4-2019

Variations in Ramsey Theory

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by

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A dissertation submitted to the Graduate College
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
Mathematics
Western Michigan University
April 2019

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ACKNOWLEDGMENTS

There are a number of individuals without whom I would never have completed my dissertation:

My advisor and chair of my dissertation committee Professor Ping Zhang has my deepest gratitude. Her guidance, effort, and dedication has been instrumental to my successes over the past five years. She has not only taught me a great deal about research but also about navigating academic processes, teamwork, writing and delivering talks, and balancing my career with my personal life. I am humbled by her depth of knowledge and wisdom, and I am incredibly thankful to have worked with her.

The rest of my committee also deserves my thanks. I thank Professor Gary Chartrand for his contributions to my research, as well as for the valuable feedback on my dissertation. He has also offered me valuable career advice that I will take with me as I continue my academic career. I am also grateful to Professor Alen Schwenk and Linda Eroh for their support and helpful comments, as well as for their earlier work developing concepts I expand on in my dissertation. Their earlier work helped lead to the expansion of this area of research and showed how interesting this topic is.

There are so many tremendous faculty members in the Mathematics Department at Western Michigan University. I thank those professors of which I've been a student, they have taught me an incredible amount of mathematics, but even more so they have taught me how to be a successful and happy mathematician.

The staff in the Department of Mathematics also have my gratitude. Their patience and endless support are greatly appreciated. Their helpful attitude and friendly demeanor has made the Department feel like an extension of my home.

I have made a great deal of close friends in the Department. They have my thanks for their support as well as for helping me stay sane while studying for preliminary exams. In particular, Alexis Byers has been an incredible source of support in my life. She has been incredibly helpful in my research and in helping me finish and format my dissertation.

Acknowledgments—continued

There were many other students in the Department with which I was fortunate to work with on research. In particular, I thank Zhenming Bi for her contributions to my dissertation. Her work-ethic and intelligence has been inspiring, and I am grateful to have worked alongside her.

My family has been a constant source of emotional support. My mother, in particular, always encouraged me and reminded me that I am capable of more than I think. There were some dark times while pursuing my degree, my family has always been there to lift me up.

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Variations in Ramsey Theory

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Western Michigan University, 2019

The Ramsey number $R(F, H)$ of two graphs F and H is the smallest positive integer n for which every red-blue coloring of the (edges of a) complete graph of order n results in a graph isomorphic to F all of whose edges are colored red (a red F) or a blue H. Beineke and Schwenk extended this concept to a bipartite version of Ramsey numbers, namely the bipartite Ramsey number $BR(F, H)$ of two bipartite graphs F and H is the smallest positive integer $r$such that every red-blue coloring of the r-regular complete bipartite graph results in either a red F or a blue H. Chartrand extended this further to a multipartite version. Bialostocki and Voxman introduced the rainbow Ramsey number $RR(G)$ of a graph G as the smallest positive integer n such that if every edge of the complete graph of order n is colored from any number of colors, then either a monochromatic G (all edges of G colored the same) or a rainbow G (no two edges of G colored the same) results. Eroh extended this concept from one graph to two graphs. These concepts are generalized even further in this work. We present results and open questions concerning several new variations of Ramsey numbers as well as their connections with some well-known concepts in chromatic graph theory.

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Introduction

The famous mathematician Ronald Graham once stated that Ramsey theory is a branch of mathematics dedicated to the proposition that *complete disorder is impossible* (a statement attributed to the mathematician Theodore S. Motzkin) in the sense that within any sufficiently large system, some regularity must occur. Ramsey theory has also been described as the study of unavoidable regularity in large structures, where the primary question is:

> *When is it the case that whenever the elements of some sufficiently large structure are partitioned into a finite number of classes, there is always at least one class within which a prescribed regular structure occurs?*

We are interested in the case where the structures in question are graphs whose edges are colored, where the classes are these subgraphs whose edges are colored the same and where one of these subgraphs contains a prescribed graph. In this case, the Ramsey theory being discussed is that in graph theory.

In a *red-blue coloring* of a graph G, every edge of G is colored either red or blue. For two graphs F and H (without isolated vertices), the *Ramsey number* $R(F, H)$ of F and H is the smallest positive integer n such that for every red-blue coloring of the complete graph K_n of order n, there is either a subgraph isomorphic to F all of whose edges are colored red (a *red* F) or a subgraph isomorphic to H all of whose edges are colored blue (a *blue* H). A graph all of whose edges are colored the same is called a *monochromatic graph.* The investigation of Ramsey numbers is one of the best known topics of study within Extremal Graph Theory. A book by Graham, Rothschild and Spencer [35] is devoted to this area of study. In addition, a chapter on Ramsey numbers by Faudree in the *Handbook of Graph Theory* [37, pp.1002-1025] is devoted, as well, to Ramsey numbers.

Ramsey numbers are named for Frank Ramsey (1903-1930), a British philosopher, economist and mathematician. The theorem for which Ramsey is known was proved only as a minor lemma in a famous paper [49] by Ramsey. This lemma became the basis of the area of graph theory called Ramsey theory.

While the study of Ramsey numbers has been a popular area of research in graph theory, over the years a number of variations of Ramsey numbers have arisen. We describe several of these here, with special emphasis on some of those which have been introduced more recently. We present several results and open questions in this area of research. While many results obtained on Ramsey numbers and their variations involve bounds, our primary emphasis here is describing some of the exact results obtained. We refer to the book [14] for graph theory notation and terminology not described in this paper.

1.1 Ramsey Numbers

When F and H are both complete graphs, the Ramsey numbers $R(F, H)$ are often referred to as *classical Ramsey numbers* as these were the orginal Ramsey numbers studied for many years. For integers $s, t \geq 3$, only a handful of classical Ramsey numbers $R(K_s, K_t)$ are known. The complete list of known classical Ramsey numbers $R(K_s, K_t)$ for $3 \leq s \leq t$ is given below.

$$\begin{array}{lll} R(K_3, K_3) = 6 & R(K_3, K_6) = 18 & R(K_3, K_9) = 36 \\ R(K_3, K_4) = 9 & R(K_3, K_7) = 23 & R(K_4, K_4) = 18 \\ R(K_3, K_5) = 14 & R(K_3, K_8) = 28 & R(K_4, K_5) = 25. \end{array}$$

In particular, the exact value of $R(K_5, K_5)$ is not known. It is only known that

$$43 \leq R(K_5, K_5) \leq 48.$$

The most elementary and best known of the Ramsey numbers listed above is $R(K_3, K_3) = 6$. One interpretation of this number is related to a well-known recreational problem with a graph theory connection:

> *Suppose that every two people at a party are either acquainted or are strangers. What is the smallest number of people who must be present at the party to be guaranteed that there are three among them who are either mutual acquaintances or mutual strangers?*

The fact is that in any group of six people every two of whom are either acquaintances or strangers, there are always three among them who are mutual acquaintances or mutual strangers. Since the red-blue coloring of K_5 whose red and blue subgraphs are both 5-cycles does not produce a monochromatic K_3, it follows that $R(K_3, K_3) \geq 6$. To verify that $R(K_3, K_3) \leq 6$, it remains to show that every red-blue coloring of K_6 produces a monochromatic K_3. Let $V(K_6) = \{u, v, w, x, y, z\}$ and let there be given a red-blue coloring of K_6. We may assume that xu, xv, xw are colored the same, say red. If one of the edges uv, vw, uw is red, then there is a red K_3; while if all three edges uv, vw, uw are blue, then there is a blue K_3. Therefore, $R(K_3, K_3) = 6$.

It is a consequence of a theorem of Ramsey [49] that $R(F, H)$ exists for every pair F, H of graphs. Furthermore, it is a result of Erdös and Szekeres [27] that if F is a graph of order s and H is a graph of order t, then

$$R(F, H) \leq R(K_s, K_t) \leq \binom{s+t-2}{s-1}.$$

The exact values of $R(F, H)$ have been determined only for pairs F, H of graphs belonging to relatively few classes. Some of these are listed below (see also [48, 50, 51]).

Theorem 1.1.1 [17] *Let T be a tree of order $p \geq 2$. For every integer $n \geq 2$,*

$$R(T, K_n) = (p-1)(n-1) + 1.$$

Theorem 1.1.2 [33] *For integers n and m with $2 \leq m \leq n$,*

$$R(P_n, P_m) = n - 1 + \lfloor m/2 \rfloor.$$

Theorem 1.1.3 [30] *Let m and n be integers with $3 \leq m \leq n$.*

(1) *If m is odd, where $(m, n) \neq (3, 3)$, then*

$$R(C_m, C_n) = 2n - 1.$$

(2) *If m and n are even, where $(m, n) \neq (4, 4)$, then*

$$R(C_m, C_n) = n + m/2 - 1.$$

(3) *If m is even and n is odd,*

$$R(C_m, C_n) = \max\{n + m/2 - 1,\ 2m - 1\}.$$

(4) $R(C_3, C_3) = R(C_4, C_4) = 6$.

Theorem 1.1.4 [19, 20] *For integers s and t with $2 \le s \le t$,*

$$R(sK_2, tK_2) = s + 2t - 1.$$

More generally, for every $k \ge 2$ graphs $F_1, F_2, \ldots, F_k$, there exists a least positive integer n such that for every edge coloring of K_n with the k colors $1, 2, \ldots, k$, there exists a subgraph of K_n isomorphic to F_i for some i with $1 \le i \le k$ such that every edge of this subgraph is colored i. This integer n is the *Ramsey number* $R(F_1, F_2, \ldots, F_k)$ of $F_1, F_2, \ldots, F_k$, which always exists. The only classical Ramsey numbers whose value are known when $k \ge 3$ and where all complete graphs have order at least 3 are $R(K_3, K_3, K_3) = 17$ (see [36]) and $R(K_3, K_3, K_4) = 30$ (see [21]).

As an illustration, we show that $R(K_3, K_3, K_3) = 17$. Since the complete graph K_{16} has an isomorphic factorization into three factors, each of which is the 5-regular triangle-free graph (called the *Clebsch graph* [18]) shown in Figure 1.1, it follows that $R(K_3, K_3, K_3) > 16$ or $R(K_3, K_3, K_3) \ge 17$.

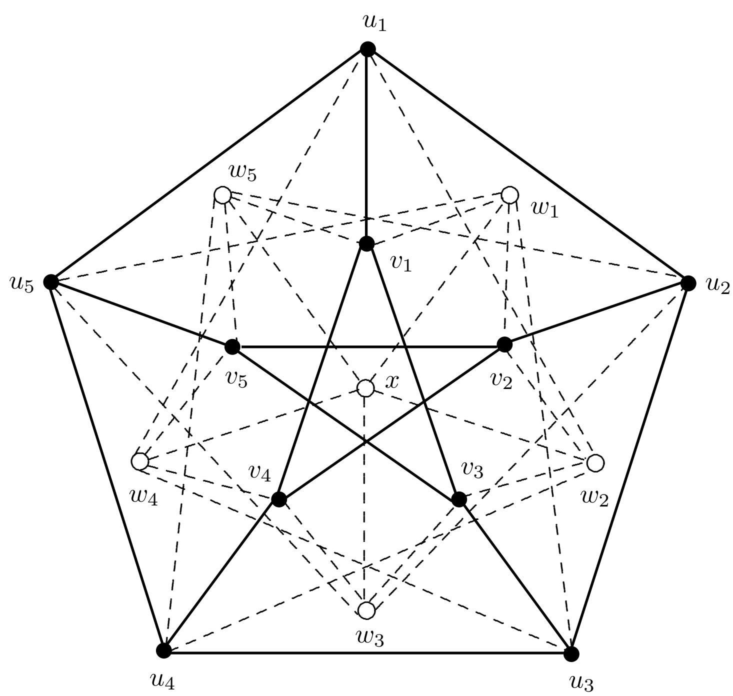


Figure 1.1: The Clebsch graph

To see that $R(K_3, K_3, K_3) \le 17$, let there be given a red-blue-green coloring of the edges of $G = K_{17}$ and let v be a vertex of G. Therefore, $\deg v = 16$. By the

Pigeonhole Principle, at least six edges incident with v are colored the same. Hence, we may assume that $vv_1, vv_2, \ldots, vv_6$ are six edges of G, all colored green. If any two vertices of $U = \{v_1, v_2, \ldots, v_6\}$ are joined by a green edge, then G contains a green K_3. Otherwise, every edge of the induced subgraph $H = G[U]$ is colored red or blue. Since $H \cong K_6$ and $R(K_3, K_3) = 6$, it follows that H, and G as well, contains either a red K_3 or a blue K_3. Therefore, $R(K_3, K_3, K_3) \leq 17$ and so $R(K_3, K_3, K_3) = 17$.

This more general Ramsey number has also been determined when all graphs F_i are stars.

Theorem 1.1.5 [12] *Let $s_1, s_2, \ldots, s_k$ be $k \geq 2$ positive integers, t of which are even, and let $s = \sum_{i=1}^{k}(s_i - 1)$. Then*

$$R(K_{1,s_1}, K_{1,s_2}, \ldots, K_{1,s_k}) = \begin{cases} s+1 & \textit{if } t \textit{ is positive and even} \\ s+2 & \textit{otherwise.} \end{cases}$$

If F and H are graphs such that $F \cong H$, then

$$R(F, H) = R(H, F) = R(F, F)$$

is the smallest positive integer n such that if each edge of K_n is colored with one of two colors, then a monochromatic F results. This leads to the following definition. For two graphs F and H, the *monochromatic Ramsey number* $MR(F, H)$ is the smallest positive integer n such that if each edge of K_n is colored with one of two colors, then a monochromatic F or a monochromatic H results. Certainly, $MR(F, H) = MR(H, F)$ for every two graphs F and H. Also, $MR(F, H) \leq R(F, H)$. Furthermore, if $F \cong H$, then $MR(F, H) = R(F, H)$ and if $F \subseteq H$, then $MR(F, H) = R(F, F)$ (see [15, pp. 315-320]). By Theorem 1.1.3, $R(C_3, C_4) = 7$. Next, we show that $MR(C_3, C_4) = 6$. Since the red-blue coloring of K_5 in which both red and blue subgraphs are C_5 avoids both a monochromatic C_3 and a monochromatic C_4, it follows that $MR(C_3, C_4) \geq 6$. Since $R(K_3, K_3) = 6$, it follows that $MR(C_3, C_4) \leq 6$ and so $MR(C_3, C_4) = 6$. Thus, $MR(C_3, C_4) < R(C_3, C_4)$.

1.2 Arrowing and Size Ramsey Numbers

While the definitions of the Ramsey number $R(F, H)$ of two graphs F and H and that of the more general Ramsey number $R(F_1, F_2, \ldots, F_k)$ of $k \geq 3$ graphs $F_1, F_2, \ldots, F_k$ concern edge colorings of complete graphs, with two colors in the first instance and k colors in the second instance, there has been research dealing with the graphs being colored that are not necessarily complete. In this case, different terminology and notation have been used.

Let F and H be two graphs. A graph G is said to *arrow* the graphs F and H, written $G \to (F, H)$, if every red-blue coloring of G results in a red F or a blue H. In this case, the primary problem concerns either determining graphs G or properties of graphs G for which $G \to (F, H)$. Obviously, one such graph G with this property is K_r where $r = R(F, H)$. Indeed, any graph G with clique number $\omega(G) \geq r$ has this property. Among the results obtained dealing with this concept are the following (see [11, 31, 46], for example).

Proposition 1.2.1 *If G is a graph for which $G \to (K_m, K_n)$, where $m, n \geq 2$, then $\omega(G) \geq \max\{m, n\}$.*

Theorem 1.2.2 *If G is a graph for which $G \to (K_m, K_n)$, where $m, n \geq 2$, then $\chi(G) \geq R(K_m, K_n)$.*

Theorem 1.2.3 *If G is a connected graph and n is a positive integer, then $G \to (K_{1,n}, K_{1,n})$ if and only if (i) $\Delta(G) \geq 2n-1$ or (ii) n is even and G is a $(2n-2)$-regular graph of odd order.*

For two graphs F and H, the *size Ramsey number* $\hat{R}(F, H)$ of F and H is the smallest size of a graph G such that $G \to (F, H)$. Bounds on the size Ramsey numbers of paths, cycles or trees have been established in terms of the order and maximum degree of the graphs (see [3, 4, 10, 25], for example).

Proposition 1.2.4 [25] *For two graphs F and H,*

$$|E(F)| + |E(H)| - 1 \leq \hat{R}(F, H) \leq \binom{R(F,H)}{2}.$$

Theorem 1.2.5 [25] *For positive integers m, n, s and t,*

(i) $\hat{R}(K_m, K_n) = \binom{R(K_m, K_n)}{2}$

(ii) $\hat{R}(sK_{1,m}, tK_{1,n}) = (m + n - 1)(s + t - 1)$.

Theorem 1.2.6 [3, 4] *There exist constants c and c' such that for any tree T_n of order n and maximum degree Δ and for n sufficiently large,*

(i) $\hat{R}(P_n, P_n) \leq cn$;

(ii) $\hat{R}(C_n, C_n) \leq c'n$;

(iii) $\hat{R}(T_n, T_n) \leq \Delta \cdot n \cdot (\log n)^{12}$.

Theorem 1.2.7 [40] *For any tree T_n with maximum degree Δ, there is a constant c such that $\hat{R}(T_n, T_n) \leq c \cdot \Delta \cdot n$.*

In general, for $k \geq 2$ graphs $F_1, F_2, \ldots, F_k$, a graph G is said to *arrow* the graphs F_1, F_2, …, F_k, written $G \to (F_1, F_2, \ldots, F_k)$, if for every k-edge coloring $c : E(G) \to [k] = \{1, 2, \ldots, k\}$ of G, there exists a subgraph G_i of G, $1 \leq i \leq k$, all of whose edges are colored i and such that $G_i \cong F_i$. In this case, the problem is to determine graphs G for which $G \to (F_1, F_2, \ldots, F_k)$. Obviously, one such graph is K_r where $r = R(F_1, F_2, \ldots, F_k)$.

1.3 Bipartite Ramsey Numbers

In 1975 Beineke and Schwenk [5] introduced a bipartite version of Ramsey numbers. For two bipartite graphs F and H, the *bipartite Ramsey number* $BR(F, H)$ is defined as the smallest positive integer r such that every red-blue coloring of the r-regular complete bipartite graph $K_{r,r}$ results in either a red F or a blue H. Consequently, if $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. The concept of bipartite Ramsey numbers of graphs is closely related to another recreational problem:

> *Suppose, for some positive integer r, that an equal number r of girls and boys are invited to a party where each girl-boy pair are either acquainted or are strangers. What is the smallest such r that guarantees that there exists a group of six people, three girls and three boys, such that either* (1) *every one of the three girls is acquainted with every one of the three boys or* (2) *every one of the three girls is a stranger of every one of the three boys?*

The answer to this question is $BR(K_{3,3}, K_{3,3}) = 17$ (see [5]).

To illustrate these concepts, we show that $BR(C_4, C_4) = 5$ (which was determined in [5]). Since the red-blue coloring of $K_{4,4}$, whose red and blue subgraphs are C_8, shown in Figures 1.2(a) and 1.2(b) which avoid a C_4, it follows that $BR(C_4, C_4) \geq 5$.

To verify that $BR(C_4, C_4) \leq 5$, it remains to show that every red-blue coloring of $K_{5,5}$ results in a monochromatic C_4. Let there be given a red-blue coloring of $G = K_{5,5}$ where U and W are the partite sets of $K_{5,5}$. Either U or W contains at least three

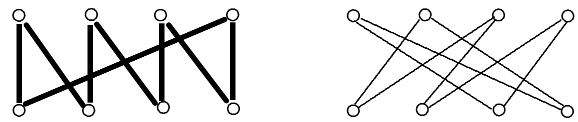


Figure 1.2: A red-blue coloring of $K_{4,4}$

vertices incident with at least three edges of the same color. Suppose that $u_1, u_2, u_3 \in U$ are three vertices incident with at least three red edges. Then two of these vertices, say u_1 and u_2, have at least two neighbors in common, producing a red C_4. Therefore, $BR(C_4, C_4) = 5$.

Beineke and Schwenk proved that $BR(F, H)$ exists for every two bipartite graphs F and H (see [5]). There are also many papers dealing with bipartite Ramsey numbers, including [16, 34, 39], for example. Indeed, if F is a bipartite graph whose largest partite set contains s vertices and H is a bipartite graph whose largest partite set contains t vertices, then $F \subseteq K_{s,s}$ and $H \subseteq K_{t,t}$, resulting in the following result of Hattingh and Henning, which also proves that $BR(F, H)$ exists for every two bipartite graphs F and H.

Theorem 1.3.1 [39] *If F and H are bipartite graphs such that $F \subseteq K_{s,s}$ and $H \subseteq K_{t,t}$, then*

$$BR(F, H) \leq BR(K_{s,s}, K_{t,t}) \leq \binom{s+t}{s} - 1.$$

The following results and conjecture were obtained on bipartite Ramsey numbers.

Theorem 1.3.2 [16] *For integers s and t with $2 \leq s \leq t$,*

$$BR(sK_2, tK_2) = s + t - 1.$$

Theorem 1.3.3 [5] *For each positive integer t,*

$$BR(K_{1,t}, K_{1,t}) = 2t - 1.$$

Conjecture 1.3.4 [5] *For integers s and t with $1 \leq s \leq t$,*

$$BR(K_{s,t}, K_{s,t}) = 2^s(t-1) + 1$$

We saw, for $k \geq 2$ graphs $F_1, F_2, \ldots, F_k$ and $r = R(F_1, F_2, \ldots, F_k)$, that $K_r \to (F_1, F_2, \ldots, F_k)$. The following is the analogue of this fact for bipartite graphs. For $k \geq 2$ bipartite graphs $F_1, F_2, \ldots, F_k$ and $r = BR(F_1, F_2, \ldots, F_k)$, it follows that

$$K_{r,r} \to (F_1, F_2, \ldots, F_k).$$

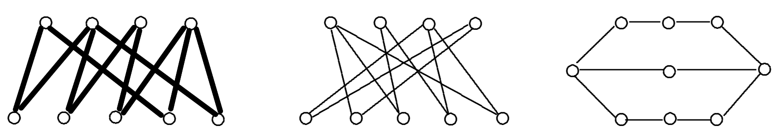
1.4 k-Ramsey Numbers

The concept of bipartite Ramsey numbers of graphs has been extended in different directions, one of which was introduced by Chartrand (see [2]). As described in [2], if $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. This brings up the question of what might occur for red-blue colorings of the intermediate graph $K_{r-1,r}$. This led to a more general concept.

For bipartite graphs F and H, the 2-*Ramsey number* $R_2(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ of order n results in a red F or a blue H. If the bipartite Ramsey number $BR(F, H)$ of two bipartite graphs F and H is r, then every red-blue coloring of $K_{r,r}$ produces a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ that produces neither. Which of these two situations occurs for the graph $K_{r-1,r}$ depends on the graphs F and H. That is, either

$$R_2(F, H) = 2BR(F, H) \text{ or } R_2(F, H) = 2BR(F, H) - 1. \tag{1.1}$$

To illustrate this concept, we show that $R_2(C_4, C_4) = 10$ (which was determined in [2]). We saw that $BR(C_4, C_4) = 5$. It then follows by (1.1) that $R_2(C_4, C_4) = 10$ or $R_2(C_4, C_4) = 9$. In fact, there is a red-blue coloring of $K_{4,5}$ that results in neither a red C_4 nor a blue C_4. To see this, consider the red-blue coloring of $K_{4,5}$ in which both the red subgraph shown in Figure 1.3(a) and the blue subgraph shown in Figure 1.3(b) are isomorphic to the graph in Figure 1.3(c). Since the graph in Figure 1.3(c) does not contain C_4 as a subgraph, this red-blue coloring of $K_{4,5}$ avoids both a red C_4 and a blue C_4. Therefore, $R_2(C_4, C_4) \geq 10$ and so $R_2(C_4, C_4) = 10$.



(a) (b) (c)

Figure 1.3: A red-blue coloring of $K_{4,5}$

The concept of the 2-Ramsey number of two bipartite graphs is a special case of a more general concept. For an integer $k \geq 2$, a *balanced complete k-partite graph* of order $n \geq k$ is the complete k-partite graph in which every partite set has either $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ vertices. So if $n = kq + r$ where $q \geq 1$ and $0 \leq r \leq k - 1$, then the balanced complete k-partite graph G of order n has r partite sets with $q + 1$ vertices and the remaining $k - r$ partite sets have q vertices. For bipartite graphs F and H and an integer k with $2 \leq k \leq R(F, H)$, the *k-Ramsey number* $R_k(F, H)$ is defined in [2] as the smallest positive integer n such that every red-blue coloring of a balanced complete k-partite graph of order n results in a red F or a blue H.

If F and H are two bipartite graphs for which $R(F, H) = n \geq 3$, then every red-blue coloring of K_n produces either a red F or a blue H. However, such is not the case for the smaller complete graphs $K_2, K_3, \ldots, K_{n-1}$. Equivalently, for every red-blue coloring of the complete n-partite graph K_n where each partite set consists of a single vertex, there is either a red F or a blue H. However, for each complete k-partite graph K_k, where $2 \leq k \leq n-1$ such that every partite set consists of a single vertex, there exists a red-blue coloring that produces neither a red F nor a blue H. On the other hand, for each of the graphs $K_2, K_3, \ldots, K_{n-1}$, we can continue to add vertices to each partite set, resulting in a balanced complete k-partite graph at each step where $2 \leq k \leq n-1$ until eventually arriving at the balanced complete k-partite graph of smallest order $R_k(F, H)$ having the property that every red-blue coloring of this graph produces a red F or a blue H. Consequently, for every two bipartite graphs F and H and every integer k with $2 \leq k \leq R(F, H)$, the k-Ramsey number $R_k(F, H)$ exists (see [2]).

For example, it is known that $R(C_4, C_4) = 6$. Furthermore, we saw that $BR(C_4, C_4) = 5$ and $R_2(C_4, C_4) = 10$. In fact, it is shown in [2] that $R_k(C_4, C_4) = 12 - k$ for $2 \leq k \leq 6 = R(C_4, C_4)$. As an illustration, we show that $R_3(C_4, C_4) = 9$. Let H be a balanced complete 3-partite graph of order 8. Then $H = K_{2,3,3}$. Figure 1.4 shows a red-blue coloring of H having neither a red C_4 nor a blue C_4, where the bold edges represent edges colored red. Thus, $R_3(C_4, C_4) \geq 9$.

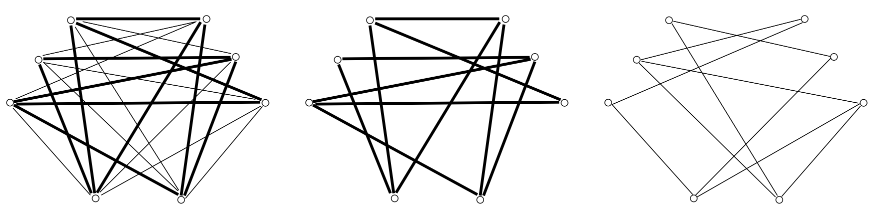


Figure 1.4: A red-blue coloring of $K_{2,3,3}$

To show that $R_3(C_4, C_4) = 9$, it remains to show that every red-blue coloring of $G = K_{3,3,3}$ results in a monochromatic C_4. Assume, to the contrary, that there is a red-blue coloring of G that produces neither a red C_4 nor a blue C_4. Let G_R and G_B denote the red and blue subgraphs of G, respectively, of sizes m_R and m_B. We may assume that $m_R \geq m_B$. Since $m_R + m_B = 27$, it follows that $m_R \geq 14$. Let V_1, V_2 and V_3 be the three partite sets of G and, for $1 \leq i < j \leq 3$, let $[V_i, V_j]$ denote the nine edges of G joining V_i and V_j. Let G'_R denote the subgraph of size m'_R in G_R with vertex set $V_1 \cup V_2$ such that $E(G'_R) \subseteq [V_1, V_2]$. The subgraphs G''_R and G'''_R with vertex sets $V_2 \cup V_3$ and $V_1 \cup V_3$ and sizes m''_R and m'''_R, respectively, are defined similarly. We may assume that $m'_R \geq m''_R \geq m'''_R$ and so $m'_R + m''_R \geq 10$. Let $V_1 = \{u_1, u_2, u_3\}$, $V_2 = \{v_1, v_2, v_3\}$ and $V_3 = \{w_1, w_2, w_3\}$. Observe that if any of u_1, u_2 and u_3 has degree 3 in G'_R, say u_1, then u_2 and u_3 have degree at most 1 in G'_R and each of w_1, w_2 and w_3 has degree at most 1 in G''_R, for otherwise, a red C_4 is produced. However then, $m'_R + m''_R \leq 8$, a contradiction. Consequently, each of u_1, u_2 and u_3 has degree at most 2 in G'_R. Therefore, $m'_R = 6$ or $m'_R = 5$. We consider these two cases.

Case 1. $m'_R = 6$. Thus, $G'_R = C_6$, say $G'_R = (u_1, v_1, u_2, v_2, u_3, v_3, u_1)$. Hence each of w_1, w_2 and w_3 has degree at most 1 in G''_R for otherwise, a red C_4 is produced. However, then $m'_R + m''_R \leq 9$, a contradiction.

Case 2. $m'_R = 5$. Hence, $m''_R = 5$ as well. We may assume that u_1 and u_2 have degree 2 in G'_R and w_1 and w_2 have degree 2 in G''_R. Neither u_1 and u_2 nor w_1 and w_2 have the same neighbors in G'_R and G''_R, respectively, for otherwise, a red C_4 is produced. This, however, implies that two of the vertices v_1, v_2 and v_3 are neighbors of both a vertex u_i in G'_R and a vertex w_j in G''_R, producing a red C_4 and a contradiction. Therefore, $R_3(C_4, C_4) = 9$.

The following three results on k-Ramsey numbers were obtained in [1].

Proposition 1.4.1 *Let F and H be two bipartite graphs. If k is an integer with $2 \leq k \leq R(F, H)$, then $R(F, H) \leq R_k(F, H)$.*

Proposition 1.4.2 *Let F and H be two bipartite graphs. If k and ℓ are positive integers with $k \geq 2$, then $R_{\ell k}(F, H) \leq R_k(F, H)$.*

Proposition 1.4.3 *Let F and H be two bipartite graphs. If k is an integer with $k \leq R(F, H)$ for which $R_k(F, H) = R(F, H)$ and $\frac{R_k(F,H)-1}{k} \leq 2$, then*

$$R_\ell(F, H) = R_k(F, H)$$

for each integer ℓ with $k \leq \ell \leq R(F, H)$.

By Theorem 1.1.5, for two integers $s, t \geq 2$,

$$R(K_{1,s}, K_{1,t}) = \begin{cases} s + t - 1 & \text{if } s \text{ and } t \text{ are both even} \\ s + t & \text{otherwise.} \end{cases} \tag{1.2}$$

Thus, if $k = R(K_{1,s}, K_{1,t})$, then $R_k(K_{1,s}, K_{1,t})$ is expressed in (1.2). The k-Ramsey number of stars have been determined for all possible values of k in [2].

