Saudi Elementary Mathematics Teachers’ Knowledge for Teaching Fractions

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SAUDI ELEMENTARY MATHEMATICS TEACHERS’ KNOWLEDGE FOR TEACHING FRACTIONS

by

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A dissertation submitted to the Graduate College in partial fulfillment of the requirements for the degree of Doctor of Philosophy
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SAUDI ELEMENTARY MATHEMATICS TEACHERS’ KNOWLEDGE FOR TEACHING FRACTIONS

Mona Khalifah A Aladil, Ph.D.

Western Michigan University, 2020

Recent reform efforts in Saudi Arabia attend to mathematics instruction with a great deal of emphasis to improve Saudi mathematics education. Studies in different countries have confirmed that teachers’ mathematical knowledge for teaching plays an important role in mathematical quality of instruction and students’ achievement (e.g., Ball, 1990; Baumert et al., 2010; Hill, Rowan, & Ball, 2005). Yet few studies about mathematics teachers’ knowledge for teaching have been conducted in the Saudi context. This study investigates Saudi elementary mathematics teachers’ knowledge for teaching in the content strand of rational numbers with an emphasis on fractions, which is an important step toward the educational reform in Saudi Arabia.

The data were first collected from 44 female in-service elementary mathematics teachers in Alahsa City. They responded to one mathematical knowledge for teaching (MKT) measure on rational numbers and a survey on teachers’ confidence in teaching fractions and beliefs about issues related to teaching and learning mathematics. Then, 12 of the teachers were selected for interviews with tasks on multiplication and division with fractions, including solving problems, generating story problems and representations, and interpreting students’ work.

The results revealed that the 44 teachers were on or below the average IRT score of the MKT measure. Also, they had high confidence in their knowledge preparation for teaching specific topics regarding fractions, and had strong beliefs that coincided with reform efforts in
school mathematics as well as beliefs that kept a traditional view of school mathematics. The 12 teachers were able to solve the problems involving multiplication and division with fractions by using the standard procedures, but overall much of their reasoning could not be considered as conceptually deep. The 12 teachers’ beliefs on teaching mathematics were not supported by their content knowledge for teaching. For example, although the 12 teachers believed representations are important in mathematics teaching and learning, many of them were not able to use representations effectively to solve multiplication and/or division with fractions.

Overall, the higher the teachers’ MKT score was, the better their performance on the interview tasks was. However, comparing the teachers’ MKT scores and interview responses within and across teachers revealed that the interview data provided more information about the teachers’ content knowledge for teaching mathematics. Although none of the teachers scored higher than an IRT score of 0.18027 (close to the mean of the norm group), some of them were good at solving the interview tasks procedurally and conceptually. Whereas some teachers with the same MKT score performed quite differently on the interview tasks, some teachers with different MKT scores had a similar performance on the tasks. These findings led to a discussion of teachers’ use of alternative strategies and representations to solve problems, and interpretation of student work. Finally, one potential factor that was notable and might have influenced the teachers’ performance on the interview tasks was the type of professional development programs they attended. This study provides several implications for teacher education, professional development, and future research in Saudi Arabia.
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CHAPTER 1

INTRODUCTION

The Purpose of the Study

This study investigates Saudi elementary mathematics teachers’ knowledge for teaching in the content strand of rational numbers with an emphasis on fractions. In Saudi Arabia, upper-level (grades 4-6) mathematics is taught by mathematics teachers.¹ The purpose of the study is to assess and examine Saudi upper-level elementary teachers’ knowledge for teaching mathematics. This study can contribute to improvement in mathematics education in Saudi Arabia by investigating teachers’ knowledge, because teacher knowledge is closely related to instructional quality and students’ learning outcomes.

Research Questions

The study has the following specific research questions:

1. What is the level of Saudi elementary mathematics teachers’ knowledge for teaching, assessed through a mathematical knowledge for teaching (MKT) measure on rational numbers?

2. How confident are Saudi elementary mathematics teachers about their mathematical knowledge for teaching in general, and fractions specifically? What are their beliefs about mathematics teaching and learning?

¹ Mathematics teachers in Saudi Arabia have at least a bachelor’s degree in mathematics or mathematics education and have general training that allows them to teach any grade levels—elementary, middle, or secondary schools.
3. How do Saudi elementary mathematics teachers reason about tasks on multiplication and division with fractions?

4. What is the relationship among Saudi elementary mathematics teachers’ confidence and beliefs, mathematical knowledge for teaching, and reasoning about multiplication and division with fractions?

**Context of the Study: Education Reform in Saudi Arabia**

The educational system in Saudi Arabia has received significant attention from the government. In the period from 1980 to 2005, the Kingdom of Saudi Arabia was among the top 10 countries in the world spending the highest percentage of GDP on public education (Maroun, Samman, Moujaes, & Abouchakra, 2008). In 2008, about 19.26% of the total Saudi government budget (UNESCO-IBE, 2011) was spent on educational projects. In 2012, more than 25% of the total budget was spent on education and vocational training. Although this reflects generous funding and educational reform efforts, Saudi students’ levels of achievement have been below average in international educational comparative studies. Trends in International Mathematics and Science Study (TIMSS), in 2003, 2007, 2011, and 2015, shows that Saudi students’ mathematics scores were one of the lowest among all participating countries. For example, TIMSS results in 2015 show that the average mathematics score for eighth-grade students in Saudi Arabia was 368, and for fourth-grade students, 383 (Mullis, Martin, Goh, & Cotter, 2016). Both were significantly lower than the average score of 500 from all participating countries. Saudi students’ performance on international examinations, such as TIMSS and PISA, became a concern for educational policymakers and national development experts in the country (Wiseman, Sadaawi, & Alromi, 2008). This sparked efforts toward educational reform in Saudi Arabia.
Efforts to Improve the Saudi Educational System

To improve the quality of the Saudi Arabia educational system, the Ministry of Education (MoE) started developing new curriculum resources (i.e., textbooks and teacher resources) for all grade levels. In 2003, the MoE intended to revise the school curriculum resources by implementing the General Project for Curriculum Development. However, this project did not result in a significant improvement in mathematics and science. Therefore, the MoE implemented a specific project for mathematics and science, known as the Development of the Mathematics and Science Curricula Project, which designed mathematics and science curricula in collaboration with experts from other countries. Alanazi (2016) translated into English some aims of this project:

1. To challenge the dominance of indoctrination and the lack of attention in developing cognitive and scientific skills needed by students, the most important of which are analysis, criticism, reasoning, problem-solving, decision-making, and understanding other perspectives.

2. To address poor educational outcomes in science and mathematics compared with many growing and developed countries.

3. To improve the teaching and learning environment in schools.

4. To enhance the professional qualifications of mathematics and science teachers.

(p. 29)

This project introduced a series of compulsory mathematics textbooks for elementary grade levels. However, the MoE found that developing curricula is not enough to improve the educational system. There was a need for a project focusing on all aspects related to improving the educational system, such as school improvement and teacher training. Therefore, the
government invested further in improving the educational system by establishing the King Abdullah Bin Abdul-Aziz Public Education Development Project (the Tatweer project) in 2007. This project aimed at “furthering teacher skills, developing curriculum, enhancing extracurricular activities, and improving school environments” (Alghamdi, 2018, p. 5). Alanazi (2016) translated some aims of the Tatweer project (2015):

1. To help those working in education to engage in continuous professional development;

2. To develop curricula and learning materials;

3. To improve the school environment to enhance learning;

4. To employ information technology (IT) to improve learning; and

5. To provide extra-curricular activities and student services. (p. 33)

Within this project, different sub-projects were implemented, such as the Program for School Development and Teacher Training Project (Tatweer, 2015). This project focused on improving teacher quality more than teaching quality (Alghamdi, 2018). In 2013, the Public Education Evaluation Commission was established and revealed that the Saudi educational system witnessed little relative improvement (Tayan, 2017).

Although these were all reform efforts, little improvement was noticed in the Saudi educational system. Saudi students’ mathematics performance in TIMSS in 2015 in the fourth and eighth grades regressed from the students’ performance in TIMSS in 2011. In 2016, the Saudi Vision 2030 was implemented. This vision is an ambitious and comprehensive reform movement in Saudi Arabia. This vision puts more emphasis on the importance of the Saudi educational system to improve students’ learning achievements. Therefore, many reform efforts are currently under development and implementation.
New Mathematics Textbooks

The Development of the Mathematics and Science Curricula Project introduced a series of compulsory mathematics textbooks for grades 1-6 that were translated and adapted from the *Math Connects* textbook series. According to the publisher, this series was developed based on the National Council of Teachers of Mathematics *Curriculum Focal Points* and was published by Macmillan/McGraw-Hill Education in the United States (McGraw-Hill, 2012).

According to the publisher, the new mathematical textbooks provide an opportunity for students to develop the five aspects critical for mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition toward mathematics (National Research Council, 2001). Also, these textbooks aim to assist teachers in encountering the challenges of helping students become mathematically proficient. For example, the textbook guides teachers to solve a problem by using different methods that enable them to open a discussion about alternative strategies to solving the problem (McGraw-Hill, 2012).

The lessons in these textbooks consist of many sections (see Appendix A). Each lesson starts with the main idea box, which summarizes the mathematical goal of the lesson. In the first section, “Get Ready,” students are required to do some thinking as a warm-up. The following section, “Real World Example,” includes two worked examples to introduce the idea of the lesson. In the next section, “Check What You Know,” students do some exercises under teacher supervision, followed by the section “Practice and Problem Solving,” in which students solve some exercises and problems from real life, either individually or in groups. The lesson ends with the “High Thinking Skills” section, in which some high-level problems challenge the
students (Al Dalan, 2015). Despite the publisher’s claims, it seems that the textbook series is largely based on a typical example-practice format of lessons.

To summarize, in Saudi Arabia significant reform efforts have been undertaken, including teacher training and textbook adoption. In particular, the recent reform effort, Saudi Vision 2030, is more comprehensive and attends to mathematics with a great deal of emphasis. In this context, investigating Saudi teachers’ knowledge for teaching mathematics is critical to the reform efforts to improve Saudi students’ mathematics learning and achievement.

**Rationale for Investigating Teacher Knowledge**

The results of international comparison studies and recent reform movements suggest additional work is needed to further improve mathematics education in Saudi Arabia. Research studies in different countries have investigated various factors that influence students’ achievements, such as students’ motivational characteristics, teachers’ mathematical knowledge, teaching quality, teaching practice, and curriculum resources. Many research studies incorporating various conceptualizations of teachers’ mathematical knowledge, instruments, and research methods confirm that teachers’ mathematical knowledge for teaching plays an important role in mathematical quality of instruction and students’ achievement (Ball, 1990; Baumert et al., 2010; Hill, Ball & Schilling, 2008; Silverman & Thompson, 2008; Tchoshanov, 2010).

Many researchers have provided evidence linking teachers’ mathematical knowledge for teaching to teaching practices. The findings have shown that teachers with strong mathematical knowledge for teaching were able to exhibit good mathematical quality of instruction (e.g., Ball, Thames, & Phelps, 2008; Charalambous, 2010; Hill, Ball, & Schilling, 2008; Tchoshanov, 2010). Other studies have identified an association between teachers’ mathematical knowledge and student outcomes in mathematics (e.g., Baumert et al., 2010; Hill, Charalambous, & Chin,
They concluded that students performed better when their teachers have a high level of mathematical knowledge for teaching.

In addition, some researchers have emphasized the importance of investigating the affective factors toward mathematics and teaching mathematics (e.g., Beswick, Callingham, & Watson, 2012; Beswick, Watson, & Brown, 2006; Ernest, 1989). These different studies suggest that teachers’ knowledge and teachers’ confidence and beliefs are closely related, affecting teaching practice and students’ learning outcomes. Therefore, affective factors, especially confidence and beliefs, are an important domain that should be taken into consideration in investigating teacher knowledge.

To summarize, teachers’ mathematical knowledge for teaching and teachers’ affective factors, specifically beliefs and confidence, play a critical role in improving teaching practices and students’ learning outcomes (see Figure 1.1). The relationship of teacher knowledge along with teacher confidence and beliefs, teaching practice, and student learning outcomes guided the design of this study. This study seeks to understand Saudi elementary mathematics teachers’ knowledge for teaching, given the context of the Saudi educational system. Examining teachers’ knowledge, along with their confidence in teaching mathematics and beliefs about mathematics teaching and learning, is an important step toward improving the educational system in the country.
Figure 1.1. Relationship among teacher affective factors, teacher knowledge, teaching practices, and students’ learning outcomes.

The Significance of the Study

One of the educational aims of all countries around the world, including Saudi Arabia, is improving students’ learning outcomes. To achieve this goal, many projects have been established to improve the educational system in different countries. Also, researchers around the world have been investigating factors that contribute to improving students’ learning outcomes. Yet few studies have been conducted in the Saudi context. One study by Dodeen, Abdelfattah, Shumrani, and Hilal (2012) emphasized the importance of improving teaching practices in Saudi schools to improve students’ achievement. This result coincides with the findings of many other international studies. For example, Hiebert and Grouws (2007) argued that improving student learning requires paying great attention to improving instructional quality.

Researchers have agreed that teachers play a key role in improving students’ learning through their teaching practices. Therefore, numerous scholars began to study teachers’ characteristics and knowledge in order to support teachers to improve their teaching practices and eventually students’ achievement. Teachers’ knowledge is one of the significant research lines that emerged after Shulman’s studies in the mid 1980s. According to Blömeke and Delaney (2012), about 50 scholars from a broad range of countries developed approaches to measuring
in-service or preservice teachers’ mathematical knowledge. They found more studies focused on preservice teachers than in-service teachers, and more elementary than secondary teachers.

Al Dalan (2015) indicated that investigating teacher knowledge is an ongoing research line in different countries, but it is less common in Arab countries, specifically in Saudi Arabia. Few studies have assessed Saudi mathematics in-service teachers’ knowledge for teaching.

Moreover, fewer researchers have investigated Saudi elementary teachers’ mathematical knowledge of fractions. Many international studies have emphasized that academic success in elementary grade levels is a strong predictor of the success in later grades (e.g., Nguyen et al., 2016). Also, researchers have concluded that fractions are the basis for proportional reasoning and many other advanced mathematical topics, such as algebra. Therefore, this dissertation will focus on examining Saudi in-service elementary mathematics teachers’ knowledge for teaching, especially the topic of fractions.

In addition, the studies on teacher knowledge of fractions in Saudi Arabia employed mainly quantitative methods (Alghazo & Alghazo, 2017; Khashan, 2014). In this research study, I used three different types of instruments to answer the research questions: an MKT measure, a survey, and task-based interviews. The MKT measure is chosen to provide an overall picture of Saudi elementary teachers’ knowledge regarding rational numbers, whereas the survey helped investigate teachers’ confidence in teaching fractions and teachers’ beliefs about issues related to teaching and learning mathematics. Also, task-based interviews were used to examine teachers’ knowledge for teaching fractions in more detail using different types of tasks, including generating story problems and representations, and interpreting students’ work. Using all three tools together (quantitative and qualitative) led to a better understanding of Saudi elementary teachers’ knowledge for teaching mathematics.
This proposed study has some potential significance by filling in the gaps in the research on teacher knowledge in Saudi Arabia. Investigating Saudi in-service upper-level elementary mathematics teachers’ knowledge for teaching can guide the design of support materials for teachers, teacher educators, and professional development (PD) providers by identifying the strengths and weaknesses in the mathematics teachers’ knowledge for teaching. Also, studying teachers’ representations for fraction operations and teachers’ interpretation of student work on fraction operations can help teacher educators and PD providers develop tools and methods to support teachers in understanding and using students’ strategies effectively in their mathematics teaching in Saudi Arabia. Also, this study may encourage researchers to study Saudi mathematics teachers’ knowledge in different mathematics areas and in different grade levels. Eventually, the foundational work of this study will be a stepping stone to further investigation and improvement in mathematics education in Saudi Arabia.
CHAPTER 2
LITERATURE REVIEW

This chapter will discuss research studies on teachers’ knowledge; its relationship with teaching practices and students’ learning; teachers’ mathematical knowledge for teaching fractions; teachers’ affective factors, specifically confidence and belief; different instruments to measure teachers’ knowledge; and guiding framework. First, I will discuss previous research on conceptualization of teacher knowledge in mathematics education. Next, I will describe research studies on the relationships among the domains of teachers’ mathematical knowledge for teaching, teaching practices, and students’ achievements. Also, I will summarize some research about teachers’ mathematical knowledge for teaching, especially in the content area of fractions. Then, I will discuss some research studies about teachers’ affective factors, specifically confidence and belief. I will also explain different instruments that were used by researchers to assess teachers’ mathematical knowledge for teaching. Finally, I will elaborate on the framework that guides the dissertation study. Overall, I attend to methods used and important findings that influenced the design of this dissertation study.

Conceptualization of Teacher Knowledge in Mathematics

Scholars around the world began investigating and assessing teachers’ knowledge largely in the late 1980s (Wang, 2017). Since the time that Shulman (1986, 1987) developed a conceptualization of teachers’ knowledge, mathematics education researchers have refined and expanded his conceptualization to investigate teachers’ mathematical knowledge for teaching. To
understand the domains of teachers’ knowledge, I will describe Shulman’s model and some other models that have been developed in mathematics education.

**Content Knowledge for Teaching: Shulman’s Conceptualization**

The seminal work of Shulman (1986) was considered the foundation for most research on teacher knowledge that followed it. Shulman contended that researchers had put more emphasis on characteristics of classroom management, teacher behavior, and effective pedagogical strategies. He argued, however, there was not enough emphasis on content knowledge for teaching. Shulman brought subject matter knowledge to the frontline of research on teachers’ knowledge. He described three different categories of knowledge necessary for teaching: content knowledge, pedagogical content knowledge, and curricular knowledge. First, content knowledge represents the structure of subject matter, both substantive and syntactic. Substantive refers to the organization of facts and ideas, and syntactic refers to the set of rules and norms that support the content. Shulman expected that

the teacher need not only understand that something is so; the teacher must further understand why it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied. (p. 9)

Second, pedagogical content knowledge links knowledge of pedagogy with knowledge of content that is taught. It is described as “the ways of representing and formulating the subject that make it comprehensible to others” (p. 9). It is knowledge of how and when to use different strategies to reorganize students’ understanding. It also includes the knowledge of student learning and thinking and what makes a subject difficult or easy for students, such as common errors, uncommon solutions, and methods for recognizing and addressing them. Finally, curricular knowledge indicates “familiarity with the topics and issues that have been and will be
taught in the same subject area during the preceding and later years in school, and the materials that embody them” (p. 10).

Shulman (1986) also proposed three forms of teacher knowledge into which the three categories of the framework can be organized. These forms are *propositional knowledge*, *case knowledge*, and *strategic knowledge*. First, propositional knowledge consists of the lessons taught to teachers as axioms based on experience. Second, case knowledge is the knowledge of prototype events and a way of analyzing or interpreting an event through theoretical construction. Third, strategic knowledge exists when teachers confront problems without a possible simple solution. Teachers can impose professional judgment and go beyond knowledge of content. Strategic knowledge is the tool that teachers use to reason when propositional and case knowledge clash.

Later, Shulman (1987) traced the biographies of four new secondary teachers in different majors: English, mathematics, biology, and social studies. He collected data through interviews and clinical tasks. As a result of this empirical study, he provided further specifications for his three categories for teachers’ professional knowledge that he discussed in his 1986 study. For example, he identified knowledge of students as an independent category, rather than a subcategory of pedagogical content knowledge. He identified seven categories of teachers’ knowledge base: (1) content knowledge; (2) general pedagogical knowledge; (3) pedagogical content knowledge; (4) curriculum knowledge; (5) knowledge of learners and their characteristics; (6) knowledge of educational contexts; and (7) knowledge of educational ends, purposes, and values, and their philosophical and historical grounds. These categories have influenced research trends in mathematics education.
Scholars around the world have largely begun investigating and assessing teachers’ knowledge since Shulman’s (1986, 1987) work (Wang, 2017). Shulman’s work focuses the attention of researchers and policymakers on the nature and types of knowledge needed for teaching a subject. The distinction he made between content knowledge for teaching and disciplinary content knowledge had “important implications for informing an emerging argument that teaching is professional work with its own unique professional knowledge base” (Ball et al., 2008, p. 392). Shulman argued that pedagogical content knowledge (PCK) is the domain distinguishing the understanding of the content specialist from the understanding of the pedagogue. Ball and her colleagues (2008) found that this distinction is valuable as a “heuristic, as a tool for helping the field to identify distinctions in teacher knowledge that could matter for effective teaching” (p. 392). Shulman’s works offer several main contributions that influence research in mathematics education, including (a) reframing the study on teacher knowledge, from focusing on general aspects of teaching to concentrating on the role of content in teaching; and (b) representing content understanding as a special type of knowledge for the professional teaching (Ball et al., 2008).

However, Shulman’s conceptualization has been criticized in several ways. Depaepe, Verschaffel, and Kelchtermans (2013) mentioned that some researchers have criticized Shulman’s conceptualization of PCK for the following reasons: (a) the lack of theoretical and empirical basis for the existence of PCK as a separate category in teachers’ knowledge base, (b) a static view on teachers’ PCK from a cognitive perspective as knowledge independent from classroom contexts, (c) the distinction between PCK and CK (content knowledge), and (d) the narrowing conceptualization of PCK in terms of teachers’ knowledge under two categories— instructional strategies and representations and students’ misconceptions. Responding to some of
the criticisms, researchers refined the conceptualization of PCK. Some scholars used their empirical research on teachers’ PCK to expand the conceptualization of PCK (e.g., Grossman, 1990; Marks, 1990). Grossman (1990) expanded the construct of PCK to include knowledge of students’ understanding, knowledge of curriculum, knowledge of instructional strategies, and knowledge of purposes for teaching. In addition, such empirical research was based on the cognitive perspective or situated perspective. The cognitive perspective provided empirical evidence for a relationship between teachers’ PCK and student achievement, and distinctions between PCK and other categories of teachers’ knowledge (Depaepe et al., 2013). The situated perspective emphasized the importance of sharing expertise for developing teachers’ PCK, explaining aspects connected to teachers’ PCK (Depaepe et al., 2013). In addition, several researchers sought to identify some aspects of mathematical knowledge needed for teaching (e.g., An, Kulm, & Wu, 2004; Ma, 1999). In mathematics education, the 1999 study by Ma is considered as an example of studies that shed light on the importance of investigating the nature of knowledge held by teachers.

**Profound Understanding of Fundamental Mathematics (PUFM)**

Ma (1999) conducted comparative research on the knowledge of elementary teachers in China and the United States and found the kind of understanding that distinguished these two groups. In this study, Ma used the interview questions from the Teacher Education and Learning to Teach (TELT) study developed by Deborah Ball. These questions were designed “to probe teachers’ knowledge of mathematics in the context of common things that teachers do in the course of teaching” (Ma, 1999, p. xxi). The four common mathematics topics that were chosen for the interviews were whole number subtraction, whole number multiplication, division with fractions, and the relationship between area and perimeter. These topics were examined with one
of four common tasks of teaching: teaching a topic, responding to a student’s mistake, generating a representation of a particular topic, and responding to a novel idea raised by a student. The sample was composed of two sets of teachers from two countries, the United States and China. Twenty-three U.S. teachers were presumed to be above average. Eleven were participants of a prestigious summer professional development program and had an average teaching experience of 11 years. Twelve of them had 1 year of teaching experience and were in a master’s program. Seventy-two Chinese teachers were selected from five schools ranging from a very high to a very low quality. Forty teachers had teaching experience of fewer than 5 years, 24 teachers had more than 5 years of teaching experience, and the remaining eight had an average teaching experience of 18 years.

Ma’s (1999) analyses of Chinese teachers’ explanation of algorithms and instructional strategies revealed profound understanding of fundamental mathematics (PUFM). According to Ma, this understanding helped them make more connections between and within topics, provide more instructional strategies, and support their students more to make sense of the mathematics than the U.S. teachers. Ma explained that a teacher with PUFM “goes beyond being able to compute correctly and to give a rationale for computational algorithms” (p. xxiv). Also, a teacher with PUFM is “not only aware of the conceptual structure and basic attitudes of the mathematics inherent in elementary mathematics but can teach them to students” (Ma, 1999, p. xxiv). According to Ma, PUFM consists of four characteristics: connectedness, multiple perspectives, basic ideas, and longitudinal coherence. Connectedness means making connections between mathematical conceptions and procedures and explaining the meaning behind these procedures. Multiple perspectives refer to solving the problem by using different strategies and providing meaning behind these different approaches. Basic ideas indicate reinforcing and revisiting the
basic idea and concepts of discipline. Longitudinal coherence involves developing knowledge packages. Ma’s framework is powerful in understanding the characteristics of PUFM teachers.

**Mathematical Knowledge for Teaching (MKT)**

In mathematics education, mathematics educators have reached a consensus on the importance of teachers’ profound understanding of the mathematics, but there has been no consensus on the kind of mathematical knowledge that teachers need to teach effectively. Recently, a set of studies expanded Shulman’s model and its domains to elaborate on mathematics teachers’ knowledge (e.g., Ball et al., 2008; Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013; Rowland, Huckstep, & Thwaites, 2005).

The work by Ball and her colleagues (e.g., Ball et al., 2008; Hill, Ball, & Schilling, 2008; Hill et al., 2005) is considered probably the most influential reconceptualization of pedagogical content knowledge (Depaepe et al., 2013). Ball and her colleagues (2008) proposed a practice-based theory of teachers’ mathematical knowledge for teaching (MKT).

The construct of mathematical knowledge for teaching (MKT) includes six domains of knowledge that are needed for mathematics teaching (see Figure 2.1). Teachers need to blend two kinds of knowledge—subject matter knowledge (SMK) and pedagogical content knowledge (PCK) (Thames & Ball, 2010). Subject matter knowledge includes three domains: common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). CCK is basic mathematical knowledge that teachers use in everyday life and in many practices, such as doing algorithms and using the mathematical concept correctly. SCK is mathematical knowledge that is needed for teaching and knowledge that exceeds what the teacher is teaching to the students. For example, teachers need to know different interpretations of the mathematical operations. HCK is knowledge of how mathematical topics in a specific
grade are related to mathematical topics in the following grades in an advanced level.

Pedagogical content knowledge (PCK) is also subdivided into three domains: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). KCS is knowledge of the ways in which students understand the mathematical content. Teachers should be able to anticipate what and how students think about the task and what they will find simple or complex. KCT is knowledge of methods for teaching mathematics in ways that support students in making sense of the mathematics they learn. For example, teachers should be able to choose an appropriate representation that is used to effectively teach a specific task. KCC is knowledge of mathematics content for the grade level that is taught and involves selecting suitable curriculum materials and understanding the goals of textbooks (Ball et al., 2008).

