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Extremal Problems on Induced Graph Colorings

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by

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Graph coloring is one of the most popular areas of graph theory, no doubt due to its many fascinating problems and applications to modern society, as well as the sheer mathematical beauty of the subject. As far back as 1880, in an attempt to solve the famous Four Color Problem, there have been numerous examples of certain types of graph colorings that have generated other graph colorings of interest. These types of colorings only gained momentum a century later, however, when in the 1980s, edge colorings were studied that led to vertex colorings of various types, led by the introduction of the irregularity strength of a graph by Chartrand and the majestic chromatic index of a graph by Harary and Plantholt. Since then, the study of such graph colorings has become a popular area of research in graph theory. Recently, two set and number theoretic graph colorings were introduced, namely royal colorings and rainbow mean colorings. These two colorings as well as variations have extended some classical graph coloring concepts. We investigate structural and extremal problems dealing with royal and rainbow mean colorings and explore relationships among the chromatic parameters resulting from these colorings and traditional chromatic parameters.

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James Hallas

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Chapter 1

Introduction

1.1 Graph Colorings

On October 23, 1852, a problem was stated that immeasurably changed the field of graph theory. A young mathematician named Francis Guthrie posed the following question. Using at most four colors, can the regions of a map be colored so that no two regions sharing a boundary are colored the same? This deceptively simple sounding question stumped problem solvers from a variety of different backgrounds for over a century. It wasn't until 1976 when Kenneth Appel and Wolfgang Haken finally found a solution to this problem, which was initially controversial due to its reliance on computer technology. Nevertheless they showed that the answer is affirmative, namely that any map can indeed be colored in the desired way using at most four colors. This famous question led to the development of a branch of graph theory focused on the study of *graph colorings*. Today, graph coloring is one of the most popular areas of graph theory because of its history as well as its many fascinating problems and applications to modern society.

A k-*edge coloring* of a graph G is a function $c : E(G) \to [k] = \{1, 2, \ldots, k\}$ where k is a positive integer. The edge coloring c is *unrestricted* if no condition is placed on how the edges may be colored. For example, in an unrestricted edge coloring, adjacent edges may be colored the same. However, if no pair of adjacent edges in G can be colored the same, then c is referred to as a *proper edge coloring*. The minimum positive integer k for which G has a proper k-edge coloring is its *chromatic index*, denoted by $\chi'(G)$. For every nonempty graph G, $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G.

The most famous theorem dealing with the chromatic index was obtained by Vizing in [43].

Theorem 1.1.1 (**Vising's Theorem**) *For every nonempty graph G,*

$$\chi'(G) \leq \Delta(G) + 1.$$

As a result of Vizing's theorem, the chromatic index of a nonempty graph G is one of two numbers, namely either $\Delta(G)$ or $\Delta(G) + 1$. A graph G with $\chi'(G) = \Delta(G)$ is called a *class one graph* while a graph G with $\chi'(G) = \Delta(G) + 1$ is called a *class two graph*.

A *vertex k-coloring* of a graph G is a function $c' : V(G) \to [k]$ where k is a positive integer. A vertex coloring of a nontrivial graph G is *vertex-distinguishing* or *rainbow* if distinct vertices of G are assigned distinct colors while a vertex coloring of G is *neighbor-distinguishing* if no two adjacent vertices are colored the same. Such a coloring is commonly called a *proper coloring*. The minimum k for which a proper (vertex) k-coloring of a graph G exists is the chromatic number of G and is denoted by $\chi(G)$. Notice that every rainbow coloring is also proper, making rainbow a stronger condition to require of a graph coloring.

We refer to the book [23] for graph theory notation and terminology not described in this dissertation. The following are some well-known results about the chromatic number of a graph. For graphs of order $n \geq 3$, it is immediate which graphs of order n have chromatic number 1, 2, or n. A graph is *empty* if it has no edges. Consequently, a *nonempty graph* has one or more edges.

Observation 1.1.2 *If G is a graph of order $n \geq 3$, then $\chi(G) = 1$ if and only if G is empty, $\chi(G) = n$ if and only if $G = K_n$, and $\chi(G) = 2$ if and only if G is a nonempty bipartite graph.*

By Observation 1.1.2, $\chi(G) \geq 3$ if and only if G contains an odd cycle (or equivalently G is not a bipartite graph).

Proposition 1.1.3 *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

The *clique number* $\omega(G)$ of a graph G is the maximum order of a complete subgraph of G. In particular, $\omega(K_n) = n$ and $\omega(G) = 2$ for every nonempty bipartite graph G.

Theorem 1.1.4 *For every graph G,*

$$\omega(G) \le \chi(G) \le \Delta(G) + 1.$$

For each odd integer $n \ge 3$, the connected graphs C_n and K_n have the property that $\chi(C_n) = 3 = \Delta(C_n)+1$ and $\chi(K_n) = n = \Delta(K_n)+1$. Brooks [9] showed that these two classes of graphs are the only connected graphs with this property.

Theorem 1.1.5 (**Brooks' Theorem**) *If G is a connected graph that is neither an odd cycle nor a complete graph, then*

$$\chi(G) \le \Delta(G).$$

During the past several decades, there have been many studies of edge labelings or edge colorings of graphs that give rise to vertex labelings or colorings where the vertex coloring is either proper or rainbow. Such colorings are often referred to as *color-induced graph colorings.* Among the colored-induced vertex colorings c' of a graph G obtained from an edge coloring c of G, the most studied are those for which the color $c'(v)$ of a vertex v of G is either (1) the set of colors of those edges incident with v, (2) the multiset of colors of the edges incident with v, or (3) the sum of the colors of the edges incident with v. The induced graph colorings studied in this work belong to one of two types, namely set-defined and sum-defined colorings. We refer to the books [15, 23, 45, 46] for definitions, applications, and results relating to these topics.

1.2 Set-Defined Colorings

First, we consider edge colorings that lead to vertex colorings by the process described in (1). An early example of such an edge coloring was introduced by Harary and Plantholt [29] in 1985. Let $c : E(G) \to [k]$ be an unrestricted edge coloring of a nontrivial connected graph G with $c'(v) = \bigcup_{e \in E_v}\{c(e)\}$ for each vertex v in G, where E_v denotes the set of edges incident to v. If c' is vertex-distinguishing, then c is called a *strong majestic edge coloring* (also called a *set irregular edge coloring*) of G. The minimum positive integer k for which a graph G has a strong majestic edge coloring is the *strong majestic index* of G. (This parameter was referred to as the *point-distinguishing chromatic index* by Harary and Plantholt.)

In 2008, an edge coloring c of a connected graph G of order 3 or more leading to a vertex coloring in the same fashion was introduced by Horňákb, Soták, Palmer and Woźniak [27] where again $c : E(G) \to [k]$ is unrestricted, but the induced set vertex coloring c' need only be proper. They referred to such a coloring c as a *neighbour-distinguishing coloring* and the minimum k for such a coloring exists is called the *general neighbour-distinguishing index* of G, denoted by $gndi(G)$. These concepts were studied further in [7, 30], under the unified terminology *majestic edge coloring* and *majestic index*, which emphasizes the relationship between the proper and rainbow cases. The majestic index of a connected graph G of order 3 or more is denoted by $\text{maj}(G)$. In this work, we will use the terminology "majestic edge coloring" and "majestic index" with the notation $\text{maj}(G)$ for this parameter of a graph G. Other concepts related to majestic edge colorings were introduced by Chartrand in 2015 [46].

While an edge coloring c of a graph G typically uses colors from the set $[k]$ for some positive integer k resulting in $c(e) = i$ for some $i \in [k]$, we might equivalently define $c(e) = \{i\}$ as well. In this case, both the edge coloring c and the induced vertex coloring c' assign subsets of $[k]$ to the edges and the vertices of G respectively, where the color assigned to an edge by c is a singleton subset of $[k]$. Looking at c in this manner suggests the idea of studying edge colorings c where both c and its induced vertex coloring c' assign nonempty subsets of $[k]$ to the elements (edges and vertices) of a graph G such that the color assigned to an edge of G by c is not necessarily a singleton subset of $[k]$. This observation gives rise to two main concepts of this work; strong royal colorings and royal colorings of graphs as well as the corresponding chromatic indexes.

1.3 Sum-Defined Colorings

Here we consider edge colorings that give rise to vertex colorings using the process described in (3). The unrestricted edge colorings inducing sum-defined vertex colorings that have attracted the most attention are those where the vertex colorings are either vertex-distinguishing or neighbor-distinguishing. A nontrivial graph has been called *irregular* if its vertices have distinct degrees. It is well known that no graph is irregular. This observation led to the concept of *irregularity strength*, introduced by Chartrand [14] at the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne (now called Purdue University Fort Wayne) in 1986. In the past 30 years, this topic has been studied extensively by many researchers, as described in [15, 45, 46].

For a connected graph G, a *weighting* w of G is an assignment of numbers (usually positive integers) to the edges of G, where $w(e)$ denotes the weight of an edge e of G. This then converts G into a weighted graph in which the (*weighted*) *degree* of a vertex v is defined as the sum of the weights of the edges incident with v. A weighted graph G is then *irregular* if the vertices of G have distinct (weighted) degrees. This concept can be viewed using graph coloring terminology.

Let G be a connected graph of order at least 3. An unrestricted edge coloring $c : E(G) \to \mathbb{N}$ induces a vertex coloring $c' : V(G) \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers, defined by $c'(v) = \sum_{e \in E_v} c(e)$ for each vertex v of G, where E_v is the set of the edges incident with v in G. Here, the induced vertex coloring c is required to be vertex-distinguishing. In this case, c is called a *vertex-distinguishing edge coloring* of G. The minimum of the largest colors used among all vertex-distinguishing edge colorings of G is called the *irregularity strength* of G. The goal was for the vertices to have distinct colors, regardless of how large the induced vertex colors may be. This observation motivates the two other primary concepts of this work, namely rainbow mean colorings and proper mean colorings of graphs as well as the corresponding chromatic indexes.

Chapter 2

Royal Colorings

Abstract: For a graph G and a positive integer k, a royal k-edge coloring of G is an assignment of nonempty subsets of the set $\{1, 2, \ldots, k\}$ to the edges of G that gives rise to a proper vertex coloring in which the color assigned to each vertex v is the union of the sets of colors of the edges incident with v. If the resulting vertex coloring is rainbow, then the edge coloring is a strong royal k-edge coloring. The minimum positive integer k for which a graph has a strong royal k-edge coloring is the strong royal index of the graph. The primary emphasis here is on strong royal colorings of trees.

2.1 Introduction

For a connected graph G of order 3 or more and a positive integer k, let $c : E(G) \to [k] = \{1, 2, \ldots, k\}$ be an unrestricted edge coloring of G. Again such a coloring allows adjacent edges of G to be assigned the same color. We write $\mathcal{P}^*([k])$ for the set consisting of the $2^k - 1$ nonempty subsets of $[k]$. The edge coloring c induces a vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ where $c'(v)$ is the set of colors of the edges incident with v. If c' is a proper vertex coloring of G, then c is a *majestic k-edge coloring* and the minimum positive integer k for which G has a majestic k-edge coloring is the *majestic index* $\text{maj}(G)$ of G. If c' is rainbow (that is, $c'(u) \neq c'(v)$ for every two distinct vertices u and v of G), then c is a *strong majestic k-edge coloring* and the minimum positive integer k for which G has a strong majestic k-edge coloring is the *strong majestic index* $\text{smaj}(G)$ of G. Majestic edge colorings were introduced by Györi, Horňák, Palmer, and Woźnick [27] under different terminology and studied further in [30, 34]. Strong majestic edge colorings were introduced by Harary and Plantholt [29] in 1985, also using different terminology, and studied further by others (see [23, 45, 46]).

The following is an immediate observation concerning these indexes.

Proposition 2.1.1 *Every connected graph G of size $m \geq 2$ has a strong majestic coloring and therefore a majestic coloring. Furthermore,*

$$2 \leq \text{maj}(G) \leq \text{smaj}(G) \leq m.$$

Proof. For a connected graph G with $E(G) = \{e_1, e_2, \ldots, e_m\}$, define an edge coloring $c : E(G) \to [m]$ by $c(e_i) = i$ for $1 \leq i \leq m$. Since the sets of edges incident with distinct vertices are distinct, it follows that c is a strong majestic m-edge coloring of G, producing the desired inequalities. ■

The following results were obtained by Harary and Plantholt [29] on complete graphs K_n, complete bipartite graphs $K_{s,t}$, paths P_n, cycles C_n, and hypercubes Q_n.

Theorem 2.1.2 [29] *For every integer $n \geq 3$,*

$$\text{smaj}(K_n) = \text{maj}(K_n) = 1 + \lceil \log_2 n \rceil.$$

Theorem 2.1.3 [29] *For integers s and t with $2 \leq s \leq t$,*

$$\text{smaj}(K_{s,t}) \leq 2 + \lceil \log_2 t \rceil.$$

In particular, $1 + \lceil \log_2 t \rceil \leq \text{smaj}(K_{t,t}) \leq 2 + \lceil \log_2 t \rceil$ for each integer $t \geq 2$.

Theorem 2.1.4 [29] *For each integer $n \geq 3$,*

$$\text{smaj}(P_n) = \min\left\{2\left\lceil \frac{1+\sqrt{8n-9}}{4} \right\rceil - 1,\ 2\left\lceil \sqrt{\frac{n-1}{2}} \right\rceil\right\},$$

$$\text{smaj}(C_n) = \min\left\{2\left\lceil \frac{1+\sqrt{8n+1}}{4} \right\rceil - 1,\ 2\left\lceil \sqrt{\frac{n}{2}} \right\rceil\right\}.$$

Theorem 2.1.5 [29] *For each integer $n \geq 2$, $\text{smaj}(Q_n) = n + 1$.*

Theorem 2.1.6 [29] *For each integer $k \geq 2$, the largest order $M(k)$ of a tree with strong majestic index k is*

$$M(k) = \begin{cases} \frac{k^2+3k-4}{2} & \text{if } k \geq 2 \text{ and } k \neq 4 \\ 11 & \text{if } k = 4. \end{cases}$$

The following is a consequence of Theorem 2.1.6.

Corollary 2.1.7 *If T is a tree of order $n \geq 3$ and $\text{smaj}(T) \neq 4$, then*

$$\text{smaj}(T) \geq \left\lceil \frac{\sqrt{8n+25}-3}{2} \right\rceil.$$

Proof. Let T be a tree of order $n \geq 3$ with $\text{smaj}(T) = k \neq 4$. It then follows by Theorem 2.1.6 that $n \leq \frac{k^2+3k-4}{2}$ and so $k^2 + 3k - 2n - 4 \geq 0$, producing the desired inequality. ■

2.2 Motivation

Recall in a majestic coloring c of a graph G that the edge colors are positive integers. As stated in the introduction, this concept can be generalized by allowing the edge coloring c to assign nonempty subsets of $[k]$ for some k. The following scenario illustrates why broadening majestic colorings in this manner is advantageous.

Consider a social media platform where users can connect with one another in some way. For example, on the popular site Facebook, users can connect as "friends." Similarly on the professional networking platform LinkedIn, users can send and accept "connection invitations." Given a collection of users from such a site, a graph (social network) can be constructed, where the vertices of the graph are the users and two users are joined by an edge if they are connected on the site. Some sites have connection processes that are not symmetric, in which case a digraph could be used, but here we assume that the social network is a graph.

Suppose common interests between friends in a social network are being tracked. For simplicity, we will refer to two connected users as friends. Let the representative graph be connected and assume further that any pair of friends has at least one common interest. If we create a list of interests for each user by compiling all the interests they share with at least one friend, what conditions must be met so that no two users who are friends share the same list of interests. Moreover, is it possible distinct users in the network always have distinct lists of interests? If either of the former questions can be answered in the affirmative, how many distinct interests must be tracked and how do those interests have to be distributed across the social network?

If we replace interests by positive integers, then these questions suggest studying a set-defined edge coloring of a graph that gives rise to a set-defined coloring of the vertices of the graph in a similar fashion to the majestic coloring problem. This type of coloring is the primary subject of this section.

2.3 The Royal Index of a Graph

In a majestic edge coloring of a graph G, the colors assigned to the edges of G are elements of some set $[k]$ for a positive integer k, which results in a proper vertex coloring of G where the color of a vertex v is the set of colors of the edges incident with v. If the vertex coloring is rainbow, then the edge coloring is a strong majestic edge coloring of G. Here, we consider edge colorings, called royal colorings and strong royal colorings, where the colors assigned to the edges of a graph are nonempty subsets of a set $[k]$ rather than elements of $[k]$.

For a connected graph G of order 3 or more, let $c : E(G) \to \mathcal{P}^*([k])$ be an unrestricted edge coloring of G for some positive integer k. The edge coloring c produces the vertex coloring $c' : V(G) \to \mathcal{P}^*([k])$ defined by

$$c'(v) = \bigcup_{e \in E_v} c(e),$$

where E_v is the set of edges of G incident with v. If c' is a proper vertex coloring of G, then c is called a *royal k-edge coloring* of G. An edge coloring c is a *royal coloring* of G if c is a royal k-edge coloring for some positive integer k. The minimum positive integer k for which a graph G has a royal k-edge coloring is the *royal index* $\text{roy}(G)$ of G. If c' is rainbow, then c is a *strong royal k-edge coloring* of G. An edge coloring c is a *strong royal coloring* of G if c is a strong royal k-edge coloring for some positive integer k. The minimum positive integer k for which a graph G has a strong royal k-edge coloring is the *strong royal index* $\text{sroy}(G)$ of G. This concept was independently introduced and studied in [8, 17]. While no royal coloring exists for the graph K_2, such a coloring exists for every connected graph of order at least 3. Since every strong majestic edge coloring is a strong royal coloring and every majestic edge coloring is a royal coloring, the following is a consequence of Proposition 2.1.1.

Proposition 2.3.1 *Every connected graph G of order 3 or more has a strong royal coloring and therefore a royal coloring. Furthermore,*

$$2 \leq \text{roy}(G) \leq \text{maj}(G) \leq \text{smaj}(G) \text{ and } \text{roy}(G) \leq \text{sroy}(G) \leq \text{smaj}(G).$$

If G is a connected graph of order 3, then either $G = P_3$ or $G = K_3$. It is easy to see that $\text{sroy}(P_3) = \text{smaj}(P_3) = 2$ and $\text{sroy}(K_3) = \text{smaj}(K_3) = 3$. Since $|\mathcal{P}^*([2])| = 3$, it follows that $\text{sroy}(G) \geq 3$ for every connected graph G of order $n \geq 4$. This implies that P_3 is the only connected graph with strong royal index 2. In what follows, we consider only connected graphs of order at least 4. For example, consider the star $G = K_{1,4}$ of size 4. Figure 2.1 shows a royal 2-edge coloring, a strong royal 3-edge coloring, and a strong majestic 4-edge coloring of G. For simplicity, we write the set $\{a\}$ as a, $\{a, b\}$ as ab, and $\{a, b, c\}$ as abc. In fact, $\text{roy}(G) = 2, \text{sroy}(G) = 3$, and $\text{smaj}(G) = 4$ for this graph G. Thus, the values of the three parameters $\text{roy}(G)$, $\text{sroy}(G)$, and $\text{smaj}(G)$ can be different for a graph G. Moreover, the value of $\text{smaj}(G) - \text{sroy}(G)$ can be arbitrarily large for a connected graph G (as we will see in Section 2.4). It can also occur that $\text{smaj}(G) = \text{sroy}(G)$ for connected graphs G of order 4 or more.

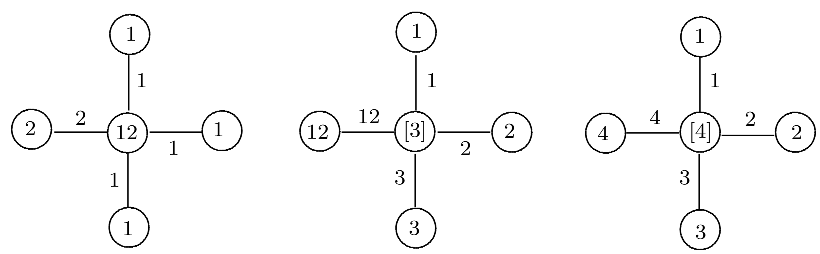

Figure 2.1: A graph G with $\text{roy}(G) = 2, \text{sroy}(G) = 3$, and $\text{smaj}(G) = 4$

Proposition 2.3.2 *For every integer $n \geq 4$,*

$$\text{sroy}(K_n) = \text{smaj}(K_n) = 1 + \lceil \log_2 n \rceil .$$

Proof. Since $\text{sroy}(K_n) \leq 1 + \lceil \log_2 n \rceil$ by Theorem 2.1.2 and Proposition 2.3.1, it remains to show that $\text{sroy}(K_n) \geq 1 + \lceil \log_2 n \rceil$. Suppose that $\text{sroy}(K_n) = k$ for an integer $n \geq 4$. Then there exists a strong royal k-edge coloring $c : E(K_n) \to \mathcal{P}^*([k])$ of K_n such that the induced vertex coloring $c' : V(K_n) \to \mathcal{P}^*([k])$ is rainbow; so $c'(u) \neq c'(v)$ for every two distinct vertices u and v of K_n. However, since $c'(u)$ and $c'(v)$ both contain the color $c(uv)$, it follows that $c'(u) \cap c'(v) \neq \emptyset$. Thus, if $A \subseteq [k]$ such that $c'(x) = A$ for some vertex x of K_n, then $c'(y) \not\subseteq \overline{A} = [k] - A$ for every vertex y of K_n distinct from x. Hence, there are at most 2^{k-1} possible colors for the n vertex colors of K_n, implying that $n \leq 2^{k-1}$ and so $\log_2 n \leq k - 1$. Therefore, $\text{sroy}(K_n) = k \geq 1 + \lceil \log_2 n \rceil$ resulting in $\text{sroy}(K_n) = 1 + \lceil \log_2 n \rceil$. ■

There are other connected graphs G for which $\text{smaj}(G) = \text{sroy}(G)$. First, we present a lower bound for the strong royal index of any connected graph of order 4 or more in terms of its order.

Proposition 2.3.3 *If G is a connected graph of order $n \geq 4$, then*

$$\text{sroy}(G) \geq \lceil \log_2(n+1) \rceil = 1 + \lfloor \log_2 n \rfloor.$$

Proof. Suppose that $\text{sroy}(G) = k$ and let $c : E(G) \to \mathcal{P}^*([k])$ be a strong royal k-edge coloring of G. Then the induced coloring $c' : V(G) \to \mathcal{P}^*([k])$ is rainbow. Since $c'(v) \neq \emptyset$ for each vertex v of G and $|\mathcal{P}^*([k])| = 2^k - 1$, it follows that $n \leq 2^k - 1$ and so $\text{sroy}(G) = k \geq \lceil \log_2(n+1) \rceil = 1 + \lfloor \log_2 n \rfloor$. ■

For the hypercubes Q_n with $n \geq 3$, we have $\text{sroy}(Q_n) \leq \text{smaj}(Q_n) = n + 1$ by Propositions 2.1.5 and 2.3.1. Since the order of Q_n is 2^n, it follows by Proposition 2.3.3 that $\text{sroy}(Q_n) \geq \lceil \log_2(2^n + 1) \rceil = n + 1$. These observations provide the following result.

Proposition 2.3.4 *For an integer $n \geq 3$,*

$$\text{sroy}(Q_n) = \text{smaj}(Q_n) = \lceil \log_2(2^n + 1) \rceil = n + 1.$$

If G is a connected graph of order 4, then

$$G \in \{K_4, K_4 - e, (K_2 + K_1) \vee K_1, C_4, P_4, K_{1,3}\}.$$

By Proposition 2.3.3, $\text{sroy}(G) \geq 3$. Figure 2.2 shows a strong royal 3-edge coloring for each of these graphs. Thus, $\text{sroy}(G) = 3 = \lceil \log_2(n+1) \rceil$ for every connected graph G of order $n = 4$. Furthermore, $\text{smaj}(G) = \text{sroy}(G) = 3$ for these six graphs G. In fact, for each integer $n \geq 4$, there is a connected graph G of order $n \geq 4$ such that $\text{sroy}(G) = \lceil \log_2(n+1) \rceil$, as we will see in Section 2.4.

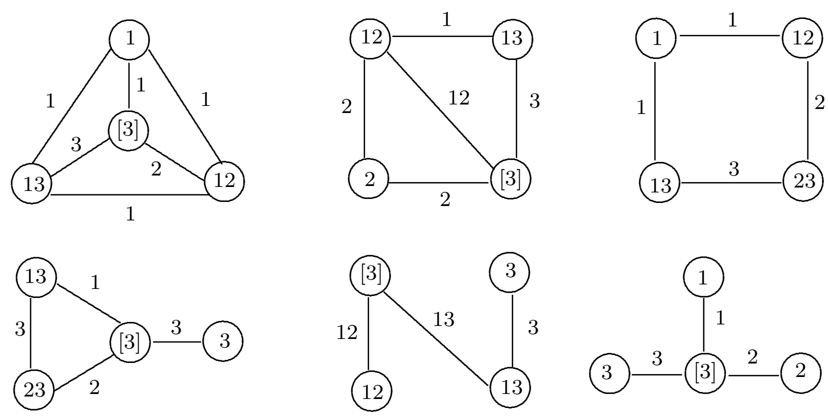

Figure 2.2: Strong royal 3-edge colorings of connected graphs of order 4

2.4 Strong Royal Colorings of Trees

In Proposition 2.3.3, a lower bound for the strong royal index of a connected graph G was presented in terms of its order. Next, we present an upper bound for the strong royal index of G in terms of the strong royal indexes of the connected spanning subgraphs of G. This bound shows the value of determining the strong royal indexes of trees.

Proposition 2.4.1 *If G is a connected graph of order 3 or more, then*

$$\text{sroy}(G) \leq 1 + \min\{\text{sroy}(H) : \ H \textit{ is a connected spanning subgraph of } G\}.$$

In particular,

$$\text{sroy}(G) \leq 1 + \min\{\text{sroy}(T) : \ T \textit{ is a spanning tree of } G\}. \tag{2.1}$$

Proof. Among all connected spanning subgraphs of G, let H be one having the minimum strong royal index, say $\text{sroy}(H) = k$. Let $c_H : E(H) \to \mathcal{P}^*([k])$ be a strong royal k-edge coloring of H. Then $c'_H(x) \neq c'_H(y)$ for every two distinct vertices x and y of H. We extend c_H to an edge coloring $c_G : E(G) \to \mathcal{P}^*([k+1])$ of G by defining

$$c_G(e) = \begin{cases} c_H(e) & \text{if } e \in E(H) \\ \{k+1\} & \text{if } e \in E(G) - E(H). \end{cases}$$

Since either $c'_G(x) = c'_H(x) \subseteq [k]$ or $c'_G(x) = c'_H(x) \cup \{k+1\}$ for each $x \in V(G)$ and c'_H is rainbow, it follows that c'_G is rainbow. Therefore, c_G is a strong royal $(k+1)$-edge coloring of G and so $\text{sroy}(G) \le k+1 = \text{sroy}(H)+1$. The inequality (2.1) follows immediately. ■

As a consequence of Proposition 2.4.1, if we know the strong royal indexes of all spanning trees of a connected graph G, then we have an upper bound for $\text{sroy}(G)$. Consequently, we now turn to investigating the strong royal indexes of trees of order 4 or more. By Proposition 2.3.3, if T is a tree of order $n \ge 4$, then $\text{sroy}(T) \ge \lceil \log_2(n+1) \rceil$. We show that there is equality for this bound when T is either a star or a path.

Proposition 2.4.2 *For every integer $n \ge 4$,*

$$\text{sroy}(K_{1,n-1}) = \lceil \log_2(n+1) \rceil .$$

Proof. Let $k = \lceil \log_2(n+1) \rceil \ge 3$ and let $G = K_{1,n-1}$ be a star of order n, where $V(G)=\{v, v_1, v_2, \ldots, v_{n-1}\}$ and $\deg_G v = n-1$. By Proposition 2.3.3, it suffices to show that G has a strong royal k-edge coloring. Since $k = \lceil \log_2(n+1) \rceil \ge 3$, it follows that

$$2^{k-1} - 1 \le n-1 \le 2^k - 2.$$

Let $S_1, S_2, \ldots, S_{2^k-2}$ be the $2^k - 2$ distinct nonempty proper subsets of $[k]$, where $S_i = \{i\}$ for $1 \le i \le k$. Define the coloring $c : E(G) \to \mathcal{P}^*([k])$ by $c(vv_i) = S_i$ for $1 \le i \le n-1$. Since $c'(v_i) = S_i$ for $1 \le i \le n-1$ and $c'(v) = [k]$, it follows that c' is rainbow. Therefore, c is a strong royal k-edge coloring of G and so $\text{sroy}(G) = \lceil \log_2(n+1) \rceil$. ■

For every path P_n of order $n \ge 4$, it was shown in [8] that there exists a strong royal coloring of P_n using colors from the set $\mathcal{P}^*([k])$ where $k = \lceil \log_2(n+1) \rceil$. Here, we provide a constructive proof that describes an appropriate strong royal coloring for each path P_n of order $n \ge 4$.

Theorem 2.4.3 *For every integer $n \geq 4$,* $\text{sroy}(P_n) = \lceil \log_2(n+1) \rceil$.

Proof. Let $k = \lceil \log_2(n+1) \rceil \geq 3$. Then $2^{k-1} \leq n \leq 2^k - 1$. By Proposition 2.3.3, it suffices to show that G has a strong royal k-edge coloring. For $4 \leq n \leq 7$, there is a strong royal 3-edge coloring of P_n (shown in Figure 2.3) and so $\text{sroy}(P_n) = 3 = \lceil \log_2(n+1) \rceil$. We may assume that $n \geq 8$.

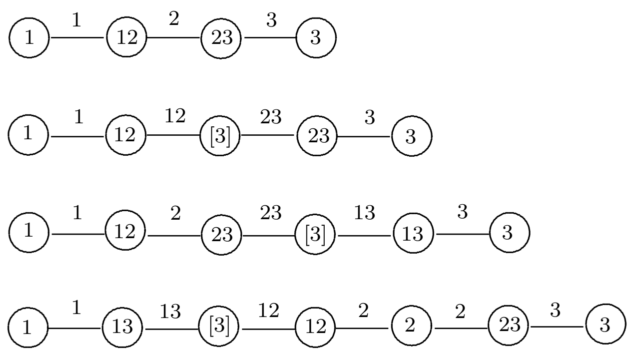

Figure 2.3: Strong royal 3-edge colorings of P_n for $4 \leq n \leq 7$

First, we construct strong royal 4-edge colorings of P_8 and P_9 from a strong royal 3-edge coloring of P_4 as follows. Let P_8 be constructed from two copies of P_4, namely (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4), by adding the edge u_4v_4 and let P_9 be obtained from P_8 by adding a new vertex v_0 and the new edge v_0v_1. (That is, P_9 is constructed from $P_4 = (u_1, u_2, u_3, u_4)$ and $P_5 = (v_0, v_1, v_2, v_3, v_4)$ by adding the edge u_4v_4.) Let c_4 be a strong royal 3-edge coloring of P_4. Define the strong royal 4-edge coloring $c_8 : E(P_8) \to \mathcal{P}^*([4])$ of P_8 as follows:

$$c_8(e) = \begin{cases} c_4(e) & \text{if } e = u_iu_{i+1} \text{ for } 1 \leq i \leq 3 \\ c_4(u_3u_4) & \text{if } e = u_4v_4 \\ c_4(u_iu_{i+1}) \cup \{4\} & \text{if } e = v_iv_{i+1} \text{ for } 1 \leq i \leq 3. \end{cases}$$

Since

$$c_8'(u_i) = c_4'(u_i) \text{ and } c_8'(v_i) = c_4'(u_i) \cup \{4\}$$

for $1 \leq i \leq 4$, it follows that c_8' is rainbow. Thus, c_8 is a strong royal 4-edge coloring of P_8. Next, we extend this strong royal 4-edge coloring c_8 of P_8 to a strong royal 4-edge coloring c_9 by assigning $\{4\}$ to the edge v_0v_1. Since $c_9'(v_0) = \{4\}$ and

$c'_9(x) = c'_8(x) \neq \{4\}$ if $x \neq v_0$, it follows that c'_9 is rainbow. Hence, c_9 is a strong royal 4-edge coloring of P_9. Thus, $\text{sroy}(P_8) = \text{sroy}(P_9) = 4$. These two colorings are illustrated in Figure 2.4. Similarly we can construct strong royal 4-edge colorings of P_{2t} and P_{2t+1} from a strong royal 3-edge coloring of P_t for $t = 5, 6, 7$. It follows that if $8 \leq n \leq 15$, then $\text{sroy}(P_n) = 4$.

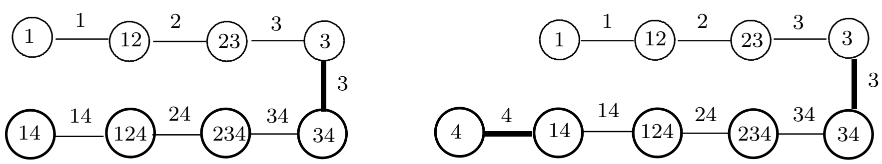

Figure 2.4: Constructing strong royal 4-edge colorings of P_8 and P_9

Suppose for an integer $n \geq 8$ such that $2^{k-1} \leq n \leq 2^k - 1$ for some integer k that $\text{sroy}(P_n) = k$. Let $c_n : E(P_n) \to \mathcal{P}^*([k])$ be a strong royal k-edge coloring of P_n. Since $2^{k-1} \leq n \leq 2^k - 1$, it follows that $2^k \leq 2n < 2^{k+1} - 1$ and $2^k < 2n + 1 \leq 2^{k+1} - 1$. Hence, $\lceil \log_2(2n + 1) \rceil = \lceil \log_2(2n + 2) \rceil = k + 1$. We construct strong royal $(k + 1)$-edge colorings of P_{2n} and P_{2n+1} from the strong royal k-edge coloring c_n of P_n as follows. Let P_{2n} be constructed from two copies of P_n, namely $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$, by adding the edge $u_n v_n$ and let P_{2n+1} be obtained from P_{2n} by adding a new vertex v_0 and joining v_0 to v_1 with the edge $v_0 v_1$. Define the edge coloring $c_{2n} : E(P_{2n}) \to \mathcal{P}^*([k + 1])$ of P_{2n} as follows:

$$c_{2n}(e) = \begin{cases} c_n(e) & \text{if } e = u_i u_{i+1} \text{ for } 1 \leq i \leq n - 1 \\ c_n(u_{n-1} u_n) & \text{if } e = u_n v_n \\ c_n(u_i u_{i+1}) \cup \{k + 1\} & \text{if } e = v_i v_{i+1} \text{ for } 1 \leq i \leq n - 1. \end{cases}$$

Since

$$c'_{2n}(u_i) = c'_n(u_i) \text{ and } c'_{2n}(v_i) = c'_n(u_i) \cup \{k + 1\}$$

for $1 \leq i \leq n$, it follows that c'_{2n} is rainbow. Thus, c_{2n} is a strong royal $(k+1)$-edge coloring of P_{2n}. Next, we extend this strong royal $(k + 1)$-edge coloring c_{2n} of P_{2n} to a strong royal $(k + 1)$-edge coloring c_{2n+1} of P_{2n+1} by assigning $\{k + 1\}$ to the edge $v_0 v_1$. Since $c'_{2n+1}(v_0) = \{k + 1\}$ and $c'_{2n+1}(x) = c'_{2n}(x) \neq \{k + 1\}$ if $x \neq v_0$, it follows that c'_{2n+1} is rainbow. Hence, c_{2n+1} is a strong royal $(k + 1)$-edge coloring of P_{2n+1}. This is illustrated in Figure 2.5 for $n = 8$ and $k = 4$, where a strong royal

5-edge coloring of P_{17} is constructed from a strong royal 4-edge coloring of P_8. Deleting the vertex labeled 5 from P_{17}, we obtain a strong royal 5-edge coloring of P_{16}.

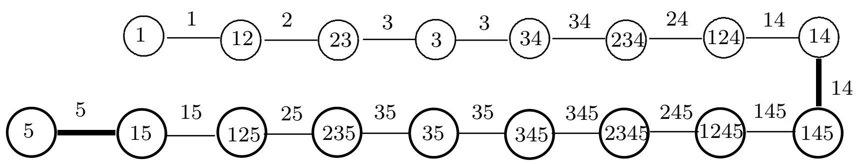

Figure 2.5: Constructing strong royal 5-edge colorings of P_{16} and P_{17}

Since the desired coloring exists, $\text{sroy}(P_n) = \lceil \log_2(n+1) \rceil$ for each integer $n \geq 4$. ■

By Proposition 2.3.3, $\text{sroy}(T) \geq 3$ for a tree T of order n where $4 \leq n \leq 7$. In fact, the following result can be readily verified.

Proposition 2.4.4 *If T is a tree of order n where $4 \leq n \leq 7$, then*

$$\text{sroy}(T) = 3.$$

Figure 2.6 shows a strong royal 3-edge coloring for each tree of order 7.

We now determine this parameter for double stars. Recall that a *double star* is a tree of diameter 3.

Theorem 2.4.5 *If T is a double star of order $n \geq 4$, then*

$$\text{sroy}(T) = \lceil \log_2(n+1) \rceil.$$

Proof. By Proposition 2.4.4, we may assume that T is a double star of order $n \geq 8$. Let $k = \lceil \log_2(n+1) \rceil \geq 4$. Then $2^{k-1} \leq n \leq 2^k - 1$. By Proposition 2.3.3, it suffices to show that T has a strong royal k-edge coloring. Let u and v be the central vertices of T where $\deg_T u = a$ and $\deg_T v = b$. Suppose that u is adjacent to the end-vertices $u_1, u_2, \ldots, u_{a-1}$ and v is adjacent to the end-vertices $v_1, v_2, \ldots, v_{b-1}$. We may assume that $2 \leq a \leq b$. Since $2^{k-1} \leq n = a + b \leq 2^k - 1$, $2 \leq a \leq b$, and $k \geq 4$, it follows that

$$1 \leq a - 1 \leq 2^{k-1} - 2 \text{ and } k - 1 \leq b - 1 \leq 2^k - a - 2. \tag{2.2}$$

We consider two cases, according to $a \leq k$ or $a \geq k + 1$.

