact, then the dual $k'$ of $k$ is compact.

**Proof:** See [11].

Let $X$ be a N.A. Banach space over $F$. Let $k$ be in $C(X,X)$ and let $a$ be in $F$ and $a \neq 0$. Then the following are known facts:

**Remarks 3.6:** (i) $a-k$ is onto implies $a-k$ is one to one.

(ii) $(a-k)X$ is closed.

(iii) ker$(a-k)$ is finite dimensional.

(For proof see [7], pp. 87-89.)
Theorem 3.7: If $X,k$ and $a$ are as in (3.6), then $\text{coker}(a-k)$ is finite dimensional.

Proof: By (2.6), $\dim(\text{coker}(a-k))' = \dim(\ker(a-k)')$
$= \dim(\ker(a-k'))$. By (3.5), $k'$ is compact. By (3.6 (iii)), $\ker(a-k')$ is finite dimensional. So, $\dim(\text{coker}(a-k))'$ is finite and hence $\dim(\text{coker}(a-k))$ is finite.

Theorem 3.8: With $X,k$ and $a$ as in (3.6), if $a-k$ is one to one, then $a-k$ is onto.

Proof: $a-k$ is one to one implies $\ker(a-k)$ is zero. So, $\dim(\ker(a-k))' = \dim(\ker(a-k)) = 0$. By (2.6), $\dim(\text{coker}(a-k')) = \dim(\ker(a-k))' = 0$. This means $a-k'$ is onto. By (3.6 (i)), $a-k'$ is one to one. $\ker(a-k')$ is zero. Again by (2.6), $\dim(\text{coker}(a-k))' = \dim(\ker(a-k')) = 0$. Hence $\dim(\text{coker}(a-k)) = 0$. This implies $a-k$ is onto.

Theorem 3.9: Let $F$ be locally compact and let $X,X_1,Y,Y_1$ be N.A. Banach spaces over $F$. If $T: X \rightarrow X_1$, $S: Y \rightarrow Y_1$ are compact, then $T \hat{\otimes} S$ is compact.

Proof: By (3.6), there exist a sequence $(T_n)$ of elements in $L(X,X_1)$ which have finite dimensional image and a sequence $(S_n)$ of elements in $L(Y,Y_1)$ which have finite dimensional image such that $\lim T_n = T$, and $\lim S_n = S$. 
Let \( i_n : \text{Im}(T_n) \to X_1 \), \( j_n : \text{Im}(S_n) \to Y_1 \) be the insertion maps. It will be shown that \( i_n \otimes j_n \) is one to one.

Let \( Z_n \) be the (algebraic) complement of \( \text{Im}(T_n) \) in \( X_1 \) and \( f_n : X_1 \to \text{Im}(T_n) \) be the natural projection. Then

\[
(f_n \otimes l \text{Im}(S_n))(i_n \otimes l \text{Im}(S_n))
\]

\[
= l \text{Im}(T_n) \otimes l \text{Im}(S_n) = l \text{Im}(T_n) \otimes \text{Im}(S_n)
\]

This implies \( i_n \otimes l \text{Im}(S_n) \) is one to one. Likewise, it can be shown that \( l X_1 \otimes j_n \) is one to one. Hence, \( (l X_1 \otimes j_n)(i_n \otimes l \text{Im}(S_n)) \) is one to one; that is, \( i_n \otimes j_n \) is one to one.

\( \text{Im}(T_n), \text{Im}(S_n) \) are finite dimensional. So, \( \text{Im}(T_n) \otimes \text{Im}(S_n) \) is finite dimensional. (For proof see [4], p. 322.) Also \( i_n \otimes j_n \) is one to one. Therefore \( \text{Im}(i_n \otimes j_n) \) is a finite dimensional subspace of \( X_1 \otimes Y_1 \).

Hence \( \text{Im}(i_n \otimes j_n) \) is closed in \( X_1 \otimes Y_1 \).

Next it will be shown that for each \( n \),

\( \text{Im}(T_n^\wedge S_n) \subset \text{Im}(i_n \otimes j_n) \). An element in \( \text{Im}(T_n^\wedge S_n) \) is of the form \( (T_n \otimes S_n)u \) where \( u = \lim_{n \to \infty} \sum (a_n, i \otimes b_n, i) \).

\[
(T_n \otimes S_n)u = \lim_{n \to \infty} \sum (T_n a_n, i \otimes S_n b_n, i) = \lim_{n \to \infty} \sum (i_n \otimes j_n) (T_n a_n, i \otimes S_n b_n, i)
\]

where
$T_n^aS_n b_n^i$ is in $\text{Im}(T_n) \otimes \text{Im}(S_n)$

$= \lim v_n$

Now $v_n$ is in $\text{Im}(i_n \otimes j_n)$ and since $\text{Im}(i_n \otimes j_n)$ is closed, $\lim v_n$ is in $\text{Im}(i_n \otimes j_n)$. This shows that $\text{Im}(T_n \otimes S_n) \subset \text{Im}(i_n \otimes j_n)$. Hence $\text{Im}(T_n \otimes S_n)$ is finite dimensional.