![Figure 2.1. Mathematical knowledge for teaching (MKT) (Ball et al., 2008).](image-url)
The construct of MKT has a few merits. It is based on empirical research, helps to
develop the measures of teachers’ mathematical knowledge for teaching, and explains the
relationship between teachers’ PCK and students’ achievement (Depaepe et al., 2013). Ball and
her colleagues (2008) argued that it is important to understand and identify the nature and types
of mathematical knowledge and skills that teachers need for teaching. They discussed some
points that explain the importance of refining a conceptual framework of content knowledge for
teaching:

1. It is useful for designing teachers’ opportunities to learn mathematics for teaching.

2. It is useful for organizing professional education to help teachers learn the range of
knowledge and skill they need in focused ways.

3. It is useful to determine the aspects of a teacher’s knowledge that influence students’
achievement.

4. It is useful to redesign the materials for teachers, teacher education, and professional
development. (Ball et al., 2008, p. 405)

Although the concept of MKT has advantages, it has also been criticized for different
reasons. For instance, critics ask to what extent the domains within MKT are theoretically
distinguished (e.g., to what extent is SCK different from PCK?). Another criticism is that it takes
a cognitive perspective on teachers’ mathematical knowledge and ignores teachers’ beliefs about
mathematics teaching (Depaepe et al., 2013).

Interestingly, Koponen, Asikainen, Viholainen, and Hirvonen (2019) studied the
hierarchical order of the six domains of MKT to understand the structure of teachers’ knowledge.
They used network analysis methods to analyze the data collected from 18 preservice teachers by
asking them to analyze face-to-face teaching and to write an essay about their perception of the
knowledge needed for teaching mathematics. The findings revealed that the six domains of MKT can be categorized into three types of knowledge: (1) foundation knowledge consisting of CCK and KCC, (2) transformation knowledge including HCK and SCK, and (3) operation knowledge involving KCS and KCT. They found that KCS and KCT are connected in many different ways to the other four MKT domains.

**Mathematics Teacher’s Specialized Knowledge (MTSK)**

Building on Shulman’s PCK and drawing on Ball and her colleagues’ MKT, Carrillo and his colleagues (Carrillo et al., 2013; Carrillo et al., 2018) developed a model of mathematics teachers’ knowledge that focuses on the specialization of mathematics teachers’ knowledge. Instead of building this model based on classroom practice (classroom observations), the authors proposed a theoretical model that can be tested in practice later (Carrillo et al., 2013). The mathematics teachers’ specialized knowledge (MTSK) model consists of six subdomains under two main domains—mathematical knowledge (MK) and pedagogical content knowledge (PCK)—and teachers’ beliefs (see Figure 2.2). Teachers’ beliefs regarding mathematics and mathematics teaching and learning are in the center of the MTSK model because of the relationship between beliefs and all other domains—teachers’ beliefs influence and are influenced by all other subdomains (Carrillo et al., 2013).
Figure 2.2. Mathematics teacher’s specialized knowledge (MTSK) (Carrillo et al., 2018).

On one hand, mathematical knowledge (MK) is subdivided into three subdomains: knowledge of topics (KOT), knowledge of the structure of mathematics (KSM), and knowledge of practices in mathematics (KPM). KOT is the knowledge of content and underlying mathematical meaning, including knowledge of procedures, definitions, properties and foundations, registers of representation, phenomenology and applications. It represents the CCK and SCK domains of the MKT model. KSM is the knowledge of connection among mathematical contents, including connections based on simplification and connections based on increased complexity, auxiliary connections, and transverse connections. It is corresponding to
HCK of the MKT model. KPM is the knowledge of how to construct new knowledge, such as knowledge about “demonstrating, justifying, defining, making deductions and inductions, giving examples and understanding the role of counterexamples” (Carrillo et al., 2018, p. 244). On the other hand, PCK is also subdivided into three subdomains: knowledge of mathematics teaching (KMT), knowledge of features of learning mathematics (KFLM), and knowledge of mathematics learning standards (KMLS). KMT is the knowledge of mathematics teaching theories; teaching resources; and strategies, techniques, tasks, and examples. KMT is similar to the domain KCT in the MKT model. KFLM is the knowledge of how mathematics is learned, similar to KCS in the MKT model, including knowledge of mathematical learning theories, strengths and weaknesses in learning mathematics, ways pupils interact with mathematical content, and emotional aspects of learning mathematics. KMLS is knowledge of what students already know and what they should or are able to achieve at each level, similar to KCC in the MKT model, such as knowledge of expected learning outcomes and level of conceptual or procedural development, and sequencing of topics (Carrillo et al., 2018).

The MTSK model has several merits. It could be used as “a tool for approaching the complexity of teachers’ knowledge” (Carrillo et al., 2018, p. 248). It could be used in analyzing and assessing teachers’ knowledge of specific mathematical topics. As with the MKT model, it could be used for developing measures of mathematics teachers’ knowledge. Also, it could be used for developing programs for preservice and in-service teachers to equip them with appropriate specialized knowledge for mathematical teachers (Carrillo et al., 2018). Moreover, the model includes teachers’ beliefs in it, whereas the MKT model is criticized for this missing element. The MTSK model, however, does not specify the interaction between the teacher beliefs and other domains clearly yet.
**Teacher Knowledge in Use**

Other researchers developed a conceptualization of teachers’ knowledge based on the situated perspective (e.g., Mason & Spence, 1999; Remillard & Kim, 2017; Rowland et al., 2005). They investigated a specific aspect of teacher knowledge that is activated in response to classroom situations, as opposed to a static entity. They have argued that “mathematical content knowledge needed for teaching is not located in the minds of teachers but rather is realized through the practice of teaching” (Rowland, 2013, p. 17). Mason and Spence (1999) termed this type of knowledge as knowing to act in the moment. They distinguished between knowing about (knowing that something is true, knowing how to do something, knowing why you do something) and knowing to do something in a particular situation — “active knowledge which is present in the moment when it is required” (Mason & Spence, 1999, p. 135). This knowledge refers to teachers’ ability to draw on different types of knowledge in response to actual situations.

Rowland and his colleagues (Rowland, 2013; Rowland et al., 2005) proposed a conceptual framework for classifying situations in which mathematical knowledge comes into play in teaching practice in classrooms. Rowland and his colleagues (2005) studied videos of preservice primary mathematics teachers and used a grounded approach to develop their domains of knowledge that are called the Knowledge Quartet. The Knowledge Quartet (KQ) coincides to a significant degree with what Shulman (1987) proposed: “processes of pedagogical reasoning and skilled action, in which knowledge is used” (Remillard & Kim, 2017, p. 68). Shulman’s model of pedagogical reasoning and action consists of six components: comprehension, transformation, instruction, evaluation, reflection, and new comprehension. Modifying Shulman’s model, Rowland et al. proposed four components of the Knowledge Quartet (KQ):
foundation, transformation, connection, and contingency. First, the foundation component is related to teachers’ mathematical content knowledge and beliefs. This component considers a basis for developing other types of knowledge, and it corresponds with the first component of Shulman’s pedagogical reasoning—comprehension. Second, the transformation component involves transforming the content in a way that is suitable for students, such as using different examples and representations. This component distinguishes between mathematics teachers’ knowledge and that of other adults. Third, the connection component is the knowledge of how to sequence and connect the topics and tasks and make a connection between mathematics and other subjects. Fourth, the contingency component is related to teachers’ ability to respond to students’ unexpected ideas in the moment. The last three components depict classroom situations in which teachers use various domains of knowledge to make instructional decisions, and the model is similar to Shulman’s cyclic process of pedagogical reasoning and action. Rowland (2013) stated that the MKT and KQ frameworks are complementary. While MKT aims to distinguish between different domains of mathematical knowledge, KQ aims to classify the situations in which mathematical knowledge surfaces in teaching (Rowland, 2013).

In addition, some researchers focused on a specific aspect of teaching to identify the forms of knowledge needed for teaching. For example, Remillard and Kim (2017) used a grounded approach to analyze teachers’ guides of elementary mathematics curricula and interviewed teachers to articulate the kind of knowledge activated by teachers when using curriculum resources. They conceptualized knowledge of curriculum embedded mathematics (KCEM), which is defined as “the mathematics knowledge activated by teachers when reading, interpreting, using mathematical tasks, instructional designs and representations in mathematics curriculum materials” (p. 65). They identified four domains of KCEM as critical when teachers
use curriculum resources to teach mathematics: (1) foundational mathematical ideas, (2) representations and connections among these ideas, (3) relative problem complexity, and (4) mathematical learning pathways.

Many mathematics education researchers investigated teachers’ mathematical knowledge for teaching from a different perspective—a cognitive perspective and situated perspective. They developed different conceptualizations of teacher knowledge, which influence the design of this dissertation investigating Saudi teachers’ mathematical knowledge for teaching. This dissertation research connects different domains of teacher knowledge and uses different tools to assess this knowledge. This study is influenced by MKT domains, specifically content knowledge (Ball et al., 2008), and uses an MKT measure to assess this domain. Also, knowledge of topic (KoT), one of the subdomains in the MTSK model (Carrillo et al., 2018), supports this study in focusing on knowledge of specific mathematical topic (fractions). Ma’s (1999) study that investigated the nature of knowledge held by teachers through teaching-scenario task-based interviews helps the design of this study. This study also adapts teacher knowledge as situated in teaching contexts by asking teachers to analyze specific situations, including problems and student work (Rowland et al., 2005), and by investigating other aspects, such as foundational mathematical ideas and representations and connections among these ideas (Remillard & Kim, 2017).

This dissertation focuses on studying teachers’ mathematical knowledge for teaching because of its significant impact on teaching practices and students’ learning outcomes. In the following section, I will discuss studies on the relationship among teachers’ knowledge, instructional quality, and students’ learning outcomes.
**Teacher Knowledge, Teaching Practice, and Students’ Achievement**

Many researchers have studied the relationship among teacher knowledge, teaching practice, and students’ achievement. Some researchers have investigated the relationship between teachers’ mathematical knowledge and their teaching practice and mathematical quality of instruction (e.g., Ball et al., 2008; Baumert et al., 2010; Charalambous, 2010; Charalambous & Hill, 2012; Copur-Gencturk, 2015; Hill, Blunk, et al., 2008; Kersting, 2008). They found a positive connection between teachers’ mathematical knowledge and their mathematical quality of instruction. Teachers’ mathematical knowledge for teaching plays an essential role in improving their instructional quality from many aspects. The studies concluded that teachers who scored lower on MKT measures exhibited more errors in their mathematics teaching. They were not able to use a different strategy to address students’ misconceptions. In contrast, teachers who scored higher on MKT measures were able to engage their students in richer mathematics, representations, explanations, and justifications; to avoid mathematical error; to use students’ mathematical thinking (e.g., Charalambous & Hill, 2012; Hill, Blunk, et al., 2008); to be precise in language and notation (Charalambous & Hill, 2012); to ask mathematically productive questions (e.g., Ball et al., 2008; Copur-Gencturk, 2015); to provide meaningful explanations; and to focus on higher-order cognitive demands at presented and enacted tasks (e.g., Charalambous, 2010). Also, the researchers showed the correlations between teachers’ mathematical knowledge for teaching and their ability to analyze teaching situations on the video clips (e.g., Kersting, 2008; Kersting, Givvin, Sotelo, & Stigler, 2010) and their noticing of students’ mathematical thinking (e.g., Jacobs, Lamb, & Philipp, 2010).

Some large-scale studies examined an association between MKT measures and student learning outcomes in mathematics. They concluded that students performed better when their
teachers had more mathematical knowledge for teaching (e.g., Baumert et al., 2010; Hill et al., 2018; Hill et al., 2005). The study by Hill et al. (2005) provides the first evidence for the practical significance of teachers’ mathematical knowledge in both common content knowledge and the specialized mathematical knowledge on students’ achievement. The results revealed a significant positive relationship between teachers’ content knowledge for teaching mathematics and students’ achievement. Similarly, the findings by Hill et al. (2018) showed a positive relationship between teachers’ knowledge and effort invested in noninstructional activities and student learning outcomes.

Other large-scale studies provided evidence of a relationship among teacher knowledge, instructional practices, and student achievement in mathematics (e.g., Baumert et al., 2010; Kelcey, Hill, & Chin, 2019; Krauss et al., 2020). Baumert et al. (2010) investigated the relationship between German in-service teachers’ CK and PCK and the significance of this knowledge for teachers’ instructional quality and student progress in mathematics. They found that PCK and CK affect instructional quality and student learning gains. However, teachers’ PCK in mathematics had higher predictive power for student progress than did CK. Also, they found that PCK was positively correlated with individual learning support, and CK had a relationship with the curricular alignment of tasks. Unlike Baumert et al., who focused on discrete instructional behaviors such as cognitive demand of the task, Kelcey et al. (2019) focused on a broader set of specific instructional behaviors, such as ambitious mathematics instruction and teachers’ ability to work with students. They found that only the ambitious mathematics instruction significantly mediated the relationship between teachers’ knowledge and students’ achievement.
The results of the above research studies explain the important role of teachers’ mathematical knowledge for teaching on instructional quality and students’ achievement. Teachers’ mathematical knowledge for teaching has significant impact on teaching practices, which influence various aspects of instructional quality and also impact student learning outcomes. As many researchers study teachers’ overall mathematical knowledge for teaching mathematics, other researchers investigate teachers’ mathematical knowledge for teaching on a specific topic.

**Knowledge for Teaching Fraction Operations**

In mathematics education, researchers from different countries around the world have studied teachers’ knowledge of various mathematical concepts, and fractions have been an important area of investigation (Lovin, Stevens, Siegfried, Wilkins, & Norton, 2018). In this section, I will first explain the rationale for investigating fractions. Then, I will discuss research studies on mathematical knowledge for teaching fractions in general, and teaching multiplication and division with fractions in particular. Finally, I will describe the research studies on mathematical knowledge for teaching fractions in the Saudi context.

**Rationale for Investigating the Topic of Fractions**

The U.S. National Mathematics Advisory Panel (NMAP, 2008) described fractions as “the most important foundational skill not presently developed” (p. 18). Also, the National Council of Teachers of Mathematics (2007) emphasized the importance of improving teachers’ and students’ understanding of fraction. Researchers have become more aware of the importance of a fundamental knowledge of fractions for both teachers and students. When explaining the rationale of selecting fractions as a specific topic of their studies, researchers give similar justifications. Like other topics, fractions have a conceptual nature and a procedural nature that
enable researchers to distinguish among the domains of teachers’ mathematical knowledge for teaching. Fractions are a fundamental concept in introductory mathematics for other advanced topics, like algebra and probability. NMAP concluded that the lack of conceptual understanding and procedural fluency of fractions as a part of a rational number is a major obstacle for further progression in advanced mathematics, including algebra. Also, both teachers and students encounter difficulty in understanding the fraction because of its multiple natures. Fraction constructs can be represented in different ways depending on the context, such as a part of a whole, measurement, division, operator, and ratio. In addition, fractions can be represented by different types of models, such as area, length or number line, or set of objects. The nature and importance of fractions for students in advanced levels were factors that led me to focus this dissertation on teacher knowledge of fractions.

**Teachers’ Mathematical Knowledge of Teaching Fractions**

Many studies have focused on understanding and measuring teachers’ knowledge in relation to the concept of fractions or the meaning of fractions, such as fraction equivalence (e.g., Ding, 2007), addition and subtraction with fractions (e.g., Lee & Lee, 2020; Marchionda, 2006), multiplication with fractions (e.g., Izsák, 2008; Luo, 2009), division with fractions (e.g., Aytekin & Şahiner, 2020; Ball, 1990; Klemer, Rapoport, & Lev, 2019; Li & Huang, 2008; Li & Kulm, 2008; Li & Smith, 2007; Lo & Luo, 2012; Ma, 1999; Simon, 1993; Son & Crespo, 2009; Tirosh, 2000), fundamental fraction constructs (e.g., Luo, Lo, & Leu, 2011), and all four operations with fractions (e.g., Depaepe et al., 2015; Kazemi & Rafiepour, 2018; Lewis, 2016; Rosli et al., 2020).

The scope of research studies on teachers’ mathematical knowledge for teaching fractions is more on preservice teachers than in-service teachers. Also, the scope of studies on in-service
teachers’ mathematical knowledge of teaching fractions has focused on middle school teachers (e.g., Bradshaw, Izsák, Templin, & Jacobson, 2014; Izsák, 2008; Izsák, Jacobson, de Araujo, & Orrill, 2012; Izsák, Orrill, Cohen, & Brown, 2010). Fewer researchers have studied in-service elementary school teachers (e.g., Chinnappan & Desplat, 2012; Kazemi & Rafiepour, 2018; Ma, 1999). The scope of studies with elementary and middle school preservice and in-service teachers revealed that participating teachers have limited mathematics knowledge for teaching fractions.

Olanoff, Lo, and Tobias (2014) reviewed 43 studies conducted from 1989 to 2013 from all over the world on preservice teachers’ fractions content knowledge, particularly CCK, SCK, and KCS. The findings showed that most preservice teachers were strong in performing procedures with fractions, but they struggled to understand the meanings behind the procedures and the reasons why the procedures work. Also, they interpreted a fraction as a part of a whole and had difficulty in making other interpretations of fractions. Most of these studies show that preservice teachers have limited mathematics knowledge for teaching fraction division conceptually. For example, preservice teachers found it difficult to generate a diagram and a story problem of a given fraction operation. They also had difficulty interpreting alternative algorithms.

Other researchers measured preservice or in-service teachers’ CK and PCK by developing instruments including all four fraction operations. Depaepe et al. (2015) evaluated CK and PCK of elementary and lower secondary preservice teachers, focusing on rational numbers. They designed 48 CK-PCK-related items involving all four operations with fractions. The CK items included fractions and decimal numbers, and PCK items consisted of two components, that is, knowledge of students’ misconceptions and knowledge of instructional
strategies and representation. These items were administered to 158 preservice elementary teachers and 34 preservice lower secondary teachers. The results showed that the teachers had a lack of CK and PCK about rational numbers. Also, there was a positive correlation between CK and PCK. Teachers lacking CK of fraction and decimal arithmetic also had limited PCK of how to teach these subjects to students. To deal adequately with students’ difficulties regarding rational numbers, it is essential for teachers to have deep-level CK and PCK (Depaepe et al., 2015). Also, Kazemi and Rafiepour (2018) developed a scale for measuring CK and PCK of in-service elementary teachers on fractions and examined the relationship between CK and PCK. The paper-and-pencil scale containing 22 items with fractions was designed based on the Cognitive Activation in the Classroom (COACTIV) study model to investigate the professional knowledge of secondary school mathematics teachers in Germany—both CK and PCK. While the MKT study by Ball and her colleagues (e.g., Ball et al., 2008) shows that CK and PCK are integrated into a single body, the COACTIV study distinguishes between CK and PCK (Kazemi & Rafiepour, 2018; Krauss, Baumert, & Blum, 2008). Fourteen items were related to PCK, including three components: (1) knowledge of representation and explanation of the problem, (2) knowledge of different methods of problem solving, and (3) knowledge of students’ misconception and difficulties. Eight items were related to CK. This scale was administered to 256 elementary teachers. The findings of this study suggested that CK and PCK are separate and correlated. Kazemi and Rafiepour concluded that measuring CK and PCK on a specific topic like fractions could give a full picture of these two types of knowledge.

**Teachers’ Mathematical Knowledge for Teaching Fraction Multiplication and Division**

Multiplication and division with fractions are the most difficult aspects in the topic of fractions. Some researchers have investigated teachers’ mathematical knowledge for teaching,
specifically multiplication with fractions (e.g., Izsák, 2008; Son & Lee, 2016) and division with fractions (Aytekin & Şahiner, 2020; Chinnappan & Desplat, 2012; Klemer et al., 2019; Lo & Luo, 2012; Ma, 1999; Simon, 1993; Tirosh, 2000). They found that preservice and in-service teachers lacked mathematical knowledge for teaching, both content knowledge and pedagogical content knowledge, in multiplication and division with fractions. They concluded that teachers had several misconceptions about the topic of fractions in multiplication and division, such as in explaining the meaning behind procedures and presenting a story problem that models fraction multiplication rather than fraction division.

Teachers’ mathematical knowledge for teaching fraction division has been investigated more than fraction multiplication. Izsák (2008) investigated two elementary teachers who had different levels of mathematical knowledge for teaching and were using the same curriculum materials on fractions. The findings revealed that a high level of knowledge supported teachers in their use of representations and in their interactions with students’ ideas on fraction multiplication. Son and Lee (2016) examined preservice teachers’ understanding of fraction multiplication in three different contexts: word problems, representation, and computational skills. Analyzing written responses from 60 preservice teachers, they concluded that preservice teachers need to improve their ability to explore different types of mathematical representations and interpret student thinking on fractions in multiplication.

Ma’s (1999) interviews with teachers on solving and generating a story problem of fraction division showed that the U.S. teachers were unable to explain the meaning beyond the invert-and-multiply strategy, whereas 90% of Chinese teachers were able to generate at least one valid explanation. Teaching fractions requires a solid understanding of a multitude of concepts and processes (Ma, 1999). Similarly, studies by Tirosh (2000) and Lo and Luo (2012) found that
preservice teachers were able to calculate fraction division correctly. However, they could not explain the meaning behind the procedure nor predict major sources of students’ incorrect responses (Tirosh, 2000), and they had difficulty representing fraction division by using word problems or pictorial diagrams (Lo & Luo, 2012).

Chinnappan and Desplat (2012) conducted a case study of four elementary teachers who differed in their teaching experience. The teachers’ knowledge was examined by analyzing teachers’ attempts to contextualize an abstract fraction problem involving division. They found that teachers’ content knowledge was procedural and algorithmic. The teachers had a conceptual weakness relating expressions with fractions to contextual situations. Also, teachers with limited pedagogical content knowledge of fractions encountered difficulty supporting their students’ understanding of fractions (Chinnappan & Desplat, 2012). Similarly, Klemer et al. (2019) investigated teachers’ SCK and KCS. They asked nine elementary teachers to respond to the questionnaire individually and then interviewed them. The questionnaire involved three tasks: solving fraction division problems, explaining the method and process they used for teaching fraction division, and discussing students’ mistakes in fraction division. They found that teachers’ specialized content knowledge was insufficient and influenced their knowledge of content and students, especially in supporting students to understand the meaning behind the procedure and to identify the sources of students’ misconceptions.

**Saudi Teachers’ Mathematical Knowledge for Teaching Fractions**

Two studies are found related to the topic of fractions in the Saudi teacher context (Alghazo & Alghazo, 2017; Khashan, 2014). One of them is with preservice teachers, and the other one with in-service teachers. Alghazo and Alghazo (2017) studied common errors and misconceptions about fractions among Saudi preservice teachers. They collected data from 107
Saudi preservice teachers by using a test and a short interview. The short interview aimed to understand students’ thought processes as they completed the test. The researchers designed the test to assess preservice teachers’ ability to perform fraction operations and their conceptual understanding of fractions, such as drawing visual representations of fractions. They found that Saudi preservice teachers had four kinds of common misconceptions about fractions:

(1) fractions are always part of 1 and are never greater than 1; (2) multiplication makes numbers bigger, and division makes them smaller; (3) cross multiplication is used to solve fraction multiplication problems; and (4) the larger the denominator, the smaller the fraction, regardless of the numerator.

Khashan (2014) investigated Saudi in-service elementary school teachers’ conceptual and procedural knowledge of fractions. He collected data from 57 elementary teachers, consisting of 27 novice teachers and 30 experienced teachers, by using an assessment based on conceptual and procedural items regarding rational numbers, designed by Faulkenberry. The assessment covered a range of concepts related to fractions, such as the concept of a part of a whole, the concept of the unit, fraction equivalence, comparing fractions, fraction operations, and story problems. He used statistical methods to calculate the mean for conceptual and procedural knowledge items. The study revealed that on both instruments the teachers correctly answered about 50% of the items that dealt with conceptual and procedural knowledge in rational numbers. The teachers performed better on procedural than on conceptual items. Also, teaching experience correlated positively with teachers’ mathematical knowledge.

To summarize, the scope of the research on teachers’ mathematical knowledge for teaching fractions has focused more on preservice teachers than in-service teachers. Furthermore, it seems that the scope of the research on preservice teachers’ knowledge of fraction operations
is limited, primarily focusing on preservice teachers’ knowledge of division with fractions. It seems that research studies on teacher knowledge for teaching fraction division are more common than studies on fraction multiplication. Most of these studies used paper-and-pencil assessments of teacher knowledge; some of them combined the assessments with interviews. Often, these research studies investigated both content knowledge and pedagogical content knowledge. Most of these studies revealed that there is a lack of teachers’ mathematical knowledge for teaching fractions. This deficiency inhibits teacher actions in instruction on fractions, such as using multiple strategies.

Many researchers have explained that the importance and nature of fractions is a foundational topic in elementary school. However, few studies can be found investigating Saudi teachers’ mathematical knowledge for teaching fractions. The two research studies that have been found focus on common content knowledge. One study (Khashan, 2014) investigated in-service teachers by using a quantitative approach, and another study (Alghazo & Alghazo, 2017) investigated preservice teachers by using a mixed-methods approach. Incorporating both quantitative and qualitative measures, this dissertation study seeks to understand in-service elementary school teachers’ mathematical knowledge for teaching in Saudi Arabia in the content strand of rational numbers with an emphasis on multiplication and division with fractions.

**Teachers’ Confidence and Beliefs**

Since the 1960s, mathematics education researchers have considered teachers’ affective, motivational characteristics, such as teachers’ attitudes, beliefs, and emotions, as important factors of teaching and learning mathematics (McLeod, 1992). Researchers studied teachers’ overall affective domains and their relationship with teachers’ knowledge and teaching practices (Philipp, 2007) and suggested that the affective domains, including confidence and beliefs,
significantly affect teaching practices (e.g., McLeod, 1992). Ernest (1989) mentioned the importance of teachers’ affective domains, including teachers’ confidence regarding mathematics and teaching mathematics and beliefs about the nature of mathematics, because of their influence on classroom environments and students’ attitudes about mathematics. Also, he believed that teachers’ knowledge influences teachers’ affective domains, especially beliefs and confidence, and that teachers’ knowledge, beliefs, and confidence play an important role in their instructional decisions.

**Definitions of Beliefs and Confidence**

Researchers have studied teachers’ beliefs much more often than teachers’ confidence. Although many researchers have studied teachers’ beliefs, only a few of them defined the term. Philipp (2007) defined beliefs as “psychologically held understandings, premises, or propositions about the world that are thought to be true” (p. 259). Confidence was considered by some researchers as a type of teachers’ attitude. Teachers’ attitudes about mathematics and teaching mathematics include different aspects, such as liking, enjoying, and being enthusiastic about teaching mathematics, as well as teachers’ confidence in mathematics and mathematics teaching (Beswick et al., 2006; Ernest, 1989). Other researchers defined confidence as one of several affect-related concepts, such as self-concept and mathematics anxiety (e.g., McLeod, 1992). Also, McLeod (1992) mentioned that confidence could be considered as beliefs about teachers’ competence of mathematics and mathematics teaching.