Case 1. $2 \leq a \leq k$. Let $p = a - 1$. Then $1 \leq p \leq k - 1$ and $b - 1 \leq 2^k - p - 3$ by (2.2). For each integer i with $1 \leq i \leq p$, let $X_i = \{i\}$ for $1 \leq i \leq p$. Next, let

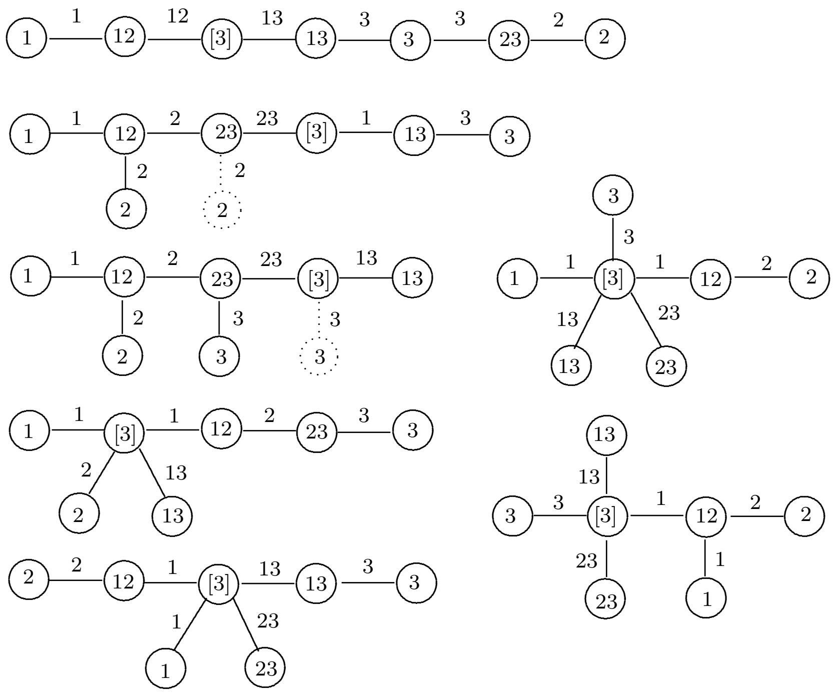

Figure 2.6: Strong royal 3-edge colorings of trees of order 7

$$Y = \mathcal{P}^*([k]) - (\{[p], [k]\} \cup \{X_i : \ 1 \le i \le p\}).$$

Then $|Y| = 2^k - p - 3$. Let $Y_1, Y_2, \ldots, Y_{2^k-p-3}$ be the $2^k - p - 3$ distinct elements of Y such that $Y_j = \{j, k\}$ for $1 \le j \le k-1$. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} X_1 & \text{if } e = uv \text{ or } e = uu_1 \\ X_i & \text{if } e = uu_i \text{ for } 2 \le i \le p \\ Y_j & \text{if } e = vv_i \text{ for } 1 \le j \le b-1 \le 2^k - p - 3. \end{cases}$$

This is shown in Figure 2.7 for a double star of order $n = 15$ where $a = 4$ and $b = 11$. Since $p = a - 1 \le k - 1$ and $b - 1 \ge k - 1$, it follows that $c'(u) = [p] \ne [k] = c'(v)$. In fact, the induced vertex coloring $c' : V(T) \to \mathcal{P}^*([k])$ of T is given by

$$c'(x) = \begin{cases} X_i & \text{if } x = u_i \text{ for } 1 \le i \le p \\ [p] & \text{if } x = u \\ [k] & \text{if } x = v \\ Y_j & \text{if } x = v_j \text{ for } 1 \le j \le b-1 \le 2^k - p - 3. \end{cases}$$

Since c' is rainbow, c is a strong royal k-edge coloring of T.

Case 2. $k+1 \le a \le 2^{k-1}-1$. Let $p = a-1$. It follows that

$$k \le p \le 2^{k-1} - 2 = |\mathcal{P}^*([k-1]) - \{[k-1]\}|.$$

Let X_1, X_2, $\ldots$, X_p be distinct elements of $\mathcal{P}^*([k-1]) - \{[k-1]\}$ such that $X_i = \{i\}$ for $1 \le i \le k-2$. Next, let

$$Y = \mathcal{P}^*([k]) - (\{[k-1],[k]\} \cup \{X_i :\ 1 \le i \le p\}).$$

Then $|Y| = 2^k - 3 - p$. Let $Y_1, Y_2, \ldots, Y_{2^k-p-3}$ be the $2^k - p - 3$ distinct elements of Y such that $Y_j = \{j, k\}$ for $1 \le j \le k-1$. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} X_1 & \text{if } e = uv \text{ or } e = uu_1 \\ X_i & \text{if } e = uu_i \text{ and } 2 \le i \le p \\ Y_j & \text{if } e = vv_i \text{ for } 1 \le i \le b-1 \le 2^{k-1} - p - 3. \end{cases}$$

This is shown in Figure 2.7 for a double star of order $n = 15$ where $a = 7$ and $b = 8$. Since $p = a-1 \ge k$ and $b - 1 \ge k-1$, it follows that $c'(u) = [k-1]$ and $c'(v) = [k]$. The induced vertex coloring $c' : V(T) \to \mathcal{P}^*([k])$ is given by

$$c'(x) = \begin{cases} X_i & \text{if } x = u_i \text{ for } 1 \le i \le p \\ [k-1] & \text{if } x = u \\ [k] & \text{if } x = v \\ Y_j & \text{if } x = v_j \text{ for } 1 \le j \le b-1 \le 2^k - p - 3. \end{cases}$$

Since c' is rainbow, c is a strong royal k-edge coloring of T. ■

There are other special classes of trees T of order $n \ge 4$ for which it has been verified that $\text{sroy}(T) = \lceil \log_2(n+1) \rceil$ (see [8, 17]). The results obtained thus far on the strong royal indexes of trees suggest the following conjecture.

Conjecture 2.4.6 *If T is a tree of order $n \ge 4$, then* $\text{sroy}(T) = \lceil \log_2(n+1) \rceil$.

For an integer $n \ge 4$, let k be the unique integer such that $2^{k-1} \le n \le 2^k - 1$. We construct a graph G_k of order $2^k - 1$ as follows. The vertices of G_k are labeled with the $2^k - 1$ distinct elements of $\mathcal{P}^*([k])$. For each $v \in V(G_k)$, let $\ell(v)$ denote the label of v. Thus,

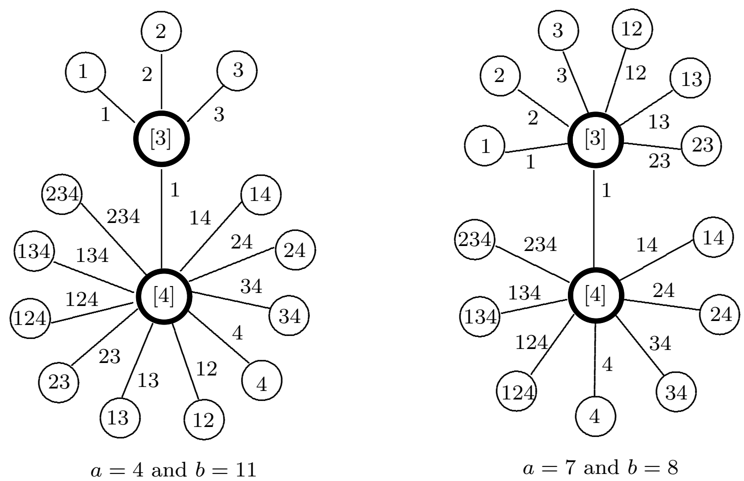

Figure 2.7: Strong royal 4-edge colorings of two double stars of order 15

$$\{\ell(v): \ v \in V(G_k)\} = \mathcal{P}^*([k]).$$

Two vertices u and v of G_k are adjacent in G_k if and only if $\ell(u) \cap \ell(v) \neq \emptyset$. The graph G_3 of order $7 = 2^3 - 1$ is shown in Figure 2.8.

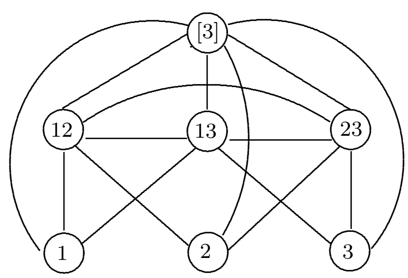

Figure 2.8: The graph G_3 of order $7 = 2^3 - 1$

Conjecture 2.4.6 is true if and only if for every tree T of order $n \geq 4$, where $2^{k-1} \leq n \leq 2^k - 1$, there is a subgraph H of G_k isomorphic to T having the property that every edge uv of H is assigned the color $c(uv) = \ell(u) \cap \ell(v)$ and every vertex v of H is assigned the color $c'(v) = \bigcup_{e \in E_H(v)} c(e)$, where $E_H(v)$ is the set of the edges of H incident with v, such that $c'(v) = \ell(v)$.

For instance, consider the tree T of order 5 in Figure 2.9 and the graph G_3 in Figure 2.8. Figure 2.9 also shows five subgraphs $G_{3,1}, G_{3,2}, G_{3,3}, G_{3,4}, G_{3,5}$ of G_3, each isomorphic to T with the corresponding edge colors and vertex colors described above. The subgraphs $G_{3,3}$ and $G_{3,5}$ result in a strong royal 3-edge coloring of T, which verifies Conjecture 2.4.6 for this tree T. This also shows that there are two distinct ways to give a strong royal 3-edge coloring of T.

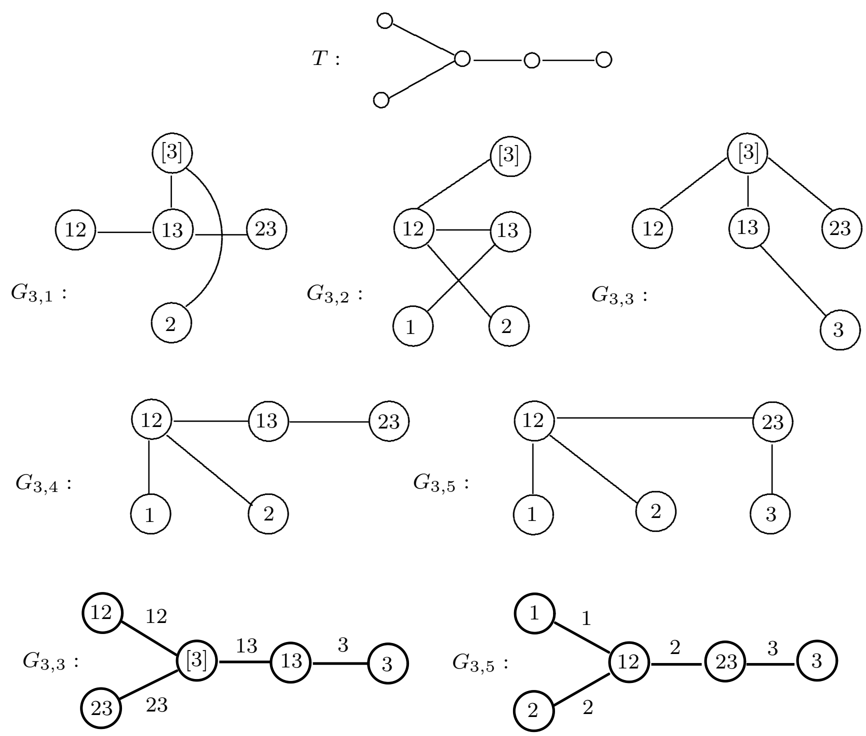

Figure 2.9: Three subgraphs of G_3 isomorphic to T

We have seen that Conjecture 2.4.6 is true for all trees of order n with $4 \leq n \leq 7$ as well as all paths, stars, and double stars. Hence, it remains to show that Conjecture 2.4.6 is true for every tree of order $n \geq 8$ that is not a path, star, or double star. A *caterpillar* is a tree T of order 3 or more, the removal of whose leaves produces a path (called the *spine* of T). A star is therefore a caterpillar of diameter 2 whose spine is a trivial path of order 1 and a double star is a caterpillar of diameter 3 whose spine is a path of order 2. We now move on to the next step by showing that Conjecture 2.4.6 is true as well if T is a caterpillar whose spine has order 3, that is, T has diameter 4. In the proof, we assume that the spine of T is (x, y, z); so T contains a path $P = (s, x, y, z, t)$, where $\deg_T s = \deg_T t = 1$ and $\deg_T x \geq 2$, $\deg_T y \geq 2$, and $\deg_T z \geq 2$.

Theorem 2.4.7 *If T is a caterpillar of order $n \geq 8$ and diameter 4, then*

$$\text{sroy}(T) = \lceil \log_2(n + 1) \rceil .$$

Proof. Let $k = \lceil \log_2(n+1) \rceil \geq 4$. Then $2^{k-1} \leq n \leq 2^k - 1$. By Proposition 2.3.3, it suffices to show that T has a strong royal k-edge coloring.

We consider three cases base on the number of internal vertices of P that have neighbors not in P.

Case 1. *Exactly one of x, y and z has degree exceeding* 2. We may assume that exactly one of x and y is adjacent to $n-5 \geq 3$ vertices not on P.

Subcase 1.1. *x is adjacent to $n-5$ vertices not on P.* Let $x_1, x_2, \ldots, x_{n-5}$ be the neighbors of x not on P and let $e_i = xx_i$ for $1 \leq i \leq n-5$. Let $S_1 = \{1, k\}$, $S_2 = [2, k]$ and let $S_3, S_4, \ldots, S_{n-5}$ be distinct nonempty proper subsets of $[k]$ different from $\{1\}$, $\{2\}$, $\{3\}$, $\{2,3\}$, $\{1,k\}$, $[2,k]$. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} \{1\} & \text{if } e = sx \\ \{2\} & \text{if } e = xy \text{ or } e = yz \\ \{3\} & \text{if } e = zt \\ S_i & \text{if } e = e_i \text{ for } 1 \leq i \leq n-5. \end{cases}$$

Then $c'(s) = \{1\}$, $c'(x) = [k]$, $c'(y) = \{2\}$, $c'(z) = \{2,3\}$, $c'(t) = \{3\}$, and $c'(x_i) = S_i$ for $1 \leq i \leq n-5$. Since c' is rainbow, c is a strong royal k-edge coloring of T.

Subcase 1.2. *y is adjacent to $n-5 \geq 3$ vertices not on P.* Let $y_1, y_2, \ldots, y_{n-5}$ be the neighbors of y not on P and let $e_i = yy_i$ for $1 \leq i \leq n-5$. Let $S_1 = \{1, k\}$, $S_2 = [2, k]$ and let $S_3, S_4, \ldots, S_{n-5}$ be distinct nonempty proper subsets of $[k]$ different from $\{1\}$, $\{3\}$, $\{1,2\}$, $\{2,3\}$, $\{1,k\}$, $[2,k]$. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} \{1\} & \text{if } e = sx \\ \{2\} & \text{if } e = xy \text{ or } e = yz \\ \{3\} & \text{if } e = zt \\ S_i & \text{if } e = e_i \text{ for } 1 \leq i \leq n-5. \end{cases}$$

Then $c'(s) = \{1\}$, $c'(x) = \{1,2\}$, $c'(y) = [k]$, $c'(z) = \{2,3\}$, $c'(t) = \{3\}$, and $c'(x_i) = S_i$ for $1 \leq i \leq n-5$. Since c' is rainbow, c is a strong royal k-edge coloring of T.

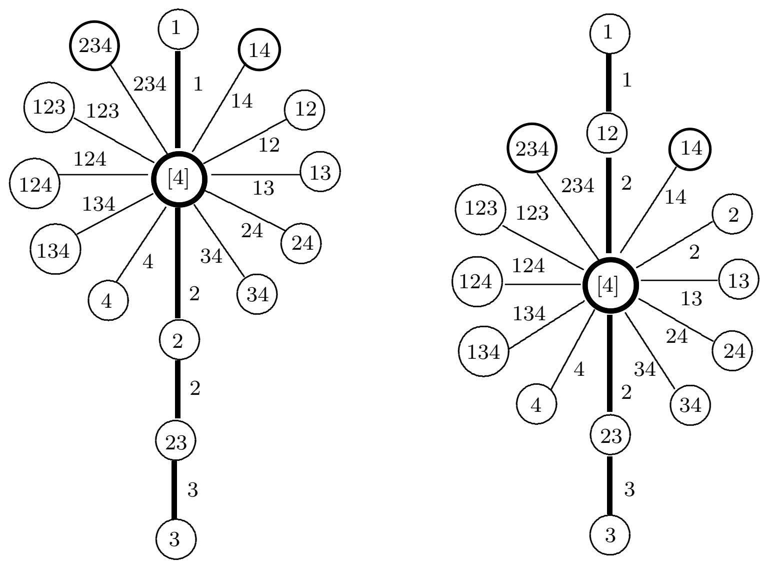

Figure 2.10: Strong royal 4-edge colorings of two caterpillars of order 15 and diameter 4 in Case 1

Case 2. *Exactly two of x, y and z have degree exceeding* 2. We may assume that x has degree exceeding 2 and exactly one of y and z has degree exceeding 2.

Subcase 2.1. *x and z have degree exceeding* 2. We may assume that x is adjacent to the p vertices $x_1, x_2, \ldots, x_p$ not on P and z is adjacent to the q vertices $z_1, z_2, \ldots, z_q$ not on P, where $1 \le p \le q$ and $p + q = n - 5$. Then $p \le \frac{1}{2}(n-5) \le \frac{1}{2}(2^k - 6)$ and so $p \le 2^{k-1} - 3$. Let $S_1, S_2, \ldots, S_p$ be distinct nonempty proper subsets of $[k-1]$ where $S_1 = [2, k-1]$ such that $S_i \ne \{1\}$ for $2 \le i \le p$. Let $T_1, T_2, \ldots, T_p$ be distinct nonempty proper subsets of $[k]$ different from $S_1, S_2, \ldots, S_p$ such that $T_1 = [2, k]$, $T_2 = \{1\} \cup [3, k]$ and $T_i \ne \{1\}, \{k\}, \{1, k\}, [k-1]$ for $3 \le i \le q$. Thus, for $1 \le i \le p$,

$$S_i \in \mathcal{P}^*([k-1]) - \{\{1\}, [k-1]\}$$

and for $1 \le i \le q$,

$$T_i \in \mathcal{P}^*([k]) - [\{S_i : 1 \le i \le p\} \cup \{\{1\}, \{k\}, \{1, k\}, [k-1], [k]\}]$$

Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} \{1\} & \text{if } e = sx \text{ or } e = xy \\ \{k\} & \text{if } e = yz \text{ or } e = zt \\ S_i & \text{if } e = xx_i \text{ for } 1 \le i \le p \\ T_i & \text{if } e = zz_i \text{ for } 1 \le i \le q. \end{cases}$$

Then $c'(s) = \{1\}$, $c'(x) = [k-1]$, $c'(y) = \{1, k\}$, $c'(z) = [k]$, $c'(t) = \{k\}$, $c'(x_i) = S_i$ for $1 \leq i \leq p$ and $c'(z_i) = T_i$ for $1 \leq i \leq q$. Since c' is rainbow, c is a strong royal k-edge coloring of T.

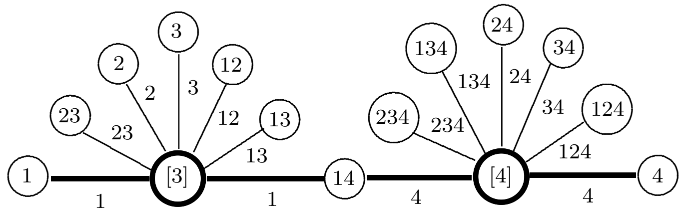

Figure 2.11: A strong royal 4-edge coloring of a caterpillar of order 15 and diameter 4 in Subcase 2.1

Subcase 2.2. *x and y have degree exceeding* 2. Suppose that x is adjacent to the p vertices $x_1, x_2, \ldots, x_p$ not on P and y is adjacent to the q vertices $y_1, y_2, \ldots, y_q$ not on P. Then $p, q \geq 1$ and $p + q = n - 5$. There are two subcases, according to whether $p \leq q$ or $p > q$. Observe that

Subcase 2.2.1. $p \leq q$. Then

$$p \leq \tfrac{1}{2}(n-5) \leq \tfrac{1}{2}(2^k - 6) = 2^{k-1} - 3 = |\mathcal{P}^*([k-1]) - \{\{1\}, [k-1]\}|.$$

Let $S_1, S_2, \ldots, S_p$ be distinct elements of $\mathcal{P}^*([k-1]) - \{\{1\}, [k-1]\}$ where $S_1 = [2, k-1]$ and let $T_1, T_2, \ldots, T_q$ be distinct elements of

$$\mathcal{P}^*([k]) - [\{S_i : \ 1 \leq i \leq p\} \cup \{\{1\}, \{1, k-1, k\}, \{k-1, k\}, [k-1], [k]\}]$$

where $T_1 = [2, k]$. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} \{1\} & \text{if } e \in \{sx, xy, yz\} \\ \{k-1, k\} & \text{if } e = zt \\ S_i & \text{if } e = xx_i \text{ for } 1 \leq i \leq p \\ T_i & \text{if } e = yy_i \text{ for } 1 \leq i \leq q. \end{cases}$$

Then $c'(s) = \{1\}$, $c'(x) = [k-1]$, $c'(y) = [k]$, $c'(z) = \{1, k-1, k\}$, $c'(t) = \{k-1, k\}$, $c'(x_i) = S_i$ for $1 \leq i \leq p$ and $c'(y_i) = T_i$ for $1 \leq i \leq q$. Since c' is rainbow, c is a strong royal k-edge coloring of T.

Subcase 2.2.2. $p > q$. Then

$$q < \frac{1}{2}(n-5) \leq \frac{1}{2}(2^k - 6) = 2^{k-1} - 3 = |\mathcal{P}^*([k-1]) - \{\{1\},[k-1]\}|.$$

Let $S_1, S_2, \ldots, S_q$ be distinct elements of $\mathcal{P}^*([k-1]) - \{\{1\},[k-1]\}$ where $S_1 = [2, k-1]$ and let $T_1, T_2, \ldots, T_p$ be distinct elements of

$$\mathcal{P}^*([k]) - [\{S_i : 1 \leq i \leq p\} \cup \{\{1\},\{1,k-1\},[k-1],[k]\}]$$

where $T_1 = [2,k]$. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} \{1\} & \text{if } e = xy \text{ or } e = zt \\ \{k-1\} & \text{if } e = yz \\ \{k\} & \text{if } e = sx \\ T_i & \text{if } e = xx_i \text{ for } 1 \leq i \leq p \\ S_i & \text{if } e = yy_i \text{ for } 1 \leq i \leq q. \end{cases}$$

Then $c'(s) = \{k\}$, $c'(x) = [k]$, $c'(y) = [k-1]$, $c'(z) = \{1, k-1\}$, $c'(t) = \{1\}$, $c'(x_i) = T_i$ for $1 \leq i \leq p$ and $c'(y_i) = S_i$ for $1 \leq i \leq q$. Since c' is rainbow, c is a strong royal k-edge coloring of T.

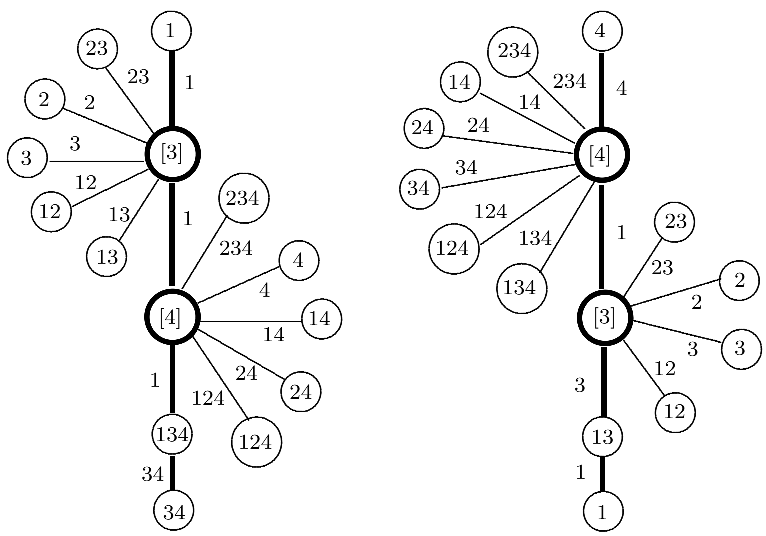

Figure 2.12: Strong royal 4-edge colorings of two caterpillars of order 15 and diameter 4 in Subcase 2.2

Case 3. *Each of x, y, and z has degree* 3 *or more.* Suppose that x is adjacent to the p vertices $x_1, x_2, \ldots, x_p$ not on P, y is adjacent to the q vertices $y_1, y_2, \ldots, y_q$ not on P, and z is adjacent to the r vertices $z_1, z_2, \ldots, z_r$ not on P. Then $p, q, r \geq 1$

and $p+q+r=n-5$. We consider three subcases, according to the values of p, q, and r.

Subcase 3.1. $1 \le p \le q \le r$. Then

$$p \le \tfrac{1}{3}(2^k-6) = \tfrac{2^k}{3} - 2 \text{ and } p+q \le \tfrac{2}{3}(2^k-6) = \tfrac{2^{k+1}}{3} - 4.$$

Since $|\mathcal{P}([k-2]) - \{[k-2]\}| = 2^{k-2}-1$, there are $2^{k-2}-1$ distinct subsets in $\mathcal{P}^*([k-2]\cup\{k\}) - \{[k-2]\cup\{k\}\}$ that contain k. (Note that it is possible that $p \ge 2^{k-2}$.) Let $S_1, S_2, \ldots, S_p$ be p distinct subsets of $\mathcal{P}^*([k-2]\cup\{k\}) - \{[k-2]\cup\{k\}\}$ such that $S_1 = [2,k-2]\cup\{k\}$, $k \in S_i$ for $2 \le i \le p$ if $p \le 2^{k-2}-1$ and $k \in S_i$ for $2 \le i \le 2^{k-2}-1$ if $p \ge 2^{k-2}$, let $T_1, T_2, \ldots, T_q$ be q distinct subsets of $\mathcal{P}^*([k-1]) - \{\{1\}, [k-1]\}$ different from $S_1, S_2, \ldots, S_p$ such that $T_1 = [2, k-1]$, and let $R_1, R_2, \ldots, R_r$ be r distinct subsets of $\mathcal{P}^*([k])$ different from

$$\{1\},\ [k-2]\cup\{k\},\ [k-1],\ [k],\ \{k-1,k\},\ S_1,\ S_2,\ \ldots,\ S_p,\ T_1,\ T_2,\ \ldots,\ T_q$$

such that $R_1 = [2,k]$. Since there are $2^{k-2}-1$ distinct subsets in $\mathcal{P}^*([k-2]\cup\{k\}) - \{[k-2]\cup\{k\}\}$ that contain k and $|\mathcal{P}^*([k-1]) - \{\{1\},[k-1]\}| = 2^{k-1}-3$, it follows that at least

$$(2^{k-2}-1) + (2^{k-1}-3) = 3\cdot 2^{k-2} - 4$$

subsets of $\mathcal{P}^*([k])$ are available for $S_1, S_2, \ldots, S_p, T_1, T_2, \ldots, T_q$. Because

$$p+q \le \tfrac{2^{k+1}}{3} - 4 \le 3\cdot 2^{k-2} - 4,$$

these $p+q$ distinct subsets $S_1, S_2, \ldots, S_p, T_1, T_2, \ldots, T_q$ of $\mathcal{P}^*([k])$ exist. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} \{1\} & \text{if } e \in \{sx, xy, yz\} \\ \{k-1,k\} & \text{if } e = zt \\ S_i & \text{if } e = xx_i \text{ for } 1 \le i \le p \\ T_i & \text{if } e = yy_i \text{ for } 1 \le i \le q \\ R_i & \text{if } e = zz_i \text{ for } 1 \le i \le r. \end{cases}$$

Then $c'(s) = \{1\}$, $c'(x) = [k-2]\cup\{k\}$, $c'(y) = [k-1]$, $c'(z) = [k]$, $c'(t) = \{k-1,k\}$, $c'(x_i) = S_i$ for $1 \le i \le p$, $c'(y_i) = T_i$ for $1 \le i \le q$ and $c'(z_i) = R_i$ for $1 \le i \le r$. Since c' is rainbow, c is a strong royal k-edge coloring of T.

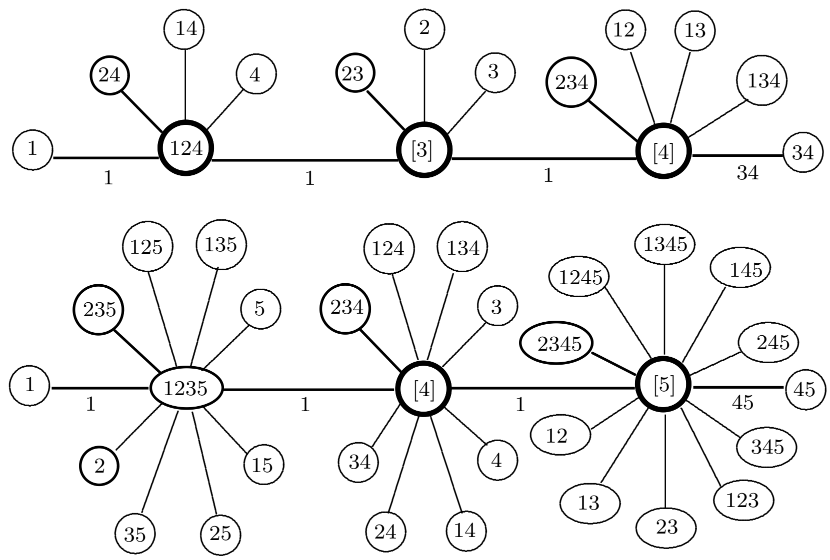

Figure 2.13: Illustrating strong royal edge colorings of caterpillars of order $2^k - 1$ for $k = 4, 5$ in Subcase 3.1

For example, Figure 2.13 shows strong royal edge colorings of caterpillars of order $2^k - 1$ for $k = 4, 5$.

Subcase 3.2. $q < \min\{p, r\}$. We may assume that $q < p \le r$. Then

$$q \le \tfrac{1}{3}(2^k - 6) = \tfrac{2^k}{3} - 2 \text{ and } p + q \le \tfrac{2}{3}(2^k - 6) = \tfrac{2^{k+1}}{3} - 4.$$

Let $S_1, S_2, \ldots, S_q$ be distinct subsets of $\mathcal{P}^*([k-2] \cup \{k\}) - \{[k-2] \cup \{k\}\}$ such that $S_1 = [2, k-2] \cup \{k\}$, $k \in S_i$ for $2 \le i \le q$ if $q \le 2^{k-2} - 1$ and $k \in S_i$ for $2 \le i \le 2^{k-2} - 1$ if $q \ge 2^{k-2}$, let $T_1, T_2, \ldots, T_p$ be distinct subsets of $\mathcal{P}^*([k-1]) - \{\{1\}, [k-1]\}$ different from $S_1, S_2, \ldots, S_q$ such that $T_1 = [2, k-1]$, and let $R_1, R_2, \ldots, R_r$ be distinct subsets of $\mathcal{P}^*([k])$ different from

$$\{1\},\ [k-2] \cup \{k\},\ [k-1],\ [k],\ \{k-1, k\},\ S_1,\ S_2,\ \ldots,\ S_q,\ T_1,\ T_2,\ \ldots,\ T_p$$

such that $R_1 = [2, k]$. Since there are $2^{k-2} - 1$ distinct subsets in $\mathcal{P}^*([k-2] \cup \{k\}) - \{[k-2] \cup \{k\}\}$ that contain k and $|\mathcal{P}^*([k-1]) - \{\{1\}, [k-1]\}| = 2^{k-1} - 3$, it follows that at least

$$(2^{k-2} - 1) + (2^{k-1} - 3) = 3 \cdot 2^{k-2} - 4$$

subsets of $\mathcal{P}^*([k])$ are available for $S_1, S_2, \ldots, S_q, T_1, T_2, \ldots, T_p$. Since

$$p + q \le \tfrac{2^{k+1}}{3} - 4 \le 3 \cdot 2^{k-2} - 4,$$

these $p+q$ distinct subsets $S_1, S_2, \ldots, S_q, T_1, T_2, \ldots, T_p$ exist. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} \{1\} & \text{if } e \in \{sx, xy, yz\} \\ \{k-1, k\} & \text{if } e = zt \\ T_i & \text{if } e = xx_i \text{ for } 1 \le i \le p \\ S_i & \text{if } e = yy_i \text{ for } 1 \le i \le q \\ R_i & \text{if } e = zz_i \text{ for } 1 \le i \le r. \end{cases}$$

Then $c'(s) = \{1\}$, $c'(x) = [k-1]$, $c'(y) = [k-2] \cup \{k\}$, $c'(z) = [k]$, $c'(t) = \{k-1, k\}$, $c'(x_i) = T_i$ for $1 \le i \le p$, $c'(y_i) = S_i$ for $1 \le i \le q$ and $c'(z_i) = R_i$ for $1 \le i \le r$. Since c' is rainbow, c is a strong royal k-edge coloring of T.

Subcase 3.3. $q > \max\{p, r\}$. We may assume that $p \le r < q$. Then

$$p \le \tfrac{1}{3}(2^k - 6) = \tfrac{2^k}{3} - 2 \text{ and } p + r \le \tfrac{2}{3}(2^k - 6) = \tfrac{2^{k+1}}{3} - 4.$$

Let $S_1, S_2, \ldots, S_p$ be distinct subsets of $\mathcal{P}^*([k-2] \cup \{k\}) - \{[k-2] \cup \{k\}\}$ such that $S_1 = [2, k-2] \cup \{k\}$, $k \in S_i$ for $2 \le i \le p$ if $p \le 2^{k-2} - 1$ and $k \in S_i$ for $2 \le i \le 2^{k-2} - 1$ if $p \ge 2^{k-2}$, let $T_1, T_2, \ldots, T_r$ be distinct subsets of $\mathcal{P}^*([k-1]) - \{\{1\}, \{1, k-1\}, [k-1]\}$ different from $S_1, S_2, \ldots, S_p$ such that $T_1 = [2, k-1]$, and let $R_1, R_2, \ldots, R_q$ be distinct subsets of $\mathcal{P}^*([k])$ different from

$$\{1\},\ [k-2] \cup \{k\},\ [k-1],\ [k],\ \{1, k-1\},\ S_1,\ S_2,\ \ldots,\ S_p,\ T_1,\ T_2,\ \ldots,\ T_r$$

such that $R_1 = [2, k]$. Since there are $2^{k-2} - 1$ distinct subsets in $\mathcal{P}^*([k-2] \cup \{k\}) - \{[k-2] \cup \{k\}\}$ that contain k and $|\mathcal{P}^*([k-1]) - \{\{1\}, \{1, k-1\}, [k-1]\}| = 2^{k-1} - 4$, it follows that at least

$$(2^{k-2} - 1) + (2^{k-1} - 4) = 3 \cdot 2^{k-2} - 5$$

subsets of $\mathcal{P}^*([k])$ are available for $S_1, S_2, \ldots, S_p, T_1, T_2, \ldots, T_r$. Since

$$p + r \le \tfrac{2^{k+1}}{3} - 4 \le 3 \cdot 2^{k-2} - 5,$$

these $p + r$ distinct subsets $S_1, S_2, \ldots, S_p, T_1, T_2, \ldots, T_r$ exist. Define an edge coloring $c : E(T) \to \mathcal{P}^*([k])$ by

$$c(e) = \begin{cases} \{1\} & \text{if } e \in \{sx, xy, yz\} \\ \{1, k-1\} & \text{if } e = zt \\ S_i & \text{if } e = xx_i \text{ for } 1 \le i \le p \\ R_i & \text{if } e = yy_i \text{ for } 1 \le i \le q \\ T_i & \text{if } e = zz_i \text{ for } 1 \le i \le r. \end{cases}$$

Then $c'(s) = \{1\}$, $c'(x) = [k-2] \cup \{k\}$, $c'(y) = [k]$, $c'(z) = [k-1]$, $c'(t) = \{k-1, k\}$, $c'(x_i) = S_i$ for $1 \le i \le p$, $c'(y_i) = R_i$ for $1 \le i \le q$ and $c'(z_i) = T_i$ for $1 \le i \le r$. Since c' is rainbow, c is a strong royal k-edge coloring of T. ■

It has been verified in [8] that if G is a connected graph of order $n \ge 4$, then $\text{sroy}(G) \le \lceil \log_2(n+1) \rceil + 2$. On the other hand, if Conjecture 2.4.6 is true, then for a connected graph G of order $n \ge 4$ there are only two possible values for $\text{sroy}(G)$ (namely $\lceil \log_2(n+1) \rceil$ and $\lceil \log_2(n+1) \rceil + 1$) by Propositions 2.3.3 and 2.4.1. Based on these observations, we make the following conjecture.

Conjecture 2.4.8 *If G is a connected graph of order $n \ge 4$, then*

$$\lceil \log_2(n+1) \rceil \le \text{sroy}(G) \le \lceil \log_2(n+1) \rceil + 1.$$

Since we know that the lower bound for $\text{sroy}(G)$ is true in Conjecture 2.4.8, the conjecture can be subsequently restated.