Theorem 1.4.4 *For each integer $t \geq 2$, $R_2(K_{1,2}, K_{1,t}) = 2t + 1$.*

Theorem 1.4.5 *Let k, s and t be integers with $3 \leq k < R(K_{1,s}, K_{1,t})$ and $s + t \geq 5$.*

(*a*) *If $s + t - 2 = (k - 1)q$ for some positive integer q, then*

$$R_k(K_{1,s}, K_{1,t}) = \begin{cases} kq & \text{if } k \text{ and } q \text{ are odd and } s \text{ and } t \text{ are even} \\ kq + 1 & \text{otherwise.} \end{cases}$$

(*b*) *If $s + t - 2 = (k - 1)q + r$ for integers q and r where $q \geq 1$ and $1 \leq r \leq k - 2$, then*

$$R_k(K_{1,s}, K_{1,t}) = \begin{cases} kq + r & \text{if } (k - r)q \text{ is odd and } s \text{ and } t \\ & \text{are of opposite parity} \\ kq + r + 1 & \text{otherwise.} \end{cases}$$

Consequently, we have the following.

> If k, s, t are integers with $3 \leq k < R(K_{1,s}, K_{1,t})$ and $s + t \geq 5$, then $R_k(K_{1,s}, K_{1,t})$ is either $s + t - 2 + \left\lfloor \frac{s+t-2}{k-1} \right\rfloor$ or $s + t - 1 + \left\lfloor \frac{s+t-2}{k-1} \right\rfloor$, depending on the values of k, s and t in Theorem 1.4.5.

A *stripe* is a 1-regular graph. The stripe of size r is denoted by rK_2 and consists of r copies of the complete graph K_2, whose edges therefore form a matching of size r. The bipartite Ramsey number of two stripes was determined in [16].

Theorem 1.4.6 [16] *For integers s and t with $2 \leq s \leq t$,*

$$BR(sK_2, tK_2) = s + t - 1.$$

In [1], the k-Ramsey numbers were determined for certain stripes F and H and for certain values of k. By Theorem 1.1.4 and Proposition 1.4.1, for integers k, s and t with $2 \leq s \leq t$ and $2 \leq k \leq R(sK_2, tK_2)$, it follows that

$$R_k(sK_2, tK_2) \geq s + 2t - 1. \tag{1.3}$$

By (1.1), if the bipartite Ramsey number $BR(F, H)$ of two bipartite graphs F and H is r, then $R_2(F, H) = 2r$ or $R_2(F, H) = 2r - 1$. In the case of stripes, $R_2(sK_2, tK_2) = 2BR(sK_2, tK_2)$, which provides the following result [1].

Proposition 1.4.7 *For integers s and t with $2 \leq s \leq t$,*

$$R_2(sK_2, tK_2) = 2s + 2t - 2.$$

The k-Ramsey numbers of $R_k(sK_2, tK_2)$ are determined in [1] for (i) all $s = 2, 3$ and $t \geq 2$ and (ii) $k = 3, 4$ and $t \geq s \geq 2$. We state these results next.

Theorem 1.4.8 *For integers k and t with $2 \leq k \leq R(2K_2, tK_2)$ and $t \geq 2$,*

$$R_k(2K_2, tK_2) = \begin{cases} 2t + 2 & \text{if } k = 2 \\ 2t + 1 & \text{otherwise.} \end{cases}$$

Theorem 1.4.9 *For integers k and t with $2 \leq k \leq R(3K_2, tK_2)$ and $t \geq 3$,*

$$R_k(3K_2, tK_2) = \begin{cases} 2t + 4 & \text{if } k = 2 \\ 2t + 2 & \text{otherwise.} \end{cases}$$

Theorem 1.4.10 *For integers s, t and k with $2 \leq s \leq t$ and $k \in \{3, 4\}$,*

$$R_k(sK_2, tK_2) = s + 2t - 1.$$

In fact, there is a conjecture on the k-Ramsey number of stripes [1].

Conjecture 1.4.11 *Let k, s and t be integers with $2 \leq s \leq t$.*

$$\text{If } 5 \leq k \leq R(sK_2, tK_2), \text{ then } R_k(sK_2, tK_2) = s + 2t - 1.$$

We have seen in (1.3) that $R_k(sK_2, tK_2) \geq s + 2t - 1$ for all integers k with $3 \leq k \leq R(sK_2, tK_2)$. Thus, by Proposition 1.4.2 and Theorem 1.4.10, to verify Conjecture 1.4.11, it suffices to establish the conjecture for primes k with $k \geq 5$.

While the k-Ramsey number $R_k(F, H)$ exists for every two bipartite graphs F and H when $2 \leq k \leq R(F, H)$, such is not the case when F and H are not bipartite. For graphs F and H that are not bipartite, it was observed in [42] that not only does $R_2(F, H)$ fail to exist but $R_3(F, H)$ and $R_4(F, H)$ also do not exist. To see this, let G be any balanced complete 3-partite graph with partite sets V_1, V_2 and V_3. Assigning the color red to every edge of $[V_1, V_2]$ and blue to all other edges of G results in G_R and G_B both being bipartite. Similarly, if G is a balanced complete 4-partite graph with partite sets V_1, V_2, V_3 and V_4 and the color red is assigned to every edge of $[V_1, V_2] \cup [V_2, V_3] \cup [V_3, V_4]$ and blue to all other edges of G, then G_R and G_B are both bipartite. Indeed, even if

$\chi(F) = \chi(H) = 3$, $R_5(F, H)$ need not exist. For example, $R_5(K_3, K_3)$ does not exist. To see this, let G be a balanced complete 5-partite graph with partite sets V_i for $1 \le i \le 5$. If the edges in $[V_1, V_2] \cup [V_2, V_3] \cup [V_3, V_4] \cup [V_4, V_5] \cup [V_5, V_1]$ are colored red and all other edges are colored blue, then G does not contain a monochromatic K_3. Consequently, $R_k(K_3, K_3)$ exists only when $k = R(K_3, K_3) = 6$. On the other hand, $R_5(F, H)$ can exist when $\chi(F) = \chi(H) = 3$ as the following result shows (see [42]).

Theorem 1.4.12 *If k and ℓ are integers with $k, \ell \ge 2$, then $R_5(C_{2\ell+1}, C_{2k+1})$ exists.*

The k-Ramsey numbers of some well-known classes of non-bipartite graphs have been investigated (see [41, 42]). A connected graph G is *unicyclic* if G contains exactly one cycle. For an integer $t \ge 3$, the unicyclic-star graph U_t is the unicyclic graph containing the star $K_{1,t}$ as a spanning subgraph. Consequently, U_t, where $t \ge 3$, is a connected graph of order and size $t + 1$, containing a single cycle, namely a triangle, two vertices of which have degree 2 in U_t and the third vertex has degree t. The unicylic-stars U_t are shown in Figure 1.5 for $t = 3, 4, 5$.

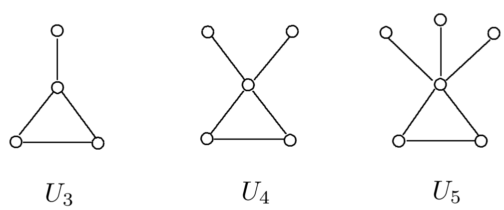


Figure 1.5: The unicylic-stars U_t for $t = 3, 4, 5$

For a graph F, we write $R(F)$ to denote the Ramsey number $R(F, F)$ and write $R_k(F)$ to denote the k-Ramsey number $R_k(F, F)$. Among the results obtained in [42] are the following:

Proposition 1.4.13 *For each integer $t \ge 3$, $R(U_t) = 2t + 1$.*

Proposition 1.4.14 *For each integer $t \ge 3$, $R_k(U_t)$ does not exist for $2 \le k \le 5$.*

For $t = 3, 4, 5$ and $6 \le k \le R(U_t)$, the k-Ramsey numbers $R_k(U_t)$ are determined in [42] as follows:

- ⋆ $R_k(U_3) = R(U_3) = 7$ for $k = 6, 7$,
- ⋆ $R_k(U_4) = R(U_4) = 9$ for $k = 6, 7, 8, 9$ and
- ⋆ $R_6(U_5) = 13$, $R_k(U_5) = 12$ for $k = 7, 8$ and $R_k(U_5) = R(U_5) = 11$ for $k = 9, 10, 11$.

Theorem 1.4.15 *For each integer $t \geq 6$,*

$$R_6(U_t) = \begin{cases} 3t-3 & \textit{if } t \textit{ is even} \\ 3t-2 & \textit{if } t \textit{ is odd.} \end{cases}$$

While the k-Ramsey numbers $R_k(U_t)$ do not exist for $2 \leq k \leq 5$ and $t \geq 3$, Theorem 1.4.15 implies that $R_k(U_t)$ exists for all integers k and t with $6 \leq k \leq R(U_t)$ and $t \geq 3$.

Theorem 1.4.16 *For integers k and t with $t \leq k \leq R(U_t)$ and $t \geq 6$,*

$$R_k(U_t) = \begin{cases} 2t+4 & \textit{if } k = t = 7 \\ 2t+3 & \textit{if (i) } t \textit{ is odd and } k = t+1 \textit{ or (ii) } k = t \neq 7 \\ 2t+2 & \textit{if } t \textit{ is odd and } t+2 \leq k \leq \lceil 3t/2 \rceil \\ 2t+1 & \textit{otherwise.} \end{cases}$$

Figure 1.6 summarizes the results obtained on the k-Ramsey numbers $R_k(U_t)$ of the graph U_t for $6 \leq k \leq R(U_t)$ and $t \geq 6$. The (k,t)-entry in row k and column t is the number $R_k(U_t)$ and the symbol $*$ indicates that this number has been determined. Where there is no (k,t)-entry, this indicates that $R_k(U_t)$ is unknown.

	6	7	8	9	10	11	...
6	*	*	*	*	*	*	*
7	*	*					
8	*	*	*				
9	*	*	*	*			
10	*	*	*	*	*		
11	*	*	*	*	*	*	
⋮	*	*	*	*	*	*	*

Figure 1.6: Known and unknown k-Ramsey numbers $R_k(U_t)$

We have seen that Ramsey numbers are defined for three or more graphs. In particular, for three graphs F_1, F_2 and F_3, the *Ramsey number* $R(F_1, F_2, F_3)$ of F_1, F_2 and F_3 is the smallest positive integer n for which every red-blue-green coloring (in which every edge is colored red, blue or green) of the complete graph K_n of order n results in a red F_1, a blue F_2 or a green F_3. This gives rise to the concept of k-Ramsey number of three (or more) graphs. For three graphs F_1, F_2 and F_3 and an integer k with $2 \le k \le R(F_1, F_2, F_3)$, the k-Ramsey number $R_k(F_1, F_2, F_3)$ of F_1, F_2 and F_3, if it exists, is the smallest order of a balanced complete k-partite graph G for which every red-blue-green coloring of the edges of G results in a red F_1, a blue F_2 or a green F_3. In particular, if $k = 2$ and $F_i \cong F$ for some graph F where $i = 1, 2, 3$, then the 2-Ramsey number $R_2(F, F, F)$ is the smallest order of a balanced complete bipartite graph G for which every red-blue-green coloring of the edges of G results in a monochromatic F (all of whose edges are colored the same). For example, it was shown in [34] that $BR(C_4, C_4, C_4) = 11$. Furthermore, it was shown in [43] that $R_2(C_4, C_4, C_4) \le 21$. Therefore, $R_2(C_4, C_4, C_4) = 21$.

1.5 Rainbow Ramsey Numbers

A subgraph F of an edge-colored graph G is said to be a *rainbow* F if no two edges of F are colored the same. For a graph G, Bialostocki and Voxman [8] defined the *rainbow Ramsey number* $RR(G)$ of G as the smallest positive integer n such that if each edge of the complete graph K_n is colored from any number of colors, then either a monochromatic G or a rainbow G results. The rainbow Ramsey number $RR(G)$ does not exist for all graphs G. While the Ramsey number $R(K_3, K_3) = 6$, the rainbow Ramsey number $RR(K_3)$ does not exist. To see this, let n be an arbitrary positive integer and let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. Consider the edge coloring $c : E(K_n) \to [n-1]$ defined by $c(v_iv_j) = j$ if $i < j$. Let T be any triangle of K_n with $V(T) = \{v_i, v_j, v_k\}$ and $i < j < k$. Since $c(v_iv_j) = j$ and $c(v_iv_k) = c(v_jv_k) = k$, the triangle T is neither monochromatic nor rainbow. Consequently, $RR(K_3)$ does not exist.

Bialostocki and Voxman [8] characterized those graphs G for which $RR(G)$ does exist.

Theorem 1.5.1 *The rainbow Ramsey number $RR(G)$ of a graph G is defined if and only if G is acyclic.*

The proof of this result follows from a theorem due to Erdös and Rado. In order to state this theorem, some additional definitions are needed. Let c be an edge coloring of a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that the colors are positive integers. In a *minimum coloring* of G, each edge v_iv_j of G is colored $\min\{i, j\}$; in a *maximum coloring* of G, each edge v_iv_j is colored $\max\{i, j\}$. An edge coloring of G that is either minimum, maximum, monochromatic or rainbow is called a *canonical coloring.* Erdős and Rado [26] proved the following result.

Theorem 1.5.2 *For every positive integer k, there exists a positive integer n such that every edge coloring of K_n contains a canonically colored complete subgraph of order k.*

Bialostocki and Voxman [8] obtained the following result.

Theorem 1.5.3 *For every positive integer n,*

$$RR(nK_2) = n(n-1) + 2.$$

Eroh [28, 29] extended the rainbow Ramsey number from one graph to two graphs. For graphs F and H, the *rainbow Ramsey number* $RR(F, H)$ is the smallest positive integer n such that if the edges of K_n are colored with an arbitrary number of colors, either a monochromatic F or a rainbow H results. As expected, $RR(F, H)$ exists only under certain conditions. The following theorem of Eroh is a consequence of Theorem 1.5.2.

Theorem 1.5.4 *The rainbow Ramsey number $RR(F, H)$ of two graphs F and H exists if and only if F is a star or H is a forest.*

Among the exact values of $RR(F, H)$ obtained by Eroh [28, 29] are the following.

Theorem 1.5.5 *For positive integers s and t,*

$$RR(K_{1,s}, K_{1,t}) = (s-1)(t-1) + 2.$$

Theorem 1.5.6 *For integers s and t with $2 \leq t < s$,*

$$RR(sK_2, tK_2) = t(s-1) + 2.$$

To describe another type of rainbow Ramsey number of graphs, let F and H be two graphs, where H has size m. For a fixed integer $k \geq m$, the *k-rainbow Ramsey number* $RR_k(F, H)$ is the smallest positive integer n such that every k-edge coloring of K_n results in either a monochromatic F or a rainbow H (see [15, pp. 319-320]). Unlike the rainbow Ramsey number $RR(F, H)$, the number $RR_k(F, H)$ always exists. For example, while $RR(K_3, K_3)$ does not exist, $RR_3(K_3, K_3) = 11$. The red-blue-green coloring of K_{10}, where the green subgraph is $K_{5,5}$ and the red and blue subgraphs are two disjoint copies of C_5 produces neither a monochromatic nor a rainbow K_3. Thus, $RR_3(K_3, K_3) \geq 11$. Showing that $RR_3(K_3, K_3) \leq 11$ is more complicated. There is a dynamic survey on this topic by Fujita, Magnant and Ozeki [32].

Chapter 2

Proper Ramsey Numbers

2.1 Introduction

While edge colorings of a graph that result in certain monochromatic or rainbow subgraphs have been the subject of much research, the edge colorings receiving the most attention are proper edge colorings, in which every two adjacent edges are assigned different colors. The minimum number of colors required of a proper edge coloring of a graph G is its *chromatic index*, denoted by $\chi'(G)$. It is an immediate observation that for every nonempty graph G, the chromatic index of G is at least as large as its maximum degree $\Delta(G)$. The best known and most useful result on edge colorings was obtained by Vizing [53].

Theorem 2.1.1 (Vizing's Theorem) *For every nonempty graph G,*

$$\chi'(G) \le \Delta(G) + 1.$$

Thus, by Vizing's theorem, for every nonempty graph G with maximum degree Δ, either $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$. A graph G is said to be of *Class* 1 if $\chi'(G) = \Delta(G)$ and of *Class* 2 if $\chi'(G) = \Delta(G) + 1$. In particular, a regular graph G is of Class 1 if and only if G is 1-factorable. Determining which graphs belong to which class is a major problem of study in this area.

For two graphs F and H, Eroh [28] defined the *edge-chromatic Ramsey number* $CR(F, H)$ of F and H as the minimum positive integer n such that if the edges of K_n are colored with an arbitrary number of colors, then there is either a monochromatic F or a properly colored H. Eroh [28] showed that the edge-chromatic Ramsey number $CR(F, H)$ exists for exactly the same pairs F, H of graphs for which rainbow Ramsey numbers exist.

Theorem 2.1.2 *The edge-chromatic Ramsey number $CR(F, H)$ of two graphs F and H exists if and only if F is a star or H is a forest.*

As is often the case for Ramey numbers and its variations, many results are bounds for these numbers. Among the exact results obtained on edge-chromatic Ramsey numbers are the following, all of which are due to Eroh [28].

Theorem 2.1.3 [28] *For integers $m \geq 2$ and $n \geq 2$,*

$$CR(C_n, P_3) = n \quad \text{and} \quad CR(C_3, P_m) = m.$$

Theorem 2.1.4 [28] *For every integer $n \geq 3$,*

$$CR(K_{1,n}, P_4) = n + 1 \quad \text{and} \quad CR(P_n, P_4) = n + 1.$$

We now consider a related Ramsey number where the number of colors assigned to edges is finite and prescribed. Let F and H be two nonempty graphs such that $\chi'(H) = t$. The *proper Ramsey number* $PR(F, H)$ of F and H is the smallest positive integer n such that every t-edge coloring of K_n results in either a monochromatic F or a properly colored H. This concept was introduced by Chartrand and first studied in [23, 24]. Since the Ramsey number $R(F_1, F_2, \ldots, F_t)$, where $F_t \cong F$ for all $1 \leq i \leq t$, exists and $PR(F, H) \leq R(F_1, F_2, \ldots, F_t)$, it follows that the proper Ramsey number $PR(F, H)$ exists for every two graphs F and H. Here, we investigate the proper Ramsey number $PR(F, H)$ for several pairs F, H of connected graphs of order at least 3 where $\chi'(H) = 2$. For each such pair then,

$$|V(F)| \leq PR(F, H) \leq R(F, F). \tag{2.1}$$

2.2 Complete Graphs Versus Paths

In this section, we determine the numbers $PR(K_n, P_k)$ for every integer $n \geq 3$ and for those integers k with $3 \leq k \leq 6$. Of course, $\chi'(P_k) = 2$ for $k \geq 3$. We begin with $PR(K_n, P_3)$.

Proposition 2.2.1 *For each integer $n \geq 3$, $PR(K_n, P_3) = n$.*

Proof. First, $PR(K_n, P_3) \geq n$ by (2.1). Let there be given a red-blue coloring of K_n. If all edges of K_n are colored the same, then a monochromatic K_n results. If not, then there are two adjacent edges of K_n whose colors are different, that is, K_n has a properly colored P_3. Therefore, $PR(K_n, P_3) \leq n$ and so $PR(K_n, P_3) = n$. ■

Theorem 2.2.2 *For each integer $n \geq 3$, $PR(K_n, P_4) = n + 1$.*

Proof. Let v be a vertex of the graph K_n. The red-blue coloring of K_n in which each edge incident with v is colored red and all other edges of K_n are colored blue has neither a monochromatic K_n nor a properly colored P_4. Hence, $PR(K_n, P_4) \geq n + 1$.

It remains to show that $PR(K_n, P_4) \leq n + 1$. Assume, to the contrary, that there is a red-blue coloring of $G = K_{n+1}$ that avoids both a monochromatic K_n and a properly colored P_4. By Proposition 2.2.1, there is a properly colored P_3, say (u, v, w), where uv is colored red and vw is colored blue. Let X be the set consisting of the remaining $n - 2$ vertices of G. Since there is no properly colored P_4 in G, the edge xu is red for each $x \in X$ and xw is blue for each $x \in X$. Assume, without loss of generality, that uw is red. Hence, xv must be blue for each $x \in X$ since there is no properly colored P_4 in G. This is illustrated in Figure 2.1, where a red edge is indicated by a solid line and a blue edge is indicated by a dashed line.

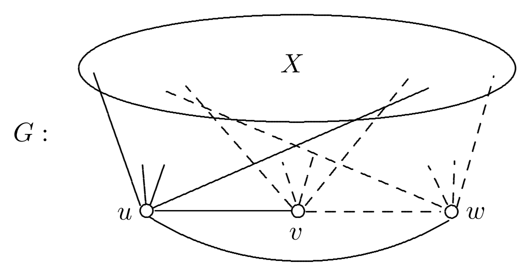


Figure 2.1: A red-blue coloring of $G = K_{n+1}$

If $n = 3$, then there is a monochromatic K_3, namely a blue K_3. So, we may assume that $n \geq 4$. If any edge of $G[X]$ is red, then there is a properly colored P_4. Thus, all such edges are blue and the subgraph $G[X \cup \{v, w\}]$ is a blue K_n, a contradiction. Therefore, $PR(K_n, P_4) \leq n + 1$ and so $PR(K_n, P_4) = n + 1$. ■

In order to evaluate $PR(K_n, P_5)$ for $n \geq 3$, we first consider the special case when $n = 3$.

Proposition 2.2.3 $PR(K_3, P_5) = 5$.

Proof. The red-blue coloring of K_4 in which the red subgraph is C_4 and the blue subgraph is $2K_2$ contains neither a monochromatic K_3 nor a properly colored P_5. Thus, $PR(K_3, P_5) \geq 5$.

Let there be given a red-blue coloring of $G = K_5$ that avoids a monochromatic K_3. Let G_R and G_B be the red and blue subgraphs, respectively, of G. Suppose that the size of G_R is at least that of G_B. Thus, G_R contains a cycle that is not C_3. If $G_R = C_5$, then $G_B = C_5$ and there is a properly colored P_5; while if G_R contains a 4-cycle C, then both of its diagonals are blue and so the vertex of G not on C is adjacent to at least one vertex on C by a red or blue edge, producing a properly colored P_5 in either case and so $PR(K_3, P_5) = 5$. ■

Theorem 2.2.4 *For every integer $n \geq 4$, $PR(K_n, P_5) = 2n - 2$.*

Proof. Since the red-blue coloring of K_{2n-3}, in which every edge of some $(n-1)$-clique is colored red and all other edges are colored blue, contains neither a monochromatic K_n nor a properly colored P_5, it follows that $PR(K_n, P_5) \geq 2n - 2$.

Next, we show that $PR(K_n, P_5) \leq 2n - 2$. Assume, to the contrary, that there is a red-blue coloring of $G = K_{2n-2}$ avoiding a monochromatic K_n and a properly colored P_5. Let G_R and G_B be the red and blue subgraphs, respectively, of G. We consider two cases.

Case 1. *$\Delta(G_R) = 2n - 3$ or $\Delta(G_B) = 2n - 3$, say the former.* Let v be a vertex of degree $2n-3$ in G_R. For each $(n-1)$-subset S of $V(G) - \{v\}$, the subgraph $G[S]$ contains a blue edge; for otherwise, $G[S \cup \{v\}]$ is a red K_n. Hence, G_B contains $\ell \geq \lfloor \frac{n}{2} \rfloor$ independent edges. Suppose that $x_i y_i$ $(1 \leq i \leq \ell)$ are independent edges in G_B. Since there is no properly colored P_5 in G, it follows $x_i y_j$ is blue for all pairs i, j with $1 \leq i \neq j \leq \ell$. Thus, the subgraph induced by $W = \{x_i, y_i : 1 \leq i \leq \ell\}$ is a blue clique of order 2ℓ. If $2\ell \geq n$, then $G[W]$ contains a blue K_n, a contradiction. Hence, we may assume that $\ell = \lfloor \frac{n}{2} \rfloor$ and n is odd. Thus, $\ell = (n-1)/2$ and $G[W]$ is a blue K_{n-1}. Let $G_1 = G[W]$ and $G_2 = G[V(G) - (\{v\} \cup W)]$. Thus, G_2 is a red K_{n-2} and $G[V(G) - W]$ is a red K_{n-1}. Since G contains no monochromatic K_n, there are two vertices p and q in G_1 and a vertex s in G_2 such that ps is red and qs is blue. Let $t \in V(G_1) - \{p, q\}$. However then, (t, p, s, q, v) is a properly colored P_5 in G, a contradiction.

Case 2. *$\Delta(G_R) \leq 2n-4$ and $\Delta(G_B) \leq 2n-4$.* We may assume that $\Delta(G_R) \geq \Delta(G_B)$ and so $\Delta(G_R) \geq n - 1$. Let v be a vertex of maximum degree in G_R. Suppose that vx_i is a red edge of G for $1 \leq i \leq \Delta(G_R)$ and vx is a blue edge of G. Let $S = \{x_i : 1 \leq i \leq \Delta(G_R)\}$. Since G contains no red K_n, the subgraph $G[S]$ contains a blue edge, say $x_1 x_2$ is blue. First, suppose that x is joined to a vertex $x_i \in S$ by a red edge. We may assume that $i \neq 1$. If $i = 2$, then (x_1, x_2, x, v, x_3) is a properly colored P_5; while if $i \neq 2$, then (x_1, x_2, v, x, x_i) is a properly colored P_5. In either case, a contradiction is produced. Thus, x is joined to every vertex in $S \cup \{v\}$ by a blue edge. However then, x has degree at least $\Delta(G_R) + 1$ in G_B, contradicting the assumption that $\Delta(G_R) \geq \Delta(G_B)$. ■

In order to determine $PR(K_n, P_6)$ for $n \geq 3$, we first consider the cases when $n = 3, 4, 5$.

Proposition 2.2.5 $PR(K_3, P_6) = PR(K_4, P_6) = 6$.

Proof. Since the red-blue coloring of K_5 resulting in a red C_5 and a blue C_5 produces neither a monochromatic K_3 nor a properly colored P_6, it follows that $PR(K_4, P_6) \geq PR(K_3, P_6) \geq 6$.

Next, we show that $PR(K_4, P_6) \leq 6$. Assume, to the contrary that, there exists a red-blue coloring of $G = K_6$ that avoids a monochromatic K_4 and a properly colored P_6. Let $V(K_6) = \{u, v, w, x, y, z\}$. Since $PR(K_4, P_5) = 6$ by Theorem 2.2.4 and G contains no monochromatic K_4, the graph G contains a properly colored P_5, say $P_5 = (u, v, w, x, y)$. We may assume that uv and wx are red and vw and xy are blue and, furthermore, that uy is blue.

- ⋆ If zu is blue, then (z, u, v, w, x, y) is a properly colored P_6; so zu is red.
- ⋆ If yz is red, then (u, v, w, x, y, z) is a properly colored P_6; so yz is blue.
- ⋆ If xz is blue, then (y, u, v, w, x, z) is a properly colored P_6; so xz is red.
- ⋆ If wy is red, then (x, z, y, w, v, u) is a properly colored P_6; so wy is blue.
- ⋆ Similarly, if vy is red, then (u, z, y, v, w, x) is a properly colored P_6; so vy is blue.
- ⋆ If ux is blue, then (v, w, x, u, z, y) is a properly colored P_6; so ux is red.
- ⋆ If both wz and vz are blue, then $G[\{v, w, y, z\}]$ is a blue K_4; so at least one is red. By symmetry, we may assume that wz is red.
- ⋆ If uw is red, then $G[\{u, w, x, z\}]$ is a red K_4; so uw is blue.
- ⋆ If vx is blue, then (v, x, w, u, z, y) is a properly colored P_6; so vx is red.
- ⋆ Now, if vz is red, then $G[\{u, v, x, z\}]$ is a red K_4; while if vz is blue, then (z, v, u, w, x, y) is a properly colored P_6. Hence, a contradiction is produced in either case.

Therefore, $PR(K_3, P_6) = PR(K_4, P_6) = 6$. ■

Proposition 2.2.6 $PR(K_5, P_6) = 8$.

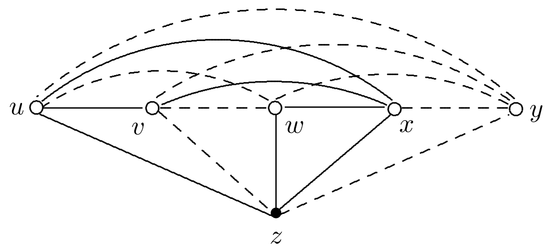


Figure 2.2: The red-blue coloring of K_6 in the proof of Proposition 2.2.5

Proof. Since the red-blue coloring of K_7, in which every edge of some 4-clique is colored red and all other edges are blue, contains neither a monochromatic K_5 nor a properly colored P_6, it follows that $PR(K_5, P_6) \geq 8$. It remains to show that $PR(K_5, P_6) \leq 8$.

Assume, to the contrary, that there exists a red-blue coloring of $G = K_8$ that avoids a monochromatic K_5 and a properly colored P_6. Let $V(K_8) = \{s, t, u, v, w, x, y, z\}$. Since $PR(K_5, P_5) = 8$ by Theorem 2.2.4 and G contains no monochromatic K_5, there is a properly colored P_5, say $P_5 = (s, t, u, v, w)$, where st and uv are red and tu and vw are blue. Furthermore, we may assume that sw is blue.

- ⋆ If sx is blue, then (x, s, t, u, v, w) is a properly colored P_6; so sx is red. Similarly, vx is red. Likewise, the edges sy, vy, sz and vz are red.
- ⋆ If wx is red, then (s, t, u, v, w, x) is a properly colored P_6; so wx is blue. Similarly, wy and wz are blue.
- ⋆ If uw is red, then (v, z, w, u, t, s) is a properly colored P_6; so uw is blue. Similarly, tw is blue.
- ⋆ If sv is blue, then (u, t, s, v, z, w) is a properly colored P_6; so sv is red.
- ⋆ If all of xy, yz, and xz are red, then $G[s, v, x, y, z]$ is a red K_5; so at least one of these three edges is colored blue, say xy is blue.
- ⋆ If all of tx, ty, ux, and uy are blue, then $G[t, u, x, y, w]$ is a blue K_5; so at least one of these four edges is colored red, say tx is red. However then, (u, t, x, y, s, w) is a properly colored P_6, a contradiction.

Therefore, $PR(K_5, P_6) = 8$. ■

Theorem 2.2.7 *For every integer $n \geq 4$, $PR(K_n, P_6) = 2n - 2$.*

Proof. By Propositions 2.2.5 and 2.2.6, we may assume that $n \geq 6$. Since

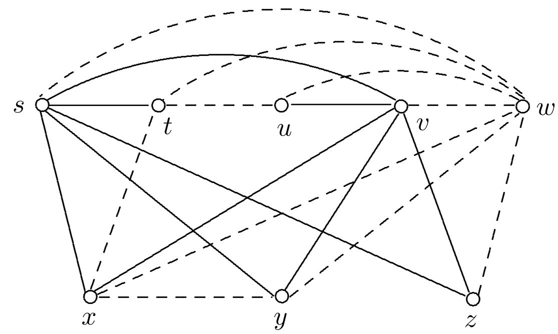


Figure 2.3: The red-blue coloring of K_8 in the proof of Proposition 2.2.6

$$PR(K_n, P_5) = 2n - 2$$

by Theorem 2.2.4, it follows that $PR(K_n, P_6) \geq 2n - 2$. It remains to show that $PR(K_n, P_6) \leq 2n - 2$.