**Research on Teacher Confidence and Beliefs**

Beswick and her colleagues (2006, 2012) specifically investigated teachers’ confidence about teaching mathematical topics and their beliefs about mathematics teaching and learning. Beswick, Watson, and Brown (2006) argued that confidence is the most important aspect of
attitudes, because of its particular relevance to teachers’ practices. Beswick et al. (2006) investigated 42 middle school mathematics teachers to determine their confidence in their ability to teach 13 topics, including fractions, decimals, percent, ratio and proportion, and mental computations, and their beliefs about numeracy and effective teaching of mathematics. They found that these middle school teachers lacked confidence in the mathematics content that they were teaching. Approximately one third of the teachers lacked confidence in relation to fractions, decimals, and percent. Also, they believed that mathematics is important, but they did not believe that mathematics is relevant to everyday life.

Researchers studied the relationship between preservice or in-service teachers’ confidence in their knowledge preparation for teaching mathematics and their mathematical knowledge for teaching specific topics (e.g., Li & Kulm, 2008; Li, Ma, & Pang, 2008). These research studies revealed different findings. Li and Kulm (2008) studied the relationship between teachers’ confidence and their mathematical knowledge for teaching. They found that U.S. preservice teachers had high confidence about their knowledge preparation for teaching, although they had limited mathematics knowledge for teaching fraction division conceptually. In contrast, Li, Ma, and Pang (2008) found that although mainland Chinese and South Korean preservice elementary teachers did not have high confidence in their knowledge preparation for teaching, their mathematical knowledge for teaching fraction division—content knowledge and pedagogical content knowledge—was actually pretty good.

Other studies investigated the relationship between preservice or in-service teachers’ beliefs about mathematics and teaching mathematics and their knowledge for teaching mathematics in general and specific mathematics topics. For example, Li and Huang (2008) studied in-service teachers’ beliefs on mathematics teaching and learning, as well as their
knowledge in mathematics and pedagogy. The results revealed that Chinese teachers had a solid mathematical knowledge for teaching, and they had strong beliefs about some aspects of teaching but not others. For example, they strongly believed in the importance of using more than one representation in teaching but not in using manipulatives when teaching. Newton, Leonard, Evans, and Eastburn (2012) examined the relationship between mathematics content knowledge and teachers’ personal efficacy beliefs about their ability to understand mathematics and teach mathematics. They found there is a strong link between preservice teachers’ content knowledge and their beliefs about their ability to teach mathematics effectively, and both positively relate to teaching practices.

In some large-scale studies, researchers investigated the correlation between preservice or in-service teachers’ beliefs about mathematics and teaching mathematics and their confidence in teaching mathematics and their teaching practices (e.g., Briley, 2012; Bobis, Way, Anderson, & Martin, 2016; Depaepe & Konig, 2018; Zee & Koomen, 2016). Some of these research studies showed a positive link between teachers’ beliefs and their teaching practices, and some of them showed a positive link between teachers’ confidence and their teaching practices. Briley (2012) found that teachers with strong beliefs in their capabilities to teach mathematics effectively have more self-efficacy beliefs in solving mathematics problems. Teachers’ mathematical self-efficacy beliefs also have a significant impact on confidence in mathematics teaching. Similarly, Depaepe and Konig (2018) studied the relationship among teachers’ general pedagogical knowledge, self-efficacy beliefs, and teaching practices. They found that teachers’ self-efficacy beliefs can positively influence teachers’ instructional practices and students’ achievement, but found no significant link between teachers’ general pedagogical knowledge and teachers’ self-efficacy beliefs. Bobis, Way, Anderson, and Martin (2016) found that teachers’
reactions to professional learning experiences were mediated by teacher beliefs about student engagement in mathematics and teacher confidence in mathematics. They discovered that teachers’ mathematical beliefs and teachers’ confidence in their ability act as barriers to their implementation of new instructional practices. Zee and Koomen (2016) synthesized the findings of 40 years of teacher self-efficacy research to explore its influence on the quality of teaching practices and students’ learning outcomes. The result of analyzing 165 articles showed that teachers’ beliefs about their capacity to affect student performance are positively related to teaching practices. Teachers with strong beliefs in their capacity to affect student performance, and especially those with more experience, tend to teach effectively. They are able to use various instructional strategies based on students’ needs, and they support students’ thinking. Beswick, Callingham, and Watson (2012) considered teachers’ confidence in teaching different topics in middle school mathematics curriculum and teachers’ beliefs about mathematics teaching and learning as parts of teacher knowledge. Also, they argued that teachers’ confidence and beliefs of teaching mathematics are related to the learning of common and specific content knowledge and pedagogical knowledge. In a large-scale study, Yang, Kaiser, König, and Blömeke (2020) found that Chinese preservice mathematics teachers’ beliefs—dynamic beliefs about the nature of mathematics and constructivist beliefs about mathematics teaching and learning—mediated the relationship between their CK, PCK, and instructional practice.

Previous studies suggest that the relationship between teachers’ knowledge and teachers’ affective factors, including confidence and beliefs, is close, and both teachers’ knowledge and teachers’ affective factors influence teaching practices. Also, investigating affect factors, such as confidence and beliefs, is important, but not many studies are found investigating both teachers’ confidence and beliefs. In this dissertation, I use the term teacher confidence to refer to teachers’
perception of their own ability to teach mathematics, which includes aspects of mathematics content and teaching of the content. Also, I use teacher beliefs to refer to teachers’ views and convictions on teaching and learning of mathematics. Part of the purpose of this dissertation is to understand the relationships between teachers’ knowledge and teachers’ confidence and beliefs in the Saudi context. I examined Saudi teachers’ confidence in teaching fractions and beliefs about issues related to teaching and learning mathematics.

**Tools to Measure Teacher Knowledge**

In the research studies on teacher knowledge that are described above, a range of different instruments has been developed and used to investigate, explore, or assess teacher mathematical knowledge for teaching.

Paper-and-pencil tests are popular instruments that have been used to assess teachers’ knowledge. They mostly include a set of items. Ball, Hill, and their colleagues (e.g., Hill, Ball, & Schilling, 2008; Hill, Rowan, & Ball, 2005) developed multiple-choice tests to measure mathematical knowledge for teaching, including CCK and SCK, of elementary and middle grades teachers. Krauss et al. (2008) introduced the Cognitive Activation in the Classroom Project (COACTIV) instrument to test the subject knowledge of secondary grade teachers in Germany—both Shulman’s CK and PCK. A paper-and-pencil instrument is useful to researchers to investigate the growth of teacher knowledge on a larger sample and to professional developers to assess teachers’ knowledge in a particular program. However, it does not provide further detail on the nature of teacher knowledge in a particular content domain (Izsák et al., 2010).

Another type of instrument is a questionnaire or survey that has been used for different purposes in research on teacher knowledge. Son and Kim (2016) developed a survey to measure teachers’ lesson plan modifications to explore whether and how teachers maintained or increased
the cognitive demand of the lesson plan. Also, teachers’ beliefs and confidence have been often measured through a survey. Li and her colleagues (Li & Huang, 2008; Li & Kulm, 2008) developed a survey of teachers’ confidence in their knowledge preparation needed for teaching elementary mathematics. As with paper-and-pencil tests, a survey is useful to collect data from a large sample, but it is limited in its ability to provide detailed information of the subject under investigation (Mayer, 1999).

Other researchers have used interviews containing a range of tasks, often with teaching scenarios. Ma (1999) assessed teachers’ content knowledge by using four specific tasks of teaching: teaching a topic, responding to a student’s mistake, generating a representation of a particular topic, and responding to a novel idea raised by a student. Tirosh (2000) interviewed preservice teachers after they completed a questionnaire on their subject matter knowledge and pedagogical content knowledge of rational numbers. Interview instruments are useful in providing the researcher more details about the nature of teachers’ knowledge.

Observation methods have been used by some researchers to identify aspects within classroom teaching—knowledge in use. Rowland et al. (2005) observed actual teaching in the classroom to investigate teaching situations that require teachers to use knowledge for teaching mathematics. Hill, Blunk, et al. (2008) used a rubric to evaluate teacher knowledge in use based on classroom video. In this rubric, they identify important aspects of the mathematical quality of instruction (MQI), such as avoiding mathematical errors; listening to students’ mathematical thinking; connecting classroom practice to mathematics; providing a rich mathematical explanation, representation, and justification; and using accurate mathematical language.

Some researchers have used analysis of student work or teaching situations to evaluate teacher knowledge. Kersting and her colleagues (Kersting, 2008; Kersting et al., 2010) used
video clips of teaching situations to identify teacher mathematical knowledge for teaching in practice, such as interpreting student thinking and making instructional decisions. Remillard and Kim (2017) used the teachers’ guides to examine teachers’ thinking in planning their lessons with curriculum resources.

Some of these researchers have used more than one type of tool. In this dissertation study, I used three different types of instruments (a paper-and-pencil test, a survey, and a task-based interview) to measure Saudi teachers’ mathematical knowledge for teaching. The paper-and-pencil test provided an overall picture of Saudi elementary teachers’ knowledge, whereas the survey and interview helped me investigate in more detail teacher confidence, beliefs, and knowledge for teaching fractions. All three tools together led to a comprehensive examination of Saudi elementary teachers’ knowledge for teaching mathematics.

**Revisiting the Guiding Framework**

The framework on the role of teacher knowledge, drawn on the research studies reviewed above, guided the design of this dissertation study. Previous studies suggest teachers’ mathematical knowledge for teaching is the foundation for improving instructional quality and student achievement. Affective factors should be taken into consideration in studying teachers’ knowledge. Teachers’ knowledge is closely related to many aspects of teaching practices and student learning outcomes, and teachers’ affective factors and teachers’ knowledge have a reciprocal relationship (see Figure 2.3). Understanding the foundation is important in any reform efforts. However, few studies have investigated in-service Saudi teachers’ knowledge and its relationship with teachers’ affective factors, including confidence and beliefs. Therefore, understanding teacher knowledge in Saudi Arabia is an important step toward improving the Saudi educational system.
As Figure 2.3 presents, teachers’ knowledge and teachers’ affective factors that are the foundation for improving teaching practices and students’ achievements include various domains and elements. For the purpose of studying in-service Saudi elementary teachers’ knowledge in depth, this study focuses on some specific aspects or domains in the two areas of mathematics teachers’ content knowledge for teaching (see Table 2.1). The first area outlined in the left column in Table 2.1 includes two significant aspects, teachers’ confidence for teaching specific topics (i.e., fractions) and teachers’ beliefs about teaching and learning mathematics. The second area focuses on some specific elements of mathematics teachers’ knowledge of content and underlying mathematical meaning, which includes procedures and strategies, generating story
problems, representations, and interpretation of student work. These elements are adapted from knowledge of topic, one of subdomains of the MTSK model (Carrillo et al., 2018), to investigate in depth Saudi teachers’ knowledge of specific mathematical topic (fractions).

Table 2.1. Focal Aspects of Investigating Content Knowledge for Teaching Mathematics

<table>
<thead>
<tr>
<th>Teachers’ confidence and belief</th>
<th>Mathematics teachers’ knowledge for teaching</th>
</tr>
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<tbody>
<tr>
<td>• Confidence for teaching specific topics – Fractions</td>
<td>• Knowledge of content and underlying mathematical meaning</td>
</tr>
<tr>
<td>• Belief about teaching mathematics</td>
<td>o Procedures and strategies</td>
</tr>
<tr>
<td>• Belief about learning mathematics</td>
<td>o Generating story problems</td>
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<tr>
<td></td>
<td>o Representations</td>
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<td></td>
<td>o Interpretation of student work</td>
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Summary

This chapter began by reviewing research studies on the conceptualization of teachers’ knowledge for teaching in mathematics education. Providing the significance of investigating teacher knowledge, I described research studies on the influence of teachers’ knowledge in teaching practice and mathematical quality of instruction, and students’ learning outcomes. In addition, I described research studies on teachers’ mathematical knowledge for teaching fractions in particular, and research studies on teachers’ affect domains, specifically confidence and beliefs. These previous research studies have demonstrated how teacher knowledge can be a powerful resource and significant foundation to support educational reform. To improve students’ achievement, it is important to understand teacher knowledge as a foundation for educational reform. This literature review shows a gap in the research, to which this study contributes, which is the lack of studies investigating Saudi in-service teachers’ knowledge,
specifically with a mixed-methods approach. This dissertation aims to provide a better understanding of in-service elementary teachers’ knowledge in Saudi Arabia, especially on the topic of fractions.
CHAPTER 3
METHODOLOGY

This study seeks to assess and understand upper-level elementary (grades 4-6) in-service teachers’ mathematical knowledge for teaching in Saudi Arabia in the content strand of rational numbers with an emphasis on fractions. I first explain the Saudi educational system and Saudi teachers in order to provide the context of the study. Next, I elaborate on the instruments and data collection procedures that were used in this study. Finally, I describe how the mixed-methods approach was used to analyze the data to answer the research questions.

The Setting

Saudi Vision 2030, the recent reform effort in Saudi Arabia that has been implemented since 2016, calls for the cultivation of a new generation that will build a strong and diversified economy. Achieving this goal will depend on the success of education by leveraging a broad range of skills and competencies. Educational institutions have a critical role to play in building the capacities the future will require to achieve the goal of the Vision. The Ministry of Education in Saudi Arabia is currently undertaking reform efforts within the educational system that require understanding and improving all aspects of instruction, including teachers’ mathematical knowledge for teaching.

Teacher Population and Sample in Saudi Arabia

The population of this study is elementary school teachers in Alahsa City that lies in eastern Saudi Arabia. The city area represents approximately 25% of the total area of Saudi Arabia and it is the largest city in the eastern region of Saudi Arabia. Alahsa has a population
exceeding 1 million, and Saudi Arabia has a total population exceeding 33 million, with more than 500,000 teachers. Alahsa was chosen as the site for this study because the government has recently paid a great deal of attention to developing this city due to its economic importance. In Saudi education, most teachers have at least a bachelor’s degree. Teachers with a major in mathematics or mathematics education can teach any grade level. Elementary grade levels in Saudi Arabia include first to sixth grades. Mathematics teachers in elementary schools can teach lower-level grades, upper-level grades, or both levels. This study focused on female Saudi elementary mathematics teachers. Male teachers are excluded from this sample because the study’s qualitative design requires direct interviews; Saudi Arabian schools are segregated by gender, and researchers cannot visit opposite-gender schools.

The data were gathered from upper elementary teachers (grades 4-6) in Alahsa. My pilot study influenced the selection of the sample for this study. The pilot study included teachers who taught lower- and/or upper-level grades, but the teachers who taught only lower-level grades found it difficult to completing interview tasks. One lower-level teacher (grades 1-3) was uncomfortable completing an interview task because her preparation and teaching experience were insufficient to complete the tasks with the specific content for students in grades 4-6, although her overall knowledge level, determined by an MKT measure, was above average. Therefore, I focused on upper-grade teachers. For the first phase of the study, the sample of 44 female elementary school teachers was selected from those who were regularly teaching or had taught grades 4-6. Then, for the next phase, 12 participants were purposefully selected from the sample of 44 teachers in order to have “information-rich cases” (Patton, 2015, as cited in Merriam & Tisdell, 2016, p. 96). Purposeful sampling allowed me to select the participants and the research site that helped to answer the research questions.
**Instruments**

This study used three kinds of instruments: (1) a mathematical knowledge for teaching (MKT) measure—a version for elementary teachers’ content knowledge of rational numbers, (2) a survey of teachers’ confidence in their knowledge preparation needed for teaching elementary mathematics and their beliefs regarding mathematics teaching and learning, and (3) teaching-scenario task-based interviews. All instruments were designed in English and translated into the Arabic language; the accuracy of the translation was checked by one Arabic person fluent in both languages, with a bachelor’s degree in English in the United States.

**MKT Measure**

MKT measures were developed by Ball and her colleagues in order to assess teachers’ mathematical knowledge for teaching in the Learning Mathematics for Teaching Project at the University of Michigan (Charalambous, 2010). This study used grades 4-8 Rational Number Content Knowledge 2008 Form B. The assessment includes 16 multiple-choice items focusing on grades 4 through 8 and on rational numbers content knowledge. The particular measure consists of 10 items related to fractions, four items on decimals, and two items about percentages. They are different in degree of difficulty, ranging from –3.160 to 0.799 (easy items overall). This measure was translated and adapted to the educational context of Saudi Arabia, following the categories of changes for translating the MKT instrument into another language by Delaney, Ball, Hill, Schilling, and Zopf (2008). These categories aim to maintain the validity of items, consisting of three aspects (a cultural context, school context, and mathematical substance). Changes made are related to the general cultural context, including change of people’s names and units, such as cents to halala. One person fluent in both the Arabic and English languages validated the translation of the MKT instrument.
Survey on Confidence and Beliefs

The survey on teacher confidence and beliefs was adapted from Li and Huang’s (2008) mathematical knowledge survey, including items from TIMSS 2003 background questionnaires and some items regarding the content of fraction division. This survey was adapted to include items on the topic of fractions rather than fraction division alone. This survey focuses on teachers’ confidence in their knowledge preparation needed for teaching elementary school mathematics in general, and rational numbers specifically. It also includes items related to mathematics teaching and learning in general. I excluded the part of the survey related to teachers’ knowledge about curriculum and grade placement of mathematics content that they teach, because Saudi Arabia has one official textbook series to use and this study does not intend to assess teachers’ knowledge in relation to curriculum. This survey was written in a tabular format rather than a bulleted matching format (see Appendix B for both the English and Arabic versions). Again, the accuracy of the translation was checked by a person fluent in both Arabic and English.

Pre-Interview Tasks

The decision to develop this instrument was made after the pilot study. The data for this dissertation study were collected in the first semester (fall semester) of school year, the same period in which I did the pilot study, and the topic of fractions is introduced in second semester (spring semester). During the pilot, one teacher said she taught this topic the year before and she needed to refresh her memory. Therefore, the pre-interview prompts were developed to remind teachers of the topic and to prepare them to talk about operations with fractions in the interview. The pre-interview tasks are distinguished from the actual interview protocol. They are grouped in two parts: the first part includes a set of story problems asking teachers to write expressions (e.g.,
that can be used to solve these story problems. The second part consists of a set of diagrams and a set of equations, asking teachers to choose the appropriate equation that can be represented by the diagram. These tasks focus on fraction multiplication and division (see Appendix C for the English and Arabic versions).

**Interview Protocol**

An interview protocol was developed based on various literature (e.g., new Saudi Arabia textbooks adapted from Math Connects [Ministry of Education in Saudi Arabia, 2014]; Empson & Levi, 2011; Ma, 1999). The initial version included three sets of tasks regarding fraction concept and comparing fraction, fraction multiplication, and fraction division. This version was tested with a Saudi elementary teacher. This interview protocol seemed too long, and the teacher needed extensive time to think about the tasks. Based on this test-interview with the teacher, the interview protocol was revised to include two sets of tasks (see Appendix D for the English and Arabic versions). The first set related to fraction division and the second set to fraction multiplication. Each set consists of two parts, and each part consists of two tasks. Part one of each set involves solving a multiplication or division problem, given in mathematical notation, and generating a story problem based on this equation. Part two includes solving a given story problem by using a representation and interpreting student work on this story problem. The first part of the first set is adapted from Ma (1999). I modified the task by asking the participant to generate a story problem, rather than generate a story problem or model, as in Ma (1999). I wanted to focus this task only on generating a story problem; the next task in the second part focuses on solving a story problem by using representation. Whether teachers can or cannot generate a story problem, they are asked to solve the given story problem in the next task. The first part of the second set (i.e., multiplication with fractions) resembles the first part of the first
set (i.e., division with fractions), but with a different operation, which is fraction multiplication. The second part of both sets includes a story problem and asks teachers to interpret student work on the story problem. Student work used in the interview protocol was adapted from Empson and Levi (2011).

**Data Collection**

Two approval processes are required for collecting the data in my study. First, I received an approval letter from the Human Subjects Institutional Review Board (HSIRB) at Western Michigan University (WMU). The HSIRB is a group formally designated to protect the rights, safety, and well-being of humans participating in research at WMU. The board reviews all aspects of the research and its appropriateness to study participants, including the risks, benefits, costs, and confidentiality of participants. Second, I received an approval letter from the Ministry of Education in Saudi Arabia. To access the participants, I contacted the Research and Project Administration department at the Ministry of Education in Saudi Arabia and discussed with them how to access participant teachers who would volunteer to share their teaching experiences and to respond to the instruments. They sent emails to teachers and schools in Alahsa to ask them to participate in this study. Then, I received a list of approximately 20 elementary schools along with teachers who were willing to participate in this study.

The data for the study were collected in two phases. In the first phase, the MKT measure and the survey were distributed and completed by 44 Saudi upper-level elementary teachers while I was visiting the schools in Alahsa. At this time, I was able to ask the participants questions about their responses to the survey and establish rapport with them that facilitated the next stage of the data collection. I introduced myself to the participants, explained the dissertation study to them, and sought their oral approval for their involvement as well as
determined their preferred time and place for an interview. Access and rapport are important factors that researchers should consider because of their influence on data collection (Creswell & Poth, 2018).

The MKT measure allowed me to answer the first research question about the level of Saudi teachers’ mathematical knowledge for teaching. Also, the survey allowed me to answer the second research question about teachers’ confidence in their knowledge preparation that is needed for teaching elementary mathematics in general, and teaching fractions specifically, and their beliefs about mathematics teaching and learning.

Teachers’ MKT scores were used to select teachers with different MKT levels for the next phase. In the beginning, I selected nine teachers who differed in their MKT levels (relatively high, middle, and low)—three teachers in each level. After I selected the nine teachers, I sent a text message to them and asked for an interview. Some of them declined to participate in the interview, so I chose other participants from the sample of 44 teachers until I obtained consent from 12 teachers who differed in their MKT levels (3 high, 4 middle, and 5 low).

Before the interviews, I contacted the 12 selected teachers about completing pre-interview tasks. This instrument was sent to the 12 teachers two or three days before I met with them. The pre-interview tasks were used to prepare teachers for the interview by asking them in advance to think about fraction operations in contexts and with representations.

In the second phase, one-on-one task-based interviews were conducted to further account for Saudi teachers’ mathematical knowledge for teaching fractions. Each teacher was interviewed for approximately 45 minutes. The interviews were captured verbally with audio recordings. The recording method was chosen based on the acceptance of the Ministry of
Education and participants’ consent, although video recording was preferred. After each interview, the interview audios were transcribed in Arabic and worthy parts of the transcripts were translated to English. The translated interviews were checked for accuracy again by the same person who verified the translation of instruments. These interviews provided insight into Saudi teachers’ mathematical knowledge for teaching fraction multiplication and division and their ability to interpret students’ work. Also, the interviews helped to collect additional demographic information about the 12 teachers, including what type of professional development programs they attended, and how and when they developed their knowledge of alternative strategies, representation, and generating story problems.

Data Analysis

The data collected through the three types of instruments were analyzed using a mixed-methods approach. The MKT measure and the survey for the first and second research questions, respectively, were analyzed using a quantitative approach. The interviews that helped answer the third research question were analyzed using a qualitative approach. The fourth research question was answered by comparing and contrasting the data from all three instruments.

Analysis of the MKT Measure

Teacher responses to the MKT measure were checked for correctness, and then individual teachers’ Item Response Theory (IRT) scores were determined based on the scaled score table provided with the measure. IRT scores are linear, ranging approximately from –3 to 3 with the mean of 0. The participant teachers were grouped into three categories based on their IRT scaled scores. Teachers with IRT scores between 0 and 0.5 were considered as having a relatively high level of MKT among the participants, teachers with IRT scores between –0.5 and 0 (a little below the mean score of 0) were considered as the middle level of MKT, and teachers
with IRT scores lower than –0.5 were considered as having a low level of MKT. Because the MKT measure includes items in a range of difficulty levels, teachers’ responses were checked in terms of item difficulty level to look for trends in teacher responses for easy, medium, and relatively difficult items.

**Analysis of the Survey**

Teachers’ responses to the confidence and beliefs survey were compiled to calculate the frequencies and percentages of each category and determine overall confidence and beliefs for each teacher. This allowed me to determine the teachers’ confidence about their knowledge preparation needed to teach elementary school mathematics, specifically, fractions. Also, the survey responses helped identify teacher beliefs in certain aspects of teaching, such as using representations and story problems in teaching mathematics.

**Analysis of the Interviews**

Teacher responses to each of the tasks in each set were analyzed based on their approach, accuracy, and reasonableness. Then, I looked for the connection among four tasks in each set within each teacher and across all teachers. The first task was analyzed based on teachers’ strategies to solve the problem, such as a standard method (formal strategy) and an alternative (informal) strategy. An example of a standard method for division with fractions is to invert and multiply (e.g., 2/3 ÷ 3/4 = 2/3 × 4/3). Examples of alternative strategies for division with fractions include common denominator (e.g., 3/2 ÷ 4/3 = 9/6 ÷ 8/6 = 9÷8 ), repeated subtraction (e.g., 4 ÷ 2/3 = 4 – 2/3 – 2/3 –……2/3 = 0), using decimals (e.g., 1¼ ÷ ½ = 1.75 ÷ 0.5), using a unit rate (e.g., 5/3 ÷ 1/2 = 5 × 2/3 × 1 = 10/3), applying the distributive law (e.g., 1¾ ÷ ½ = (1 + 3/4) ÷ ½ = (1 ÷ ½) + (¾ ÷ ½)), and dividing numerators and denominators (e.g., 2/9 ÷ 1/3 = 2 ÷ 1/9 ÷ 3 = 2/3) (Ma, 1999; Son & Crespo, 2009). An example of a standard method to solve
multiplication fraction is multiplying numerators and denominators (e.g., $2/3 \times 4/3 = 2 \times 4/3 \times 3$). Examples of alternative strategies to solve multiplication fraction are applying the distributive law (e.g., $1\frac{3}{4} \times \frac{1}{2} = (1 + \frac{3}{4}) \times \frac{1}{2} = (1 \times \frac{1}{2}) + (\frac{3}{4} \times \frac{1}{2})$) and partitioning composite (e.g., $2/3 \times 3/5 = 2/5$) (Mack, 2001). I looked for what common strategies Saudi elementary teachers used and whether they were able to solve the problem to find the answer.

In the second task of each set, story problems generated by the teachers were analyzed based on reasonableness and type of story problem. I examined whether the story problems were plausible by focusing on the context and the numbers—whether they were appropriate to present the meaning of the given fraction operation. Also, fraction division story problems were analyzed in terms of two types: measurement division (i.e., group size known and number of groups unknown) and partitive division (i.e., number of groups known and group size unknown). Fraction multiplication story problems were analyzed based on two different meanings of equal groups, and area or array.

The third task of each interview set was analyzed based on the appropriateness of representations generalized by the teachers. I looked for whether the teachers could produce an appropriate representation, whether this representation accurately presented the meaning for the story problems, and whether the teachers were able to solve the story problem by using the representation they generated. Also, the representations were categorized based on types of visual model or diagram, such as an array, area, and number line.