$$|T_n \otimes S_n - T_n^a S_n| = |T_n \otimes S_n - T_n \otimes S_n + T_n \otimes S_n - T_n^a \otimes S_n|$$

$\leq \max(|T_n \otimes S_n - T_n^a \otimes S_n|, |T_n \otimes S_n - T_n \otimes S_n|)$

$= \max(|(T_n - T_n^a) \otimes S_n|, |T_n \otimes (S_n - S_n)|)$

$= \max(|(T_n - T_n^a) \otimes S|, |T_n \otimes (S_n - S_n)|)$ (by (1.33))

$\leq \max(|T_n| |S|, |T_n| |S_n - S_n|)$ (by (1.30))

Since $\lim T_n = T$ and $\lim S_n = S$, it follows that $\lim (T_n \otimes S_n) = T \otimes S$. Also, it was shown that for each $n$, $T_n \otimes S_n$ is compact. Hence, by (3.2), $T \otimes S$ is compact.
IV FREDHOLM OPERATORS

This section deals with Fredholm operators in the case of N.A. Banach spaces over a valued field which is complete. Certain results on Fredholm operators in the classical case, that is, in the case of Banach spaces over the reals or the complex numbers are investigated and are found to hold in the case of N.A. Banach spaces over a spherically complete field. The results in the classical case can be found in [8].

In this section it is further proved that if the N.A. valued field $F$ is locally compact, $T$ is a Fredholm operator and $M$ is a finite dimensional N.A. Banach space over $F$, then $T \oplus_{M} 1$ is Fredholm and furthermore $\text{ind}(T \oplus_{M} 1) = \text{ind}(T) \cdot \dim(M)$.

Throughout this section $F$ denotes a N.A. valued complete field.

**Definition 4.1:** Let $X, Y$ be Banach spaces over $F$ and let $T$ be in $L(X, Y)$. If $\ker(T)$ is finite dimensional, $T(X)$ is closed in $Y$ and $\text{coker}(T)$ is finite dimensional, then $T$ is called a Fredholm operator.

**Definition 4.2:** If $X, Y$ are Banach spaces over $F$ and $T$ in $L(X, Y)$ is Fredholm, then $\dim(\ker(T)) - \dim(\text{coker}(T))$ is called the index of $T$ and is
denoted by \( \text{ind}(T) \).

**Notation 4.3:** The set of all Fredholm operators in \( L(X,Y) \) is denoted by \( H(X,Y) \). The set of all Fredholm operators in \( L(X,Y) \) of index \( n \) is denoted by \( H_n(X,Y) \).

**Proposition 4.4:** If \( F' \) is spherically complete and if \( T \) is in \( H_n(X,Y) \), then \( T' \) is in \( H_{-n}(Y',X') \).

**Proof:** \( F \) is spherically complete. \( T \), being Fredholm has closed range. Hence by (2.6), we have the following linear topological isomorphisms:

\[
\ker(T') \cong (\text{coker}(T))' \text{ and } \quad \text{coker}(T') \cong (\ker(T))'.
\]

So, \( \ker(T') \) and \( \text{coker}(T') \) have finite dimension.

\[
dim(\ker(T')) = \dim(\text{coker}(T))' = \dim(\text{coker}(T)) \text{ and } \quad \dim(\text{coker}(T')) = \dim(\ker(T))' = \dim(\ker(T)).
\]

It follows that \( \text{ind}(T') = -\text{ind}(T) \).

**Theorem 4.5:** (i) Let \( X,Y \) be Banach spaces over \( F \). Let \( T \) be in \( L(X,Y) \). If there exist \( S,S_1 \) in \( L(Y,X) \) such that \( ST^{-1}X \) is in \( C(X,X) \) and \( TS_1^{-1}Y \) is in \( C(Y,Y) \), then \( T \) is in \( H(X,Y) \).

(ii) If \( F \) is spherically complete and if \( T \) is in \( H(X,Y) \), then there exists \( S \) in \( L(Y,X) \) such that \( ST^{-1}X \), \( TS_1^{-1}Y \) have finite dimensional images.
Proof: (i) $ST - l_x \in C(X, X)$ implies $ST = l_x + k$ where $k \in C(X, X)$. So, by (3.6 (iii)), $\ker(ST)$ is finite dimensional. Now $\ker(T) \subset \ker(ST)$ and so $\ker(T)$ is finite dimensional. $TS_1 - l_y \in C(Y, Y)$. Hence, by (3.6 (ii)), $TS_1(Y)$ is closed in $Y$, and by (3.7), $\dim(\text{coker}(TS_1(Y)))$ is finite. Now $Y \supset TX \supset TS_1(Y)$ implies $\dim(TX/TS_1(Y))$ is finite. $F$ being complete, $TX/TS_1(Y)$ is closed in $Y/TS_1(Y)$. From the continuity of the natural projection $Y \rightarrow Y/TS_1(Y)$ we get that $TX$ is closed in $Y$.

Since $TX \supset TS_1(Y)$, we have the linear map $f: Y/TS_1(Y) \rightarrow Y/TX$, where $f(y+TS_1(Y)) = y+TX$, and this map is onto. $\dim(Y/TS_1(Y))$ is finite. So, $\dim(Y/TX)$ is finite and in fact, $\dim(Y/TX) \leq \dim(Y/TS_1(Y))$, that is, $\text{coker}(T)$ is finite dimensional.