Assume, to the contrary, that there is a red-blue coloring of $G = K_{2n-2}$ avoiding both a monochromatic K_n and a properly colored P_6. By Theorem 2.2.4, there is a properly colored P_5 in G, say $P = (v_1, v_2, v_3, v_4, v_5)$, where v_1v_2 and v_3v_4 are red and v_2v_3 and v_4v_5 are blue. Furthermore, we may assume that v_1v_5 is red. Let $X = V(G) - V(P)$ where then $|X| = 2n - 7$. Necessarily, v_1x is red and v_5x is blue for each $x \in X$; for otherwise, either $(x, v_1, v_2, v_3, v_4, v_5)$ or $(x, v_5, v_4, v_3, v_2, v_1)$ is a properly colored P_6, which is impossible. Likewise, v_2x is blue for each $x \in X$. This is illustrated in Figure 2.4, where a red edge is indicated by a solid line and a blue edge is indicated by a dashed line.

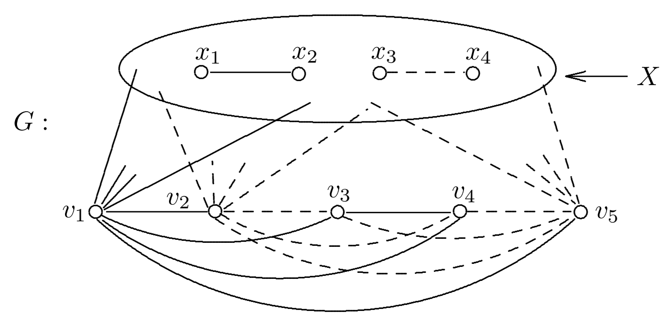


Figure 2.4: Illustrating a step in a red-blue coloring of $G = K_{2n-2}$

Since $n \geq 6$, it follows that $2n-7 \geq n-1$. This implies that $G[X]$ contains a red edge and a blue edge, for otherwise, either $G[X \cup \{v_1\}]$ or $G[X \cup \{v_2\}]$ is a monochromatic K_n. Then $G[X]$ contains nonadjacent edges x_1x_2 and x_3x_4, where x_1x_2 is red and x_3x_4 is blue.

⋆ If v_1v_4 is blue, then $(x_3, x_4, v_1, v_4, v_3, v_2)$ is a properly colored P_6; so v_1v_4 is red.

⋆ If v_2v_5 is red, then $(x_1, x_2, v_5, v_2, v_3, v_4)$ is a properly colored P_6; so v_2v_5 is blue.

⋆ If v_1v_3 is blue, then $(v_5, v_4, v_3, v_1, v_2, x_1)$ is a properly colored P_6; so v_1v_3 is red.

⋆ If v_3v_5 is red, then $(v_1, v_2, v_3, v_5, x_1, x_2)$ is a properly colored P_6; so v_3v_5 is blue.

⋆ If v_2v_4 is red, then $(v_1, v_5, v_4, v_2, x_1, x_2)$ is a properly colored P_6; so v_2v_4 is blue.

Consequently, every edge incident with v_1 is red and, with the exception of the edges v_1v_2 and v_1v_5, every edge incident with v_2 or v_5 is blue. (See Figure 2.4).

We now consider the set $S_2 = V(G) - \{v_1, v_2, v_5\}$ where $|S_2| = 2n - 5 \geq n + 1$. Certainly, if $G[S_2]$ is monochromatic, then G contains a monochromatic K_n, a contradiction. Thus, $G[S_2]$ contains a properly colored P_3, say $P_3 = (y_1, y_2, y_3)$, where y_1y_2 is red and y_2y_3 is blue. Then $(v_1, v_5, y_1, y_2, y_3)$ is a properly colored P_5, so, except for v_1y_3, every edge incident with y_3 is blue (see Figure 2.5). Next, let $S_3 = S_2 - \{y_3\}$, where $|S_3| = 2n - 6 \geq n$. Again, if $G[S_3]$ is monochromatic, then G contains a monochromatic K_n, a contradiction. Hence, $G[S_3]$ contains a properly colored P_3. Applying the argument above, there is a vertex in S_3 that is joined to every vertex in $V(G) - \{v_1\}$ by a blue edge. Deleting this vertex from S_3, we obtain the set S_4.

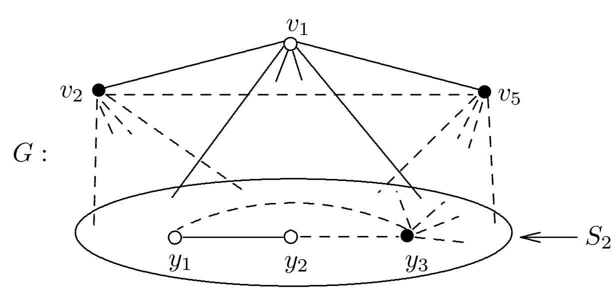


Figure 2.5: Selecting the vertex y_3 in $G = K_{2n-2}$

In general, for each integer k with $2 \leq k \leq n - 2$, let

$$S_k = (V(G) - \{v_1\}) - \{w_1, w_2, \ldots, w_k\}$$

(where $\{w_1, w_2, w_3\} = \{v_2, v_5, y_3\}$). Since $|S_k| = (2n - 3) - k \geq n - 1$ and G contains no monochromatic K_n, it follows that $G[S_k]$ contains a properly colored P_3 by Proposition 2.2.1. Thus, there is a vertex $w_{k+1} \in S_k$ such that w_k is joined to every vertex in $V(G) - \{v_1\}$ by a blue edge. Let

$$S_{k+1} = S_k - \{w_k\}.$$

In particular, $|S_{n-2}| = n-1$. Since G contains no monochromatic K_n, it again follows by Proposition 2.2.1 that $G[S_{n-2}]$ contains a properly colored P_3. Hence, there is $w_{n-1} \in S_{n-2}$ such that w_{n-1} is joined to every vertex in $V(G) - \{v_1\}$ by a blue edge. Let $S_{n-1} = S_{n-2} - \{w_{n-1}\}$ and let $w_n \in S_{n-1}$. However then, the subgraph $G[\{w_1, w_2, \ldots, w_n\}]$ is a blue K_n in G, a contradiction. Therefore, $PR(K_n, P_6) = 2n - 2$. ∎

Since $PR(K_n, P_k) = 2n - 2$ for $k = 5, 6$ by Theorems 2.2.4 and 2.2.7, it follows that $PR(K_n, P_7) \geq 2n-2$. From the results obtained above, we have the following conjecture.

Conjecture 2.2.8 *For every integer $n \geq 4$ and $k = 7, 8$,*

$$PR(K_n, P_k) = 2n - 2.$$

Proposition 2.2.9 *For integers n and k with $n \geq 4$ and $k \geq 9$,*

$$PR(K_n, P_k) \geq 2n - 1.$$

Proof. Partition the vertex set of $G = K_{2n-2}$ into two sets U and W with $|U| = |W| = n - 1$. Then $G[U] = G[W] = K_{n-1}$. Let e be an edge of $G[U]$. The red-blue coloring of K_{2n-2}, in which every edge in $E(G[W]) \cup \{e\}$ is colored red and all other edges are blue, avoids both a monochromatic K_n and a properly colored P_9. (In fact, the largest properly colored path is P_8.) Thus, $PR(K_n, P_k) \geq 2n - 1$ for every integer $k \geq 9$. ∎

Problem 2.2.10 *Is $PR(K_n, P_9) = 7$ for each integer $n \geq 4$?*

Proposition 2.2.9 should be improved for n and k sufficiently large. The following is an example.

Proposition 2.2.11 *For integers n and k with $n \geq 7$ and $k \geq 13$,*

$$PR(K_n, P_k) \geq 2n.$$

Proof. Partition the vertex set of $G = K_{2n-1}$ into two sets U and W with $|U| = n$ and $|W| = n - 1$. Then $G[U] = K_n$ and $G[W] = K_{n-1}$. Let e be an edge of $G[U]$ and let f_1 and f_2 be two nonadjacent edges of $G[W]$. The red-blue coloring of K_{2n-1}, in which every edge in $E(G[U] - e) \cup \{f_1, f_2\}$ is colored red and all other edges are blue, avoids both a monochromatic K_n and a properly colored P_{13}. (In fact, the largest properly colored path is P_{12}.) Thus, $PR(K_n, P_k) \geq 2n$ for every integer $k \geq 13$. ∎

2.3 Complete Graphs Versus Even Cycles

We have seen that $PR(K_n, H) = 2n - 2$ for $n \geq 4$, where $H = P_5$ and $H = P_6$. We now show that this proper Ramsey number has the same value when H is the 2-chromatic graph C_4. In fact, $PR(K_n, C_4) = 2n - 2$ when $n = 3$ as well.

Proposition 2.3.1 $PR(K_3, C_4) = 4$.

Proof. Since a red-blue coloring of K_3 in which not all edges are colored the same avoids both a monochromatic K_3 and a properly colored C_4, it follows that $PR(K_3, C_4) \geq 4$. Next, let there be given a red-blue coloring of $G = K_4$ that contains no monochromatic K_3. We may assume that the size of the red subgraph G_R is at least 3. Thus, G_R either contains $K_{1,3}$ or P_4. If G_R contains $K_{1,3}$, then G has a monochromatic K_3, a contradiction; while if G_R contains $P_4 = (v_1, v_2, v_3, v_4)$, then $(v_1, v_2, v_4, v_3, v_1)$ is a properly colored C_4. Therefore, $PR(K_3, C_4) = 4$. ∎

Theorem 2.3.2 *For each integer $n \geq 3$, $PR(K_n, C_4) = 2n - 2$.*

Proof. We proceed by induction on $n \geq 3$. By Proposition 2.3.1, the statement holds for $n = 3$. Assume that $PR(K_{n-1}, C_4) = 2n - 4$ for some integer $n \geq 4$. We show that $PR(K_n, C_4) = 2n - 2$.

Since the red-blue coloring of K_{2n-3} in which every edge of some $(n-1)$-clique is colored red and all other edges are blue, contains neither a monochromatic K_n nor a properly colored C_4, it follows that $PR(K_n, C_4) \geq 2n - 2$. It remains to show that $PR(K_n, C_4) \leq 2n - 2$. Assume to the contrary, that there is a red-blue coloring of $G = K_{2n-2}$ that avoids a monochromatic K_n and a properly colored C_4. By the induction hypothesis, G contains a monochromatic K_{n-1}. We may assume that G contains a red K_{n-1} with vertex set $X = \{x_1, x_2, \ldots, x_{n-1}\}$. Let

$$Y = V(G) - X = \{y_1, y_2, \ldots, y_{n-1}\}.$$

We claim that $G[Y]$ is a blue K_{n-1}. If this were not the case, then $G[Y]$ contains a red edge, say y_1y_2 is red. Since there is no red K_n, it follows that each vertex in Y is joined to at least one vertex in X by a blue edge. We may assume that x_1y_1 is blue where $x_1 \in X$. If x_iy_2 is blue for some $i \in \{2, 3, \ldots, n-1\}$, then $(x_1, y_1, y_2, x_i, x_1)$ is a properly colored C_4. Thus, x_iy_2 is red for each $i \in \{2, 3, \ldots, n-1\}$. Since there is no red K_n, it follows that x_1y_2 is blue. Furthermore, y_1x_i is red for $2 \leq i \leq n-1$; for otherwise, $(y_1, x_i, x_1, y_2, y_1)$ is a properly colored C_4. So, each edge in $[\{y_1, y_2\}, \{x_2, x_3, \ldots, x_{n-1}\}]$

is red. However then, $G[\{x_2, x_3, \ldots, x_{n-1}, y_1, y_2\}]$ is a red K_n, a contradiction. Thus, as claimed, $G[Y]$ is a blue K_{n-1}.

Next, we claim that the vertices of X can be labeled as $u_1, u_2, \ldots, u_{n-1}$ and the vertices of Y can be labeled as $v_1, v_2, \ldots, v_{n-1}$ in such a way that for each integer k with $1 \le k \le n-1$, the edge $u_i v_j$ $(1 \le i, j \le k)$ is red if and only if $1 \le i \le j$. We verify this statement by induction on k.

Since $G[Y]$ is a blue K_{n-1}, every vertex in X must be joined to some vertex in Y by a red edge. Let $u_1 v_1$ is a red edge where $u_1 \in X$ and $v_1 \in Y$. Hence the statement holds for $k = 1$. Assume for some integer k with $1 \le k < n-1$ that X contains k vertices $u_1, u_2, \ldots, u_k$ and Y contains k vertices $v_1, v_2, \ldots, v_k$ such that $u_i v_j$ is red if $1 \le i \le j \le k$ and $u_i v_j$ is blue if $1 \le j < i \le k$.

We now show that the statement is true for $k+1$. By assumption, v_k is joined to $u_1, u_2, \ldots, u_k$ by red edges. Since v_k cannot be joined to each vertex of X by a red edge, there must be a vertex $u_{k+1} \in X$ such that $u_{k+1} v_k$ is blue. If $u_{k+1} v_i$ were red for some i with $1 \le i < k$, then $(v_i, u_{k+1}, v_k, u_k, v_i)$ would be a properly colored C_4, which is impossible. Thus, $u_{k+1} v_i$ is blue for all i with $1 \le i < k$. However, u_{k+1} must be joined to some vertex of Y by a red edge, say $u_{k+1} v_{k+1}$ is red, where $v_{k+1} \in Y$. If $u_i v_{k+1}$ were blue for some i with $1 \le i \le k$, then $(v_{k+1}, u_i, v_i, u_{k+1}, v_{k+1})$ would be a properly colored C_4, again impossible. Thus, $u_i v_{k+1}$ is red for all i with $1 \le i \le k$ (see Figure 2.6). This verifies the claim. In particular then, v_{n-1} is joined to every vertex of X by a red edge. However then, $G[X \cup \{v_{n-1}\}]$ is a red K_n, a contradiction. Therefore, $PR(K_n, C_4) = 2n - 2$. ∎

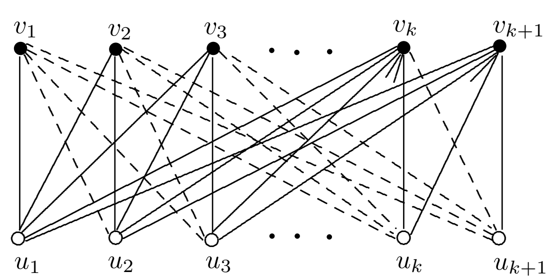


Figure 2.6: A step in the proof of Theorem 2.3.2

Proposition 2.3.3 $PR(K_3, C_6) = 6$.

Proof. Since the red-blue coloring of K_5 in which each of the red and blue subgraphs is C_5 avoids both a monochromatic K_3 and a properly colored C_6, it follows that $PR(K_3, C_6) \ge 6$. Since $PR(K_3, C_6) \le R(K_3, K_3) = 6$, it follows that $PR(K_3, C_k) = 6$. ∎

Proposition 2.3.4 *For every integer $n \geq 4$, $PR(K_n, C_6) \geq 2n-1$.*

Proof. Partition the vertex set of $G = K_{2n-2}$ into two sets U and W with $|U| = |W| = n-1$. Then $G[U] = G[W] = K_{n-1}$. Let e be an edge of $G[U]$. The red-blue coloring of K_{2n-2}, in which every edge in $E(G[W]) \cup \{e\}$ is colored red and all other edges are blue, avoids both a monochromatic K_n and a properly colored C_6. Thus, $PR(K_n, C_6) \geq 2n-1$. ■

Remarks

1. By Proposition 2.3.4, $PR(K_4, C_6) \geq 7$. What is $PR(K_4, C_6)$?

2. In the red-blue coloring of K_{2n-2} in the proof of Proposition 2.3.4, the largest properly colored cycle is a 4-cycle. Hence, if $n \geq 4$ and $k \geq 3$, then

$$PR(K_n, C_{2k}) \geq 2n-1.$$

   Find better bounds.

2.4 Stars Versus Even Cycles

We now turn our attention to proper Ramsey numbers of stars versus even cycles. We begin by determining the value of $PR(K_{1,n}, C_4)$ for each integer $n \geq 3$.

Theorem 2.4.1 *For every integer $n \geq 3$, $PR(K_{1,n}, C_4) = n+1$.*

Proof. Since the order of $K_{1,n}$ is $n+1$, it follows by (2.1) that

$$PR(K_{1,n}, C_4) \geq n+1.$$

It remains to show that $PR(K_{1,n}, C_4) \leq n+1$. We proceed by induction on n. For $n = 3$, let there be given a red-blue coloring of K_4 that avoids a monochromatic $K_{1,3}$. Thus, each vertex of K_4 is incident with at least one red edge and at least one blue edge. So, there is a $2K_2, P_4$ or C_4 in each color, which implies that there is a properly colored C_4. Therefore, $PR(K_{1,3}, C_4) \leq 4$, establishing the base step.

Next, suppose that $PR(K_{1,n-1}, C_4) \leq n$ for some integer $n \geq 4$. We show that $PR(K_{1,n}, C_4) \leq n+1$. Assume, to the contrary, that there is a red-blue coloring of $G = K_{n+1}$ avoiding both a monochromatic $K_{1,n}$ and a properly colored C_4. Let $u \in V(G)$. By the induction hypothesis, $G[V(G) - \{u\}] = K_n$ contains either a monochromatic $K_{1,n-1}$ or a properly colored C_4. Since G has no properly colored C_4, there is a monochromatic $F = K_{1,n-1}$. We may assume that F is a red $K_{1,n-1}$ whose central vertex is v. Because

G has no monochromatic $K_{1,n}$, it follows that uv is blue and u is incident with at least one red edge, say ux. Necessarily, x is incident with at least one blue edge, say xy is blue. However then, (u, v, y, x, u) is a properly colored C_4, which is impossible. Thus, $PR(K_{1,n}, C_4) \leq n + 1$.

Therefore, $PR(K_{1,n}, C_4) = n + 1$ for each $n \geq 3$. ■

Next, we turn our attention to proper Ramsey numbers $PR(K_{1,n}, C_6)$ for $n \geq 3$. The following result was obtained by Burr and Roberts [12] in 1973.

Theorem 2.4.2 [12] *For integers $s, t \geq 2$,*

$$R(K_{1,s}, K_{1,t}) = \begin{cases} s + t - 1 & \text{if } s \text{ and } t \text{ are both even} \\ s + t & \text{otherwise.} \end{cases}$$

Since $R(K_{1,n}, K_{1,n}) = 2n - 1$ for all even integers $n \geq 4$ by Theorem 2.4.2 and $PR(K_{1,n}, C_6) \leq R(K_{1,n}, K_{1,n})$ by (2.1), it follows that $PR(K_{1,n}, C_6) \leq 2n - 1$ for all even integers $n \geq 4$. In fact, $PR(K_{1,n}, C_6) = 2n - 1$ for each integer $n \geq 4$, as we show next. First, we introduce some terminology. Let G be a graph each of whose edges is colored red or blue. For a vertex v of G, the *red neighborhood* $N_R(v)$ is the set of vertices each of which is joined to v by a red edge and the *blue neighborhood* $N_B(v)$ of v is the set of vertices joined to v by blue edges. Because the next result can be readily verified, its proof is omitted. Nevertheless, it is useful so that a more complete result can be presented.

Proposition 2.4.3 *$PR(K_{1,3}, C_6) = 6$, $PR(K_{1,4}, C_6) = 7$ and $PR(K_{1,5}, C_6) = 9$.*

Theorem 2.4.4 *For every integer $n \geq 4$, $PR(K_{1,n}, C_6) = 2n - 1$.*

Proof. By Proposition 2.4.3, we may assume that $n \geq 6$. Since the red-blue coloring of K_{2n-2}, in which the red subgraph is $2K_{n-1}$ and the blue subgraph is $K_{n-1,n-1}$, avoids both a monochromatic $K_{1,n}$ and a properly colored C_6, it follows that $PR(K_{1,n}, C_6) \geq 2n - 1$.

It remains to show that every red-blue coloring of K_{2n-1} produces either a monochromatic $K_{1,n}$ or a properly colored C_6. Assume, to the contrary, that there is a red-blue coloring of $G = K_{2n-1}$ that avoids both a monochromatic $K_{1,n}$ and a properly colored C_6. Necessarily, each vertex is incident with exactly $n-1$ red edges and exactly $n-1$ blue edges. Thus, both the red subgraph G_R and the blue subgraph G_B are $(n - 1)$-regular graphs of order $2n - 1$. We first verify three claims.

Claim 1. There is no monochromatic K_n.

Proof of Claim 1. Assume, to the contrary, that G contains a monochromatic $F = K_n$. We may assume that F is a red K_n. Let $x \in V(G) - V(F)$. Since $|V(G) - V(F)| = n - 1$ and x is incident with exactly $n - 1$ red edges, it follows that x is joined to at least one vertex y in F by a red edge. However then, y is incident with at least n red edges, producing a red $K_{1,n}$. This is impossible; so Claim 1 holds.

Claim 2. There is no monochromatic K_{n-1}.

Proof of Claim 2. Assume, to the contrary, that G contains a monochromatic $F = K_{n-1}$. We may assume that F is a red K_{n-1}. Let $X = V(F)$ and let $Y = V(G) - X$; so $|X| = n - 1$ and $|Y| = n$. Since each $x \in X$ is incident with exactly $n - 1$ red edges, it follows that each x is joined to exactly one vertex in Y by a red edge; so $[X, Y]$ contains exactly $n - 1$ red edges. This implies that at least one of the n vertices in Y, say y, is incident with exactly $n - 1$ blue edges in $[X, Y]$. Thus, y is joined to each vertex in Y by a red edge (see Figure 2.7). Consider the subgraph $H = G[Y - \{y\}]$ of order $n - 1$ in G. Either H is a monochromatic K_{n-1} or H contains a properly colored P_3.

- ⋆ If H is a red K_{n-1}, then $G[Y]$ is a red K_n, which is impossible by Claim 1.
- ⋆ If H is a blue K_{n-1}, then each vertex in H is adjacent to exactly $n - 2$ vertices in X by red edges. This implies that $[X, Y]$ contains $(n - 1)(n - 2)$ red edges. However then, $(n - 1)(n - 2) = n - 1$; so $n = 3$, which is impossible since $n \geq 6$.
- ⋆ If H contains a properly colored $P_3 = (u, v, w)$, where say uv is red and vw is blue, then (u, v, w, y) is a properly colored P_4 (see Figure 2.7). First, suppose that u is joined to a vertex $x \in X$ by a blue edge. Let $x' \in X - \{x\}$. Then (x', x, u, v, w, y, x') is a properly colored C_6, which is impossible. Hence, u is joined to all vertices in X by red edges. However then, $G[X \cup \{u\}]$ is a red K_n, which is impossible by Claim 1.

Therefore, Claim 2 holds.

Claim 3. There is a monochromatic K_{n-2}.

Proof of Claim 3. Since $PR(K_n, P_5) = 2n - 2$ by Theorem 2.2.4, it follows that G contains either a monochromatic K_n or a properly colored P_5. By Claim 1, the graph G contains a properly colored $P_5 = (u_1, u_2, u_3, u_4, u_5)$. We may assume that u_1u_2 and u_3u_4 are red and u_2u_3 and u_4u_5 are blue and, furthermore, u_1u_5 is red (see Figure 2.8).

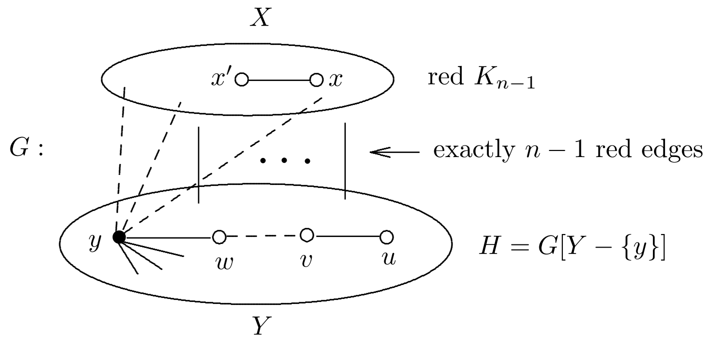


Figure 2.7: A step in the proof of Claim 2

Let $S = \{v_1, v_2, \ldots, v_{2n-6}\} = V(G) - V(P_5)$. Since (i) u_1 is incident with exactly $n-1$ blue edges and (ii) u_1u_2 and u_1u_5 are red, it follows that u_1 is adjacent to at least $n-3$ vertices in S by blue edges. Hence, $|N_B(u_1) \cap S| \geq n-3$. If u_5 is joined to some vertex $v \in N_B(u_1) \cap S$ by a red edge, then $(u_5, v, u_1, u_2, u_3, u_4, u_5)$ is a properly colored C_6, which is impossible. Hence, u_5 is joined to all vertices in $N_B(u_1) \cap S$ by a blue edge. Hence, $N_B(u_1) \cap S \subseteq N_B(u_5) \cap S$ and so $|N_B(u_5) \cap S| \geq n-3$ (see Figure 2.8). Likewise, since (i) u_5 is incident with exactly $n-1$ red edges and (ii) u_1u_5 is red, it follows that u_5 is joined to at least $n-4$ vertices in S by red edges. That is, $|N_R(u_5) \cap S| \geq n-4 \geq 2$. Furthermore, since $N_B(u_1) \cap S \subseteq N_B(u_5) \cap S$, it follows that $N_B(u_1) \cap S$ and $N_R(u_5) \cap S$ are disjoint. If u_1 is joined to some vertex $w \in N_R(u_5) \cap S$ by a blue edge, then $(u_1, w, u_5, u_4, u_3, u_2, u_1)$ is a properly colored C_6, which is impossible. Thus, u_1 is joined to all vertices in $N_R(u_5) \cap S$ by red edges (see Figure 2.8).

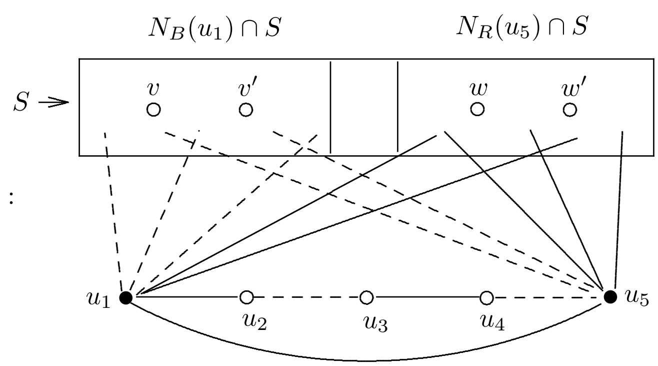


Figure 2.8: A step in the proof of Claim 3

First, suppose that there is a red edge vv' in $G[N_B(u_1) \cap S]$. If there is also a blue edge in $G[N_R(u_5) \cap S]$, say ww', then $(v, v', u_1, w, w', u_5, v)$ is a properly colored C_6, which is impossible. Hence, $G[N_R(u_5) \cap S]$ is a red clique of order at least $n-4$. Thus, $G_R[N_R(u_5) \cup \{u_5\}]$ contains a red K_{n-2}. Next, suppose that each edge in $G[N_B(u_1) \cap S]$

is blue. Then $G[N_B(u_1) \cap S]$ is a blue clique of order at least $n-3$. Thus, $G[(N_B(u_1) \cap S) \cup \{u_1\}]$ contains a blue K_{n-2}.

Therefore, there is a monochromatic K_{n-2} and so Claim 3 holds.

By Claim 3, the graph $G = K_{2n-1}$ contains a monochromatic K_{n-2}. Assume, without loss of generality, that G contains a red K_{n-2} with vertex set $X = \{u_1, u_2, \dots, u_{n-2}\}$. Let $Y = V(G) - X$, where then $|Y| = n+1$. Since $PR(K_{1,n}, C_4) = n+1$ by Theorem 2.4.1 and G contains no monochromatic $K_{1,n}$, it follows that $G[Y]$ contains a properly colored $C_4 = (v_1, v_2, v_3, v_4, v_1)$, where say v_1v_2 and v_3v_4 are blue and v_2v_3 and v_1v_4 are red. Consider the vertex u_1. Since u_1 is incident with exactly $n-1$ blue edges, u_1 is joined to $n-1$ vertices in Y by blue edges. Thus, u_1 is joined to at least two vertices of C_4 by blue edges. We may assume, without loss of generality, that u_1v_1 is blue.

- $\star$ If there is $x \in X - \{u_1\}$ such that v_2x is blue, then $(v_1,\ u_1,\ x,\ v_2,\ v_3,\ v_4,\ v_1)$ is a properly colored C_6, which is impossible. Thus, v_2x is red for all $x \in X - \{u_1\}$. Since there is no red K_{n-1} by Claim 2, it follows that v_2u_1 is blue.

- $\star$ If there is $x \in X - \{u_1\}$ such that v_1x is blue, then $(v_1,\ v_4,\ v_3,\ v_2,\ u_1,\ x,\ v_1)$ is a properly colored C_6, which is impossible. Thus, v_1x is red for all $x \in X - \{u_1\}$.

In particular, v_1u_2, v_1u_3, v_2u_2 and v_2u_3 are red (see Figure 2.9).

G :

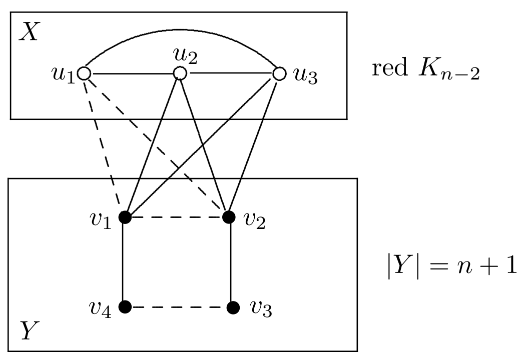


Figure 2.9: A step in the proof of Theorem 2.4.4

Since v_1u_2 and v_2u_2 are red, it follows that u_2 is joined to each of the $n-1$ vertices in $Y - \{v_1, v_2\}$ by a blue edge. In particular, u_2v_3 and u_2v_4 are blue. Likewise, u_3v_3 and u_3v_4 are blue. However then, $(u_2, u_3, v_3, v_2, v_1, v_4, u_2)$ is a properly colored C_6, which is impossible. Therefore, $PR(K_{1,n}, C_6) \le 2n-1$ and so $PR(K_{1,n}, C_6) = 2n-1$. ■

Problem 2.4.5 *For integers $k, n \ge 3$, establish bounds for $PR(K_{1,n}, C_{2k})$.*

2.5 Stars Versus Paths

We begin with a preliminary result concerning stars and the path P_3.

Proposition 2.5.1 *For each integer $n \geq 3$, $PR(K_{1,n}, P_3) = n + 1$.*

Proof. Since the coloring of K_n in which each edge is colored red avoids both a monochromatic $K_{1,n}$ and a properly colored P_3, it follows that $PR(K_{1,n}, P_3) \geq n+1$. For any red-blue coloring of K_{n+1}, if all edges are colored same, then there is a monochromatic $K_{1,n}$; otherwise, there are adjacent edges that are colored differently, producing a properly colored P_3. Therefore, $PR(K_{1,n}, P_3) = n + 1$. ■

Next, we show that $PR(K_{1,n}, P_k) = n + 1$ when $n \geq k - 1 \geq 3$ for $k \leq 6$.