In the fourth task of each set, teachers’ interpretation of student work was analyzed based on three criteria: validity, generalizability, and efficiency. These criteria were articulated by Campbell, Rowan, and Suarez (1998) to help teachers to support students’ invented strategies. Validity indicates that teachers consider whether or not students’ invented strategy works in a
given problem. Generalizability indicates that teachers examine whether or not students’
invented strategy is reasonable to use in any problem of the type. Efficiency means that teachers
can provide guidance about efficiency of the strategy (Son & Crespo, 2009). They are able to
evaluate and determine whether their students’ strategy is reasonable to use in various situations
and could be taught to students. Also, teachers’ interpretation was analyzed based on the two
types of evidence they provided—procedural interpretation and/or conceptual interpretation.

Finally, I looked for similarities and differences across all teachers for each task. For
example, what were common strategies Saudi elementary teachers used to solve division and
multiplication with fractions? Were they able to generate an appropriate story problem and a
representation? And, what is a common type of story problem or representation they produced?
Then, I looked for the connection among these four tasks for each teacher. Was there a common
pattern? For example, was there a relationship among teachers’ ability to solve the problem by
using a strategy (traditional strategy and alternative strategy), their ability to generate a
reasonable story problem, their ability to generate an appropriate representation, and their
interpretation of students’ work? I also looked for patterns across teachers in the relationship
among different aspects, especially patterns in the three different groups of MKT levels.

After I analyzed each set of data from the three instruments to answer the first three
research questions, I looked for the relationship, if any, among these three sets of data to answer
the fourth research question. I looked for whether there is the relationship among Saudi teachers’
mathematical knowledge for teaching that was assessed by the MKT measure; teacher
confidence in their preparation for teaching elementary school mathematics in general and
teaching fractions specifically, and their beliefs about mathematics teaching and learning that
were assessed by the survey; and their responses to teaching scenario-based tasks on two
different operations with fractions. The interview tasks helped identify significant examples in same or different levels of teachers’ knowledge assessed by the MKT measure. I described two or three interesting teacher examples in different MKT levels and within the same MKT levels—how the teachers within each MKT level and across different MKT levels responded to interview tasks and the survey. Also, the additional demographic information that I collected in the interviews helped me infer potential factors that might have influenced the teachers’ responses to interview tasks.

Trustworthiness

During this research, several strategies discussed by Creswell and Poth (2018) and Merriam and Tisdell (2016) were used to ensure the trustworthiness of this study. These strategies include careful record keeping, accurate translation, rich descriptions, triangulation of data, and clarifying researcher bias. Throughout the data collection and analysis, I kept careful records and then ensured careful translation to maintain accuracy. Merriam and Tisdell stated that the methods of cautious recording and rich description can be used by researchers to establish the credibility and quality of the study. Also, data collected were triangulated to confirm emerging findings of Saudi elementary teachers’ mathematical knowledge for teaching. Triangulation of data is a way to support the validity and reliability of the study (Creswell & Poth, 2018).

Because the setting of this study is my hometown, Alahsa, in Saudi Arabia, where I received my education, the participant teachers and I are from the same culture, which helped me in several respects. In selecting the interview tasks, I was able to identify the context that is appropriate for Saudi teachers. Through the interviews, I was able to understand the participants and ask them to clarify their answers, if needed, during the interviewing. The awareness of my
biases and identity and understanding about Saudi teachers’ education and culture supported me in avoiding any negative impact on the analysis of the data and interpretation of the findings. My identity has been shaped initially by my own learning experiences in schooling and through my academic journey in master’s courses in Saudi Arabia and Ph.D. courses in the United States. To manage my personal bias, I took a neutral stance. I did not let my prior knowledge about Saudi teachers influence the assessment of teachers’ knowledge through collecting and analyzing data. Also, to convey the participant teachers’ voice/thinking, I provide specific direct quotations from their responses in describing my findings.
CHAPTER 4
RESULTS

The data for this exploratory and mixed-method study were collected and analyzed sequentially, as described in Chapter 3. The results will be described in three chapters—this chapter and the next two chapters. The first stage of the study, elaborated in this chapter, focused on quantitative data collection and analysis using one form of MKT measures and the confidence and beliefs survey described in Chapter 3. The second stage was qualitative analysis using interview data (see Chapter 5). Lastly, the results from both stages were integrated to find the overall trends and the relations in Saudi elementary teachers’ knowledge for teaching mathematics (see Chapter 6). In the three chapters (4, 5, and 6), I describe the findings regarding the four research questions asked in this study. In this chapter, I elaborate on the emergent findings from the Saudi teachers’ MKT scores, and their confidence and beliefs about mathematics teaching and learning.

MKT Measure Results

Participating Teachers’ MKT Levels

The MKT measure for content knowledge of rational numbers, including fractions, decimals, and percentages, was used in this study to address the first research question: What is the level of Saudi elementary mathematics teachers’ knowledge for teaching, assessed through a mathematical knowledge for teaching (MKT) measure on rational numbers? The number of correct answers (raw scores) and Item Response Theory (IRT) scale scores are summarized in Table 4.1 for each participating teacher. The 44 participating teachers’ raw scores ranged from 6
to 20 correct answers out of 32 items in total (see Table 4.1 and Figure 4.1). The mean and median of their correct responses were 14, and the mode was 14 and 17 (bimodal). The teachers’ IRT scores were from –1.854452 to 0.18027 (see Table 4.1). The result shows most of the teachers (91%) were below the IRT scaled score of 0 (the average teacher score of the norm group), which equated to roughly 19 correct responses out of 32 items. These results reveal the weakness in Saudi in-service elementary teachers’ mathematical knowledge for teaching, specifically teaching rational numbers.

Table 4.1. Teachers’ Scores on MKT Measure

<table>
<thead>
<tr>
<th>Teacher</th>
<th># of correct answers</th>
<th>IRT score</th>
<th>Teacher</th>
<th># of correct answers</th>
<th>IRT score</th>
</tr>
</thead>
<tbody>
<tr>
<td>LA</td>
<td>20</td>
<td>0.18027</td>
<td>AS</td>
<td>14</td>
<td>-0.652215</td>
</tr>
<tr>
<td>SR</td>
<td>19</td>
<td>0.038132</td>
<td>GA</td>
<td>14</td>
<td>-0.652215</td>
</tr>
<tr>
<td>MM</td>
<td>19</td>
<td>0.038132</td>
<td>ZD</td>
<td>14</td>
<td>-0.652215</td>
</tr>
<tr>
<td>SM</td>
<td>19</td>
<td>0.038132</td>
<td>MH</td>
<td>14</td>
<td>-0.652215</td>
</tr>
<tr>
<td>FR</td>
<td>18</td>
<td>-0.101753</td>
<td>FT</td>
<td>13</td>
<td>-0.790757</td>
</tr>
<tr>
<td>SA</td>
<td>18</td>
<td>-0.101753</td>
<td>TH1</td>
<td>13</td>
<td>-0.790757</td>
</tr>
<tr>
<td>HY</td>
<td>18</td>
<td>-0.101753</td>
<td>FA2</td>
<td>13</td>
<td>-0.790757</td>
</tr>
<tr>
<td>SH</td>
<td>17</td>
<td>-0.240082</td>
<td>DA</td>
<td>13</td>
<td>-0.790757</td>
</tr>
<tr>
<td>ZA</td>
<td>17</td>
<td>-0.240082</td>
<td>AS</td>
<td>12</td>
<td>-0.930975</td>
</tr>
<tr>
<td>MA</td>
<td>17</td>
<td>-0.240082</td>
<td>RA</td>
<td>12</td>
<td>-0.930975</td>
</tr>
<tr>
<td>SS</td>
<td>17</td>
<td>-0.240082</td>
<td>AZ</td>
<td>11</td>
<td>-1.073587</td>
</tr>
<tr>
<td>ZB</td>
<td>17</td>
<td>-0.240082</td>
<td>BN</td>
<td>11</td>
<td>-1.073587</td>
</tr>
<tr>
<td>RO</td>
<td>17</td>
<td>-0.377512</td>
<td>HN</td>
<td>11</td>
<td>-1.073587</td>
</tr>
<tr>
<td>KH</td>
<td>16</td>
<td>-0.377512</td>
<td>RH</td>
<td>11</td>
<td>-1.073587</td>
</tr>
<tr>
<td>LL</td>
<td>16</td>
<td>-0.377512</td>
<td>HO</td>
<td>11</td>
<td>-1.073587</td>
</tr>
<tr>
<td>ZC</td>
<td>16</td>
<td>-0.377512</td>
<td>RE</td>
<td>10</td>
<td>-1.219381</td>
</tr>
<tr>
<td>AZ</td>
<td>16</td>
<td>-0.377512</td>
<td>EL</td>
<td>10</td>
<td>-1.219381</td>
</tr>
<tr>
<td>MN</td>
<td>16</td>
<td>-0.377512</td>
<td>TH2</td>
<td>9</td>
<td>-1.369244</td>
</tr>
<tr>
<td>FA1</td>
<td>15</td>
<td>-0.51468</td>
<td>RY</td>
<td>9</td>
<td>-1.369244</td>
</tr>
<tr>
<td>AM</td>
<td>15</td>
<td>-0.51468</td>
<td>HN</td>
<td>8</td>
<td>-1.524199</td>
</tr>
<tr>
<td>AA</td>
<td>14</td>
<td>-0.652215</td>
<td>HA</td>
<td>6</td>
<td>-1.854452</td>
</tr>
<tr>
<td>ZH</td>
<td>14</td>
<td>-0.652215</td>
<td>NO</td>
<td>6</td>
<td>-1.854452</td>
</tr>
</tbody>
</table>

Note. Teachers are designated by pseudo-initials.
**Figure 4.1.** Teachers’ scores on MKT measure.

**Item Difficulty and Teacher Responses**

The MKT measure on rational numbers used in this study included items in a range of difficulty levels distributed from $-3.160$ to $0.799$. These items covered different concepts and properties of rational numbers, such as comparing fractions, equivalent forms, and word problems. The most difficult item, choosing a word problem that represents the subtraction fraction equation (difficulty level of 0.799), was correctly answered by only 11% of the participant teachers. The easiest item, choosing the expressions that are equivalent forms of the same number (difficulty level of $-3.160$), was correctly answered by 86% of the participating teachers. Almost all of the participating teachers (98%) responded correctly to the item on comparing fractions (difficulty level of $-1.175$). Also, there are items about interpretation of division with fractions that have different difficulty levels. The hardest one (difficulty level of 0.579) was correctly answered by 25% of the participating teachers, and the easiest one (difficulty level $-1.926$) was correctly answered by 64% of the participating teachers. These results reveal Saudi in-service teachers’ weakness in word problems and interpretation of fraction operations and strength in equivalent forms of same number and comparing fractions. In fact, the easiest item for the participating teachers (comparing fractions) was not the easiest for the norm group, as shown above. Also, whereas the item on division with fractions (difficulty
level of −1.926) was easier than the item on comparing fractions (difficulty level of −1.175), according to the norm group, the Saudi teachers performed much better on the item on comparing fractions.

The MKT measure included 15 items about fraction concepts and operations with fractions in different contexts, representations, and interpretations of operations or students’ work. It seems that the participating teachers did better on items using symbols only, and they had difficulty answering the items related to representations, interpretation of concepts or students’ work, and reasons for students’ errors. For example, 16% of the participating teachers answered correctly the item about representations of a proper fraction. As anticipated, the number of teachers who gave correct answers decreased as the difficulty level of items increased overall. For example, about 70% of participating teachers answered correctly the easiest word problems and just 11% of them answered correctly the hardest one.

Clusters of Teachers Based on MKT Scores

The participating teachers’ scores on the MKT measure were used to identify the target teachers for interviews. For the purpose of further analysis of the Saudi teachers’ mathematical knowledge, I clustered the teachers into three groups (high, middle, low) based on their MKT IRT scores, as described in Chapter 3. Based on this categorization, 55% of the participating teachers were at the lower level of MKT (below the score of −0.5), 36% at the middle level (scores between −0.5 and 0), and just 9% at the high level of MKT (above the score of 0). Overall, about half of the participating Saudi in-service teachers were around the average IRT score (between −0.5 and 0.5) (see Figure 4.2).
Correlation Among Teachers’ MKT, Teaching Experience, and Professional Development

To examine the correlation between teachers’ MKT level and the years of teaching experience and professional development programs, all data were transferred to SPSS. Statistical analyses show that the years of teaching experience and the number of professional development programs the participating teachers attended had essentially no effect on their level of MKT. Correlation analysis (linear regression) was conducted to determine possible correlations among these variables. Examining the scatterplot (Figure 4.3), it seems that teachers with fewer years of teaching experience have better mathematical knowledge for teaching than teachers with more years of teaching experience. However, when examining the correlation between scores of MKT and years of teaching experience, no statistically significance in the correlation ($p > 0.05$) between these two variables ($r = –0.205$, $p = 0.183$) was found. Also, correlation analysis shows no significant correlation between scores of MKT and participation in the number of professional development programs the teachers attended ($r = –0.154$, $p = 0.320$, $p > 0.05$) (see Figure 4.4).
These results indicate that years of teaching experience and the number of professional development programs attended did not have a relationship with the participant teachers’ knowledge for teaching mathematics.

Figure 4.3. Relationship between teachers’ MKT level and years of teaching experiences.

Figure 4.4. Relationship between teachers’ MKT level and professional development programs.
The Confidence and Beliefs Survey Results

To address the second research question (How confident are Saudi elementary mathematics teachers about their mathematical knowledge for teaching in general, and fractions specifically? What are their beliefs about mathematics teaching and learning?), the Confidence and Beliefs Survey was used. This survey included items related to teachers’ confidence in their knowledge preparation for teaching elementary mathematics, specifically, fractions, and items regarding teachers’ beliefs in certain aspects of teaching and learning mathematics.

Participating Teachers’ Confidence in Teaching Mathematics

To study teachers’ confidence in teaching mathematics, teachers were asked about their readiness for teaching mathematics in general and teaching specific content areas. The analysis of teachers’ responses to the confidence items (see Figure 4.5) shows that 59% of the participating teachers were very confident in their knowledge needed for teaching elementary mathematics in general, whereas 36% were very confident in their knowledge for teaching rational number in general. In contrast, 36% of the participating teachers were confident in their knowledge needed for teaching elementary mathematics in general, and 57% were confident in their knowledge for teaching rational number in general. Also, 5% of the participating teachers were not confident in their knowledge needed for teaching elementary mathematics in general, and 7% were not confident in their knowledge for teaching rational number in general.
Figure 4.5. Percentages of teachers’ confidence in their preparation to teach mathematics.

Regarding the selected content topics of the four operations with fractions, the survey results revealed two particular aspects. First, the participating teachers seemed more confident in explaining the operations with fractions than representing the operations with fractions (see Figure 4.6). For example, 55% of the participating teachers were very confident in explaining addition with fractions by using words, numbers, and models, but 48% were very confident in their ability to represent addition with fractions by using models.

Second, the participating teachers were more confident in their knowledge needed for teaching addition and subtraction than teaching multiplication and division with fractions (see Figure 4.6). About 50% of the teachers showed high confidence in explaining addition and subtraction with fractions, whereas 36% were very confident in explaining multiplication and division with fractions. Also, about 50% of teachers showed high confidence in their ability to represent addition and subtraction with fractions, whereas 34% of them were very confident in their ability to represent multiplication and division with fractions.
Figure 4.6. Percentages of teachers’ confidence in their preparation to teach specific topics.

In addition, the analysis of the survey data indicates that approximately half of the participating teachers rated themselves as having high understanding of the mathematical knowledge for teaching fractions. Precisely 80% of the teachers with an average level of MKT (IRT scores between −0.5 and 0.5) perceived themselves as having a high level of mathematical knowledge for teaching fractions, whereas 54% of the teachers at the below average level of MKT (IRT scores lower than −0.5) perceived themselves as having a middle level of mathematical knowledge for teaching fractions (see Figure 4.7).
Participating Teachers’ Beliefs About Teaching and Learning Mathematics

The results indicate that the participating teachers seemed to hold common beliefs on certain aspects of teaching and learning mathematics (e.g., a strong agreement on some statements and a strong disagreement on other statements) (see Figure 4.8). All the teachers agreed (or agreed a lot) that more than one representation should be used in teaching (Statement 1), that modeling real-world problems is essential in teaching mathematics (Statement 9), and that teachers need to know students’ common misconceptions (Statement 4). Also, most teachers agreed (or agreed a lot) that mathematics should be learned as sets of algorithms or rules (Statement 3, 88%), and disagreed (or disagreed a lot) that learning mathematics mainly involves memorizing (Statement 7, 86%).
Figure 4.8. Percent of teachers’ responses related to their beliefs on mathematics teaching and learning.

On the other hand, the results show that the participating teachers seemed to hold conflicting beliefs on other certain aspects of teaching and learning mathematics. For example, the participating teachers’ beliefs about preventing students from making errors in their learning of mathematics (Statement 5) varied (23% agree a lot, 32% agree, 36% disagree, 9% disagree a lot). On one hand, their beliefs coincide with reform efforts in school mathematics; for example, most teachers agreed on using manipulatives to avoid abstract mathematics (Statement 2, 95%) and disagreed on learning mathematics by memorizing (Statement 7, 86%). On the other hand, teachers still kept a traditional view of school mathematics, as reflected in their agreement with the statement that mathematics should be learned as sets of algorithms or rules (Statement 3, 88%).
Although the participating teachers differ in their MKT levels, they hold similar beliefs regarding many aspects of teaching and learning mathematics. In both groups of teachers with an average level of MKT (IRT scores between −0.5 and 0.5) and those with a below average level of MKT (IRT scores below −0.5), the teachers had a strong belief about using more than one representation in teaching a mathematics topic (see Figure 4.9), and they agreed that mathematics should be learned as sets of algorithms or rules (see Figure 4.10).

Figure 4.9. Using more than one representation in teaching a math topic.
Figure 4.10. Learning math as set of algorithms or rules.

The results described in this chapter provide an overall picture of the 44 Saudi elementary teachers’ knowledge for teaching rational numbers, and their confidence and beliefs in mathematics teaching and learning. In order to explain in depth the nature of the teachers’ knowledge, the next chapter provides details about the Saudi elementary mathematics teachers’ content knowledge for teaching division and multiplication with fractions in particular, using the task-based interviews with the participants.
CHAPTER 5

TEACHERS’ REASONING ON DIVISION AND MULTIPLICATION WITH FRACTIONS

In this chapter, I elaborate on the results from the interviews with 12 of the 44 participant teachers—teacher responses to the tasks in the interview protocol across all teachers—to address the third research question: How do Saudi elementary mathematics teachers reason about tasks on multiplication and division with fractions? The interviews included two sets of tasks, one on division and the other on multiplication. Each set included solving a division or multiplication problem, generating a story problem, using a representation to solve a story problem, and interpreting student work.

Overall, the interview results reveal that the participant Saudi in-service elementary teachers had strengths in their procedural knowledge of operations with fractions—in division with fractions and multiplication with fractions—and weaknesses in their conceptual knowledge of these two operations with fractions. Moreover, their conceptual understanding of fraction division is weaker than their conceptual understanding of fraction multiplication. Despite teachers’ strong proficiency in solving fraction division and multiplication problems by using standard methods, a significant number of teachers were unable to pose an appropriate story problem for given fraction operations, specifically, fraction division. Although most teachers were able to recognize proper operations with fractions for given story problems and generated an appropriate representation underlying the meaning of the story problem, many of them had difficulty using their representations to solve the story problem. Also, the interview results reveal that the teachers’ MKT scores were related to their ability to generate a story problem for
fraction division, to use their representations effectively to solve the story problem involving fraction division or fraction multiplication, and to interpret students’ alternative strategies about fraction division. Appendix E summarizes the interview results by teacher, and the following two sections explain the results in more detail.

**Teachers’ Reasoning on the Fraction Division Tasks**

This section includes three subsections that detail the Saudi elementary mathematics teachers’ reasoning and thinking on division with fractions. Table 5.1 gives a brief summary of the participating teachers’ performance on three interview tasks about division with fractions in two different MKT levels: around the average (i.e., IRT scores between –0.5 and 0.5) and below the average (IRT scores below –0.5), as described previously.

Table 5.1. Summary of Teacher Responses to the Fraction Division Tasks

<table>
<thead>
<tr>
<th>Approaches to solving $1 \frac{3}{4} \div \frac{1}{2}$</th>
<th>Teachers around the average level of MKT (7 teachers)</th>
<th>Teachers below the average level of MKT (5 teachers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Valid standard method (7 teachers)</td>
<td>• Valid standard method (5 teachers)</td>
<td></td>
</tr>
<tr>
<td>• Informal strategy (4 teachers)</td>
<td>• Informal strategy (2 teachers)</td>
<td></td>
</tr>
<tr>
<td>o Valid distributive law (2 teachers)</td>
<td>o Valid representation (1 teacher)</td>
<td></td>
</tr>
<tr>
<td>o Valid representations (2 teachers)</td>
<td>o Incomplete representation (1 teacher)</td>
<td></td>
</tr>
<tr>
<td>o Incomplete representation (1 teacher)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Generating a story problem for $1 \frac{3}{4} \div \frac{1}{2}$</th>
<th>Teachers around the average level of MKT (7 teachers)</th>
<th>Teachers below the average level of MKT (5 teachers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Valid measurement division story problem (6 teachers)</td>
<td>• Valid measurement division story problem (1 teacher)</td>
<td></td>
</tr>
<tr>
<td>• Invalid story problem</td>
<td>• Invalid story problem</td>
<td></td>
</tr>
<tr>
<td>o Multiplying by 2 (1 teacher)</td>
<td>o Dividing by 2 (4 teachers)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Representation of a story problem involving $6 \frac{3}{4} \div \frac{3}{5}$</th>
<th>Teachers around the average level of MKT (7 teachers)</th>
<th>Teachers below the average level of MKT (5 teachers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Appropriate representation (7 teachers)</td>
<td>• Appropriate representation (4 teachers)</td>
<td></td>
</tr>
<tr>
<td>• Solving the problem</td>
<td>• Solving the problem</td>
<td></td>
</tr>
<tr>
<td>o Using the representation effectively (5 teachers)</td>
<td>o Using the representation effectively (1 teacher)</td>
<td></td>
</tr>
<tr>
<td>o Solving symbolically with the representation (1 teacher)</td>
<td>o Solving symbolically with the representation (2 teacher)</td>
<td></td>
</tr>
<tr>
<td>o Using invert and multiple strategy (1 teacher)</td>
<td>o Using invert and multiple strategy (1 teacher)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>o Incorrect solution (1 teacher)</td>
<td></td>
</tr>
</tbody>
</table>
Approaches to Solving $1\frac{3}{4} \div \frac{1}{2}$

The teachers’ responses to the task indicate the strength in using procedures of fraction division and the weakness in their conceptual knowledge of division with fractions. All 12 participating teachers demonstrated proper procedural knowledge of how to calculate the division with fractions by using a standard method, i.e., invert and multiply. Teacher responses to this task shown in Table 5.1 are summarized in Table 5.2 again, with teacher initials and more detail.

Table 5.2. Summary of Teacher Responses to Solving $1\frac{3}{4} \div \frac{1}{2}$

<table>
<thead>
<tr>
<th>Teachers around the average level of MKT (7 teachers)</th>
<th>Teachers below the average level of MKT (5 teachers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Valid standard method (7 teachers)</td>
<td>• Valid standard method (5 teachers)</td>
</tr>
<tr>
<td>• Informal strategy</td>
<td>• Informal strategy</td>
</tr>
<tr>
<td>o Valid distributive law (2 teachers: LA, SH)</td>
<td>o Valid representation (1 teacher: RA)</td>
</tr>
<tr>
<td>o Valid representations (2 teachers: MA, ZH)</td>
<td>o Incomplete solution with representation (1 teacher: AA)</td>
</tr>
<tr>
<td>o Incomplete solution with representation (1 teacher: SH)</td>
<td>o Other ways may be possible but don’t know them (1 teacher: MR)</td>
</tr>
<tr>
<td>o Other ways may be possible but don’t know them (1 teacher: MN)</td>
<td>o No other way (2 teachers: HU, ZM)</td>
</tr>
<tr>
<td>o No other way (2 teachers: SA, AZ)</td>
<td></td>
</tr>
</tbody>
</table>

The 12 teachers verbalized the specific steps of the calculation procedure accurately. Here is a typical example of how teachers described clearly and explicitly the entire process and arrived at the correct answer.

$$1\frac{3}{4} \div \frac{1}{2}$$

First, I convert the mixed number one and three fourths into an improper fraction by using [one of these methods below] $1 + \frac{3}{4} = \frac{4}{4} + \frac{3}{4} = \frac{7}{4}$. 

$$1 + \frac{3}{4} = \frac{4}{4} + \frac{3}{4} = \frac{7}{4}$$
Or
\[
\frac{3}{4} = \frac{(1 \times 4) + 3}{4} = \frac{7}{4}
\]

To divide seven fourths by one half, I would invert \( \frac{1}{2} \) into 2 and multiply
\[
\frac{7}{4} \div \frac{1}{2} = \frac{7}{4} \times 2 =
\]

Now, we can simplify and then multiply
\[
\frac{7}{4} \times \frac{1}{2} = \frac{3}{2}
\]

or multiply and then simplify
\[
\frac{7}{4} \times 2 = \frac{14}{4} = \frac{3}{2}
\]

Six of the 12 teachers who were interviewed also tried to solve the problem by using alternative approaches (informal strategies) in addition to the standard method (invert and multiply) when they were asked to solve it in a different way. Four of these six teachers were around the average score of MKT (IRT scores between –0.5 and 0.5), and two were below the average score of MKT (IRT scores below –0.5). They were able to propose one or two other approaches: using the distributive law or/and representations. All four around the average score of MKT were able to solve the problem correctly by using the distributive law, i.e., \( (1 + \frac{3}{4}) \div \frac{1}{2} = \left( 1 \div \frac{1}{2} \right) + \left( \frac{3}{4} + \frac{1}{2} \right) \) or using a representation; just one teacher below the average score of MKT was able to solve the problem correctly by using a representation.

Two teachers used the distributive law, but they calculated the answer differently. One of them (Teacher LA) explained it symbolically by using the invert-and-multiply method, and the other one (Teacher SH) conceptually by using the meaning of the division, \( 1\frac{3}{4} \div \frac{1}{2} \). Here are the solution processes by the two teachers.
Teacher LA: I separated the mixed numbers into one and three fourths, then divide each by $\frac{1}{2}$

\[
\begin{align*}
1 \div \frac{1}{2} &= 2 \\
\frac{3}{4} \div \frac{1}{2} &= \frac{3}{4} \times 2 = \frac{6}{4} = 1 \frac{1}{2} = 1 \frac{1}{2}
\end{align*}
\]

Then add the two products 2 and $1 \frac{1}{2}$, we get $3 \frac{1}{2}$

Teacher SH: I read it how many $\frac{1}{2}$ in $1 \frac{3}{4}$, how many $\frac{1}{2}$ in 1 and how many $\frac{1}{2}$ in $\frac{3}{4}$. We know that 1 consists of two halves and $\frac{3}{4}$ has one half, and a half of one half, this means we have three halves and a half ($3 \frac{1}{2}$).