$\ker(T)$, $\text{coker}(T)$ being finite dimensional, $T$ is in $H(X, Y)$.

(ii) If $T$ is in $H(X, Y)$, then $\dim(\ker(T))$ is finite, $TX$ is closed in $Y$ and $\dim(\text{coker}(T))$ is finite.

$\text{Ker}(T)$ is finite dimensional and $F$ is spherically complete. So, by (2.9), there exists a closed subspace $V$ of $X$ complementary to $\ker(T)$. $X = \ker(T) + V$ where $\ker(T) \cap V = \{0\}$.

Define $T_1: V \rightarrow TX$ by $T_1(x) = Tx$. $V$, being
closed in \(X\), is complete and hence is a Banach space. For the same reason \(TX\) is a Banach space.

Now, \(T_1\) is clearly linear, continuous and onto. Also, \(Tx = Ty\) and \(x, y \in V\) imply \(x - y \in \ker(T) \cap V\). So \(x - y = 0\). This shows \(T_1\) is one to one. \(T_1\) is a continuous linear bijection from a Banach space into another. By the open mapping theorem, \(T_1\) has a continuous inverse \(T_1^{-1}\).

\(TX\) is closed and \(\text{coker}(T)\) is finite dimensional.

By (2.9), there exists a finite dimensional closed subspace \(W\) of \(Y\) complementary to \(TX\) and an isomorphism of topological vector spaces \(f: TX + W \to TX \oplus W\) defined by \(f(x + y) = (x, y)\) where \(x \in TX, y \in W\).

Define \(S_1: TX \oplus W \to X\) by \(S_1(y, w) = T_1^{-1}(y)\).

\(S_1\) is linear.

\[
|S_1(y, w)| = |T_1^{-1}(y)| \leq |T_1^{-1}||y| \\
\leq |T_1^{-1}| \max(|y|, |w|) \\
= |T_1^{-1}| |(y, w)|
\]

So \(S_1\) is continuous. Let \(S = S_1f\). \(S\) is in \(L(X, Y)\). Each element of \(X\) can be uniquely written in the form \(x + z\) with \(x\) in \(\ker(T)\) and \(z\) in \(V\).

\[
(ST_1^{-1})(x+z) = (S_1f)(Tz) - (x+z) \\
= S_1(Tz, 0) - x - z \\
= T_1^{-1}(Tz) - x - z = -x \text{ in } \ker(T)
\]
So, \((ST_1X)X \subseteq ker(T)\). Since \(ker(T)\) has finite dimension \((ST_1X)X\) has finite dimension.

Let \(y \in Y\) and \(y = Tx+w\) where \(x \in X\) and \(w \in W\).

\[
(TS_1y)(Tx+w) = TS_1f(Tx+w) - (Tx+w) \\
= TS_1(Tx,w) - Tx - w \\
= TS_1(Tx) - Tx - w \\
= Tx - Tx - w \\
= -w \text{ (in } W)\).
\]

This shows \((TS_1y)y \subseteq W\). Since \(W\) has finite dimension, \((TS_1y)y\) has finite dimension.

(By (3.3), it follows that \(ST_1X, TS_1Y\) are compact, provided \(F\) is locally compact.)

**Corollary 4.6:** Let \(X, Y\) be N.A. Banach spaces over \(F\). If \(T \in H(X,Y)\) and \(k \in C(X,Y)\), then \(T+k \in H(X,Y)\).

**Proof:** If \(k = 0\), the result is true. Let \(k \neq 0\).

By (3.2(i)), \(F\) is locally compact. Hence \(F\) is spherically complete. By (4.5), there exists \(S\) in \(L(Y,X)\) such that \(ST_1X \subseteq C(X,X)\) and \(TS_1Y \subseteq C(Y,Y)\).

Hence, by (3.2 (ii), (iii)), \(S(T+k) - L_X\) is in \(C(X,X)\) and \((T+k)S - L_Y\) is in \(C(Y,Y)\). By (4.5), \(T+k\) is in \(H(X,Y)\).
Corollary 4.7: Let $F$ be spherically complete and $X, Y$ be Banach spaces over $F$. Then $H(X, Y)$ is open in $L(X, Y)$.

Proof: If $H(X, Y)$ is empty, it is open in $L(X, Y)$. If not, let $T \in H(X, Y)$. By (4.5), there exists $S$ in $L(Y, X)$ such that $ST^{-1}_X = k$ in $C(X, X)$ and $TS^{-1}_Y = k_1$ in $C(Y, Y)$.

Let $T_1 \in L(X, Y)$ such that $|T - T_1| < |S|^{-1}$. Let $A = T - T_1$. $|SA| < |S||S|^{-1} = 1$. Hence $l_X - SA$ has an invertible element in $L(X, X)$. (See proof of (2.10).) Let $B$ be the inverse of $l_X - SA$. Therefore $B - BSA = l_X$. Let $S_1 = BS$. $S_1T_1 = BST_1 = BS(T - A) = BST - BSA = B(l_X + k) - B + l_X = Bk + l_X$. Hence $S_1T_1 - l_X \in C(X, X)$.