Proposition 2.5.2 *For each integer $n \geq 3$, $PR(K_{1,n}, P_4) = n + 1$.*

Proof. Since the coloring of K_n in which each edge is colored red avoids both a monochromatic $K_{1,n}$ and a properly colored P_3 (and so a properly colored P_4), it follows that $PR(K_{1,n}, P_4) \geq n + 1$. To show that $PR(K_{1,n}, P_4) \leq n + 1$, let there be given a red-blue coloring of $G = K_{n+1}$ that avoids a monochromatic $K_{1,n}$. Then every vertex of G is incident with at least one edge of each color and there is a properly colored P_3 in G. Suppose that $P_3 = (u_1, u_2, u_3)$, where u_1u_2 is red and u_2u_3 is blue. We may assume that u_1u_3 is red. Since u_1 is incident with at least one blue edge, there is $x \in V(G) - \{u_1, u_2, u_3\}$ such that u_1x is blue. Then (x, u_1, u_2, u_3) a properly colored P_4. Therefore, $PR(K_{1,n}, P_4) = n + 1$. ■

Proposition 2.5.3 *For each integer $n \geq 4$, $PR(K_{1,n}, P_5) = n + 1$.*

Proof. By Proposition 2.5.2, $PR(K_{1,n}, P_5) \geq n + 1$. It remains to show that

$$PR(K_{1,n}, P_5) \leq n + 1.$$

Let there be a red-blue coloring of $G = K_{n+1}$ that avoids a monochromatic $K_{1,n}$. Then every vertex of G is incident with at least one edge of each color. Furthermore, by Proposition 2.5.2, there is a properly colored $P_4 = (u_1, u_2, u_3, u_4)$. We may assume that u_1u_2 and u_3u_4 are red and u_2u_3 is blue. Let $X = V(K_{n+1}) - V(P_4)$, where then $|X| = n + 1 - 4 = n - 3 \geq 1$. If u_1 or u_4 is joined to a vertex in X by a blue edge, then there is a properly colored P_5. Thus, we may assume that each edge in $[\{u_1, u_4\}, X]$ is red. Since each of u_1 and u_4 is incident with at least one blue edge, it follows that either u_1u_4 is blue or both u_1u_3 and u_2u_4 are blue. If u_1u_4 is blue, then for each $x \in X$, the path (x, u_1, u_4, u_3, u_2) is a properly colored P_5; while if u_1u_3 and u_2u_4 are blue, then, for each $x \in X$, the path (x, u_1, u_3, u_4, u_2) is a properly colored P_5. Therefore, $PR(K_{1,n}, P_5) = n + 1$. ■

In fact, for $k \in \{6, 7, 8\}$, $PR(K_{1,n}, P_k) = n + k - 5$ when $n \geq k - 1$. We verify this next.

Proposition 2.5.4 *For each integer $n \geq 5$, $PR(K_{1,n}, P_6) = n + 1$.*

Proof. By Proposition 2.5.3, $PR(K_{1,n}, P_6) \geq n + 1$. It remains to show that

$$PR(K_{1,n}, P_6) \leq n + 1.$$

Let there be given a red-blue coloring of $G = K_{n+1}$ that avoids a monochromatic $K_{1,n}$. Then every vertex of G is incident with at least one edge of each color. Furthermore, by Proposition 2.5.3, there is a properly colored $P_5 = (u_1, u_2, u_3, u_4, u_5)$. We may assume that u_1u_2 and u_3u_4 are red, u_2u_3 and u_4u_5 blue and furthermore u_1u_5 is red. Let $X = V(G) - V(P_5)$, where then $|X| = n + 1 - 5 = n - 4 \geq 1$. If

(i) u_1 is joined to a vertex in X by a blue edge or

(ii) one of u_2 and u_5 is joined to a vertex in X by a red edge, then there is a properly colored P_6.

Thus, we may assume that each edge in $[\{u_1\}, X]$ is red and each edge in $[\{u_2, u_5\}, X]$ is blue. Since u_1 is incident with at least one blue edge, it follows that either u_1u_3 or u_1u_4 is blue, say u_1u_3. Now let $x \in X$. Then $(u_2, x, u_1, u_3, u_4, u_5)$ is a properly colored P_6. Therefore, $PR(K_{1,n}, P_6) = n + 1$. ■

Proposition 2.5.5 *For each integer $n \geq 6$, $PR(K_{1,n}, P_7) = n + 2$.*

Proof. Since the red-blue coloring of K_{n+1}, in which the red subgraph is $K_{n-1} + K_2$ and the blue subgraph $K_{2,n-1}$, avoids both a monochromatic $K_{1,n}$ and a properly colored P_7, it follows that $PR(K_{1,n}, P_7) \geq n + 2$.

Next, we show that $PR(K_{1,n}, P_7) \leq n + 2$. Assume, to the contrary, that there exists a red-blue coloring of $G = K_{n+2}$ that avoids both a monochromatic $K_{1,n}$ and a properly colored P_7. Thus,

$$\text{each vertex of } G \text{ is incident with at least two red and two blue edges.} \tag{2.2}$$

By Proposition 2.5.4 , there is a properly colored $P_6 = (u_1, u_2, u_3, u_4, u_5, u_6)$. We may assume that u_1u_2, u_3u_4 and u_5u_6 are red and u_2u_3 and u_4u_5 are blue. Let $X = V(G) -$

$V(P_6)$, where then $|X| = n+2-6 = n-4 \geq 2$. Since there is no properly colored P_7, each edge in $[\{u_1, u_6\}, X]$ is red. Furthermore, if u_1u_6 is blue, then for $x \in X$, the path $(x, u_1, u_6, u_5, u_4, u_3, u_2)$ is a properly colored P_7, a contradiction. Thus u_1u_6 is red. By (2.2), u_1 is joined to at least two vertices in $\{u_3, u_4, u_5\}$ by blue edges and u_6 is joined to at least two vertices in $\{u_2, u_3, u_4\}$ by blue edges. Hence, at least one of u_1u_3 and u_1u_4 is blue. If $G[X]$ contains a blue edge, say x_1x_2 is blue, then either $(u_6, x_2, x_1, u_1, u_3, u_4, u_5)$ or $(u_6, x_2, x_1, u_1, u_4, u_3, u_2)$ is a properly colored P_7. Hence, $G[X]$ is a red K_{n-4}.

First, suppose that at least one of u_1u_3 and u_4u_6 is blue, say u_1u_3.

- ⋆ If u_6u_2 is blue, then, for $x \in X$, $(x, u_1, u_3, u_4, u_5, u_6, u_2)$ is a properly colored P_7; so u_6u_2 is red. By (2.2), both u_6u_3 and u_6u_4 are blue.
- ⋆ If u_1u_5 is blue, then, for $x \in X$, $(x, u_6, u_4, u_3, u_2, u_1, u_5)$ is a properly colored P_7; so u_1u_5 is red. By (2.2), u_1u_4 is blue.
- ⋆ If there exists $x \in X$ such that xu_2 or xu_5 is blue, say xu_2, then $(x, u_2, u_6, u_4, u_3, u_1, u_5)$ is a properly colored P_7; so each edge in $[\{u_2, u_5\}, X]$ is red. By (2.2) then, each edge in $[\{u_3, u_4\}, X]$ is blue. Again, by (2.2), both u_3u_5 and u_4u_2 are red and so u_2u_5 is blue. However then, $(x, u_2, u_5, u_1, u_3, u_4, u_6)$ is a properly colored P_7.

Next, both u_1u_3 and u_4u_6 are red. It follows by (2.2) that each of $u_1u_4, u_1u_5, u_6u_3, u_6u_2$ is blue. If there exists $x \in X$ such that xu_2 or x_1u_5 is blue, say xu_2, then $(x, u_2, u_1, u_4, u_3, u_6, u_5)$ is a properly colored P_7. Hence, each edge in $[\{u_2, u_5\}, X]$ is red. By (2.2) then, each edge in $[\{u_3, u_4\}, X]$ is blue. Now let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then $(x_2, x_1, u_3, u_1, u_4, u_6, u_2)$ is a properly colored P_7, a contradiction. Therefore, $PR(K_{1,n}, P_7) = n + 2$. ■

Proposition 2.5.6 *For each integer $n \geq 7$, $PR(K_{1,n}, P_8) = n + 3$.*

Proof. Since the red-blue coloring of K_{n+2}, in which the red subgraph is $K_{n-1} + K_3$ and the blue subgraph $K_{3,n-1}$, avoids both a monochromatic $K_{1,n}$ and a properly colored P_8, it follows that $PR(K_{1,n}, P_8) \geq n + 3$.

Next, we show that $PR(K_{1,n}, P_8) \leq n + 3$. Assume, to the contrary, that there exists a red-blue coloring of $G = K_{n+3}$ that avoids both a monochromatic $K_{1,n}$ and a properly colored P_8. Thus, each vertex of G is incident with at least three red edges and three blue edges. Furthermore, by Proposition 2.5.5 , there is a properly colored $P_7 = (u_1, u_2, u_3, u_4, u_5, u_6, u_7)$. We may assume that u_iu_{i+1} is red for $i = 1, 3, 5$ and u_iu_{i+1} is blue for $i = 2, 4, 6$; furthermore, u_1u_7 is red. Let $X = V(G) - V(P_7)$, where then $|X| = n + 3 - 7 = n - 4 \geq 3$. Since there is no properly colored P_8, each edge in

$[\{u_1\}, X]$ is red and each edge in $[\{u_2, u_7\}, X]$ is blue. Since u_1 is incident with at least three blue edges, it follows that u_1 is joined to at least three vertices in $\{u_3, u_4, u_5, u_6\}$ by blue edges. Hence, u_1 is joined to u_3 or u_6 by a blue edge. Let $x \in X$. If u_1u_3 is blue, then $(u_2, x, u_1, u_3, u_4, u_5, u_6, u_7)$ is a properly colored P_8; while if u_1u_6 is blue, then $(u_7, x, u_1, u_6, u_5, u_4, u_3, u_2)$ is a properly colored P_8. In each case, a contradiction is produced. ∎

The results obtained in this section suggest the following conjecture.

Conjecture 2.5.7 *For integers m and n with $m \geq 4$ and $n \geq \lceil \frac{m}{2} \rceil + 1$,*

$$PR(K_{1,n}, P_m) = n + \left\lfloor \frac{m-3}{4} \right\rfloor + \left\lceil \frac{m-3}{4} \right\rceil.$$

2.6 Problems and Comments

In this section, we first present some problems on proper Ramsey numbers.

1. Stars Versus Stars

   Problem 2.6.1 *For integers s and t, study $PR(K_{1,s}, K_{1,t})$.*

2. Paths Versus Paths

   The following result is known [33]

   Theorem 2.6.2 *For integers n, m with $2 \leq m \leq n$,*

   $$R(P_n, P_m) = n - 1 + \left\lfloor \frac{m}{2} \right\rfloor.$$

   Proposition 2.6.3 *For integers n and m where $5 \leq m \leq n$,*

   $$PR(P_n, P_m) = n - 1 + \left\lfloor \frac{n}{2} \right\rfloor.$$

   Proof. Let $N = n - 1 + \lfloor \frac{n}{2} \rfloor$. First, the red-blue coloring of K_{N-1}, in which every edge of a $(n-1)$-clique is colored red and all other edges are blue, avoids a monochromatic P_n and a properly colored P_5. Hence, $PR(P_n, P_m) \geq N$. Since $PR(P_n, P_m) \leq R(P_n, P_n) = N$ by Theorem 2.6.2, it follows that $PR(P_n, P_m) = N$. ∎

   Proposition 2.6.4 $PR(P_5, P_6) = 6$.

Proof. First, consider the red-blue coloring of K_5 in which every edge incident with a vertex v is red and all other edges are blue. Since this coloring avoids both a monochromatic P_5 and a properly colored P_6, it follows that $PR(P_5, P_6) \geq 6$. We now show that $PR(P_5, P_6) \leq 6$.

Assume, to the contrary, that there is a red-blue coloring of $G = K_6$ that avoids both a monochromatic P_5 and a properly colored P_6. Since $PR(K_5, P_4) = 6$ by Theorem 2.2.2, it follows that G contains a properly colored P_4, say (v_1, v_2, v_3, v_4) where v_1v_2 and v_3v_4 red and v_2v_3 blue. Let x and y be the remaining two vertices of G. If xv_1 and xv_4 are both red, then (v_2, v_1, x, v_4, v_3) is a red P_5, which is impossible. Thus, at least one of xv_1 and xv_4 is blue, say xv_1 is blue. Hence, (x, v_1, v_2, v_3, v_4) is a properly colored P_5. We may assume, without loss of generality, that xv_4 is red. If xy is red or v_4y is blue, then G contains a properly colored P_6. Thus, xy is blue and v_4y is red.

- ⋆ If v_1y is red, then (v_2, v_1, y, v_4, v_3) is a red P_5; so v_1y is blue.
- ⋆ If v_2y is blue, then (v_1, x, y, v_2, v_3) is a blue P_5; so v_2y is red.

However then, (v_1, v_2, y, v_4, v_3) is a red P_5, a contradiction. Thus, $PR(P_5, P_6) \leq 6$ and so $PR(P_5, P_6) = 6$. ■

Problem 2.6.5 *For integers n and m where $m > n$, study $PR(P_n, P_m)$.*

3. Paths Versus Bipartite Graphs

Problem 2.6.6 *For integers n and m where $4 \leq m \leq n$ and m is even, study $PR(P_n, C_m)$.*

Problem 2.6.7 *Let n and m be integers with $4 \leq m \leq n$ and let G be a connected bipartite graph of order m. Study $PR(P_n, G)$.*

There is a general setting for Ramsey numbers. Let S=$\{G_1, G_2, G_3, \ldots\}$ be an infinite set of graphs with the property that G_i is a proper induced subgraph of G_{i+1} for $i = 1, 2, 3, \ldots$. Let F and H be two graphs with the property that $F \subseteq G_k$ and $H \subseteq G_k$ for some $k \in \mathbb{N}$. Therefore, $F \subseteq G_n$ and $H \subseteq G_n$ for every $n \geq k$.

- ⋆ If $G_i = K_i$ for each $i \in \mathbb{N}$, then for every two graphs F and H, there exist positive integers n such that for every red-blue coloring of G_n, there is either a red F in G_n or a blue H in G_n. Of course, the smallest such positive integer n with this property is the Ramsey number $R(F, H)$.

- $\star$ If $G_i = K_{i,i}$ for each $i \in \mathbb{N}$, then for every two *bipartite* graphs F and H, there exist positive integers r such that for every red-blue coloring of G_r, there is either a red F in G_r or a blue H in G_r. The smallest such positive integer r with this property is the bipartite Ramsey number $BR(F, H)$.

- $\star$ If $G_2 = K_{1,1}, G_3 = K_{1,2}, G_4 = K_{2,2}, G_5 = K_{2,3}, G_6 = K_{3,3}, \ldots$, that is, if $G_i = K_{\lfloor \frac{i}{2} \rfloor, \lceil \frac{i}{2} \rceil}$ for each integer $i \geq 2$, then for every two *bipartite* graphs F and H, there exist positive integers n such that for every red-blue coloring of G_n, there is either a red F in G_n or a blue H in G_n. The smallest such positive integer n with this property is the 2-Ramsey number $R_2(F, H)$. In a similar way, the k-Ramsey number $R_k(F, H)$ of two *bipartite* graphs F and H can be defined for every integer $k \geq 2$. For example, if $k = 3$, then let $G_3 = K_{1,1,1}, G_4 = K_{1,1,2}, G_5 = K_{1,2,2}, G_6 = K_{2,2,2}, G_7 = K_{2,2,3}, \ldots$ and so on.

This suggests looking at other collections S of graphs G_i and pairs F, H of graphs that are subgraphs of $G_i \in S$ for some $i \in \mathbb{N}$ and study the *S-Ramsey number $R_S(F, H)$ of F and H* defined as the smallest positive integer n such that for every red-blue coloring of G_n, there is either a red F in G_n or a blue H in G_n. Furthermore, there are also corresponding concepts of monochromatic S-Ramsey number, rainbow S-Ramsey number and proper S-Ramsey number of graphs.

Chapter 3

On s-Bipartite Ramsey Numbers

3.1 Introduction

First, let's review some concepts that we discussed in Chapter 1. Recall, for two bipartite graphs F and H that the *bipartite Ramsey number* $BR(F, H)$ of F and H is the smallest positive integer r such that every red-blue coloring of the r-regular complete bipartite graph $K_{r,r}$ results in either a red F or a blue H. These concepts were introduced and studied by Beineke and Schwenk [5]. If $BR(F, H) = r$ for bipartite graphs F and H, then every red-blue coloring of $K_{r,r}$ results in a red F or a blue H, while there exists a red-blue coloring of $K_{r-1,r-1}$ for which there is neither a red F nor a blue H. Red-blue colorings of the intermediate graph $K_{r-1,r}$ were considered in [2], which led to the concept of the 2-Ramsey number of two bipartite graphs.

For bipartite graphs F and H, the 2-*Ramsey number* $R_2(F, H)$ of F and H is the smallest positive integer n such that every red-blue coloring of the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ of order n results in a red F or a blue H. In [6, 7], red-blue colorings of complete bipartite graphs were considered where the numbers of vertices in the two partite sets need not differ by at most 1. For two bipartite graphs F and H and a positive integer s, the s-*bipartite Ramsey number* $BR_s(F, H)$ of F and H is the smallest integer t with $t \geq s$ such that every red-blue coloring of $K_{s,t}$ results in a red F or a blue H. It was observed in [6] that this concept has a connection with another recreational problem:

> *There are eight girls at a party. What is the minimum number of boys who must be invited to the party to guarantee that there exists a group of six people, three girls and three boys, such that either* (1) *every one of the three girls is acquainted with every one of the three boys or* (2) *every one of the three girls is a stranger of every one of the three boys?*

In [6, 7], $BR_s(F, H)$ was studied for bipartite graphs $F = K_{2,2}$ and $H \in \{K_{2,2}, K_{2,3}, K_{3,3}\}$

and for $F = H = K_{3,3}$. If $F = H$, then $BR_s(F, H)$ is often denoted by $BR_s(F)$. For $F = H \in \{K_{2,2}, K_{3,3}\}$, the following results were obtained in [6].

Theorem 3.1.1 *For each integer $s \geq 2$,*

$$BR_s(K_{2,2}) = \begin{cases} \textit{does not exist} & \textit{if } s = 2 \\ 7 & \textit{if } s = 3, 4 \\ s & \textit{if } s \geq 5. \end{cases}$$

Theorem 3.1.2 *For each integer $s \geq 2$,*

$$BR_s(K_{3,3}) = \begin{cases} \textit{does not exist} & \textit{if } s = 2, 3, 4 \\ 41 & \textit{if } s = 5, 6 \\ 29 & \textit{if } s = 7, 8. \end{cases}$$

The answer to the aforementioned party problem is therefore the s-bipartite Ramsey number $BR_8(K_{3,3}) = 29$. That is, if there are eight girls at a party, then we must invite at least 29 boys to the party to be certain that there are three girls and three boys where all three girls are acquaintances of all three boys or all three girls are strangers of all three boys. For the case $F = K_{2,2}$ and $H = K_{3,3}$, the s-bipartite Ramsey number $BR(F, H)$ was obtained in [7].

Theorem 3.1.3 *For each integer $s \geq 2$,*

$$BR_s(K_{2,2}, K_{3,3}) = \begin{cases} \textit{does not exist} & \textit{if } s = 2, 3 \\ 15 & \textit{if } s = 4 \\ 12 & \textit{if } s = 5, 6 \\ 9 & \textit{if } s = 7, 8 \\ s & \textit{if } s \geq 9. \end{cases}$$

Another recreational problem whose solution is the number $BR_s(K_{2,2}, K_{3,3})$ can now be stated.

> *For a gathering of people, exactly six of whom are women, what is smallest number of men who must also be present at the gathering so that either* (1) *there are four people among them, two women and two men, where each woman is an acquaintance of each man, or* (2) *there are six people among them, three women and three men, where each woman is a stranger of each man.*

Since $BR_6(K_{2,2}, K_{3,3}) = 12$, the minimum number of men to be present is 12 (see [7]).

Although the numbers $BR_s(K_{3,3})$ have been determined for $1 \leq s \leq 8$, these numbers are not known for $9 \leq s < BR(K_{3,3})$. It was conjectured in [6], however, that $BR_9(K_{3,3}) = BR_{10}(K_{3,3})$ and shown that $17 \leq BR_{10}(K_{3,3}) \leq 23$.

In [6], a general formula was established for the s-bipartite Ramsey numbers $BR_s(K_{r,r})$ when $s \in \{2r-1, 2r\}$ and all integers $r \geq 2$.

Theorem 3.1.4 *For each integer $r \geq 2$,*

$$BR_{2r-1}(K_{r,r}) = BR_{2r}(K_{r,r}) = (2r-2)\binom{2r-1}{r} + 1.$$

As was noted in [6, 7], it is convenient to make use of the so-called *Zarankiewicz number* $Z_{s,t}(m,n)$, defined in [38] as the maximum size of a subgraph of $K_{m,n}$ not containing $K_{s,t}$. This number was named after the Polish mathematician Kazimierz Zarankiewicz, who proposed the *Zarankiewicz Problem* in 1951 of determining these numbers [9, 54]. In particular, $Z_{2,2}(3,7) = 10$ (see [38]). More generally, Čůlík obtained the following result (see [22]).

Theorem 3.1.5 *For integers s, t, m, n with $1 \leq s \leq m$ and $n > (t-1)\binom{m}{s}$,*

$$Z_{s,t}(m,n) = (s-1)n + (t-1)\binom{m}{s}.$$

We will see that the techniques used to determine s-bipartite Ramsey numbers have a connection with the concept of Steiner triple systems. For this reason, we briefly discuss this topic here. A *Steiner triple system* of order n is a set S with n elements and a collection T of 3-element subsets of S, called *triples*, such that every two distinct elements of S belong to a unique triple in T. A primary question here is that of determining those integers n for which a Steiner triple system of order n exists. An immediate observation is that there exists a Steiner triple system of order n if and only if K_n is K_3-decomposable. While it is not difficult to see that if there is a Steiner triple system of order n, then $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$, Kirkman [44] verified the converse in 1846, resulting in the following result.

Theorem 3.1.6 *A Steiner triple system of order $n \geq 3$ exists if and only if $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.*

For example, there is a Steiner triple system of order 7. In this case where $S = \{1, 2, \ldots, 7\}$, one Steiner triple system of order 7 has the following set of triples:

$$T = \{\{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 6\}, \{2, 4, 5\}, \{2, 6, 7\}, \{3, 4, 6\}, \{3, 5, 7\}\}. \tag{3.1}$$

Consequently, every pair of elements of S belongs to exactly one element of T. That is, no two triples of T have two elements of S in common. However, every two triples of T have exactly one element of S in common. To see that this is the case for every Steiner triple system of order 7, suppose that there is a Steiner triple system $S = \{a, b, c, d, e, f, g\}$ of order 7 with a set T of triples containing two disjoint triples, say $\{a, b, c\}$ and $\{d, e, f\}$. That is, there is a K_3-decomposition of K_7 with vertex set S, containing disjoint triangles with vertex sets $\{a, b, c\}$ and $\{d, e, f\}$. The vertex g belongs to three triples, where each triple contains one vertex (element) of $\{a, b, c\}$ and one vertex of $\{d, e, f\}$. We may assume that these three triples are $\{g, a, d\}$, $\{g, b, e\}$ and $\{g, c, f\}$ (see Figure 3.1). The element a belongs to one other triple, namely $\{a, e, f\}$. However, e and f already belong to the triple $\{d, e, f\}$, which is impossible.

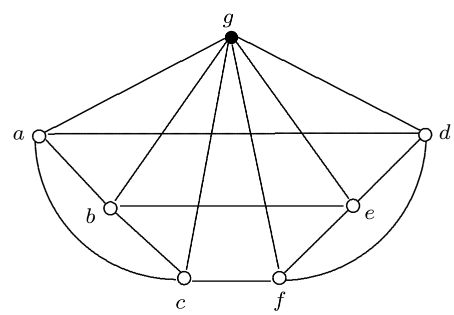


Figure 3.1: Showing that no K_3-decomposition of K_7 contains disjoint triangles

Let's return to the Steiner triple system $S = \{1, 2, \ldots, 7\}$ of order 7 and the set T of triples shown in (3.1). Let G be the bipartite graph with partite sets U and W where $U = \{u_1, u_2, \ldots, u_7\}$, $W = \{w_1, w_2, \ldots, w_7\}$ and u_iw_j is an edge of G if and only if the ith triple in T contains the element j. Hence, G is 3-regular as shown in Figure 3.2. Notice for each integer j with $1 \leq j \leq 7$ that in the graph G of Figure 3.2, the vertex w_j is adjacent to the elements in the jth triple of $\{u_1, u_2, \ldots, u_7\}$ below, resulting in the set T' of triples of subscripts as follows:

$$T' = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{3, 4, 7\}, \{3, 5, 6\}, \{2, 5, 7\}\}.$$

Observe that this set T' of triples is also a Steiner triple system of order 7 of the set $S = \{1, 2, \ldots, 7\}$.

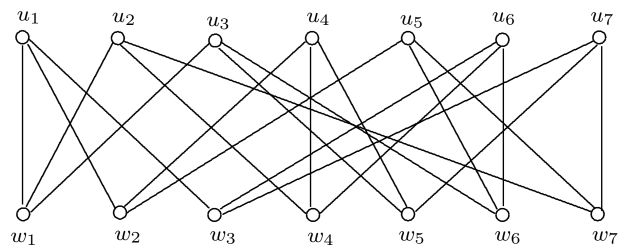


Figure 3.2: A 3-regular bipartite graph G

3.2 The s-Bipartite Ramsey Numbers $BR_s(K_{2,3})$

Here, we determine $BR_s(K_{2,3})$ for each positive integer s, beginning with an observation when $s = 1, 2$.

Proposition 3.2.1 *The numbers $BR_s(K_{2,3})$ do not exist when $s = 1, 2$.*

Proof. For an arbitrary integer $t \geq 2$, the graph $K_{1,t}$ does not contain $K_{2,3}$ as a subgraph. Thus, $BR_1(K_{2,3})$ does not exist. Since the red-blue coloring of $K_{2,t}$ in which both red and blue subgraphs are $K_{1,t}$ produces no monochromatic $K_{2,3}$, the number $BR_2(K_{2,3})$ does not exist. ■

It is convenient to denote a set $\{v_a, v_b, v_c, \ldots, v_\alpha\}$ of vertices of a graph by $abc\ldots\alpha$ or $a, b, c, \ldots, \alpha$.

Theorem 3.2.2 *If $s = 3, 4$, then $BR_s(K_{2,3}) = 13$.*

Proof. First, we show that there exists a red-blue coloring of $K_{4,12}$ that avoids a monochromatic $K_{2,3}$. For $G = K_{4,12}$, let $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, w_2, \ldots, w_{12}\}$ be the partite sets of G. There are $\binom{4}{2} = 6$ distinct 2-element subsets of U and so there are twelve 2-element subsets of U when each such subset occurs twice. We denote them by $U_1, U_2, \ldots, U_{12}$, which are shown below. For $i = 1, 2, \cdots, 12$, let $\overline{U}_i = U - U_i$.

U_i :	12	12	13	13	14	14	23	23	24	24	34	34
$\overline{U}_i$:	34	34	24	24	23	23	14	14	13	13	12	12

Consider the red-blue coloring of G where w_i $(1 \leq i \leq 12)$ is joined to each vertex in U_i by red edges and to the vertices in $\overline{U}_i$ by blue edges. The resulting red subgraph of this red-blue coloring is shown in Figure 3.3.

For this red-blue coloring of G, the red-neighborhood of w_i is $N_R(w_i) = U_i$ and the blue-neighborhood of w_i is $N_B(w_i) = \overline{U}_i$ for $1 \leq i \leq 12$. Furthermore, for each integer j

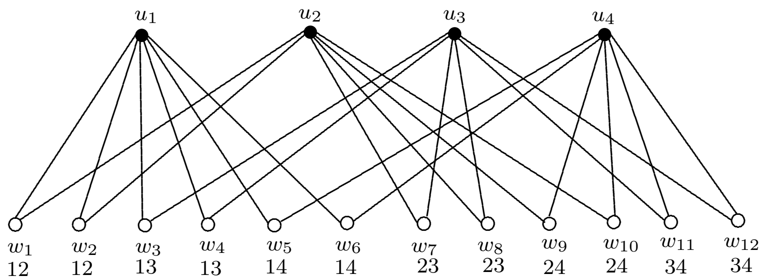


Figure 3.3: The red subgraph in a red-blue coloring of $K_{4,12}$

with $1 \le j \le 4$, let $W_j = N_R(u_j)$ and $\overline{W}_j = N_B(u_j) = W - W_j$. Then the sets W_j and $\overline{W}_j$ are listed below:

W_j :	$1, 2, 3, 4, 5, 6$	$1, 2, 7, 8, 9, 10$	$3, 4, 7, 8, 11, 12$	$5, 6, 9, 10, 11, 12$
$\overline{W}_j$:	$7, 8, 9, 10, 11, 12$	$3, 4, 5, 6, 11, 12$	$1,\ 2,\ 5,\ 6,\ 9,\ 10$	$1,\ 2,\ 3,\ 4,\ 7,\ 8$

Since $|U_i| = |\overline{U}_i| = 2$ for $1 \le i \le 12$ and each 2-element subset of U appears exactly twice in the sets U_i and $\overline{U}_i$, it follows that there is no monochromatic $K_{2,3}$ in which the two vertices of degree 3 belong to U. Furthermore, since each 3-element subset of W appears at most once in W_j and in $\overline{W}_j$ for $1 \le j \le 4$, it follows that there is no monochromatic $K_{2,3}$ in which the two vertices of degree 3 belong to U. Since $|U_i| = |\overline{U}_i| = 2$ for $1 \le i \le 12$, there is no monochromatic $K_{2,3}$ in which the two vertices of degree 3 belong to W. Therefore, there is no monochromatic $K_{2,3}$ in G and so $BR_4(K_{2,3}) \ge 13$. This also implies that there is a red-blue coloring of $K_{3,12}$ that avoids a monochromatic $K_{2,3}$ and so $BR_3(K_{2,3}) \ge 13$.

Next, we show that every red-blue coloring of $H = K_{3,13}$ results in a monochromatic $K_{2,3}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, \ldots, w_{13}\}$ be the partite sets of H. Each of the vertices w_i $(1 \le i \le 13)$ is incident with at least two edges of the same color and at least seven of these 13 vertices are incident with at least two red edges or two blue edges, say the former. Since there are only three distinct 2-element subsets of $\{u_1, u_2, u_3\}$, it follows that three of these seven vertices are joined to the same pair of vertices in $\{u_1, u_2, u_3\}$ by red edges, producing a red $K_{2,3}$. Thus, every red-blue coloring of $K_{3,13}$ results in a monochromatic $K_{2,3}$ and so $BR_3(K_{2,3}) \le 13$. This also implies that every red-blue coloring of $K_{4,13}$ results in a monochromatic $K_{2,3}$ and so $BR_4(K_{2,3}) \le 13$. Therefore, $BR_s(K_{2,3}) = 13$ for $s = 3, 4$. ∎

Theorem 3.2.3 *If $s = 5, 6$, then $BR_s(K_{2,3}) = 11$.*

Proof. First, we show that there exists a red-blue coloring of $K_{6,10}$ that avoids a monochromatic $K_{2,3}$. Let $U = \{u_1, u_2, \ldots, u_6\}$ and $W = \{w_1, w_2, \ldots, w_{10}\}$ be the partite sets of $G = K_{6,10}$. Consider the ten subsets $U_1, U_2, \ldots, U_{10}$ of U shown below and let $\overline{U}_i = U - U_i$ for $1 \le i \le 10$.