When the participant teachers were asked to solve $1 \frac{3}{4} \div \frac{1}{2}$ in a different way, five teachers solved it by using a representation. One teacher represented the problem by drawing seven cubes, and two teachers represented the problem by drawing one and three fourths of pizzas; then they completed the calculation differently. Here are the explanations by the three teachers about how they solved the problem by using the representations appropriately:

Teacher ZH: I represent the problem by using cubes where the 4 cubes represent 1, so 7 cubes represent $1 \frac{3}{4}$. Now I divide [group them] by one half. The 2 cubes represent $\frac{1}{2}$ then I have three halves that are 3 whole number, and one cube is half of one whole, so is one half. The result is $3 \frac{1}{2}$.

Teacher RA: I divided the pizzas into halves, so we get three halves and a half of a half. Because the half considers one whole, the result is three and one half ($3 \frac{1}{2}$).

Teacher MA: the one whole pizza has two halves, and another pizza has $\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times 2 = \frac{6}{4} = 1 \frac{1}{2}$, then the result is 2 and $1 \frac{1}{2}$, which equals $3 \frac{1}{2}$.

Two of the five teachers who drew appropriate diagrams of the problem $\frac{3}{4} \div \frac{1}{2}$ were unable to use their representation appropriately to solve the problem. Teacher AA drew one and three fourths of pizzas, and then she divided the first pizza into two halves and the other pizza
into three quarters. Then she tried to calculate the answer by adding one half of the first pizza to one fourth of the other pizza. Then, she stopped and she said, “I do not know how I get the answer by using this diagram.” Teacher SH drew one and three fourths slices, and she said, “We divide them into halves.” She stopped and she said, “We need to use manipulatives to see how many halves in these slices.” Although, as shown above, Teacher SH was able to solve the problem verbally (without writing anything on paper) and conceptually by using the distributive law, she struggled to solve it by using the representation she drew.

The remaining six teachers’ responses varied when they were asked to solve the problem in a different way. Two teachers said, “The problem could be solved by other strategies, such as using a representation, but I do not know how.” Four teachers said, “There are no other ways to solve the problem. This is the way [invert and multiply] I know, and this is the strategy included in the curriculum.”

Although all participating teachers solved \((1 \frac{3}{4} + \frac{1}{2})\) by the standard method, only six of them were able to solve it in a different way, such as dividing 1 by \(\frac{1}{2}\) and \(\frac{3}{4}\) by \(\frac{1}{2}\) and combining the results, and drawing a diagram (cubes or pizza). Other teachers were not able to come up with another way to solve the problem.

**Generating a Story Problem**

During the interviews, the participant teachers were asked to generate a story problem to make the problem \((1 \frac{3}{4} + \frac{1}{2})\) meaningful for students. It seems that the teachers’ MKT scores were related to their ability to generate an appropriate story problem. The result shows that six of the seven participating teachers around the average score of MKT (IRT scores between –0.5 and 0.5) were able to generate appropriate story problems representing division with fractions, and
just one out of the five teachers below the average score of MKT (IRT scores below –0.5) was able to generate a meaningful story problem. All these teachers generated a measurement division story problem, that is, how many \( \frac{1}{2} \)s are in \( 1 \frac{3}{4} \)? The most common context of the story problems generated was food, specifically pizza shared among people. Teachers realized that it is impossible to get a fraction of a person, so they said, “The answer is 3 friends, and there is a remainder.” Here are the seven teachers’ story problems:

- Maha has \( 1 \frac{3}{4} \) pizza; she wants to divide them for her friends, such that each friend takes \( \frac{1}{2} \) pizza. How many friends will take the pizza?
- Hand buys fabric whose length is one-and-three-quarters meter. How many halves in the fabric’s length? Or, how many halves in one-and three-quarters meter?
- I have one-and-three-quarters cups of flour, and one pizza needs \( \frac{1}{2} \) cup of flour. How many pizzas I can make?
- You buy one-and three-quarters of pizza; you want to give each friend \( \frac{1}{2} \) pizza. How many one-half of pizza will be distributed?
- Fatimah jumps for 1.75 meters. How many jumps did she jump if the length of one jump is 0.50 meter?
- Maryam has one-and-three-quarters of cakes. She wants to divide them into halves [and distribute them among people]. How many people can she distribute the cakes to?
- Norah buys 2 pizzas that cut into 4 pieces. Her brother takes a quarter of one pizza. Now, she distributed the rest among her friends, \( \frac{1}{2} \) each. How many will take the pizza?

The remaining five teachers were not able to come up with a story problem matching the given fraction division problem. One teacher generated a story problem that holds the meaning of multiplying by 2 rather than dividing by \( \frac{1}{2} \).
• Mohamed has $1\frac{3}{4}$ kilograms of dates. If Khaled has twice as much as Mohamed, how many kilograms of dates does Khaled have?

The other four teachers were confused between dividing by $\frac{1}{2}$ and dividing by 2. They generated story problems about dividing $1\frac{3}{4}$ equally into two parts. Here are examples of their story problems:

• You have $1\frac{3}{4}$ of chocolate; you want to divide them into two halves equally.

• Mohamed has $1\frac{3}{4}$ meters of fabric; he wants to divide it into two pieces. What is the length for each piece?

• One-and-three-quarters of cakes; I will divide them into 2 halves. What will be the quantity of the $\frac{1}{2}$?

• I have one whole pizza and another pizza is missing a quarter of it. Could you distribute them into two $\frac{1}{2}$’s equally?

As shown in the first, third, and fourth examples above, the teachers were not clearly indicating appropriate units associated with the quantities/numbers they were using, which indicates their confusion and lack of understanding of division with fractions. After being asked to solve the story problem they generated, they noticed that their story problems were dividing by 2, not dividing by $\frac{1}{2}$. They were able to see that there is a difference between dividing by 2 and dividing by $\frac{1}{2}$, but they struggled to generate a story problem that represented the meaning of dividing by $\frac{1}{2}$. One of these teachers commented after she read her story problem, “No, this story problem is for dividing $1\frac{3}{4}$ by 2, not by $\frac{1}{2}$.” (silence). How do we write a story problem for dividing $1\frac{3}{4}$ by $\frac{1}{2}$?”
Using a Representation to Solve a Story Problem

The teachers were asked to use a representation to solve a story problem involving a partitive division with fractions ($\frac{3}{5}$ of a bag of candy weighs $6\frac{3}{4}$ pounds, how much does 1 bag of candy weigh?). Overall, the results show that the teachers conceptualized the story problem correctly by drawing an appropriate representation, but they had difficulty using their representation to solve the problem. Eleven of the 12 teachers interviewed provided appropriate representations, but only six teachers were able to use their representations to solve the problem. About half of the participating teachers reverted to memorized procedures instead of making sense of the representation they generated when they were solving the story problem. Also, it seems that the teachers’ MKT levels were related to their ability to use a representation to solve the fraction division. Five out of the seven teachers around the average score of MKT were able to use the representations they generated effectively to solve the story problem, and just one out of the five teachers below the average score of MKT was able to solve the story problem by using a representation. The teachers’ responses to this task, shown in Table 5.1, are summarized in Table 5.3 again, with more details and teacher initials.
All the 12 participating teachers recognized the story problem correctly as requiring division with fractions \(6\frac{3}{4} \div \frac{3}{5}\), and 11 of them provided an appropriate representation that could be used to solve the story problem. Here is a typical example of common representation (either vertical or horizontal):

I draw one box of candy and divided it into five parts where each part presents one fifth, and then I color three parts, which presents the three fifths. The weight of the three parts is six and three fourths

![Diagram of candy box divided into five parts, three parts colored, and the weight of three parts shown as 6 3/4]

Although these 11 participating teachers were able to generate an appropriate representation, about half of them (6 out of 11 teachers) were able to use their representation effectively to solve the problem. Here is a typical example of how the teachers used the typical representation above effectively to get the correct answer:
I want to find the weight of the one part that represents one fifth, so I distributed 6 into, 2, 2, and 2 then $\frac{3}{4}$ into $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{4}$.

This means the weight of each part or one fifth is $2\frac{1}{4}$ kg, then:

We add

$2 + 2 + 2 + 2 + 2 = 10$ and $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1\frac{1}{4}$

Then the answer is 10 and $1\frac{1}{4}$ equal $11\frac{1}{4}$ kg

Or we multiply the $2\frac{1}{4}$ by 5,

$2\frac{1}{4} \times 5 = 11\frac{1}{4}$ kg

As shown in the examples above, the teachers represented the meaning of the story problem and understood that they needed to find the weight of the one entire bag. One fifth of a bag is $2\frac{1}{4}$ kg and therefore one bag is five times $2\frac{1}{4}$ kg, which is $11\frac{1}{4}$ kg. Overall, the six teachers knew what they were doing and why, and were able to use their representation to solve the problem.

In contrast, five out of the 11 teachers who were able to generate an appropriate representation were unable to use their representation effectively to solve the problem. They completed the solution for the story problem symbolically with or without using a
representation—invert and multiply. Two teachers were not able to use the representation to illustrate how to solve the story problem, and they reverted to the invert-and-multiply strategy to solve the problem. Three teachers solved the problem symbolically with reversion to the representation to complete the solution. Here is a typical example of how the three teachers completed the solution process symbolically along with a representation to get the correct answer:

I want to find the weight of the one part that represents one fifth, so I divide $6 \frac{3}{4}$ by 3

$$
6 \frac{3}{4} \div 3 = \frac{27}{4} \div 3 = \frac{27}{4} \times \frac{1}{3} = \frac{9}{4} = 2 \frac{1}{4}
$$

This means the weight of one fifth is $2 \frac{1}{4}$

Then we add $2 \frac{1}{4}$ five times or multiply $2 \frac{1}{4}$ by 5

$$
2 \frac{1}{4} \times 5 = \frac{9}{4} \times 5 = \frac{45}{4} = 11 \frac{1}{4}
$$

In the example above, although finding $6 \frac{3}{4} \div 3$ and $2 \frac{1}{4} \times 5$ was done symbolically, the overall process was based on the conceptual meaning of the problem.

Below is an example from a teacher who drew a different type of representation (measurement model for division).

**Teacher ZH:** It is a fraction division problem. We want to find $6 \frac{3}{4} \div \frac{3}{5}$
We draw $6 \frac{3}{4}$, then we bring manipulatives (e.g. magnetic slices) that represent $\frac{3}{5}$ and put and repeat them on $6 \frac{3}{4}$ [the diagram above]. To calculate the answer, we count how many slices of $\frac{3}{5}$ we used. [She was asked how to find the answer in case we do not have the manipulatives.] We need to draw an accurate diagram, then we divide each part into 5 parts and take 3 parts each time until the end. Then we count how many $\frac{3}{5}$ we have. Maybe we get 11. Like this

[She was asked, “You said you divided each part into 5 parts and take 3 parts each time until the end. What about the three fourths? How do you divide it?”] We find how many one fifth in three fourths:

$$\frac{3}{4} \div \frac{1}{5} = \frac{3}{4} \times 5 = \frac{15}{4} = 3 \frac{1}{2}$$

Teacher ZH’s representation involved measurement division (finding how many $\frac{3}{5}$ are in $6 \frac{3}{4}$), and it did not illustrate the underlying meaning of the story problem (partitive division – finding the size of one whole). Teacher ZH was not able to recognize which type the division story problem was. Despite her representation that could be used to determine the right numerical answer for $6 \frac{3}{4} \div \frac{3}{5}$ (although the representation does not match the story problem), she was unable to finish the problem because she thought she needed manipulatives or a precise drawing to get the answer. Whereas the story problem involved a partitive division, the teacher
was approaching the problem \(6 \frac{3}{4} \div \frac{3}{5}\) from the perspective of measurement division and did not recognize the discrepancy.

Here is another example of an incorrect solution:

Teacher ZM: If the weight of \(\frac{3}{5}\) of a bag of candy is \(6 \frac{3}{4}\), then the weight of \(\frac{2}{5}\) of a bag of candy is less than \(6 \frac{3}{4}\). Let us say approximately is \(6 \frac{1}{2}\). Then the weight of the whole bag of candy is about \(12 \frac{1}{2}\).

In this example, not giving an accurate answer, Teacher ZM tried to estimate the answer to the given story problem. She estimated the weight of \(\frac{2}{5}\) of a bag of candy as \(6 \frac{1}{2}\) pounds, which indicates \(\frac{1}{5}\) of a bag of candy becomes \(3 \frac{1}{4}\) pounds. Since \(\frac{3}{5}\) of a bag of candy is \(6 \frac{3}{4}\) pounds, however, her estimation of \(\frac{2}{5}\) of a bag being \(6 \frac{1}{2}\) pounds makes \(\frac{1}{5}\) of a bag \(\frac{1}{4}\) pound. It seems that she did not see these differences.

**Summary of Division Tasks**

In sum, the Saudi in-service elementary teachers were able to solve the given fraction division problem by using the invert-and-multiply method, but many of them struggled to generate a story problem that matched the required operation. Although the teachers were able to draw a diagram that represented the meaning for the given story problem, some of them used a symbolic procedure to complete the problem. Overall, the Saudi in-service elementary teachers’ MKT levels were related to their conceptual understanding revealed in their responses to fraction division tasks.

**Teachers’ Responses to Fraction Multiplication Tasks**

This section includes three subsections that describe in detail the Saudi elementary mathematics teachers’ reasoning and thinking on multiplication with fractions. Table 5.4
provides a brief summary of the participating teachers’ responses to three interview tasks about multiplication with fractions.

Table 5.4. **Summary of Results on Fraction Multiplication Tasks**

<table>
<thead>
<tr>
<th>Approaches to solving $2 \frac{1}{2} \times \frac{3}{4}$</th>
<th>Teachers around the average level of MKT (7 teachers)</th>
<th>Teachers below the average level of MKT (5 teachers)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Valid standard method (7 teachers)</td>
<td>Valid standard method (5 teachers)</td>
</tr>
<tr>
<td></td>
<td>Informal strategy (3 teachers)</td>
<td>Informal strategy (1 teacher)</td>
</tr>
<tr>
<td></td>
<td>o Valid distributive law (1 teacher)</td>
<td>o Valid distributive law (1 teacher)</td>
</tr>
<tr>
<td></td>
<td>o Invalid distributive law (1 teacher)</td>
<td>o Invalid distributive law (1 teacher)</td>
</tr>
<tr>
<td></td>
<td>o Valid representation (1 teacher)</td>
<td>o Valid representation (1 teacher)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generating a story problem for $2 \frac{1}{2} \times \frac{3}{4}$</td>
<td>Valid equal groups story problem (5 teachers)</td>
<td>Valid equal groups story problem (5 teachers)</td>
</tr>
<tr>
<td></td>
<td>Invalid story problem</td>
<td></td>
</tr>
<tr>
<td></td>
<td>o Different numerical meaning (2 teachers)</td>
<td></td>
</tr>
<tr>
<td>Representation of a story problem involving $6 \frac{3}{4} \times \frac{2}{3}$</td>
<td>Appropriate representation (6 teachers)</td>
<td>Appropriate representation (4 teachers)</td>
</tr>
<tr>
<td></td>
<td>Solving the problem</td>
<td>Solving the problem</td>
</tr>
<tr>
<td></td>
<td>o Using representation effectively (4 teachers)</td>
<td>o Using representation effectively (1 teacher)</td>
</tr>
<tr>
<td></td>
<td>o Using representation with some use of symbolic</td>
<td>o Using representation with some use of symbolic</td>
</tr>
<tr>
<td></td>
<td>manipulation (3 teachers)</td>
<td>manipulation (1 teacher)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>o Multiplying denominators and numerators, respectively</td>
</tr>
<tr>
<td></td>
<td></td>
<td>o Multiplying denominators and numerators, respectively</td>
</tr>
<tr>
<td></td>
<td></td>
<td>o Incomplete solution with representation and symbolic</td>
</tr>
<tr>
<td></td>
<td></td>
<td>o Incomplete solution with representation and symbolic</td>
</tr>
</tbody>
</table>


Approaches to Solving $2\frac{1}{2} \times \frac{3}{4}$

The Saudi teachers’ responses to the task indicate that the teachers had strong procedural knowledge and deficiency in their conceptual understanding of multiplication with fractions. All 12 participating teachers demonstrated proper procedural knowledge of how to calculate multiplication with fractions by using the standard method: multiplying numerators and denominators, respectively. Teachers’ responses to this task, shown in Table 5.4, are summarized in Table 5.5 again, with more details and teacher initials.

Table 5.5. Summary of Teacher Responses to Solving $2\frac{1}{2} \times \frac{3}{4}$

<table>
<thead>
<tr>
<th>Teachers around the average level of MKT (7 teachers)</th>
<th>Teachers below the average level of MKT (5 teachers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Valid standard method (7 teachers)</td>
<td>• Valid standard method (5 teachers)</td>
</tr>
<tr>
<td>• Informal strategy</td>
<td>• Informal strategy</td>
</tr>
<tr>
<td>o Valid distributive law (1 teacher: LA)</td>
<td>o Valid distributive law (1 teacher: RA)</td>
</tr>
<tr>
<td>o Invalid distributive law (1 teacher: SH)</td>
<td>o No other way (4 teachers: AA, MR, HU, ZM)</td>
</tr>
<tr>
<td>o Valid representation with symbolic manipulation (1 teacher: ZH)</td>
<td></td>
</tr>
<tr>
<td>o Invalid representation (1 teacher: SH)</td>
<td></td>
</tr>
<tr>
<td>o No other way (4 teachers: SA, MA, MN, AZ)</td>
<td></td>
</tr>
</tbody>
</table>

The 12 participating teachers verbalized the specific steps of the calculation procedure correctly. Here is a typical example of how the teachers described the solution process explicitly and reached the correct answer.

\[2\frac{1}{2} \times \frac{3}{4}\]

I convert this mixed number $[2 \frac{1}{2}]$ into an improper fraction.

\[2\frac{1}{2} = \frac{(2 \times 2) + 1}{2} = \frac{5}{2}\]

Then I multiply a numerator by a numerator, and a denominator by a denominator.
Then I write the product in a simpler way by convert the improper fraction to a mixed number

\[ \frac{5}{2} \times \frac{3}{4} = \frac{15}{8} \]

When the teachers were asked to solve the problem $2\frac{1}{2} \times \frac{3}{4}$ in a different way, four of them used alternative approaches (informal strategies) in addition to the standard method, and the remaining eight teachers thought that there were no other ways to solve the problem. Three of the four teachers were around the average score of MKT, and one of them was below the average score of MKT. Two teachers solved the problem by using the distributive law, and one teacher solved it symbolically along with a representation. One teacher attempted to solve it by using the distributive law and a representation, but she was not able to complete the solution process. Here is an example of a complete distributive law strategy:

**Teacher LA:** I separated the mixed number into $2$ and $\frac{1}{2}$, then multiply each by $\frac{3}{4}$.

\[
\begin{align*}
2 \times \frac{3}{4} &= \frac{3}{2} \\
\frac{1}{2} \times \frac{3}{4} &= \frac{3}{8}
\end{align*}
\]

I add the products together.

\[
\frac{3}{2} + \frac{3}{8} = \frac{12}{8} + \frac{3}{8} = \frac{15}{8}
\]

Then I convert the improper fraction to a mixed number.

\[ \frac{15}{8} = 1\frac{7}{8} \]

In the example above, Teacher LA multiplied $2$ and $\frac{1}{2}$ by $\frac{3}{4}$, respectively. She relied solely on the symbolic procedure.
The teacher who solved the problem symbolically along with a representation started with an appropriate idea in her diagram, but she did not use the representation to complete the problem appropriately. Here is her strategy.

**Teacher ZH:**

We divide each piece into four parts, and we take 3 of every 4 parts, so we have 4 parts represent 1, and 2 parts are $\frac{1}{2}$. To find the $\frac{3}{4}$ of $\frac{1}{2}$, we multiply them

$$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$$

Then

$$\frac{3}{8} + \frac{1}{2} = \frac{7}{8}$$

The answer is $1 \frac{7}{8}$.

As shown in this example, the teacher drew an appropriate representation, but she switched to symbolic manipulation to complete the problem. She found $\frac{3}{4}$ of $\frac{1}{2}$ symbolically without using her representation.

Here is an example of incomplete use of the distributive law and representation strategy:

**Teacher SH:** The three fourths of one is three fourths, and the three fourths of one is three fourths, but it is hard to find the three fourths of one half. … I think I put $2 \frac{1}{2}$ here (length) and $\frac{3}{4}$ here (width), but I need manipulatives (ex, Deniz slice) to cover this area and then find the answer.
As shown in Teacher SH’s response above, she started to solve the problem by using the distributive law, but she struggled to find \( \frac{3}{4} \) of \( \frac{1}{2} \). Then she drew an area model to find the answer for \( \frac{3}{4} \) of \( \frac{1}{2} \), but she also struggled to find it without using manipulatives to cover the area. This indicates her strategies of solving the fraction multiplication problem were mainly symbolic manipulation.

Although all of the participant teachers solved \( 2 \frac{1}{2} \times \frac{3}{4} \) by the standard method, only four of them were able to solve it in a different way, such as multiplying \( 2 \) by \( \frac{3}{4} \) and \( \frac{1}{2} \) by \( \frac{3}{4} \), and using a diagram to find \( 2 \times \frac{3}{4} \) and symbolic manipulation to find \( \frac{1}{2} \times \frac{3}{4} \). Other teachers were not able to come up with another way to solve the problem.

**Generating a Story Problem**

During the interviews, the participant teachers were asked to generate a story problem to make multiplication with fractions (\( 2 \frac{1}{2} \times \frac{3}{4} \)) meaningful for students. Most of the participating teachers were able to come up with an appropriate equal groups story problem that matched the given multiplication with fractions (either \( \frac{3}{4} \) of \( 2 \frac{1}{2} \), or \( 2 \frac{1}{2} \) groups of \( \frac{3}{4} \)). The results show that five of the seven teachers around the average score of MKT and all the five teachers below the average score of MKT were able to provide a meaningful story problem. Common contexts used were weight (measurement) and money. Here are their story problems:
• I want to make two and half cakes for a party. If one cake needs \( \frac{3}{4} \) cup of water, how many cups of water do I need to make the cakes?

• If you have two and half pieces of pizza and you want to put on each pizza \( \frac{3}{4} \) can of olives, how many cans do you need?

• If the weight of one kg of apples is 0.75 kg, how much will be the weight of \( 2 \frac{1}{2} \) kg of apples?

• Reem has \( 2 \frac{1}{2} \) boxes of apples; the weight of one box is \( \frac{3}{4} \) kg. How much is the weight of \( 2 \frac{1}{2} \) boxes?

• I received \( \frac{3}{4} \) Ryal [Saudi monetary unit] in an hour. How much I will get in two hours and half?

• If the price for one piece of cake is \( \frac{3}{4} \) Ryal and I want to buy \( 2 \frac{1}{2} \) pieces of the cake, how much money do I need?

• Mohmad wants to buy \( 2 \frac{1}{2} \) kg of sugar. How much is the sugar if the price for one kg is \( \frac{3}{4} \) Ryal or 0.75 Halal [Saudi monetary unit]?

• I have two and half pieces of candy. How much will three quarters of the candy be?

• The price for 1 kg of candy is \( 2 \frac{1}{2} \) Ryal. How much will be \( \frac{3}{4} \) kg of candy?

• The train whose length is \( 2 \frac{1}{2} \) m is exchanged for another train that equals \( \frac{3}{4} \) of its length. Find the length for second train.

As seen in the first seven examples above, seven out of the 10 teachers who gave an appropriate story problem were thinking about a story context with \( 2 \frac{1}{2} \) groups of \( \frac{3}{4} \) rather than \( \frac{3}{4} \) of \( 2 \frac{1}{2} \).

All the teachers generated appropriate story problems except two teachers. One of them was not able to generate a story problem, and the other one generated a story problem that had a context different from the given multiplication with fractions. This story problem consisted of two calculation parts: first multiplying by 3, then dividing by 4.
• I have fabric whose length is two and a half meters. I take the same length three times then divide it among four students. How much each student will have?

Representation of a Story Problem

The participant teachers were asked to provide a representation that conveyed the meaning of a given story problem (Mary has $\frac{3}{4}$ boxes of candy. One box of candy weighs $\frac{2}{3}$ pounds. How many pounds of candy does she have?) and solve the problem by using the representation they generated. Overall, the results show that the teachers conceptualized the story problem appropriately, and they had a conceptual understanding needed to use representations properly to solve the story problem involving fraction multiplication. Three fourths of the participating teachers (9 out of 12 teachers) used representation along with no or some symbolic manipulation, and the other teachers reverted to symbolic procedures instead of making sense of the representation they generated. Also, it seems that the teachers’ MKT levels were closely related to their reasoning on the fraction multiplication. All of the seven teachers around the average score of MKT and just two out of the five teachers below the average score of MKT were able to use their representation effectively to solve the problem. The teachers’ responses to this task, shown in Table 5.4, are summarized in Table 5.6 again, with teacher initials and more details.
Table 5.6. Summary of Teacher Responses to Solve the Story Problem Involving $6\frac{3}{4} \times \frac{2}{3}$

<table>
<thead>
<tr>
<th>Teachers around the average level of MKT (7 teachers)</th>
<th>Teachers below the average level of MKT (5 teachers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Appropriate representation (6 teachers)</td>
<td>• Appropriate representation (4 teachers)</td>
</tr>
<tr>
<td>• Solving the problem (7 teachers)</td>
<td>• Solving the problem (2 teachers)</td>
</tr>
<tr>
<td>o Using representation effectively</td>
<td>o Using representation effectively (1 teacher; MR)</td>
</tr>
<tr>
<td>(4 teachers; LA, SA, MN, AZ)</td>
<td>o Using representation effectively with some</td>
</tr>
<tr>
<td>o Using representation effectively with some</td>
<td>symbolic manipulation (1 teacher; HU)</td>
</tr>
<tr>
<td>symbolic manipulation (2 teachers; MA, SH)</td>
<td>o Incomplete solution with representation and</td>
</tr>
<tr>
<td>o Using inappropriate representation</td>
<td>symbolic manipulation (2 teachers; RA, AA)</td>
</tr>
<tr>
<td>effectively with some symbolic manipulation (1</td>
<td>o Multiplying denominators and numerators,</td>
</tr>
<tr>
<td>teacher: ZH)</td>
<td>respectively (2 teachers; RA, ZM)</td>
</tr>
</tbody>
</table>

All of the 12 participating teachers recognized the story problem correctly as requiring multiplication with fractions ($6\frac{3}{4} \times \frac{2}{3}$), and 10 of them provided an appropriate representation that conveyed the meaning of the story problem. Here is an example of common appropriate representations:

I have six whole boxes of candy and I divided this box [the seventh box] into four parts. I take three parts that represent the three quarters of the box of candy. Each box weighs two thirds.