Since $|AS| < 1$, $l_Y - AS$ has an inverse $E$ in $L(Y, Y)$. Letting $S_2 = SE$, we can, as before, show that $T_1S_2 - l_Y \in C(Y, Y)$.

By (4.5), we conclude that $T_1 \in H(X, Y)$. This proves that $H(X, Y)$ is open in $L(X, Y)$.

Corollary 4.8: If $F$ is spherically complete and $X, Y, Z$ are N.A. Banach spaces over $F$, then $T_1 \in H(X, Y)$ and $T_2 \in H(Y, Z)$ imply that $T_2T_1 \in H(X, Z)$.

Proof: By (4.5), there exist $S_1$ in $L(Y, X)$ and $S_2$ in $L(Z, Y)$ such that $T_1S_1 - l_Y = k_1$ in $C(Y, Y)$.
\[ \begin{align*}
S_1 T_1 - l_x &= k_2 \quad \text{in} \quad C(X,X); \quad T_2 S_2 - l_z = k_3 \quad \text{in} \quad C(Z,Z) \\
\text{and} \quad S_2 T_2 - l_Y &= k_4 \quad \text{in} \quad C(Y,Y). 
\end{align*} \]

\[ (T_2 T_1)(S_1 S_2) = T_2 S_2 + T_2 k_1 S_2 = l_Z + k_3 + T_2 k_1 S_2. \]

By (3.2 (ii), (iii)),
\[ (T_2 T_1)(S_1 S_2) - l_z \]
is in \( C(Z,Z) \).

It can be shown similarly that \( (S_1 S_2)(T_2 T_1) - l_x \)
is in \( C(X,X) \). It follows from (4.5) that \( T_2 T_1 \) is in \( H(X,Z) \).

**Theorem 4.9:** Let \( F \) be spherically complete and let \( X, Y, Z \) be N.A. Banach spaces over \( F \). If \( T \) is in \( H(X,Y) \) and \( S \) is in \( H(Y,Z) \), then \( \text{ind}(ST) = \text{ind}(S) + \text{ind}(T) \).

**Proof:** \( \text{Ker}(S) \) is closed in \( Y \) and \( \text{ker}(T) \) and \( \text{ker}(ST) \) are closed in \( X \). Also, \( T \) is in \( H(X,Y) \) implies \( TX \) is closed in \( Y \). Hence, we have the following S.E.S. of Banach spaces:

\[ \begin{align*}
0 &\rightarrow \text{ker}(T) \rightarrow \text{ker}(ST) \rightarrow \text{ker}(S) \cap TX \rightarrow 0 \\
\end{align*} \]

\((\text{ker}(T) \rightarrow \text{ker}(ST))\) is the insertion and \( T_1 \) is defined by \( T_1(x) = Tx \) for every \( x \) in \( \text{ker}(ST) \).

Now \( T \) and \( S \) being Fredholm operators, \( ST \) is Fredholm. Hence \( \text{ker}(T), \text{ker}(ST) \) and \( \text{ker}(S) \cap TX \) are finite dimensional. It follows that:

\[ \text{dim}(\text{ker}(ST)) = \text{dim}(\text{ker}(T)) + \text{dim}(\text{ker}(S) \cap TX). \]
(1) \( \dim(\ker(ST)) = \dim(\ker(T)) + \dim(\ker(S)) - \dim(\ker(S)/(\ker(S) \cap TX)) \)

\( T \in H(X,Y), S \in H(Y,Z) \) imply, by (4.8), that \( ST \in H(X,Z) \). So, \( ST(X) \) is closed in \( Z \). Also, \( SY \) is closed in \( Z \). Therefore \( SY \) is a Banach space, \( ST(X) \) is closed in \( SY \). We have the S.E.S. of Banach spaces:

\[
0 \rightarrow SY/ST(X) \rightarrow Z/ST(X) \xrightarrow{f} Z/SY \rightarrow 0
\]

where \( SY/ST(X) \rightarrow Z/ST(X) \) is the insertion and \( f \) is defined by \( f(z+ST(X)) = z+SY \) for every \( z \in Z \). Clearly \( f \) is continuous. Since \( \text{coker}(ST) \) has finite dimension, it follows that

(2) \( \dim(\text{coker}(ST)) = \dim(\text{coker}(S)) + \dim(\text{SY/ST}(X)) \).

\( TX \) is closed in \( Y \) and \( \ker(S) \) is finite dimensional. Hence \( TX + \ker(S) \) is closed in \( X \) (For proof, see [7], pp. 70,71). Hence \( TX + \ker(S) \) is a Banach space. Consider the S.E.S. of Banach spaces,

\[
0 \rightarrow TX + \ker(S) \rightarrow Y \xrightarrow{\overline{S}} SY/ST(X) \rightarrow 0,
\]

where \( TX + \ker(S) \rightarrow Y \) is the insertion map and \( \overline{S} \) is defined by \( \overline{S}(y) = Sy + STX \). For every \( y \) in \( Y \),

\[
|Sy + ST(X)| = \inf\{|Sy + z| \mid z \in ST(X)\} \leq |Sy| \leq |S||y|.
\]

This shows that \( \overline{S} \) is continuous.
By (2.3), \( \dim(SY/ST(X)) = \dim(Y/(TX+\ker(S))) \)

\[ \begin{align*}
&= \dim((Y/TX)/(TX+\ker(S))/TX)) \\
&= \dim(\coker(T)) - \\
&\quad \dim((TX+\ker(S))/TX)
\end{align*} \]

Now, from (2), it follows that

(3) \( \dim(\coker(ST)) = \dim(\coker(S)) + \dim(\coker(T)) - \dim((TX+\ker(S))/TX) \).