U_i :	123	124	135	146	156	236	245	256	345	346
$\overline{U}_i$:	456	356	246	235	234	145	136	134	126	125

We now define a red-blue coloring of G where w_i $(1 \le i \le 10)$ is joined to the three vertices in U_i by red edges and to the remaining three vertices in $\overline{U}_i$ by blue edges. The resulting red subgraph of this red-blue coloring is shown in Figure 3.4.

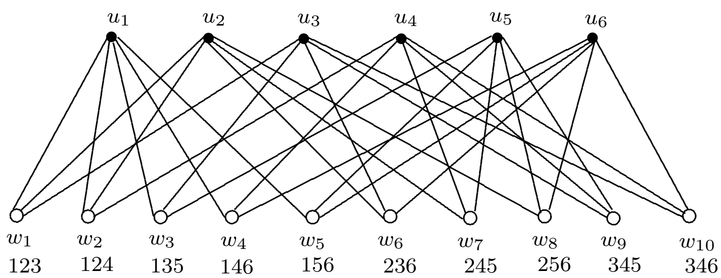


Figure 3.4: The red subgraph in a red-blue coloring of $K_{6,10}$

For this red-blue coloring of G, the red-neighborhood of w_i is $N_R(w_i) = U_i$ and the blue-neighborhood of w_i is $N_B(w_i) = \overline{U}_i$ for $1 \le i \le 10$. Furthermore, for each integer j with $1 \le j \le 6$, let $W_j = N_R(u_j)$ and $\overline{W}_j = N_B(u_j) = W - W_j$. Then the sets W_j and $\overline{W}_j$ are listed below, where 10 is indicated by 0:

W_j :	12345	12678	13690	24790	35789	45680
$\overline{W}_j$:	67890	34590	24578	13568	12460	12379

Since each pair of vertices of U appears exactly twice in the sets U_i for $1 \le i \le 10$, there is no red $K_{2,3}$ in G in which the two vertices of degree 3 belong to U. Furthermore, each pair of vertices of W appears at most twice in the sets W_j for $1 \le j \le 6$ and so there is no red $K_{2,3}$ in G in which the two vertices of degree 3 belong to W. Thus, there is no red $K_{2,3}$ in G. Since the red subgraph and the blue subgraph are isomorphic in this red-blue coloring, there is no blue $K_{2,3}$ in G as well. Therefore, there is no monochromatic $K_{2,3}$ in G and so $BR_6(K_{2,3}) \ge 11$.

Next, we show that every red-blue coloring of $H = K_{5,11}$ results in a monochromatic $K_{2,3}$. Let there be given a red-blue coloring of H where, say, there are more red edges than blue edges. Since the size of H is 55, this coloring has at least 28 red edges. Denote the partite sets of H by $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $W = \{w_1, w_2, \ldots, w_{11}\}$. Since the average degree of the vertices of W in the resulting red subgraph H_R is 2.5, the maximum degree of W in H_R is at least 3. We may assume that at least a_i edges incident with w_i $(1 \leq i \leq 11)$ are red, where $a_1 \geq a_2 \geq \cdots \geq a_{11}$ and $\sum_{i=1}^{11} a_i = 28$. Hence, for each integer i with $1 \leq i \leq 11$, the remaining $5 - a_i$ edges incident with w_i may be red or blue. Denote the sequence $a_1, a_2, \ldots, a_{11}$ by $\{a_i\}$. We consider two cases.

Case 1. *$a_i = 3$ for $1 \leq i \leq 6$ and $a_i = 2$ for $7 \leq i \leq 11$.* Therefore, for each integer i with $1 \leq i \leq 6$, the vertex w_i is joined to at least $\binom{a_i}{2} = 3$ distinct pairs of vertices in U by red edges; while for $7 \leq i \leq 11$, the vertex w_i is joined to at least $\binom{a_i}{2} = 1$ pair of vertices in U by red edges. Hence, the vertices $w_1, w_2, \ldots, w_6$ are joined to at least $6 \cdot 3 = 18$ pairs of vertices in U by red edges and the vertices $w_7, w_8, \ldots, w_{11}$ are joined to at least 5 pairs of vertices in U by red edges. Thus, the vertices of W are joined to at least 23 pairs of vertices in U by red edges. However, since there are only $\binom{5}{2} = 10$ distinct pairs of vertices in U, there are three vertices of W that are joined to the same pair of vertices of U by red edges, producing a red $K_{2,3}$.

Case 2. *$\{a_i\}$ is not the sequence as described in Case* 1. Now, let $\{b_i\} = \{b_1, b_2, \ldots, b_{11}\}$ be the sequence in Case 1, where then $b_i = 3$ for $1 \leq i \leq 6$ and $b_i = 2$ for $7 \leq i \leq 11$. Then $\{a_i\}$ can be obtained from $\{b_i\}$ by replacing two terms b_i and b_j with $i < j$ by $b_i + 1$ and $b_j - 1$ (perhaps multiple times). Then $\sum_{i=1}^{11} a_i = \sum_{i=1}^{11} b_i = 28$. However, since

$$\binom{b_i + 1}{2} - \binom{b_j - 1}{2} > \binom{b_i}{2} - \binom{b_j}{2},$$

it follows that $\sum_{i=1}^{11} \binom{a_i}{2} > 23$, also producing a red $K_{2,3}$.

Therefore, every red-blue coloring of $H = K_{5,11}$ results in a monochromatic $K_{2,3}$. Thus, $BR_5(K_{2,3}) \leq 11$ and so $BR_5(K_{2,3}) = 11$. This also implies that every red-blue coloring of $K_{6,11}$ results in a monochromatic $K_{2,3}$ and so $BR_6(K_{2,3}) = 11$. ■

In [5], Beineke and Schwenk showed that $BR(K_{2,n}) = 4n - 3$ if $n \geq 3$ is odd and there exists a Hadamard matrix of order $2(n-1)$. This result implies that $BR(K_{2,3}) = 9$. Their proof techniques involve some known results on the existence of Hadamard matrices and a form of Jensen's inequality for convex functions. Here, we present an independent and constructive proof of the next result, which also implies that $BR(K_{2,3}) = 9$.

Theorem 3.2.4 *If $s = 7, 8$, then $BR_s(K_{2,3}) = 9$.*

Proof. First, we show that there exists a red-blue coloring of $K_{8,8}$ that avoids a monochromatic $K_{2,3}$. Let $U = \{u_1, u_2, \ldots, u_8\}$ and $W = \{w_1, w_2, \ldots, w_8\}$ be the partite sets of $G = K_{8,8}$. Consider the eight 4-element subsets $U_1, U_2, \ldots, U_8$ of U shown below, where $\{u_a, u_b, u_c, u_d\}$ is denoted by $abcb$, and let $\overline{U}_i = U - U_i$ for $1 \le i \le 8$.

U_i :	1234	1256	1357	1467	5678	3478	2468	2358
$\overline{U}_i$:	5678	3478	2468	2358	1234	1256	1357	1467

Then for each integer i with $5 \le i \le 8$, the set U_i is the complement of U_{i-4}. Furthermore, $\{U_1, U_2, \ldots, U_8\} = \{\overline{U}_1, \overline{U}_2, \ldots, \overline{U}_8\}$. We now define a red-blue coloring of G where w_i $(1 \le i \le 8)$ is joined to the four vertices in U_i by red edges and to the remaining four vertices in $\overline{U}_i$ by blue edges. The resulting red subgraph of this red-blue coloring is shown in Figure 3.5. Then the red subgraph G_R and the blue subgraph G_B are both 4-regular and $G_R \cong G_B$. The red-neighborhood of w_i is $N_R(w_i) = U_i$ and the blue-neighborhood of w_i is $N_B(w_i) = \overline{U}_i$ for $1 \le i \le 8$. None of the 2-element subsets $18, 27, 36, 46$ of U is a subset of any of $U_1, U_2, \ldots, U_8$; while every other 2-element subset of U is a subset of exactly two of these eight sets. Therefore, each 3-element subset of U containing any of $18, 27, 36, 46$ is not a subset of any of $U_1, U_2, \ldots, U_8$, while every other 3-element subset of U is a subset of exactly one of $U_1, U_2, \ldots, U_8$. Hence, there is no red $K_{2,3}$. Furthermore, because $G_R \cong G_B$, there is no blue $K_{2,3}$ either. Therefore, there is no monochromatic $K_{2,3}$ in G and so $BR_8(K_{2,3}) \ge 9$. This also implies that there is a red-blue coloring of $K_{7,8}$ that avoids a monochromatic $K_{2,3}$ and so $BR_7(K_{2,3}) \ge 9$ as well.

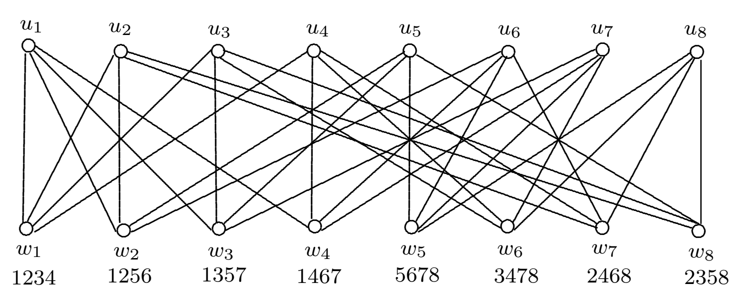


Figure 3.5: The red subgraph in a red-blue coloring of $K_{8,8}$

To verify that $BR_s(K_{2,3}) \le 9$ for $s = 7, 8$, we show that every red-blue coloring of $K_{7,9}$ results in a monochromatic $K_{2,3}$. Assume, to the contrary, that there exists a red-blue coloring of $H = K_{7,9}$, resulting in a red subgraph H_R and a blue subgraph H_B, with sizes m_R and m_B, respectively, but with no monochromatic $K_{2,3}$. Let $U = \{u_1, u_2, \ldots, u_7\}$ and $W = \{w_1, w_2, \ldots, w_9\}$ be the partite sets of H. Since the size of H is 63, we may assume, without loss of generality, that $m_R \ge 32$ and $\deg_{H_R} w_1 \ge \deg_{H_R} w_2 \ge \cdots \ge \deg_{H_R} w_9$.

First, suppose that $\deg_{H_R} w_i = 4$ for $1 \le i \le 5$ and $\deg_{H_R} w_i = 3$ for $6 \le i \le 9$; that is, the degree sequence of the vertices of W in H_R is $4, 4, 4, 4, 4, 3, 3, 3, 3$. Thus,

$$\sum_{i=1}^{9} \deg_{H_R} w_i = 32 \quad \text{and} \quad \sum_{i=1}^{9} \binom{\deg_{H_R} w_i}{2} = 42.$$

Since there is no red $K_{2,3}$ and the number of pairs of vertices in U is $\binom{7}{2} = 21$, it follows that each pair of vertices of U must be joined to exactly two vertices of W by red edges. Since each vertex of U belongs to six distinct pairs of vertices of U, it follows that each vertex of U belongs to 12 pairs of vertices of U that are joined to vertices of W by red edges. For example, the vertex u_1 belongs to $\{u_1, u_i\}$ for $2 \le i \le 7$. Let $W_1 = \{w_1, w_2, w_3, w_4, w_5\}$ be the vertices of W having degree 4 in H_R and $W_2 = \{w_6, w_7, w_8, w_9\}$ be the vertices of degree 3 in H_R. We now verify two claims.

> **Claim 1.** *No vertex of U can be joined to an odd number of vertices in W_1 by red edges.*

If Claim 1 is false, then there is $u_j \in U$ where $1 \le j \le 7$ such that u_j is joined to an odd number of vertices of W_1 by red edges. Since each vertex of W_1 joined to u_j by a red edge is joined to three pairs of vertices of U including u_j by red edges, it follows that u_j belongs to an odd number of pairs of vertices of U that are joined to vertices of W_1 by red edges. If u_j is joined to a vertex in W_2 by a red edge, then u_j belongs to an even number of pairs of vertices of U that are joined to vertices of W_2 by red edges. Hence, u_j belongs to an odd number of pairs of vertices of U that are joined to vertices of W by red edges. This is impossible since each vertex of U belongs to 12 such pairs of vertices of U. For example, suppose that a vertex u_j $(1 \le j \le 7)$ is joined to a vertices of W_1 and b vertices of W_2 by red edges. Then the vertices of W are joined to $3a + 2b$ pairs of vertices of U that are joined to vertices of W by red edges. However, if a is odd, then $3a + 2b \ne 12$, which is impossible.

To illustrate this fact, suppose that u_1 is joined to exactly three vertices of W_1 by red edges, say $u_1 w_i$ is red for $i = 1, 2, 3$ and $u_1 w_i$ is blue for $i = 4, 5$. Since $\deg_{H_R} w = 4$ for each $w \in W_1$, we may assume that $N_{H_R}(w_1) = \{u_1, u_{a_1}, u_{a_2}, u_{a_3}\}$, $N_{H_R}(w_2) = \{u_1, u_{b_1}, u_{b_2}, u_{b_3}\}$ and $N_{H_R}(w_3) = \{u_1, u_{c_1}, u_{c_2}, u_{c_3}\}$, where $u_{a_i}, u_{b_i}, u_{c_i}$ $(i = 1, 2, 3)$ may not be distinct. Thus, u_1 belongs to 9 pairs of vertices of U that are joined to vertices of W_1 by red edge, namely, $\{u_1, u_{a_i}\}$, $\{u_1, u_{b_i}\}$, $\{u_1, u_{b_i}\}$ for $i = 1, 2, 3$. Furthermore, suppose that u_1 is also joined to w_6 of W_2 by a red edge and u_1 is joined to the remaining vertices of W_2 by blue edges. Since $\deg_{H_R} w_6$, we may assume that $N_{H_R}(w_6) = \{u_1, u_{d_1}, u_{d_2}\}$

and so u_1 belongs to additional two pairs of vertices of U, namely $\{u_1, u_{d_1}\}$ and $\{u_1, u_{d_2}\}$, that are joined to vertices of W_2 by red edge. Thus, u_1 belongs to $9 + 2 = 11$ pairs of vertices of U that are joined to vertices of W by red edges. Since u_1 belongs to 12 pairs of vertices of U that are joined to vertices of W by red edges, this is impossible. Therefore, Claim 1 holds.

Claim 2. *Evey vertex of U is joined to some vertex in W_1 by a red edge.*

If Claim 2 is false, then there is $u_j \in U$ where $1 \leq j \leq 7$ such that u_j is not joined to any vertex in W_1 by a red edge. Thus, u_j is joined to at most four vertices of W by red edges, namely the vertices in W_2. Thus, u_j belongs to at most eight pairs of vertices of U that are joined to vertices of W by red edges. Again, since u_j belongs to 12 such pairs of vertices of U, this is impossible. Therefore, Claim 2 holds.

By Claims 1 and 2, every vertex of U is either joined to four vertices in W_1 or to two vertices in W_1 by red edges. Suppose that x vertices in U are joined to four vertices of W_1 by red edges and so $7-x$ vertices in U are joined to two vertices of W_1 by red edges. Since each of the five vertices of W_1 has degree 4 in H_R, it follows that $4x+2(7-x) = 20$ and so $x = 3$. Suppose that u_s and u_t, where $1 \leq s < t \leq 7$, are joined to four vertices of W_1 by red edges. Since $|W_1| = 5$, it follows that u_s and u_t are joined to the same three vertices of W_1 by red edges and so H contains a red $K_{2,3}$, which is a contradiction.

Next, suppose that the degree sequence of the vertices of W in H_R is not 4, 4, 4, 4, 4, 3, 3, 3, 3. Since $m_R \geq 32$, we may assume that at least a_i edges incident with w_i $(1 \leq i \leq 9)$ are red such that $a_1 \geq a_2 \geq \cdots \geq a_9$ and $\sum_{i=1}^{11} a_i = 32$. Hence, for each integer i with $1 \leq i \leq 9$, the remaining $7 - a_i$ edges incident with w_i may be red or blue. If $a_i = 4$ for $1 \leq i \leq 5$ and $a_i = 3$ for $6 \leq i \leq 9$, then H contains a red $K_{2,3}$ by the discussion above, which is a contradiction. Thus, we may assume that this not the case. Now, let $b_1, b_2, \ldots, b_9$ be the sequence where $b_i = 4$ for $1 \leq i \leq 5$ and $b_i = 3$ for $6 \leq i \leq 9$. Then $\{a_i\}$ can be obtained from $\{b_i\}$ by replacing two terms b_i and b_j with $i < j$ by $b_i + 1$ and $b_j - 1$ (perhaps multiple times). Then

$$\sum_{i=1}^{9} a_i = \sum_{i=1}^{9} b_i = 32.$$

However, since

$$\binom{b_i+1}{2} - \binom{b_j-1}{2} > \binom{b_i}{2} - \binom{b_j}{2},$$

it follows that $\sum_{i=1}^{11} \binom{a_i}{2} > 42$. This implies that there are three vertices of W that are joined to the same pair of vertices of U by red edges, producing a red $K_{2,3}$, a contradiction.

Therefore, every red-blue coloring of $K_{7,9}$ results in a monochromatic $K_{2,3}$. Thus, $BR_7(K_{2,3}) \leq 9$ and so $BR_7(K_{2,3}) = 9$. This also implies that every red-blue coloring of $K_{8,9}$ results in a monochromatic $K_{2,3}$ and so $BR_8(K_{2,3}) = 9$. ∎

Suppose, for a bipartite graph F, that every red-blue coloring of the graph $K_{k,k+1}$ results in a monochromatic F. For each integer $s \geq k+1$, since the graph $K_{s,s}$ contains $K_{k,k+1}$ as a subgraph, it follows that every red-blue coloring of $K_{s,s}$ results in a monochromatic F. This gives rise to the following useful observation.

Observation 3.2.5 *If F is a bipartite graph such that $BR_k(F) = k+1$ for some positive integer k, then $BR_s(F) = s$ for all integers $s \geq k+1$.*

By Theorem 3.2.4 and Observation 4.4.2, it follows that $BR_s(K_{2,3}) = s$ for each integer $s \geq 9$. In summary, we have the following result.

Theorem 3.2.6 *For each integer $s \geq 2$,*

$$BR_s(K_{2,3}) = \begin{cases} \textit{does not exist} & \textit{if } s = 2 \\ 13 & \textit{if } s = 3, 4 \\ 11 & \textit{if } s = 5, 6 \\ 9 & \textit{if } s = 7, 8 \\ s & \textit{if } s \geq 9. \end{cases}$$

As a consequence of Theorem 3.2.6, it follows that $BR(K_{2,3}) = 9$. While $BR(K_{3,3}) = 17$ (see [5]), the bipartite Ramsey numbers $BR(K_{r,s})$ of the complete bipartite graphs $K_{r,s}$ are not known in general for $3 \leq r \leq s$.

3.3 The Numbers $BR_s(K_{2,3}, K_{3,3})$ for $1 \leq s \leq 7$

We now determine $BR_s(K_{2,3}, K_{3,3})$ for each positive integer s, beginning with an observation when $1 \leq s \leq 3$.

Proposition 3.3.1 *For $1 \leq s \leq 3$, the number $BR_s(K_{2,3}, K_{3,3})$ does not exist.*

Proof. Certainly, $BR_1(K_{2,3}, K_{3,3})$ does not exist. For $s = 2, 3$ and an integer $t \geq s$, the red-blue coloring of $K_{s,t}$, in which the red subgraph is $K_{1,t}$ and the blue subgraph is $K_{s-1,t}$ produces neither a red $K_{2,3}$ nor a blue $K_{3,3}$. ∎

We now present two results that give the exact values of $BR_s(K_{2,3}, K_{3,3})$ for four values of s.

Theorem 3.3.2 *If $s = 4, 5$, then $BR_s(K_{2,3}, K_{3,3}) = 21$.*

Proof. First, we show that there exists a red-blue coloring of $K_{5,20}$ that avoids both a red $K_{2,3}$ and a blue $K_{3,3}$. Let $U = \{u_1, u_2, \ldots, u_5\}$ and $W = \{w_1, w_2, \ldots, w_{20}\}$ be the partite sets of $G = K_{5,20}$. There are twenty 2-element subsets of U when each such subset occurs exactly twice. We denote them by $U_1, U_2, \ldots, U_{20}$. For $i = 1, 2, \cdots, 20$, let $\overline{U}_i = U - U_i$. We now define a red-blue coloring of G where w_i $(1 \leq i \leq 20)$ is joined to the vertices in U_i by red edges and to the vertices in $\overline{U}_i$ by blue edges. Denote the spanning subgraph of G with red edges by G_R and the spanning subgraph of G with blue edges by G_B. The red-neighborhood of w_i is therefore $N_R(w_i) = U_i$ and the blue-neighborhood of w_i is $N_B(w_i) = \overline{U}_i$ for $1 \leq i \leq 20$. Since every vertex of W has degree 2 in G_R, there is no red $K_{2,3}$ where the partite set of order 3 belongs to U. Since, for every two vertices of U, only two vertices of W are joined to these two vertices of U in G_R, and so there is no red $K_{2,3}$ where the partite set of order 3 belongs to W. Thus, G_R does not contain $K_{2,3}$ as a subgraph. Since each 3-element subset $\overline{U}_i$ occurs exactly twice among $\{\overline{U}_1, \overline{U}_2, \ldots, \overline{U}_{20}\}$, there is no blue $K_{3,3}$ in G. Hence, G contains neither a red $K_{2,3}$ nor a blue $K_{3,3}$ and so $BR_5(K_{2,3}, K_{3,3}) \geq 21$. This red-blue coloring also shows that there is a red-blue coloring of $K_{4,20}$ that produces neither a red $K_{2,3}$ nor a blue $K_{3,3}$ and so $BR_4(K_{2,3}, K_{3,3}) \geq 21$.

Next, we show that every red-blue coloring of $H = K_{4,21}$ produces a red $K_{2,3}$ or a blue $K_{3,3}$. Let $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, w_2, \ldots, w_{21}\}$ be the partite sets of H. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. We consider two cases.

Case 1. *At least nine vertices of W have degree at least 3 in H_B, say $\deg_{H_B} w_i \geq 3$ for $1 \leq i \leq 9$.* Since there are $\binom{4}{3} = 4$ distinct 3-element subsets of U, there are at least three vertices in $\{w_1, w_2, \ldots, w_9\}$ joined to the same three vertices in U by blue edges. Thus, H_B contains $K_{3,3}$ as a subgraph.

Case 2. *At most eight vertices of W have degree at least 3 in H_B.* Therefore, at least 13 vertices of W have degree at most 2 in H_B, say $\deg_{H_B} w_i \leq 2$ for $9 \leq i \leq 21$. Therefore, $\deg_{H_R} w_i \geq 2$ for $9 \leq i \leq 21$. Since there are $\binom{4}{2} = 6$ distinct 2-element subsets of U, there are at least three vertices in $\{w_9, w_{10}, \ldots, w_{21}\}$ joined to the same two vertices in U by red edges. Thus, H_R contains $K_{2,3}$ as a subgraph.

Consequently, $BR_4(K_{2,3}, K_{3,3}) \leq 21$. Since every red-blue coloring of $K_{4,21}$ produces either a red $K_{2,3}$ or a blue $K_{3,3}$, the same is true of $K_{5,21}$ and so $BR_5(K_{2,3}, K_{3,3}) \leq 21$. Therefore, $BR_s(K_{2,3}, K_{3,3}) = 21$ for $s = 4, 5$. ■

Theorem 3.3.3 *If $s = 6, 7$, then $BR_s(K_{2,3}, K_{3,3}) = 15$.*

Proof. First, we show that there exists a red-blue coloring of $K_{7,14}$ that avoids both a red $K_{2,3}$ and a blue $K_{3,3}$. Let $U = \{u_1, u_2, \ldots, u_7\}$ and $W = \{w_1, w_2, \ldots, w_{14}\}$ be the partite sets of $G = K_{7,14}$. Let $\{U_1, U_2, \ldots, U_7\}$ and $\{U_8, U_9, \ldots, U_{14}\}$ be the two Steiner triple systems of the set $\{1, 2, \ldots, 7\}$ indicated below. Here, we denote a set $\{u_a, u_b, u_c, u_d\}$ by $abcd$, for example. For $1 \leq i \leq 14$, let $\overline{U}_i = U - U_i$.

U_i $(1 \leq i \leq 7)$:	124	137	156	235	267	346	457
$\overline{U}_i$ $(1 \leq i \leq 7)$:	3567	2456	2347	1467	1345	1257	1236
U_i $(8 \leq i \leq 14)$:	125	136	147	237	246	345	567
$\overline{U}_i$ $(8 \leq i \leq 14)$:	3467	2457	2356	1456	1357	1267	1234

The two Steiner triple systems above have the properties that any triple of $\{1, 2, \ldots, 7\}$ is at most one U_i $(1 \leq i \leq 14)$ and every pair of elements in $\{1, 2, \ldots, 7\}$ necessarily belongs to exactly one triple U_i for $1 \leq i \leq 7$ and exactly one triple U_i for $8 \leq i \leq 14$.

We now define a red-blue coloring of G where w_i $(1 \leq i \leq 14)$ is joined to the vertices in U_i by red edges and to the vertices in $\overline{U}_i$ by blue edges. Since each 2-element subset of U appears exactly twice in the triples U_i for $1 \leq i \leq 14$ and each triple of U occurs at most once among the triples U_i for $1 \leq i \leq 14$, it follows that there is no red $K_{2,3}$ in G. It remains to show that there is no blue $K_{3,3}$ in G. Let abc be any triple of U. We claim that abc occurs at most twice among the triples in $\overline{U}_i$ $(1 \leq i \leq 14)$, for suppose that abc occurs three times, say $\overline{U}_x = abci$, $\overline{U}_y = abcj$ and $\overline{U}_z = abck$ where x, y and z are distinct integers in $\{1, 2, \ldots, 14\}$. Let $\{\ell\} = U - \{a, b, c, i, j, k\}$. Thus, $U_x = jk\ell$, $U_y = ik\ell$ and $U_z = ij\ell$. Since $k\ell \subseteq U_x$ and $k\ell \subseteq U_y$, it follows that U_x and U_y belong to distinct Steiner triples systems, say $1 \leq x \leq 7$ and $8 \leq y \leq 14$. However, $j\ell = U_x \cap U_z$ and so $8 \leq z \leq 14$. On the other hand, $i\ell = U_y \cap U_z$; so $1 \leq z \leq 7$, which is impossible. Hence, there is no blue $K_{3,3}$ in G. Therefore, there is neither a red $K_{2,3}$ nor a blue $K_{3,3}$ and so $BR_7(K_{2,3}, K_{3,3}) \geq 15$. This red-blue coloring also shows that there is a red-blue coloring of $K_{6,14}$ that produces neither a red $K_{2,3}$ nor a blue $K_{3,3}$ and so $BR_6(K_{2,3}, K_{3,3}) \geq 15$.

Next, we show that every red-blue coloring of $H = K_{6,15}$ produces a red $K_{2,3}$ or a blue $K_{3,3}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R

and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_6\}$ and $W = \{w_1, w_2, \ldots, w_{15}\}$ be the partite sets of H. First, we verify the three claims below.

Claim 1. *If* $\displaystyle\sum_{i=1}^{15} \binom{\deg_{H_R} w_i}{2} \geq 31$*, then* H *contains a red* $K_{2,3}$*.*

If $\displaystyle\sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} \geq 41$*, then* H *contains a blue* $K_{3,3}$*.*

If $\sum_{i=1}^{15} \binom{\deg_{H_R} w_i}{2} \geq 31$, then the vertices of W are joined to at least 31 pairs of vertices of U (with repetitions) by red edges. Since the number of distinct 2-element subsets in U is $\binom{6}{2} = 15$, at least three vertices of W are joined to the same two vertices of U by red edges, producing a red $K_{2,3}$. If $\sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} \geq 41$, then the vertices of W are joined to at least 41 triples of vertices of U (with repetitions) by blue edges. Since the number of distinct 3-element subsets in U is $\binom{6}{3} = 20$, at least three vertices of W are joined to the same three vertices of U by blue edges, producing a blue $K_{3,3}$. Thus, Claim 1 holds.

Claim 2. *If* W *contains at least three vertices of degree at most* 1 *in* H_R*, then* H *contains a blue* $K_{3,3}$*.*

To verify Claim 2, we may assume that $\deg_{H_R} w_i \leq 1$ for $1 \leq i \leq 3$. Thus, the neighborhood $N_R(\{w_1, w_2, w_3\})$ of $\{w_1, w_2, w_3\}$ contains at most three vertices of U, say $u_4, u_5, u_6 \notin N_R(\{w_1, w_2, w_3\})$. However then, H contains a blue $K_{3,3}$ with partite sets $\{w_1, w_2, w_3\}$ and $\{u_4, u_5, u_6\}$ and so Claim 2 holds.

Claim 3. *If* W *contains at least eleven vertices of degree at most* 2 *in* H_R*, then* H *contains a blue* $K_{3,3}$*.*

To verify Claim 3, we may assume that $\deg_{H_R} w_i \leq 2$ for $1 \leq i \leq 11$ and so $\deg_{H_B} w_i \geq 4$. Since

$$\sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} \geq \sum_{i=1}^{11} \binom{\deg_{H_B} w_i}{3} \geq 11\binom{4}{3} = 44,$$

it follows by Claim 1 that H contains a blue $K_{3,3}$ and so Claim 3 holds.

By Claim 3, if the maximum degree of the vertices of W in H_R is at most 2, then H contains a blue $K_{3,3}$. If the maximum degree of the vertices of W in H_R is 6 and H contains no red $K_{2,3}$, then there are 14 vertices of W having degree at most 2 in H_R. Again, by Claim 3, there is a blue $K_{3,3}$. Hence, we may assume that the maximum degree of the vertices of W in H_R is 5, 4 or 3. We consider these three cases. In all three cases, we assume that there is no red $K_{2,3}$ in H and show that there is a blue $K_{3,3}$ in H.

Case 1. *The maximum degree of the vertices of W in H_R is* 5. Since there is no red $K_{2,3}$, it follows that W contains exactly one vertex of degree 5 and no vertex of degree 4 in H_R, say $\deg_{H_R} w_1 = 5$ and $\deg_{H_R} w_i \leq 3$ for $2 \leq i \leq 15$. By Claim 3, we may assume that W contains at least four vertices of degree 3 in H_R, for otherwise, H contains a blue $K_{3,3}$. By Claim 1, W has at most six vertices of degree 3 in H_R, for otherwise, H contains a blue $K_{3,3}$. Let x denote the number of vertices of degree 3 in H_R, Thus, $4 \leq x \leq 6$. By Claim 2, we may assume that at most two vertices of W have degree at most 1 in H_R. If W has exactly one vertex of degree 1 in H_R or no vertex of degree 1 in H_R, then

$$\textstyle\sum_{i=2}^{15} \binom{\deg_{H_R} w_i}{2} \geq \binom{5}{2} + x\binom{3}{2} + (13 - x) = 23 + 2x \geq 31.$$

Thus, by Claim 1, there is a red $K_{2,3}$, which is impossible. Therefore, W has exactly two vertices of degree 1 in H_R and so $12 - x$ vertices of degree 2 in H_R. Since

$$\binom{5}{2} + x\binom{3}{2} + (12 - x) = 22 + 2x \leq 30 \text{ and } 4 \leq x \leq 6,$$

it follows that $x = 4$ and exactly four vertices of W have degree 3 in H_R by Claim 1. However then,

$$\textstyle\sum_{i=2}^{15} \binom{\deg_{H_B} w_i}{3} \geq 4\binom{3}{3} + 10\binom{4}{3} = 44,$$

producing a blue $K_{3,3}$ by Claim 1.