![Image of candy boxes](image)

Eight of these 10 participating teachers who were able to generate an appropriate representation that conveyed the meaning of the story problem were able to use their representation effectively to solve the problem. They solved the problem by using a representation with no or some use of symbolic manipulation. Four of them had a solution process that relied heavily on representations, specifically to find the product of two fractions
(\(\frac{3}{4} \times \frac{2}{3}\)), and other four teachers used some symbolic manipulation with representations to solve the problem.

Here are examples of how the teachers effectively used the representation above to get the right answer:

**Teacher SH:** The weight of the six boxes is 
\[
\frac{2}{3} \times 6 = 4
\]

Now, I want to find the weight of box of candy. We know the weight of one box is \(\frac{2}{3}\).

I divided \(\frac{2}{3}\) by 4 to find the weight of one fourth.

\[
\frac{2}{3} \div 4 = \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}
\]

The weight of one fourth is \(\frac{1}{6}\)

Then, the weight of box of candy is 
\[
\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}
\]

Then the weight of 6\(\frac{3}{4}\) boxes of candy is \(4\frac{1}{2}\) kg.

**Teacher LA:** The weight of the six boxes is 
\[
2 \times 6 = 12, \text{ then } \frac{12}{3} = 4 \text{ kg}
\]

Now, I want to find the weight of the box of candy. The box is 3 parts of 4 parts.

Also, I divide the box into three parts, and I color these two parts that represent the \(\frac{2}{3}\).

Then, I look for the intersection of the two colors, that is \(\frac{6}{12} = \frac{1}{2}\).

Now, the weight of the box of candy is \(\frac{1}{2}\) [kg], and the weight of 6\(\frac{3}{4}\) boxes of candy is \(4\frac{1}{2}\) kg.
Teacher AZ: I find the weight of one box of candy. I divide the box into four parts and I take all the parts, then I split the box into three parts, and I took two parts that represent $\frac{2}{3}$.

Then, I look for the intersection of the two colors that is $\frac{8}{12}$, then the weight of six boxes is

$$\frac{8}{2} \times \frac{6}{12} = 4 \text{ kg}$$

Now, I want to find the weight of $\frac{3}{4}$ box of candy. I divide the box into three parts, and I color these two parts that represent the $\frac{2}{3}$.

Then, I look for the intersection of the two colors, that is $\frac{6}{12} = \frac{1}{2}$. The weight of $\frac{3}{4}$ box of candy is $\frac{1}{2}$ [kg]. Then the weight of $6\frac{3}{4}$ boxes of candy is $4\frac{1}{2}$ kg.

The examples above indicate that the teachers relied on their appropriate representations to solve the problem with no or some use of symbolic manipulation. Overall, the teachers knew what they were doing and why and were able to use their representation to solve the problem.

Here is another example of using a representation with some symbolic manipulation to solve the problem. Teacher ZH drew a representation that did not convey the meaning of the story problem, but it helped her to determine what calculation was needed to solve the problem.

She drew six and three fourths boxes connected as if they were one whole.

Teacher ZH: I drew $6\frac{3}{4}$. We divide each piece into three parts, and we take two parts of them. Now, we have three fourths. We split each one fourth into three parts, and we take two parts.
Now, we want to calculate the parts that have the intersection of two colors. We have 12 of \( \frac{1}{3} \).

\[
12 \times \frac{1}{3} = 4
\]

Now, we have \( \frac{2}{3} \) of \( \frac{1}{4} \) is repeated three times

The \( \frac{2}{3} \) of \( \frac{1}{4} \) is

\[
\frac{2}{3} \times \frac{1}{4} = \frac{2}{12} = \frac{1}{6}
\]

The \( \frac{2}{3} \) of \( \frac{3}{4} \) is

\[
\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}
\]

The answer is \( 4 \frac{1}{2} \) [kg].

The remaining two teachers who had an appropriate representation of the story problem were able to solve it partially. One of them was able to calculate the weight of six boxes, but she struggled to find the weight of three fourths of a box. Another teacher was able to find the weight of three fourths of a box by using her representation, but she struggled to find the weight of six boxes. Then she reverted to the standard method (multiplying numerators and denominators, respectively) to solve the problem. Finally, the teacher who had inappropriate representation reverted to the standard method to solve the problem.
Summary of Multiplication Tasks

In sum, most of the Saudi in-service elementary teachers showed one common strategy (i.e., multiplying numerators and denominators, respectively) to solve the fraction multiplication, and many thought that this standard method was the sole strategy that could be used for solving the problem. Few of them were able to propose an alternative strategy to solve the fraction multiplication problem, and most teachers generated one type of a story problem using the equal groups meaning. Most of the teachers were able to draw a representation that conveyed the meaning of a given story problem for fraction multiplication and completed the solution by using the representation they generated. Some struggled either in drawing an appropriate representation or in using their representation to complete the solution process. Overall, it seems that the Saudi in-service elementary teachers’ MKT levels were related to their ability to use a representation to solve a multiplication problem.

Teachers’ Interpretations of and Reasoning About Students’ Alternative Strategies

In this section, I explain the knowledge that the participating teachers demonstrated in interpreting students’ alternative approaches to two tasks involving fraction division and fraction multiplication, respectively. The teachers were asked to interpret students’ alternative strategies to solving the story problems that they attempted to solve in the earlier part of the interview. The two student strategies are as follows:

Reem’s alternative strategy to solving a division story problem (If \( \frac{3}{5} \) of a bag of candy weighs \( 6\frac{3}{4} \) pounds, how much does 1 bag of candy weigh?):

\[
6 \frac{3}{4} \div 3 = 2 \frac{1}{4}
\]

\[
6 \div 3 = 2
\]
Dana’s alternative strategy to solving a multiplication story problem (Mary has $\frac{3}{4}$ boxes of candy. One box of candy weighs $\frac{2}{3}$ bounds, how many pounds of candy do she has?):

\[
\frac{3}{4} \div 3 = \frac{1}{4}
\]

\[
2\frac{1}{4} \times 5 = 11 \frac{1}{4}
\]

The results reveal differences in teachers’ responses to and reasoning about the two alternative strategies (see Table 5.7). Regarding the division alternative strategy, the teachers around the average score of MKT (IRT scores between –0.5 and 0.5) were able to explicitly interpret it more than the teachers below the average score of MKT (IRT scores below –0.5). Whereas six out of the seven teachers around the average score of MKT were able to recognize the validity, generalizability, and efficiency of the division alternative strategy, just two out of the five teachers below the average score of MKT were able to do so. As for the multiplication alternative strategy, all 12 participating teachers were able to recognize the validity, generalizability, and efficiency of the strategy.
Table 5.7. Summary of Teachers’ Reasoning on Students’ Alternative Strategies

<table>
<thead>
<tr>
<th>Students’ alternative approach to division with fractions</th>
<th>Teachers around the average level of MKT (7 teachers)</th>
<th>Teachers below the average level of MKT (5 teachers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Validity</td>
<td>7 teachers</td>
<td>4 teachers</td>
</tr>
<tr>
<td>Generalizability</td>
<td>6 teachers</td>
<td>2 teachers</td>
</tr>
<tr>
<td>Efficiency</td>
<td>6 teachers</td>
<td>2 teachers</td>
</tr>
<tr>
<td>Teachability</td>
<td>Could be taught?</td>
<td>6 teachers</td>
</tr>
<tr>
<td></td>
<td>Like to teach?</td>
<td>4 teachers</td>
</tr>
<tr>
<td>Students’ alternative approach to multiplication with fractions</td>
<td>6 teachers</td>
<td>2 teachers</td>
</tr>
<tr>
<td>Validity</td>
<td>7 teachers</td>
<td>5 teachers</td>
</tr>
<tr>
<td>Generalizability</td>
<td>7 teachers</td>
<td>5 teachers</td>
</tr>
<tr>
<td>Efficiency</td>
<td>7 teachers</td>
<td>5 teachers</td>
</tr>
<tr>
<td>Teachability</td>
<td>Could be taught?</td>
<td>7 teachers</td>
</tr>
<tr>
<td></td>
<td>Like to teach?</td>
<td>3 teachers</td>
</tr>
</tbody>
</table>

It seems that the participating teachers’ interpretations of students’ rational thinking in the two alternative strategies were influenced by their method of solving the story problem in Task 3 (using a representation to solve the given story problem) in each set of interview tasks. Pointing to their responses in previous tasks, they mentioned that these alternative strategies were like what they did in those tasks, but with symbolic manipulation instead of using a representation. After prompted to elaborate more, they provided short answers that involved different types of interpretations. Teachers’ interpretations were either solely procedural, or both procedural and conceptual. Procedural interpretations indicate that teachers described and interpreted students’ approaches based on mathematical concepts and properties. Conceptual interpretations indicate that teachers described and interpreted students’ approaches based on potential meaning and representations.
Teachers’ Responses to Reem’s Strategy (Division)

Although most of the teachers were able to recognize that this strategy worked for the given problem, about one third of them were not able to recognize that this strategy could be generalized to any problem and could be more efficient than the standard method (invert and multiply).

Validity. Eleven out of the 12 teachers provided a procedural interpretation, and six of them provided a conceptual interpretation as well, when they were asked whether the student’s alternative approach to the division with fractions worked for the given problem. The teachers described the student’s action and potential meaning and representations. Here is a typical example of such an interpretation that was given by one teacher:

Part1: The student wrote a mixed number as an integer and a fraction (6 3/4 = 6 + 3/4), and divided it by 3 and used the distributive property to find the weight of one fifth of a bag as what I did in the previous task. Then the student multiplied the answer by 5 to find the weight of one whole bag.
Part2: It seems that the student built this strategy based on appropriate internal representations of the problem like this [pointing to their representation in task 3], and this student has a high level of thinking, which is based on imagination of the diagram and using it to solve the problem.

As shown in the example above, the participant teachers first described the computation procedures that the student went through (part 1), and then they tried to describe and interpret the student’s potential meaning and representation (part 2). They mentioned that the student might have created the same representation they had in task 3, and based on their mental image of this representation, the student might have written the solution steps of the strategy.

Generalizability. Eight of the 11 teachers who recognized the validity of the strategy examined each step, and they said, “Each step built on rational thinking and conceptual understanding of fractions and the story problem, so this strategy could be generalizable for
solving any fraction division problem with mixed numbers.” It seems that the other three teachers lacked a conceptual understanding of division as an inverse operation of multiplication. They said that this strategy could not be generalizable because there was no distributive property of division over addition.

**Efficiency.** The eight teachers who recognized the validity and generalizability of Reem’s strategy elaborated on its efficiency. Below are some comments from the teachers:

- This strategy is more efficient if the numbers are easily divided by the divisor.
- In case the numbers are not divisible [by other numbers], this strategy will take longer to complete the problem.
- It is more efficient if we provide it with a representation.

As shown in these comments, the teachers were able to determine the conditions in which this strategy could work efficiently. They explained when, why, and/or how this strategy could be used efficiently. In the first and second comments above, the teachers identified when the strategy would work best. In the second comment, the teachers also explained the strategy would not be efficient when the numbers were not divisible by other numbers. As shown in the third comment, the teachers mentioned how this strategy could be used efficiently.

**Teachability.** When the eight teachers were asked whether this strategy could be taught to students, they said it could be taught because it was reasonable. However, when the eight teachers were asked if they could use this strategy in their teaching, six of them concluded that this strategy was easier and more understandable than the standard method, so they would like to teach it. The other two teachers thought that this strategy was long and involved many mathematical properties that would confuse students, so they preferred to teach the standard method (i.e., invert and multiply).
It seems that the six teachers clearly understood Reem’s alternative strategy, so they were able to see that this strategy was valuable for teaching. The other two teachers, despite their ability to recognize the validity, generalizability, and efficiency of the strategy, thought that teaching this strategy might confuse students. This indicates that these two teachers might have interpreted the reasonable steps in the strategy but not understood it conceptually, or they preferred the procedural knowledge and the method they were familiar with.

**Teachers’ Responses to Dana’s Strategy (Multiplication)**

Unlike the alternative strategy on division with fractions, all the participating teachers were able to recognize that the alternative strategy on multiplication with fractions worked for the given problem, could be generalizable to any other problem, and could be more efficient than the standard method (multiplying numerators and denominators, respectively).

**Validity.** All participating teachers provided a procedural interpretation when they were asked whether the student’s alternative approach to multiplication with fractions worked for the given problem. They explained the solution process and claimed that it was reasonable because it built on correct mathematical concepts and properties, such as writing a mixed number as an integer and a fraction \(6 \frac{3}{4} = 6 + \frac{3}{4}\), decomposition of a fraction \(\frac{2}{3} = \frac{1}{3} \times 2\), and the distributive property of multiplication over addition.

**Generalizability.** All 12 teachers also agreed that this strategy could be generalizable to solve “any fraction multiplication problem with mixed numbers.” They examined each step in the strategy, and they said each step built on rational thinking and correct mathematical properties of fractions, so this strategy could be generalizable.
Efficiency. Also, all the participating teachers provided various comments on the efficiency of this alternative strategy. Below are some of the comments about the fraction multiplication strategy:

- This strategy is more efficient if the numbers have common factors so that they could be simplified.
- It is more efficient if we provide it with a representation.
- It is an easier calculation than the standard method.

As shown in these comments, teachers were able to determine some situations in which this strategy could be efficient. They explained when, why, or how this strategy could be worked efficiently. In the first comment, the teachers identified a specific condition in which the strategy could be used efficiently. In the second comment, the teachers mentioned how this strategy worked more efficiently. In the third comment, the teachers compared this alternative strategy with the standard method (i.e., multiplying numerators and denominators, respectively) and explained it was efficient because it involved an easier calculation than the standard method.

Teachability. When the teachers were asked whether this strategy could be taught to students, they were in agreement that this strategy could be taught because it built on reasonable steps and included easy and understandable calculations. However, when the teachers were asked if they would use this strategy in their teaching, most of them preferred to teach the standard method (multiplying numerators and denominators, respectively). Seven out of the 12 teachers concluded that they liked to teach the standard method because it had shorter steps and was more accurate than this alternative strategy.

Although all participant teachers were able to recognize the validity, generalizability, and efficiency of the strategy and they were in agreement that the strategy involved understandable
calculations, most of them still preferred the standard method. It seems that the teachers focused on the number of steps more than on the reasoning. This indicates that the teachers might believe in procedural knowledge more than the conceptual foundation.

**Summary of Teacher Interpretations of Student Work**

In sum, all of the Saudi in-service elementary teachers were able to recognize the validity, generalizability, and efficiency of the alternative strategy on the multiplication with fractions, and two thirds of them were able to recognize these three aspects of the alternative strategy on the division with fractions. Also, their interpretations of these two alternative strategies relied heavily on procedural interpretations more than conceptual interpretations. Although the participating teachers who recognized the three aspects of the strategies concluded the two alternative strategies included easy and understandable calculations, most of them preferred to teach the standard method. Overall, the Saudi in-service elementary teachers’ MKT levels were related to their responses and reasoning to alternative strategies to fraction division problems more than to multiplication problems.
CHAPTER 6
RELATIONSHIPS IN TEACHERS’ RESPONSES TO MKT MEASURE, SURVEY, AND INTERVIEW TASKS

In the previous two chapters, I described the findings from the participant teachers’ responses to the MKT measure, the survey, and the interviews. Teacher responses to each instrument were described and some relationships revealed in teacher responses between instruments. This chapter centers on the relationships in responses to the three instruments from the 12 participating teachers who completed all three instruments. I compared and contrasted those teachers’ MKT levels, i.e., teachers’ MKT scores (specifically content knowledge for teaching rational numbers), their confidence for teaching the content of fractions and their beliefs on some particular aspects of mathematics teaching and learning, and their reasoning and thinking on the tasks involving division and multiplication with fractions (see Figure 6.1). Examining relationships among the teachers’ responses to the three instruments led to several interesting findings that are discussed in detail in this section, which helps to addresses the fourth research question: What is the relationship among Saudi elementary mathematics teachers’ confidence and beliefs, mathematical knowledge for teaching, and reasoning about multiplication and division with fractions?
Figure 6.1. Relationships in teachers’ responses to MKT measure, survey, and interview tasks.

First, I will describe relationship 1 in Figure 6.1 between teachers’ confidence and beliefs on teaching mathematics and their mathematical content knowledge for teaching (MKT level, and thinking and reasoning on interview tasks). Then, I will describe relationship 2 in Figure 6.1 between teachers’ MKT levels and their thinking and reasoning on tasks about division and multiplication with fractions, which includes some interesting findings.

(Mis)Match Between Teachers’ Confidence and Beliefs on Teaching Mathematics and Their Mathematical Content Knowledge for Teaching

In Chapter 4, I explained some overall patterns in 44 participant teachers’ MKT levels and survey responses. In this chapter, exploring the relationships between teachers’ beliefs and confidence and their content knowledge, I focus on the 12 teachers who were interviewed with
tasks on fraction division and multiplication. Examining the relationships between the 12 participating teachers’ confidence and beliefs on teaching mathematics and their mathematical content knowledge for teaching rational numbers (specifically fractions)—their MKT levels and their content knowledge for teaching division and multiplication with fractions—reveals two aspects of the relationships: (1) the mismatch between teachers’ beliefs on some aspects of teaching mathematics and their mathematical content knowledge for teaching, and (2) the match between teachers’ confidence on teaching specific topics and their mathematical content knowledge for teaching.

Mismatch Between Teachers’ Beliefs on Teaching Mathematics and Their Mathematical Content Knowledge for Teaching

The study results reveal the disparity between teachers’ perceptions or beliefs on teaching mathematics and their mathematical content knowledge for teaching (teachers’ MKT levels and teachers’ thinking and reasoning on division and multiplication with fractions). All participating teachers had pedagogical beliefs that aligned with reform efforts that seek to equip teachers to work in ways conducive to conceptual thinking, but the inclination alone seems insufficient. Although all participating teachers had a strong belief in the importance of using representations and story problems to make mathematical ideas meaningful, many of them did not have sufficient knowledge to respond to MKT items regarding representations (11 out of 12 teachers) and story problems (6 out of 12 teachers), and they struggled to provide an appropriate representation and/or story problem on tasks dealing with fraction division and multiplication during the interviews (8 out of 12 teachers).
**Match Between Teachers’ Confidence in Teaching and Their Mathematical Content Knowledge for Teaching**

The results show a match between the teachers’ confidence in teaching specific mathematical fractional topics and their mathematical content knowledge for teaching (teachers’ MKT levels and teachers’ thinking and reasoning on tasks about division and multiplication with fractions). As seen in Figure 6.2, the participating teachers’ level of MKT and their confidence in their ability for teaching rational numbers and representing fractions by using words, numbers, and models are positively related overall. Note that the levels of MKT are relative among the participating teachers in this analysis: the highest group refers to those who received IRT scores over 0 (between 0 and 0.5); the middle group, those who scored between –0.5 and 0; and the lowest group, below –0.5. All participating teachers in the high level of MKT were very confident teaching rational number and representing fractions; three out of the four teachers in the middle level were confident, whereas three out of the five teachers in the low level of MKT were not confident.

Also, teachers’ confidence in their ability to explain and represent division and multiplication with fractions was somewhat related to the number of valid responses to the six interview tasks for each teacher, as can be seen in Figures 6.3 and 6.4. When teachers’ confidence in explaining and representing division and multiplication with fractions increased, the extent of the misconception or error decreased. However, such relationships seem not very strong because, as shown in Figures 6.3 and 6.4, there were still teachers who performed well on the interview tasks, but their confidence was moderate (confident as opposed to very confident).
Figure 6.2. Teachers’ level of MKT and their confidence in teaching rational numbers and representing fractions.

Figure 6.3. Teachers’ confidence in explaining division and multiplication with fractions and the number of valid responses to interview tasks.
Thus far, I have described the relationships between teachers’ confidence and beliefs on teaching mathematics and their mathematical content knowledge for teaching. Now, I will describe the relationships between teachers’ MKT levels and their thinking and reasoning on the interview tasks about division and multiplication with fractions (i.e., relationship 2 in Figure 6.1).

**Teachers’ MKT Levels and Their Thinking and Reasoning on Tasks About Division and Multiplication With Fractions**

Teachers’ responses to the six interview tasks (the first three tasks in each set) revealed that most of the participating teachers (8 out of 12 teachers) had confusion or errors in understanding division with fractions and/or multiplication with fractions. Not surprisingly, the frequency of teachers’ confusion or errors in the six interview tasks (valid responses were
counted based on the six main tasks, not on additional related tasks) was closely related to their levels on the MKT measure (see Figure 6.5). That is, the frequency of teachers’ confusion or errors increased as their MKT level decreased. Most teachers who were in the middle and high levels (between –0.5 and 0.5) had 5 to 6 appropriate responses. In contrast, most teachers who were in the low MKT level (below –0.5) had 4 to 5 valid responses.

**Figure 6.5.** The number of valid responses to the six interview tasks for each teacher.

Figure 6.5 also shows the variations in teachers’ responses to interview tasks within and across their MKT levels, along with some teachers’ pseudo initials. On the one hand, teachers at the same MKT level performed quite differently on the tasks (e.g., MR, AA, and ZM). On the other hand, teachers at different MKT levels had a similar performance on the tasks (e.g., SA and MR). These two variations are described below in detail in different themes, along with examples of teacher responses to some interview tasks.
Variations in Teacher Reasoning and Thinking Within the Same MKT Levels

Examining and comparing individual teachers’ data in Figure 6.5 revealed that there were groups of teachers receiving the same score on the MKT measure who performed differently on the interview tasks. Two groups of teachers (Teachers MR, AA, and ZM, and Teachers SH and ZH), who had the same score on the MKT measure, are selected as examples because of differences in their reasoning, and the variations in their reasoning and thinking on selected tasks about division and multiplication with fractions are described within each group and between the two groups. Note that these teachers’ initials are indicated in Figure 6.5. Similarities and differences in each of the two groups of teachers’ responses are summarized in Table 6.1.

Table 6.1. Similarities and Differences in Each Group of Teachers’ Responses

<table>
<thead>
<tr>
<th>MKT raw scores (IRT scores)</th>
<th>Teachers MR, AA, and ZM</th>
<th>Teachers SH and ZH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Similarities in each group</td>
<td>Solving correctly the fraction division and multiplication problems ([\frac{3}{4} \div \frac{1}{2}; \ 2 \times \frac{3}{4}]) by using the standard methods (invert and multiply; multiply numerators and denominators, respectively)</td>
<td>Solving correctly the fraction division and multiplication problems ([\frac{3}{4} \div \frac{1}{2}; \ 2 \times \frac{3}{4}]) by using the standard methods (invert and multiply; multiply numerators and denominators, respectively)</td>
</tr>
<tr>
<td></td>
<td>Generating an invalid story problem for (\frac{3}{4} + \frac{1}{2})</td>
<td>Generating a valid measurement division story problem for (\frac{3}{4} + \frac{1}{2})</td>
</tr>
<tr>
<td></td>
<td>Generating a valid equal group story problem for (2 \times \frac{3}{4})</td>
<td>Generating a valid equal group story problem for (2 \times \frac{3}{4})</td>
</tr>
<tr>
<td>Differences in each group in the area of</td>
<td>Solving the fraction division problem ([\frac{3}{4} + \frac{1}{2}]) by using alternative strategies</td>
<td>Solving the fraction division and multiplication problems ([\frac{3}{4} + \frac{1}{2}; \ 2 \times \frac{3}{4}]) by using alternative strategies</td>
</tr>
<tr>
<td></td>
<td>Using a representation to solve story problems involving fraction division and multiplication</td>
<td>Using a representation to solve story problems involving fraction division and multiplication</td>
</tr>
</tbody>
</table>
As shown in Table 6.1, all five teachers—Teachers MR, AA, and ZM (the low level among the participant teachers) and Teachers SH and ZH (the middle level of MKT scores among the participants)—were able to solve the division and multiplication with fractions problem by using the standard methods accurately, and to generate a valid equal group story problem for $2 \frac{1}{2} \times \frac{3}{4}$. Whereas Teachers MR, AA, and ZM were not able to generate a story problem that represents the meaning of the fraction division problem ($1 \frac{3}{4} \div \frac{1}{2}$)—their story problem represented the meaning of dividing by 2 rather than dividing by $\frac{1}{2}$, Teachers SH and ZH were able to generate a valid measurement division story problem for $1 \frac{3}{4} \div \frac{1}{2}$. (See Chapter 5: Teachers’ Reasoning on Division and Multiplication with Fractions.)

Although teachers’ reasoning and thinking within each group were similar on some interview tasks, their reasoning and thinking on other interview tasks were rather diverse. The significant variations that appeared in both groups of teachers (Teachers MR, AA, and ZM, and Teachers SH and ZH) were related to conceptual understanding of division and multiplication with fractions, specifically knowledge of alternative strategies and knowledge of representations. Now, teachers’ alternative strategies and teachers’ representational fluency will be described in detail within each group of teachers and between the two groups.

**Teachers’ Alternative Strategies**

Whereas the participating teachers demonstrated strong procedural knowledge in solving the given division and multiplication with fractions, deficiencies and variations surfaced in their conceptual understanding of the operations with fractions. Variations were apparent in the participant teachers’ ability to solve the problems using alternative strategies between teachers
below the average score of MKT (below \(-0.5\)) and those around the average score of MKT (between \(-0.5\) and 0.5), and also between teachers at the same level of MKT.

Teachers MR, AA, and ZM, who were below the average score of MKT, struggled to solve the problems by using alternative approaches, and their responses to the question “Do you know other strategies to solve the problem?” varied. Teacher MR mentioned that there were different ways of solving division and multiplication with fractions such as a representation, but she did not know how to do them. Teacher AA stated that it was possible to solve the problem about fraction division by using a representation, but she was not able to complete the solution process. She drew one and three fourths pizzas but was not able to use the representation to find the answer. (See Chapter 5: Teachers’ Reasoning on the Fraction Division Tasks.) Teacher ZM thought that there was no other way to solve the problems involving division and multiplication with fractions. Unlike Teachers MR, AA, and ZM, Teachers SH and ZH were able to propose one or two alternative approaches. Teacher SH was able to provide an accurate description of two alternative strategies, the distributive law and using representations. Her struggle lay in how to use a visual representation to find the answer (see Chapter 5). Teacher ZH was able to solve the problems by using an alternative strategy (using representations). Their ways of solving the division and multiplication problems by using a representation will be discussed in detail in the next section on representational fluency.

The teachers’ alternative strategies and representation use seemed to influence their ability to interpret students’ alternative approaches to the division with fractions. Teachers SH, ZH, and MR, who demonstrated understanding of representation, were able to give suitable explanations regarding the validity, generalizability, and efficiency of students’ work, based on conceptual and procedural interpretations. Teacher AA, who had some issues with using
representation to solve the problems, gave adequate explanations about the validity of the students’ work based on procedural interpretation, but she was not able to discuss the generalizability and efficiency of the strategies. In contrast, Teacher ZM, who did not demonstrate a good use of representations and alternative strategies, was not able to interpret students’ work appropriately. (See Appendix E and Teachers’ Interpretations of and Reasoning About Students’ Alternative Strategies section in Chapter 5.) Now, I turn to the teachers’ use of representation in the two groups.