Conjecture 2.4.9 *If G is a connected graph of order $n \ge 4$ where $2^{k-1} \le n \le 2^k - 1$ for some integer k, then there exists a strong royal $(k+1)$-edge coloring of G.*

We have seen numerous examples of connected graphs G of order $n \ge 4$ where $\text{sroy}(G) = \lceil \log_2(n+1) \rceil$. Indeed, every tree of order $n \ge 4$ has either been shown to have strong royal index $\lceil \log_2(n+1) \rceil$ or has been conjectured to have this value for its strong royal index. By Proposition 2.3.2, if $n \ge 4$ is an integer with $2^k < n < 2^{k+1}$ for some integer $k \ge 2$, then $\text{sroy}(K_n) = \lceil \log_2(n+1) \rceil + 1$. Thus, both bounds in Conjecture 2.4.8 are attainable. Hence, if Conjectures 2.4.8 and 2.4.9 are true, then the resulting theorem cannot be improved. The only question that would remain then is for a given connected graph G of order $n \ge 4$, which of these two values is the strong royal index of G?

Chapter 3

Royal-Zero & Royal-One Graphs

Abstract: It was conjectured that if G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ for a positive integer k, then the strong royal index of G is either k or $k+1$. A connected graph G of order $n \geq 3$ where $2^{k-1} \leq n \leq 2^k - 1$ is a royal-zero graph if $\mathrm{sroy}(G) = k$ and is a royal-one graph if $\mathrm{sroy}(G) = k + 1$. We investigate this conjecture for several well-known classes of graphs along with other information concerning royal-zero and royal-one graphs. A sufficient condition for a graph to be royal-one is presented.

3.1 Introduction

We have conjectured that the strong royal index of every connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ is either k or $k + 1$. A connected graph G of order $n \geq 3$ where $2^{k-1} \leq n \leq 2^k - 1$ is a *royal-zero graph* if $\mathrm{sroy}(G) = k$ and is a *royal-one graph* if $\mathrm{sroy}(G) = k + 1$. Using this framework, Conjecture 2.4.9 can be rephrased using this terminology.

Conjecture 3.1.1 *Every connected graph of order at least* 4 *is either royal-zero or royal-one.*

In this chapter, we investigate conditions concerning the size and minimum degree for which a connected graph is royal-zero or royal-one.

3.2 Some Well-Known Classes of Graphs

We begin by considering the strong royal index of every cycle C_n of order $n \geq 3$. Note that the size of C_n is n and $\delta(C_n) = 2$. While the strong royal index of each cycle was stated in [8], we present a proof that provides a strong royal coloring of each cycle C_n of order $n \geq 3$.

Theorem 3.2.1 *For every integer $n \geq 3$,*

$$\mathrm{sroy}(C_n) = \begin{cases} 1 + \lceil \log_2(n+1) \rceil & \text{if } n = 3, 7 \\ \lceil \log_2(n+1) \rceil & \text{if } n \neq 3, 7. \end{cases}$$

That is, if C_n is a cycle of length $n \geq 3$ where $2^{k-1} \leq n \leq 2^k - 1$ for some integer k, then $\mathrm{sroy}(C_n) = k$ *unless $n = 3$ or $n = 7$, in which case,* $\mathrm{sroy}(C_3) = 3$ *and* $\mathrm{sroy}(C_7) = 4$.

Proof. Let $k = \lceil \log_2(n+1) \rceil \geq 2$. Then $2^{k-1} \leq n \leq 2^k - 1$. We show that $\mathrm{sroy}(C_3) = 3$, $\mathrm{sroy}(C_7) = 4$, and $\mathrm{sroy}(C_n) = k$ if $n \neq 3, 7$. Figure 3.1 shows a strong royal 3-edge coloring of C_3 and a strong royal 4-edge coloring of C_7, which implies that $\mathrm{sroy}(C_3) \leq 3$ and $\mathrm{sroy}(C_7) \leq 4$. (As before, we write the set $\{a\}$ as a, $\{a, b\}$ as ab, and $\{a, b, c\}$ as abc.) If $\mathrm{sroy}(C_3) = 2$, then because $|\mathcal{P}^*([2])| = 3$, there are vertices of C_3 colored 1 and 2, implying the existence of two edges that are colored 1 and two edges that are colored 2, which is impossible. If $\mathrm{sroy}(C_7) = 3$, then because $|\mathcal{P}^*([3])| = 7$, there are vertices of C_7 colored 1, 2, and 3, implying that these colors are each assigned to at least two edges of C_7. Regardless of how the seventh edge of C_7 is colored, the resulting set of vertex colors is not $\mathcal{P}^*([3])$. As a result, $\mathrm{sroy}(C_3) = 3$ and $\mathrm{sroy}(C_7) = 4$. By Proposition 2.3.3, it suffices to show that C_n has a strong royal k-edge coloring if $n \neq 3, 7$. Figure 3.1 also shows a strong royal 3-edge coloring for each of C_4, C_5, and C_6 yielding $\mathrm{sroy}(C_n) = 3$ for $n = 4, 5, 6$.

Next, suppose that $n \geq 8$, where $2^{k-1} \leq n \leq 2^k - 1$ for a unique integer $k \geq 4$. We show that C_n has a strong royal k-edge coloring by considering two cases, depending on whether n is even or n is odd. Let $P_n = (v_1, v_2, \ldots, v_n)$ where $e_i = v_i v_{i+1}$ for $1 \leq i \leq n - 1$.

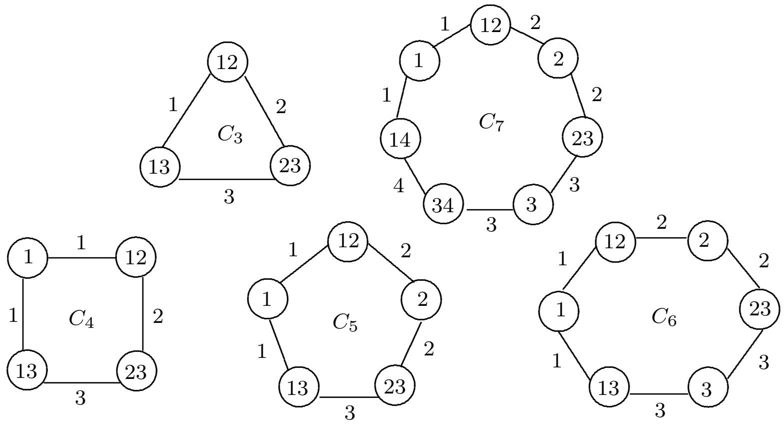

Figure 3.1: Strong royal colorings of C_n where $3 \leq n \leq 7$

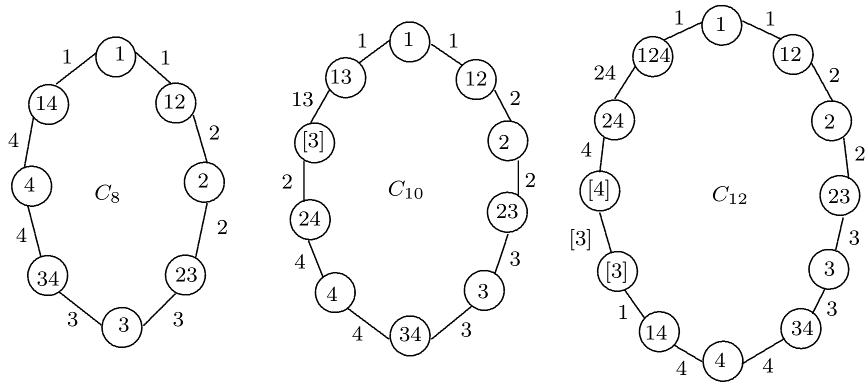

Figure 3.2: Strong royal 4-edge colorings of C_n for $n = 8, 10, 12$

Case 1. *$n \geq 8$ is even.* Figure 3.2 shows strong royal 4-edge colorings of C_8, C_{10}, and C_{12}. It follows that $\text{sroy}(C_n) = 4$ for $n = 8, 10, 12$.

We may assume that $n = 2r \geq 14$ where $r \geq 7$ is an integer such that $2^{k-2} \leq r \leq 2^{k-1} - 1$. If $r = 7$, then $k - 1 = 3$. However, if $8 \leq r \leq 15$, then $k - 1 = 4$. A strong royal $(k-1)$-edge coloring c for each path P_r ($7 \leq r \leq 15$) is shown in Figure 3.3.

For $7 \leq r \leq 15$, let $P_r = (v_1, v_2, \ldots, v_r)$ and let $P_r^* = (u_1, u_1, \ldots, u_r)$. The path P_{2r} is constructed from P_r and P_r^* by adding the edge $v_r u_r$ and the cycle C_{2r} is constructed from P_{2r} by adding the edge $v_1 u_1$. The edge coloring c is first extended to an edge coloring c of P_{2r} by defining $c(u_i u_{i+1}) = c(v_i v_{i+1}) \cup \{k\}$ (where $k = 4$ if $r = 7$ and $k = 5$ if $8 \leq r \leq 15$) for $1 \leq i \leq r - 1$ and $c(v_r u_r) = c(v_{r-1} v_r)$. The resulting coloring is extended to an edge coloring c of C_{2r} by defining $c(v_1 u_1) = c(v_1 v_2)$. Note that no vertex of P_{2r} is colored $\{k\}$.

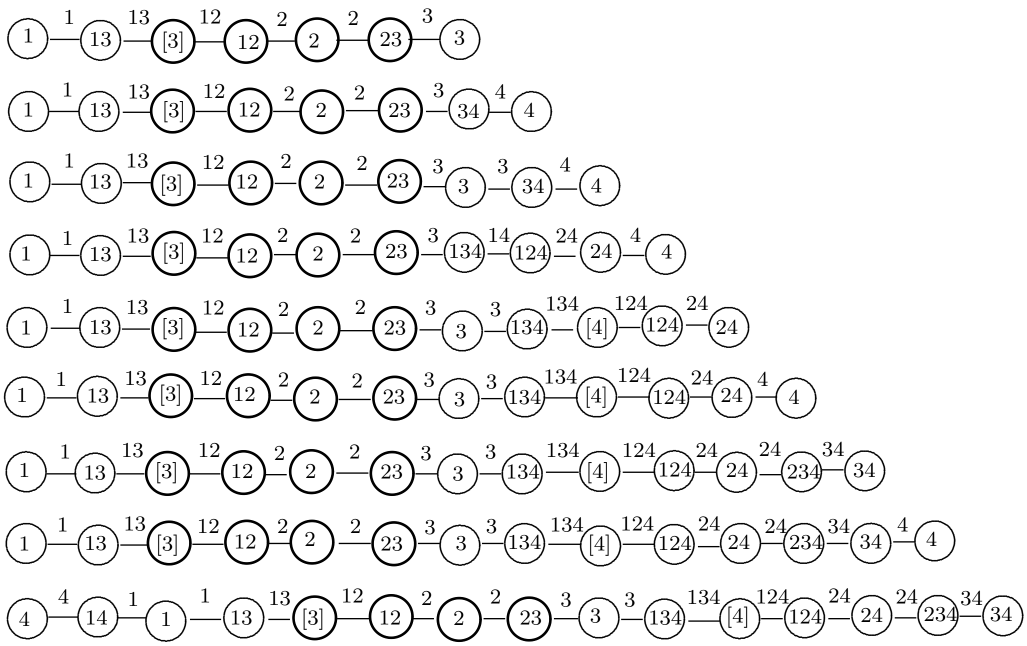

Figure 3.3: Strong royal $(k-1)$-edge colorings of P_r for $7 \le r \le 15$

Since this edge coloring is a strong royal k-edge coloring of C_{2r}, it follows that $\text{sroy}(P_{2r}) = \text{sroy}(C_{2r}) = k$ for $7 \le r \le 15$, where $k = 4$ if $r = 7$ and $k = 5$ if $8 \le r \le 15$. Figure 3.4 shows the construction of a strong royal 4-edge coloring of C_{14} from the paths P_7 and P_7^*.

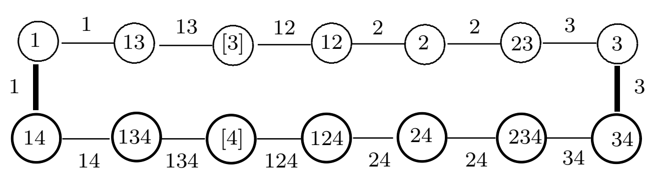

Figure 3.4: Constructing a strong royal 4-edge coloring of C_{14}

For each such path P_{2r} $(7 \le r \le 15)$, we construct the path P_{2r+1} by adding a new vertex u_0 and the edge u_0u_1 and coloring the edge u_0u_1 by $\{k\}$, where $k = 4$ if $r = 7$ and $k = 5$ if $8 \le r \le 15$. Then u_0 is colored $\{k\}$, resulting in a strong royal k-edge coloring of P_{2r+1} for $7 \le r \le 15$. Next, we repeat this procedure by beginning with the paths P_{14}, P_{15}, ..., P_{31}—that is, we use P_{14} and P_{15} to create strong royal 5-edge colorings of C_{28} and C_{30} respectively and use $P_{16}, \ldots, P_{31}$ to create a strong royal 6-edge coloring of C_{2r} for $15 \le r \le 31$. Iteratively applying this process produces the desired coloring for all even cycles. Therefore, $\text{sroy}(C_n) = k$ for all even integers $n \geq 4$ with $2^{k-1} \leq n \leq 2^k - 1$.

Case 2. *$n \geq 9$ is odd.* Figure 3.5 shows a strong royal 4-edge coloring for each of C_9, C_{11}, and C_{13} and so $\text{sroy}(C_n) = 4$ for $n = 9, 11, 13$. We may assume that $n = 2r + 1 \geq 15$, where $r \geq 7$.

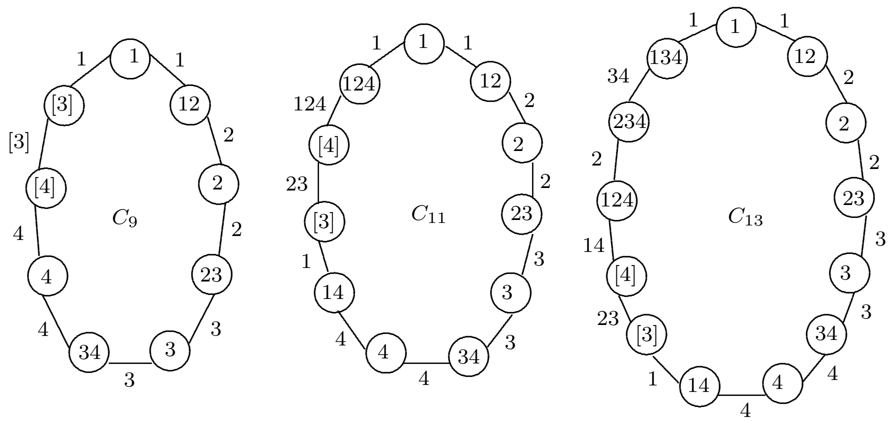

Figure 3.5: Strong royal 4-edge colorings of C_n for $n = 9, 11, 13$

For each path P_r, there is a subpath $Q = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$, where $3 \leq i < i + 4 \leq r$ such that

$$c'(v_{i+1}) = \{1, 2\},\ c(v_{i+1}v_{i+2}) = \{2\},\ \text{and}\ c'(v_{i+2}) = \{2\}.$$

From the manner in which each even cycle C_{2r} was constructed and a strong royal k-edge coloring c of C_{2r} was defined in Case 1, the path $Q^* = (u_i, u_{i+1}, u_{i+2}, u_{i+3})$ is a subapth in C_{2r} such that

$$c'(u_{i+1}) = \{1, 2, k\},\ c(u_{i+1}u_{i+2}) = \{2, k\},\ \text{and}\ c'(u_{i+2}) = \{2, k\}.$$

Furthermore, $c'(x) \neq \{k\}$ for each vertex x of C_{2r}. We now construct the cycle C_{2r+1} from C_{2r} by deleting the edge $u_{i+1}u_{i+2}$ from C_{2r} and adding a new vertex u along with the two new edges $u_{i+1}u$ and uu_{i+2}. We define an edge coloring c of C_{2r+1} from the strong royal k-edge coloring of C_{2r} (as described in Case 1) by assigning the color $\{k\}$ to the edges $u_{i+1}u$ and uu_{i+2} where the colors of remaining edges of C_{2r+1} are the same as in C_{2r}. Thus, $c'(u) = \{k\}$ and $c'(x)$ is the same as in C_{2r} for all other vertices x of C_{2r+1}. Figure 3.6 shows the construction of such a strong royal 4-edge coloring of C_{15} from the strong royal 4-edge coloring of C_{14} of Figure 3.4. Since this edge coloring is a strong royal k-edge coloring of C_{2r+1},

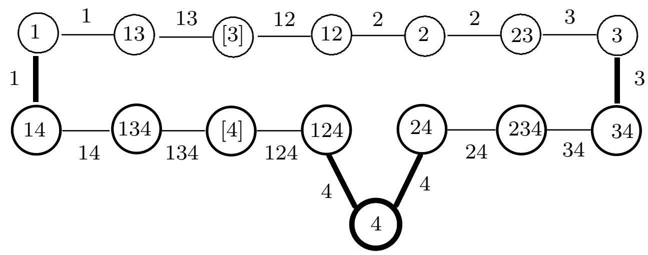

Figure 3.6: Constructing a strong royal 4-edge coloring of C_{15}

it follows that $\text{sroy}(C_n) = k$ for all odd integers $n \geq 3$ with $2^{k-1} \leq n \leq 2^k - 1$ and the exception of $n = 3$ and $n = 7$. ■

Theorem 3.2.1 demonstrates that C_3 and C_7 are royal-one, but all other cycles are royal-zero. For complete graphs, the following result is a consequence of Proposition 2.3.2.

Proposition 3.2.2 *For an integer $n \geq 4$, the complete graph K_n is a royal-zero graph if n is a power of 2 and royal-one otherwise.*

The *corona* $\text{cor}(G)$ of a graph G is the graph obtained from G by adding a pendant edge at each vertex of G. Note that if the order of G is n, then the order of $\text{cor}(G)$ is $2n$. The strong royal index of $\text{cor}(G)$ never exceeds $\text{sroy}(G)$ by more than 1.

Proposition 3.2.3 *If G is a connected graph of order $n \geq 4$, then*

$$\text{sroy}(\text{cor}(G)) \leq \text{sroy}(G) + 1.$$

Consequently, if G is a royal-zero graph, then so is $\text{cor}(G)$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $H = \text{cor}(G)$ be obtained from G by adding the pendant edge u_iv_i at v_i for $1 \leq i \leq n$. Suppose that $\text{sroy}(G) = k$. Then there is a strong royal k-edge coloring $c_G : E(G) \to \mathcal{P}^*([k])$ of G. Define an edge coloring $c_H : E(H) \to \mathcal{P}^*([k+1])$ by

$$c_H(e) = \begin{cases} c_G(e) \cup \{k+1\} & \text{if } e \in E(G) \\ c'_G(v_i) & \text{if } e = u_iv_i \text{ for } 1 \leq i \leq n. \end{cases}$$

Then the induced vertex coloring c'_H is given by

$$c'_H(u_i) = c'_G(v_i) \text{ and } c'_H(v_i) = c'_G(v_i) \cup \{k+1\} \text{ for } 1 \le i \le n.$$

Since c'_H is rainbow, it follows that c_H is a strong royal $(k+1)$-edge coloring of $\mathrm{cor}(G)$ and so $\mathrm{sroy}(H) \le k+1 = \mathrm{sroy}(G)+1$.

If G is a connected royal-zero graph of order $n \ge 4$ where $\mathrm{sroy}(G) = k$, say, then $2^{k-1} \le n \le 2^k - 1$. Since $\mathrm{cor}(G)$ is a connected graph of order $2n \ge 8$ where $2^k \le 2n \le 2^{k+1} - 2$, it follows that $\mathrm{sroy}(\mathrm{cor}(G)) \ge k+1$. On the other hand, there is a strong royal $(k+1)$-edge coloring of $\mathrm{cor}(G)$ and so $\mathrm{sroy}(\mathrm{cor}(G)) = k+1$, which implies that $\mathrm{cor}(G)$ is royal-zero as well. ■

In fact, a stronger statement can be made, regarding the strong royal index of the corona of any connected graph.

Proposition 3.2.4 *If G is a connected graph of order $n \ge 2$, then* $\mathrm{cor}(G)$ *is royal-zero.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $H = \mathrm{cor}(G)$ be obtained from G by adding the pendant edge u_iv_i at v_i for $1 \le i \le n$. Let k be an integer such that $2^{k-1} \le n \le 2^k - 1$. Then $2^k \le 2n < 2^{k+1} - 1$. Denote n distinct subsets of $[k]$ by S_i with $1 \le i \le n$. Define an edge coloring $c : E(H) \to \mathcal{P}^*([k+1])$ by

$$c(e) = \begin{cases} \{k+1\} & \text{if } e \in E(G) \\ S_i & \text{if } e = u_iv_i \text{ for } 1 \le i \le n. \end{cases}$$

Since G is a connected graph, the induced vertex coloring c' is given by

$$c'(u_i) = S_i \text{ and } c'(v_i) = S_i \cup \{k+1\} \text{ for } 1 \le i \le n.$$

Since c' is rainbow, it follows that c is a strong royal $(k+1)$-edge coloring of $\mathrm{cor}(G)$ and so $H = \mathrm{cor}(G)$ is royal-zero. ■

A tree T is called *cubic* if every vertex of T that is not an end-vertex has degree 3. The following result makes use of the proof of Proposition 3.2.3.

Corollary 3.2.5 *If T is a cubic caterpillar of order at least 4, then T is royal-zero.*

Proof. Let T be a cubic caterpillar. Since the statement is true if T has four vertices, we may assume that T has six or more vertices. For an integer $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$, let $H = P_n = (v_1, v_2, \ldots, v_n)$ be a longest path in T, where then $\text{diam}(T) = n - 1 \geq 3$ and the order of T is $2n - 2$. As noted earlier, it was shown in [17] that all paths of order 4 or more are royal-zero and so $\text{sroy}(H) = k$. Let u_iv_i be the pendant edges at v_i for $2 \leq i \leq n-1$. We consider two cases, according to whether $2^{k-1} < n \leq 2^k - 1$ or $n = 2^{k-1}$. In the first case, we apply the same procedure used in the proof of Proposition 3.2.3.

Case 1. $2^{k-1} < n \leq 2^k - 1$. Then $2^k \leq 2n - 2 \leq 2^{k+1} - 4$. Thus, it suffices to show that $\text{sroy}(T) \leq k + 1$. Since $\text{sroy}(H) = k$, there is a strong royal k-edge coloring $c_H : E(H) \to \mathcal{P}^*([k])$. Define an edge coloring $c_T : E(T) \to \mathcal{P}^*([k+1])$ by

$$c_T(e) = \begin{cases} c_H(e) \cup \{k+1\} & \text{if } e \in E(H) \\ c'_H(v_i) & \text{if } e = u_iv_i \text{ for } 2 \leq i \leq n-1. \end{cases}$$

Then the induced vertex coloring c'_T is given by

$$c'_T(u_i) = c'_H(v_i) \text{ for } 2 \leq i \leq n-1 \text{ and } c'_T(v_i) = c'_H(v_i) \cup \{k+1\} \text{ for } 1 \leq i \leq n.$$

Since c'_T is rainbow, it follows that c_T is a strong royal $(k+1)$-edge coloring of T and $\text{sroy}(T) \leq k + 1$. Thus, T is royal-zero.

Case 2. $n = 2^{k-1}$. Then $2n - 2 = 2^k - 2$. Here, we show that $\text{sroy}(T) = \text{sroy}(H) = k$. First, we consider the case where $n = 4$ and $k = 3$. A strong royal 3-edge coloring c of $H = P_4 = (v_1, v_2, v_3, v_4)$ is shown in Figure 3.7, namely $c(v_1v_2) = 1$, $c(v_2v_3) = \{1, 2\}$, and $c(v_3v_4) = \{1, 3\}$. Observe that the induced vertex colors of the vertices of H are all subsets of $[3]$ containing 1 and $c'(v_1) = \{1\}$. The tree T is constructed from H by attaching the pendant edges u_2v_2 and u_3v_3 to v_2 and v_3, respectively. The colors of u_iv_i, $i = 2, 3$, are defined by $c(u_iv_i) = c'(v_i) - \{1\}$, which results in a strong royal 3-edge coloring of T. In the case where $n = 8$ and $k = 4$, we begin with the path $H = P_8 = (v_1, v_2, \ldots, v_8)$, where the edges v_1v_2, v_2v_3, v_3v_4 of P_8 are colored as in the case when $n = 4$, and define $c(v_4v_5) = c'(v_4)$ and $c(v_iv_{i+1}) = c(v_{8-i}v_{9-i}) \cup \{4\}$ for $i = 5, 6, 7$. Again, each edge color and induced vertex color contains 1 and $c'(v_1) = \{1\}$. The tree T in this case is constructed from H by attaching the pendant edges u_iv_i for $2 \leq i \leq 7$. The color of u_iv_i is defined by $c(u_iv_i) = c'(v_i) - \{1\}$ for $2 \leq i \leq 7$, which results in a strong royal

4-edge coloring of T. This is illustrated in Figure 3.7. Continuing in this manner gives the desired result. ■

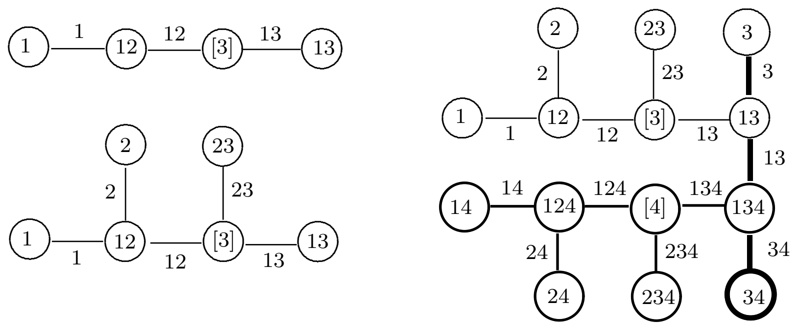

Figure 3.7: Constructing strong royal colorings of the cubic caterpillars

As stated in Proposition 3.2.3, if G is a connected graph of order 4 or more, then $\text{sroy}(\text{cor}(G)) \leq \text{sroy}(G) + 1$ and if G is a royal-zero graph, then $\text{cor}(G)$ is a royal-zero graph. On the other hand, it is possible that G is a royal-one graph and $\text{cor}(G)$ is a royal-zero graph. By Proposition 3.2.2, every complete graph K_n where n is not a power of 2 is a royal-one graph. Thus, if $2^{k-1} + 1 \leq n \leq 2^k - 1$ for some integer $k \geq 3$, then $\text{sroy}(K_n) = k + 1$. If one were to assign distinct nonempty subsets of $[k]$ to the n pendant edges of $\text{cor}(K_n)$ and assign the color $\{k + 1\}$ to the remaining $\binom{n}{2}$ edges of $\text{cor}(K_n)$, then we have a strong royal $(k + 1)$-edge coloring of $\text{cor}(K_n)$ and so $\text{sroy}(\text{cor}(K_n)) = k + 1$. Therefore, $\text{cor}(K_n)$ is a royal-zero graph for each integer $n \geq 5$ where n is not a power of 2. For a more interesting example, Figure 3.8 shows a strong royal 4-edge coloring of $\text{cor}(C_7)$ and so $\text{sroy}(\text{cor}(C_7)) = \text{sroy}(C_7) = 4$ (by Theorem 3.2.1). Thus, C_7 is royal-one, while $\text{cor}(C_7)$ is royal-zero.

A graph operation somewhat related to the corona of a graph G is the *Cartesian product* of G with K_2. In fact, we have the following result that corresponds to Proposition 3.2.3.

Proposition 3.2.6 *If G is a connected graph of order $n \geq 4$, then*

$$\text{sroy}(G \,\Box\, K_2) \leq \text{sroy}(G) + 1.$$

Consequently, if G is a royal-zero graph, then $G \,\Box\, K_2$ is a royal-zero graph.

Proof. Let G be a connected graph of order $n \geq 4$ where $\text{sroy}(G) = k$ for some positive integer k. Let $H = G \,\Box\, K_2$ where G_1 and G_2 are the two copies

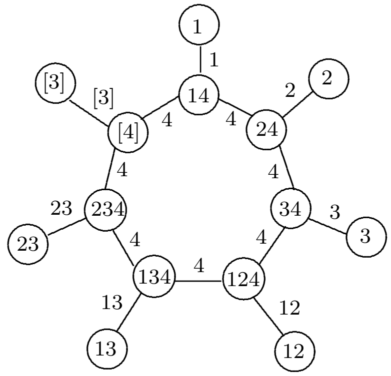

Figure 3.8: A strong royal 4-edge coloring of cor(C_7)

of G. Suppose that $V(G_1) = \{u_1, u_2, \ldots u_n\}$ where u_i is labeled v_i in G_2. Thus, $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ and $E(H) = E(G_1) \cup E(G_2) \cup \{u_iv_i : 1 \leq i \leq n\}$. Since $\text{sroy}(G) = k$, there is a strong royal k-edge coloring $c_{G_1} : E(G_1) \to \mathcal{P}^*([k])$ of G_1. Define an edge coloring $c_H : E(H) \to \mathcal{P}^*([k+1])$ by

$$c_H(e) = \begin{cases} c_{G_1}(e) & \text{if } e \in E(G_1) \\ c_{G_1}(u_iu_j) \cup \{k+1\} & \text{if } e = v_iv_j \in E(G_2) \text{ for } 1 \leq i, j \leq n \text{ and } i \neq j \\ c'_{G_1}(u_i) & \text{if } e = u_iv_i \text{ for } 1 \leq i \leq n. \end{cases}$$

The induced coloring $c'_H : V(H) \to \mathcal{P}^*([k+1])$ is then given by $c'_H(u_i) = c'_{G_1}(u_i)$ and $c'_H(v_i) = c'_{G_1}(u_i) \cup \{k+1\}$. Since c'_H is rainbow, it follows that c'_H is a strong royal $(k+1)$-edge coloring of H. Thus, $\text{sroy}(H) \leq k+1 = \text{sroy}(G)+1$. Therefore, if G is a royal-zero graph, then $G \,\square\, K_2$ is a royal-zero graph. ■

The *hypercube* Q_k is K_2 if $k = 1$, while for $k \geq 2$, Q_k is defined recursively as the Cartesian product $Q_{k-1} \,\square\, K_2$ of Q_{k-1} and K_2. Since $Q_2 = C_4$ is royal-zero by Theorem 3.2.1, the following is a consequence of Proposition 3.2.6.

Corollary 3.2.7 *For each integer $k \geq 2$, the hypercube Q_k is a royal-zero graph.*

As stated in Proposition 3.2.6, if G is a royal-zero graph, then $G \,\square\, K_2$ is a royal-zero graph. On the other hand, it is possible that G is a royal-one graph and $G \,\square\, K_2$ is a royal-zero graph. To see an example of this, we return to the 7-cycle C_7, which we saw (in Theorem 3.2.1) is a royal-one graph. Figure 3.9 shows a strong royal 4-edge coloring of $C_7 \,\square\, K_2$ and so $\text{sroy}(C_7) = \text{sroy}(C_7 \,\square\, K_2) = 4$. Thus, C_7

is royal-one, while $C_7 \square K_2$ is royal-zero. Two other royal-one graphs G for which $G \square K_2$ are royal-zero are K_5 and K_6; that is, $\text{sroy}(K_5 \square K_2) = \text{sroy}(K_6 \square K_2) = 4$. A strong royal 4-edge coloring c of $H = K_6 \square K_2$ can be defined as follows. Let H_1 and H_2 be two copies of K_6 in H, where $V(H_1) = \{u_1, u_2, \dots u_6\}$ and $V(H_2) = \{v_1, v_2, \dots v_6\}$ such that $u_i v_i \in E(H)$. First, we define the rainbow coloring $c' : V(H) \to \mathcal{P}^*([4])$ by

$$\begin{gathered} c'(u_1) = \{1,4\},\ c'(u_2) = \{1\},\ c'(u_3) = \{1,2,4\}, \\ c'(u_4) = \{1,2,3\},\ c'(u_5) = \{1,3\},\ c'(u_6) = \{1,2\}, \\ c'(v_1) = \{4\},\ c'(v_2) = \{1,3,4\},\ c'(v_3) = [4], \\ c'(v_4) = \{2,4\},\ c'(v_5) = \{3,4\},\ c'(v_6) = \{2,3,4\}. \end{gathered}$$

The edge coloring $c : V(H) \to \mathcal{P}^*([4])$ is then defined by $c(xy) = c'(x) \cap c'(y)$ for each edge $xy \in E(H)$. Since c' is the induced vertex coloring of c, it follows that c is a strong royal 4-edge coloring of H. Thus, H is royal-zero.

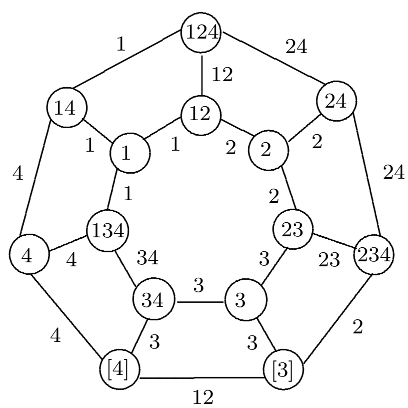

Figure 3.9: A strong royal 4-edge coloring of $C_7 \square K_2$

As noted in Proposition 3.2.2, the complete graph K_7 is a royal-one graph. However, $H = K_7 \square K_2$ is royal-one as well. That there is a strong royal 5-edge coloring of H is straightforward. To show that $\text{sroy}(K_7 \square K_2) = 5$, however, it is necessary to show that there is no strong royal 4-edge coloring of H, for assume that such an edge coloring c of H exists. Since the order of H is 14, the induced vertex colors of H must consist of 14 elements of $\mathcal{P}^*([4])$. In particular, at least three of the four singleton subsets of [4] must be vertex colors of H. Suppose that H_1 and H_2 are the two copies of K_7 in the construction of H. Therefore, at least one of H_1 and H_2 has at least two singleton subsets as its vertex colors, say

$c'(u_1) = \{1\}$ and $c'(u_2) = \{2\}$ where $u_1, u_2 \in V(H_1)$, which is impossible since u_1 and u_2 are adjacent. Hence, $\text{sroy}(K_7 \,\square\, K_2) = 5$.

3.3 Conditions for Royal-One Graphs

We have seen that many graphs are royal-zero graphs. We now present a sufficient condition for a connected graph G of order $n \geq 4$ to be a royal-one graph. Let k be the unique integer such that $2^{k-1} \leq n \leq 2^k - 1$. Recall that a graph G_k of order $2^k - 1$ can be constructed as follows. The vertices of G_k are labeled with the $2^k - 1$ distinct elements of $\mathcal{P}^*([k])$. For each vertex v of G_k, let $\ell(v)$ denote its label. Thus, $\{\ell(v) : \ v \in V(G_k)\} = \mathcal{P}^*([k])$. Two vertices u and v of G_k are adjacent in G_k if and only if $\ell(u) \cap \ell(v) \neq \emptyset$. The vertex set $V(G_k)$ is partitioned into k subsets $V_1, V_2, \ldots, V_k$ where $V_i = \{v \in V(G_k) : |\ell(v)| = i\}$ for $1 \leq i \leq k$. Therefore, $G_k[V_k] = K_1$ and $G_k[V_1] = \overline{K}_k$ is empty. If $k = 2p + 1$ is odd, then $G_k\left[V_{p+1} \cup V_{p+2} \cup \cdots \cup V_k\right] = K_{2^{k-1}}$. If $k = 2p$ is even, then let V'_p be the subset consisting of those elements S in V_p for which $1 \in S$. Then $|V'_p| = \frac{1}{2}\binom{k}{p}$ and $G_k\left[V'_p \cup V_{p+1} \cup V_{p+2} \cup \cdots \cup V_k\right] = K_{2^{k-1}}$. Let m_k be the size of G_k. The graph G_3 of order $7 = 2^3 - 1$ has size $m_3 = 15$ and is shown in Figure 3.10. [Note that this graph is the same graph shown in Figure 2.8.]

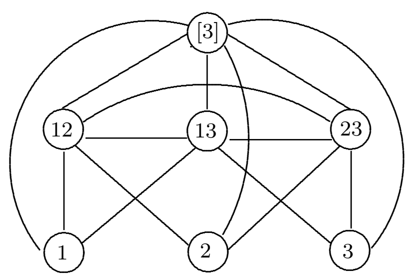

Figure 3.10: The graph G_3 of order $7 = 2^3 - 1$ and size $m_3 = 15$

There is an immediate condition under which a connected graph cannot be a royal-zero graph.

Observation 3.3.1 *Let G be a connected graph of order $n \geq 4$ and size m where $2^{k-1} \leq n \leq 2^k - 1$ for an integer k. If G is not a subgraph of the graph G_k, then* $\text{sroy}(G) \geq k + 1$. *Consequently, if $m \geq m_k + 1$, then* $\text{sroy}(G) \geq k + 1$.

Since $\text{sroy}(T) = 3$ for each tree T of order n where $4 \leq n \leq 7$, it follows by Observation 2.4.1 that if G is a connected graph of order n where $4 \leq n \leq 7$, then $\text{sroy}(G)$ is either 3 or 4. If G is a connected graph of order 7 that is not isomorphic to a subgraph of G_3 of Figure 2.8, then $\text{sroy}(G) \neq 3$ and so $\text{sroy}(G) = 4$. Since the size of G_3 is 15, it follows that if G is a connected graph of order 7 with size at least 16, then $\text{sroy}(G) = 4$. Figure 3.11 shows the graphs H_4, H_5, and H_6 of order 4, 5, and 6, respectively, of greatest size that are subgraphs of G_3. For each graph H_i where $i = 4, 5, 6$, if every edge uv of H_i is assigned the color $c(uv) = \ell(u) \cap \ell(v)$, then $c'(v) = \bigcup_{e \in E_{H_i}(v)} c(e) = \ell(v)$, resulting in a strong royal 3-edge coloring of H_i. Hence, $\text{sroy}(H_i) = 3$ for $i = 4, 5, 6$. The graph $H_4 = K_4$, while H_5 has size 9 and H_6 has size 12. So, if G is a connected graph of order 5 whose size is at least 10 (that is, $G = K_5$) or if G is a connected graph of order 6 whose size is at least 13, then $\text{sroy}(G) = 4$.