From (1) and (3), it follows that

(4) \( \text{ind}(ST) = \text{ind}(T) + \text{ind}(S) + \dim((TX+\ker(S))/TX) - \dim(\ker(S)/(\ker(S)\cap TX)) \).

Consider the S.E.S. of Banach spaces:

0 \to \ker(S)\cap TX \to \ker(S) \to (TX+\ker(S))/TX \to 0.

It follows that:

(5) \( \dim((TX+\ker(S))/TX) = \dim(\ker(S)/(\ker(S)\cap TX)) \)

Equations (4) and (5) imply \( \text{ind}(ST) = \text{ind}(S) + \text{ind}(T) \).

**Theorem 4.10:** Let \( F \) be spherically complete and let \( X,Y \) be N.A. Banach spaces. The function \( \text{Ind} : H(X,Y) \to \mathbb{Z} \), which maps each Fredholm operator to its index, is continuous.
Proof: \( T \) is in \( H(X,Y) \). So, \( T \) has finite dimensional kernel. Since \( F \) is spherically complete, by (2.9), there exists a closed subspace of \( X \) complementary to \( \ker(T) \). Furthermore \( T(X) \) is closed and has finite codimension in \( Y \). Again, by (2.9), there exists a (closed) finite dimensional subspace \( W \) of \( Y \) complementary to \( T(X) \). Thus \( X = \ker(T) + V \) where \( \ker(T) \cap V = \{0\} \), and \( Y = TX + W \) where \( TX \cap W = \{0\} \).

If \( S \in L(X,Y) \), define \( f_S: V \oplus W \to Y \) by
\[
f_S(v,w) = Sv + w.
\]
\( f_S \) is linear. \(|f_S(v,w)| = |Sv + w| \leq \max(|Sv|, |w|) \). If \(|S| \leq 1\), then
\[
\max(|Sv|, |w|) \leq \max(|S||v|, |w|) \\
\leq \max(|v|, |w|) \\
= |(v,w)|.
\]

If \(|S| > 1\), then
\[
\max(|Sv|, |w|) \leq \max(|S||v|, |S||w|) \\
= |S||(v,w)|.
\]

So, in either case, \(|f_S(v,w)| \leq \max\{1, |S|\} |(v,w)|\).

This implies that \( f_S \) is continuous.

Since \( Y = TX + W = TV + W \) and \( TV \cap W \) is zero, it is clear that \( f_T: V \oplus W \to Y \) is one to one and onto. \( f_T \) is a linear continuous bijection of a Banach space into another. Hence, by the open mapping theorem, \( f_T \) is
an isomorphism of topological vector spaces. By (2.10), there is a $\delta > 0$ such that if $S$ is in $L(X,Y)$ and $|f_S-f_T| < \delta$, then $f_S$ is an isomorphism of topological vector spaces. Since $T$ is in $H(X,Y)$, by (4.7), there exists $\varepsilon > 0$, such that if $S$ is in $L(X,Y)$ and $|S-T| < \varepsilon$, then $S$ is in $H(X,Y)$. Furthermore, $|(f_S-f_T)(v,w)| = |Sv-Tv| < |S-T||v,w|$.

Therefore, $|f_S-f_T| < |S-T|$. Let $S$ be in $L(X,Y)$ such that $|S-T| < \min\{\varepsilon, \delta\}$. $S$ is in $H(X,Y)$ and $f_S$ is an isomorphism of topological vector spaces. Since $V$ is closed in $V\oplus W$, $f_S(V) = SV$ is closed in $Y$. $f_S$ is onto implies $Y = SV+W$. $f_S$ is one to one implies $SV\cap W = \{0\}$. Hence $\text{codim}(SV) = \text{dim}(W)$.

$X = \ker(T)+V$ where $\ker(T)\cap V = \{0\}$. This implies $\text{codim}(V) = \text{dim}(\ker(T))$. Now $V\subseteq V+\ker(S)$, $V$ is closed in $X$ and $\text{codim}(V)$ is finite. Hence $V+\ker(S)$ is closed in $X$ and has finite codimension in $X$. (For proof, see (4.5).) By (2.9), there is a closed subspace $Z$ of $X$ such that $X = V+\ker(S)+Z$ where $(V+\ker(S))\cap Z = \{0\}$. Also, from the linear (topological) isomorphism $V \rightarrow SV$ induced by $f_S$, it follows that $\ker(S)\cap V = \{0\}$. So $X = V+\ker(S)+Z$, a direct sum of linear subspaces. So, $SX = SV+SZ$ and $SV\cap SZ = \{0\}$. 
Now $S \in H(X,Y)$ implies $SX$ is closed in $Y$ and has finite codimension in $Y$. So, there exists a closed finite dimensional subspace $U$ of $Y$ complementary to $SX$; that is, $Y = U + SX$ where $U \cap SX = \{0\}$. Therefore $Y = U + SV + SZ$ where $U \cap (SV + SZ) = \{0\}$. Also $SV \cap SZ = \{0\}$. It follows that $(U + SV) \cap SZ = (U + SZ) \cap SV = \{0\}$. Hence $Y = U + SV + SZ$, a direct sum of linear subspaces.