**Teachers’ Representational Fluency**

Significant differences in teachers’ ability to draw an appropriate representation and use it to solve the problem were noticeable between teachers below the average score of MKT (below $-0.5$) and those around the average score of MKT (between $-0.5$ and $0.5$), and also between teachers in the same level of MKT.

Teachers MR, AA, and ZM, who had the same MKT score of $-0.652215$ (the low level among the participant teachers), exhibited varied representational fluency (see Table 6.2). Unlike Teachers AA and ZM, Teacher MR had proper representations and used them effectively to solve the two given story problems about division and multiplication with fractions. Although Teacher AA had appropriate representations, she was not able to use them effectively to solve the given story problems. She struggled in symbolic manipulation with representation to solve the division story problem and in using a representation to solve the multiplication story problem. She was not able to find the weight of three fourths of a box. Teacher ZM had inappropriate representations. She had errors in generating appropriate representations and using them to solve the two story problems about division and multiplication with fractions.
As shown in Table 6.2, there were variations in the three teachers’ responses to the two tasks about using representation in division and multiplication with fractions. As for division
with fractions, Teachers MR and AA drew appropriate representations that conveyed the meaning of the story problem and reported the same two steps of the solution process: (1) finding the weight of \( \frac{1}{5} \) of a bag of candy by dividing the \( 6 \frac{3}{4} \) by 3, and then (2) multiplying the result (the weight of the \( \frac{1}{5} \)) by 5 or adding the result 5 times. However, they approached the first step differently. MR distributed mentally 6 and \( \frac{3}{4} \) into 3 parts to find the weight of the \( \frac{1}{5} \). AA used standard methods to find the weight of \( \frac{1}{5} \) of a bag. She first converted the mixed number to an improper fraction, then converted the division into multiplication, then multiplied numerators and denominators, respectively. It seems that MR was able to see that mixed numbers could be written as the whole number 6 and the proper fraction \( \frac{3}{4} \), and these two numbers could be distributed easily mentally into three parts (dividing by 3) to find the weight for the \( \frac{1}{5} \). AA was not able to approach this step conceptually by using a representation, so she changed her approach and addressed it procedurally (standard methods). Whereas Teachers MR and AA followed reasonable steps that built on either conceptual understanding or rational thinking, Teacher ZM used a different approach, that is, estimating the answer to the given story problem. She had an inaccurate estimation of the weight of \( \frac{2}{5} \) of a bag of candy (see Teachers’ Reasoning on the Fraction Division Tasks section in Chapter 5).

Regarding the multiplication with fractions, Teachers MR and AA drew proper representations for the story problem (see Table 6.2). To find the weight of \( 6 \frac{3}{4} \) boxes of candy, they followed two steps: (1) finding the weight of the 6 boxes, and (2) finding the weight of the \( \frac{3}{4} \) box. Both teachers approached the first step by multiply 6 by \( \frac{2}{3} \). However, they approached the second step differently. MR approached the second step correctly by drawing the diagram (area)
that represented \( \left( \frac{3}{4} \times \frac{2}{3} \right) \) to find the weight of the \( \frac{3}{4} \) box. AA struggled to find the weight of the \( \frac{3}{4} \) box. She used standard methods, and then she divided \( \frac{2}{3} \) by \( \frac{3}{4} \). She realized that this was not reasonable, and she did not complete the solution. It seems that MR had a conceptual understanding of the story problem and representation in multiplication with fractions. She used the representation effectively to solve the problem. Although AA began with appropriate representation and reported reasonable steps, she struggled to find the weight of \( \frac{3}{4} \) of a box by using her representation. So, she employed standard methods, but she struggled again. It seems that she might have some confusion in understanding fraction division and multiplication: she divided \( \frac{3}{4} \) by \( \frac{2}{3} \) rather than multiplying them. Whereas Teacher MR was able to use her representation effectively to solve the problem, and Teacher AA was able to use her proper representation partially, Teacher ZM was not able to use a representation at all to solve the problem, so she used the standard method (multiply numerators and denominators, respectively).

Now, we turn to the other two teachers, who also had the same MKT score of –0.240082 (the middle level of MKT scores among the participants). Although Teachers SH and ZH proposed appropriate representations, they were not able to use their representations effectively to solve the fraction division or fraction multiplication problems (either with numbers only or in story contexts). They introduced representations that were different from the common representations proposed by the rest of the participating teachers. Teacher SH proposed an area model to solve the fraction multiplication problem, and Teacher ZH proposed a partitive model for the fraction division problem. Although they showed strength in procedural knowledge and fluency in physical representation, their weakness surfaced in visual representations. Whereas Teacher SH was not able to use visual representations to solve the division and multiplication
problems *without* a story context, Teacher ZH was not able to solve the problems given *with* a story context by using visual representations.

Teacher SH was able to use representations to solve the division and multiplication problems in context, but she was not able to use a representation to find the answer to the problems given numerically. As seen in Table 6.3, she was able to solve the two story problems by using symbolic manipulation along with representations, and she tried to solve the problems given without a context by using representations. She understood what the division and multiplication problems were about and was able to draw an appropriate diagram, but she struggled to complete the solution process without using physical manipulatives.

Table 6.3. *Two Teachers’ Use of Representations in Solving Problems*

<table>
<thead>
<tr>
<th>Teacher</th>
<th>SH</th>
<th>ZH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approaches to solving $\frac{3}{4} ÷ \frac{1}{2}$</td>
<td>I drew one and three fourths slices; I divide them into halves. [She stopped drawing her picture and said] We need manipulatives to see how many halves are in these slices.</td>
<td>I represent the problem by using cubes where the 4 cubes represent 1, so 7 cubes represent $1\frac{3}{4}$. Now I divide by one half. The 2 cubes represent $\frac{1}{2}$, then I have three halves that are 3 whole numbers, and one cube is half of a whole, so it is one half. The result is $3 \frac{1}{2}$.</td>
</tr>
<tr>
<td></td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
</tbody>
</table>
Table 6.3—Continued

<table>
<thead>
<tr>
<th>Teacher</th>
<th>SH</th>
<th>ZH</th>
</tr>
</thead>
</table>
| **Representation of a story problem involving** $\frac{3}{4} \div \frac{3}{5}$ | I want to find the weight of one part that represents one fifth, so I divide $6\frac{3}{4}$ by 3.  

$$\frac{3}{4} \div \frac{3}{5} = \frac{27}{4} \div 3 = \frac{27}{4} \times \frac{5}{3} = \frac{9}{2} = 2\frac{1}{2}$$

This means the weight of one fifth is $2\frac{1}{2}$ [kg].

Then we add $2\frac{1}{4}$ five times or multiply $2\frac{1}{4}$ by 5.  

$$2\frac{1}{4} \times 5 = \frac{9}{4} \times 5 = \frac{45}{4} = 11\frac{1}{4}$$

<table>
<thead>
<tr>
<th></th>
<th><img src="image1.png" alt="Diagram" /></th>
<th><img src="image2.png" alt="Diagram" /></th>
</tr>
</thead>
</table>

We draw $6\frac{2}{4}$ then we bring manipulatives (e.g., magnetic slices) that represent $\frac{3}{5}$ and put them on the diagram and repeat them on $6\frac{3}{4}$ [diagram above]. To find the answer, we count how many slices of $\frac{3}{5}$ we are using.  

[She was asked how to find the answer in case we do not have the manipulative tool.]

We need to draw an accurate diagram, then we divide each part into 5 parts and take 3 parts each time until the end. Then we count how many $\frac{3}{5}$ we have. Maybe we get 11. Like this

[She was asked, “You divided each part into 5 parts and take 3 parts. What about the three fourths?”]

We find how many one fifths in three fourths.

$$\frac{3}{4} \div \frac{1}{5} = \frac{3}{4} \times 5 = \frac{15}{4} = 3\frac{1}{2}$$

[Image 164x339 to 308x384]
### Table 6.3—Continued

<table>
<thead>
<tr>
<th>Teacher</th>
<th><strong>Approaches to solving</strong></th>
<th><strong>SH</strong></th>
<th><strong>ZH</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2 \times \frac{1}{2} \times \frac{3}{4}$</td>
<td>I think I put $2 \frac{1}{2}$ here (length) and $\frac{3}{4}$ here (width), but I need manipulatives (ex, Deniz slice) to cover this area and then find the answer.</td>
<td>We divide each piece into four parts, and we take 3 of every 4 parts. So, we have 4 parts [medium gray shade in the diagram above] representing 1, and 2 parts [dark gray shade in the diagram above] are $\frac{1}{2}$. To find $\frac{3}{4}$ of $\frac{1}{2}$, we multiply them.</td>
</tr>
<tr>
<td></td>
<td><img src="image" alt="Diagram of a rectangular area divided into parts" /></td>
<td><img src="image" alt="Diagram of a rectangular area divided into parts" /></td>
<td>$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$</td>
</tr>
<tr>
<td></td>
<td><img src="image" alt="Diagram of a rectangular area divided into parts" /></td>
<td><img src="image" alt="Diagram of a rectangular area divided into parts" /></td>
<td>Then $\frac{3}{8} + \frac{1}{2} = \frac{7}{8}$</td>
</tr>
<tr>
<td></td>
<td><img src="image" alt="Diagram of a rectangular area divided into parts" /></td>
<td><img src="image" alt="Diagram of a rectangular area divided into parts" /></td>
<td>The answer is $1 \frac{7}{8}$.</td>
</tr>
</tbody>
</table>
Table 6.3—Continued

<table>
<thead>
<tr>
<th>Teacher</th>
<th>SH</th>
<th>ZH</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Representation of a story problem involving</strong> $\frac{3}{6} \times \frac{2}{4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mary has $6 \frac{3}{4}$ boxes of candy. One box of candy weighs $\frac{2}{3}$ pounds. How many pounds of candy does she have?</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

![Diagram](image)

The weight of six boxes is $\frac{2}{3} \times 6 = 4$ [kg].

Now, I want to find the weight of $\frac{3}{4}$ box of candy. We know the weight for one box is $\frac{2}{3}$. I divided $\frac{2}{3}$ into 4 parts to find the weight of one fourth $\frac{1}{4}$.

$$\frac{2}{3} \div 4 = \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$$

The weight of one fourth [box] is $\frac{1}{6}$ [kg].

Then, the weight of $\frac{3}{4}$ box of candy is $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$ [kg].

So, the weight of $6 \frac{3}{4}$ boxes of candy is $4 \frac{1}{2}$ kg.

![Diagram](image)

I drew $6 \frac{3}{4}$. We divide each piece into three parts, and we take two parts of them. Now, we have three fourths. We divide each one fourth into three parts [see the last row in the diagram below], and we take two parts.

Now, we want to find the parts that are shaded. We have 12 of $\frac{1}{3}$.

$$12 \times \frac{1}{3} = 4$$

Now, we have $\frac{2}{3}$ of $\frac{1}{4}$ and repeat that three times.

The $\frac{2}{3}$ of $\frac{1}{4}$ is

$$\frac{2}{3} \times \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$$

The $\frac{2}{3}$ of $\frac{3}{4}$ is

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

The answer is $4 \frac{1}{2}$ [kg].

Teacher ZH was able to solve the problems given without a context by using representations. She used an internal representation to solve the problem involving fraction division. She imagined and then drew cubes to solve the problem effectively. Also, she was able to solve the fraction multiplication problem by symbolic manipulation along with a representation (see Table 6.3). However, her responses revealed a misconception in using
representation to solve the story problems involving fraction division and fraction multiplication. To solve the story problem involving fraction division \( \left( \text{If} \frac{3}{5} \text{ of a bag of candy weighs} \ 6\frac{3}{4} \text{ pounds, how much does 1 bag of candy weigh?} \right) \), she drew an inappropriate representation for division (measurement model) that did not represent the meaning of the story problem. She was able to recognize the story problem correctly as requiring division with fractions, but she was not able to draw an appropriate diagram to solve the problem. In addition, she struggled to complete the solution without using physical manipulatives. She asked for manipulatives to complete the problem. When the numbers were small and simple in the problem \( (1\frac{3}{4} \div \frac{1}{2}) \), she was able to imagine the manipulative and drew a representation and then completed the solution, but she struggled to complete the solution when the numbers were bigger and more complex in the story problem \( (6\frac{3}{4} \div \frac{3}{5}) \). Also, as seen in Table 6.3, she understood what both multiplication problems were about and was able to solve the story problem involving fraction multiplication by a representation with some use of symbolic manipulation, but her representation of the story problem was conceptually inappropriate. She drew \( 6\frac{3}{4} \) boxes of candy all connected as if one whole. She understood that these were \( 6 \) and \( \frac{3}{4} \) separate boxes of candy, but students, when they see this diagram, might think about it as one whole box.

Thus far, I have described the variations in teachers within the same MKT level. The significant variations in the teachers below the average score of MKT and those around the average score of MKT appeared to be in knowledge of alternative strategies and knowledge of representation. Now, I will describe the similar performances by teachers across different MKT levels to illustrate how they tried to make sense of the division and multiplication with fractions.
Similarities in Teacher Reasoning and Thinking Across Different MKT Levels

Scrutinizing individual teachers’ data in Figure 6.5 again revealed that groups of teachers who received different scores on the MKT measure performed similarity on the interview tasks. Teachers SA and MR, who had different MKT scores, are selected as examples, and similarities in their reasoning and thinking on selected tasks about division and multiplication with fractions are described.

Although Teachers SA and MR were at different levels of MKT (Teacher SA had an IRT score of 0.038132, grouped as high level of MKT among the participant teachers, and Teacher MR had a score of –0.652215, at the low level of MKT), their performances on the interview tasks revealed their similar understanding of division and multiplication with fractions. Both teachers had one invalid response on the six interview tasks (see Figure 6.5). Both teachers exhibited procedural knowledge to solve the fraction division and multiplication problems. They used the standard methods (invert and multiply, and multiply numerators and denominators, respectively, and simplify) to solve the problems, but they were not able to come up with other strategies to solve the problems. (See Chapter 5: Teachers’ Reasoning on the Division and Multiplication with Fractions.)

Both Teachers SA and MR struggled to make the meaning of the division or multiplication with fractions. The story problems they generated for $1 \frac{3}{4} \div \frac{1}{2}$ and $2 \frac{1}{2} \times \frac{3}{4}$ revealed that they had difficulty generating story contexts appropriate for the given problems in a symbolic form. Whereas Teacher SA generated an appropriate story problem for $1 \frac{3}{4} \div \frac{1}{2}$ (Hand buys fabric whose length is one-and-three-quarters meter. How many halves are in the fabric’s length? Or how many halves in one-and-three-quarters meters?) that made the fraction division
problem meaningful after several attempts, she struggled to make meaning of the fraction multiplication problem. She created a story problem (*I have fabric whose length is two and a half meters. I take its triple then divide it among four students. How much will each student have?*) whose meaning differed from the given multiplication with fractions (*2 \( \frac{1}{2} \times \frac{3}{4} \)*). Her story problem consisted of two calculation parts: first multiplying *2 \( \frac{1}{2} \)* by 3, then dividing the result by 4, rather than multiplying *2 \( \frac{1}{2} \)* by the proper fraction *\( \frac{3}{4} \)*. Also, Teacher MR was able to generate an appropriate story problem (*1 \( \frac{3}{4} \) \( \div \) \( \frac{1}{2} \) (If the price for one piece of cake is \( \frac{3}{4} \) riyal and I want to buy \( 2 \frac{1}{2} \) cakes, how much money do I need?)*) that represented the meaning of the fraction multiplication problem, but she struggled to make meaning of the fraction division problem. She posed a story problem (*You have \( 1 \frac{2}{4} \) of chocolate; you want to divide them into two halves equally.*) whose meaning differed from the given division with fractions. She mixed the meaning between dividing by 2 and dividing by \( \frac{1}{2} \) when she generated a story problem for *1 \( \frac{3}{4} \) \( \div \) \( \frac{1}{2} \).*

Although Teachers SA and MR struggled in making meaning of given operations with fractions in symbolic form, they demonstrated fluency in making meaning of operations with fractions in story contexts. Both teachers were able to use representations to make sense of the story problems and division and multiplication with fractions. They provided the same reasoning and thinking on fraction division and multiplication story problems (see Table 6.2). They drew same appropriate representations that conveyed the meaning of the story problems, and they used the representations effectively to solve the problems. Their solution process that built on reasonable steps indicated their conceptual understanding and rational thinking of division and multiplication with fractions (see Teachers’ Representational Fluency section above).
Possible Influence of Professional Development Programs

As shown in the demographic information of each of the six teachers in the examples above (see Table 6.4), the teachers had a varied number of years of teaching experience, and they had various amounts and types of professional development programs. I checked to see if the number and type of professional development programs the participant teachers attended and years of teaching experience in grades 4-6 were potential factors related to teacher reason on the interview tasks.

Table 6.4. The Five Teachers’ Demographic Information

<table>
<thead>
<tr>
<th>Teacher</th>
<th>SA</th>
<th>MR</th>
<th>AA</th>
<th>ZM</th>
<th>SH</th>
<th>ZH</th>
</tr>
</thead>
<tbody>
<tr>
<td>MKT raw scores (IRT scores)</td>
<td>19 (0.038132)</td>
<td>14 (–0.652215)</td>
<td>14 (–0.652215)</td>
<td>14 (–0.652215)</td>
<td>17 (–0.240082)</td>
<td>17 (–0.240082)</td>
</tr>
<tr>
<td>Valid responses to 6 main tasks</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>Years of teaching in grades 4-6</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>8</td>
<td>11</td>
</tr>
<tr>
<td>Professional development</td>
<td>6, including representation, manipulatives, and solving problems (in numbers and in story contexts)</td>
<td>9, including representation, manipulatives, and solving problems (in numbers and in story contexts)</td>
<td>2, including manipulatives and solving problems (in numbers and in story contexts)</td>
<td>5</td>
<td>20, including manipulatives and solving problems (in numbers and in story contexts)</td>
<td>20, including manipulatives and solving problems (in numbers and in story contexts)</td>
</tr>
</tbody>
</table>

Note. The three colors of shading indicate the teachers’ levels of MKT scores: green for high, pink for low, and yellow for middle levels in the participant group. Note that SA is next to MR for easier comparison.

First, it seems that years of teaching experiences did not influence teachers’ mathematical content knowledge for teaching mathematics. Teachers SH and ZH had more teaching experience (8 and 11 years, respectively) than Teachers SA and MR (2 and 3 years, respectively). However, these four teachers responded quite well to the six interview tasks. They
had a higher frequency of valid responses to the six interview tasks than the other participating teachers. This confirms the result in Chapter 4 that shows that there is no relationship between the 44 participating teachers’ years of teaching experiences and their knowledge for teaching mathematics— their level of MKT.

Second, in Table 6.4, it appears there was some relationship between the number of professional development programs attended and teacher reasoning in the interviews. Teachers SA, MR, SH, and ZH had a higher number of professional development programs attended than Teachers AA and ZM, and they had a higher frequency of valid responses to the six interview tasks. However, statistical analyses of the large sample (all 44 participating teachers in Chapter 4) showed that the number of professional development programs the participating teachers attended had essentially no strong effect on their level of MKT.

Finally, a more plausible factor related to teacher reasoning seems to be the type of professional development programs they attended. For example, Teachers MR, AA, and ZM had never participated in professional development programs that explored story problems, and these teachers had difficulty generating appropriate story problems involving fraction divisions. Unlike Teachers AA and ZM, Teacher MR had attended more than one professional development program regarding representational fluency, and her representation fluency was more advanced than Teachers AA and ZM. In contrast, Teachers SH and ZH had attended professional development programs using only physical representations (manipulatives), and their representation fluency was limited. They tried to transfer their knowledge of physical representations into visual representations (diagrams and pictures), but they struggled and were not able to complete the solution using visual representations. Also, both Teachers SA and MR had attended professional development programs including representations, manipulatives, and
solving the problem (either in contexts or without contexts). Neither of the two teachers had attended a professional development program exploring modeling problems using real-world contexts, and they struggled to make sense of the given division and multiplication problems by using a story context. Such data suggest that the nature of professional development programs teachers attended can be an important factor that influences teachers’ conceptual understanding of division and multiplication with fractions.

**Summary**

This chapter described in detail the relationships that emerged in the analysis of teacher responses to three instruments (MKT measure, survey, and interview). Despite the participating teachers’ desire to teach conceptually, their conceptual knowledge of the content (fraction division and multiplication) seemed to limit their thinking about the problems given. Most teachers struggled in at least one task. The participating teachers’ confidence in teaching specific topics, such as teaching rational numbers and representing fractions, was closely related to their mathematical content knowledge for teaching (MKT level, and thinking and reasoning on interview tasks).

Also, teachers’ responses to the MKT measure and interview tasks revealed some interesting relationships between the participating teachers’ MKT scores and their thinking and reasoning about division and multiplication with fractions. Overall, the frequency of teacher confusion or errors in the six interview tasks was closely related to their levels on the MKT measure. Whereas some teachers with the same MKT score performed quite differently on the tasks, other teachers across different MKT scores had a similar performance on the tasks. The significant variations among teachers at same MKT level appeared in their understanding of the operations with fractions, specifically, knowledge of alternative strategies and knowledge of
representation. The similar understanding and similar deficiency of making meaning of division and multiplication with fractions surfaced among teachers across different MKT levels. Finally, the nature of professional development programs seemed to have an effect on the teachers’ mathematical content knowledge for teaching.
CHAPTER 7
CONCLUSION

This study aimed to investigate a group of Saudi upper elementary (grades 4-6) mathematics teachers’ knowledge for teaching in the content strand of rational numbers with an emphasis on fractions. In this closing chapter, I first summarize some important findings in terms of the nature of these teachers’ mathematical knowledge for teaching rational numbers, investigated through one MKT measure and interviews, and a survey on their confidence and beliefs on teaching and learning mathematics. Also, relationships that surfaced among teacher responses to the three instruments (i.e., the MKT measure, the survey, and the interviews), and factors (e.g., professional development experience) that might have influenced their knowledge are summarized. The results are then discussed in relation to the existing literature and the framework explained previously. Finally, this chapter ends with the limitations of the study, and implications and recommendations for future research.

Summary of Findings

Findings From Teacher Responses to MKT Measure and Survey

Overall, the 44 Saudi elementary mathematics teachers scored lower than their counterparts in the United States in content knowledge for teaching rational numbers (MKT measure). The participating teachers were below and around the average IRT score of MKT; none of these teachers scored above 0.18027 in the MKT measure. About half of the participating teachers were around the average IRT score of the norm group (between −0.5 and 0.5), and about half of them were below the average teacher score of the norm group
(below −0.5). Also, it seems that MKT items that were difficult for the Saudi teachers were different from those for the norm group. For example, Saudi teachers performed well on the item on comparing fractions, whereas this item was more difficult than the item on division with fractions for the norm group. The easiest item for the Saudi teachers was not the easiest for the norm group in the U.S., and the Saudi teachers performed very well on some items that were difficult for the norm group.

Although the Saudi teachers’ level of MKT was not high, approximately half of them rated themselves as having high understanding of the mathematical knowledge for teaching fractions, and high confidence in their knowledge preparation for teaching specific topics regarding fractions. Also, the Saudi teachers had mixed beliefs about reform and traditional teaching and learning. They had strong beliefs that are compatible to reform efforts in school mathematics (e.g., relational thinking and conceptual understanding should be encouraged in teaching and learning mathematics), as well as beliefs that kept a traditional view of school mathematics. For example, most teachers held beliefs on using manipulatives to avoid abstract mathematics and learning mathematics by memorizing, along with beliefs that mathematics should be learned as sets of algorithms or rules.

Findings From Teacher Responses to Interview Tasks

Twelve Saudi teachers’ responses to the tasks about division and multiplication with fractions revealed that these teachers had procedural knowledge that is needed for solving typical school mathematics problems, such as solving problems in context or without context by using the standard procedures. As explained in Chapter 5, all 12 teachers who were interviewed were able to solve the given division and multiplication problems without a story context

\[
\left(1 \frac{3}{4} \div \frac{1}{2}; 2 \frac{1}{2} \times \frac{3}{4}\right)
\]

by using standard methods (invert and multiply, and multiplying numerators
and denominators, respectively). However, it seemed that the teachers had a limited conceptual understanding of division and multiplication with fractions. Some of them used alternative strategies for solving the problems. Eight out of the 12 teachers thought that there was no other way to solve the fraction multiplication problem \((2 \frac{1}{2} \times \frac{3}{4})\); six teachers mentioned either that there was no other way to solve the division problem \((1 \frac{3}{4} \div \frac{1}{2})\), or that other strategies to solve \(1 \frac{3}{4} \div \frac{1}{2}\) might be possible, but they did not know these strategies. Many teachers generated an appropriate story problem after a few attempts. The teachers who were good at generating story problems for the given operations with fractions generated one type of story problem (measurement division problem for fraction division, and equal groups multiplication problem for fraction multiplication). No one generated more than one story problem or posed a story problem involving partitive division for the fraction division or area for the fraction multiplication. Also, most teachers were able to draw a diagram that represented the meaning of the given story problems, and some of them had difficulty using their representations properly to solve the given story problems. Also, the teachers’ interpretation of student alternative strategies was mostly procedural and much less on the conceptual aspect. Overall, teachers’ struggles with the interview tasks increased as their level of MKT decreased. Regarding the operations, teachers demonstrated similar levels of understanding of division and multiplication with fractions.

**Relationships Among Teacher Knowledge Measured by MKT, Teacher Beliefs and Confidence, and Teacher Knowledge Revealed in Interviews**

Twelve of the 44 participant teachers responded to all three instruments, and relationships in their responses across the instruments were examined. The results of the study revealed a gap between the 12 Saudi in-service elementary mathematics teachers’ general beliefs about teaching and learning mathematics conceptually, and their mathematics content knowledge needed for
teaching division and multiplication with fractions conceptually. Also, the 12 teachers’ confidence in their knowledge preparation for teaching the topic was somewhat tied to their reasoning with the interview tasks and MKT scores, although there was no significant relationship between the 44 participating teachers’ confidence and their MKT scores.

Comparing the 12 teachers’ MKT scores and interview responses revealed that the teachers’ MKT level was related to their thinking and reasoning on interview tasks. Overall, teachers with higher MKT scores performed better in the interviews than those with lower scores. It seems that the MKT measure provided important information about the teachers’ mathematics knowledge for teaching, and yet it did not provide a complete picture of the teachers’ mathematical knowledge for teaching. Although none of the participating teachers scored higher than an IRT score of 0.18027, some of them were good at solving the interview tasks procedurally and conceptually (e.g., Teachers LA, MA, SH, and AZ). Also, Teachers MR and RA were at the lower level of MKT among participating teachers, but they performed on interview tasks as well as Teachers SA and ZH, who were at high (scores from 0 to 0.5) and middle (scores from –0.5 to 0.5) levels of MKT, respectively. Also, some interesting observations were made between the participating teachers’ MKT levels, and their thinking and reasoning about division and multiplication with fractions. Some teachers at the same MKT score performed differently on some fraction division and fraction multiplication tasks, specifically on alternative strategies and representations. Some teachers across different MKT scores made good sense of the operations with fractions conceptually.