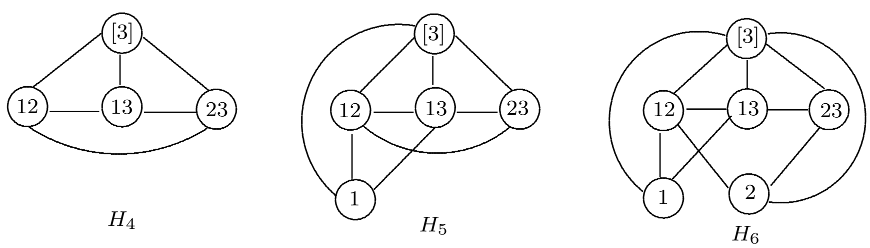

Figure 3.11: Subgraphs of G_3

By Observation 3.3.1, if G is a connected graph of order $n \geq 4$ and size m where $2^{k-1} \leq n \leq 2^k - 1$ such that $m > m_k$, which implies that $G \not\subseteq G_k$, then $\text{sroy}(G) \geq k+1$. In fact, if G possesses any property that implies that $G \not\subseteq G_k$, then $\text{sroy}(G) \geq k+1$. For example, if the order of G is $n = 2^k - 1$ and $\delta(G) \geq \delta(G_k)+1$ or G has more than one vertex of degree $n-1$, then $\text{sroy}(G) \geq k+1$. On the other hand, even though $C_7 \subseteq G_3$ (where $n = 2^3 - 1$ and $k = 3$), $|E(C_7)| = 7 < m_3$, and $\delta(C_7) < \delta(G_3)$, we saw that $\text{sroy}(C_7) = 4 = k+1$. Furthermore, for every chord e of C_7, $\text{sroy}(C_7 + e) = 3$ (see Figure 3.12). Consequently, even though one might suspect that $\text{sroy}(G + uv) \geq \text{sroy}(G)$ for every connected graph G and every pair u, v of nonadjacent vertices of G, such is not the case.

What we have seen is that if G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ having a sufficiently large size, then $\text{sroy}(G) \neq k$. However, if G is a connected graph of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$ having a small

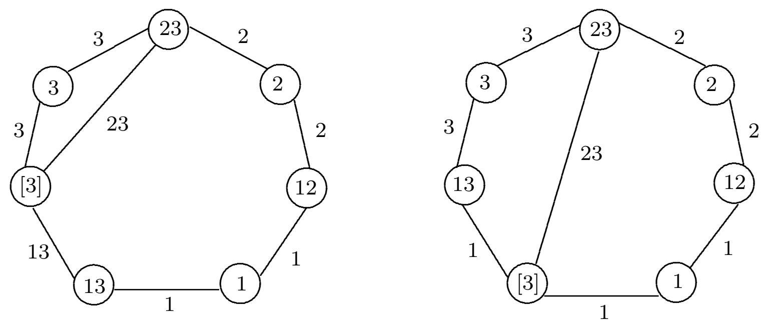

Figure 3.12: Showing that $\text{sroy}(C_7 + e) = 3$ for each $e \notin E(C_7)$

size, then we are not guaranteed that $\text{sroy}(G) = k$. Indeed, even the strong royal index of trees is in doubt.

If Conjecture 3.1.1 is true, then for every connected graph G of order $n \geq 4$ where $2^{k-1} \leq n \leq 2^k - 1$, either $\text{sroy}(G) = k$ or $\text{sroy}(G) = k+1$. In order to present a sufficient condition for $\text{sroy}(G) \neq k$ in terms of the size and minimum degree of G, we describe an expression for the size m_k of the graph G_k (as it is easier in general to compare two numbers than to determine whether a graph contains a subgraph isomorphic to a given graph).

Recall that we label the $2^k - 1$ vertices of G_k with the distinct elements of $\mathcal{P}^*([k])$. The label of each vertex v of G_k is denoted by $\ell(v)$ and so $\{\ell(v) : v \in V(G_k)\} = \mathcal{P}^*([k])$. Let $\{V_1, V_2, \ldots, V_k\}$ be the partition of of $V(G_k)$ described earlier, where then $V_i = \{v \in V(G_k) : |\ell(v)| = i\}$ for $1 \leq i \leq k$. Let $v \in V_i$ for some integer i with $1 \leq i \leq k$. Then $\ell(v) = S$ is some i-element subset of $[k]$. There are $2^i - 1$ nonempty subsets of S and 2^{k-i} subsets of $[k] - S$. For each nonempty subset S' of S and each subset T of $[k] - S$, the vertex v is adjacent to that vertex w of G_k for which $\ell(w) = S' \cup T$. Since v is not adjacent to itself, however, it follows that $\deg_{G_k} v = (2^i - 1)2^{k-i} - 1$. Furthermore, there are $\binom{k}{i}$

vertices in V_i for $1 \le i \le k$. Therefore,

$$\begin{aligned}
m_k &= \frac{1}{2}\sum_{i=1}^{k}\binom{k}{i}\left[(2^i-1)2^{k-i}-1\right] = \frac{1}{2}\sum_{i=1}^{k}\binom{k}{i}(2^k-2^{k-i}-1) \\
&= \frac{1}{2}\left[\sum_{i=1}^{k}\binom{k}{i}2^k - \sum_{i=1}^{k}\binom{k}{i}2^{k-i} - \sum_{i=1}^{k}\binom{k}{i}\right] \\
&= \frac{1}{2}\left[2^k\sum_{i=1}^{k}\binom{k}{i} - 2^k\sum_{i=1}^{k}\binom{k}{i}\left(\frac{1}{2}\right)^i - \sum_{i=1}^{k}\binom{k}{i}\right] \\
&= \frac{1}{2}\left\{2^k(2^k-1) - 2^k\left[\left(1+\frac{1}{2}\right)^k - 1\right] - (2^k-1)\right\} \\
&= \frac{1}{2}(4^k-3^k-2^k+1).
\end{aligned}$$

In particular, if $k = 3$, then the size of G_3 is $m_3 = 15$, as we saw in Figure 2.8.

Proposition 3.3.2 *Let G be a graph of order $n \ge 4$ and size m where $2^{k-1} \le n \le 2^k - 1$ for some integer $k \ge 3$. If $m > \frac{1}{2}(4^k - 3^k - 2^k + 1)$, then* $\text{sroy}(G) \ge k+1$.

For each integer $k \ge 3$, the minimum degree $\delta(G_k)$ of the graph G_k is $2^{k-1} - 1$. Consequently, if G is a graph of order $n \ge 4$ and size m where $2^{k-1} \le n \le 2^k - 1$ for which $\delta(G) \ge 2^{k-1}$, then it may occur that $m < m_k$ but yet G is not a subgraph of G_k, and so (by Observation 3.3.1) $\text{sroy}(G) \ge k+1$. However, in this case, more can be said. It is useful to recall that every path P_n for $n \ge 4$ is royal-zero (see [8, 17]).

Proposition 3.3.3 *Let G be a connected graph of order $n \ge 4$ where $2^{k-1} \le n \le 2^k - 1$ for some integer $k \ge 2$. If $\delta(G) \ge 2^{k-1}$, then* $\text{sroy}(G) = k+1$.

Proof. We have already observed that $\text{sroy}(G) \ge k+1$ for such a graph. Since $\delta(G) \ge 2^{k-1}$ and $n \le 2^k - 1$, it follows that $\delta(G) \ge (n+1)/2$ and therefore G has a Hamiltonian path (in fact, a Hamiltonian cycle). Since $\text{sroy}(P_n) = k$ for every path P_n of order n, it follows by Observation 2.4.1 that $\text{sroy}(G) \le k+1$ and so $\text{sroy}(G) = k+1$. ■

3.4 Open Questions

We have seen that both K_7 and C_7 (a spanning subgraph, or factor, of K_7) are royal-one graphs. The complement $\overline{C}_7$ of C_7 is a 4-regular graph of order 7 and so it is not a subgraph of the graph G_3 shown in Figure 2.8. Hence, $\overline{C}_7$ is also a royal-one graph. The size of $\overline{C}_7$ is 14 which is less than the size 15 of G_3 (the graph of order 7 having the maximum size that is royal-zero). This brings up the problem of determining for each integer $n \geq 3$, the minimum size of a graph of order n that is royal-one. Of course, the minimum size is 7 when $n = 7$.

Another related concept is the minimum number $\chi_{\text{sroy}}(G)$ of elements of the set $\mathcal{P}^*([k])$ for some integer k (where k could be very large) needed to color of the edges of G so that the resulting vertex coloring is rainbow. So, this concept minimizes *the number of edge colors* in an edge coloring that produces a rainbow vertex coloring. For example, if we only use 5 elements in the set $\mathcal{P}^*([100])$ as edge colors in an edge coloring for some graph G to produce a rainbow vertex coloring, then the minimum number of edge colors needed is at most 5 for G or $\chi_{\text{sroy}}(G) \leq 5$. But $\text{sroy}(G) \leq 100$. The natural question here is to investigate how the values of these two parameters are related.

Consequently, there is a host of additional problems that arise with strong royal colorings of graphs.

Chapter 4

Rainbow Mean Colorings I

Abstract: For an edge coloring of a connected graph G of order 3 or more with positive integers, the chromatic mean of a vertex v of G is the sum of the colors of the edges incident with v divided by the degree of v. We only consider edge colorings c for which the chromatic mean of every vertex is a positive integer. If distinct vertices have distinct chromatic means, then c is called a rainbow mean coloring of G. The maximum vertex color in a rainbow mean coloring c of G is the rainbow chromatic mean index of c and the rainbow chromatic mean index of the graph G is the minimum chromatic mean index among all rainbow mean colorings of G. It is shown that the rainbow chromatic mean index exists for every connected graph of order 3 or more. The rainbow chromatic mean index is determined for paths, cycles, complete graphs, and stars.

4.1 Introduction

It is a well-known fact in graph theory that in every nontrivial graph, there are always two vertices having the same degree. Indeed, this fact is listed (indirectly) among the 24 theorems in the article by David Wells [44], asking which of these theorems is the most beautiful. A graph G was initially called *perfect* and then called *irregular* if the degrees of all vertices of G are distinct. Consequently, no nontrivial graph is perfect. While there is no nontrivial graph all of whose vertices have distinct degrees, there are nontrivial graphs in which only two vertices have the same degree. In fact, for every integer $n \geq 2$, there is exactly one connected graph of order n having only two vertices of the same degree.

Over the years, "irregular graphs" have been looked at in a variety of ways (see [14, 15, 16, 23], for example). While no nontrivial graph is irregular, there are irregular multigraphs of each order $n \geq 3$. A multigraph M can be looked at as a labeled graph G_M where each edge uv of G_M is labeled with the positive integer equal to the number of parallel edges joining u and v in M. The degree of v in M is then the sum of the labels of the edges in G_M that are incident with v.

In 1986, at the 250th Anniversary of Graph Theory Conference held at Indiana University-Purdue University Fort Wayne (now called Purdue University Fort Wayne), the concept of "irregular strength" was introduced by Chartrand, which is the smallest positive integer k for which an edge labeling from the set $[k] = \{1, 2, \ldots, k\}$ exists giving rise to vertex labels, all of which are distinct (see [21]). Consequently, the problem was to determine the smallest positive integer k such that each edge of a graph can be labeled with an element of $[k]$ in such a way that the chromatic sum of all vertices are distinct. Later each edge label was considered as an edge color and each chromatic sum was interpreted as a vertex color so that a "rainbow vertex coloring" of the graph resulted. Rather than coloring edges so that distinct vertices have distinct chromatic sums, we now consider coloring edges so that distinct vertices have distinct integral averages.

4.2 Rainbow Mean Index

A *mean coloring* of a connected graph G of order 3 or more is an edge coloring $c : E(G) \to \mathbb{N}$ of G such that for every vertex v of G, its vertex color

$$\mathrm{cm}(v) = \frac{\sum_{e \in E_v} c(e)}{\deg v}, \text{ where } E_v \text{ is the set of edges incident with } v,$$

is an integer, called the *chromatic mean* of v. Clearly, every nontrivial connected graph G has mean colorings. For example, if every edge of G is assigned the same positive integer a, the resulting edge coloring is a mean coloring in which $\mathrm{cm}(v) = a$ for every vertex v of G. If distinct vertices have distinct chromatic means, then the edge coloring c is called a *rainbow mean coloring* of G. The following result shows that, for every connected graph of order 3 or more, such an edge coloring always exists.

Theorem 4.2.1 *Every connected graph of order* 3 *or more has a rainbow mean coloring.*

Proof. Suppose that G is a connected graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$ where $m \geq 2$. Thus, $\Delta(G) = \Delta \geq 2$. Let $k = 2\Delta$ and $t = \Delta! k^m$. Define the edge coloring $c : E(G) \to [t]$ by

$$c(e_i) = \Delta! k^i \text{ for } 1 \leq i \leq m.$$

We show that the coloring c has the desired property. Assume, to the contrary, that there are two distinct vertices u and v of G such that $\text{cm}(u) = \text{cm}(v)$. Let $\deg u = r$ and $\deg v = s$, where $r \leq s$ say, and let

$$E_u = \{e_{i_1}, e_{i_2}, \ldots, e_{i_r}\} \text{ and } E_v = \{e_{j_1}, e_{j_2}, \ldots, e_{j_s}\}$$

where $1 \leq i_1 < i_2 < \cdots < i_r \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_s \leq m$. If $uv \notin E(G)$, then $E_u \cap E_v = \emptyset$; while if $uv \in E(G)$, then $E_u \cap E_v = \{uv\}$. Consequently,

$$\begin{aligned} \text{cm}(u) &= \frac{\Delta!}{r}\left(k^{i_1} + k^{i_2} + \cdots + k^{i_r}\right) \\ \text{cm}(v) &= \frac{\Delta!}{s}\left(k^{j_1} + k^{j_2} + \cdots + k^{j_s}\right), \end{aligned}$$

where both $\text{cm}(u)$ and $\text{cm}(v)$ are positive integers. We consider two cases, according to whether $r = s$ or $r < s$.

Case 1. $r = s$. Then $k^{i_1} + k^{i_2} + \cdots + k^{i_r} = k^{j_1} + k^{j_2} + \cdots + k^{j_r}$.

- First, suppose that $i_r \neq j_r$. We may assume that $i_r < j_r$. Let $p = j_r \geq 2$. Since $k = 2\Delta \geq 4$, it follows that

  $$1 > \tfrac{1}{k^{p-1}} + \tfrac{1}{k^{p-2}} + \cdots + \tfrac{1}{k}$$

  and so $k^p > k + k^2 + \ldots + k^{p-1}$. However then,

  $$\begin{aligned} k^{j_1} + k^{j_2} + \cdots + k^{j_r} \geq k^{j_r} &= k^p > k + k^2 + \ldots + k^{p-1} \\ &\geq k^{i_1} + k^{i_2} + \cdots + k^{i_r}, \end{aligned}$$

  which is a contradiction.

- Next, suppose that $i_r = j_r$. Then

$$k^{i_1} + k^{i_2} + \cdots + k^{i_{r-1}} = k^{j_1} + k^{j_2} + \cdots + k^{j_{r-1}}$$

and $i_{r-1} \neq j_{r-1}$. We can apply the argument above to produce a contradiction.

Case 2. $r < s$. Then $s\,[k^{i_1} + k^{i_2} + \cdots + k^{i_r}] = r\,[k^{j_1} + k^{j_2} + \cdots + k^{j_s}]$.

- First, suppose that $i_r < j_s$. Let $p = j_s \geq 2$. Since

$$1 > \tfrac{1}{k^{p-1}} + \tfrac{1}{k^{p-2}} + \cdots + \tfrac{1}{k},$$

it follows that

$$2 > \tfrac{1}{k^{p-1}} + \tfrac{1}{k^{p-2}} + \cdots + \tfrac{1}{k} + 1 > \tfrac{1}{k^{p-2}} + \tfrac{1}{k^{p-3}} + \cdots + \tfrac{1}{k} + 1.$$

Hence, $k = 2\Delta > \Delta\left(\tfrac{1}{k^{p-2}} + \tfrac{1}{k^{p-3}} + \cdots + 1\right)$. Because $\Delta \geq s/r$, it follows that

$$\begin{aligned} k^{j_1} + k^{j_2} + \cdots + k^{j_s} &\geq k^{j_s} = k^p = k(k^{p-1}) \\ &> \Delta\left(\frac{1}{k^{p-2}} + \frac{1}{k^{p-3}} + \cdots + 1\right) k^{p-1} \\ &= \Delta(k + k^2 + \cdots + k^{p-1}) \\ &\geq \frac{s}{r}(k + k^2 + \cdots + k^{p-1}) \\ &\geq \frac{s}{r}\left[k^{i_1} + k^{i_2} + \cdots + k^{i_r}\right], \end{aligned}$$

which is a contradiction.

- Next, suppose that $i_r \geq j_s$. The argument in Case 1 shows that

$$k^{i_1} + k^{i_2} + \cdots + k^{i_r} > k^{j_1} + k^{j_2} + \cdots + k^{j_s}.$$

Since $r < s$, it follows that $1 > r/s$ and so

$$k^{i_1} + k^{i_2} + \cdots + k^{i_r} > k^{j_1} + k^{j_2} + \cdots + k^{j_s} > \tfrac{r}{s}\,[k^{j_1} + k^{j_2} + \cdots + k^{j_s}],$$

which is a contradiction. ∎

For a rainbow mean coloring c of a graph G, the maximum vertex color is the *rainbow chromatic mean index* (or simply, the *rainbow mean index*) $\mathrm{rm}(c)$ of c. That is,

$$\mathrm{rm}(c) = \max\{\mathrm{cm}(v) : \ v \in V(G)\}.$$

The *rainbow chromatic mean index* (or the *rainbow mean index*) $\mathrm{rm}(G)$ of the graph G itself is defined as

$$\mathrm{rm}(G) = \min\{\mathrm{rm}(c) : \quad c \text{ is a rainbow mean coloring of } G\}.$$

First, we present some useful observations.

Observation 4.2.2 *If G is a connected graph of order $n \geq 3$, then $\mathrm{rm}(G) \geq n$.*

Observation 4.2.3 *If c is a rainbow mean coloring of a connected graph G, then*

$$\sum_{v \in V(G)} \deg v \cdot \mathrm{cm}(v) = 2 \sum_{e \in E(G)} c(e).$$

Furthermore, if the order of G is n and $\mathrm{rm}(c) = n$, then $\sum_{v \in V(G)} \mathrm{cm}(v) = \binom{n+1}{2}$.

4.3 The Rainbow Mean Index of Paths and Cycles

First, we determine the rainbow mean index of every path P_n of order $n \geq 3$. The path P_4 is a special case.

Proposition 4.3.1 $\mathrm{rm}(P_4) = 5$.

Proof. The edge coloring in Figure 4.1 shows that $\mathrm{rm}(P_4) \leq 5$. Next, we show that $\mathrm{rm}(P_4) \geq 5$. Assume, to the contrary, that there is a rainbow mean coloring c of P_4 such that $\mathrm{rm}(c) = 4$. Let $P_4 = (v_1, v_2, v_3, v_4)$. Since $\{\mathrm{cm}(v_i) : 1 \leq i \leq 4\} = [4]$, no two edges can be colored the same. Consequently, since only one vertex is colored 1, this implies that $\mathrm{cm}(v_1) = 1$ or $\mathrm{cm}(v_4) = 1$. We may assume that $\mathrm{cm}(v_1) = 1$ and so $c(v_1v_2) = 1$. Hence, the edges of P_4 are colored with distinct odd

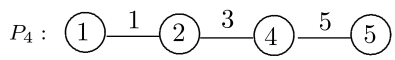

Figure 4.1: A rainbow mean coloring of P_4

integers. If some edge of P_4 is colored 7 or more, then some vertex of P_4 is colored 5 or more, which is impossible. Thus, $\{c(v_iv_{i+1}) : i = 1, 2, 3\} = \{1, 3, 5\}$ and so $\{c(v_2v_3), c(v_3v_4)\} = \{3, 5\}$. In either case, it follows that $\{\text{cm}(v_i) : 1 \leq i \leq 4\} \neq [4]$, a contradiction. Thus, $\text{rm}(P_4) \geq 5$ and so $\text{rm}(P_4) = 5$. ■

Theorem 4.3.2 *For each integer $n \geq 3$ and $n \neq 4$, $\text{rm}(P_n) = n$.*

Proof. Since $\text{rm}(P_n) \geq n$ for all integers $n \geq 3$, it remains to show that there is a rainbow mean coloring c of P_n such that $\text{rm}(c) = n$. First, suppose that $n \geq 3$ is odd. Define the edge coloring $c : E(P_n) \to [n]$ of P_n by $c(e) = i$ if e is incident with v_i where $1 \leq i \leq n$ and i is odd. Figure 4.2 shows such an edge coloring of P_n for $n = 3, 5, 7$. Since $\text{cm}(v_i) = i$ for $1 \leq i \leq n$, it follows that c is a rainbow mean coloring of P_n with $\text{rm}(c) = n$. Therefore, $\text{rm}(P_n) = n$ for each odd integer $n \geq 3$.

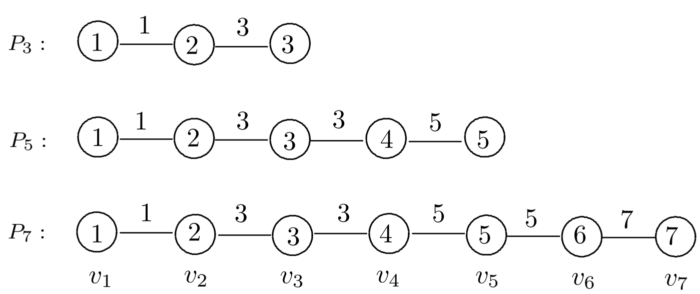

Figure 4.2: Rainbow mean colorings of P_3, P_5, and P_7

We may therefore assume that $n \geq 6$ is even. Let $P_n = (v_1, v_2, \ldots, v_n)$ and let $e_i = v_iv_{i+1}$ for $1 \leq i \leq n - 1$. Since $n \geq 6$ is even, it follows that either $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$. We proceed by induction to prove the following statements.

- ⋆ If $n \equiv 2 \pmod 4$, then there is a rainbow mean coloring c_n of P_n such that $c_n(e_{n-1}) = 3$ and $\text{rm}(c_n) = n$.
- ⋆ If $n \equiv 0 \pmod 4$ and $n \geq 8$, then there is a rainbow mean coloring c_n of P_n such that $c_n(e_{n-1}) = 5$ and $\text{rm}(c_n) = n$.

The edge colorings of P_6 and P_8 in Figure 4.3 show that the statements are true for $n = 6, 8$. Suppose that the statement is true for an arbitrary even integer $n \geq 6$. Next, we show that the statement is true for $n+4$ by considering two cases, according to whether $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$. We use $\mathrm{cm}_t(v)$ to denote the chromatic mean of a vertex v with respect to an edge coloring c_t of the path P_t of order t.

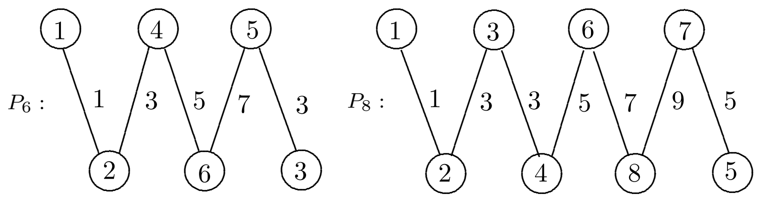

Figure 4.3: Rainbow mean colorings of P_6 and P_8

Case 1. $n \equiv 2 \pmod 4$. By the induction hypothesis, there is a rainbow mean coloring c_n of P_n such that $c_n(e_{n-1}) = \mathrm{cm}(v_n, c_n) = 3$ and $\{\mathrm{cm}(v_i, c_n) : 1 \leq i \leq n\} = [n]$. We now extend c_n to an edge coloring c_{n+4} of P_{n+4} by defining $c_{n+4}(e_n) = 2n+1$, $c_{n+4}(e_{n+1}) = 1$, $c_{n+4}(e_{n+2}) = 2n+5$, and $c_{n+4}(e_{n+3}) = 3$. Then $\mathrm{cm}_{n+4}(v_i) = \mathrm{cm}_n(v_i)$ for $1 \leq i \leq n-1$ and $\mathrm{cm}_{n+4}(v_n) = n+2$, $\mathrm{cm}_{n+4}(v_{n+1}) = n+1$, $\mathrm{cm}_{n+4}(v_{n+2}) = n+3$, $\mathrm{cm}_{n+4}(v_{n+3}) = n+4$, and $\mathrm{cm}_{n+4}(v_{n+4}) = 3$. It follows that $\{\mathrm{cm}_{n+4}(v_i) : 1 \leq i \leq n+4\} = [n+4]$. Figure 4.4 illustrates the construction of such an edge coloring for $n = 6$, where a rainbow mean coloring c_{10} of P_{10} is constructed from the given rainbow mean coloring c_6 of P_6.

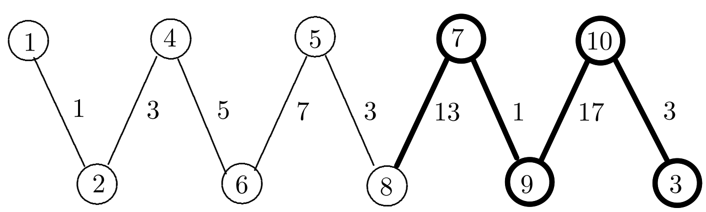

Figure 4.4: The construction of the rainbow mean coloring c_{10} of P_{10} in Case 1

Case 2. $n \equiv 0 \pmod 4$ *and* $n \geq 8$. By the induction hypothesis, there is a rainbow mean coloring c_n of P_n such that $c_n(e_{n-1}) = \mathrm{cm}(v_n, c_n) = 5$ and $\{\mathrm{cm}(v_i, c_n) : 1 \leq i \leq n\} = [n]$. We now extend c_n to an edge coloring c_{n+4} of P_{n+4} by defining $c_{n+4}(e_n) = 2n-3$, $c_{n+4}(e_{n+1}) = 7$, $c_{n+4}(e_{n+2}) = 2n+1$, and $c_{n+4}(e_{n+3}) = 5$. Then $\mathrm{cm}_{n+4}(v_i) = \mathrm{cm}_n(v_i)$ for $1 \leq i \leq n-1$ and $\mathrm{cm}_{n+4}(v_n) =$

$n+1$, $\mathrm{cm}_{n+4}(v_{n+1}) = n+2$, $\mathrm{cm}_{n+4}(v_{n+2}) = n+4$, $\mathrm{cm}_{n+4}(v_{n+3}) = n+3$, and $\mathrm{cm}_{n+4}(v_{n+4}) = 5$. Thus, $\{\mathrm{cm}_{n+4}(v_i) : 1 \le i \le n+4\} = [n+4]$. Figure 4.5 illustrates the construction of such an edge coloring for $n = 8$, where a rainbow mean coloring c_{12} of P_{12} is constructed from the given rainbow mean coloring c_8 of P_8. ■

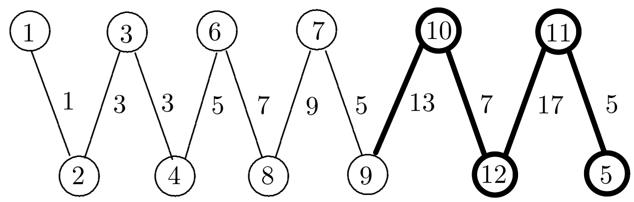

Figure 4.5: The construction of the a rainbow mean coloring c_{12} of P_{12} in Case 2

Next, we determine the rainbow mean index of every cycle.

Theorem 4.3.3 *For each integer $n \ge 4$,*

$$\mathrm{rm}(C_n) = \begin{cases} n & \text{if } n \equiv 0, 1 \pmod 4 \\ n+1 & \text{if } n \equiv 2, 3 \pmod 4 \end{cases}$$

Proof. We consider two cases, according to whether $n \equiv 0, 1 \pmod 4$ or $n \equiv 2, 3 \pmod 4$.

Case 1. $n \equiv 0 \pmod 4$ *or* $n \equiv 1 \pmod 4$. In this case, it suffices to show that there is a rainbow mean coloring c of C_n such that $\mathrm{rm}(c) = n$. First, suppose that $n \equiv 0 \pmod 4$. Then $n = 4k$ for some positive integer k. Let C_{4k} be the cycle obtained from the paths $P = (u_1, u_2, \ldots, u_{2k})$ and $P' = (v_1, v_2, \ldots, v_{2k})$ by adding the two edges u_1v_1 and $u_{2k}v_{2k}$. The edge coloring $c : E(C_{4k}) \to [4k+1]$ is defined by

$$c(e) = \begin{cases} 1 & \text{if } e = u_1v_1 \\ 4k+1 & \text{if } e = u_{2k}v_{2k} \\ 2i+1 & \text{if } e = u_iu_{i+1} \text{ for } 1 \le i \le 2k-1 \\ 2i-1 & \text{if } e \in V(P') \text{ and } e \text{ is incident with } v_i \text{ where} \\ & i \text{ is odd and } 1 \le i \le 2k-1. \end{cases}$$

A rainbow mean coloring c_n of C_n is given in Figure 4.6 for $n = 4, 8, 12$. Note that there is exactly one edge $e = uv$ colored $n+1$ in C_n and $\{\mathrm{cm}(u), \mathrm{cm}(v)\} = \{n-1, n\}$.

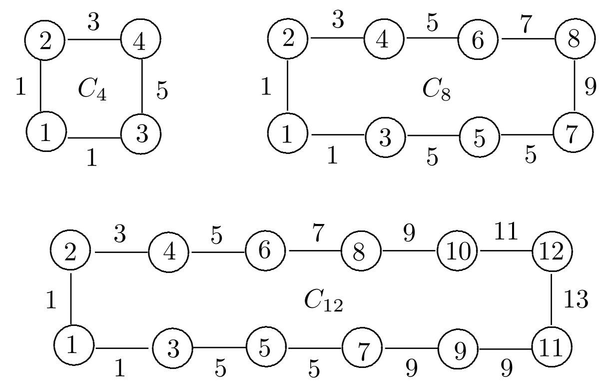

Figure 4.6: Rainbow mean colorings of C_4, C_8, and C_{12}

Then $\text{cm}(u_i) = 2i$ for $1 \le i \le 2k$ and $cm(v_i) = 2i - 1$ for $1 \le i \le 2k$. Since cm is rainbow, $\text{rm}(C_{4k}) = 4k$ for each positive integer k.

Next, suppose that $n \equiv 1 \pmod 4$. Thus, $n = 4k+1$ where $k \in \mathbb{N}$. Then C_n can be obtained by subdividing exactly one edge of C_{n-1}, where then $n-1 \equiv 0 \pmod 4$. A rainbow mean coloring c_n of C_n can be constructed from the rainbow mean coloring c_{n-1} of C_{n-1} described above by subdividing the edge $u_{2k}v_{2k}$ colored n by a new vertex w and coloring the two edges $u_{2k}w$ and wv_{2k} in C_n by n. This is illustrated in Figure 4.7 for $n = 5, 9, 13$. Therefore, $\text{rm}(C_{4k+1}) = 4k + 1$ for each positive integer k.

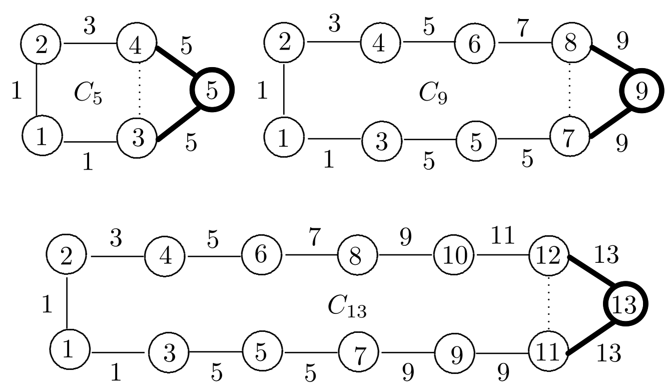

Figure 4.7: Rainbow mean colorings of C_5, C_9, and C_{13}

Case 2. *$n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.* Let $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $e_i = v_iv_{i+1}$ for $1 \le i \le n$. First, we show that $\text{rm}(C_n) \ge n + 1$. Assume, to the contrary, that $\text{rm}(C_n) = n$. Then there is a rainbow mean coloring c of C_n such that $\{\text{cm}(v) : v \in V(C_n)\} = [n]$. Since the color of some vertex of C_n is 1, the color of each edge incident with said vertex is also 1. This implies that $c(e)$ is odd for each $e \in E(C_n)$. Thus, $c(e_i) = 2a_i + 1$ for some nonnegative integer a_i where $1 \leq i \leq n$. First, suppose that $n \equiv 2 \pmod 4$. Then $n = 4k + 2$ for some positive integer k. We have that

$$2 \sum_{v \in V(C_n)} \text{cm}(v) = 2\binom{4k+3}{2} = (4k+3)(4k+2) = 16k^2 + 20k + 6.$$

Hence, $2\sum_{v\in V(C_n)} \text{cm}(v) \equiv 2 \pmod 4$. On the other hand,

$$\begin{aligned} 2 \sum_{v \in V(C_n)} \text{cm}(v) &= 2\sum_{i=1}^{4k+2} c(e_i) = 2\sum_{i=1}^{4k+2}(2a_i+1) = \sum_{i=1}^{4k+2}(4a_i+2) \\ &= \left[\sum_{i=1}^{4k+2} 4a_i\right] + (8k+4) \equiv 0 \pmod 4, \end{aligned}$$

which is impossible. Next, suppose that $n \equiv 3 \pmod 4$. Thus, $n = 4k + 3$ for some positive integer k. Then

$$2 \sum_{v \in V(C_n)} \text{cm}(v) = 2\binom{4k+4}{2} = (4k+4)(4k+3) = 4(k+1)(4k+3).$$

Hence, $2\sum_{v\in V(C_n)} \text{cm}(v) \equiv 0 \pmod 4$. Contrariwise,

$$\begin{aligned} 2 \sum_{v \in V(C_n)} \text{cm}(v) &= 2\sum_{i=1}^{4k+3} c(e_i) = 2\sum_{i=1}^{4k+3}(2a_i+1) = \sum_{i=1}^{4k+3}(4a_i+2) \\ &= \left[\sum_{i=1}^{4k+3} 4a_i\right] + (8k+6) \equiv 2 \pmod 4, \end{aligned}$$

which is impossible. Therefore, $\text{rm}(C_n) \geq n + 1$ if $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.

It remains to show that there exists a rainbow mean coloring c of C_n such that $\text{rm}(c) = n + 1$. First, suppose that $n = 4k + 2$ for some positive integer k. Define $c : E(C_n) \to [n + 1]$ by

$$c(e) = \begin{cases} i & \text{if } e \text{ is incident with } v_i,\ i \text{ is odd and } i \in [1, 2k-1] \\ i+2 & \text{if } e \text{ is incident with } v_i,\ i \text{ is odd and } i \in [2k+1, n-1] \end{cases}$$

Consequently, the chromatic means of the vertices of C_n are given by

$$\text{cm}(v_i) = \begin{cases} i & \text{if } i \text{ is odd and } i \in [1, 2k-1] \\ i+2 & \text{if } i \text{ is odd and } i \in [2k+1, n-1] \\ i & \text{if } i \text{ is even, } i \in [2, 2k-2] \text{ and } k \geq 2 \\ 2k+1 & \text{if } i = 2k \\ i+2 & \text{if } i \text{ is even and } i \in [2k+2, n-2] \\ 2k+2 & \text{if } i = n. \end{cases}$$

This is illustrated in Figure 4.8 for C_{18} where $k = 4$.

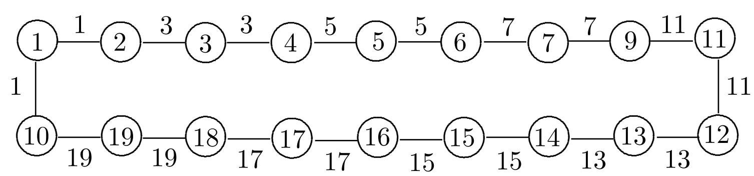

Figure 4.8: A rainbow mean coloring of C_{18}

Next, suppose that $n \equiv 3 \pmod 4$ and so $n + 1 \equiv 0 \pmod 4$. Then C_n can be obtained from C_{n+1} (colored as described above) by deleting a vertex v and joining the two neighbors u and w of v by the edge uw. A rainbow mean coloring c_n of C_n with $\text{rm}(c_n) = n + 1$ can be constructed from the rainbow mean coloring c_{n+1} of C_{n+1} with $\text{rm}(c_{n+1}) = n + 1$ in Case 1 by deleting the vertex v colored 1 and coloring the edge uw with 1. This is illustrated in Figure 4.7 for $n = 7, 11$. ■

4.4 The Rainbow Mean Index of Complete Graphs

Let G be a connected graph of order $n \geq 3$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $c : E(G) \to \mathbb{N}$ be an edge coloring of G. The *matrix representation* M of G with

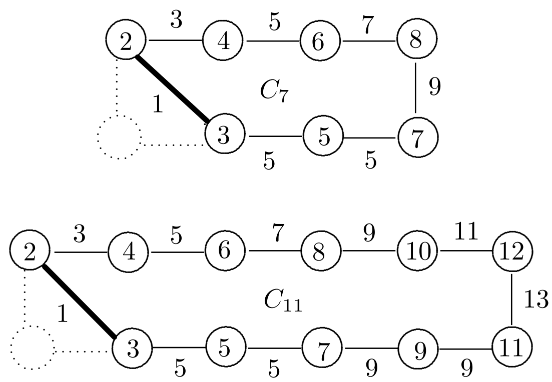

Figure 4.9: Rainbow mean colorings of C_7 and C_{11}

the edge coloring c is the $n \times n$ matrix $[m_{i,j}]$ where

$$m_{i,j} = \begin{cases} c(v_i v_j) & \text{if } 1 \le i \ne j \le n \\ 0 & \text{if } 1 \le i = j \le n. \end{cases}$$

There are several observations that can be made about the matrix representation M of a graph G of order n with an edge coloring c. First, all entries along the main diagonal of M are 0 since no vertex of G is adjacent to itself. Second, M is a symmetric matrix, that is, row i of M is identical to column i of M for every integer i with $1 \le i \le n$. Also, if we were to add the entries in row i (equivalently, in column i), then we obtain $\deg v_i \cdot \text{cm}(v_i)$ for $1 \le i \le n$. Using this framework, we determine the rainbow mean index for complete graphs. We begin with the complete graphs K_n where either n is odd or n is divisible by 4.