Case 2. *The maximum degree of the vertices of W in H_R is* 4. By Claim 1 then, W contains at most five vertices of degree 4 in H_R. If W contains exactly five vertices of degree 4 in H_R, then these five vertices of W are joined to $5\binom{4}{2} = 30$ pairs of vertices of U by red edges. By Claim 1 then, W contains no vertex of degree 2 or 3 in H_R. This implies that W contains ten vertices of degree at most 1 in H_R, producing a blue $K_{3,3}$ by Claim 2. If W contains exactly four vertices of degree 4 in H_R, then these four vertices of W are joined to $4\binom{4}{2} = 24$ pairs of vertices of U by red edges. By Claim 1 then, W contains at most six vertices of degree 2 or 3 in H_R. This implies that W contains at least five vertices of degree at most 1 in H_R, producing a blue $K_{3,3}$ by Claim 2. Therefore, the number of vertices of W having degree 4 in H_R is 3, 2 or 1. We consider these three subcases.

Subcase 2.1. *W contains exactly three vertices of degree* 4 *in H_R*. Let x be the number of vertices of W having degree 3 in H_R. By Claim 2, we may assume that W contains at most two vertices of degree at most 1 in H_R. Then W contains at least $9 - x$ vertices of degree 2 in H_R. Then the vertices of W are joined to at least

$$\sum_{i=1}^{15} \binom{\deg_{H_R} w_i}{2} = 3\binom{4}{2} + x\binom{3}{2} + (9-x)\binom{2}{2}$$

pairs of vertices of U by red edges. By Claim 1, we may assume that

$$3\binom{4}{2} + x\binom{3}{2} + (9-x)\binom{2}{2} \leq 30$$

and so $x = 0$ or $x = 1$. Then W contains at least $12 - x \geq 11$ vertices of degree at most 2 in H_R, producing a blue $K_{3,3}$ by Claim 3.

Subcase 2.2. *W contains exactly two vertices of degree* 4 *in H_R.* By Claim 3, W contains at least three vertices of degree 3 in H_R. Let y be the number of vertices of W having degree 3 in H_R and so $y \geq 3$. By Claim 2, we may assume that W contains at most two vertices of degree at most 1 in H_R. Then W contains at least $11 - y$ vertices of degree 2 in H_R. Then the vertices of W are joined to at least

$$\sum_{i=1}^{15} \binom{\deg_{H_R} w_i}{2} = 2\binom{4}{2} + y\binom{3}{2} + (11-y)\binom{2}{2}$$

pairs of vertices of U by red edges. By Claim 1, we may assume that

$$2\binom{4}{2} + y\binom{3}{2} + (11-y)\binom{2}{2} \leq 30$$

and so $y \leq 3$. Since $y \geq 3$, it follows that $y = 3$. So W contains ten vertices of degree at most 2 in H_R and W contains ten vertices of degree at least 4 in H_B. Thus,

$$\sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} \geq 3\binom{3}{3} + 10\binom{4}{3} = 43,$$

producing a blue $K_{3,3}$ by Claim 1.

Subcase 2.3. *W contains exactly one vertex of degree* 4 *in H_R.* Let z be the number of vertices of W having degree 3 in H_R and let w be the number of vertices of W having degree at most 1 in H_R. Then W contains $14 - z - w$ vertices of degree 2 in H_R. Thus,

$$\sum_{i=1}^{15} \binom{\deg_{H_R} w_i}{2} = \binom{4}{2} + z\binom{3}{2} + (14-z-w)\binom{2}{2} \text{ and}$$

$$\sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} \geq w\binom{5}{3} + (14-z-w)\binom{4}{3} + z\binom{3}{3}.$$

By Claim 1, we may assume that

$$\sum_{i=1}^{15} \binom{\deg_{H_R} w_i}{2} \leq 30 \text{ and } \sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} \leq 40.$$

This implies that $2z - w \leq 10$ and $3z - 6w \geq 16$. However then,

$$\tfrac{16}{3} + 2w \leq z \leq 5 + \tfrac{w}{2} \text{ and so } \tfrac{16}{3} + 2w \leq 5 + \tfrac{w}{2},$$

which is impossible for a nonnegative integer w.

Case 3. *The maximum degree of the vertices of* W *in* H_R *is* 3. Let x be the number of vertices of W having degree 3 in H_R and let y be the number of vertices of W having degree 2 in H_R; so $15-x-y$ vertices of W have degree at most 1 in H_R. Then $y \leq 10$ by Claim 3 and $x+y \geq 13$ by Claim 2. Since $15-x$ vertices of W have degree at most 2 in H_R, there are $15-x$ vertices of W having degree at least 4 in H_B. Hence,

$$\textstyle\sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} \geq x\binom{3}{3} + (15-x)\binom{4}{3}.$$

It then follows by Claim 1 that

$$x\tbinom{3}{3} + (15-x)\tbinom{4}{3} = x + (15-x)4 \leq 40 \text{ and so } x \geq 7.$$

On the other hand, also by Claim 1,

$$\textstyle\sum_{i=1}^{15} \binom{\deg_{H_R} w_i}{2} = x\binom{3}{2} + y\binom{2}{2} = 3x + y \leq 30$$

and

$$\textstyle\sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} = x\binom{3}{3} + y\binom{4}{3} + (15-x-y)\binom{5}{3} = 150 - 9x - 6y \leq 40.$$

This implies that $\frac{110-9x}{6} \leq y \leq 30-3x$ and so $x \leq 7$. Therefore, $x = 7$. Since $x+y \geq 13$, it follows that $y \geq 6$. By Claim 1 then,

$$\textstyle\sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} \geq 7\binom{3}{3} + y\binom{4}{3} + (15-7-y)\binom{5}{3}.$$

Hence, $7 + 4y + 80 - 10y \leq 40$ and so $y \geq 8$. Since $x + y \leq 15$, it follows that $y = 8$. Therefore, W contains exactly seven vertices of degree 3 in H_R and W contains exactly eight vertices of degree 2 in H_R. We may assume that $\deg_{H_R} w_i = 3$ for $1 \leq i \leq 7$ and $\deg_{H_R} w_i = 2$ for $8 \leq i \leq 15$. Then the degree sequence of the vertices of W in H_B is

$$3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4.$$

Thus, $\sum_{i=1}^{15} \binom{\deg_{H_B} w_i}{3} = 39$. This implies that

(a) exactly one 3-element subset of U belongs to the neighborhood of exactly one vertex of W in H_B and

(b) each of the remaining 3-element subsets of U belongs to the neighborhoods of exactly two vertices of W in H_B.

Next, we verify two additional claims.

Claim 4. *No two vertices of W having degree 3 in H_B can have the same neighborhoods in H_B.*

To verify Claim 4, we may assume that $N_{H_B}(w_1) = N_{H_B}(w_2) = \{u_1, u_2, u_3\}$. However then, $N_{H_R}(w_1) = N_{H_R}(w_2) = \{u_4, u_5, u_6\}$, producing a red $K_{2,3}$, which is a contradiction. Thus, Claim 4 holds.

Claim 5. *If W contains two vertices of degree 4 in H_B that have the same neighborhoods, then H contains a blue $K_{3,3}$.*

To verify Claim 5, we may assume that

$$N_{H_B}(w_8) = N_{H_B}(w_9) = \{u_1, u_2, u_3, u_4\}.$$

If $|N_{H_B}(w_i) \cap \{u_1, u_2, u_3, u_4\}| \geq 3$ for $10 \leq i \leq 15$, then there is a blue $K_{3,3}$. Hence, we may assume that $|N_{H_B}(w_i) \cap \{u_1, u_2, u_3, u_4\}| = 2$ and so $N_{H_B}(w_i) = \{u_s, u_t, u_5, u_6\}$ where $10 \leq i \leq 15$, $s, t \in \{1, 2, 3, 4\}$ and $s \neq t$. Since there are exactly six distinct 2-element subsets of $\{u_1, u_2, u_3, u_4\}$, each of these 2-element subsets belongs to exactly one $N_{H_B}(w_i)$ with $10 \leq i \leq 15$, say $N_{H_B}(w_{10}) = \{u_1, u_2, u_5, u_6\}$, $N_{H_B}(w_{11}) = \{u_1, u_3, u_5, u_6\}$ and $N_{H_B}(w_{12}) = \{u_1, u_4, u_5, u_6\}$. Then there is a blue $K_{3,3}$ with partite sets $\{u_1, u_5, u_6\}$ and $\{w_{10}, w_{11}, w_{12}\}$. Thus, Claim 5 holds.

By Claim 4 then, $|N_{H_B}(w_i) \cap N_{H_B}(w_j)| \leq 2$ for $1 \leq i, j \leq 7$ and $i \neq j$. If $|N_{H_B}(w_i) \cap N_{H_B}(w_j)| = 0$, then by Claim 4, each w_k ($1 \leq k \leq 7$, $k \neq i, j$) has the property that $|N_{H_B}(w_i) \cap N_{H_B}(w_k)| = 2$ or $|N_{H_B}(w_j) \cap N_{H_B}(w_k)| = 2$. Therefore, $|N_{H_B}(w_i) \cap N_{H_B}(w_j)| \in \{1, 2\}$ for $1 \leq i, j \leq 7$ and $i \neq j$. We claim that $|N_{H_B}(w_i) \cap N_{H_B}(w_j)| = 2$ for some i, j with $1 \leq i, j \leq 7$ and $i \neq j$. Suppose that this is not the case and $N_{H_B}(w_1) = \{u_1, u_2, u_3\}$, say. There are three 2-element subsets of $U' = \{u_4, u_5, u_6\}$. For each w_i ($2 \leq i \leq 7$), $|N_{H_B}(w_i) \cap U'| = 2$. Thus, there are two distinct vertices w_i and w_j ($2 \leq i, j \leq 7$) such that $N_{H_B}(w_i) \cap U' = N_{H_B}(w_j) \cap U'$, which implies by Claim 4 that $|N_{H_B}(w_i) \cap N_{H_B}(w_j)| = 2$. Therefore, as claimed, $|N_{H_B}(w_i) \cap N_{H_B}(w_j)| = 2$ for some i, j with $1 \leq i, j \leq 7$ and $i \neq j$. Thus, we may assume that

$$N_{H_B}(w_1) = \{u_1, u_2, u_3\} \text{ and } N_{H_B}(w_2) = \{u_1, u_2, u_4\}. \tag{3.2}$$

If $|N_{H_B}(w_i) \cup N_{H_B}(w_j) \cup N_{H_B}(w_k)| \leq 4$ for three distinct vertices w_i, w_j, w_k of W with $1 \leq i, j, k \leq 15$, then there is a red $K_{2,3}$, one of whose partite sets is $\{w_i, w_j, w_k\}$ and the other is a 2-element subset of $U - [N_{H_B}(w_i) \cup N_{H_B}(w_j) \cup N_{H_B}(w_k)]$. This implies that

(c) neither $\{u_1, u_3, u_4\}$ nor $\{u_2, u_3, u_4\}$ can be the neighborhood of a vertex of W having degree 3 in H_B and $\{u_1, u_2, u_3, u_4\}$ cannot be the neighborhood of a vertex of W having degree 4 in H_B.

Hence, each of $\{u_1, u_3, u_4\}$ and $\{u_2, u_3, u_4\}$ must be a proper subset of the neighborhood of a vertex in W of degree 4 in H_B. It then follows by (a) and (b) that at least one of $\{u_1, u_3, u_4\}$ and $\{u_2, u_3, u_4\}$ is a subset of the neighborhoods of exactly two vertices of degree 4 in H_B. We consider these two possibilities.

Subcase 3.1. *Both* $\{u_1, u_3, u_4\}$ *and* $\{u_2, u_3, u_4\}$ *are subsets of the neighborhoods of exactly two vertices of degree* 4 *in* H_B. By Claim 5, we may assume that

$$N_{H_B}(w_8) = \{u_1, u_3, u_4, u_5\},\ N_{H_B}(w_9) = \{u_1, u_3, u_4, u_6\},$$

$$N_{H_B}(w_{10}) = \{u_2, u_3, u_4, u_5\},\ N_{H_B}(w_{11}) = \{u_2, u_3, u_4, u_6\}.$$

By (3.2), (a), (b) and Claim 4, at least one of $\{u_1, u_2, u_3\}$ and $\{u_1, u_2, u_4\}$ is a subset of the neighborhood of exactly one vertex of degree 4 in H_B, say the former. We may assume that

$$N_{H_B}(w_{12}) = \{u_1, u_2, u_3, u_5\}.$$

Next, we show that $\{u_3, u_5, u_6\}$ cannot belong to the neighborhood of any vertex of degree 4 in H_B. If this were the case, then $\{u_3, u_5, u_6\} \cup \{u_i\}$ is the neighborhood of some vertex of degree 4 for some $i \in \{1, 2, 4\}$. Since each of $\{u_1, u_3, u_5\}$, $\{u_2, u_3, u_5\}$ and $\{u_3, u_4, u_5\}$ has already appeared twice in the neighborhoods of vertices of W of degree 4 in H_B, there is a blue $K_{3,3}$. Hence, by Claim 4, the 3-element subset $\{u_3, u_5, u_6\}$ of U belongs to exactly one vertex of W having degree 3 in H_B, say

$$N_{H_B}(w_3) = \{u_3, u_5, u_6\}.$$

Again by Claim 4, $\{u_1, u_2, u_4\}$ is a subset of the neighborhood of exactly one vertex of degree 4 in H_B, say w_{13}. Then

$$N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_5\} \text{ or } N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_6\}.$$

Now, since $\{u_4, u_5, u_6\}$ appears twice in the neighborhoods of the vertices of W in H_B, it must belong to the neighborhood of at least one vertex of W of degree 4 in H_B by Claim 4, say$\{u_4, u_5, u_6\} \subseteq N_{H_B}(w_{14})$. Then $N_{H_B}(w_{14}) = \{u_i, u_4, u_5, u_6\}$ for some $i \in \{1, 2, 3\}$.

First, suppose that $N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_5\}$.

- If $i = 1$, then there is a blue $K_{3,3}$ with partite sets $\{u_1, u_4, u_5\}$ and $\{w_8, w_{13}, w_{14}\}$.
- If $i = 2$, then there is a blue $K_{3,3}$ with partite sets $\{u_2, u_4, u_5\}$ and $\{w_{10}, w_{13}, w_{14}\}$.
- If $i = 3$, then there is a blue $K_{3,3}$ with partite sets $\{u_3, u_4, u_5\}$ and $\{w_8, w_{10}, w_{14}\}$.

Next, suppose that $N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_6\}$.

- If $i = 1$, then there is a blue $K_{3,3}$ with partite sets $\{u_1, u_4, u_6\}$ and $\{w_9, w_{13}, w_{14}\}$.
- If $i = 2$, then there is a blue $K_{3,3}$ with partite sets $\{u_2, u_4, u_6\}$ and $\{w_{11}, w_{13}, w_{14}\}$.
- If $i = 3$, then there is a blue $K_{3,3}$ with partite sets $\{u_3, u_4, u_5\}$ and $\{w_8, w_{10}, w_{14}\}$.

Subcase 3.2. *Exactly one of* $\{u_1, u_3, u_4\}$ *and* $\{u_2, u_3, u_4\}$ *is the* 3*-element subset of* U *that belongs to the neighborhood of exactly one vertex of* W *in* H_B*, say the former.* Then $\{u_1, u_3, u_4\}$ is a subset of the neighborhood of exactly one vertex of degree 4 in H_B and $\{u_2, u_3, u_4\}$ is a subset of the neighborhoods of exactly two vertices of degree 4 in H_B. We may assume that

- ⋆ $N_{H_B}(w_8) = \{u_2, u_3, u_4, u_5\}$, $N_{H_B}(w_9) = \{u_2, u_3, u_4, u_6\}$ and
- ⋆ $N_{H_B}(w_{10}) = \{u_1, u_3, u_4, u_5\}$ or $N_{H_B}(w_{10}) = \{u_1, u_3, u_4, u_6\}$.

We can assume that $N_{H_B}(w_{10}) = \{u_1, u_3, u_4, u_5\}$, as the argument for the case when $N_{H_B}(w_{10}) = \{u_1, u_3, u_4, u_6\}$ is similar. In this subcase, each of $\{u_1, u_2, u_3\}$ and $\{u_1, u_2, u_4\}$ is the neighborhood of exactly one vertex of W having degree 4 in H_B. Hence, we may assume that

- ⋆ $N_{H_B}(w_{12}) = \{u_1, u_2, u_3, u_5\}$ or $N_{H_B}(w_{12}) = \{u_1, u_2, u_3, u_6\}$ and
- ⋆ $N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_5\}$ or $N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_6\}$.

First, suppose that $N_{H_B}(w_{12}) = \{u_1, u_2, u_3, u_5\}$.

- ⋆ If $N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_5\}$, then we consider the 3-element subset $\{u_4, u_5, u_6\}$ and we may assume that $N_{H_B}(w_{14}) = \{u_i, u_4, u_5, u_6\}$ for $i \in \{1, 2, 3\}$. By a similar argument in Case 3.1, there is a blue $K_{3,3}$.
- ⋆ If $N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_6\}$, then we consider the 3-element subset $\{u_3, u_5, u_6\}$ and we may assume that $N_{H_B}(w_{14}) = \{u_i, u_3, u_5, u_6\}$ for some $i \in \{1, 2, 4\}$. By a similar argument in Case 3.1, there is a blue $K_{3,3}$.

Next, suppose that $N_{H_B}(w_{12}) = \{u_1, u_2, u_3, u_6\}$.

- ⋆ If $N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_5\}$, then we consider the 3-element subset $\{u_4, u_5, u_6\}$ and we may assume that $N_{H_B}(w_{14}) = \{u_i, u_4, u_5, u_6\}$ for some $i \in \{1, 2, 3\}$. By a similar argument in Case 3.1, there is a blue $K_{3,3}$.
- ⋆ If $N_{H_B}(w_{13}) = \{u_1, u_2, u_4, u_6\}$, then we consider the 3-element subset $\{u_2, u_5, u_6\}$ and we may assume that $N_{H_B}(w_{14}) = \{u_i, u_2, u_5, u_6\}$ for some $i \in \{1, 3, 4\}$. By a similar argument in Case 3.1, there is a blue $K_{3,3}$. ■

3.4 On the Numbers $BR_s(K_{2,3}, K_{3,3})$ for $s \geq 8$

In this section, we show that $BR_s(K_{2,3}, K_{3,3})$ is either 13 or 14 when s is 8 or 9. First, we present an upper bound for $BR_s(K_{2,3}, K_{3,3})$ for all s with $8 \leq s \leq BR(K_{2,3}, K_{3,3})$. It is well known for integers r and n with $1 \leq r \leq n-1$ that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ or, equivalently, $\binom{n}{r} - \binom{n-1}{r} = \binom{n-1}{r-1}$. In particular, if $r = 3$ and $n \geq 4$, then

$$\binom{n}{3} - \binom{n-1}{3} = \binom{n-1}{2} \text{ is an increasing function of } n. \tag{3.3}$$

Theorem 3.4.1 *If $8 \leq s \leq BR(K_{2,3}, K_{3,3})$, then $BR_s(K_{2,3}, K_{3,3}) \leq 14$.*

Proof. It suffices to show that every red-blue coloring of $H = K_{8,14}$ produces a red $K_{2,3}$ or a blue $K_{3,3}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B having sizes m_R and m_B, respectively. Let $U = \{u_1, u_2, \ldots, u_8\}$ and $W = \{w_1, w_2, \ldots, w_{14}\}$ be the partite sets of H. First, we verify the following.

(a) If $\displaystyle\sum_{i=1}^{14} \binom{\deg_{H_R} w_i}{2} \geq 57$, then H contains a red $K_{2,3}$.

(b) If $\displaystyle\sum_{i=1}^{14} \binom{\deg_{H_B} w_i}{3} \geq 113$, then H contains a blue $K_{3,3}$.

If $\sum_{i=1}^{14} \binom{\deg_{H_R} w_i}{2} \geq 57$, then the vertices of W are joined to at least 57 pairs of vertices of U (with repetitions) by red edges. Since the number of distinct 2-element subsets in U is $\binom{8}{2} = 28$, at least three vertices of W are joined to the same two vertices of U by red edges. Thus, there is a red $K_{2,3}$ and so (a) holds. If $\sum_{i=1}^{14} \binom{\deg_{H_B} w_i}{3} \geq 113$, then the vertices of W are joined to at least 113 triples of vertices of U (with repetitions) by blue edges. Since the number of distinct 3-element subsets in U is $\binom{8}{3} = 56$, at least three vertices of W are joined to the same three vertices of U by blue edges. Thus, there is a blue $K_{3,3}$ and so (b) holds.

We may assume that

$$\deg_{H_R} w_1 \geq \deg_{H_R} w_2 \geq \cdots \geq \deg_{H_R} w_{14}.$$

First, suppose that W has five vertices of degree 4 in H_R and nine vertices of degree 3 in H_R. Thus, the degree sequence of the vertices of W in H_R is

$$4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3. \tag{3.4}$$

Then $m_R = 5 \cdot 4 + 9 \cdot 3 = 20 + 27 = 47$. Since

$$\sum_{i=1}^{14} \binom{\deg_{H_R} w_i}{2} = 5\binom{4}{2} + 9\binom{3}{2} = 5 \cdot 6 + 9 \cdot 3 = 57,$$

there is a red $K_{2,3}$ by (a). Next, we show that if $m_R \geq 47$ and the degree sequence of the vertices of W in H_R is not that in (3.4), then $\sum_{i=1}^{15} \binom{\deg_{H_R} w_i}{2} > 57$. It suffices to verify this fact when $m_R = 47$. Let $\deg_{H_R} w_i = a_i$ for $1 \leq i \leq 14$, where then $a_1 \geq a_2 \geq \cdots \geq a_{14}$ and $\sum_{i=1}^{14} a_i = 47$. Suppose that $\{a_i\}$ is not the sequence as described in (3.4). Now, let $\{b_i\} = \{b_1, b_2, \ldots, b_{14}\}$ be the sequence in (3.4), where then $b_i = 4$ for $1 \leq i \leq 5$ and $b_i = 3$ for $6 \leq i \leq 14$. Then $\{a_i\}$ can be obtained from $\{b_i\}$ by replacing two terms b_i and b_j with $i < j$ by $b_i + 1$ and $b_j - 1$ (perhaps multiple times). Then $\sum_{i=1}^{11} a_i = \sum_{i=1}^{11} b_i = 47$. However, since

$$\binom{b_i+1}{2} - \binom{b_j-1}{2} > \binom{b_i}{2} - \binom{b_j}{2},$$

it follows that

$$\sum_{i=1}^{11} \binom{a_i}{2} > \sum_{i=1}^{11} \binom{b_i}{2} = 57.$$

By (a), there is a red $K_{2,3}$.

Hence, we may assume that $m_R \leq 46$ and so

$$m_B = \sum_{i=1}^{14} \deg_{H_B} w_i = 112 - m_R \geq 66.$$

First, suppose that $m_B = 66$ and W has four vertices of degree 4 and ten vertices of degree 5. Thus, the degree sequence of the vertices of W in H_B is

$$4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5. \tag{3.5}$$

Since

$$\sum_{i=1}^{14} \binom{\deg_{H_B} w_i}{3} = 4\binom{4}{3} + 10\binom{5}{3} = 4 \cdot 4 + 10 \cdot 10 = 116,$$

there is a blue $K_{3,3}$ by (b). Next, we show that if $m_B \geq 66$ and the degree sequence of the vertices of W in H_B is not that in (3.5), then $\sum_{i=1}^{14} \binom{\deg_{H_R} w_i}{3} > 113$. It suffices to verify this fact when $m_B = 66$. Let $\deg_{H_B} w_{14-i} = a_i$ for $0 \leq i \leq 13$, where then $a_1 \geq a_2 \geq \cdots \geq a_{14}$ and $\sum_{i=1}^{14} a_i = 66$. Suppose that $\{a_i\}$ is not $5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 4, 4, 4, 4$, namely, $\{a_i\}$ is not the reverse sequence of the sequence in (3.5). Now, let $\{b_i\} = \{b_1, b_2, \ldots, b_{14}\}$ be the reverse sequence of the sequence in (3.5), where then $b_i = 5$ for $1 \leq i \leq 10$ and $b_i = 4$ for $11 \leq i \leq 14$. Then $\{a_i\}$ can be obtained from $\{b_i\}$ by replacing two terms b_i and b_j with $i < j$ by $b_i + 1$ and $b_j - 1$ (perhaps multiple times). Then

$$\sum_{i=1}^{14} a_i = \sum_{i=1}^{14} b_i = 66.$$

However, since $\binom{b_i+1}{3} - \binom{b_j-1}{3} > \binom{b_i}{3} - \binom{b_j}{3}$ by (3.3), it follows that

$$\sum_{i=1}^{14} \binom{a_i}{3} > \sum_{i=1}^{14} \binom{b_i}{3} = 116 > 113.$$

Therefore, there is a blue $K_{3,3}$ by (b). ■

Next, we establish a lower bound for the s-bipartite Ramsey numbers of $K_{2,3}$ and $K_{3,3}$ where $s = 8, 9$.

Proposition 3.4.2 *If $s = 8, 9$, then $BR_s(K_{2,3}, K_{3,3}) \geq 13$.*

Proof. First, we show that there exists a red-blue coloring of $K_{9,12}$ that avoids both a red $K_{2,3}$ and a blue $K_{3,3}$. Let $U = \{u_1, u_2, \ldots, u_9\}$ and $W = \{w_1, w_2, \ldots, w_{12}\}$ be the partite sets of $G = K_{9,12}$. Consider the following twelve subsets $U_1, U_2, \ldots, U_{12}$ of U and let $\overline{U}_i = U - U_i$ for $1 \leq i \leq 12$. Here again, we denote a set $\{u_a, u_b, u_c, u_d, u_e\}$ by $abcde$, for example.

U_i $(1 \leq i \leq 6)$:	12349	1256	1357	1467	56789	3478
$\overline{U}_i$ $(7 \leq i \leq 12)$:	5678	34789	24689	23589	1234	12569
U_i $(1 \leq i \leq 6)$:	2468	2358	189	279	369	459
$\overline{U}_i$ $(7 \leq i \leq 12)$:	13579	14679	234567	134568	124578	123678

We now define a red-blue coloring of G where w_i $(1 \leq i \leq 12)$ is joined to the vertices in U_i by red edges and to the remaining vertices in $\overline{U}_i$ by blue edges. The red-neighborhood of w_i is $N_R(w_i) = U_i$ and the blue-neighborhood of w_i is $N_B(w_i) = \overline{U}_i$ for $1 \leq i \leq 12$. Since each 2-element subset of U appears at most twice in U_i for $1 \leq i \leq 12$ and each 3-element subset of U appears at most twice in $\overline{U}_i$ for $1 \leq i \leq 12$, it follows that

(1) there is no red $K_{2,3}$ in which the two vertices of degree 3 belong to W and

(2) there is no blue $K_{3,3}$.

Thus, it remains to show that there is no red $K_{2,3}$ in which the two vertices of degree 3 belong to U. For each integer j with $1 \leq j \leq 9$, let $W_j = N_R(u_j)$. These seven subsets of W are listed below, where $10, 11, 12$ are denoted by A, B, C.

12349	1278A	1368B	1467C	2358C
2457B	3456A	56789	15ABC	

Since each 2-element subset of W appears at most twice in W_j for $1 \leq j \leq 8$, there is no red $K_{2,3}$ in which the two vertices of degree 3 belong to U. Therefore, this coloring of $K_{9,12}$ avoids both a red $K_{2,3}$ and a blue $K_{3,3}$ and so $BR_9(K_{2,3}, K_{3,3}) \geq 13$. This also implies that $BR_8(K_{2,3}, K_{3,3}) \geq 13$. ■

It is therefore a consequence of Theorem 3.4.1 and Proposition 3.4.2 that the number $BR_s(K_{2,3}, K_{3,3})$ is either 13 or 14, when $s \in \{8, 9\}$. For the number $BR_{10}(K_{2,3}, K_{3,3})$, we are able to present a lower bound.

Proposition 3.4.3 $BR_{10}(K_{2,3}, K_{3,3}) \geq 11$.

Proof. There exists a red-blue coloring of $K_{10,10}$ that avoids both a red $K_{2,3}$ and a blue $K_{3,3}$. Let $U = \{u_1, u_2, \ldots, u_{10}\}$ and $W = \{w_1, w_2, \ldots, w_{10}\}$ be the partite sets of $G = K_{10,10}$. Consider the following twelve subsets $U_1, U_2, \ldots, U_{10}$ of U and let $\overline{U}_i = U - U_i$ for $1 \leq i \leq 10$, where 10 is denoted by A. Recall that we denote a set $\{u_a, u_b, u_c, u_d, u_e\}$ by $abcde$, for example.

U_i $(1 \leq i \leq 5)$:	12349	1256	1357A	1467	56789
$\overline{U}_i$ $(1 \leq i \leq 5)$:	5678A	34789A	24689	23589A	12569A
U_i $(6 \leq i \leq 10)$:	3478	2468A	2358	189A	279A
$\overline{U}_i$ $(6 \leq i \leq 10)$:	2569A	13579	14679A	234567	134568

We now define a red-blue coloring of G where w_i $(1 \leq i \leq 10)$ is joined to the vertices in U_i by red edges and to the remaining vertices in $\overline{U}_i$ by blue edges. The red-neighborhood of w_i is $N_R(w_i) = U_i$ and the blue-neighborhood of w_i is $N_B(w_i) = \overline{U}_i$ for $1 \leq i \leq 10$. Since each 2-element subset of U appears at most twice in U_i for $1 \leq i \leq 10$ (in fact, 18, 27 appear once, 36, 45 don't appear and all other pairs appear exactly twice), each 3-element subset of U appears at most twice in $\overline{U}_i$ for $1 \leq i \leq 10$. This implies that (1) there is no blue $K_{3,3}$ and (2) there is no red $K_{2,3}$ in which the two vertices of degree 3 belong to W. Thus, it remains to show that there is no red $K_{2,3}$ in which the two vertices of degree 3 belong to U. For each integer j with $1 \leq j \leq 10$, let $W_j = N_R(u_j)$. These ten subsets of W are listed below.

12349	1278A	1368	1467	2358
2457	3456A	56789	159A	379A

Since each 2-element subset of W appears at most twice in W_j for $1 \leq j \leq 8$ (in fact, 15, 37 appear once, 26, 48 don't appear and all other pairs appear exactly twice), there is no red $K_{2,3}$ in which the two vertices of degree 3 belong to U. Therefore, this coloring of $K_{10,10}$ avoids both a red $K_{2,3}$ and a blue $K_{3,3}$ and so $BR_{10}(K_{2,3}, K_{3,3}) \geq 11$. ■

While the value of the bipartite Ramsey number $BR(K_{2,3}, K_{3,3})$ is unknown, Proposition 3.4.3 implies that $BR(K_{2,3}, K_{3,3}) \geq 11$. In summary then, we have the following result.