$Y = U + SX$ where $U \cap SX = \{0\}$ implies $\dim(\text{coker}(S)) = \dim(U)$. Also $Y = U + SV + SZ$, a direct sum of linear spaces and $\text{codim}(SV)$ is finite. It follows that $\text{codim}(SV) = \dim(U) + \dim(SZ)$. But it was proved before that $\text{codim}(SV) = \dim(W)$. Also, $\ker(S) \cap Z = \{0\}$ implies $S$ is one to one on $Z$. Hence $\dim(SZ) = \dim(Z)$. $\dim(W) = \dim(U) + \dim(Z)$. Therefore, $\dim(U) = \dim(W) - \dim(Z)$. It follows that $\dim(\text{coker}(S)) = \dim(W) - \dim(Z) = \dim(\text{coker}(T)) - \dim(Z)$.

$X = V + \ker(T)$ and $V \cap \ker(T) = \{0\}$ imply $\text{codim}(V) = \dim(\ker(T))$. Also, $X = V + \ker(S) + Z$, a direct sum of linear spaces. Hence, $\text{codim}(V) = \dim(\ker(S)) + \dim(Z)$; that is, $\dim(\ker(T)) = \dim(\ker(S)) + \dim(Z)$. So, $\dim(\ker(S)) = \dim(\ker(T)) - \dim(Z)$.

\[
\text{ind}(S) = \dim(\ker(S)) - \dim(\text{coker}(S)) = \dim(\ker(T)) - \dim(Z) - \dim(\text{coker}(T)) + \dim(Z)
\]
\[
= \dim(\ker(T)) - \dim(\coker(T))
= \text{ind}(T)
\]

This completes the proof.

**Lemma 4.11**: Let \( F \) be spherically complete and \( X \) be a N.A. normed space over \( F \). If \( x_1, x_2, \ldots, x_n \) are linearly independent elements of \( X \), then there exist \( g_1, g_2, \ldots, g_n \) in \( X' \) such that if \( 1 \leq i, j \leq n \) and \( i \neq j \), then \( g_i(x_j) = 0 \) and \( g_i(x_i) = 1 \).

**Proof**: Let \( V \) be the vector space generated by \( \{x_1, x_2, \ldots, x_n\} \). If \( 1 \leq i \leq n \), define \( h_i : V \to F \) by \( h_i(a_1x_1 + a_2x_2 + \ldots + a_nx_n) = a_i \), where \( a_1, a_2, \ldots, a_n \) are in \( F \). Clearly \( h_i \) is linear and continuous and \( h_i(x_j) = 0 \) if \( i \neq j \) and \( h_i(x_i) = 1 \). Since \( F \) is spherically complete, it is possible, by the Hahn Banach theorem for N.A. normed spaces, to extend each \( h_i \) to \( g_i : X \to F \) so that \( g_i \) is continuous and linear. If \( 1 \leq i, j \leq n \) and \( i \neq j \), then \( g_i(x_j) = h_i(x_j) = 0 \) and \( g_i(x_i) = h_i(x_i) = 1 \) for all \( i \) \( (i = 1, 2, \ldots, n) \).

**Definition 4.12**: If \( X \) is a N.A. normed space over \( F \) and if \( S \subseteq X' \), then the annihilator of \( S \) is the set \( \text{ann}(S) = \{x \mid x \text{ is in } X \text{ and for every } f \text{ in } S, f(x) = 0 \} \).
Lemma 4.13: If $X$ is a N.A. normed space over $F$ and are linearly independent elements of $X'$, then there exist $x_1, x_2, ..., x_n$ in $X$ such that $f_i(x_j) = 0$ if $1 \leq i, j \leq n$ and $i \neq j$; $f_i(x_i) = 1$ if $1 \leq i \leq n$.

Proof: The following claim is made: If $f_1, f_2, ..., f_n$ are $n$ linearly independent elements of $X'$ and if $g$ is in $X'$ such that $\{f_1, f_2, ..., f_n\} \subseteq \{g\}$, then $g$ is a linear combination of $f_1, f_2, ..., f_n$.

This is proved by induction on $n$ as follows:

Let $\{f_1\} \subseteq \{g\}$. $f_1 \neq 0$ and hence there is $x_1$ in $X$ such that $f_1(x_1) = 1$. For each $x$ in $X$, $f_1(x - f_1(x)x_1) = 0$. Hence $g(x - f_1(x)x_1) = 0$.

$g(x) = f_1(x)g(x_1) = (g(x_1)f_1)x$. This proves the claim for $n = 1$.