Theorem 4.4.1 *For an integer $n \ge 4$ with $n \equiv 0, 1, 3 \pmod 4$, $\text{rm}(K_n) = n$.*

Proof. By Observation 4.2.2, it suffices to show that there is a rainbow mean coloring of K_n having rainbow mean index n. We consider three cases.

Case 1. *$n \ge 4$ and $n \equiv 0 \pmod 4$.* Thus, $n = 4k$ for some positive integer k. In order to describe a rainbow mean coloring c_n of K_n with $\text{rm}(c_n) = n$, we construct an $n \times n$ symmetric matrix M_n. First, we define, recursively, a sequence $B_1, B_2, \ldots, B_k$ of 4×4 symmetric matrices. For $a = n - 1$, let

$$B = \begin{bmatrix} 0 & a & a & 2a \\ a & 0 & 2a & a \\ a & 2a & 0 & a \\ 2a & a & a & 0 \end{bmatrix} \text{ and } B_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & a+1 \\ 1 & 1 & 0 & 2a+1 \\ 1 & a+1 & 2a+1 & 0 \end{bmatrix}.$$

For $2 \le i \le k$, define $B_i = B_{i-1} + B = B_1 + (i-1)B$. Thus,

$$\begin{aligned} B_i &= B_1 + (i-1)B \\ &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & a+1 \\ 1 & 1 & 0 & 2a+1 \\ 1 & a+1 & 2a+1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & (i-1)a & (i-1)a & 2(i-1)a \\ (i-1)a & 0 & 2(i-1)a & (i-1)a \\ (i-1)a & 2(i-1)a & 0 & (i-1)a \\ 2(i-1)a & (i-1)a & (i-1)a & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (i-1)a+1 & (i-1)a+1 & 2(i-1)a+1 \\ (i-1)a+1 & 0 & 2(i-1)a+1 & ia+1 \\ (i-1)a+1 & 2(i-1)a+1 & 0 & (i+1)a+1 \\ 2(i-1)a+1 & ia+1 & (i+1)a+1 & 0 \end{bmatrix}. \end{aligned}$$

To describe the $n \times n$ matrix M_n, we begin with a $k \times k$ matrix $A = [a_{i,j}]$ and then replace the entry $a_{i,i}$ on the main diagonal of A by the 4×4 matrix B_i for $1 \le i \le k$ and each entry off the main diagonal of A by the 4×4 matrix J, each of whose entries is 1. That is, $M_n = [M_{i,j}]$ is an $n \times n$ matrix, where $M_{i,j}$ is a 4×4 matrix such that

$$M_{i,j} = \begin{cases} B_i & \text{if } 1 \le i = j \le k \\ J & \text{if } 1 \le i \ne j \le k. \end{cases}$$

Thus,

$$M_4 = B_1,\ M_8 = \begin{bmatrix} B_1 & J \\ J & B_2 \end{bmatrix}, \text{ and } M_{12} = \begin{bmatrix} B_1 & J & J \\ J & B_2 & J \\ J & J & B_3 \end{bmatrix}.$$

If we were to add the entries in row i (or in column i) in M_n, then we obtain ia for $1 \le i \le n$. That is, if $M_n = [m_{i,j}]$, then

$$\textstyle\sum_{j=1}^{n} m_{i,j} = ia = i(n-1) \text{ for } 1 \le i \le n. \tag{4.1}$$

We now define an edge coloring $c : E(K_n) \to \mathbb{N}$ by $c(v_iv_j) = m_{i,j}$ for each pair i, j of integers with $1 \le i \le j \le n$ and $i \ne j$. Since $\text{cm}(v_i) = \frac{1}{n-1}\sum_{j=1}^{n} m_{i,j} = i$ for $1 \le i \le n$ by (4.1), it follows that c is a rainbow mean coloring of K_n with $\text{rm}(c) = n$. For example,

$$M_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 7 \\ 1 & 4 & 7 & 0 \end{bmatrix} \text{ and } M_8 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 8 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 15 & 1 & 1 & 1 & 1 \\ 1 & 8 & 15 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 8 & 8 & 15 \\ 1 & 1 & 1 & 1 & 8 & 0 & 15 & 15 \\ 1 & 1 & 1 & 1 & 8 & 15 & 0 & 22 \\ 1 & 1 & 1 & 1 & 15 & 15 & 22 & 0 \end{bmatrix}$$

The matrices M_4 and M_8 give rise to rainbow mean colorings of K_4 and K_8 as shown in Figure 4.10, respectively, where each edge drawn in a thin line is colored by 1.

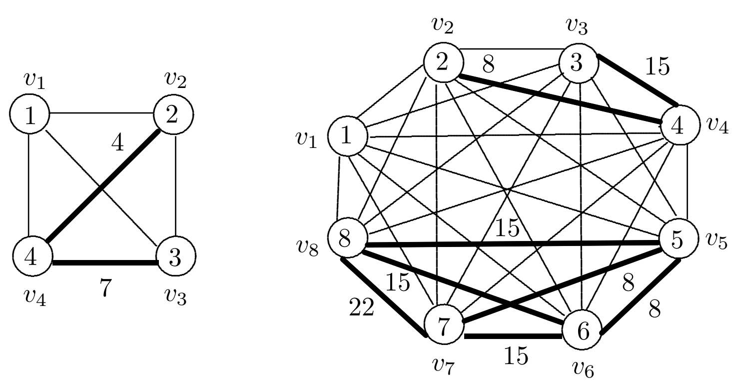

Figure 4.10: Rainbow mean colorings of K_4 and K_8

Case 2. *$n \ge 5$ and $n \equiv 1 \pmod 4$. Then $n = 4k+1$ for some positive integer k.* First, we define, recursively, a sequence $B_1, B_2, \ldots, B_k$ of symmetric matrices, where B_1 is a 5×5 matrix and B_i is a 4×4 matrix for $2 \le i \le k$. For $a = n - 1$, define

$$B_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a+1 & 1 & 1 \\ 1 & a+1 & 0 & 1 & a+1 \\ 1 & 1 & 1 & 0 & 3a+1 \\ 1 & 1 & a+1 & 3a+1 & 0 \end{bmatrix} \text{ and}$$

$$B_2 = \begin{bmatrix} 0 & 2a+1 & a+1 & 2a+1 \\ 2a+1 & 0 & 2a+1 & 2a+1 \\ a+1 & 2a+1 & 0 & 4a+1 \\ 2a+1 & 2a+1 & 4a+1 & 0 \end{bmatrix}.$$

For $3 \le i \le k$, define

$$B_i = B_{i-1} + B = B_2 + (i-2)B,$$

where

$$B = \begin{bmatrix} 0 & a & a & 2a \\ a & 0 & 2a & a \\ a & 2a & 0 & a \\ 2a & a & a & 0 \end{bmatrix}$$

was defined in Case 1. To describe the $n \times n$ matrix M_n, we begin with a $k \times k$ matrix $A = [a_{i,j}]$ and then replace the entry $a_{i,i}$ on the main diagonal of A by the matrix B_i for $1 \le i \le k$ and each entry off the main diagonal of A by the matrix J, each of whose entries is 1. Thus, $a_{1,1}$ is replaced by the 5×5 matrix B_1 and $a_{i,i}$ for $2 \le i \le k$ is replaced by the 4×4 matrix B_i. That is, $M_n = [M_{i,j}]$ is an $n \times n$ matrix, where

$$M_{i,j} = \begin{cases} B_i & \text{if } 1 \le i = j \le k \\ J & \text{if } 1 \le i \ne j \le k. \end{cases}$$

Thus,

$$M_5 = B_1,\ M_9 = \begin{bmatrix} B_1 & J \\ J & B_2 \end{bmatrix}, \text{ and } M_{13} = \begin{bmatrix} B_1 & J & J \\ J & B_2 & J \\ J & J & B_3 \end{bmatrix}.$$

In particular,

$$M_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 5 & 1 & 1 \\ 1 & 5 & 0 & 1 & 5 \\ 1 & 1 & 1 & 0 & 13 \\ 1 & 1 & 5 & 13 & 0 \end{bmatrix} \text{ and } M_9 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 9 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 9 & 0 & 1 & 9 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 25 & 1 & 1 & 1 & 1 \\ 1 & 1 & 9 & 25 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 17 & 9 & 17 \\ 1 & 1 & 1 & 1 & 1 & 17 & 0 & 17 & 17 \\ 1 & 1 & 1 & 1 & 1 & 9 & 17 & 0 & 33 \\ 1 & 1 & 1 & 1 & 1 & 17 & 17 & 33 & 0 \end{bmatrix}$$

We now define an edge coloring $c : E(K_n) \to \mathbb{N}$ by $c(v_iv_j) = m_{i,j}$ for each pair i, j of integers with $1 \le i \le j \le n$ and $i \ne j$. Since

$$\text{cm}(v_i) = \frac{1}{n-1} \sum_{j=1}^{n} m_{i,j} = i$$

for $1 \le i \le n$, it follows that c is a rainbow mean coloring of K_n with $\text{rm}(c) = n$. For example, the matrices M_5 and M_9 give rise to rainbow mean colorings of K_5 and K_9 as shown in Figure 4.11, respectively, where again each edge drawn in a thin line is colored by 1.

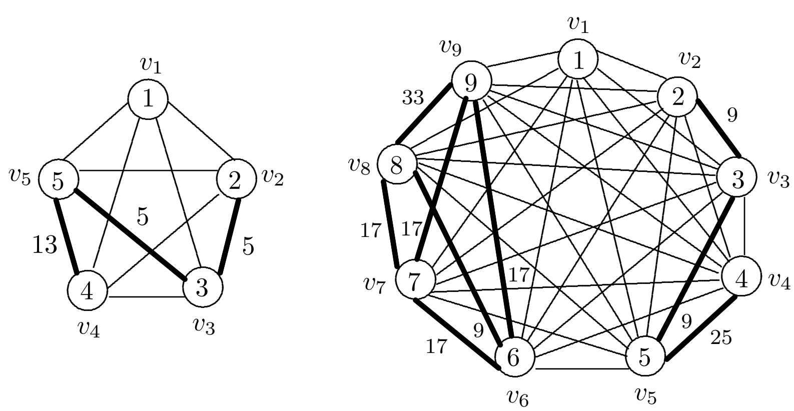

Figure 4.11: Rainbow mean colorings of K_5 and K_9

Case 3. $n \ge 7$ *and* $n \equiv 3 \pmod 4$. Thus, $n = 4k + 3$ for some positive integer k. Again, we construct an $n \times n$ symmetric matrix M_n. For $a = \frac{n-1}{2}$, let

$$C = \left[\begin{array}{cccc} 0 & 2a & 2a & 4a \\ 2a & 0 & 4a & 2a \\ 2a & 4a & 0 & 2a \\ 4a & 2a & 2a & 0 \end{array} \right].$$

First, we define, recursively, a sequence $C_1, C_2, \ldots, C_k$ of symmetric matrices, where C_1 is a 7×7 matrix and C_i is a 4×4 matrix for $2 \leq i \leq k$. Define

$$C_1 = \left[\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2a+1 \\ 1 & 1 & 0 & 1 & 1 & 2a+1 & 2a+1 \\ 1 & 1 & 1 & 0 & 2a+1 & 2a+1 & 2a+1 \\ 1 & 1 & 1 & 2a+1 & 0 & 3a+1 & 3a+1 \\ 1 & 1 & 2a+1 & 2a+1 & 3a+1 & 0 & 3a+1 \\ 1 & 2a+1 & 2a+1 & 2a+1 & 3a+1 & 3a+1 & 0 \end{array} \right] \text{ and}$$

$$C_2 = \left[\begin{array}{cccc} 0 & 3a+1 & 5a+1 & 6a+1 \\ 3a+1 & 0 & 6a+1 & 7a+1 \\ 5a+1 & 6a+1 & 0 & 7a+1 \\ 6a+1 & 7a+1 & 7a+1 & 0 \end{array} \right].$$

For $3 \leq i \leq k$, define $C_i = C_{i-1} + C = C_2 + (i-2)C$. To describe the $n \times n$ matrix M_n, we begin with a $k \times k$ matrix $A = [a_{i,j}]$ and then replace the entry $a_{i,i}$ on the main diagonal of A by the matrix C_i for $1 \leq i \leq k$ and each entry off the main diagonal of A by the matrix J, each of whose entries is 1. That is, $M_n = [M_{i,j}]$ is an $n \times n$ matrix, where

$$M_{i,j} = \begin{cases} C_i & \text{if } 1 \leq i = j \leq k \\ J & \text{if } 1 \leq i \neq j \leq k. \end{cases}$$

Thus, $M_7 = C_1$ where $a = 3$, $M_{11} = \left[\begin{array}{cc} C_1 & J \\ J & C_2 \end{array} \right]$, where $a = 5$, and

$$M_{15} = \left[\begin{array}{ccc} C_1 & J & J \\ J & C_2 & J \\ J & J & C_3 \end{array} \right]$$

where $a = 7$. In particular, $M_7 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 7 \\ 1 & 1 & 0 & 1 & 1 & 7 & 7 \\ 1 & 1 & 1 & 0 & 7 & 7 & 7 \\ 1 & 1 & 1 & 7 & 0 & 10 & 10 \\ 1 & 1 & 7 & 7 & 10 & 0 & 10 \\ 1 & 7 & 7 & 7 & 10 & 10 & 0 \end{bmatrix}$ and

$$M_{11} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 11 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 11 & 11 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 11 & 11 & 11 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 11 & 0 & 16 & 16 & 1 & 1 & 1 & 1 \\ 1 & 1 & 11 & 11 & 16 & 0 & 16 & 1 & 1 & 1 & 1 \\ 1 & 11 & 11 & 11 & 16 & 16 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 16 & 26 & 31 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 16 & 0 & 31 & 36 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 26 & 31 & 0 & 36 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 31 & 36 & 36 & 0 \end{bmatrix}.$$

We now define a rainbow mean coloring $c : E(K_n) \to \mathbb{N}$ by $c(v_i v_j) = m_{i,j}$ for each pair i, j of integers with $1 \leq i \leq j \leq n$ and $i \neq j$. For example, the matrix M_7 gives rise to the rainbow mean coloring of K_7 as shown in Figure 4.11, where again each edge drawn in a thin line is colored by 1. Since $\text{rm}(c) = n$, it follows that $\text{rm}(K_n) = n$ for each integer $n \geq 7$ with $n \equiv 3 \pmod 4$. ■

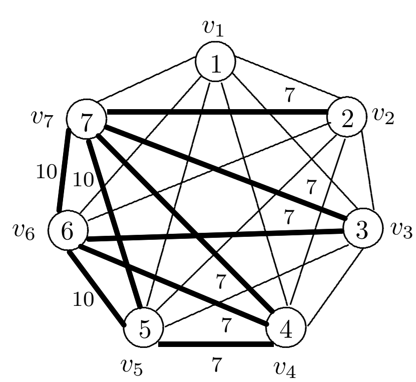

Figure 4.12: A rainbow mean coloring of K_7

The rainbow mean index of each remaining complete graph of order $n \geq 3$ is $n + 1$.

Theorem 4.4.2 *For an integer $n \geq 6$ with $n \equiv 2 \pmod 4$, $\mathrm{rm}(K_n) = n+1$.*

Proof. Since $n \geq 6$ and $n \equiv 2 \pmod 4$, it follows that $n = 4k+2$ for some positive integer k. First, we show that $\mathrm{rm}(K_n) \geq n+1$. Assume, to the contrary, that there is a rainbow mean coloring c of K_n with $\mathrm{rm}(c) = n$. Since

$$\{\mathrm{cm}(v) : v \in V(K_n)\} = [n]$$

for the coloring c, by Observation 4.2.3 we have that

$$\begin{aligned} \sum_{e \in E(K_n)} 2c(e) &= (n-1) \sum_{v \in V(K_n)} \mathrm{cm}(v) \\ &= (n-1)\binom{n+1}{2} = (2k+1)(4k+1)(4k+3) \end{aligned}$$

is an odd integer, a contradiction. Therefore, $\mathrm{rm}(K_n) \geq n+1$.

It remains to show that there is a rainbow mean coloring c_n of K_n with $\mathrm{rm}(c_n) = n+1$. In order to do this, we construct an $n \times n$ symmetric matrix M_n using a sequence $A_1, A_2, \ldots, A_k$ of symmetric matrices, where A_1 is a 6×6 matrix and A_i is a 4×4 matrix for $2 \leq i \leq k$. For $a = n-1$, let

$$B = \begin{bmatrix} 0 & a & a & 2a \\ a & 0 & 2a & a \\ a & 2a & 0 & a \\ 2a & a & a & 0 \end{bmatrix}.$$

Define

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a+1 & 1 & 1 & 1 \\ 1 & a+1 & 0 & 1 & 1 & a+1 \\ 1 & 1 & 1 & 0 & a+1 & 2a+1 \\ 1 & 1 & 1 & a+1 & 0 & 3a+1 \\ 1 & 1 & a+1 & 2a+1 & 3a+1 & 0 \end{bmatrix} \text{ and}$$

$$A_2 = \begin{bmatrix} 0 & a+1 & 3a+1 & 3a+1 \\ a+1 & 0 & 3a+1 & 4a+1 \\ 3a+1 & 3a+1 & 0 & 3a+1 \\ 3a+1 & 4a+1 & 3a+1 & 0 \end{bmatrix}.$$

For $3 \le i \le k$, define

$$A_i = A_{i-1} + B = A_2 + (i-2)B.$$

To describe the $n \times n$ matrix M_n, we begin with a $k \times k$ matrix $A = [a_{i,j}]$ and then replace the entry $a_{i,i}$ on the main diagonal of A by the matrix A_i for $1 \le i \le k$ and each entry off the main diagonal of A by the matrix J, each of whose entries is 1. Thus, $a_{1,1}$ is replaced by the 6×6 matrix A_1 and $a_{i,i}$ for $2 \le i \le k$ is replaced by the 4×4 matrix A_i. That is, $M_n = [M_{i,j}]$ is an $n \times n$ matrix where

$$M_{i,j} = \begin{cases} A_i & \text{if } 1 \le i = j \le k \\ J & \text{if } 1 \le i \ne j \le k. \end{cases}$$

Thus, $M_6 = A_1$, $M_{10} = \begin{bmatrix} A_1 & J \\ J & A_2 \end{bmatrix}$ and $M_{14} = \begin{bmatrix} A_1 & J & J \\ J & A_2 & J \\ J & J & A_3 \end{bmatrix}$. In particular,

$$M_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 6 & 1 & 1 & 1 \\ 1 & 6 & 0 & 1 & 1 & 6 \\ 1 & 1 & 1 & 0 & 6 & 11 \\ 1 & 1 & 1 & 6 & 0 & 16 \\ 1 & 1 & 6 & 11 & 16 & 0 \end{bmatrix} \text{ and } M_{10} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 10 & 0 & 1 & 1 & 10 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 10 & 19 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 10 & 0 & 28 & 1 & 1 & 1 & 1 \\ 1 & 1 & 10 & 19 & 28 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 10 & 28 & 28 \\ 1 & 1 & 1 & 1 & 1 & 1 & 10 & 0 & 28 & 37 \\ 1 & 1 & 1 & 1 & 1 & 1 & 28 & 28 & 0 & 28 \\ 1 & 1 & 1 & 1 & 1 & 1 & 28 & 37 & 28 & 0 \end{bmatrix}.$$

We now define a rainbow mean coloring $c : E(K_n) \to \mathbb{N}$ by $c(v_i v_j) = m_{i,j}$ for each pair i, j of integers with $1 \le i \le j \le n$ and $i \ne j$. For example, the matrix M_6 gives rise to the rainbow mean coloring of K_6 as shown in Figure 4.13, where again each edge drawn in a thin line is colored by 1. Since $\text{rm}(c) = n+1$, it follows that $\text{rm}(K_n) = n+1$ for each integer $n \ge 6$ with $n \equiv 2 \pmod 4$. ■

From Theorems 4.4.1 and 4.4.2, we then have the following result.

Corollary 4.4.3 *For an integer $n \ge 3$,*

$$\text{rm}(K_n) = \begin{cases} n & \text{if } n \ge 4 \text{ and } n \equiv 0,1,3 \pmod 4 \\ n+1 & \text{if } n = 3 \text{ or } n \equiv 2 \pmod 4. \end{cases}$$

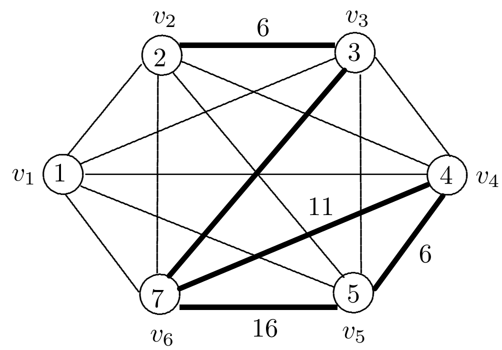

Figure 4.13: A rainbow mean coloring of K_6

4.5 The Rainbow Mean Index of Stars

For each connected graph G of order $n \geq 3$ that we have considered thus far, either $\text{rm}(G) = n$ or $\text{rm}(G) = n + 1$. While this observation may suggest a conjecture, the following result indicates that the value of $\text{rm}(G)$ for a connected graph G of order $n \geq 3$ can be one of at least *three* integers.

Theorem 4.5.1 *If G is a star of order $n \geq 3$, then*

$$\text{rm}(G) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $G = K_{1,n-1}$ where $V(G) = \{v, v_1, v_2, \ldots, v_{n-1}\}$ and $\deg v = n - 1$. First, suppose that n is odd. Thus, $n = 2t + 1$ for some positive integer t. Define the coloring $c : E(G) \to [n]$ by $c(vv_i) = i$ for $1 \leq i \leq t$ and $c(vv_i) = i + 1$ for $t+1 \leq i \leq 2t$. Since $\text{cm}(v) = \frac{1}{2t}\left[\sum_{i=1}^{2t+1} i - (t+1)\right] = t+1$ and $\text{cm}(v_i) = c(vv_i)$ for $1 \leq i \leq 2t$, it follows that c is a rainbow mean coloring with $\text{rm}(c) = n$. Therefore, $\text{rm}(G) = n$ if n is odd.

Next, suppose that $n \geq 4$ is even. Then $n = 2t$ for some integer $t \geq 2$. First, we show that there is a rainbow mean coloring c of G with $\text{rm}(c) = n + 2$. Define $c : E(G) \to \mathbb{N}$ such that $\{c(vv_i) : 1 \leq i \leq 2t - 1\} = [2t + 2] - \{t + 1, t + 2, 2t + 1\}$.

Since

$$\begin{aligned}
\text{cm}(v) &= \frac{1}{2t-1}\sum_{i=1}^{2t-1} c(vv_i) = \frac{1}{2t-1}\left[\binom{2t+3}{2} - (t+1) - (t+2) - (2t+1)\right] \\
&= \frac{1}{2t-1}\left[(2t+3)(t+1) - (4t+4)\right] = t+1
\end{aligned}$$

and $\text{cm}(v_i) = c(vv_i)$ for $1 \le i \le 2t-1$, it follows that c is a rainbow mean coloring of G with $\text{rm}(c) = 2t+2$. Therefore, $\text{rm}(G) \le n+2$.

It remains to show that $\text{rm}(G) \ge n+2 = 2t+2$. Assume, to the contrary, that there is a rainbow mean coloring c of G such that $\text{rm}(c) \in \{2t, 2t+1\}$. We consider two cases, according to whether $\text{rm}(c) = 2t$ or $\text{rm}(c) = 2t+1$.

Case 1. $\text{rm}(c) = 2t$. Then $\{\text{cm}(u) : u \in V(G)\} = [2t]$. Since $\text{cm}(v_i) = c(vv_i)$ for $1 \le i \le 2t-1$, it follows that $\{c(vv_i) : 1 \le i \le 2t-1\} = [2t] - \{a\}$ for some integer $a \in [2t]$. Thus,

$$\text{cm}(v) = \frac{1}{2t-1}\left[\binom{2t+1}{2} - a\right] = \frac{1}{2t-1}[t(2t+1) - a] = \frac{1}{2t-1}(2t^2 + t - a).$$

If $a = 1$, then $\text{cm}(v) = t+1$; while if $a = 2t$, then $\text{cm}(v) = t$. In either case, $\text{cm}(v) = \text{cm}(v_i)$ for some integer i with $1 \le i \le 2t-1$, which is impossible. On the other hand, if $1 < a < 2t$, then $\text{cm}(v)$ is not an integer, which is also impossible.

Case 2. $\text{rm}(c) = 2t+1$. Then $\{\text{cm}(u) : u \in V(G)\} \subseteq [2t+1]$. Since $\text{cm}(v_i) = c(vv_i)$ for $1 \le i \le 2t-1$, it follows that $\{c(vv_i) : 1 \le i \le 2t-1\} = [2t+1] - \{a, b\}$ for some $a, b \in [2t+1]$ and $a \ne b$. Thus,

$$\begin{aligned}
\text{cm}(v) &= \frac{1}{2t-1}\left[\binom{2t+2}{2} - (a+b)\right] = \frac{1}{2t-1}[(t+1)(2t+1) - (a+b)] \\
&= \frac{1}{2t-1}[(2t^2+3t+1) - (a+b)]
\end{aligned}$$

- ⋆ If $a = 1$ and $b = 2$, then $\text{cm}(v) = t+2$;
- ⋆ If $a = 2t$ and $b = 2t+1$, then $\text{cm}(v) = t$;

⋆ If $\mathrm{cm}(v) = t + 1$, then $a + b = 2t + 2$, where $1 \leq a < t + 1 < b \leq 2t + 1$.

In any of these situations, $\mathrm{cm}(v) = \mathrm{cm}(v_i)$ for some integer i with $1 \leq i \leq 2t - 1$, which is impossible. For any other choice of a and b, it follows that $\mathrm{cm}(v)$ is not an integer, which is also impossible. ■

Since each edge of a connected graph G of order at least 3 is to be assigned a positive integer color in such a way that every vertex color is an integer and all vertex colors are distinct, one may suspect that vertex colors considerably larger than the order of the graph may be required for some graphs. However, no such graph has been found yet. Indeed, the value of $\mathrm{rm}(G)$ has always been either n or $n+1$ for connected graphs G of order $n \geq 3$ studied thus far, with the exception of stars of even order $n \geq 4$. This observation suggests the following conjecture.

Conjecture 4.5.2 *For every connected graph G of order $n \geq 3$,*

$$n \leq \mathrm{rm}(G) \leq n + 2.$$

Chapter 5

Rainbow Mean Colorings II

Abstract: It was conjectured that if G is a connected graph of order $n \geq 3$, then $n \leq \text{rm}(G) \leq n+2$. In this chapter, we investigate this conjecture for some well-known classes of connected bipartite graphs and verify it for prisms, hypercubes, and complete bipartite graphs.

5.1 Introduction

Let c be a rainbow mean coloring of a connected graph G. For a vertex v of G, recall that the *chromatic sum* $\text{cs}(v)$ of v is defined as the sum of the colors of the edges incident with v. Hence, $\text{cs}(v) = \sum_{e \in E_v} c(e) = \deg v \cdot \text{cm}(v)$. Consequently, Observation 4.2.3 can be thought of as an extension of the First Theorem of Graph Theory when stated in the following way.

Observation 5.1.1 *If c is a rainbow mean coloring of a connected graph G, then*

$$\sum_{v \in V(G)} \text{cs}(v) = 2 \sum_{e \in E(G)} c(e).$$

Furthermore, if the order of G is n and $\text{rm}(c) = n$, then $\sum_{v \in V(G)} \text{cm}(v) = \binom{n+1}{2}$.

A connected graph of order 3 or more with a rainbow mean coloring is referred to as a *mean-colored graph*. A vertex v in a mean-colored graph G is *chromatically odd* if $\text{cs}(v) = \deg v \cdot \text{cm}(v)$ is an odd integer; otherwise, v is *chromatically even*. The following is an immediate consequence of Observation 5.1.1 and is a generalization of the well-known fact that every graph has and an even number of odd vertices.

Corollary 5.1.2 *Every mean-colored graph contains an even number of chromatically odd vertices.*

Corollary 5.1.3 *If G is a connected graph of order $n \geq 6$ with $n \equiv 2 \pmod 4$ all of whose vertices are odd, then $\mathrm{rm}(G) \geq n+1$.*

Proof. Assume, to the contrary, that $\mathrm{rm}(G) = n$. Then there exists a rainbow mean coloring $c : E(G) \to \mathbb{N}$ of G such that $\{\mathrm{cm}(v) : v \in V(G)\} = [n] = \{1, 2, \ldots, n\}$. Since $n \equiv 2 \pmod 4$, it follows that $n = 4k+2$ for some positive integer k. Thus, the set $[n]$ contains $2k+1$ odd integers, namely $1, 3, \ldots, 4k+1$. Suppose that $u_1, u_2, \ldots, u_{2k+1}$ are the vertices of G such that $\mathrm{cm}(u_i) = 2i-1$ for $i = 1, 2, \ldots, 2k+1$. Since every vertex of G has odd degree, the vertices $u_1, u_2, \ldots, u_{2k+1}$ are the only chromatically odd vertices, implying that there is an odd number of chromatically odd vertices, a contradiction. ■

For example, the Petersen graph P is a connected cubic graph of order $10 \equiv 2 \pmod 4$. Figure 5.1 shows a rainbow mean coloring c of P with $\mathrm{rm}(c) = 11$. Thus, $\mathrm{rm}(P) = 11$ by Corollary 5.1.3.

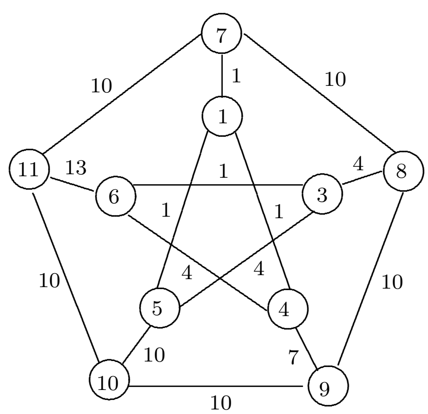

Figure 5.1: A rainbow mean coloring of the Petersen graph P

Here, we will be dealing primarily with connected bipartite graphs G of order 3 or more having partite sets U and W. Because each of $\sum_{u \in U} \mathrm{cs}(u)$ and $\sum_{w \in W} \mathrm{cs}(w)$ counts the sum of the colors of the edges of G, we have the following fact.

Observation 5.1.4 *Let G be a connected bipartite graph with partite sets U and W. If c is a rainbow mean coloring of G, then*

$$\sum_{u \in U} \mathrm{cs}(u) = \sum_{w \in W} \mathrm{cs}(w).$$

Recall that the rainbow mean index was determined for each complete graph and cycle, which we restate below.

Theorem 5.1.5 *For an integer $n \geq 3$,*

$$\text{rm}(K_n) = \begin{cases} n & \text{if } n \geq 4 \text{ and } n \equiv 0, 1, 3 \pmod 4 \\ n+1 & \text{if } n = 3 \text{ or } n \equiv 2 \pmod 4 \end{cases}$$

$$\text{rm}(C_n) = \begin{cases} n & \text{if } n \equiv 0, 1 \pmod 4 \\ n+1 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

In this chapter, we determine the rainbow mean index of every prism, hypercube, and complete bipartite graph as well as for some star-related trees.

5.2 Prisms, Hypercubes, and Complete Bipartite Graphs

The *prism* $C_n \,\square\, K_2$, $n \geq 3$, is the Cartesian product of the n-cycle C_n and K_2. Of course, $C_n \,\square\, K_2$ is bipartite if and only if n is even. The two smallest prisms are shown in Figure 5.2.

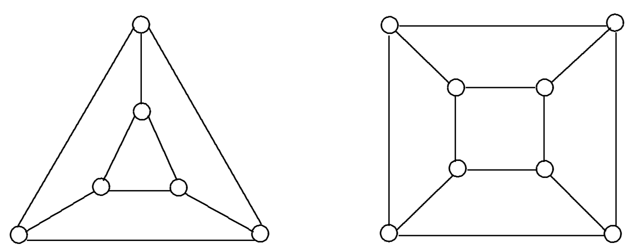

Figure 5.2: The prisms $C_3 \,\square\, K_2$ and $C_4 \,\square\, K_2$

We now determine the rainbow mean index of every prism.

Theorem 5.2.1 *For each integer $n \geq 3$,*

$$\text{rm}(C_n \,\square\, K_2) = \begin{cases} 2n & \text{if } n \text{ is even} \\ 2n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $G = C_n \square K_2$ be constructed from the two n-cycles $(u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ and $(v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ and the edges u_iv_i for $1 \leq i \leq n$. Thus, G is a cubic graph of order $2n$. We consider two cases depending on whether n is even or n is odd.

Case 1. *n is even.* By Observation 4.2.2, it suffices to show that there is a rainbow mean coloring c of G with $\text{rm}(c) = 2n$. Define the edge coloring $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} i & \text{if } e \in \{u_iu_{i-1}, u_iu_{i+1}\} \text{ where } i \text{ is odd and } 1 \leq i \leq n-1 \\ i+1+n & \text{if } e = v_iv_{i+1} \text{ where } i \text{ is odd and } 1 \leq i \leq n-1 \\ i+2n & \text{if } e = v_iv_{i+1} \text{ where } i \text{ is even and } 1 \leq i \leq n-2 \\ j & \text{if } e = u_jv_j \text{ where } 1 \leq j \leq n-1 \\ 2n & \text{if } e \text{ is incident with } v_n. \end{cases}$$

Since $\text{cm}(u_i) = i$ for $1 \leq i \leq n$ and $\text{cm}(v_i) = n + i$ for $1 \leq i \leq n$, it follows that c is a rainbow mean coloring of G with $\text{rm}(c) = 2n$. This is illustrated in Figure 5.3 for $n = 4, 6$.

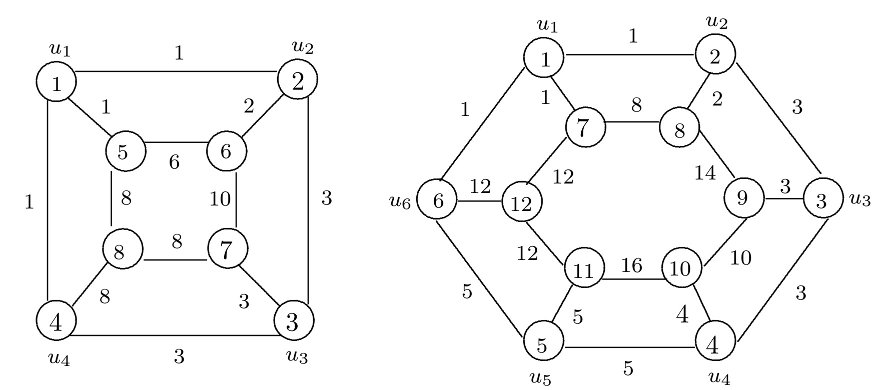

Figure 5.3: Rainbow mean colorings of $C_4 \square K_2$ and $C_6 \square K_2$

Case 2. *n is odd.* By Corollary 5.1.3, $\text{rm}(G) \geq 2n + 1$. It remains to show that there is a rainbow mean coloring c of G with $\text{rm}(c) = 2n+1$. Define the edge coloring $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} 1 & \text{if } e = u_i u_{i+1} \text{ where } i \text{ is odd and } 1 \leq i \leq n \\ 4 & \text{if } e = u_i u_{i+1} \text{ where } i \text{ is even and } 2 \leq i \leq n-1 \\ 3i-2 & \text{if } e = u_i v_i \text{ for } 1 \leq i \leq n \\ \frac{3n+5}{2} & \text{if } e = v_i v_{i+1} \text{ where } 1 \leq i \leq n. \end{cases}$$

Since $\text{cm}(u_1) = 1$, $\text{cm}(u_i) = i + 1$ for $2 \leq i \leq n$, and $\text{cm}(v_i) = n + 1 + i$ for $1 \leq i \leq n$, it follows that c is a rainbow mean coloring of G with $\text{rm}(c) = 2n + 1$. This is illustrated in Figure 5.4 for $n = 3, 5$. ■

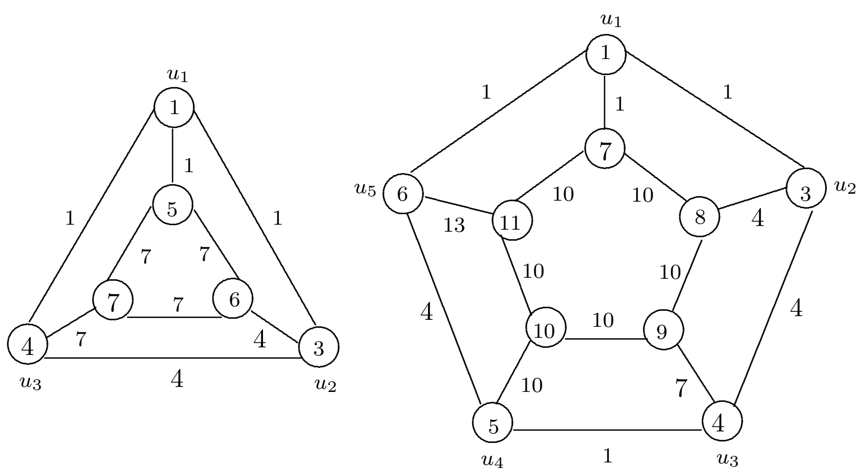

Figure 5.4: Rainbow mean colorings of $C_3 \,\square\, K_2$ and $C_5 \,\square\, K_2$

Another well-known class of bipartite graphs defined by means of Cartesian products is that of the hypercubes. The *hypercube* Q_n is K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as the Cartesian product $Q_{n-1} \,\square\, K_2$ of Q_{n-1} and K_2. For each integer $n \geq 2$, the hypercubes Q_n is an n-regular bipartite graph of order 2^n. In order to determine the rainbow mean index of every hypercube, we first recall a well-known result (see [23]).