Theorem 3.4.4 *For each positive integer s,*

$$BR_s(K_{2,3}, K_{3,3}) = \begin{cases} \textit{does not exist} & \textit{if } 1 \leq s \leq 3 \\ 21 & \textit{if } s = 4, 5 \\ 15 & \textit{if } s = 6, 7 \\ 13 \textit{ or } 14 & \textit{if } s = 8, 9 \\ 11, 12, 13 \textit{ or } 14 & \textit{if } 10 \leq s \leq BR(K_{2,3}, K_{3,3}). \end{cases}$$

Furthermore, $11 \leq BR(K_{2,3}, K_{3,3}) \leq 14$.

We conclude with the following conjecture.

Conjecture 3.4.5 *If $s = 10, 11$, then $BR_{10}(K_{2,3}, K_{3,3}) = 11$.*

If Conjecture 3.4.5 is true, then $BR_s(K_{2,3}, K_{3,3}) = s$ for $s \geq 11$.

Chapter 4

On s-Bipartite Ramsey Numbers Of Forests

4.1 Introduction

A *forest* is a graph whose components are trees. Therefore, a tree is itself a forest. In particular, stars, stripes and paths are all forests. In this chapter, we study The s-bipartite Ramsey numbers $BR_s(F, H)$ where F and H are special classes of forests.

4.2 Stars and Matchings

In this section, we determine the s-bipartite Ramsey numbers $BR_s(F, H)$ of bipartite graphs F and H for all positive integers s where each of F and H is either a star or a matching (also referred to as *stripes*). We beginning with the case when F and H are both stars. For an integer $n \geq 2$, a *star of size* n is denoted by $K_{1,n}$.

Theorem 4.2.1 *For integers $m, n, s \geq 2$,*

$$BR_s(K_{1,m}, K_{1,n}) = \begin{cases} m+n-1 & \text{if } 2 \leq s \leq m+n-2 \\ s & \text{otherwise.} \end{cases}$$

Proof. We consider two cases, according to whether $2 \leq s \leq m+n-2$ or $s \geq m+n-1$.

Case 1. $2 \leq s \leq m+n-2$. Let there be given a red-blue coloring of $H = K_{s,m+n-1}$ resulting in the red subgraph H_R and the blue subgraph H_B. Let U and W be the partite sets of H with $|U| = s$ and $|W| = m+n-1$. Now let $u \in U$. If $\deg_{H_R} u \geq m$, then H contains a red $K_{1,m}$; while if $\deg_{H_R} u \leq m-1$, then $\deg_{H_B} u \geq (m+n-1)-(m-1) = n$ and so H contains a blue $K_{1,n}$. Therefore, $BR_s(K_{1,m}, K_{1,n}) \leq m+n-1$.

Next, we show that there exists a red-blue coloring of $G = K_{s,m+n-2}$ that avoids both a red $K_{1,m}$ and a blue $K_{1,n}$. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{m+n-2}\}$ be the partite sets of G. For each integer i with $1 \leq i \leq s$, assign the color red to the $m-1$ edges $u_iw_i, u_iw_{i+1}, \ldots, u_iw_{i+m-2}$ incident with u_i, where the subscripts of vertices are expressed as integers modulo $m+n-2$, and assign the color blue to the remaining edges of G. For $m = 5$, $n = 3$ and $s = 4$, this red-blue coloring of $K_{4,6}$ is illustrated in Figure 4.1 where only the red edges of $K_{4,6}$ are shown.

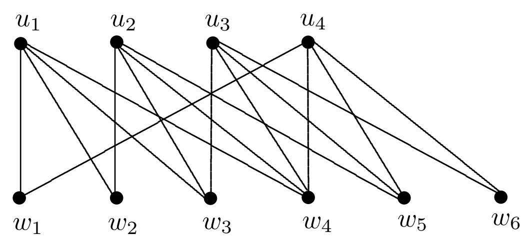


Figure 4.1: The red subgraph of $K_{4,6}$

Let G_R and G_B be the resulting red and blue subgraphs of G, respectively. If $u \in U$, then $\deg_{G_R} u = m-1$ and $\deg_{G_B} u = n-1$. Thus, this red-blue coloring of G produces neither a red $K_{1,m}$ nor a blue $K_{1,n}$ whose central vertex belongs to U. By this construction,

$$\begin{aligned} \max\{\deg_{G_R} w : w \in W\} &= \deg_{G_R} w_s \leq m-1 & (4.1)\\ \min\{\deg_{G_R} w : w \in W\} &= \deg_{G_R} w_{m+n-2} \geq 0. & (4.2) \end{aligned}$$

Hence, $0 \leq \delta(G_R) \leq \Delta(G_R) \leq m-1$ and so there is no red $K_{1,m}$ whose central vertex belongs to W. Let $\deg_{G_R} w_{m+n-2} = k$. Since $\deg_{G_R} w + \deg_{G_B} w = s$ for each $w \in W$, it follows that

$$\deg_{G_B} w_{m+n-2} = s - \deg_{G_R} w_{m+n-2} = s - k. \tag{4.3}$$

First, suppose that $k \geq 1$. Since $N_{G_R}(u_{s-k+1}) = \{w_{s-k+1}, w_{s-k+2}, \ldots, w_{m+n-2}\}$ and $\deg_{G_R} u_{s-k+1} = m-1$, it follows that $(m+n-2)-(s-k+1)+1 = m-1$ and so $s = n+k-1$. It then follows by (4.3) that $\deg_{G_B} w_{m+n-2} = n-1$ and so $\deg_{G_B} w \leq n-1$ for each $w \in W$ by (4.2). Next, suppose that $k = 0$. Since u_s is adjacent to $w_s, w_{s+1}, \ldots, w_{s+(m-2)}$, it follows that $s+(m-2) < m+n-2$ and so $s \leq n-1$. Hence, $\deg_{G_B} w \leq \deg_G w = s \leq n-1$ for each $w \in W$. Therefore, there is no blue $K_{1,n}$ whose central vertex belongs to W.

Hence, there is neither a red $K_{1,m}$ nor a blue $K_{1,n}$ in G. Therefore, $BR_s(K_{1,m}, K_{1,n}) \geq m+n-1$ and so $BR_s(K_{1,m}, K_{1,n}) = m+n-1$ when $2 \leq s \leq m+n-2$.

Case 2. $s \geq m + n - 1$. We show that every red-blue coloring of $H = K_{s,s}$ produces either a red $K_{1,m}$ or a blue $K_{1,n}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let U and W be the partite sets of H with $|U| = |W| = s$. Let v be any vertex of H, say $v \in U$. If $\deg_{H_R} v \geq m$, then H contains a red $K_{1,m}$; while if $\deg_{H_R} v \leq m - 1$, then $\deg_{H_B} v \geq s - (m - 1) \geq (m + n - 1) - (m - 1) = n$ and so H contains a blue $K_{1,n}$. Therefore, $BR_s(K_{1,m}, K_{1,n}) = s$. ■

Next, we determine the s-bipartite Ramsey numbers $BR_s(F, H)$ when F and H are both matchings. For an integer $n \geq 2$, a *matching of size n* is denoted by nK_2, which consists of n independent edges. For two disjoint sets X and Y of vertices of a graph G, the set of edges joining a vertex of X and a vertex of Y in G is denoted by $G[X, Y]$ or, more simply, by $[X, Y]$ if the graph G under discussion is clear.

Theorem 4.2.2 *For integers $m, n, s \geq 2$,*

$$BR_s(mK_2, nK_2) = \begin{cases} \text{does not exist} & \text{if } 2 \leq s \leq m+n-2 \\ s & \text{otherwise.} \end{cases}$$

Proof. First, suppose that $2 \leq s \leq m + n - 2$. Let t be an integer where $t \geq s$. We show that there is a red-blue coloring of $G = K_{s,t}$ that produces neither a red mK_2 nor a blue nK_2. If $s \leq m - 1$, then assign the color red to each edge of G. This produces a red-blue coloring that avoids both a red mK_2 and a blue nK_2. Thus, we may assume that $s \geq m$. Let U and W be the partite sets of G with $|U| = s$ and $|W| = t$. Now partition the set U into two subsets U_1 and U_2 where $|U_1| = m-1$ and $|U_2| = s-m+1$. Assign the color red to each edge in $[U_1, W]$ and the color blue to each edge in $[U_2, W]$. This red-blue coloring results in the red subgraph $G_R = K_{m-1,t}$ and the blue subgraph $G_B = K_{s-m+1,t} \subseteq K_{n-1,t}$ (since $s-m+1 \leq (m+n-2)-m+1 = n-1$). Hence, there is neither a red mK_2 nor a blue nK_2 and so $BR_s(mK_2, nK_2)$ does not exist.

Next, suppose that $s \geq m+n-1$. We show that every red-blue coloring of $H = K_{s,s}$ produces either a red mK_2 or a blue nK_2. Let there be given a red-blue coloring of H and let M be a perfect matching in H. Thus, $|M| = s$. If there are m edges in M that are colored red, then there is a red mK_2; otherwise, at most $m - 1$ edges in M are red and so at least $s-(m-1) \geq (m+n-1)-(m-1) = n$ edges in M are blue, producing a blue nK_2. Therefore, $BR_s(mK_2, nK_2) = s$. ■

Bipartite Ramsey numbers $BR(F, H)$ when one of F and H is a star and the other is a matching were studied in [16] and the following was obtained.

Theorem 4.2.3 [16] *For integers $m, n \geq 2$, $BR(K_{1,m}, nK_2) = m + \left\lfloor \frac{n-1}{2} \right\rfloor$.*

We now determine $BR_s(F,H)$ when one of F and H is a star and the other is a matching, beginning with conditions under which these numbers do not exist.

Proposition 4.2.4 *For integers $m, n, s \geq 2$, if $s \leq n-1$ or $s \leq m-1 \leq n-1$, then $BR_s(K_{1,m}, nK_2)$ does not exist.*

Proof. Suppose that $2 \leq s \leq n-1$. For an arbitrary integer t, the red-blue coloring of $K_{s,t}$ that assigns the color blue to each edge of $K_{s,t}$ produces neither a red $K_{1,m}$ nor a blue nK_2. Therefore, $BR_s(K_{1,m}, nK_2)$ does not exist when $s \leq n-1$ as well as when $s \leq m-1 \leq n-1$. ■

Under any other conditions, the numbers $BR_s(K_{1,m}, nK_2)$ always exist.

Proposition 4.2.5 *If m, n, s are integers with $2 \leq n \leq s \leq m-1$, then*

$$BR_s(K_{1,m}, nK_2) = m+n-1.$$

Proof. First, we show that $BR_s(K_{1,m}, nK_2) \geq m+n-1$; that is, there is a red-blue coloring of $G = K_{s,m+n-2}$ that produces neither a red $K_{1,m}$ nor a blue nK_2. Let U and W be the partite sets of G with $|U| = s$ and $|W| = m+n-2$. Partition the partite set W into two subsets W_1 and W_2 with $|W_1| = m-1$ and $|W_2| = n-1$. Define a red-blue coloring of G by assigning the color red to each edge in $[U, W_1]$ and the color blue to each edge in $[U, W_2]$. Then the red subgraph is $G_R = K_{s,m-1}$ and the blue subgraph is $G_B = K_{s,n-1}$. Since $s \leq m-1$ and the maximum matching in G_B has size $n-1$, there is neither a red $K_{1,m}$ in G_R nor a blue nK_2 in G_B. Therefore, $BR_s(K_{1,m}, nK_2) \geq m+n-1$.

To verify that $BR_s(K_{1,m}, nK_2) \leq m+n-1$, we show every red-blue coloring of $H = K_{s,m+n-1}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{m+n-1}\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \geq n$, then we obtain a blue nK_2. So we may assume that $|M| \leq n-1$. Suppose that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let $U_1 = \{u_1, u_2, \ldots, u_{|M|}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{|M|}\}$. Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that $H[U_2, W_2] = K_{s-|M|,\ m+n-1-|M|} \subseteq H_R$. Since $|M| \leq n-1$, it follows that $m+n-1-|M| \geq m+n-1-(n-1) = m$. So there is a red $K_{1,m}$ in H. Thus, every red-blue coloring of $K_{s,m+n-1}$ results in a red $K_{1,m}$ or a blue nK_2 and so $BR_s(K_{1,m}, nK_2) \leq m+n-1$. Therefore, $BR_s(K_{1,m}, nK_2) = m+n-1$. ■

Proposition 4.2.6 *If m, n, s are integers with $m, n \geq 2$ and $s \geq m + \left\lfloor \frac{n-1}{2} \right\rfloor$, then*

$$BR_s(K_{1,m}, nK_2) = s.$$

Proof. By the definition of s-bipartite Ramsey number, $BR_s(K_{1,m}, nK_2) \geq s$. Hence, we need only show that $BR_s(K_{1,m}, nK_2) \leq s$, that is, every red-blue coloring of $H = K_{s,s}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \geq n$, then we obtain a blue nK_2. If $|M| \leq \lfloor \frac{n-1}{2} \rfloor$, then we may assume that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let $U_1 = \{u_1, u_2, \ldots, u_{|M|}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{|M|}\}$. Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that $H[U_2, W_2] = K_{s-|M|,\ s-|M|} \subseteq H_R$. Since $|M| \leq \lfloor \frac{n-1}{2} \rfloor$ and $s \geq m + \lfloor \frac{n-1}{2} \rfloor$, it follows that $s - |M| \geq m + \lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{2} \rfloor = m$. So there is a red $K_{1,m}$ in H. Thus, we may assume that $\lfloor \frac{n-1}{2} \rfloor + 1 \leq |M| \leq n - 1$. For each vertex $w \in W_2$,

$$\deg_{H_R} w \geq s - |M| \geq m + \left\lfloor \tfrac{n-1}{2} \right\rfloor - |M| = m - 1 - (|M| - \left\lfloor \tfrac{n-1}{2} \right\rfloor - 1).$$

If w is joined to at least $|M| - \lfloor \frac{n-1}{2} \rfloor$ vertices in U_1 by red edges, then there is a red $K_{1,m}$ in H. Otherwise, each vertex in W_2 is joined to at most $|M| - \lfloor \frac{n-1}{2} \rfloor - 1$ vertices in U_1 by red edges; so each vertex in W_2 is joined to at least $\lfloor \frac{n-1}{2} \rfloor + 1$ vertices in U_1 by blue edges. Assume, without loss of generality, that $u_iw_{|M|+1}$ is blue for each i with $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor + 1$. If there is an integer j with $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor + 1$ such that $u_{|M|+1}w_j$ is blue, say $u_{|M|+1}w_1$ is blue, then there is a matching

$$M' = \{u_{|M|+1}w_1, u_1w_{|M|+1}\} \cup \{u_iw_i : 2 \leq i \leq |M|\}$$

whose size is larger than M, a contradiction. Hence, $u_{|M|+1}w_j$ is red for all j with $1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor + 1$. This implies that

$$\begin{aligned}
\deg_{H_R} u_{|M|+1} &\geq s - |M| + \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \geq (m + \left\lfloor \frac{n-1}{2} \right\rfloor) - (n-1) + \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \\
&= m - n + 2 + 2\left\lfloor \frac{n-1}{2} \right\rfloor \geq m - n + 2 + (n-2) = m.
\end{aligned}$$

Thus, there is a red $K_{1,m}$ whose central vertex is $u_{|M|+1}$ in H. Consequently, every red-blue coloring of $K_{s,s}$ results in a red $K_{1,m}$ or a blue nK_2 and so $BR_s(K_{1,m}, nK_2) \leq s$. Therefore, $BR_s(K_{1,m}, nK_2) = s$. ■

For two vertex-disjoint graphs G and H, let $G + H$ denote the *union* of G and H.

Theorem 4.2.7 *If m, n, s are integers with $3 \le n < m \le s \le m + \left\lfloor \frac{n-1}{2} \right\rfloor - 1$, then*

$$BR_s(K_{1,m}, nK_2) = 2(m-1) + n - s.$$

Proof. Since $m \le s \le m + \left\lfloor \frac{n-1}{2} \right\rfloor - 1$, we can write $s = m + j$ for some integer j with $0 \le j \le \left\lfloor \frac{n-1}{2} \right\rfloor - 1$. Then $2(m-1) + n - s = m + n - 2 - j$.

First, we show that $BR_s(K_{1,m}, nK_2) \ge m+n-2-j$; that is, we show that there is a red-blue coloring of $G = K_{s,m+n-3-j}$ that produces neither a red $K_{1,m}$ nor a blue nK_2. Let U and W be the partite sets of G with $|U| = s = m + j$ and $|W| = m + n - 3 - j$. Partition the partite set U into three subsets U_1, U_2 and U_3 and the partite set W into three subsets W_1, W_2 and W_3, where

$$\begin{aligned}
&|U_1| = |W_1| = n - 1 - (j+1) = n - j - 2 \\
&|U_2| = |W_2| = j + 1 \\
&|U_3| = s - (n-1) = m + j - (n-1) = m + j - n + 1 \\
&|W_3| = m + n - 3 - j - (n-1) = m - j - 2.
\end{aligned}$$

Define a red-blue coloring of G by assigning the color blue to each edge in the set $[U_1 \cup U_3, W_1] \cup [U_2, W_2 \cup W_3]$ and the color red to the remaining edges of G. Let G_B and G_R be the resulting blue and red subgraphs of G. Observe that

$$\begin{aligned}
G_B &= G[U_1 \cup U_3, W_1] + G[U_2, W_2 \cup W_3] = K_{n-1-(j+1),\ m-1} + K_{j+1,\ m-1} \\
G_R &= G[U_1 \cup U_3, W_2 \cup W_3] + G[U_2, W_1] = K_{m-1,\ m-1} + K_{n-1-(j+1),\ j+1}.
\end{aligned}$$

Since there is neither a red $K_{1,m}$ in G_R nor a blue nK_2 in G_B, it follows that

$$BR_s(K_{1,m}, nK_2) \ge m + n - 2 - j.$$

To verify that $BR_s(K_{1,m}, nK_2) \le m+n-2-j$, we show that every red-blue coloring of $H = K_{s,m+n-2-j}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s = u_{m+j}\}$ and $W = \{w_1, w_2, \ldots, w_{m+n-2-j}\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \ge n$, then we obtain a blue nK_2. If $|M| \le n - j - 2$, then we may assume that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let $U_1 = \{u_1, u_2, \ldots, u_{|M|}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{|M|}\}$. Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that $H[U_2, W_2] = K_{s-|M|,\ m+n-2-j-|M|} \subseteq H_R$. Since $|M| \le n-j-2$ and $j \le \left\lfloor \frac{n-1}{2} \right\rfloor - 1$, it follows that

$$m + n - 2 - j - |M| \ge m + n - 2 - j - (n - j - 2) = m.$$

So there is a red $K_{1,m}$ in H. Thus, we may assume that $n-j-1 \leq |M| \leq n-1$. For each vertex $u \in U_2$, it follows that $\deg_{H_R} = m+n-2-j-|M|$. If u is joined to at least $|M|-(n-j-1)+1$ vertices in W_1 by red edges, then there is a red $K_{1,m}$ in H. Thus, each vertex in U_2 is joined to at most $|M|-(n-j-1)$ vertices in W_1 by red edges; so each vertex in U_2 is joined to at least $n-j-1$ vertices in W_1 by blue edges. Assume, without loss of generality, that $u_{|M|+1}w_i$ is blue for each i with $1 \leq i \leq n-j-1$. If there is an integer i with $1 \leq i \leq n-j-1$ such that $w_{|M|+1}u_i$ is blue, say $w_{|M|+1}u_1$ is blue, then there is a matching $M' = \{w_{|M|+1}u_1, u_{|M|+1}w_1\} \cup \{u_iw_i : 2 \leq i \leq |M|\}$ whose size is larger than $|M|$, a contradiction. Hence, $w_{|M|+1}u_i$ is red for all i with $1 \leq i \leq n-j-1$. This implies that

$$\begin{aligned}\deg_{H_R} w_{|M|+1} &\geq m+j-|M|+n-j-1 = m+n-1-|M| \\ &\geq m+n-1-(n-1) = m.\end{aligned}$$

So there is a red $K_{1,m}$ whose central vertex is $w_{|M|+1}$ in H. Thus, $BR_s(K_{1,m}, nK_2) \leq m+n-2-j$, and therefore, $BR_s(K_{1,m}, nK_2) = m+n-2-j$. ■

Theorem 4.2.8 *Let n and m be integers with $n \geq m \geq 3$. If s is an integer with $n \leq s \leq m + \left\lfloor \frac{n-1}{2} \right\rfloor - 1$, then*

$$BR_s(K_{1,m}, nK_2) = \begin{cases} 2(m-1)+n-s & \textit{if } m \leq n \leq 2m-3 \\ s & \textit{if } n \geq 2m-2. \end{cases}$$

Proof. We consider two cases, according to whether $m \leq n \leq 2m-3$ or $n \geq 2m-2$.

Case 1. $m \leq n \leq 2m-3$. First, observe that since $m + \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \geq s$, it follows that $2(m-1)+n-s \geq s+1$. Suppose that $s = n+j$ for some integer j with $0 \leq j \leq m + \left\lfloor \frac{n-1}{2} \right\rfloor - 1 - n$. We show that $BR_s(K_{1,m}, nK_2) = 2m-2-j$. First, we show that $BR_s(K_{1,m}, nK_2) \geq 2m-2-j$; that is, we show that there is a red-blue coloring of $G = K_{s,2m-3-j}$ resulting in neither a red $K_{1,m}$ nor a blue nK_2. Let U and W be the partite sets of G with $|U| = s = n+j$ and $|W| = 2m-3-j$. Partition the partite set U into three subsets U_1, U_2 and U_3 and the partite set W into three subsets W_1, W_2 and W_3, where

$$\begin{aligned}&|U_1| = |W_1| = |W_3| = m-j-2 \\ &|U_2| = j+1+n-m \\ &|U_3| = |W_2| = j+1.\end{aligned}$$

Define a red-blue coloring of G by assigning the color blue to each edge in the set $[U_1 \cup U_3, W_1] \cup [U_2, W_2 \cup W_3]$ and the color red to the remaining edges of G. Let G_B and G_R be the resulting blue and red subgraphs of G. Observe that

$$\begin{aligned} G_B &= G[U_1 \cup U_3, W_1] + G[U_2, W_2 \cup W_3] = K_{m-j-2,\ m-1} + K_{j+1+n-m,\ m-1} \\ G_R &= G[U_1 \cup U_3, W_2 \cup W_3] + G[U_2, W_1] = K_{m-1,\ m-1} + K_{m-j-2,\ j+1+n-m}. \end{aligned}$$

Since $j \leq m + \left\lfloor \frac{n-1}{2} \right\rfloor - 1 - n$ and $n \leq 2m - 3$, it follows that

$$\begin{aligned} j + 1 + n - m &\leq m + \left\lfloor \frac{n-1}{2} \right\rfloor - 1 - n + 1 + n - m \\ &= \left\lfloor \frac{n-1}{2} \right\rfloor \leq \left\lfloor \frac{2m-4}{2} \right\rfloor = \lfloor m - 2 \rfloor < m. \end{aligned}$$

Thus, there is neither a red $K_{1,m}$ in G_R nor a blue nK_2 in G_B. Therefore,

$$BR_s(K_{1,m}, nK_2) \geq 2m - 2 - j.$$

To verify that $BR_s(K_{1,m}, nK_2) \leq 2m - 2 - j$, we show that every red-blue coloring of $H = K_{s,2m-2-j}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s = u_{n+j}\}$ and $W = \{w_1, w_2, \ldots, w_{2m-2-j}\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \geq n$, then we obtain a blue nK_2. If $|M| \leq j + n - m$, then we may assume that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let $U_1 = \{u_1, u_2, \ldots, u_{|M|}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{|M|}\}$. Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that $H[U_2, W_2] = K_{s-|M|,\ 2m-2-j-|M|} \subseteq H_R$. Since $|M| \leq j + n - m$, it follows that $s - |M| = n + j - |M| \geq n + j - (j + n - m) = m$. So there is a red $K_{1,m}$ in H. Thus, we may assume that $j + 1 + n - m \leq |M| \leq n - 1$. If there is $w \in W_2$ such that w is joined to at least $|M| - (j + 1 + n - m) + 1$ vertices in U_1 by red edges, then there is a red $K_{1,m}$ in H. Thus, each vertex in W_2 is joined to at most $|M| - (j + 1 + n - m)$ vertices in U_1 by red edges; so each vertex in W_2 is joined to at least $j + 1 + n - m$ vertices in U_1 by blue edges. Assume, without loss of generality, that $u_iw_{|M|+1}$ is blue for each i with $1 \leq i \leq j + 1 + n - m$. If there is an integer i with $1 \leq i \leq j + 1 + n - m$ such that $u_{|M|+1}w_i$ is blue, say $u_{|M|+1}w_1$ is blue, then there is a matching

$$M' = \{u_{|M|+1}w_1, u_1w_{|M|+1}\} \cup \{u_iw_i : 2 \leq i \leq |M|\}$$

whose size is larger than $|M|$, a contradiction. Hence, $u_{|M|+1}w_i$ is red for all i with $1 \le i \le j+1+n-m$. This implies that

$$\begin{aligned}\deg_{H_R} u_{|M|+1} &\ge 2m-2-j-|M|+j+1+n-m = m-1-|M|+n \\ &\ge m-1-(n-1)+n = m.\end{aligned}$$

Thus, there is a red $K_{1,m}$ whose central vertex is $u_{|M|+1}$ in H. Hence, $BR_s(K_{1,m}, nK_2) \le 2m-2-j$. Therefore, if $n \le s \le m + \left\lfloor \frac{n-1}{2} \right\rfloor - 1$ and $s = n+j$, then $BR_s(K_{1,m}, nK_2) = 2m-2-j$.

Case 2. $n \ge 2m-2$. Since $BR_s(K_{1,m}, nK_2) \ge s$, we need only show that $BR_s(K_{1,m}, nK_2) \le s$, that is, every red-blue coloring of $H = K_{s,s}$ results in a red $K_{1,m}$ or a blue nK_2. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of H. Let M be a maximum matching in H_B. If $|M| \ge n$, then we obtain a blue nK_2. If $|M| \le m-2$, then we may assume that $M = \{u_1w_1, u_2w_2, \ldots, u_{|M|}w_{|M|}\}$. Let $U_1 = \{u_1, u_2, \ldots, u_{|M|}\}$ and $W_1 = \{w_1, w_2, \ldots, w_{|M|}\}$. Now, let $U_2 = U - U_1$ and $W_2 = W - W_1$. If there is a blue edge in $[U_2, W_2]$, then we obtain a matching by adding this blue edge to M, which contradicts the maximality of M. Hence, we may assume that $H[U_2, W_2] = K_{s-|M|,\, s-|M|} \subseteq H_R$. Since $|M| \le m-2$ and $s \ge n \ge 2m-2$, $s - |M| \ge 2m-2-(m-2) = m$. So there is a red $K_{1,m}$ in H. Thus, we may assume that $m-1 \le |M| \le n-1$. For each vertex $w \in W_2$, it follows that

$$\deg_{H_R} w \ge s - |M| \ge n - |M| \ge 2m-2-|M|.$$

If w is joined to at least $|M|-m+2$ vertices in U_1 by red edges, then there is a red $K_{1,m}$ in H. Thus, each vertex in W_2 is joined to at most $|M|-m+1$ vertices in U_1 by red edges; so each vertex in W_2 is joined to at least $m-1$ vertices in U_1 by blue edges. Assume, without loss of generality, that $u_iw_{|M|+1}$ is blue for each i with $1 \le i \le m-1$. If there is an integer i with $1 \le i \le m-1$ such that $u_{|M|+1}w_i$ is blue, say $u_{|M|+1}w_1$ is blue, then there is a matching

$$M' = \{u_{|M|+1}w_1, u_1w_{|M|+1}\} \cup \{u_iw_i : 2 \le i \le |M|\}$$

whose size is larger than $|M|$, a contradiction. Hence, $u_{|M|+1}w_i$ is red for all i with $1 \le i \le m-1$. This implies that

$$\begin{aligned}\deg_{H_R} u_{|M|+1} &\ge s-|M|+m-1 \ge n-|M|+m-1 \\ &\ge n-(n-1)+m-1 = m.\end{aligned}$$

Thus, there is a red $K_{1,m}$ whose central vertex is $u_{|M|+1}$ in H. Thus, every red-blue coloring of $K_{s,s}$ results in a red $K_{1,m}$ or a blue nK_2 and so $BR_s(K_{1,m}, nK_2) \leq s$. Therefore, $BR_s(K_{1,m}, nK_2) = s$. ■

The following result summarizes the values of $BR_s(F, H)$ for all positive integers s when F is a star and H is a matching.

Theorem 4.2.9 *Let m, n and s be integers with $m, n, s \geq 2$.*

1. *If $s \leq n-1$ or $s \leq m-1 \leq n-1$, then $BR_s(K_{1,m}, nK_2)$ does not exist.*
2. *If $n \leq s \leq m-1$, then $BR_s(K_{1,m}, nK_2) = m+n-1$.*
3. *If (i) $3 \leq n < m \leq s \leq m + \lfloor \frac{n-1}{2} \rfloor - 1$ or (ii) $n \leq s \leq m + \lfloor \frac{n-1}{2} \rfloor - 1$ and $3 \leq m \leq n \leq 2m-3$, then $BR_s(K_{1,m}, nK_2) = 2(m-1) + n - s$.*
4. *If (i) $s \geq m + \lfloor \frac{n-1}{2} \rfloor$ or (ii) $m \geq 3$ and $2m-2 \leq n \leq s \leq m + \lfloor \frac{n-1}{2} \rfloor - 1$, then $BR_s(K_{1,m}, nK_2) = s$.*

4.3 Double Stars

For integers $a, b \geq 2$ where $a \leq b$, let $S_{a,b}$ be the double star whose central vertices have degrees a and b. In this section, we determine the values of $BR_s(F, H)$ for all positive integers s when $F = H$ is a double star. In this case, we write $BR_s(S_{a,b}, S_{a,b})$ as $BR_s(S_{a,b})$.

Proposition 4.3.1 *Let a, b, s be integers with $a, b, s \geq 2$ and $a \leq b$. If $s \leq 2a-2$, then $BR_s(S_{a,b})$ does not exist.*

Proof. For an integer t where $t \geq 2a-2$, the red-blue coloring of $K_{2a-2,t}$, in which both red and blue subgraphs are $K_{a-1,t}$, produces no monochromatic $S_{a,b}$. Since $K_{s,t} \subseteq K_{2a-2,t}$ for each integer s with $2 \leq s \leq 2a-2$, there is a red-blue coloring of $K_{s,t}$ that avoids a monochromatic $S_{a,b}$. Therefore, $BR_s(S_{a,b})$ does not exist. ■

We now show that $BR_s(S_{a,b})$ exists otherwise, beginning with the case where $2a-1 \leq s \leq 2b-2$.