Assume the claim holds for an arbitrary positive integer $n$. Let now $f_1, f_2, ..., f_{n+1}$ be linearly independent elements of $X'$ and let $g$ be in $X'$ such that $\{f_1, f_2, ..., f_{n+1}\} \subseteq \{g\}$. Now $f_1, f_2, ..., f_{n+1}$ are linearly independent. Hence for each $j (1 \leq j \leq n+1)$, $f_j$ is not a linear combination of $f_1, f_2, ..., f_{j-1}, f_{j+1}, ..., f_{n+1}$. It follows from induction hypotheses that $\{f_1, f_2, ..., f_{j-1}, f_{j+1}, ..., f_{n+1}\} \
\not\subseteq \{f_j\}$. So there exists $x_j$ in $X$ such that $f_j(x_j) = 1$ and $f_i(x_j) = 0$ if $1 \leq i \leq n+1$, and $i \neq j$.

Now for each $x$ in $X$ and $1 \leq i \leq n+1$,
\[ f_i(x - \sum_{j=1}^{n+1} f_j(x) x_j) = f_i(x) - f_i(x) = 0. \] Therefore,

\[ g(x - \sum_{j=1}^{n+1} f_j(x) x_j) = 0. \] This shows that \( g = \sum_{j=1}^{n+1} g(x_j) f_j \).

This proves the claim for all positive integers \( n \).

If \( f_1, f_2, \ldots, f_n \) being linearly independent, it follows that for each \( j \), \( \{f_1, f_2, \ldots, f_j-1, f_{j+1}, \ldots, f_n\} \not= \{f_j\} \). Hence there exists \( x_i \) in \( X \) such that \( f_i(x_j) = 0 \) for \( 1 \leq i \leq n \), \( i \neq j \) and \( f_j(x_j) = 1 \).

Using lemmas (4.11) and (4.13), the following theorem can be proved:

**Theorem 4.14**: Let \( X \) be a N.A. Banach space over \( F \) and let \( k \) be a nontrivial compact operator on \( X \). If \( a \) is in \( F \) and \( a \neq 0 \), then \( a-k \) is a Fredholm operator of index zero.

**Proof**: See [7], pp. 90, 91.

**Corollary 4.15**: If \( X, Y \) are N.A. Banach spaces over \( F \) and \( u, k \in L(X, Y) \) where \( u \) is invertible and \( k \) is compact then \( \text{ind}(u+k) = 0 \).

**Proof**: \( u+k = u(l+u^{-1}k) \). \( k \) is compact and, by the open mapping theorem \( u^{-1} \) is continuous. So, by (3.2(iii)), \( u^{-1}k \) is compact. Hence by (4.14), \( \text{ind}(1+u^{-1}k) = 0 \). By (4.9), \( \text{ind}(u(l+u^{-1}k)) = \text{ind}(u) + \text{ind}(1+u^{-1}k) = 0 \). Hence \( \text{ind}(u+k) = 0 \).
In the case of Banach spaces over $\mathbb{R}$ (reals) or $\mathbb{C}$ (the set of complex numbers), it has been proved that every Fredholm operator of index 0 is a sum of an invertible operator and an operator of finite rank. The proof can be found in [6]. The following theorem shows that the same is true in the case of N.A. Banach spaces over $F$, provided $F$ is locally compact.

**Theorem 4.16:** Let $F$ be spherically complete. Let $X,Y$ be N.A. Banach spaces over $F$. Every operator in $H_0(X,Y)$ is a sum of an invertible operator and an operator of finite rank.

**Proof:** Let $T$ be in $H_0(X,Y)$. $\ker(T)$ is finite dimensional, $TX$ is closed and $\text{coker}(T)$ is finite dimensional. By (2.9), $\ker(T)$ has a closed complementary subspace $V$ and $TX$ has a closed finite dimensional complementary subspace $W$. This means, $X = \ker(T) + V$ where $\ker(T) \cap V = \{0\}$.

$T$ is of index 0 and hence $\dim(\ker(T)) = \dim(\text{coker}(T)) = \dim(W)$. There exists an isomorphism of topological vector spaces $A_1 : \ker(T) \rightarrow W$.

Define $A : X \rightarrow Y$ such that if $x \in X$ and $x = a + b$ where $a \in \ker(T)$ and $b \in V$, then $A(x) = A_1(a)$.

$A$ is clearly linear. Furthermore the mapping $f : \ker(T) \oplus V \rightarrow Y$ defined by $f(a,b) = A_1(a)$ is linear and continuous. (Notice that $|A_1(a)| \leq |A_1||a|$
\[ |A_1| \max(|a|, |b|) = |A_1|(a,b). \] Hence, by (2.8), A is continuous. Also A has finite dimensional range. F being locally compact, it follows that A is compact.

Define \( u: X \to Y \) by \( u(a+b) = A_1(a) + T(b) \) where \( a \) is in \( \ker(T) \) and \( b \) is in \( V \). \( u \) is clearly linear. As before, it can be shown that \( u \) is continuous. Also, \( u \) is a bijection. Hence, by the open mapping theorem, \( u \) is an isomorphism of topological vector spaces.