Theorem 5.2.2 *Every regular bipartite graph contains a* 1*-factor* (*and is* 1*-factorable*)*.*

Theorem 5.2.3 *For each integer* $n \geq 2$, $\text{rm}(Q_n) = 2^n$.

Proof. By Observation 4.2.2, it suffices to show that there is a rainbow mean coloring c of Q_n with $\mathrm{rm}(c) = 2^n$. We proceed by induction on $n \geq 2$. Since $\mathrm{rm}(Q_2) = \mathrm{rm}(C_4) = 4$ and $\mathrm{rm}(Q_3) = \mathrm{rm}(C_4 \,\square\, K_2) = 8$ by Theorems 5.1.5 and 5.2.1, it follows that the statement is true for $n = 2, 3$. Suppose that there is a rainbow mean coloring of Q_n with rainbow mean index 2^n for some integer $n \geq 3$. We show that $G = Q_{n+1} = Q_n \,\square\, K_2$ has a rainbow mean coloring c with $\mathrm{rm}(c) = 2^{n+1}$.

Let H and H' be the two copies of Q_n in G where each vertex v in H is adjacent to the vertex v' in H'. Since Q_n is a regular bipartite graph, it follows by Theorem 5.2.2 that Q_n has a 1-factor. Let F be a 1-factor of H and let F' be the corresponding 1-factor in H'. By the induction hypothesis, there is a rainbow mean coloring $c_H : E(H) \to \mathbb{N}$ of H with $\mathrm{rm}(c_H) = 2^n$. Thus,

$$\{\mathrm{cm}_{c_H}(v) : v \in V(H)\} = [2^n]. \tag{5.1}$$

We now extend the coloring c_H of H to an edge coloring $c : E(G) \to \mathbb{N}$ of G by defining

$$c(e) = \begin{cases} c_H(e) & \text{if } e \in E(H) \cup [E(H') - E(F')] \\ c_H(e) + (n+1)2^n & \text{if } e \in E(F') \\ i & \text{if } e = vv' \text{ and } \mathrm{cm}_{c_H}(v) = i \text{ for } 1 \leq i \leq 2^n \end{cases}$$

It remains to show that c is a rainbow mean coloring with $\mathrm{rm}(c) = 2^{n+1}$.

- ⋆ Let $v \in V(H)$, where $\mathrm{cm}_{c_H}(v) = i \in [2^n]$. Since

  $$(n+1)\,\mathrm{cm}_c(v) = ni + i = (n+1)i,$$

  it follows that $\mathrm{cm}(v) = \mathrm{cm}_{c_H}(v)$. Hence, $\{\mathrm{cm}_c(v) : v \in V(H)\} = [2^n]$ by (5.1).

- ⋆ Let $v' \in V(H')$, where v is the neighbor of v' in H. Then $\mathrm{cm}_{c_H}(v) = i$ for some integer $i \in [2^n]$. By the defining property of c, it follows that

  $$\begin{aligned}(n+1)\mathrm{cm}_c(v') &= n\mathrm{cm}_{c_H}(v) + i + (n+1)2^n = ni + i + (n+1)2^n \\ &= (n+1)(i + 2^n).\end{aligned}$$

  Since $\deg_G v' = n+1$, it follows that $\mathrm{cm}_c(v') = i + 2^n$. Hence, $\{\mathrm{cm}_c(v') : v' \in V(H')\} = [2^n + 1, 2^{n+1}]$.

This implies that c is a rainbow mean coloring of G with $\text{rm}(c) = 2^{n+1}$. Hence, by mathematical induction, $\text{rm}(Q_n) = 2^n$ for each integer $n \geq 2$. ■

To illustrate the proof of Theorem 5.2.3, we construct a rainbow mean coloring c of Q_4 with $\text{rm}(c) = 2^4 = 16$ from a rainbow mean coloring c_H of $H = Q_3$ with $\text{rm}(c_H) = 2^3 = 8$. This coloring c is shown in Figure 5.5, where the four edges in the 1-factor F in H and the four edges in the 1-factor F' is H' are drawn in bold and each edge e' of F' is colored by $c_H(e) + 4 \cdot 2^3 = c_H(e) + 32$ where $e \in E(F)$. Thus, $\{c(e) : e \in E(F)\} = \{1, 3, 6, 8\}$ and $\{c(e') : e' \in E(F')\} = \{33, 35, 38, 40\}$.

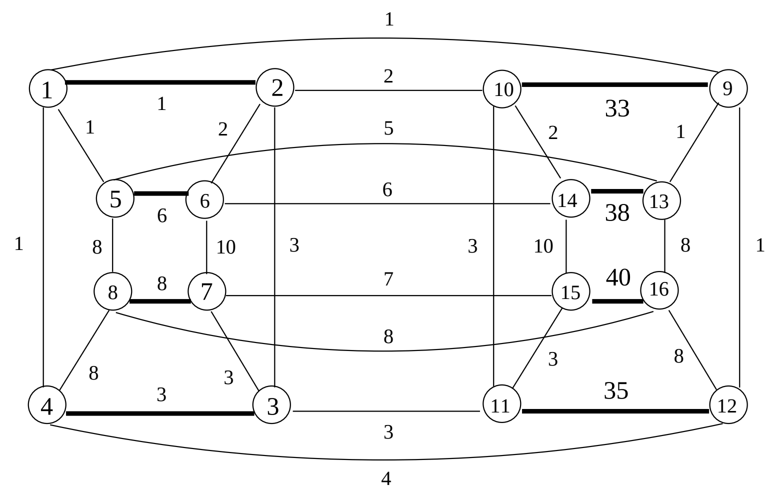

Figure 5.5: Constructing a rainbow mean coloring of Q_4 from a rainbow mean coloring of Q_3

A proof similar to that of Theorem 5.2.3 can be used to prove the following:

Theorem 5.2.4 *If G is a connected regular bipartite graph of order $n \geq 3$ with $\text{rm}(G) = n$, then $\text{rm}(G \,\square\, K_2) = 2n$.*

Recall the rainbow mean index was determined for all stars $K_{1,t}$, with $t \geq 2$.

Theorem 5.2.5 *For an integer* $t \geq 2$, $\mathrm{rm}(K_{1,t}) = \begin{cases} t+1 & \text{if } t \text{ is even} \\ t+3 & \text{if } t \text{ is odd.} \end{cases}$

We begin with an observation concerning $\mathrm{rm}(K_{s,t})$ when s and t are both odd.

Proposition 5.2.6 *If s and t are odd integers with $s, t \geq 3$, then*

$$\mathrm{rm}(K_{s,t}) \geq s + t + 1.$$

Proof. Let $G = K_{s,t}$ where $s, t \geq 3$. Since s and t are both odd, it follows that $s = 2a+1$ and $t = 2b+1$ for some nonnegative integers a and b with $a+b \geq 2$. Note that if $s \equiv t \pmod 4$, then the statement follows by Corollary 5.1.3. However, we verify the statement independently from this observation. Assume, to the contrary, that there is a rainbow mean coloring $c : E(G) \to \mathbb{N}$ of G with $\mathrm{rm}(c) = s+t$. Thus,

$$\sum_{v \in V(G)} \mathrm{cm}(v) = \binom{s+t+1}{2} = \frac{(s+t+1)(s+t)}{2} = (2a+2b+3)(a+b+1).$$

Let $\{X, Y\}$ be a partition of the set $[s+t]$ where $|X| = t$ and $|Y| = s$ such that the sum of elements in X is x and the sum of elements in Y is y. Since $x + y = (2a+2b+3)(a+b+1)$ and $sx = ty$, it follows that

$$sx = ty = t[(2a+2b+3)(a+b+1) - x] = t(2a+2b+3)(a+b+1) - xt.$$

Thus, $sx + tx = x(s+t) = t(2a+2b+3)(a+b+1)$ or $(2a+2b+2)x = t(2a+2b+3)(a+b+1)$. However then, $2x = t(2a+2b+3)$, which is an odd integer. This is a contradiction. Therefore, $\mathrm{rm}(G) \geq s+t+1$. ■

Proposition 5.2.6 establishes lower bounds on the rainbow mean index for many complete bipartite graphs. However, the construction of a rainbow mean coloring for these graphs resulting in the desired upper bound involves extensive case by case analysis and so we simply state the value of this parameter for this class of graph without proof.

Theorem 5.2.7 *If s and t are positive integers with $\min\{s,t\} \geq 2$, then*

$$\operatorname{rm}(K_{s,t}) = \begin{cases} s+t & \text{if at least one of } s \text{ and } t \text{ is even} \\ s+t+1 & \text{if both } s \text{ and } t \text{ are odd} \end{cases}$$

Figure 5.6 exhibits minimizing rainbow mean colorings with respect to the rainbow mean indexes for $K_{4,6}$ and $K_{4,7}$. These examples represent the pattern involved in constructing such a coloring for a few general cases.

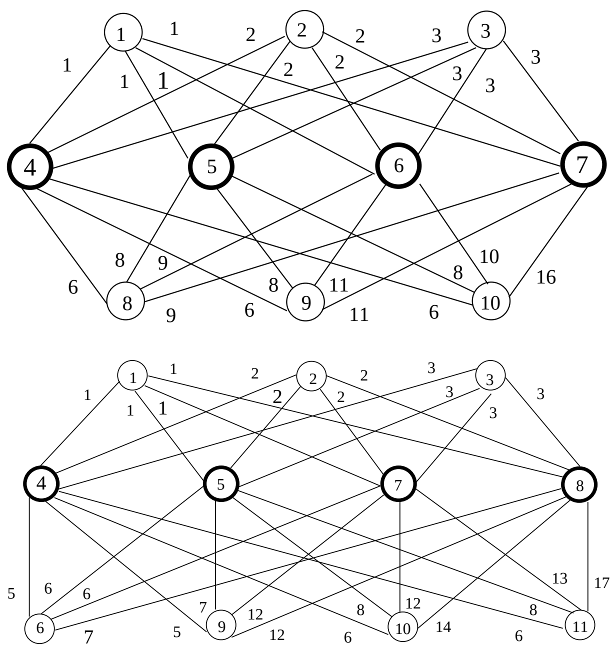

Figure 5.6: Rainbow mean colorings of $K_{4,6}$ and $K_{4,7}$

As we saw, it was conjectured in that if G is a connected graph of order $n \geq 3$, then $\operatorname{rm}(G) \leq n+2$. All of the bipartite graphs considered in this article substantiate this conjecture. Indeed, the only connected bipartite graphs G of order $n \geq 3$ found thus far having $\operatorname{rm}(G) = n+2$ are stars of even order. Consequently, not only may this conjecture be true but those connected graphs G for which $\operatorname{rm}(G) = n+2$ may be rare.

5.3 Star-Related Trees

Since only stars of even order $n \geq 4$ have been shown to have rainbow mean index different from n or $n+1$, this suggests studying the rainbow mean index of trees related to stars in some manner. In this section, we determine the rainbow mean index of three classes of trees, namely cubic caterpillars, subdivided stars, and double stars.

5.3.1 Cubic Caterpillars

A tree T is referred to as *r-regular* for some integer $r \geq 2$ if every non-leaf of T has degree r. A *caterpillar* T is a tree of order 3 or more, the removal of whose leaves produces a path called the *spine* of T. A star is therefore a caterpillar with a trivial spine. A caterpillar T is *cubic* if $\deg v = 3$ for every non-leaf v of T. We now consider the class of cubic caterpillars T_n of even order $n = 2\ell \geq 6$ consisting of the path $(u_0, u_1, \ldots, u_\ell)$ of order $\ell + 1$ and $\ell - 1$ additional pendent edges u_iv_i where $1 \leq i \leq \ell - 1$. The vertices u_i, $1 \leq i \leq \ell$, thus have degree 3 and all other vertices of T_n are leaves.

Theorem 5.3.1 *For each integer $n \geq 6$,*

$$\mathrm{rm}(T_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4 \\ n+1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Assume first that $n \equiv 0 \pmod 4$. Then $n = 4k$ for some integer $k \geq 2$. To show that $\mathrm{rm}(T_n) = n$ in this case, it suffices to show that there is a rainbow mean coloring c of T_n with $\mathrm{rm}(c) = n$. Then T_n consists of the path $P = (u_0, u_1, \ldots, u_{2k})$ of order $2k + 1$ and $2k - 1$ additional pendent edges u_iv_i where $1 \leq i \leq 2k - 1$. Let c be the edge coloring of T_n defined by

$$c(e) = \begin{cases} 2i & \text{if } e = u_iv_i \text{ for } 1 \leq i \leq 2k-2 \\ 4k-3 & \text{if } e = u_{2k-1}v_{2k-1} \\ 1 & \text{if } e = u_0u_1 \\ 2i+4 & \text{if } e = u_iu_{i+1} \text{ where } 1 \leq i \leq 2k-3 \text{ and } i \text{ is odd} \\ 2i+1 & \text{if } e = u_iu_{i+1} \text{ where } 2 \leq i \leq 2k-4 \text{ and } i \text{ is even} \\ 4k & \text{if } e = u_{2k-2}u_{2k-1}, u_{2k-1}u_{2k}. \end{cases}$$

Then the chromatic means of the vertices of T_n are given by

$$\mathrm{cm}(u_i) = \begin{cases} 2i+1 & \text{if } 0 \le i \le 2k-3 \text{ or } i = 2k-1 \\ 2i+2 & \text{if } i = 2k-2 \\ 2i & \text{if } i = 2k \end{cases}$$

$$\mathrm{cm}(v_i) = \begin{cases} 2i & \text{if } 1 \le i \le 2k-2 \\ 2i-1 & \text{if } i = 2k-1. \end{cases}$$

Hence, c is a rainbow mean coloring with $\mathrm{rm}(c) = n$ and so $\mathrm{rm}(T_n) = n$ if $n \equiv 0 \pmod 4$. This coloring is illustrated in Figure 5.7 for the cubic caterpillar T_{16} where $k = 4$.

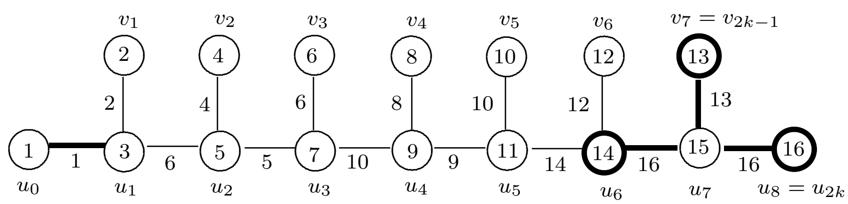

Figure 5.7: A rainbow mean coloring of T_{16}

Next, suppose that $n \equiv 2 \pmod 4$. Then $n = 4k + 2$ for a positive integer k. Then T_n consists of the path $P = (u_0, u_1, \ldots, u_{2k+1})$ of order $2k + 2$ and $2k$ additional pendent edges u_iv_i where $1 \le i \le 2k$. Since $n \equiv 2 \pmod 4$ and each vertex of T_n is odd, it follows by Corollary 5.1.3 that $\mathrm{rm}(T_n) \ge n + 1$. It suffices to show that there is a rainbow mean coloring c of T_n with $\mathrm{rm}(c) = n + 1$. Let c be the edge coloring of T_n defined by

$$c(e) = \begin{cases} 2 & \text{if } e = u_1v_1 \\ 2i+1 & \text{if } e = u_iv_i \text{ for } 2 \le i \le 2k \\ 1 & \text{if } e = u_0u_1 \\ 2i+4 & \text{if } e = u_iu_{i+1} \text{ where } 1 \le i \le 2k-1 \text{ and } i \text{ is odd} \\ 2i+3 & \text{if } e = u_iu_{i+1} \text{ where } 2 \le i \le 2k \text{ and } i \text{ is even.} \end{cases}$$

Then the chromatic means of the vertices of T_n are given by

$$\text{cm}(u_i) = \begin{cases} 2i+1 & \text{if } i = 0, 1, 2k+1 \\ 2i+2 & \text{if } 2 \leq i \leq 2k \end{cases}$$

$$\text{cm}(v_i) = \begin{cases} 2 & \text{if } i = 1 \\ 2i+1 & \text{if } 2 \leq i \leq 2k. \end{cases}$$

Hence, c is a rainbow mean coloring with $\text{rm}(c) = n+1$ and so $\text{rm}(T_n) = n+1$ if $n \equiv 2 \pmod 4$. This coloring is illustrated in Figure 5.7 for the cubic caterpillar T_{16} where $k = 4$. ■

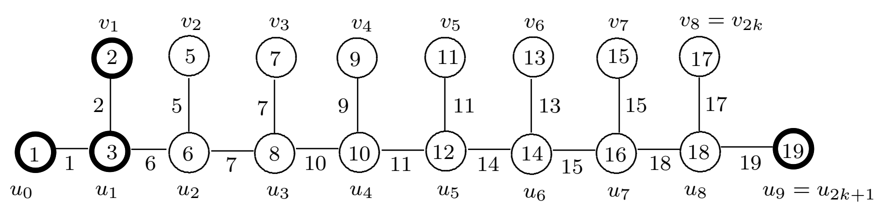

Figure 5.8: A rainbow mean coloring of T_{18}

5.3.2 Subdivided Stars

The *subdivision graph* $S(G)$ of a graph G is the graph obtained from G by subdividing each edge of G exactly once (that is, by replacing each edge $e = uv$ of G by a new vertex w_e and the two new edges uw_e and vw_e, where w_e is called the *subdivision vertex* of e). If G is a graph of order n and size m, then the order of $S(G)$ is $n + m$ and its size is $2m$.

Theorem 5.3.2 *For each integer $t \geq 3$, $\text{rm}(S(K_{1,t})) = 2t + 1$.*

Proof. Let $G = S(K_{1,t})$ be the subdivision graph of the star $K_{1,t}$, where $t \geq 3$. Then the order of G is $n = 2t + 1$. By Observation 4.2.2, it suffices to show that there is a rainbow mean coloring c of G with $\text{rm}(c) = n$. We consider two cases, according to whether t is even or t is odd.

Case 1. *$t \geq 4$ is even.* Then $t = 2k$ for some integer $k \geq 2$. Let

$$V(K_{1,2k}) = \{u_1, u_2, \ldots, u_k\} \cup \{x_1, x_2, \ldots, x_k\} \cup \{w\},$$

where w is the central vertex of $K_{1,2k}$. For each integer i with $1 \leq i \leq k$, let v_i be the subdivision vertex of $u_i w$ and let y_i be the subdivision vertex of $x_i w$. Define the edge coloring $c : E(G) \to [4k+1]$ as follows: For $1 \leq i \leq k$,

$$c(u_i v_i) = 2i - 1,\ c(v_i w) = 2i + 1,$$
$$c(x_i y_i) = 2k + 2i + 1, \text{ and } c(y_i w) = 2k + 2i - 1.$$

Then the chromatic means of the vertices of G are given by

$$\text{cm}(u_i) = 2i - 1 \text{ and } \text{cm}(v_i) = 2i \text{ for } 1 \leq i \leq k,$$
$$\text{cm}(w) = 2k + 1,$$
$$\text{cm}(x_i) = 2k + 2i + 1 \text{ and } \text{cm}(y_i) = 2k + 2i \text{ for } 1 \leq i \leq k.$$

Thus, c is a rainbow mean coloring of G with $\text{rm}(c) = 4k + 1$. This coloring is illustrated in Figure 5.9 for the subdivision graph $S(K_{1,8})$ of the star $K_{1,8}$.

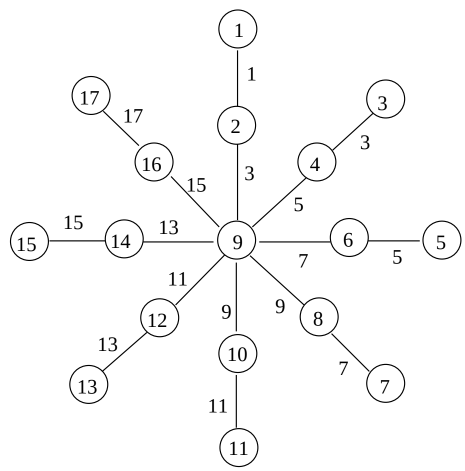

Figure 5.9: A rainbow mean coloring of $S(K_{1,8})$

Case 2. $t \geq 3$ *is odd.* Then $t = 2k + 1$ for some positive integer k. Let

$$V(K_{1,2k+1}) = \{u_1, u_2, \ldots, u_k\} \cup \{x_1, x_2, \ldots, x_{k-1}\} \cup \{w_1, z_1\} \cup \{w\},$$

where w is the central vertex of $K_{1,2k+1}$. For each integer i with $1 \leq i \leq k$, let v_i be the subdivision vertex of $u_i w$ for $1 \leq i \leq k$, let y_i be the subdivision vertex of $x_i w$ for $1 \leq i \leq k - 1$, let w_2 be the subdivision vertex of $w_1 w$, and let z_2 be the subdivision vertex of $z_1 w$. Define the edge coloring $c : E(G) \to [4k+3]$ by

$$c(u_iv_i) = 2i-1 \text{ and } c(v_iw) = 2i+1 \text{ for } 1 \le i \le k$$
$$c(w_1w_2) = 2k+3,\ c(w_2w) = 2k-1,$$
$$c(z_1z_2) = 2k+4,\ c(z_2w) = 2k+6,$$
$$c(x_iy_i) = 2k+2i+5, \text{ and } c(y_iw) = 2k+2i+3 \text{ for } 1 \le i \le k-1.$$

Then the chromatic means of the vertices of G are given by

$$\text{cm}(u_i) = 2i-1 \text{ and } \text{cm}(v_i) = 2i \text{ for } 1 \le i \le k,$$
$$\text{cm}(w) = 2k+2,\ \text{cm}(w_1) = 2k+3,\ \text{cm}(w_2) = 2k+1,$$
$$\text{cm}(z_1) = 2k+4,\ \text{cm}(z_2) = 2k+6$$
$$\text{cm}(x_i) = 2k+2i+5 \text{ and } \text{cm}(y_i) = 2k+2i+4 \text{ for } 1 \le i \le k-1.$$

Thus, c is a rainbow mean coloring of G with $\text{rm}(c) = 4k+3$. This coloring is illustrated in Figure 5.10 for the subdivision graph $S(K_{1,9})$ of the star $K_{1,9}$. ■

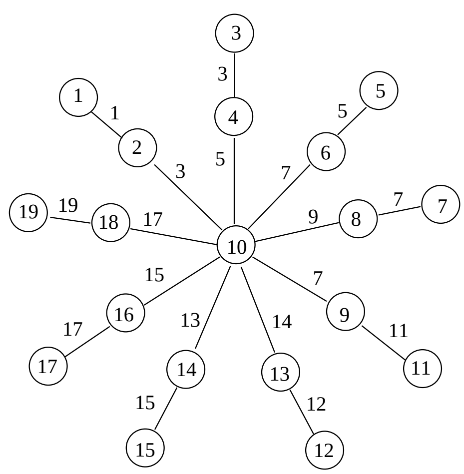

Figure 5.10: A rainbow mean coloring of $S(K_{1,9})$

5.3.3 Double Stars

We saw in Theorem 5.2.7 that the rainbow mean index of the star $K_{1,t}$, $t \ge 2$, is $t+1$ if t even and is $t+3$ if t is odd. In fact, the stars of even order 4 or more are the only connected graphs whose rainbow mean index has been shown to be neither the order nor one plus the order of the graph. This suggests investigating the rainbow mean index of the related double stars class of graphs. For integers a and b with $2 \le a \le b$, the double star $S_{a,b}$ is that tree of order $a+b$ (and size $a+b-1$)

and diameter 3 whose central vertices u and v have degrees a and b, respectively. The vertex u is thus adjacent to $a-1$ end-vertices, denoted by $u_1, u_2, \ldots, u_{a-1}$, while v is adjacent to $b-1$ end-vertices, denoted by $v_1, v_2, \ldots, v_{b-1}$.

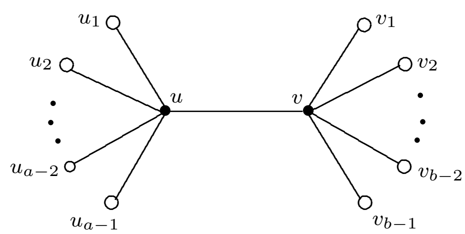

Figure 5.11: The double star $S_{a,b}$ of order $a+b$ where $a, b \geq 2$

First, we determine $\text{rm}(S_{a,b})$ where $a = b$. Since $\text{rm}(S_{2,2}) = \text{rm}(P_4) = 5$ by Theorem 4.3.2, we may assume that $a \geq 3$.

Theorem 5.3.3 *For each integer $a \geq 3$,*

$$\text{rm}(S_{a,a}) = \begin{cases} 2a & \text{if } a \text{ is even} \\ 2a+1 & \text{if } a \text{ is odd.} \end{cases}$$

Proof. Suppose that u and v are the central vertices of $G = S_{a,a}$ where u is adjacent to the $a-1$ end-vertices $u_1, u_2, \ldots, u_{a-1}$ and v is adjacent to the $a-1$ end-vertices $v_1, v_2, \ldots, v_{a-1}$. We consider two cases, according to whether a is even or a is odd.

Case 1. *$a \geq 4$ is even.* Then $a = 2k$ for some integer $k \geq 2$. Since the order of G is $4k$, it suffices to show that there is a rainbow mean coloring c of G with $\text{rm}(c) = 4k$ by Observation 4.2.2. Define the edge coloring c such that

$$\begin{aligned} \{c(uu_i) : 1 \leq i \leq 2k\} &= [k] \cup [3k+1, 4k-1] \\ c(uv) &= k \\ \{c(vv_i) : 1 \leq i \leq 2k\} &= ([k+1, 3k] \cup \{4k\}) - \{2k-1, 2k+1\}. \end{aligned}$$

Then the chromatic means of the vertices of G are given by

$$\text{cm}(u_i) = c(uu_i) \text{ and } \text{cm}(v_i) = c(vv_i) \text{ for } 1 \leq i \leq 2k,$$
$$\text{cm}(u) = 2k-1 \text{ and } \text{cm}(v) = 2k+1.$$

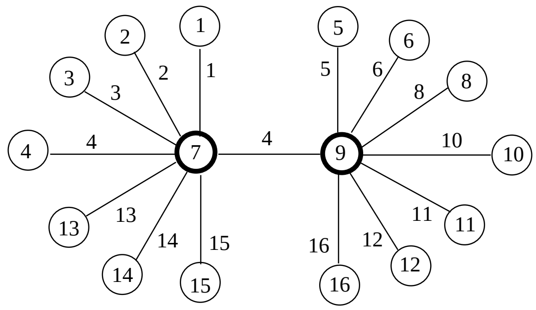

Figure 5.12: A rainbow mean coloring of $S_{8,8}$

This coloring c is illustrated in Figure 5.12 for the double star $S_{8,8}$ where $k = 4$. Thus, c is a rainbow mean coloring of G with $\text{rm}(c) = 4k$.

Case 2. *$a \geq 3$ is odd.* Then $a = 2k + 1$ for some positive integer k. Since the order of G is $4k + 2$ and every vertex of G is odd, it follows by Corollar 5.1.3 that $\text{rm}(G) \geq 4k + 3$. Thus, it remains to show that there is a rainbow mean coloring c of G with $\text{rm}(c) = 4k + 3$. Define the edge coloring c as follows:

$$c(u_i u) = 2i - 1 \text{ for } 1 \leq i \leq k \text{ and } c(u_i u) = 2i + 1 \text{ for } k + 1 \leq i \leq 2k$$
$$c(v_i v) = 2i \text{ for } 1 \leq i \leq k \text{ and } c(v_i v) = 2i + 2 \text{ for } k + 1 \leq i \leq 2k - 1,$$
$$c(uv) = 2k + 1 \text{ and } c(v_{2k} v) = 4k + 3.$$

Then the chromatic means of the vertices of G are given by

$$\text{cm}(u_i) = c(u_i u) \text{ for } 1 \leq i \leq 2k \text{ and } \text{cm}(v_i) = c(v_i v) \text{ for } 1 \leq i \leq k,$$
$$\text{cm}(u) = 2k + 1 \text{ and } \text{cm}(v) = 2k + 2.$$

This coloring c is illustrated in Figure 5.13 for the double star $S_{9,9}$ where $k = 4$. Thus, c is a rainbow mean coloring of G with $\text{rm}(c) = 4k + 3$. ∎

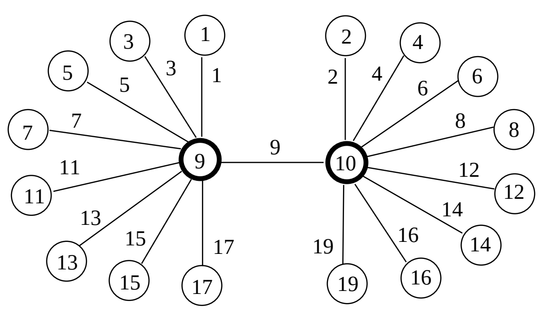

Figure 5.13: A rainbow mean coloring of $S_{9,9}$

If $a, b \geq 3$ are odd and $a \equiv b \pmod 4$, then it follows by Corollary 5.1.3 that $\text{rm}(S_{a,b}) \geq a+b+1$. In fact, $\text{rm}(S_{a,b}) = a+b+1$ as we show next.

Theorem 5.3.4 *If a and b are odd integers with $a, b \geq 3$ and $a \equiv b \pmod 4$, then*

$$\text{rm}(S_{a,b}) = a+b+1.$$

Proof. By Theorem 5.3.3, we may assume that $a < b$. Since a and b are odd integers and $a \equiv b \pmod 4$, it follows that either a and b are both congruence to 1 modulo 4 or a and b are both congruence to 3 modulo 4. In each case, $a+b \equiv 2 \pmod 4$ and every vertex of G is odd. Hence, $\text{rm}(G) \geq a+b+1$ by Corollary 5.1.3. Thus, it remains to show that there is a rainbow mean coloring c of G with $\text{rm}(c) = a+b+1$. We consider these two cases.

Case 1. $a \equiv 1 \pmod 4$ *and* $b \equiv 1 \pmod 4$. Then $a = 4j+1$ and $b = 4k+1$ for some integers j, k with $1 \leq j < k$. Let u and v be the central vertices of $G = S_{4j+1,4j+1}$ where u is adjacent to the $a-1 = 4j+2$ end-vertices $u_1, u_2, \ldots u_{4j}$ and v is adjacent to the $b-1 = 4k+2$ end-vertices $v_1, v_2, \ldots, v_{4k}$. Define the edge coloring c by

$$\{c(uu_i) : 1 \leq i \leq 2j\} = [4j+1] - \{2j+1\},$$
$$c(uv) = 2j+1$$
$$\{c(vv_i) : 1 \leq i \leq 2j+2\} = [4j+2, 4j+4k+3] - \{2k+2j+2, 2k+4j+2\}.$$

Then the chromatic means of the vertices of G are given by

$$\text{cm}(u_i) = c(uu_i) \text{ for } 1 \leq i \leq 4j,$$
$$\text{cm}(u) = 2j+1,\ \text{cm}(v) = 2k+4j+2.$$
$$\text{cm}(v_i) = c(vv_i) \text{ for } 1 \leq i \leq 4k.$$

Thus, c is a rainbow mean coloring of G with $\text{rm}(c) = 4j+4k+3$.

Case 2. $a \equiv 3 \pmod 4$ *and* $b \equiv 3 \pmod 4$. Then $a = 4j+3$ and $b = 4k+3$ for some integers j, k with $0 \leq j < k$. Let u and v be the central vertices of $G = S_{4j+3,4j+3}$ where u is adjacent to the $a-1 = 4j+2$ end-vertices $u_1, u_2, \ldots u_{4j+2}$ and v is adjacent to the $b-1 = 4k+2$ end-vertices $v_1, v_2, \ldots, v_{4k+2}$. Define the edge coloring c by

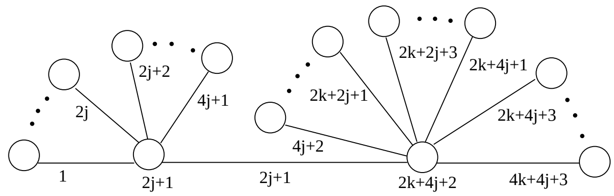

Figure 5.14: A rainbow mean coloring of $S_{4j+1,4k+1}$

$$\{c(uu_i) : 1 \leq i \leq 2j\} = [4j+3] - \{2j+2\},$$
$$c(uv) = 2j+2$$
$$\{c(vv_i) : 1 \leq i \leq 2j+2\} = [4j+4, 4j+4k+7] - \{2k+2j+4, 2k+4j+5\}.$$

Then the chromatic means of the vertices of G are given by

$$\text{cm}(u_i) = c(uu_i) \text{ for } 1 \leq i \leq 4j+2,$$
$$\text{cm}(u) = 2j+2, \text{ cm}(v) = 2k+4j+5.$$
$$\text{cm}(v_i) = c(vv_i) \text{ for } 1 \leq i \leq 4k+2.$$

Thus, c is a rainbow mean coloring of G with $\text{rm}(c) = 4j+4k+7$. ■

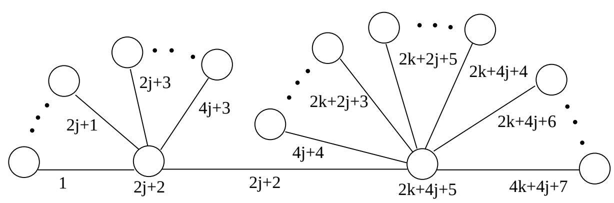

Figure 5.15: A rainbow mean coloring of $S_{4j+3,4k+3}$

We now turn our attention to the double stars $S_{a,b}$ where $2 \leq a < b$ and at least one of a and b is even.

Theorem 5.3.5 *If a and b are integers with $2 \leq a < b$ such that ab is even, then*

$$\text{rm}(S_{a,b}) = a + b.$$

Proof. Let $G = S_{a,b}$ where $2 \leq a < b$ and ab is even. By Observation 4.2.2, it suffices to show that there is a rainbow mean coloring c of G with $\text{rm}(c) = a + b$. We consider three cases, according to the parities of a and b.

Case 1. *a and b are both even.* Then $a = 2j$ and $b = 2k$ where j and k are integers and $1 \leq j < k$. Let u and v be the central vertices of $G = S_{2j,2k}$ where

u is adjacent to the $a-1=2j-1$ end-vertices $u_1, u_2, \dots u_{2j-1}$ and v is adjacent to the $b-1=2k-1$ end-vertices $v_1, v_2, \dots, v_{2k-1}$. It suffices to show that there exists a rainbow mean coloring c with $\mathrm{rm}(c)=a+b$. Define the edge coloring c by

$$\{c(uu_i): 1 \le i \le 2j-1\} = [j+1, 3j-1],$$
$$c(uv) = 2j(j+1)$$
$$\{c(vv_i): 1 \le i \le 2k-1\} = [j] \cup [3j+1, 2j+2k] - \{2j+k\}.$$

Then the chromatic means of the vertices of G are given by

$$\mathrm{cm}(u_i) = c(uu_i) \text{ for } 1 \le i \le 2j-1,$$
$$\mathrm{cm}(u) = 3j,\ \mathrm{cm}(v) = 2j+k.$$
$$\mathrm{cm}(v_i) = c(vv_i) \text{ for } 1 \le i \le 2k-1.$$

Since $j+1 \le k$, it follows that $\mathrm{cm}(u) \ne \mathrm{cm}(v)$. Thus, c is a rainbow mean coloring of G with $\mathrm{rm}(c) = 2j+2k$.

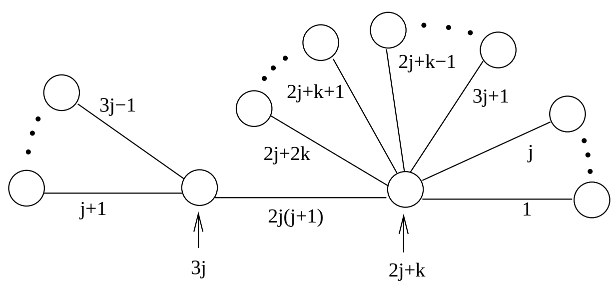

Figure 5.16: A rainbow mean coloring of $S_{2j,2k}$ where $j<k$

Case 2. *$a \ge 3$ is odd and $b \ge 4$ is even.* Then $a=2j+1$ and $b=2k$ for some integers j,k with $1 \le j < k$. Let u and v be the central vertices of G where u is adjacent to the $a-1=2j$ end-vertices $u_1, u_2, \dots, u_{2j}$ and v is adjacent to the $b-1=2k-1$ end-vertices $v_1, v_2, \dots, v_{2k-1}$. Define the edge coloring c by

$$c(uu_i) = i \text{ for } 1 \le i \le 2j,\ c(uv) = 2jk+2j+k+1$$
$$\{c(vv_i): 1 \le i \le 2k-1\} = [2j+1, 2k+2j+1] - \{k+j+1, k+3j+1\}.$$

Then the chromatic means of the vertices of G are given by

$$\mathrm{cm}(u_i) = c(uu_i) \text{ for } 1 \le i \le 2j,\ \mathrm{cm}(u) = k+j+1,\ \mathrm{cm}(v) = k+3j+1.$$
$$\mathrm{cm}(v_i) = c(vv_i) \text{ for } 1 \le i \le 2k-1.$$

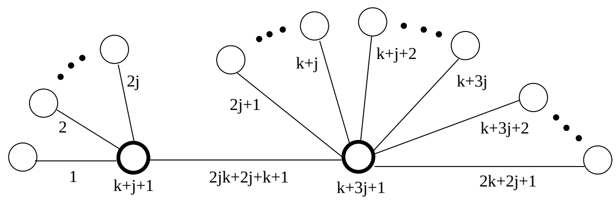

Figure 5.17: A rainbow mean coloring of $S_{2j+1,2k}$ where $j < k$

Thus, c is a rainbow mean coloring of G with $\text{rm}(c) = 2j + 2k + 1$.