Theorem 4.3.2 *Let a, b and s be integers with $2 \leq a \leq b$. If $2a-1 \leq s \leq 2b-2$, then $BR_s(S_{a,b}) = 2b-1$.*

Proof. First, we show that $BR_s(S_{a,b}) \geq 2b-1$; that is, we show that there is a red-blue coloring of $G = K_{s,2b-2}$ that produces no monochromatic $S_{a,b}$. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{2b-2}\}$ be the partite sets of G. Partition the set U into two subsets U_1 and U_2 with $|U_1| = \lceil s/2 \rceil$ and $|U_2| = \lfloor s/2 \rfloor$ and partition the set W into two subsets W_1 and W_2 with $|W_1| = |W_2| = b-1$. Define a red-blue coloring of G by assigning the color blue to each edge in $[U_1, W_1] \cup [U_2, W_2]$ and the color red to each edge in $[U_1, W_2] \cup [U_2, W_1]$. Let G_R and G_B be the resulting red and blue subgraphs of G, respectively. For each vertex x of G, it follows that $\deg_{G_R} x \leq b-1$ and $\deg_{G_B} x \leq b-1$. Therefore, there is no monochromatic $S_{a,b}$ in G and so $BR_s(S_{a,b}) \geq 2b-1$.

To show that $BR_s(S_{a,b}) \leq 2b-1$, we proceed by induction on $a \geq 2$. First, suppose that $a = 2$ and so $3 \leq s \leq 2b-2$. Let there be given a red-blue coloring of $H = K_{s,2b-1}$ resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{2b-1}\}$ be the partite sets of H. Since u_1 is incident with $2b-1$ edges, at least b edges are colored the same, say u_1w_i is red for $1 \leq i \leq b$. Let $S = U - \{u_1\}$ and $T = \{w_1, w_2, \ldots, w_b\}$. If there is a red edge in $[S, T]$, then there is a red $S_{2,b}$; otherwise, $H[S, T]$ is a blue $K_{s-1,b}$. Since $s-1 \geq 2$, it follows that $H[S, T]$ contains a blue $S_{2,b}$. Hence, there is a monochromatic $S_{2,b}$ in H. Therefore, the statement is true when $a = 2$.

Next, suppose that the inequality $BR_s(S_{a,b}) \leq 2b-1$ holds for an integer $a-1 \geq 2$. Thus, for every integer c with $c \geq a-1$ and every integer s with $2a-3 \leq s \leq 2c-2$, it follows that $BR_s(S_{a-1,c}) \leq 2c-1$. We show next that the inequality holds for a. So, let b and s be integers such that $b \geq a$ and $2a-1 \leq s \leq 2b-2$. We show that $BR_s(S_{a,b}) \leq 2b-1$. Since $b \geq a$, it follows that $b \geq a-1$. Because $2a-1 \leq s \leq 2b-1$, it follows that $2a-3 \leq s \leq 2b-2$. Hence, $BR_s(S_{a-1,b}) \leq 2b-1$. Consequently, every red-blue coloring of $K_{s,2b-1}$ results in a monochromatic $S_{a-1,b}$. We show that every such coloring also results in monochromatic $S_{a,b}$.

Let there be given a red-blue coloring of $H = K_{s,2b-1}$ resulting in the red subgraph H_R and the blue subgraph H_B. Assume, to the contrary, that there is no monochromatic $S_{a,b}$. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{2b-1}\}$ be the partite sets of H. Since $2a-1 \leq s \leq 2b-2$ and so $2a-3 \leq s \leq 2b-2$, it follows by the induction hypothesis that H contains a monochromatic $F = S_{a-1,b}$. We may assume, without loss of generality, that F is a blue $S_{a-1,b}$ whose central vertices are u_1 and w_1 such that u_1 is adjacent w_i for $1 \leq i \leq b$ and w_1 is adjacent u_j for $1 \leq j \leq a-1$. Thus, $\deg_F u_1 = b$ and $\deg_F w_1 = a-1$. Let $U_1 = \{u_a, u_{a+1}, \ldots, u_s\}$.

- ⋆ If there is a blue edge in $[\{w_1\}, U_1]$, then there is a blue $S_{a,b}$, a contradiction. Thus, each edge in $[\{w_1\}, U_1]$ is red. Hence,

$$\deg_{H_R} w_1 = s - (a-1) \geq (2a-1) - (a-1) = a. \tag{4.4}$$

- ⋆ If there is $u \in U_1$ such that $\deg_{H_R} u \geq b$, then there is a red $S_{a,b}$ whose central vertices are u and w_1, a contradiction. Thus, $\deg_{H_R} u \leq b-1$ for each $u \in U_1$ and so

$$\deg_{H_B} u \geq (2b-1) - (b-1) = b \text{ for each } u \in U_1. \tag{4.5}$$

- ⋆ If there is $w \in W$ such that $\deg_{H_B} w \geq a$, then w must be adjacent to some vertex $u \in U_1$ (as $|U - U_1| = a-1$). Since $\deg_{H_B} u \geq b$ by (4.5), there is a blue $S_{a,b}$ whose central vertices are u and w, a contradiction. Thus,

$$\deg_{H_B} w \leq a-1 \text{ for each } w \in W. \tag{4.6}$$

It then follows by (4.6) that the size m_{H_B} of H_B is at most $(a-1)(2b-1)$ and

$$\deg_{H_R} w \geq s - (a-1) \geq a \text{ for each } w \in W. \tag{4.7}$$

- ⋆ If there is $u \in U$ such that $\deg_{H_R} u \geq b$, it then follows by (4.7) that there is a red $S_{a,b}$, a contradiction. Thus, $\deg_{H_R} u \leq b-1$ for each $u \in U$ and so

$$\deg_{H_B} u \geq (2b-1) - (b-1) = b \text{ for each } u \in U. \tag{4.8}$$

It then follows by (4.8) that $m_{H_B} \geq sb$.

Therefore, $sb \leq m_{H_B} \leq (2b-1)(a-1)$. Since $s \geq 2a-1 > 2a-2$, it follows that $(2a-2)b < sb \leq (2b-1)(a-1)$ and so $a < 1$, which is impossible.

It then follows by the Principle of Mathematical Induction that there is a monochromatic $S_{a,b}$ in H and so $BR_s(S_{a,b}) \leq 2b-1$. Therefore, $BR_s(S_{a,b}) = 2b-1$ when $2a-1 \leq s \leq 2b-2$. ■

Theorem 4.3.3 *Let a and b be integers with $2 \leq a \leq b$. If s is an integer with $s \geq 2b-1$, then $BR_s(S_{a,b}) = s$.*

Proof. Since $BR_s(S_{a,b}) \geq s$, we need only show that $BR_s(S_{a,b}) \leq s$. We proceed by induction on $a \geq 2$ to show that every red-blue coloring of $H = K_{s,s}$ produces a monochromatic $S_{a,b}$ for integers a and b with $2 \leq a \leq b$ where $s \geq 2b-1$.

First, suppose that $a = 2$. We show that H contains a monochromatic $S_{2,b}$. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of H. Since u_1 is incident with $s \geq 2b-1$ edges, at least b edges are colored the same, say $u_1 w_i$ is red for $1 \leq i \leq b$. Let $S = U - \{u_1\}$ and $T = \{w_1, w_2, \ldots, w_b\}$. If there is a red edge in $[S, T]$, then there is a red $S_{2,b}$; for otherwise, $H[S,T]$ is a blue $K_{s-1,b}$. Since $s - 1 \geq 2b - 2 \geq 2$, it follows that $K_{2,b} \subseteq H[S,T]$ and so $H[S,T]$ contains a blue $S_{2,b}$. Hence, there is a monochromatic $S_{2,b}$ in H.

Next, assume for an integer $a - 1 \geq 2$ that for all integers c and s with $c \geq a - 1$ and $s \geq 2c - 1$, we have $BR_s(S_{a,c}) \leq s$. We now show for integers b and s with $b \geq a$ and $s \geq 2b-1$ that $BR_s(S_{a,b}) \leq s$. Assume, to the contrary, that there exists a red-blue coloring of $H = K_{s,s}$ for which there is no monochromatic $S_{a,b}$. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of H. Since $b \geq a$, it follows that $b \geq a-1$ and so by the induction hypothesis there is a monochromatic $F = S_{a-1,b}$ in H. We may assume, without loss of generality, that F is a blue $S_{a-1,b}$ whose central vertices are u_1 and w_1 such that u_1 is adjacent w_i for $1 \leq i \leq b$ and w_1 is adjacent u_j for $1 \leq j \leq a-1$. Thus, $\deg_F u_1 = b$ and $\deg_F w_1 = a - 1$. Let $U_1 = \{u_a, u_{a+1}, \ldots, u_s\}$.

- ⋆ If there is a blue edge in $[\{w_1\}, U_1]$, then there is a blue $S_{a,b}$, a contradiction. Thus, each edge in $[\{w_1\}, U_1]$ is red. Hence,

$$\deg_{H_R} w_1 = s - (a-1) \geq (2b-1) - (a-1) = 2b - a \geq b. \tag{4.9}$$

- ⋆ If there is $u \in U_1$ such that $\deg_{H_R} u \geq a$, then there is a red $S_{a,b}$ whose central vertices are u and w_1, a contradiction. Thus, $\deg_{H_R} u \leq a-1$ for each $u \in U_1$ and so

$$\deg_{H_B} u \geq s - (a-1) \geq b \text{ for each } u \in U_1. \tag{4.10}$$

- ⋆ If there is $w \in W$ such that $\deg_{H_B} w \geq a$, then w must be adjacent to some vertex $u \in U_1$ (as $|U - U_1| = a - 1$). Since $\deg_{H_B} u \geq b$ by (4.10), there is a blue $S_{a,b}$ whose central vertices are u and w, a contradiction. Thus, $\deg_{H_B} w \leq a - 1$ for each $w \in W$ and so

$$\deg_{H_R} w \geq s - (a-1) \geq b \text{ for each } w \in W. \tag{4.11}$$

  This implies that the size m_{H_R} of H_R is at least $s(s - a + 1)$.

- ⋆ If there is $u \in U$ such that $\deg_{H_R} u \geq a$, it then follows by (4.11) that there is a red $S_{a,b}$, a contradiction. Thus, $\deg_{H_R} u \leq a-1$ for each $u \in U$ and so $m_{H_R} \leq s(a-1)$.

Therefore, $s(s-a+1) \leq m_{H_R} \leq s(a-1)$ or $s \leq 2a-2$. Since $s \geq 2b-1$, it follows that $b \leq a - 1/2 < a$, which is impossible.

It then follows by the Principle of Mathematical Induction that there is a monochromatic $S_{a,b}$ in H and so $BR_s(S_{a,b}) \leq s$. Therefore, $BR_s(S_{a,b}) = s$ when $s \geq 2b - 1$. ■

In summary, we have the following theorem which provides the values of $BR_s(S_{a,b})$ for all integers $a, b, s \geq 2$.

Theorem 4.3.4 *Let a, b, s be integers with $a, b, s \geq 2$ and $a \leq b$.*

1. *If $s \leq 2a - 2$, then $BR_s(S_{a,b})$ does not exist.*
2. *If $2a - 1 \leq s \leq 2b - 2$, then $BR_s(S_{a,b}) = 2b - 1$.*
3. *If $s \geq 2b - 1$, then $BR_s(S_{a,b}) = s$.*

4.4 Paths

In this section, we study the s-bipartite Ramsey numbers of paths $BR_s(P_n)$ for $n \geq 3$. Since P_3 is a star and P_4 is a double star, we may assume that $n \geq 5$ by Theorems 4.2.1 and 4.3.4.

Proposition 4.4.1 *Let n and s be integers with $n \geq 5$ and $s \geq 2$.*

(1) *If n is odd, then $BR_s(P_n)$ exists only when $s \geq n - 2$.*

(2) *If n is even, then $BR_s(P_n)$ exists only when $s \geq n - 1$.*

Proof. For an odd integer $n \geq 5$, it suffices to show that $BR_{n-3}(P_n)$ does not exist. Let $n = 2k + 1$ for some integer $k \geq 2$. For an arbitrarily large integer t, the red-blue coloring of $K_{2k-2,t}$, in which both red and blue subgraphs are $K_{k-1,t}$, produces no monochromatic P_{2k+1}. Therefore, $BR_s(P_{2k+1})$ does not exist for $2 \leq s \leq 2k-2 = n-3$.

For an even integer $n \geq 6$, it suffices to show that $BR_{n-2}(P_n)$ does not exist. Let $n = 2k$ for some integer $k \geq 2$. For an arbitrarily large integer t, the red-blue coloring of $K_{2k-2,t}$, in which both red and blue subgraphs are $K_{k-1,t}$, produces no monochromatic P_{2k}. Therefore, $BR_s(P_{2k})$ does not exist for $2 \leq s \leq 2k - 2 = n - 2$. ■

Next, we determine $BR_s(P_n)$ for $5 \leq n \leq 8$ and for all possible values of s. In order to do this, we first present a useful observation.

Observation 4.4.2 *If F is a bipartite graph such that $BR_k(F) = k+1$ for some positive integer k, then $BR_s(F) = s$ all integers $s \geq k + 1$.*

Proposition 4.4.3 *For each integer $s \geq 2$,*

$$BR_s(P_5) = \begin{cases} \textit{does not exist} & \textit{if } s = 2 \\ 5 & \textit{if } s = 3, 4 \\ s & \textit{if } s \geq 5. \end{cases}$$

Proof. By Proposition 4.4.1, $BR_2(P_5)$ does not exist. Since the red-blue coloring of $K_{4,4}$ of Figure 4.2 in which both the red and blue subgraphs are $2C_4$ produces no monochromatic P_5, it follows that $BR_4(P_5) \geq 5$. This also implies that there is a red-blue coloring of $K_{3,4}$ that avoids a monochromatic P_5 and so $BR_3(P_5) \geq 5$.

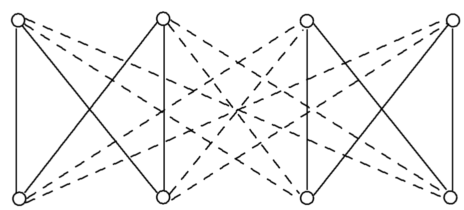


Figure 4.2: A red-blue coloring of $K_{4,4}$ without a monochromatic P_5

To verify that $BR_3(P_5) \leq 5$, we show that every red-blue coloring of $H = K_{3,5}$ produces a monochromatic P_5. Let $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the partite sets of H. Now let there be given a red-blue coloring of H. Each vertex u_i $(1 \leq i \leq 3)$ is incident with at least three edges of the same color and at least two of the vertices u_i $(1 \leq i \leq 3)$ are incident with at least three edges that are either red or blue. We may assume, without loss of generality, that u_1 and u_2 are incident with three red edges and u_1w_i is red for $1 \leq i \leq 3$. Furthermore, we may assume that u_2w_3 is also red. If either u_2w_4 or u_2w_5 is red, say the former, then $(w_1, u_1, w_3, u_2, w_4)$ is a red P_5; otherwise, u_2w_4 and u_2w_5 are both blue and so u_2w_i is red for $1 \leq i \leq 3$. Thus, $(w_1, u_1, w_2, u_2, w_3)$ is a red P_5 and so $BR_3(P_5) \leq 5$. This also implies that every red-blue coloring of $K_{4,5}$ produces a monochromatic P_5 and so $BR_4(P_5) \leq 5$. Therefore, $BR_s(P_5) = 5$ for $s = 3, 4$. It then follows by Observation 4.4.2 that $BR_s(P_5) = s$ for each integer $s \geq 5$. ■

For two disjoint sets X and Y of vertices of a graph G, the set of edges joining a vertex of X and a vertex of Y in G is denoted by $G[X, Y]$ or, more simply, by $[X, Y]$ if the graph G under discussion is clear.

Proposition 4.4.4 *For each integer $s \geq 2$,*

$$BR_s(P_6) = \begin{cases} \textit{does not exist} & \textit{if } s = 2, 3, 4 \\ s & \textit{if } s \geq 5. \end{cases}$$

Proof. By Proposition 4.4.1, if $s = 2, 3, 4$, then $BR_s(P_6)$ does not exist. It remains to show that $BR_s(P_6) = s$ for $s \geq 5$. To verify this, it suffices to show that every red-blue coloring of $K_{5,5}$ produces a monochromatic P_6. Let there be given a red-blue coloring of $H = K_{5,5}$. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $W = \{w_1, w_2, w_3, w_4, w_5\}$ be the partite sets of H. It then follows by Proposition 4.4.3 that H contains a monochromatic P_5, say $(w_1, u_1, w_2, u_2, w_3)$ is a red P_5. If there is a red edge joining a vertex of $S = \{u_3, u_4, u_5\}$ and a vertex of $T = \{w_1, w_3\}$, then there is a red P_6. Thus, we may assume that $H[S, T] = K_{2,3}$ is blue. We now consider the blue $P_5 = (u_3, w_1, u_4, w_3, u_5)$ in $H[S, T]$. If either w_4u_3 or w_4u_5 is a blue edge, then there is a blue P_6. Hence, we may assume that both w_4u_3 are w_4u_5 are red. If w_2u_3 is blue, then $(w_2, u_3, w_3, u_4, w_1, u_5)$ is a blue P_6; while if w_2u_3 is red, then $(w_1, u_1, w_2, u_3, w_4, u_5)$ is a red P_6. Therefore, there is a monochromatic P_6 in H and so $BR_s(P_6) = s$ for $s \geq 5$. ■

Proposition 4.4.5 *For each integer $s \geq 2$,*

$$BR_s(P_7) = \begin{cases} \textit{does not exist} & \textit{if } 2 \leq s \leq 4 \\ 7 & \textit{if } s = 5, 6 \\ s & \textit{if } s \geq 7. \end{cases}$$

Proof. By Proposition 4.4.1, if $2 \leq s \leq 4$, then $BR_s(P_7)$ does not exist. Since the red-blue coloring of $K_{6,6}$ in which both red and blue subgraphs are $2K_{3,3}$ produces no monochromatic P_7, it follows that $BR_6(P_7) \geq 7$. This also implies that there is a red-blue coloring of $K_{5,6}$ avoiding a monochromatic P_7 and so $BR_5(P_7) \geq 7$.

To verify that $BR_5(P_7) \leq 7$, we show that every red-blue coloring of $H = K_{5,7}$ produces a monochromatic P_7. Let $U = \{u_1, u_2, \ldots, u_5\}$ and $W = \{w_1, w_2, \ldots, w_7\}$ be the partite sets of H. Now let there be given a red-blue coloring of H. By Proposition 4.4.4, H contains a monochromatic P_6. We may assume, without loss of generality, that $(u_1, w_1, u_2, w_2, u_3, w_3)$ is a red P_6 in H. If (i) u_1 is joined to a vertex in $T = \{w_4, w_5, w_6, w_7\}$ by a red edge or (ii) w_3 is joined to a vertex in $\{u_4, u_5\}$ by a red edge, then there is a red P_7 in H. Thus, we may assume that every edge in $[\{u_1\}, T] \cup [\{w_3\}, \{u_4, u_5\}]$ is blue. Let $S = \{u_3, u_4, u_5\}$. Consider the subgraph $H[S, T] = K_{3,4}$ of H. If every edge in $H[S, T]$ is red, then there is a red P_7 in H. Thus, we may assume that there is a blue edge in $H[S, T]$ and so at least one vertex of T is joined to a vertex of S by a blue edge.

Case 1. *A vertex of T is joined to u_4 or u_5 by a blue edge, say u_5w_7 is blue.* Then $(u_4, w_3, u_5, w_7, u_1, w_4)$ is a blue P_6 in H. If (i) u_4 is joined to a vertex of $\{w_1, w_2, w_5, w_6\}$ by a blue edge or (ii) w_4 is joined to a vertex in $\{u_2, u_3\}$ by a blue edge, then there is a blue P_7 in H. Thus, we may assume that (i^*) u_4 is joined to every vertex of $\{w_1, w_2, w_5, w_6\}$ by a red edge and (ii^*) w_4 is joined to every vertex in $\{u_2, u_3\}$ by a red edge. Then $(u_1, w_1, u_2, w_4, u_3, w_2, u_4)$ is a red P_7 in H.

Case 2. *A vertex of T is joined to u_3 by a blue edge, say u_3w_7 is blue.* By Case 1, we may assume that every vertex of T is joined to both u_4 and u_5 by red edges. Thus, $H[\{u_4, u_5\}, T] = K_{2,4}$ is red. If there is a red edge joining a vertex of $S' = \{u_1, u_2, u_3\}$ and a vertex of T, then there is a red P_7 in H. Hence, $H[S', T] = K_{3,4}$ is blue and so H contains a blue P_7.

Therefore, there is a monochromatic P_7 in H and so $BR_5(P_7) \le 7$. This also implies every red-blue coloring of $K_{6,7}$ produces a monochromatic P_7 and so $BR_6(P_7) \le 7$. Therefore, $BR_s(P_7) = 7$ for $s = 5, 6$. Since every red-blue coloring of $K_{5,7}$ produces a monochromatic P_7, it follows that every red-blue coloring of $K_{7,7}$ produces a monochromatic P_7. Thus, $BR_s(P_7) = s$ for $s \ge 7$. ■

Proposition 4.4.6 *For each integer $s \ge 2$,*

$$BR_s(P_8) = \begin{cases} \text{does not exist} & \text{if } 2 \le s \le 6 \\ s & \text{if } s \ge 7. \end{cases}$$

Proof. By Proposition 4.4.1, if $2 \le s \le 6$, then $BR_s(P_8)$ does not exist. To verify that $BR_s(P_8) = s$ if $n \ge 7$, it suffices to show that every red-blue coloring of $K_{7,7}$ produces a monochromatic P_8. Let there be given a red-blue coloring of $H = K_{7,7}$. Let $U = \{u_1, u_2, \ldots, u_7\}$ and $W = \{w_1, w_2, \ldots, w_7\}$ be the partite sets of H. Since $BR_7(P_7) = 7$, it follows that H contains a monochromatic P_7, say $(u_1, w_1, u_2, w_2, u_3, w_3, u_4)$ is a red P_7. If there is a red edge joining a vertex of $S = \{u_1, u_4\}$ and a vertex of $T = \{w_4, w_5, w_6, w_7\}$, then there is a red P_8. Thus, $H[S, T] = K_{2,4}$ is blue. Consider the subgraph $F = K_{3,3}$ of H induced by $\{u_5, u_6, u_7\} \cup \{w_5, w_6, w_7\}$. Since $BR_3(P_4) = 3$, it follows that F contains a monochromatic P_4, say $P = (u_6, w_6, u_7, w_7)$ is a monochromatic P_4. If P is a blue P_4, then $(u_6, w_6, u_7, w_7, u_4, w_5, u_1, w_4)$ is a blue P_8. Thus, we may assume that P is a red P_4. If there is a red edge joining a vertex in $\{u_1, u_2, u_3, u_4\}$ and a vertex in $\{w_6, w_7\}$, then there is a red P_8. Thus, we may assume that every edge joining a vertex in $\{u_1, u_2, u_3, u_4\}$ and a vertex in $\{w_6, w_7\}$ is blue. Then $(u_3, w_7, u_2, w_6, u_1, w_4, u_4, w_5)$ is a blue P_8. Therefore, there is a monochromatic P_8 in H and so $BR_s(P_8) = s$ for $s \geq 7$. ■

Propositions 4.4.3-4.4.6 suggest the following conjecture.

Conjecture 4.4.7 *Let n and s be integers with $n \geq 5$ and $s \geq 2$. If $BR_s(P_n)$ exists, then*

$$BR_s(P_n) = \begin{cases} n & \text{if } n \text{ is odd and } s \in \{n-2, n-1\} \\ s & \text{otherwise.} \end{cases}$$

That is, if $BR_s(P_n)$ exists, then $BR_s(P_n) = \max\{n, s\}$ or

(1) *if n is odd and $s \geq n-2$, then $BR_s(P_n) = \max\{n, s\}$.*

(2) *if n is even and $s \geq n-1$, then $BR_s(P_n) = s$.*

As we saw, Conjecture 4.4.7 is true for $5 \leq n \leq 8$ by Propositions 4.4.3-4.4.6. It is an obvious upper bound for $BR_s(P_m, P_n)$. Since $P_k \subseteq K_{\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil}$ for each integer $k \geq 2$, it follows that if $n, m \geq 2$ are integers, then

$$BR_s(P_m, P_n) \leq BR_s(K_{\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil}, K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}).$$

Problem 4.4.8 *Establish better bounds for $BR_s(P_m, P_n)$.*

4.5 Paths Versus Stars

We now investigate $BR_s(K_{1,m}, P_n)$ for integers $m, n, s \geq 2$. First, we show that the s-bipartite Ramsey number $BR_s(K_{1,m}, P_n)$ does not exist for each integer s when $s \leq \lfloor \frac{n-2}{2} \rfloor$ or $s \leq m \leq \lfloor \frac{n-2}{2} \rfloor$.

Proposition 4.5.1 *Let $m, n, s \geq 2$ be integers. If either $s \leq \lfloor \frac{n-2}{2} \rfloor$ or $s \leq m \leq \lfloor \frac{n-2}{2} \rfloor$, then $BR_s(K_{1,m}, P_n)$ does not exist.*

Proof. Suppose that $2 \leq s \leq \lfloor \frac{n-2}{2} \rfloor$. Then $2s + 1 < n$. Since the order of a longest path in $K_{s,t}$ is $2s + 1$, there is no P_n in $K_{s,t}$. For an arbitrary integer t, the red-blue coloring of $K_{s,t}$ that assigns the color blue to each edge of $K_{s,t}$ produces neither a red

$K_{1,m}$ nor a blue P_n. Therefore, $BR_s(K_{1,m}, P_n)$ does not exist when $s \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ as well as when $s \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. ■

Next, we show that $BR_s(K_{1,m}, nK_2) = s$ if s is sufficiently large.

Proposition 4.5.2 *If m, n, s are integers with $m, n \geq 2$ and $s \geq m + \left\lfloor \frac{n}{2} \right\rfloor$, then*

$$BR_s(K_{1,m}, P_n) = s.$$

Proof. By the definition of s-bipartite Ramsey number,

$$BR_s(K_{1,m}, P_n) \geq s.$$

Hence, it remains to show that $BR_s(K_{1,m}, P_n) \leq s$, that is, every red-blue coloring of $H = K_{s,s}$ results in a red $K_{1,m}$ or a blue P_n. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of H. Next, let P_k be a longest path in H_B. If $k \geq n$, then there is a blue P_n. So, we may assume that $k \leq n - 1$. We consider two cases according to whether k is even or k is odd.

Case 1. *k is even.* We may assume that

$$P_k = (u_1, w_2, u_2, w_2, \ldots, u_{\frac{k}{2}}, w_{\frac{k}{2}}).$$

Since P_k is a longest path in H_B, it follows that $w_{\frac{k}{2}}$ must be joined to every vertex in $U - \{u_1, u_2, \ldots, u_{\frac{k}{2}}\}$ by red edges. Since $k \leq n - 1$ and $s \geq m + \left\lfloor \frac{n}{2} \right\rfloor$, it follows that

$$\begin{aligned} \deg_{H_R} w_{\frac{k}{2}} &= s - \frac{k}{2} \geq \left(m + \left\lfloor \frac{n}{2} \right\rfloor\right) - \frac{k}{2} \\ &\geq m + \frac{n-1}{2} - \frac{n-1}{2} = m. \end{aligned}$$

Thus, there is a red $K_{1,m}$ whose central vertex is $w_{\frac{k}{2}}$ in H.

Case 2. *k is odd.* We may assume that

$$P_k = (u_1, w_2, u_2, w_2, \ldots, u_{\frac{k-1}{2}}, w_{\frac{k-1}{2}}, u_{\frac{k}{2}}).$$

Since P_k is a longest path in H_B, it follows that $u_{\frac{k}{2}}$ must be joined to every vertex in $W - \{w_1, w_2, \ldots, w_{\frac{k-1}{2}}\}$ by red edges. Since $k \leq n - 1$ and $s \geq m + \left\lfloor \frac{n}{2} \right\rfloor$, it follows that

$$\begin{aligned} \deg_{H_R} w_{\frac{k}{2}} &= s - \frac{k-1}{2} \geq \left(m + \left\lfloor \frac{n}{2} \right\rfloor\right) - \frac{n-2}{2} \\ &\geq m + \frac{n-1}{2} - \frac{n-2}{2} = m + \frac{1}{2}. \end{aligned}$$

Thus, there is a red $K_{1,m}$ whose central vertex is $u_{\frac{k}{2}}$ in H.

Consequently, every red-blue coloring of $K_{s,s}$ results in a red $K_{1,m}$ or a blue P_n and so $BR_s(K_{1,m}, P_n) \leq s$. Therefore, $BR_s(K_{1,m}, P_n) = s$. ■

We now consider the case when $\left\lfloor \frac{n-2}{2} \right\rfloor < s \leq m + \left\lfloor \frac{n}{2} \right\rfloor - 1$. There is an obvious lower bound for $BR_s(K_{1,m}, P_n)$ since

$$BR_s(K_{1,m}, P_n) \geq BR_s(K_{1,m}, \left\lfloor \tfrac{n}{2} \right\rfloor K_2)$$

and $BR_s(K_{1,m}, \left\lfloor \frac{n}{2} \right\rfloor K_2)$ was determined in Theorem 4.2.9. Next, we establish an upper bound for $BR_s(K_{1,m}, P_n)$ when $\left\lfloor \frac{n-2}{2} \right\rfloor < s \leq m + \left\lfloor \frac{n}{2} \right\rfloor - 1$.

Proposition 4.5.3 *If m, n, s are integers with $\left\lfloor \frac{n-2}{2} \right\rfloor < s \leq m + \left\lfloor \frac{n}{2} \right\rfloor - 1$, then*

$$BR_s(K_{1,m}, P_n) \leq s(m-1) + \left\lceil \tfrac{n}{2} \right\rceil.$$

Proof. To verify that $BR_s(K_{1,m}, P_n) \leq s(m-1) + \left\lceil \frac{n}{2} \right\rceil$, we show that every red-blue coloring of $H = K_{s,s(m-1)+\lceil \frac{n}{2} \rceil}$ results in a red $K_{1,m}$ or a blue P_n. Let there be given a red-blue coloring of H resulting in the red subgraph H_R and the blue subgraph H_B. Let $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_{s(m-1)+\lceil \frac{n}{2} \rceil}\}$ be the partite sets of H. Suppose that there is no red $K_{1,m}$ in H. We will show that there is a blue P_n in H. Since there is no red $K_{1,m}$, it follows that each vertex in U is joined to at most $m-1$ vertices in W by red edges. Thus, $|N_{H_R}(U)| \leq s(m-1)$. Since $N_{H_R}(U) \subset W$ and $|W| = s(m-1) + \left\lceil \frac{n}{2} \right\rceil$, it follows that there are $\left\lceil \frac{n}{2} \right\rceil$ vertices in W that are joined to every vertex in U by blue edges. Hence, there is a blue $K_{s,\lceil \frac{n}{2} \rceil}$ in H_B. Since $s > \left\lfloor \frac{n-2}{2} \right\rfloor$, it follows that $s \geq \frac{n-1}{2}$ and so $K_{s,\lceil \frac{n}{2} \rceil}$ contains a path of order n. Thus, there is a blue P_n in H_B. Therefore, $BR_s(K_{1,m}, P_n) \leq s(m-1) + \left\lceil \frac{n}{2} \right\rceil$. ■

Problem 4.5.4 *Establish better bounds for $BR_s(K_{1,m}, P_n)$.*

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