If \( x \in X \) and \( x = a+b \) where \( a \in \ker(T) \) and \( b \in V \), then \( T(a+b) = T(b) \). Also \( (u-A)(a+b) = A_1(a) + T(b) - A_1(a) = T(b) \). So, \( T = u-A \).

**Remark 4.17:** From (4.15) and (4.16), it is seen that if \( F \) is locally compact, then in the case of N.A. Banach spaces over \( F \), the Fredholm operators of index 0 are precisely those which can be expressed as a sum of an invertible operator and an operator of finite dimensional image.

**Theorem 4.18:** Let \( F \) be spherically complete. Let \( X,Y \) be N.A. Banach spaces over \( F \). If \( T:X \to Y \) is a Fredholm operator, then for every finite dimensional N.A. Banach space \( M \) over \( F \), \( \mathring{T}l_M \) is Fredholm and \( \text{ind} (\mathring{T}l_M) = \text{ind}(T) \text{dim}(M) \).
Proof: $X$ is a Banach space and $\ker(T)$ is closed in $X$. Hence $\ker(T)$ is a Banach subspace of $X$. $T$ being Fredholm, $Tx$ is closed in $Y$ and so $\text{coker}(T)$ is a Banach space. If $i: \ker(T) \rightarrow X$ is the insertion and $p: Y \rightarrow \text{coker}(T)$ is the natural projection, then the following is an exact sequence of Banach spaces:

$$0 \rightarrow \ker(T) \xrightarrow{i} X \xrightarrow{T} Y \xrightarrow{p} \text{coker}(T) \rightarrow 0.$$ 

Let $q: X \rightarrow TX$ be defined by $q(x) = Tx$ and let $j: TX \rightarrow Y$ be the insertion. It is clear that $jq = T$ and that the following are short exact sequences of Banach spaces:

$$0 \rightarrow \ker(T) \xrightarrow{i} X \xrightarrow{q} TX \rightarrow 0$$

$$0 \rightarrow TX \xrightarrow{j} Y \xrightarrow{\text{coker}(T)} 0$$

$F$ is spherically complete and $\ker(T)$ is finite dimensional. Hence by (2.9), $X = \ker(T) + V$ where $\ker(T) \cap V = \{0\}$ and $f: X \rightarrow \ker(T) \oplus V$ defined by $f(x+z) = (x,z)$ for every $x$ in $\ker(T)$, $z$ in $V$, is an isomorphism of topological vector spaces. Let $u: \ker(T) \oplus V \rightarrow \ker(T)$ be the natural projection. $uf: X \rightarrow \ker(T)$ is continuous linear and $ufi = 1_{\ker(T)}$. Hence the S.E.S. (2) splits. By (1.38), the following is a S.E.S. of Banach spaces:
Now $TX$ is closed and has finite codimension in $Y$. Hence, by (2.9), $Y = TX + W$ where $TX \cap W = \{0\}$.

As before, it can be shown that the S.E.S. (3) splits. By (1.38), the following is a S.E.S. of Banach spaces:

(5) $0 \rightarrow TX \otimes M \xrightarrow{j \otimes l} Y \otimes M \xrightarrow{p \otimes l} \text{coker}(T) \otimes M \rightarrow 0$

$T = jq$ and therefore $T \otimes l = (j \otimes l)(q \otimes l)$. By (1.34(i)), $T \otimes l = (j \otimes l)(q \otimes l)$. From the short exact sequences of Banach spaces (4) and (5), it follows that $q \otimes l$ is onto and $j \otimes l$ is one to one. Hence $\ker(T \otimes l) = \ker(q \otimes l) = \text{Im}(i \otimes l)$, and $\text{Im}(T \otimes l) = \text{Im}(j \otimes l) = \ker(p \otimes l)$.

The following is an exact sequence of Banach spaces:

(6) $0 \rightarrow \ker(T) \otimes M \xrightarrow{i \otimes l} X \otimes M \xrightarrow{T \otimes l} Y \otimes M \xrightarrow{p \otimes l} \text{coker}(T) \otimes M \rightarrow 0$.

$\ker(T)$, $\text{coker}(T)$ and $M$ are finite dimensional. Hence $\text{Ker}(T) \otimes M$, $\text{coker}(T) \otimes M$ are finite dimensional and $\dim(\ker(T) \otimes M) = \dim(\ker(T)) \dim(M)$ and $\dim(\text{coker}(T) \otimes M) = \dim(\text{coker}(T)) \dim(M)$. (For proof, see [4], p. 322.)

From the exact sequence (6), it is now clear that $T \otimes l$ is Fredholm and that
\[ \text{ind}(T^\dagger) = \dim(\ker(T^\dagger)) - \dim(\text{coker}(T^\dagger)) \]
\[ = \dim(\ker(T) \hat{\otimes} M) - \dim(\text{coker}(T) \hat{\otimes} M) \]
\[ = \dim(\ker(T)) \dim(M) - \dim(\text{coker}(T)) \dim(M) \]
\[ = (\dim(\ker(T)) - \dim(\text{coker}(T))) \dim(M) \]
\[ = \text{ind}(T) \dim(M). \]
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