Case 3. *$a \geq 2$ is even, and $b \geq 3$ is odd.* Then $a = 2j$ and $b = 2k + 1$ where $1 \leq j \leq k$. Let u and v be the central vertices of G where u is adjacent to the $a - 1 = 2j - 1$ end-vertices $u_1, u_2, \ldots u_{2j-1}$ and v is adjacent to the $b - 1 = 2k$ end-vertices $v_1, v_2, \ldots, v_{2k}$. Define the edge coloring c by

$$\{c(uu_i) : 1 \leq i \leq 2j - 1\} = [j + k + 2, 3j + k],$$
$$c(uv) = 2j(j + 1) + k + 1$$
$$\{c(vv_i) : 1 \leq i \leq 2k\} = [j + k] \cup [3j + k + 2, 2j + 2k + 1].$$

Then the chromatic means of the vertices of G are given by

$$\text{cm}(u_i) = c(uu_i) \text{ for } 1 \leq i \leq 2j - 1,$$
$$\text{cm}(u) = 3j + k + 1, \text{ cm}(v) = j + k + 1.$$
$$\text{cm}(v_i) = c(vv_i) \text{ for } 1 \leq i \leq 2k.$$

Thus, c is a rainbow mean coloring of G with $\text{rm}(c) = 2j + 2k + 1$. ∎

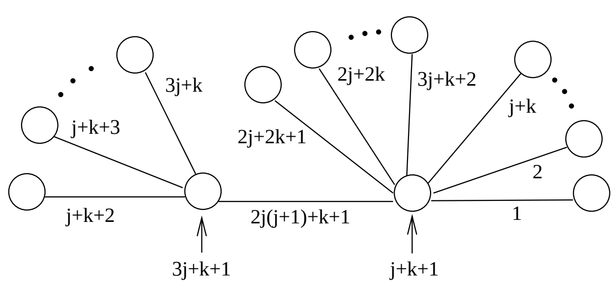

Figure 5.18: A rainbow mean coloring of $S_{2j,2k+1}$ where $j \leq k$

The one remaining class of double stars $S_{a,b}$ for which the rainbow mean index has not yet been determined is that where a and b are both odd and $a \not\equiv b \pmod 4$. In order to present a result dealing with this class, it is convenient to establish the following two lemmas.

Lemma 5.3.6 *For positive integers a and b with $a \leq b$ and the set*

$$X = [4a + 4b + 4] - \{2a + 2b + 1, 2a + 2b + 3\},$$

let $s_1 = \sum_{i=1}^{4a} i$ *and* $s_2 = \sum_{i=1}^{4a}(4b + 4 + i)$. *For every integer s with $s_1 \leq s \leq s_2$, there exists a $(4a)$-element subset S of X such that* $\sum_{x \in S} x = s$.

Proof. First, we show that there exists a $(4a)$-element subset $S \subseteq [4a + 4b + 4]$ such that $\sum_{x \in S} x = s$. If $s = s_1$ or $s = s_2$, then the result holds. Thus, we may assume that $s_1 < s < s_2$. Let m be the minimum integer in $[4b + 4]$ such that

$$[m + (m + 1) + \cdots + (m + 4a - 1)] < s < [(m + 1) + (m + 2) + \cdots + (m + 4a)].$$

Let $t = (m+1)+(m+2)+\cdots+(m+4a-1)$. Therefore, $m+t < s < t+(m+4a)$. Thus, $s = m + t + r$ for some integer r with $1 \leq r \leq 4a - 1$. Consequently, by adding 1 to the last r terms in the sum $m+(m+1)+\cdots+(m+4a-1)$, we obtain the $(4a)$-element set

$$T = \{m, m+1, \ldots, m+4a-r-1\} \cup \{m+4a-r+1, m+4a-r+2, \ldots, m+4a\}$$

such that $\sum_{x \in T} x = s$.

It remains to show that there are $4a$ distinct integers in X whose sum is s. Of course, if neither $2a + 2b + 1$ nor $2a + 2b + 3$ belongs to T, then T has the desired property. Thus, we may assume that at least one of $2a + 2b + 1$ and $2a + 2b + 3$ belongs to T, say $2a + 2b + 1 \in T$.

- ⋆ If $2a+2b+3 \in T$ as well, then we remove $2a+2b+1$ and $2a+2b+3$ from T and replace them by 1 and $4a + 4b + 3$, obtaining the set $T_1 \subseteq X$ such that the sum of elements in T_1 is s.

- ⋆ If $2a+2b+3 \notin T$, then either $2a+2b \in T$ or $2a+2b+2 \in T$, say the former. Hence, we remove $2a+2b$ and $2a+2b+1$ from T and replace them by 1 and $4a+4b$, obtaining the set $T_2 \subseteq X$ such that the sum of elements in T_2 is s. ■

Lemma 5.3.7 *For positive integers a and b with $a \leq b$ and the set*

$$X = [4a + 4b + 4] - \{2a + 2b + 1, 2a + 2b + 3\},$$

let $s_1 = \sum_{i=1}^{4a+2} i$ *and* $s_2 = \sum_{i=1}^{4a+2} (4b + 2 + i)$*. For every integer* s *with* $s_1 \le s \le s_2$*, there exists a* $(4a + 2)$*-element subset* S *of* X *such that* $\sum_{x \in S} x = s$*.*

Proof. First, we show that there exists a $(4a+2)$-element subset $S \subseteq [4a+4b+4]$ such that $\sum_{x \in S} x = s$. If $s = s_1$ or $s = s_2$, then the result holds. Thus, we may assume that $s_1 < s < s_2$. Let m be the minimum integer in $[4b + 2]$ such that

$$[m + (m + 1) + \cdots + (m + 4a + 1)] < s < [(m + 1) + (m + 2) + \cdots + (m + 4a + 2)].$$

Let $t = (m+1)+(m+2)+\cdots+(m+4a+1)$. Therefore, $m+t < s < t+(m+4a+2)$. Thus, $s = m + t + r$ for some integer r with $1 \le r \le 4a + 1$. Consequently, by adding 1 to the last r terms in the sum $m + (m + 1) + \cdots + (m + 4a + 1)$, we obtain the $(4a + 2)$-element set

$$T = \{m, m+1, \ldots, m+4a-r+1\} \cup \{m+4a-r+3, m+4a-r+4, \ldots, m+4a+2\}$$

such that $\sum_{x \in T} x = s$.

It remains to show that there are $4a + 2$ distinct integers in X whose sum is s. Of course, if neither $2a+2b+1$ nor $2a+2b+3$ belongs to T, then T has the desired property. Thus, we may assume that at least one of $2a + 2b + 1$ and $2a + 2b + 3$ belongs to T, say $2a + 2b + 1 \in T$.

- ⋆ If $2a+2b+3 \in T$ as well, then we remove $2a+2b+1$ and $2a+2b+3$ from T and replace them by 1 and $4a + 4b + 3$, obtaining the set $T_1 \subseteq X$ such that the sum of elements in T_1 is s.

- ⋆ If $2a+2b+3 \notin T$, then either $2a+2b \in T$ or $2a+2b+2 \in T$, say the former. Hence, we remove $2a+2b$ and $2a+2b+1$ from T and replace them by 1 and $4a+4b$, obtaining the set $T_2 \subseteq X$ such that the sum of elements in T_2 is s. ■

We are now prepared to present the following result.

Theorem 5.3.8 *If* a *and* b *are odd integers with* $3 \le a < b$ *such that* $a \not\equiv b \pmod 4$*, then* $\mathrm{rm}(S_{a,b}) = a + b$*.*

Proof. Let $G = S_{a,b}$. We show that there is a rainbow mean coloring $c : E(G) \to [a+b]$ of G with $\mathrm{rm}(c) = a+b$ such that $\mathrm{cm}(u)$ and $\mathrm{cm}(v)$ have certain prescribed values. We consider two cases. In each case, we let

$$\begin{aligned} A &= \sum_{i=1}^{a-1} c(uu_i) = \sum_{i=1}^{a-1} \mathrm{cm}(u_i) \\ B &= \sum_{i=1}^{b-1} c(vv_i) = \sum_{i=1}^{b-1} \mathrm{cm}(v_i) \\ x &= c(uv). \end{aligned}$$

Observe that $A + x = \mathrm{cm}(u) \cdot a$ and $B + x = \mathrm{cm}(v) \cdot b$. Furthermore,

$$A + B + \mathrm{cm}(u) + \mathrm{cm}(v) = 1 + 2 + \cdots + (a+b) = \binom{a+b+1}{2}.$$

Case 1. $a \equiv 3 \pmod 4$ *and* $b \equiv 1 \pmod 4$. Then $a = 4j+3$ and $b = 4k+1$ where $0 \le j < k$. We show that there is a rainbow mean coloring $c : E(G) \to [4k+4j+4]$ of G with $\mathrm{rm}(c) = 4j+4k+4$ such that $\mathrm{cm}(u) = 2k+2j+1$ and $\mathrm{cm}(v) = 2k+2j+3$. For such an edge coloring c of G, we have

$$\begin{aligned} A + x &= (2k + 2j + 1)(4j + 3) = 8kj + 8j^2 + 6k + 10j + 3 \\ B + x &= (2k + 2j + 3)(4k + 1) = 8kj + 8j^2 + 14k + 2j + 3 \\ A + B &= 1 + 2 + \cdots + (4k + 4j + 5) - (\mathrm{cm}(u) + \mathrm{cm}(v)) \\ &= (16kj + 8k^2 + 8j^2 + 18k + 18j + 10) - (4k + 4j + 4) \\ &= 16kj + 8k^2 + 8j^2 + 14k + 14j + 6. \end{aligned}$$

Hence,

$$\begin{aligned} A &= 8kj + 8j^2 + 3k + 9j + 3 \\ B &= 8kj + 8k^2 + 11k + 3j + 3 \\ x &= 3k - j. \end{aligned}$$

Therefore, such an edge coloring c of G exists if there are $4a + 2$ distinct elements in the set $X = [4k + 4j + 4] - \{2k + 2j + 1, 2k + 2j + 3\}$ whose sum is $A = 8kj+8j^2+3k+9j+3$. The sum of the $4j+2$ smallest integers in the set $[4k+4j+4]$ is

$$\binom{4j+3}{2} = (2j+1)(4j+3) = 8j^2 + 10j + 3;$$

while the sum of the $4j+2$ largest integers in the set $[4k+4j+4]$ is

$$(2j+1)(8k+4j+7) = 16kj + 8j^2 + 8k + 18j + 7.$$

Since

$$8j^2 + 10j + 3 \le A \le 16kj + 8j^2 + 8k + 18j + 7,$$

it follows by Lemma 5.3.7 that there is a $(4a+2)$-element subset S of X such that $\sum_{x \in S} x = S$. Observe that the sum of integers in $X - S$ is therefore B.

Case 2. $a \equiv 1 \pmod 4$ *and* $b \equiv 3 \pmod 4$. Then $a = 4j+1$ and $b = 4k+3$ where $1 \le j \le k$. We show that there is a rainbow mean coloring $c : E(G) \to [4k+4j+4]$ of G with $\text{rm}(c) = 4j+4k+4$ such that $\text{cm}(u) = 2k+2j+1$ and $\text{cm}(v) = 2k+2j+3$. For such an edge coloring c of G, we have

$$\begin{aligned} A + x &= (2k+2j+1)(4j+1) = 8kj + 8j^2 + 2k + 6j + 1 \\ B + x &= (2k+2j+3)(4k+3) = 8kj + 8j^2 + 18k + 6j + 9 \\ A + B &= 16kj + 8k^2 + 8j^2 + 14k + 14j + 6. \end{aligned}$$

Hence,

$$\begin{aligned} A &= 8kj + 8j^2 - k + 7j - 1 \\ B &= 8kj + 8k^2 + 15k + 7j + 7 \\ x &= 3k - j + 2. \end{aligned}$$

Therefore, such an edge coloring c of G exists if there are $4a$ distinct elements in the set $X = [4k+4j+4] - \{2k+2j+1, 2k+2j+3\}$ whose sum is $A = 8kj + 8j^2 - k + 7j - 1$. The sum of the $4j$ smallest integers in the set $[4k+4j+4]$ is

$$\binom{4j+1}{2} = 2j(4j+1) = 8j^2 + 2j;$$

while the sum of the $4j$ largest integers in the set $[4k+4j+4]$ is

$$2j(8k+4j+9) = 16kj + 8j^2 + 18j.$$

Since $8j^2 + 2j \le A \le 16kj + 8j^2 + 18j$, it follows by Lemma 5.3.6 that there is a $4a$-element subset S of X such that $\sum_{x \in S} x = A$. Again, the sum of integers in $X - S$ is therefore B. ■

In summary, we have the following result.

Theorem 5.3.9 *For integers a and b where $a, b \geq 2$,*

$$\mathrm{rm}(S_{a,b}) = \begin{cases} a+b & \textit{if } ab \textit{ is even or } ab \textit{ is odd and } a+b \not\equiv 2 \pmod 4 \\ a+b+1 & \textit{if } ab \textit{ is odd and } a+b \equiv 2 \pmod 4. \end{cases}$$

All trees that have been studied thus far lead us to the following conjecture.

Conjecture 5.3.10 *Let T be a tree of order $n \geq 5$ that is not a star. Then* $\mathrm{rm}(T) = n$ *if and only if (i) $n \not\equiv 2 \pmod 4$ or (ii) $n \equiv 2 \pmod 4$ and T has at least one even vertex; while* $\mathrm{rm}(T) = n + 1$ *if $n \equiv 2 \pmod 4$ and all vertices of T have odd degrees.*

Chapter 6

Proper Mean Colorings

Abstract: For an edge coloring of a connected graph G of order 3 or more with positive integers, the chromatic mean of a vertex v of G is the sum of the colors of the edges incident with v divided by the degree of v. Only edge colorings c are considered for which the chromatic mean of every vertex is a positive integer. If adjacent vertices have distinct chromatic means, then c is a proper mean coloring of G. The maximum vertex color in a proper mean coloring c of G is the proper mean index of c and the proper mean index $\mu(G)$ of G is the minimum proper mean index among all proper mean colorings of G. The proper mean index is determined for complete graphs, cycles, stars, double stars, and paths. The non-leaf minimum degree $\delta^*(T)$ of a tree T is the minimum degree among the non-leaves of T. It is shown that if T is tree with $\delta^*(T) \geq 10$ or a caterpillar with $\delta^*(T) \geq 6$, then $\mu(T) \leq 4$. Furthermore, it is conjectured that $\chi(G) \leq \mu(G) \leq \chi(G) + 2$ for every connected graph G of order 3 or more.

6.1 Introduction

First, we recall some basic definitions and notation on mean colorings of a graph. For every connected graph G of order 3 or more, there are edge colorings c with positive integers that induce a positive integer vertex coloring cm defined for each vertex v of G by

$$\text{cm}(v) = \frac{\sum_{e \in E_v} c(e)}{\deg v}, \text{ where } E_v \text{ is the set of edges incident with } v.$$

Edge colorings with this property are called *mean colorings*. The induced vertex color $\mathrm{cm}(v)$ of a vertex v of G is called the *chromatic mean* of v. Consequently, only edge colorings c are considered for which $\mathrm{cm}(v)$ is a positive integer for every vertex v of G. For a vertex v in a graph G with a mean coloring c, the *chromatic sum* of v is defined as $\mathrm{cs}(v) = \sum_{e \in E_v} c(e)$. Note that $\mathrm{cs}(v) = \deg v \cdot \mathrm{cm}(v)$. Thus, the sum of the chromatic sums of all vertices in a graph satisfies the identity

$$\sum_{v \in V(G)} \mathrm{cs}(v) = 2 \sum_{e \in E(G)} c(e).$$

Also, recall that if distinct vertices have distinct chromatic means, then the edge coloring c is called a *rainbow mean coloring* of G. It was shown that every connected graph of order 3 or more has a rainbow mean coloring. For a rainbow mean coloring c of a graph G, the maximum vertex color is the *rainbow mean index* $\mathrm{rm}(c)$ of c. That is, $\mathrm{rm}(c) = \max\{\mathrm{cm}(v) : v \in V(G)\}$. The *rainbow mean index* $\mathrm{rm}(G)$ of G itself is defined as $\mathrm{rm}(G) = \min\{\mathrm{rm}(c) : c \text{ is a rainbow mean coloring of } G\}$. We saw that if G is a connected graph of order $n \geq 3$, then $\mathrm{rm}(G) \geq n$.

A mean coloring of a connected graph G of order 3 or more is defined to be a *proper mean coloring* of G if no two adjacent vertices in G have the same chromatic mean. The maximum vertex color in a proper mean coloring c is the *proper mean index* $\mu(c)$ of c and the minimum proper mean index among all proper mean colorings of G is the *proper mean index* $\mu(G)$ of G. Since every such graph has a rainbow mean coloring, each such graph has a proper mean coloring as well. In addition, the proper mean index of a graph G is at least its chromatic number $\chi(G)$. Thus, $\chi(G) \leq \mu(G) \leq \mathrm{rm}(G)$ for every connected graph G of order at least 3. As an illustration of these concepts, Figure 6.1 shows proper meaning colorings of the cycles C_5 and C_6. In fact, $\mu(C_5) = 4 = \chi(C_5) + 1$ and $\mu(C_6) = 4 = \chi(C_6) + 2$.

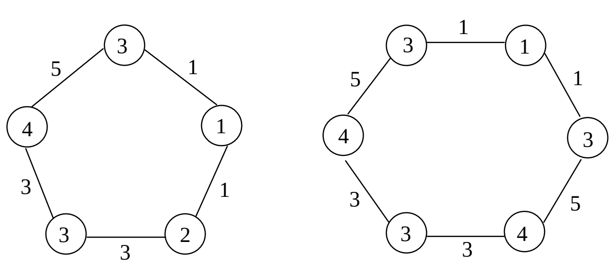

Figure 6.1: Proper mean colorings of C_5 and C_6

While $\chi(G) = 2$ for every nontrivial connected bipartite graph G, $\mu(G) \neq \chi(G)$ for every bipartite graph G. Indeed, $\mu(G) \neq 2$ for every connected graph G of order at least 3. In order to verify this fact, we first present a useful observation.

Observation 6.1.1 *Let G be a connected bipartite graph with partite sets U and W. If c is an edge coloring of G, then*

$$\sum_{u\in U} \mathrm{cs}(u) = \sum_{w\in W} \mathrm{cs}(w).$$

Proposition 6.1.2 *If G is a connected graph of order at least 3, then $\mu(G) \geq 3$. Furthermore, if c is a proper mean coloring of G with $\mu(c) = 3$, then*

$$\{\mathrm{cm}(v) : v \in V(G)\} = [3].$$

Proof. We show that if c is a proper mean coloring of G, then the induced vertex coloring cm of c uses at least three distinct colors. Assume, to the contrary, that the chromatic mean cm obtained from c uses only two distinct colors $a, b \in [3]$. Then G is a connected bipartite graph. Let U and W be the partite sets of G. We may assume that $\mathrm{cm}(u) = a$ for each $u \in U$ and $\mathrm{cm}(w) = b$ for each $w \in W$. Since

$$\begin{gathered}\textstyle\sum_{u\in U} \mathrm{cs}(u) = a\sum_{u\in U} \deg u = a|E(G)| \text{ and}\\ \textstyle\sum_{w\in W} \mathrm{cs}(w) = b\sum_{w\in W} \deg w = b|E(G)|,\end{gathered}$$

it follows by Observation 6.1.1 that $a = b$, which is a contradiction. As a result, $\mu(G) \geq \mu(c) \geq 3$. Moreover, if $\mu(c) = 3$, then cm must use all three colors in [3]. ■

6.2 The Proper Mean Index of Some Well-Known Graphs

As we saw, the rainbow mean indexes of complete graphs were determined. Since $\mu(K_n) = \mathrm{rm}(K_n)$ for each integer $n \geq 3$, we have the following result.

Theorem 6.2.1 *For an integer $n \geq 3$,*

$$\mu(K_n) = \mathrm{rm}(K_n) = \begin{cases} n & \text{if } n \geq 4 \text{ and } n \not\equiv 2 \pmod 4\\ n+1 & \text{if } n = 3 \text{ or } n \equiv 2 \pmod 4. \end{cases}$$

Theorem 6.2.1 also shows that there are graphs G for which $\mu(G) = \chi(G)$. We now determine the proper mean index of all paths and cycles, beginning with paths.

Theorem 6.2.2 *For each integer $n \geq 3$,*

$$\mu(P_n) = \begin{cases} 3 & \textit{if } n \textit{ is odd} \\ 4 & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. First, suppose that n is odd. By Proposition 6.1.2, it suffices to show that there is a proper mean coloring c such that $\mu(c) = 3$. Let $P = (u_1, u_2, \ldots u_n)$. For each integer i with $1 \leq i \leq n$, define

$$c(e) = \begin{cases} 1 & \text{if } e \text{ is incident with } u_i \text{ for } i \equiv 1 \pmod 4 \\ 3 & \text{if } e \text{ is incident with } u_i \text{ for } i \equiv 3 \pmod 4. \end{cases}$$

Then the vertex color $\text{cm}(u_i)$, $1 \leq i \leq n$, is given by

$$\text{cm}(u_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 4 \\ 2 & \text{if } i \equiv 0, 2 \pmod 4 \\ 3 & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

This edge coloring is illustrated in Figure 6.2 for P_n when $n = 3, 5, 7$. Since cm is a proper coloring of P_n, it follows that $\mu(c) = 3$. Thus, $\mu(P_n) = 3$ for every odd integer $n \geq 3$.

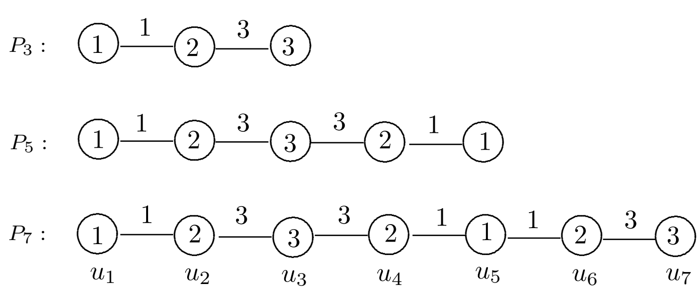

Figure 6.2: Proper mean colorings of P_n for $n = 3, 5, 7$

Next, suppose that n is even. We show that $\mu(P_n) = 4$. First, we show that $\mu(P_n) \geq 4$. Assume, to the contrary, that there is a proper mean coloring c of P_n

with $\mu(c) = 3$. By Proposition 6.1.2, there is a vertex v in P_n such that $\text{cm}(v) = 1$. Hence, each edge incident with v is colored 1, which implies that $c(e)$ is odd for all $e \in E(P_n)$. Suppose that there is an edge e such that $c(e) = 5$. If e is a pendant edge of P_n, then $\mu(c) \geq 5$, which is a contradiction. Thus, we may assume that e is adjacent to two edges e_1 and e_2 of P_n. Since $c(e_1) \neq c(e_2)$ and $c(e_1)$ and $c(e_2)$ are both odd, at least one of $c(e_1)$ and $c(e_2)$ is 3 or more. However then, $\text{cm}(u) \geq 4$ for at least one vertex u incident with e, which is a contradiction. Thus, all edges are colored 1 or 3. We may assume, without loss of generality, that $c(u_1u_2) = 1$. This implies that $c(e) = 1$ if e is incident with u_i where $i \equiv 1 \pmod 4$ and $c(e) = 3$ if e is incident with u_i where $i \equiv 3 \pmod 4$. Since $n \geq 4$ is even, it follows that $\text{cm}(u_{n-1}) = \text{cm}(u_n) \in \{1, 3\}$, which is a contradiction. Hence, $\mu(P_n) \geq 4$.

To verify that $\mu(P_n) \leq 4$, it remains to show that there is a proper mean coloring c with $\mu(c) = 4$ for each even integer $n \geq 4$. For the path $P_{n-1} = (v_1, v_2, v_3, \ldots, v_{n-1})$, where $n - 1 \geq 3$ is odd, let c_0 be the proper mean coloring of P_{n-1} with $\mu(c_0) = 3$ defined in Case 1. Subdividing the edge v_2v_3 of P_{n-1}, we obtain the path $P_n = (v_1, v_2, w, v_3, \ldots, v_{n-1})$ of order n. Now, define the edge coloring c of P_n by $c(v_2w) = 5$, $c(wv_3) = 3$, and $c(e) = c_0(e)$ if e is not incident with w. If we denote P_n by $(u_1, u_2, \ldots, u_n)$, then the vertex color $\text{cm}(u_i)$, $1 \leq i \leq n$, is given by

$$\text{cm}(u_i) = \begin{cases} 1 & \text{if } i = 1 \text{ or } i \equiv 2 \pmod 4 \text{ for } i \neq 2 \\ 2 & \text{if } i \equiv 1, 3 \pmod 4 \text{ for } 5 \leq i \leq n-1 \\ 3 & \text{if } i = 2 \text{ or } i \equiv 0 \pmod 4 \\ 4 & \text{if } i = 3. \end{cases}$$

This edge coloring is illustrated in Figure 6.3 for P_n when $n = 4, 6, 8$. Since cm is a proper coloring of P_n, it follows that $\mu(c) = 4$. Therefore, $\mu(P_n) = 4$ for each even integer $n \geq 4$. ■

Theorem 6.2.3 *For each integer $n \geq 4$,*

$$\mu(C_n) = \begin{cases} 3 & \textit{if } n \equiv 0 \pmod 4 \\ 4 & \textit{if } n \not\equiv 0 \pmod 4. \end{cases}$$

Proof. Let $C_n = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ be a cycle of order $n \geq 3$, where $e_i = u_iu_{i+1}$ for $1 \leq i \leq n$. First, suppose that $n \geq 4$ and $n \equiv 0 \pmod 4$. By

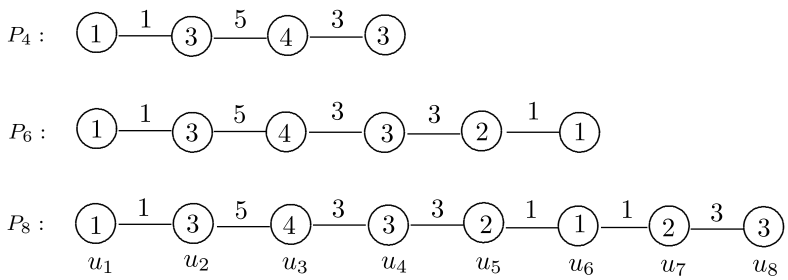

Figure 6.3: Proper mean colorings of P_n for $n = 4, 6, 8$

Proposition 6.1.2, it suffices to show that there exists a proper mean coloring c with $\mu(c) = 3$. Define the edge coloring c by

$$c(e) = \begin{cases} 1 & \text{if } e \text{ is incident with } u_i \text{ where } i \equiv 2 \pmod 4 \\ 3 & \text{if } e \text{ is incident with } u_i \text{ where } i \equiv 0 \pmod 4. \end{cases} \tag{6.1}$$

Then the vertex color $\text{cm}(u_i)$, $1 \leq i \leq n$, is given by

$$\text{cm}(u_i) = \begin{cases} 1 & \text{if } i \equiv 2 \pmod 4 \\ 2 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \equiv 0 \pmod 4. \end{cases}$$

Since the vertex coloring cm is proper, it follows that $\mu(c) = 3$ and so $\mu(C_n) = 3$ if $n \equiv 0 \pmod 4$.

Next, suppose that $n \not\equiv 0 \pmod 4$. First, we show that $\mu(C_n) \geq 4$. Assume, to the contrary, that there exists a proper mean coloring c of C_n with $\mu(c) = 3$. By Proposition 6.1.2, there is a vertex v such that $\text{cm}(v) = 1$. Hence, each edge incident with v of C_n is colored 1, which implies that $c(e)$ is odd for all $e \in E(C_n)$. First, suppose that there is an edge e such that $c(e) = 5$. Let e_1 and e_2 be the two edges adjacent to e. Since $c(e_1) \neq c(e_2)$ and $c(e_1)$ and $c(e_2)$ are both odd, at least one of $c(e_1)$ and $c(e_2)$ is 3 or more. However then, $\text{cm}(u) \geq 4$ for at least one vertex u incident with e, which is a contradiction. Hence, we may assume that each edge of C_n is colored 1 or 3. Since the resulting vertex coloring cm is proper, no edge is adjacent to two edges having the same color. Without loss of generality, we may conclude that the edges incident with u_i with $i \equiv 1 \pmod 4$ are colored 1 and the edges incident with u_i with $i \equiv 3 \pmod 4$ are colored 3. This, in turn, implies that $n \equiv 0 \pmod 4$, a contradiction.

It remains then to show that there is a proper mean coloring c of C_n with $\mu(c) = 4$. We consider three cases.

Case 1. $n \equiv 1 \pmod 4$. Then $n-1 \equiv 0 \pmod 4$ and so $\mu(C_{n-1}) = 3$. Let c_0 be the proper mean coloring of $C_{n-1} = (u_1, u_2, \ldots, u_{n-1}, u_1)$ defined in (6.1) with $\mu(c_0) = 3$. Now, let C_n be obtained from C_{n-1} by subdividing the edge $u_{n-1}u_1$ with the vertex u_n. We now extend the coloring c_0 to a proper mean coloring c of C_n by defining $c(u_{n-1}u_n) = 3$ and $c(u_nu_1) = 5$. This is illustrated in Figure 6.4 for $n = 5, 9$.

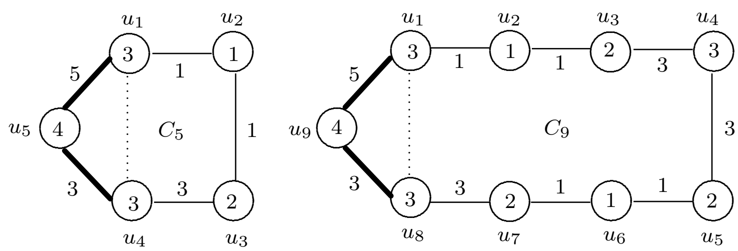

Figure 6.4: Proper mean colorings of C_5 and C_9

Case 2. $n \equiv 2 \pmod 4$. Then $n-1 \equiv 1 \pmod 4$. Let c_1 be the proper mean coloring of $C_{n-1} = (u_1, u_2, \ldots, u_{n-1}, u_1)$ defined in Case 1 and let C_n be obtained from C_{n-1} by subdividing the edge $u_{n-2}u_{n-3}$ with the vertex w. We now extend the coloring c_1 to a proper mean coloring c of C_n with $\mu(c) = 4$ by defining $c(u_{n-2}w) = 3$ and $c(wu_{n-3}) = 5$. This is illustrated in Figure 6.5 for $n = 10$.

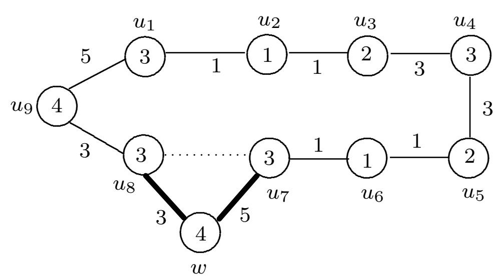

Figure 6.5: A proper mean coloring of C_{10}

Case 3*:* $n \equiv 3 \pmod 4$. Then $n-3 \equiv 0 \pmod 4$ and so $\mu(C_{n-3}) = 3$. Let c_0 be the proper mean coloring of $C_{n-3} = (u_1, u_2, \ldots, u_{n-3}, u_1)$ defined in (6.1) with $\mu(c_0) = 3$. Now, let C_n be obtained from C_{n-3} by replacing the edge u_1u_{n-3} (a 2-path) by the 5-path $(u_1, u_n, u_{n-1}, u_{n-2}, u_{n-3})$. We now extend the coloring c_0 to a proper mean coloring c of C_n with $\mu(c) = 4$ by defining $c(u_1u_n) = 3$, $c(u_nu_{n-1}) = 5$, $c(u_{n-1}u_{n-2}) = 1$ and $c(u_{n-2}u_{n-3}) = 3$. This is illustrated in Figure 6.6 for $n = 11$. ■

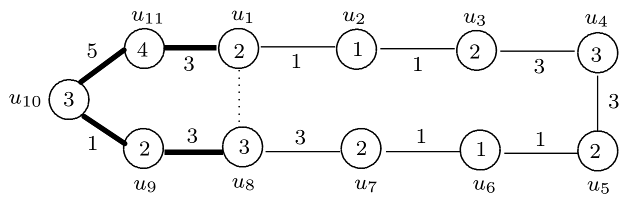

Figure 6.6: A proper mean coloring of C_{11}

Next, we determine the proper mean index of all complete bipartite graphs.

Theorem 6.2.4 *For positive integers s and t with $s + t \geq 3$,*

$$\mu(K_{s,t}) = \begin{cases} 3 & \textit{if } st \textit{ is even} \\ 4 & \textit{if } st \textit{ is odd.} \end{cases}$$

Proof. Let $G = K_{s,t}$ with partite sets $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. First, suppose that st is even. We may assume that s is even. Then $s = 2a$ for some positive integer a. By Proposition 6.1.2, it suffices to show that there is a proper mean coloring $c : E(G) \to \mathbb{N}$ such that $\mu(c) = 3$. For each $w \in W$, define

$$c(u_iw) = \begin{cases} 1 & \text{if } 1 \leq i \leq a \\ 3 & \text{if } a + 1 \leq i \leq 2a. \end{cases}$$

Then $\text{cm}(u_i) = 1$ for $1 \leq i \leq a$, $\text{cm}(u_i) = 3$ for $a + 1 \leq i \leq 2a$, and $\text{cm}(w) = 2$ for each $w \in W$. Since cm is a proper coloring of G, it follows that $\mu(c) = 3$. This implies that $\mu(G) = 3$ if st is even.

Next, suppose that st is odd. We may assume that $1 \leq s \leq t$. Then $s = 2a + 1$ and $t = 2b + 1$ for some integers a and b with $0 \leq a \leq b$ and $b \geq 1$. First, we show that there is a proper mean coloring c of G with $\mu(c) = 4$. If $a = 0$, then define

$$c(u_1w_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq b + 1 \\ 3 & \text{if } b + 2 \leq i \leq 2b \\ 4 & \text{if } i = 2b + 1. \end{cases}$$

Then $\mathrm{cm}(u_1) = 2$, $\mathrm{cm}(w_i) = 1$ for $1 \le i \le b+1$, $\mathrm{cm}(w_i) = 3$ for $b+2 \le i \le 2b$, and $\mathrm{cm}(w_{2b}) = 4$. If $a = 1$, then define

$$c(u_i w) = \begin{cases} 1 & \text{if } i = 1, 2 \\ 4 & \text{if } i = 3. \end{cases}$$

Then $\mathrm{cm}(u_1) = \mathrm{cm}(u_2) = 1$, $\mathrm{cm}(u_3) = 4$, and $\mathrm{cm}(w) = 2$ for each $w \in W$. If $a \ge 2$, then define

$$c(u_i w) = \begin{cases} 1 & \text{if } 1 \le i \le a+1 \\ 3 & \text{if } a+2 \le i \le 2a \\ 4 & \text{if } i = 2a+1. \end{cases}$$

Then $\mathrm{cm}(u_i) = 1$ for $1 \le i \le a+1$, $\mathrm{cm}(u_i) = 3$ for $a+2 \le i \le 2a$, $\mathrm{cm}(u_{2a+1}) = 4$, and $\mathrm{cm}(w) = 2$ for each $w \in W$. Therefore, $\mu(G) \le 4$.

It remains to show that $\mu(G) \ne 3$. Assume, to the contrary, that there is a proper mean coloring c of G with $\mu(G) = 3$. Thus, $\{\mathrm{cm}(v) : v \in V(G)\} = \{1, 2, 3\}$ by Observation 6.1.2. First, suppose that $s = 1$. Since $\mathrm{cm}(u_1) \ne 1$, it follows that $\mathrm{cm}(u_1) = 2$ or $\mathrm{cm}(u_1) = 3$.

- ⋆ First, suppose that $\mathrm{cm}(u_1) = 2$. Thus, $\mathrm{cm}(w) \in \{1, 3\}$ for each $w \in W$. Let x be the number of the vertices $w \in W$ such that $\mathrm{cm}(w) = 1$. Then there are $2b + 1 - x$ vertices $w \in W$ such that $\mathrm{cm}(w) = 3$. By Observation 6.1.1,

$$x \cdot 1 + (2b + 1 - x) \cdot 3 = 2(2b + 1).$$

However then, $2x = 2b + 1$, which is impossible.

- ⋆ Next, suppose that $\mathrm{cm}(u_1) = 3$. Thus, $\mathrm{cm}(w) \in \{1, 2\}$ for each $w \in W$. Let x be the number of the vertices $w \in W$ such that $\mathrm{cm}(w) = 1$. Then there are $2b + 1 - x$ vertices $w \in W$ such that $\mathrm{cm}(w) = 2$. By Observation 6.1.1,

$$x \cdot 1 + (2b + 1 - x) \cdot 2 = 3(2b + 1).$$

However then, $2b + 1 + x = 0$, which is impossible.

Next, suppose that $s \geq 3$. We may assume that $\mathrm{cm}(u_1) = 1$ (as the argument for $\mathrm{cm}(w_1) = 1$ is similar). Hence, there is $u \in U$ such that $\mathrm{cm}(u) \neq 1$; if this were not the case, then $c(e) = 1$ for every edge e of G and so $\mathrm{cm}(v) = 1$ for every vertex v of G. Since G is a complete bipartite graph, it follows that $\mathrm{cm}(u) \neq \mathrm{cm}(w)$ for every $u \in U$ and $w \in W$. Thus, either $\{\mathrm{cm}(u) : u \in U\} = \{1, 2\}$ or $\{\mathrm{cm}(u) : u \in U\} = \{1, 3\}$.

- ⋆ If $\{\mathrm{cm}(u) : u \in U\} = \{1, 2\}$, then $\mathrm{cm}(w) = 3$ for each $w \in W$. Let x be the number of the vertices $u \in U$ such that $\mathrm{cm}(u) = 1$. Then there are $2a+1-x$ vertices $u \in U$ such that $\mathrm{cm}(u) = 2$. By Observation 6.1.1,

$$x(2b+1) \cdot 1 + (2a+1-x)(2b+1) \cdot 2 = (2b+1)(2a+1) \cdot 3.$$

  However then, $2a + x + 1 = 0$, which is impossible.

- ⋆ If $\{\mathrm{cm}(u) : u \in U\} = \{1, 3\}$, then $\mathrm{cm}(w) = 2$ for each $w \in W$. Let x be the number of vertices $u \in U$ such that $\mathrm{cm}(u) = 1$. Then there are $2a + 1 - x$ vertices $u \in U$ such that $\mathrm{cm}(u) = 3$. By Observation 6.1.1,

$$x(2b+1) \cdot 1 + (2a+1-x)(2b+1) \cdot 3 = (2b+1)(2a+1) \cdot 2.$$

  However then, $2a + 1 = 2x$, which is impossible. ■

In each of the examples we've seen, the proper mean index of a graph has not exceeded its chromatic number by more than 2. This leads to the following conjecture.

Conjecture 6.2.5 *For every connected graph G of order 3 or more,*

$$\chi(G) \leq \mu(G) \leq \chi(G) + 2.$$

6.3 Trees

In the case of trees, Conjecture 6.2.5 states that $\mu(T) \leq 4$ for every tree T of order at least 3. We thus turn our attention to investigate this conjecture for various classes of trees. By Theorems 6.2.2 and 6.2.4, Conjecture 6.2.5 is true for paths and stars. We now show that if the edges of a nontrivial star are subdivided *in any manner*, then the proper mean index of the resulting tree is at most 4. In order to verify this fact, we first present a lemma.

Lemma 6.3.1 *If P_n is a path of order $n \geq 3$, then there is a proper mean coloring c of P_n such that $\mu(c) \leq 4$ and the chromatic mean of an end-vertex of P_n is* 3.

Proof. Let $G = P_n = (v_1, v_2, \ldots, v_n)$. We consider two cases based on the parity of n.

Case 1. *n is odd.* We handle the cases of $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$ separately.

Subcase 1.1. *$n \equiv 1 \pmod 4$.* We may assume that $n \geq 5$. Notice that $|E(G)| \equiv 0 \pmod 4$. Let a proper mean coloring $c : E(P_n) \to \mathbb{N}$ be given by the color sequence

$$\mathcal{S}_c(P_n) = (3, 1, 1, 3, 3, 1, 1, 3, \ldots, 3, 1, 1, 3).$$

The vertex coloring cm induced by c is given by the sequence

$$\mathcal{S}_{\rm cm}(P_n) = (3, 2, 1, 2, 3, 2, 1, 2, 3, \ldots, 2, 1, 2, 3).$$

It follows that $\mu(c) = 3$ and so $\text{cm}(v_1) = 3$.

Subcase 1.2. *$n \equiv 3 \pmod 4$.* We may assume that $n \geq 3$. Notice that $|E(G)| \equiv 2 \pmod 4$. Let a proper mean coloring $c : E(P_n) \to \mathbb{N}$ be given by the color sequence

$$\mathcal{S}_c(P_n) = (3, 1, 1, 3, 3, 1, 1, 3, 3, 1 \ldots, 1, 3, 3, 1).$$

The vertex coloring cm induced by c is given by the sequence

$$\mathcal{S}_{\rm cm}(P_n) = (3, 2, 1, 2, 3, 2, 1, 2, 3, 2, 1 \ldots, 2, 3, 2, 1).$$

It follows that $\mu(c) = 3$ and so $\text{cm}(v_1) = 3$.

Case 2. *n is even.* We handle the cases of $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$ separately.

Subcase 2.1. *$n \equiv 0 \pmod 4$.* We may assume that $n \geq 4$. Notice that $|E(G)| \equiv 3 \pmod 4$. Let a proper mean coloring $c : E(P_n) \to \mathbb{N}$ be given by the color sequence

$$\mathcal{S}_c(P_n) = (3, 5, 1, 1, 3, 3, 1, 1, 3, 3, 1 \ldots, 1, 3, 3, 1).$$

The vertex coloring cm induced by c is given by the sequence

$$\mathcal{S}_{\rm cm}(P_n) = (3, 4, 3, 1, 2, 3, 2, 1, \ldots, 2, 3, 2, 1).$$

It follows that $\mu(c) = 4$ and so $\text{cm}(v_1) = 3$.

Subcase 2.2. $n \equiv 2 \pmod 4$. We may assume that $n \geq 6$. Notice that $|E(G)| \equiv 1 \pmod 4$. Let a proper mean coloring $c : E(P_n) \to \mathbb{N}$ be given by the color sequence

$$\mathcal{S}_c(P_n) = (3, 5, 1, 1, 3, 3, 1, 1, 3, \ldots, 3, 1, 1, 3).$$

The vertex coloring cm induced by c is given by the sequence

$$\mathcal{S}_{\rm cm}(P_n) = (3, 4, 3, 1, 2, 3, 2, 1, 2, 3 \ldots, 2, 1, 2, 3).$$

It follows that $\mu(c) = 4$ and so $\text{cm}(v_1) = 3$. ∎

Theorem 6.3.2 *If T is a subdivided nontrivial star, then $\mu(T) \leq 4$.*

Proof. Let T be the tree obtained from the star $K_{1,t}$ by subdividing at least one edge of $K_{1,t}$. Since the proper mean index of every path of order 3 or more is at most 4, we may assume that $t \geq 3$. Suppose, in constructing the tree T, that r edges of $K_{1,t}$ are subdivided and s edges of $K_{1,t}$ are not subdivided, where then $r \geq 1$, $s \geq 0$, and $r + s = t$. We show that there is a proper mean coloring c of T with $\text{cm}(c) \leq 4$.

Let v be the central vertex of $K_{1,t}$, let $U = \{v_1, v_2, \ldots, v_r\}$ be the set of vertices adjacent to v with degree at least 2 in T, and let $W = \{w_1, w_2, \ldots, w_s\}$ be the set of end-vertices adjacent to v in T. We consider two cases.

Case 1. *s is even.* Then define a proper mean coloring $c : E(T) \to \mathbb{N}$ by $c(vw_i) = 2$ if i is even, $c(vw_i) = 4$ if i is odd, and color each subpath of of T starting at v with order at least 3 using the coloring defined in Lemma 6.3.1, with $c(vv_i) = 3$ for all i where $1 \leq i \leq r$. It follows that $\text{cm}(v) = 3$ and $\text{cm}(w_i) \in \{2, 4\}$, which implies that cm is a proper k-coloring of T where $k \leq 4$ and so $\mu(c) \leq 4$.

Case 2. *s is odd.* We consider the case of $s = 1$ separately.

Subcase 2.1. $s = 1$. Then define a proper mean coloring $c : E(T) \to \mathbb{N}$ as follows. First let $c(vw_1) = 1$. Consider a subpath starting at v with order at least 3, say the subpath containing v_1. Let $c(vv_1) = 5$. If the path has odd order, let $c(v_1v_2) = 3$. Then iteratively color the remaining edges of the path by alternating between the color sequences $(3,1)$ and $(1,3)$. If the path has even order, iteratively color the remaining edges starting with $e = v_1v_2$ by alternating between the color sequences $(3,1)$ and $(1,3)$. Color the edges of any remaining subpaths using the coloring described in Lemma 6.3.1 so that $c(vv_i) = 3$ for $2 \le i \le r$. Then $\text{cm}(v) = 3$, $\text{cm}(w_1) = 1$, $\text{cm}(v_1) = 4$, and each subpath starting at v is colored properly by cm. It follows that cm is a proper k-coloring of T where $k \le 4$, implying that $\mu(c) \le 4$.

Subcase 2.2. $s \ge 3$. Then define a proper mean coloring $c : E(T) \to \mathbb{N}$ by $c(vw_2) = 1$, $c(vw_i) = 4$ if i is odd, and $c(vw_i) = 2$ if i is even for $i \ge 4$, and color each subpath of of T starting at v with order at least 3 using the coloring defined in Lemma 6.3.1, with $c(vv_i) = 3$ for all i where $1 \le i \le r$. It follows that $\text{cm}(v) = 3$ and $\text{cm}(w_i) \in \{1,2,4\}$, which implies that cm is a proper k-coloring of T where $k \le 4$ and so $\mu(c) \le 4$. ■

By Theorems 6.2.2, 6.2.4, and 6.3.2, Conjecture 6.2.5 is true for all trees having at most one vertex of degree greater than 2. We now show that Conjecture 6.2.5 is true as well for trees all of whose non-leaves have sufficiently large degree. The *non-leaf minimum degree* $\delta^*(T)$ of a tree T of order 3 or more is the minimum degree among the non-leaves of T.

Lemma 6.3.3 *Let x be a vertex in a tree T such that* $\deg x \ge 10$.

(a) *There exists a coloring of the edges of T incident with x using colors from* $[4]$ *such that* $\text{cm}(x) = 2$, *where* (i) *exactly one edge incident with x is colored* 2 *or* (ii) *no edges incident with x are colored* 2.

(b) *There exists a coloring of the edges of T incident with x using colors from* $[4]$ *such that* $\text{cm}(x) = 3$, *where* (i) *exactly one edge incident with x is colored* 3 *or* (ii) *no edges incident with x are colored* 3.

Proof. We begin with (a). First, suppose that x has even degree. Then $\deg x = 10 + 2k$ where $k \ge 0$. For (i), we color $k+5$ edges incident with x by 1, one edge by 2, $k+3$ edges by 3, and one edge by 4. For (ii), we color $k+6$ edges incident with x by 1, $k+2$ edges by 3, and two edges by 4. If (i) occurs, then x is referred to as a *Type* 2.1 *vertex*; while if (ii) occurs, then x is referred to as a *Type* 2.2 *vertex*. It is convenient to represent these colorings of the edges incident with x as follows:

Type 2.1:

$k+5$	1	$k+3$	1
1	2	3	4

Type 2.2:

$k+6$	0	$k+2$	2
1	2	3	4

.

Now suppose that x has odd degree. Then $\deg x = 11 + 2k$ where $k \geq 0$. The following colorings of the edges incident with x have the desired properties (i) or (ii). Here, the vertex x is referred to as a *Type* 2.3 *vertex* if (i) occurs or as a *Type* 2.4 *vertex* if (ii) occurs.

Type 2.3:

$k+6$	1	$k+2$	2
1	2	3	4

Type 2.4:

$k+6$	0	$k+4$	1
1	2	3	4

.

Next, we verify (b). First, suppose that x has even degree. Then $\deg x = 10+2k$ where $k \geq 0$. The following colorings of the edges incident with x have the desired properties (i) or (ii) and the vertex x is referred to as a *Type* 3.1 *vertex* if (i) occurs or as a *Type* 3.2 *vertex* if (ii) occurs.

Type 3.1:

1	$k+3$	1	$k+5$
1	2	3	4

Type 3.2:

2	$k+2$	0	$k+6$
1	2	3	4

.

Next, suppose that x has odd degree. Then $\deg x = 11 + 2k$ where $k \geq 0$. The following colorings of the edges incident with x have the desired properties (i) or (ii). Similarly, the vertex x is called a *Type* 3.3 *vertex* if (i) occurs or a *Type* 3.4 *vertex* if (ii) occurs.

Type 3.3:

2	$k+2$	1	$k+6$
1	2	3	4

Type 3.4:

1	$k+4$	0	$k+6$
1	2	3	4

.

Therefore, (a) and (b) both hold. ■

Theorem 6.3.4 *If T is a tree with $\delta^*(T) \geq 10$, then $\mu(T) \leq 4$.*

Proof. By Theorem 6.2.4, the statement is true if T is a star. Hence, we may assume that T is not a star. Let v be a vertex of T that is not a leaf. Thus, $\deg v = d \geq 10$. Let T be a tree rooted at v, where $V_i = \{u \in V(T) : d(u,v) = i\}$ for $i = 0, 1, \ldots, e(v)$, where $e(v)$ is the eccentricity of v. Hence, $V_0 = \{v\}$, $V_1 =$

$N(v)$, and $V_i \neq \emptyset$ for $0 \leq i \leq e(v)$. Furthermore, for each vertex $x \in V_i$, where $1 \leq i \leq e(v)$, there is exactly one vertex $y \in V_{i-1}$ such that $xy \in E(T)$. Next, we construct a proper mean coloring c of T recursively such that $\mu(c) = 4$.

Let $V_1 = \{v_1, v_2, \ldots, v_d\}$. Since T is not a star, at least one vertex of V_1 has degree 10 or more. By Lemma 6.3.3, we can color the edges incident with v so that v is a Type 2.2 vertex if v has even degree or a Type 2.4 vertex if v has odd degree. Thus, $\text{cm}(v) = 2$ and no edge incident with v is colored 2. Hence, if $v_i \in V_1$, $1 \leq i \leq d$, is a leaf, then $\text{cm}(v_i) \neq 2$. On the other hand, one or more vertices in V_1 has degree 10 or more. Let $v_j \in V_1$, $1 \leq j \leq d$, such that $\deg v_j \geq 10$. Then $c(vv_j) \in \{1, 3, 4\}$. If $c(vv_j) \in \{1, 4\}$, then we color the edges incident with v_j so that v_j is a Type 3.2 vertex if $\deg v_j$ is even or color these edges so that v_j is a Type 3.4 vertex if $\deg v_j$ is odd. If $c(vv_j) = 3$, then we color the edges incident with v_j so that v_j is a Type 3.1 vertex if $\deg v_j$ is even or color these edges so that v_j is a Type 3.3 vertex if $\deg v_j$ is odd. In either case, $\text{cm}(v_j) = 3$ and for any leaf x (necessarily in V_2) adjacent to v_j, it follows that $\text{cm}(x) \neq 3$. We perform such a coloring for each vertex $v_j \in V_1$ of degree 10 or more such that $\text{cm}(v_j) = 3$ where no edge joining v_j and a vertex in V_2 is colored 3.

Next, suppose that y is a vertex in V_2 such that $\deg y \geq 10$. Let x be the vertex of V_1 such that $xy \in E(T)$. Then $c(xy) \in \{1, 2, 4\}$. If $c(xy) \in \{1, 4\}$, then we color the remaining $\deg y - 1$ edges incident with y so that y is a Type 2.2 vertex if $\deg y$ is even or color these edges so that y is a Type 2.4 vertex if $\deg y$ is odd. If $c(xy) = 2$, then we color the remaining $\deg y - 1$ edges incident with y so that v_j is a Type 2.1 vertex if $\deg y$ is even or color these edges so that y is a Type 2.3 vertex if $\deg y$ is odd. In either case, $\text{cm}(y) = 2$ and $\text{cm}(z) \neq 2$ for all leaves $z \in V_3$ adjacent to y. We perform such a coloring for each vertex $y \in V_2$ of degree 10 or more such that $\text{cm}(y) = 2$ where no edge joining y and a vertex in V_3 is colored 2.

Proceeding in this manner for each vertex x in V_i for $3 \leq i \leq e(v) - 1$ with $\deg x \geq 10$, we arrive at a proper mean coloring c of T with $\mu(c) = 4$. Therefore, $\mu(T) \leq 4$. ■

If the tree T being considered is a caterpillar (the removal of all leaves produces a path, called the *spine* of T), then a result similar to Theorem 6.3.4 can be obtained with a weaker hypothesis. Once again, we begin with a lemma.

Lemma 6.3.5 *Let x be a vertex in a caterpillar T such that* $\deg x \geq 6$.

(a) *There exists a coloring of the edges of T incident with x with colors from* $[4]$ *such that* $\mathrm{cm}(x) = 2$*, where* (i) *exactly one edge incident with x is colored* 2 *or* (ii) *exactly two edges incident with x are colored* 2*.*

(b) *There exists a coloring of the edges of T incident with x with colors from* $[4]$ *for which* $\mathrm{cm}(x) = 3$ *such that no edges incident with x are colored* 3*.*

Proof. We begin with (a). First, suppose that x has even degree. Then $\deg x = 6 + 2k$ where $k \geq 0$. For (i), we color $k+3$ edges incident with x by 1, one edge by 2, $k+1$ edges by 3, and one edge by 4. For (ii), we color $k+2$ edges incident with x by 1, two edges by 2, and $k+2$ edges by 3. If (i) occurs, then x is referred to as a *Type* 2*a vertex*; while if (ii) occurs, then x is referred to as a *Type* 2*b vertex*.

Type 2a:

$k+3$	1	$k+1$	1
1	2	3	4

Type 2b:

$k+2$	2	$k+2$	0
1	2	3	4

.

Next, suppose that x has odd degree. Then $\deg x = 7 + 2k$ where $k \geq 0$. The following colorings of the edges incident with x have the desired properties (i) or (ii). Here, the vertex x is referred to as a *Type* 2*c vertex* if (i) occurs or as a *Type* 2*d vertex* if (ii) occurs.

Type 2c:

$k+3$	1	$k+3$	0
1	2	3	4

Type 2d:

$k+3$	2	$k+1$	1
1	2	3	4

.

Next, we verify (b). If x has even degree, then $\deg x = 6+2k$ where $k \geq 0$. The following coloring of the edges incident with x (labeled Type 3a) has the desired properties and the vertex x is referred to as a *Type* 3*a vertex*. If x has odd degree, then $\deg x = 7 + 2k$ where $k \geq 0$. The following coloring of the edges incident with x (labeled Type 3b) has the desired properties and the vertex x is referred to as a *Type* 3*b vertex*.

Type 3a:

0	$k+3$	0	$k+3$
1	2	3	4

Type 3b:

1	$k+2$	0	$k+4$
1	2	3	4

.

Therefore, (a) and (b) hold. ■

Theorem 6.3.6 *If T is a caterpillar with $\delta^*(T) \geq 6$, then $\mu(T) \leq 4$.*

Proof. Let $(v_1, v_2, \ldots, v_d)$ be the spine of T. Since the statement is true if T is a star, we may assume that $d \geq 2$. With the aid of Lemma 6.3.5, we construct a proper mean coloring c of T such that $\mu(c) = 4$.

First, we color the edges incident with v_1 so that v_1 is a Type 2a vertex if v_1 has even degree or a Type 2c vertex if v_1 has odd degree where v_1v_2 is colored 2. Thus, $\text{cm}(v_1) = 2$ and no leaf incident with v_1 is colored 2. Next, we color the remaining $\deg v_2 - 1$ edges incident with v_2 so that v_2 is a Type 3a vertex if $\deg v_2$ is even or color these edges so that v_2 is a Type 3b vertex if $\deg v_2$ is odd. If $d \geq 3$, then v_2v_3 is colored 2. Thus, $\text{cm}(v_2) = 3$ and no leaf incident with v_2 is colored 3.

We now proceed to v_3 if $d \geq 3$. First, suppose that $d = 3$. Since $c(v_2v_3) = 2$ and v_3 is adjacent to $\deg v_3 - 1$ leaves, we color the edges incident with v_3 so that v_3 is a Type 2a vertex if v_3 has even degree or a Type 2c vertex if v_3 has odd degree. Thus, $\text{cm}(v_3) = 2$ and no leaf incident with v_3 is colored 2. Next, suppose that $d \geq 4$. We color the remaining $\deg v_3 - 1$ edges incident with v_3 so that v_3 is a Type 2b vertex if v_3 has even degree or a Type 2d vertex if v_3 has odd degree. If $d \geq 4$, then v_3v_4 is colored 2. Thus, $\text{cm}(v_3) = 2$ and no leaf incident with v_3 is colored 2.

We now proceed to v_4 if $d \geq 4$. Since $c(v_3v_4) = 2$ and $\text{cm}(v_3) = 2$, we color the remaining $\deg v_4 - 1$ edges incident with v_4 so that v_2 is a Type 3a vertex if $\deg v_2$ is even or color these edges so that v_2 is a Type 3b vertex if $\deg v_2$ is odd so that $\text{cm}(v_4) = 3$ and no leaf incident with v_4 is colored 3. Furthermore, if $d \geq 5$, we color the edge v_4v_5 by 2.

In general, if i is odd and $5 \leq i \leq d$, then we color the remaining $\deg v_i - 1$ edges incident v_i in the same manner as the coloring of the edges incident with v_3; while if i is even and $6 \leq i \leq d$, then we color the remaining $\deg v_i - 1$ edges incident v_i in the same manner as the coloring of the edges incident with v_4. Proceeding in this manner, we arrive at a proper mean coloring c of T with $\mu(c) = 4$. Therefore, $\mu(T) \leq 4$. ■

If the caterpillar T being considered has small diameter, then it can be shown that $\mu(T) \leq 4$ regardless of the non-leaf minimum degree of T.

Theorem 6.3.7 *If T is a caterpillar of diameter 4, then $\mu(T) \leq 4$.*

Proof. Let T be a caterpillar of diameter 4 whose spine is (u, v, w). We may assume that $2 \le \deg u \le \deg w$. We consider two cases, according to the parities of the degrees of u and w.

Case 1. *Both* $\deg u$ *and* $\deg w$ *are odd.* First, suppose that T is one of the trees T' and T'' shown in Figure 6.7. Since each of T' and T'' has a proper mean coloring with proper mean index 3 (as shown in Figure 6.7), it follows that $\mu(T) = 3$ if $T \in \{T', T''\}$. Next, suppose that $T \notin \{T', T''\}$. Then T contains either T' or T'' as a subtree. We show that the proper mean coloring of T' or of T'' in Figure 6.7 can be extended to a proper mean coloring c of T such that $\mu(c) = 3$.

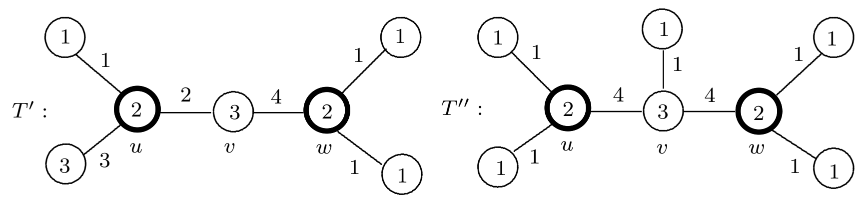

Figure 6.7: Proper mean colorings of T' and T''

- ⋆ If $\deg v \ge 4$ is even, say $\deg v = 2 + 2k$ for some positive integer k, then we begin with the coloring of T' and color each of the additional k pairs of pendant edges at v by 2 and 4. If $\deg v \ge 5$ is odd, say $\deg v = 3 + 2k$ for some positive integer k, then we begin with the coloring of T'' and color each of the additional k pairs of pendant edges at v by 2 and 4.

- ⋆ If $\deg u \ge 5$ or $\deg w \ge 5$, say $\deg u = 3 + 2\ell$ for some positive integer ℓ, then we begin with the coloring of T' (if $\deg v$ is even) or the coloring of T'' (if $\deg v$ is odd) and color each of the additional ℓ pairs of pendant edges at u by 1 and 3.

Since the resulting coloring c of T is a proper mean coloring with $\mu(c) = 3$, it follows that $\mu(T) = 3$ if both $\deg u$ and $\deg w$ are odd.

Case 2. *At least one of* $\deg u$ *and* $\deg w$ *is even, say* $\deg u$ *is even.* There are two subcases, according to whether $\deg u = 2$ or $\deg u \ge 4$.

Subcase 2.1. $\deg u = 2$. First, suppose that T is one of the seven caterpillars T_1, T_2, …, T_7 of diameter 4 shown in Figure 6.8. Since each of these seven caterpillars has a proper mean coloring with proper mean index at most 4 (as shown in Figure 6.8), it follows that $\mu(T_i) \le 4$ for $1 \le i \le 7$.

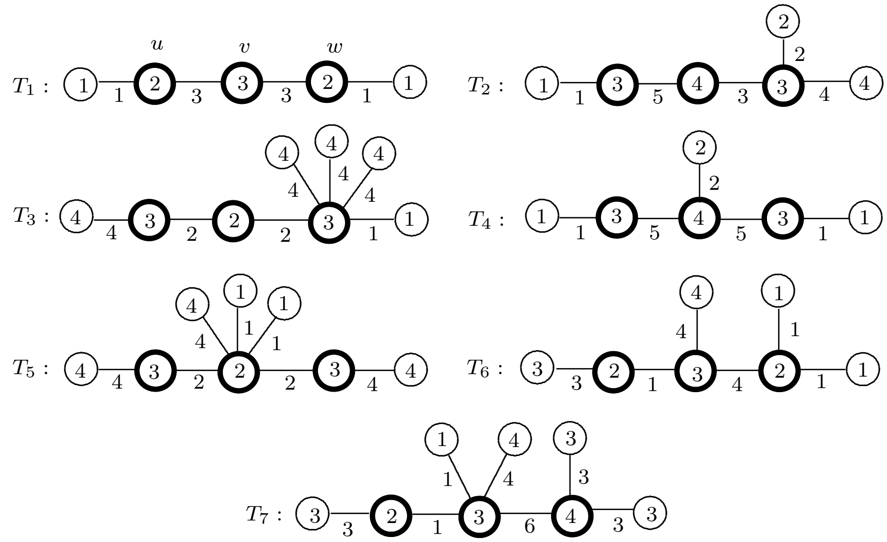

Figure 6.8: Proper mean colorings of T_i for $1 \le i \le 7$

Next, suppose that $T \neq T_i$ for $1 \le i \le 7$. Then T contains T_i as a subtree for some $i \in [7]$. We show that the proper mean coloring c_i of T_i in Figure 6.8 can be extended to a proper mean coloring c of T such that $\mu(c) = \mu(c_i)$.

- ⋆ Suppose that $\deg v$ and $\deg w$ are both even. Then $\deg v = 2 + 2k$ and $\deg w = 2 + 2\ell$ for some nonnegative integers k and ℓ. Since $T \neq T_1$, it follows that $\max\{k, \ell\} \ge 1$. Beginning with the coloring of T_1, we color each of the additional k pairs of pendant edges at v (not in T_1) by 2 and 4 and color each of the additional ℓ pairs of pendant edges at w (not in T_1) by 1 and 3.

- ⋆ Suppose that $\deg v$ and $\deg w$ are both odd. Then $\deg v = 3 + 2k$ and $\deg w = 3 + 2\ell$ for some nonnegative integers k and ℓ with $\max\{k, \ell\} \ge 1$ (since $T \neq T_6$). Beginning with the coloring of T_6, we color each of the additional k pairs of pendant edges at v (not in T_6) by 2 and 4 and color each of the additional ℓ pairs of pendant edges at w (not in T_6) by 1 and 3.

- ⋆ Suppose that $\deg v$ is even and $\deg w$ is odd. Then $\deg v = 2 + 2k$ and $\deg w = 3 + 2\ell$, where $k \ge 1$ and $\ell \ge 0$. If $\ell = 0$, then we may assume that $k \ge 2$ (since $T \neq T_2, T_7$). Beginning with the coloring of T_7, we color each of the additional $k - 1$ pairs of pendant edges at v (not in T_7) by 2 and 4. If $\ell \ge 1$, then we begin with the coloring of T_3, color each of the additional

k pairs of pendant edges at v (not in T_3) by 1 and 3, and color each of the additional $\ell - 1$ pairs of pendant edges at w (not in T_3) by 2 and 4.

- ⋆ Suppose that $\deg v$ is odd and $\deg w$ is even. Then $\deg v = 3 + 2k$ and $\deg w = 2 + 2\ell$ where $k, \ell \geq 0$. If $k = 0$, then we may assume that $\ell \geq 1$ (since $T \neq T_4$). Beginning with the coloring of T_4, we color each of the additional ℓ pairs of pendant edges at w (not in T_4) by 2 and 4. If $k \geq 1$, then we begin with the coloring of T_5, color each of the additional $k-1$ pairs of pendant edges at v (not in T_5) by 1 and 3, and color each of the additional ℓ pairs of pendant edges at w (not in T_5) by 2 and 4.

In each situation, the resulting coloring c is a proper mean coloring of T with $\mu(c) \leq 4$.

Subcase 1.2. $\deg u \geq 4$ *is even.* Let $\deg u = 2 + 2p$ for some positive integer p. Then T is obtained from a caterpillar T_0 of diameter 4 of Subcase 2.1 by adding $2p$ pendant edges at u. We begin with the coloring c_0 of T_0 as described in Subcase 2.1. Then $\text{cm}_{c_0}(u) \in \{2, 3\}$. If $\text{cm}_{c_0}(u) = 2$, then we color each of the additional p pairs of the pendant edges at u (not in T_0) by 1 and 3; while if $\text{cm}_{c_0}(u) = 3$, then we color each of the additional p pairs of the pendant edges at u (not in T_0) by 2 and 4. In each case, the resulting coloring c is a proper mean coloring of T with $\text{cm}(c) = \text{cm}(c_0) \leq 4$. ■

For caterpillar of diameter 3 (that is double stars), the proper mean index has been determined exactly.

Theorem 6.3.8 *If a and b are integers with $2 \leq a \leq b$, then*

$$\mu(S_{a,b}) = \begin{cases} 3 & \text{if } a \neq b \\ 4 & \text{if } a = b. \end{cases}$$

Proof. Let $G = S_{a,b}$ where $2 \leq a \leq b$. Suppose that u and v are the central vertices of G with $\deg u = a$ and $\deg v = b$ where u is adjacent to the $a-1$ end-vertices $u_1, u_2, \ldots, u_{a-1}$ and v is adjacent to the $b-1$ end-vertices $v_1, v_2, \ldots, v_{b-1}$. First assume $a \neq b$. It suffices to show that there exists a proper mean coloring c of $S_{a,b}$ with $\mu(c) = 3$. Since $a - 1 < b - 1$, there exist integers q and r such that $b - 1 = q(a-1) + r$ where $0 \leq r < b - 1$ and $q \geq 1$. We consider two cases.

Case 1. *q is odd.* We handle the case of $q = 1$ separately.

Subcase 1.1. $q = 1$. Since $a \neq b$ it follows that $r > 0$. Partition the end-vertices adjacent to u into sets $U_1 = \{u_1, u_2, \ldots, u_r\}$ and $U_2 = \{u_{r+1}, u_{r+2}, \ldots, u_{a-1}\}$ and partition the end-vertices adjacent to v into sets $V_1 = \{v_1, v_2, \ldots, v_r\}$, $V_2 = \{v_{r+1}, v_{r+2}, \ldots, v_{a-1}\}$, and $V_3 = \{v_a, \ldots, v_{b-1}\}$. Since $r > 0$, U_1 is nonempty. Define $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} 1 & \text{if } e = uw \text{ for } w \in U_1 - \{u_1\} \text{ and } w \in V_i, \text{ for } i \in [3] \\ 2 & \text{if } e = uw \text{ for } w \in U_2 \text{ and } w = u_1 \\ a + r + 1 & \text{if } e = uv. \end{cases}$$

Then $\text{cm}(u) = 3$, $\text{cm}(v) = 2$, $\text{cm}(u_i) \in \{1, 2\}$ for $1 \leq i \leq a - 1$, and $\text{cm}(v_i) = 1$ for $1 \leq i \leq b - 1$. Since cm is a proper coloring of G, it follows that $\mu(c) = 3$.

Subcase 1.2. $q \geq 3$. Let $q = 2k + 1$. Partition the end-vertices adjacent to u into sets $U_1 = \{u_1, u_2, \ldots, u_r\}$ and $U_2 = \{u_{r+1}, u_{r+2}, \cdots, u_{a-1}\}$ and partition the end-vertices adjacent to v into q sets $V_1, V_2, \ldots V_q$ each containing $a-1$ end-vertices and W containing r end-vertices. Let $x \in V_2$ and let $y \in U_2$. Define $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} 1 & \text{if } e = uw \text{ for } w \in \{x, y\},\ w \in U_1,\ w \in W, \text{ and } w \in V_i \text{ for odd } i \\ 2 & \text{if } e = uw \text{ for } w \in U_2 \text{ with } w \neq y \\ 3 & \text{if } e = uw \text{ for } w \in V_i \text{ for even } i \text{ with } w \neq x \\ a + r + 3 & \text{if } e = uv. \end{cases}$$

Then $\text{cm}(u) = 3$, $\text{cm}(v) = 2$, $\text{cm}(u_i) \in \{1, 2\}$ for $1 \leq i \leq a-1$, and $\text{cm}(v_i) \in \{1, 3\}$ for $1 \leq i \leq b - 1$. Since cm is a proper coloring of G, it follows that $\mu(c) = 3$.

Case 2. *q is even.* Let $q = 2k$ with $k \geq 1$. Partition the end-vertices adjacent to u into sets $U_1 = \{u_1, u_2, \ldots, u_r\}$ and $U_2 = \{u_{r+1}, u_{r+2}, \cdots, u_{a-1}\}$ and partition the end-vertices adjacent to v into q sets $V_1, V_2, \ldots V_q$ each containing $a - 1$ end vertices and W containing r end-vertices. Let $x \in U_2$. Define $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} 1 & \text{if } e = uw \text{ for } w \in U_2 - \{x\} \text{ or } w \in V_i \text{ for } i \in \{1, 2\} \text{ or odd } i \text{ with } i \geq 3 \\ 2 & \text{if } e = uw \text{ for } w = x \text{ or } w \in U_1 \\ 3 & \text{if } e = uw \text{ for } w \in W \text{ or } w \in V_i \text{ for even } i \text{ with } i \geq 2 \\ 2a - r & \text{if } e = uv. \end{cases}$$

Then $\text{cm}(u) = 3$, $\text{cm}(v) = 2$, $\text{cm}(u_i) \in \{1, 2\}$ for $1 \le i \le a-1$, and $\text{cm}(v_i) \in \{1, 3\}$ for $1 \le i \le b-1$. Since cm is a proper coloring of G, it follows that $\mu(c) = 3$.

Next, assume that $a = b$. First, we show $\mu(G) \ge 4$. Assume, to the contrary, that there exists a proper mean coloring $c : V(G) \to \mathbb{N}$ of G with $\mu(c) = 3$. Since $\text{cm}(u) \ge 2$ and $\text{cm}(v) \ge 2$, we may assume that $\text{cm}(u) = 3$ and $\text{cm}(v) = 2$. Let $A = \sum_{i=1}^{a-1} c(u_i)$ and $B = \sum_{i=1}^{a-1} c(v_i)$. Since cm is proper, $c(u_i) \le 2$ for all i with $1 \le i \le a-1$. Note that $c(v_i) \ge 1$ for all i with $1 \le i \le a-1$. Observe that

$$3a = A + c(uv) \le 2(a-1) + c(uv)$$

and

$$2a = B + c(uv) \ge (a-1) + c(uv).$$

It follows that $a + 2 \le c(uv) \le a + 1$, which is impossible.

Next we exhibit a proper mean coloring c of G with $\mu(c) = 4$. We consider two cases.

Case 1. *a is even.* Then $b = a = 2k$ for some positive integer k. We may assume that $k \ge 2$. Define $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} 1 & \text{if } e = uu_1 \text{ or } e = vv_i \text{ for } 1 \le i \le k \\ 2 & \text{if } e = uu_i \text{ for } 2 \le i \le k-1 \text{ when } k \ge 3 \\ 3 & \text{if } e = uv \text{ or } e = vv_i \text{ for } k+1 \le i \le 2k-1 \\ 4 & \text{if } e = uu_i \text{ for } k \le i \le 2k-1. \end{cases}$$

Then $\text{cm}(u) = 3$, $\text{cm}(v) = 2$, $\text{cm}(u_i) \in \{1, 2, 4\}$ for $1 \le i \le a-1$, and $\text{cm}(v_i) \in \{1, 3\}$ for $1 \le i \le a-1$. Since cm is a proper coloring of G, it follows that $\mu(c) = 4$.

Case 2. *a is odd.* Then $b = a = 2k+1$ for some positive integer k. We may assume that $k \ge 1$. Define $c : E(G) \to \mathbb{N}$ by

$$c(e) = \begin{cases} 1 & \text{if } e = uv \text{ or } e = vv_i \text{ for } 2 \le i \le k+1 \\ 2 & \text{if } e = uu_i \text{ for } k+2 \le i \le 2k \text{ when } k \ge 2 \\ 3 & \text{if } e = vv_i \text{ for } k+2 \le i \le 2k \text{ when } k \ge 2 \\ 4 & \text{if } e = vv_1 \text{ or } e = uu_i \text{ for } 1 \le i \le k+1. \end{cases}$$

Then $\text{cm}(u) = 3$, $\text{cm}(v) = 2$, $\text{cm}(u_i) \in \{2, 4\}$ for $1 \le i \le a-1$, and $\text{cm}(v_i) \in \{1, 3, 4\}$ for $1 \le i \le a-1$. Since cm is a proper coloring of G, it follows that $\mu(c) = 4$. ■

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