Finally, one potential factor that might have influenced the variations and similarities in teachers’ performance on the interview tasks was the type of professional development programs the teachers attended. For example, Teacher MR attended professional development programs
about representational fluency, and her use of representations was more advanced than other teachers’. Also, teachers who did not have professional development programs about using real-world context for modeling problems tended to struggle in generating story contexts for division and multiplication with fractions problems.

**Discussion**

This study drew on the existing mathematics education literature focusing on teachers’ mathematical knowledge for teaching in general and teaching fractions in particular. Now, the results of this study are discussed in relation to the subdomains of teacher knowledge of mathematics content as shown in the framework and compared to the existing literature in order to affirm and extend what is understood in this study about Saudi elementary mathematics teachers’ content knowledge for teaching fractions.

**The Framework and Instruments**

Before discussing the results, I will step back to make some comments about the guiding framework and instruments. The framework, as shown in Figure 2.3 and explained in Chapter 2, was useful in my dissertation study, which sought to understand the nature of Saudi in-service elementary teachers’ content knowledge for teaching a specific mathematical topic (fractions).

The framework portrays teachers’ knowledge and teachers’ affective factors as the foundation for improving teaching practices and students’ achievements. It helped me focus on and investigate in depth the focal aspects of the study in the two areas of content knowledge for teaching mathematics, as shown in Table 2.1 and explained in Chapter 2.

In the beginning, the focal aspects of the study in investigating content knowledge for teaching mathematics guided the design of the study, including the choice of the instruments (MKT measure, survey, interview) that were helpful to address the research questions. For
example, the survey provided a general picture about Saudi teachers’ confidence and beliefs about mathematics teaching and learning; the MKT measure revealed an overall picture of the mathematics teachers’ content knowledge for teaching rational numbers; and the interviews helped study in depth the focal aspects of mathematics teachers’ knowledge of content and underlying mathematical meaning, which include procedures and strategies, generating story problems, representations, and interpretation of student work. Later, the focal aspects were helpful in discussing the results based on subdomains within mathematics teachers’ content knowledge for teaching, as shown in Table 2.1 in Chapter 2.

In the following section I will discuss the results of this study in relation to the study’s focal aspects of content knowledge for teaching mathematics within each area (teachers’ confidence and beliefs, and mathematics teachers’ knowledge for teaching), and between the two areas. Also, I will compare the results from the existing literature to understand the results on Saudi elementary mathematics teachers’ content knowledge for teaching fractions in a broad context.

**Mathematical Knowledge for Teaching**

Prior research studies in many countries showed that in-service teachers (e.g., Chinnappan & Desplat, 2012; Izsák, 2008; Ma, 1999) and preservice teachers (e.g., Ball, 1990; Depaepe et al., 2015; Tirosh, 2000) had limited mathematical knowledge for teaching fractions, which is foundational content for many advanced mathematics topics (NMAP, 2008). The results of this dissertation study revealed that the Saudi elementary teachers also had limited mathematical knowledge for teaching fractions. Also, this study indicated that the participating teachers who were around the average score of MKT (IRT scores between −0.5 and 0.5) demonstrated a better understanding of the mathematics content regarding division and multiplication with fractions.
than those who were below the average score of MKT. The Saudi teachers with higher IRT scores in the MKT measure on rational numbers performed better than other participating teachers on interview tasks, such as generating and using representations. This result is consistent with Izsák’s (2008) finding that revealed a relationship between the teachers’ level of MKT and their use of representations and interaction with students’ thinking on fraction multiplication.

**Teacher Beliefs and Confidence**

The high percentage of the participating Saudi elementary teachers indicated confidence in their preparation for and readiness in teaching elementary mathematics in general and rational numbers in particular. However, none of them scored higher than 0.18027 on the MKT measure. In fact, previous studies on teachers’ confidence and knowledge had mixed findings. The result of this study is similar to what Li and Kulm (2008) found regarding U.S. preservice teachers who had high confidence about their knowledge preparation for teaching mathematics but had limited mathematics knowledge for teaching fraction division. In contrast, Li, Ma, and Pang (2008) found that mainland Chinese and South Korean preservice elementary teachers did not have high confidence in their knowledge preparation for teaching, although they had adequate mathematical knowledge for teaching fraction division.

In addition, the results of this study indicated that the participating teachers’ beliefs about teaching and learning mathematics did not reflect their mathematical content knowledge for teaching. Although most teachers held beliefs on the importance of relational thinking and conceptual understanding in teaching and learning mathematics, such as using a representation and generating a story problem, many of them struggled to respond to the tasks related to representations and/or story problems, either in MKT items and/or interview tasks. It seems that
the Saudi teachers accepted reform efforts in school mathematics (e.g., developing conceptual foundation), and yet they needed to be equipped with knowledge for teaching mathematics in such a way. This result is consistent with Charalambous’s (2015) finding that preservice teachers’ belief about the importance of supporting students to develop a conceptual understanding is insufficient to offer a conceptual explanation. This result is, however, different from the findings by Newton et al. (2012) that referred to a secure link between preservice teachers’ content knowledge and their beliefs about their ability to teach mathematics effectively.

**Conceptual and Procedural Knowledge**

The Saudi teachers’ performance on the mathematics tasks in the interviews revealed that their mathematics content knowledge for teaching division and multiplication with fractions was procedurally stronger than conceptually. This result matches and extends the statistical finding by Khashan’s (2014) quantitative study that showed Saudi in-service elementary teachers performed better on procedural items than conceptual items related to fractions and operations with fractions. Also, this finding is consistent with several research studies that concluded teachers had a weak conceptual understanding of operations with fractions (e.g., Chinnappan & Desplat, 2012; Lo & Luo, 2012; Tirosh, 2000). For example, teachers could not explain the meaning behind the procedure (Tirosh, 2000), had difficulty representing fraction division by using word problems or pictorial diagrams (Lo & Luo, 2012), and had a conceptual weakness in relating expressions with fractions to contextual situations (Chinnappan & Desplat, 2012). Overall, the Saudi teachers’ competence in procedural knowledge was not sufficient in supporting their reasoning on operations with fractions conceptually. This result suggests that Saudi teachers need to prepare to teach division and multiplication with fractions conceptually as well as procedurally.
Moreover, the results of this study showed varied levels of teacher knowledge in three contexts: alternative strategies, generating story problems, and using representations effectively to solve the problems (with or without context). These aspects of content knowledge for teaching mathematics will be discussed in the following sub-sections.

**Alternative strategies.** The Saudi teachers demonstrated limited knowledge of informal strategies for solving the fraction division and fraction multiplication problems. Although the operations with fractions could be solved in several different ways, two types of informal approaches appeared in the interviews with the Saudi teachers: using the distributive property and using a representation. During or after the interviews, all the teachers mentioned that their knowledge of alternative strategies was influenced by what they had learned in school as students, what was included in the curriculum that was supposed to be taught to students, and what subjects (PD content) were explored in professional development programs. This result sheds light on the importance of Saudi teachers’ exploration of different alternative strategies through their education.

As explained in Chapter 1, the new school mathematics textbook series in Saudi Arabia aimed to guide teachers to solve a problem by using different methods that could enable them to open a discussion about alternative strategies for solving a problem. This means that the new textbook series in Saudi Arabia encouraged teachers to use different strategies in problem solving, but the teachers were not yet able to do this productively. Also, this indicates that the teachers might not have seen the intention of the curriculum to consider alternative strategies. Curriculum change does not ensure changes in teaching practice (Wright, 2020). Kim (2019) concluded that teachers’ use of curriculum resources will not be productive or support students’ learning if teachers interpret the curriculum in a way that is not aligned with the curriculum
designers’ intention. Saudi teachers might need to better understand the aims and guidance of the curriculum. Also, curriculum designers need to make their expectations and aims of mathematical lessons accessible to teachers through appropriate forms of communication (Remillard & Kim, 2020). In addition, there is a possibility that high-stake assessments in Saudi Arabia are not aligned with the curriculum aims yet, and teachers do not see the need for change.

Examining the courses offered in teacher preparation at some universities and professional development programs offered in Saudi Arabia through personal contact, I found that limited courses and programs were incorporating alternative strategies. Although the importance of teachers’ understanding of different strategies for solving problems in supporting their students’ learning was specified by the teachers’ guides, Saudi teachers were not supported and prepared for using/teaching these strategies. It seems that teachers were still influenced by the traditional mathematics teaching and learning in Saudi Arabia, such as using standard algorithms and procedures. Therefore, teachers who have a traditional education need to be supported in being flexible to draw on different strategies, not to be restricted to standard methods to solve problems, in order to support their students to build different strategies based on the concepts they learned. Developing such teacher knowledge and capacity is important for better instruction in Saudi Arabia.

**Story problems.** The Saudi elementary teachers demonstrated suitable knowledge in generating story problems for fraction multiplication and limited knowledge in generating story problems for fraction division. As noted, the Saudi teachers generated one type of story problem in multiplication and division, respectively: equal groups multiplication story problems and measurement division story problems. These findings suggest that teachers in Saudi Arabia need
to be aware of different mathematical and contextual meanings of the operations with fraction to support their students’ learning.

During or after the interviews with the Saudi teachers, they mentioned that they developed their ability to pose a story problem based on their experiences in reading and solving the story problems included in the textbooks in Saudi Arabia. Using story problems is suggested for meaningful mathematics teaching and learning in Saudi Arabia. However, elementary students in Saudi Arabia are not required to pose story problems. In addition, the study showed that Saudi teachers are not prepared for modeling real-world problems, even though developing such teacher knowledge and capacity is an avenue for developing students’ understanding of mathematical concepts and for engaging students in meaningful mathematics (e.g., Luo, 2009).

**Representations.** As the interview results indicated, there were variations in the Saudi teachers’ knowledge of using representations effectively to solve problems involving multiplication and division with fractions. During the interviews, the teachers mentioned that representations were included in the curriculum that they used, but they used representations during the introduction to topics or to explain the final results/answers, and using representation were not required in the curriculum guidelines as an essential proficiency with fraction operations that students should develop. Examining the school mathematics curriculum in Saudi Arabia, I found the new textbook series had multiplication and division with fractions by using representations in one short lesson, and then moved to standard methods in the following lesson. Again, it seems that using representations are incorporated, but not emphasized, in the new textbook series, and therefore teachers did not give enough attention to representations. It seems that Saudi teachers use representations to explain the concepts, not treat them as essential tools
for solving problems, even though representations are versatile tools that play an important role in mathematics teaching and learning (Kim & Remillard, 2020).

In personal contacts with mathematics educators and professional developers in Saudi Arabia, I found that representations are offered in a few lessons in some courses, and professional development programs deal with physical representations (manipulatives) more than visual representations. Saudi teachers need to improve their ability in using visual representations to explain the solution process and to make sense of operations with fractions conceptually as well as procedurally. Also, mathematics educators and professional development providers in Saudi Arabia need to be aware of the importance of representations, specifically visual ones, and support teachers to improve such an ability regarding representations.

Also, the demographic information of six teachers in Chapter 6 indicates that the content of professional development programs might have had a significant impact on teachers’ knowledge of representations. That is, Saudi teachers who attended PD including representations as its content performed better than those who did not. I will discuss the relationship between teacher knowledge and PD a little more in the following section.

**Teachers’ Content Knowledge and Professional Development**

It was evident that the teachers with better responses to the interview tasks had PD programs different from those experienced by the other teachers. This means that the nature of professional development programs might have influenced the Saudi elementary mathematics teachers’ content knowledge. This finding is consistent with several research studies (e.g., Hill & Ball, 2004; Jacob, Hill, & Corey, 2017; Jacobs et al., 2010) that concluded that professional development programs have an impact on teachers’ mathematical knowledge for teaching. For
example, Jacobs and her colleagues (2010) observed that teachers’ knowledge and skills in describing and interpreting students’ work improved with professional development support.

In Saudi Arabia, the Ministry of Education has provided various professional development programs (PDs) for elementary mathematics teachers, which include conceptual understanding of concepts, using manipulatives in teaching mathematics, solving problems in context or without context, differentiated instruction, lesson study, active learning strategies, and student-centered learning. It seems that these programs have influenced the Saudi teachers’ beliefs about mathematics teaching and learning that align with reform-based conceptual understanding of mathematics. Yet, these programs were insufficient to support teachers to improve their mathematical content knowledge needed for teaching specific topics in such ways. As described in Chapter 2, both teachers’ mathematical knowledge for teaching and teachers’ affective factors are important for the quality of teaching practices. Therefore, the PDs that support teachers to improve their content knowledge for teaching, for example, operations with fractions, are needed as well as PDs that contribute to teacher beliefs on the importance of teaching conceptually.

Upon examining the content of the PD programs the teacher attended, it seems that there have been no programs related to improving the teachers’ content knowledge for teaching specific topics, such as fractions. Therefore, the professional development in Saudi Arabia should allot time for teachers to explore specific mathematics content to support them in improving their content knowledge. This corresponds to the characteristics of professional development programs that are critical for improving teachers’ knowledge and skills (Desimone, 2009). According to Desimone (2009), focusing on content is one of the key features of effective professional development programs.
Limitations

Exploring a group of Saudi upper elementary mathematics teachers’ content knowledge and their confidence and belief in teaching mathematics led to significant findings. Yet, the study has some limitations. Upon reflecting on the research findings, these limitations should be considered, which are related to the methodology chosen for this study.

First, the sample did not represent all in-service elementary mathematics teachers in Saudi Arabia, not even in Alahsa City from where the participant teachers were selected. The sample is too small to make claims about all elementary mathematics teachers in Alahsa City or Saudi Arabia. In addition, the participant teachers were all female. Therefore, this study cannot be generalized to all in-service elementary mathematics teachers in Alahsa City, let alone those in Saudi Arabia. In addition, the teachers who were selected for this study taught different upper grade levels (grades 4-6), and the content of the rational numbers was taught differently based on the mathematics curriculum expectations at each grade level. This could have influenced the content knowledge of rational numbers each teacher demonstrated.

Second, the instruments employed in this study have some limitations. The survey is limited in its ability to provide detailed information about the teachers’ beliefs and confidence. For example, teachers had mixed beliefs about reform and traditional teaching practice, and the survey could not help investigate teachers’ beliefs further. The MKT measure used in the study has two limitations. The length of the MKT measure impacted teachers’ decisions regarding whether to participate in the interview of this research study. It took approximately 40 to 60 minutes for the teachers to complete the MKT measure, and some teachers mentioned that it was long for them. Also, the MKT measure alone did not provide a complete picture about mathematical knowledge for teaching; other tools were needed. Therefore, the participant
teachers’ content knowledge was examined deeply in two specific areas, fraction division and fraction multiplication, through interviews, and this did not represent the teachers’ content knowledge in other mathematics topic areas regarding rational numbers or fractions. Another limitation is related to the order of interview tasks. The order of the tasks influenced and limited the data collected on Task 4 in each set of problems. It seemed that the teachers’ responses to Task 4, in which the teachers were asked to interpret students’ alternative strategies to solve the story problems, were influenced by their responses to Task 3. In fact, Task 4 asked teachers to interpret an alternative strategy provided for the problem they had just solved in Task 3.

The third limitation is related to the method of recording interviews. Video recording was not allowed (only audio recording was permitted) during the interview for cultural reasons, which might have influenced the analysis and interpretation of the data. Also, audio recording of interviews influenced teachers’ decisions to participate in the interview.

Fourth, investigating teachers’ knowledge includes many significant aspects, as shown in the guiding framework (Figure 2.3) in Chapter 2. As seen in the framework, there are four significant areas in investigating teachers’ mathematical knowledge for teaching—teachers’ knowledge, teachers’ affective factors, teaching practices, and students’ achievements, and each aspect included many elements or domains, which cannot be investigated in one single research study. Therefore, as described earlier in the chapter and Chapter 2, my study focused on the particular aspects, that is, teacher beliefs and confidence, and knowledge of content in terms of procedures, strategies, story problems, representations, and student work. Also, teachers’ mathematical knowledge for teaching included many domains and elements that can be investigated based on the cognitive perspective and situated perspective (Sfard, 1998). I focused
on teacher knowledge more from the cognitive perspective and did not attend to teachers’ knowledge activated in teaching practice or knowledge revealed in classroom teaching.

Future Research and Implications

This study contributed to understanding Saudi elementary mathematics teachers’ content knowledge for teaching rational numbers, specifically division and multiplication with fractions. It has several implications for professional development, teacher education, and future research, particularly in Saudi Arabia. This study can inform the design of support materials for teachers, professional developers, and teacher educators by identifying the strengths and weaknesses in Saudi teachers’ mathematical knowledge for teaching for further improvement.

Professional Development

The results presented in this study underscore the importance of professional development programs because of their potential influence on teachers’ conceptual understandings of the operations with fractions. In fact, most of the participant teachers who struggled with MKT items and interview tasks on rational numbers and fractions had no professional development on the content.

Professional developers should provide teachers with further opportunities that promote the growth and development of conceptual understandings of content. As mentioned above, in addition to the reform pedagogy, professional development programs in Saudi Arabia should also focus on specific mathematical topics to support teachers on the content that is taught to students. Since fractions are one of the important mathematical topics and, at the same time, a challenging topic for many teachers and students, it should be explored extensively in PD programs.
Teacher Education

The participating teachers pointed out that their learning journey in K-12 and teacher preparation had significantly impacted what they knew and understood about fraction topics. This indicates the need to develop preservice teachers’ content knowledge for teaching important topics, like operations with fractions, during teacher education as well. Therefore, Saudi mathematics teacher educators should work to design and organize courses focusing on conceptual understanding of mathematics content in new and/or existing courses. This could be achieved through various means that engage preservice teachers with different mathematical tasks regarding alternative strategies, different types of the story problems, multiple representations, and students’ correct and incorrect strategies.

Future Research

This study has several implications for future research, particularly in Saudi Arabia. First, more studies on teacher knowledge in Saudi Arabia are needed, including various content and domains of knowledge, since there have not been many studies on Saudi teachers’ knowledge for teaching mathematics. For example, additional work needs to be done to examine Saudi teachers’ mathematical content knowledge for teaching rational numbers in relation to specific topics, such as fractions concepts, comparing fractions, equivalence fractions, addition and subtraction with fractions, decimal and percentage, etc. Also, future research is needed to study teachers’ mathematical content knowledge for teaching other topics, such as proportional reasoning, geometrical concepts, etc. In addition, research is required to examine Saudi mathematical teachers’ knowledge that is activated in response to classroom situations, such as how they react to student questions or incorrect answers, and how they use curriculum resources and guides to make in-the-moment decisions. Also, future research is needed to investigate other
domains of teachers’ mathematical knowledge for teaching, such as Saudi teachers’ pedagogical content knowledge.

Second, studies on Saudi teachers’ knowledge need to include diverse samples. This study investigated a small sample of female Saudi elementary mathematics teachers in one city. More small-scale and large-scale research studies are needed to provide a more accurate picture of Saudi teachers’ mathematical knowledge for teaching. Also, future research is required with teachers in different grades levels, such as lower elementary and middle/secondary levels. Future research may replicate this study with male in-service elementary mathematics teachers to examine similarities and differences between the genders, with other female in-service elementary mathematics teachers in different cities in Saudi Arabia, or with preservice teachers to investigate the nature of the content knowledge that the new generation of teachers has.

Third, studies on Saudi teachers’ knowledge need to be methodologically diverse. This study used a survey to investigate Saudi teachers’ confidence and beliefs about mathematics teaching and learning. The survey gave an overall picture about the Saudi teachers’ beliefs and confidence. Future studies can study extensively teachers’ affective factors by using other instruments, such as interviews and other forms of data collection. The MKT measure could be used by different course instructors or professional development programs to examine when and how Saudi preservice and in-service teachers’ mathematical knowledge for teaching is improved. Moreover, instruments other than the MKT measure can be developed and used to assess Saudi teachers’ mathematics knowledge.

Finally, the findings of this study referred to a deficiency in some aspects of teachers’ mathematical content knowledge and the importance of professional development programs to improve these aspects. Therefore, designing and studying different PD offerings and courses that
support the improvement of Saudi elementary mathematics teachers’ mathematical content knowledge are needed for further investigations. For example, studying teachers’ reasoning and thinking on different mathematics tasks might help teacher educators and professional developers in Saudi Arabia to design the courses and programs that support in-service teachers and preservice teachers. To address the Saudi teachers’ limited knowledge of representations and story problems, it is important for mathematics education researchers to explore which materials and learning opportunities help teachers to develop their abilities, for example, to generate story problems for fraction division and fraction multiplication, and to use appropriate representations effectively for explaining the solution process of story problems involving those operations.
REFERENCES


Appendix A

Lessons from Saudi Mathematics Textbooks
Appendix B

Survey
Questionnaire:

- How many years of teaching experience do you have in elementary (grades 1-6) school?
- How many years of teaching experience do you have in upper elementary levels (grades 4-6)?
- What grades have you taught?
- What is your highest education degree? In what area?
- How many professional development programs do you attend to? How many days or hours are these equivalent to?
- Do you enjoy teaching mathematics?
  (a) Yes  (b) Neutral  (c) No

1. How would you rate yourself in terms of the degree of your understanding of the mathematical knowledge for teaching fractions?
  (a) High  (b) medium  (c) Low

2. Considering your training and experience in both mathematics and instruction, how ready do you feel you are to teach primary school mathematics in general and the following topics in specific?

<table>
<thead>
<tr>
<th></th>
<th>Very ready</th>
<th>Ready</th>
<th>Not ready</th>
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</thead>
<tbody>
<tr>
<td>Primary school mathematics in general</td>
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<tr>
<td>Rational numbers in general (ex. Fractions, decimals, percentages)</td>
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<tr>
<td>Representing fractions using words, numbers, or models</td>
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<tr>
<td>Representing addition with fractions using words, numbers, and models</td>
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<td>Explaining addition with fractions using words, numbers, and models</td>
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<td>Representing subtraction with fractions using words, numbers, and models</td>
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<td>Explaining subtraction with fractions using words, numbers, and models</td>
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<td>Explaining division with fractions using words, numbers, and models</td>
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</table>
3. To what extent do you agree or disagree with each of the following statements?

<table>
<thead>
<tr>
<th>Statement</th>
<th>Agree a lot</th>
<th>Agree</th>
<th>Disagree</th>
<th>Disagree a lot</th>
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<tbody>
<tr>
<td>More than one representation (picture, concrete material, symbols, etc.) should be used in teaching a mathematics topic</td>
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<td>Use of manipulatives can help students avoid abstract mathematics.</td>
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<tr>
<td>Mathematics should be learned as sets of algorithms or rules that cover all possibilities.</td>
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<tr>
<td>Teacher needs to know students’ common misconception/difficulty in teaching a mathematics topic.</td>
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<tr>
<td>Teacher should prevent students from making errors in their learning of mathematics.</td>
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<tr>
<td>Solving mathematics problems often involves hypothesizing, estimating, testing, and modifying findings.</td>
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<tr>
<td>Learning mathematics mainly involves memorizing.</td>
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<tr>
<td>There are different ways to solve most mathematical problems.</td>
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<tr>
<td>Modeling real-world problems is essential to teaching mathematics.</td>
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الملحق ب

اسألة الدراسة

استبيان:

كم عدد سنوات الخبرة في تدريس المرحلة الابتدائية (1-6)؟
كم عدد سنوات الخبرة في تدريس الصفوف العليا من المرحلة الابتدائية (4-6)؟
المراحل التي درستها؟
ما درجتك العلمية؟ متخصصة؟
كم عدد البرامج التدريبية التي حضرت؟ كم تعادل بالأيام أو الساعات؟
هل تستمتع بتدريس الرياضيات؟

(أ) لا (ب) محيد (ج) نعم

1 - كيف تقيم درجة فهمك للمعرفة الرياضية اللازمة لتدريس الكسور؟
(أ) عالية (ب) متوسطة (ج) منخفضة

2 - بالنظر إلى تدريبك وخبرتك في كل من الرياضيات وطرق تدريسه، ما مدى استعدادك لتدريس الرياضيات في المدارس الابتدائية بشكل عام والمواضيع التالية على وجه التحديد؟

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<th>الرياضيات للمرحلة الابتدائية بشكل عام</th>
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<th>مستعد جدا</th>
<th>مستعد</th>
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إلى أي مدى توافق أو لا توافق على كل عبارة من العبارات التالية؟

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<th>موافق جدا</th>
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<td>حل المشكلات أو المسائل الرياضية غالبا يتضمن سلسلة من الخطوات: افتراض، تقدير، تجرب، تعديل الحل</td>
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<tr>
<td>تعلم الرياضيات بنطوي أساسا على الحفظ يوجد طرق مختلفة لحل معظم مسال الرياضيات</td>
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<tr>
<td>نمذجة مسائل واقع الحياة اللفظية (استخدام مشكلات رياضية من واقع الحياة) هو أساسي لتدريس الرياضيات</td>
<td></td>
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</tbody>
</table>
Appendix C

Pre-interview
Part 1:

Determine what equation can be used to solve the following story problem (e.g. \(3 \frac{3}{4} + \frac{1}{3}\))

- You have \(1 \frac{3}{4}\) cups of jelly worms and a recipe that calls for \(\frac{1}{3}\) cup of jelly worms. How many batches of your recipe can you make?

- If \(\frac{1}{3}\) pound of candy costs $1, then how many dollars should you expect to pay for \(1 \frac{3}{4}\) pounds of candy?

- If \(1 \frac{3}{4}\) pounds of candy costs \(\frac{1}{3}\) of a dollar, then how many pounds should you be able to buy for 1 dollar?

- If you have \(1 \frac{3}{4}\) pounds of candy and you divide the candy in thirds then how much candy will you have in each portion?
Part 2:

Choose the appropriate expression that can be represented by each diagram.

a. A.

\[ 4 \div \frac{1}{3} \]

b. B.

\[ \frac{1}{3} \div 4 \]

c. C.

\[ \frac{2}{3} \times \frac{1}{2} \]
الملحق ج

قبل المقابلة

الجزء الأول:

حدد المعادلة التي يمكن استخدامها لحل المسألة اللغوية التالية (على سبيل المثال

- لديك كوب من قطع الشوكولاتة الصغيرة والوصفة تتطلب 1/3 كوب من قطع الشوكولاتة. كم عدد الوصفات التي

يمكنك صناعتها؟

- إذا كانت تكلفة 1/3 كيلو غرام من الحلوى ريال واحد، فكم ريال تتوقع أن تدفعه مقابل 3/4 كيلو غرام من الحلوى؟

- إذا كانت تكلفة 3/4 كيلو غرام من الحلوى هو 1/3 ريال، فكم كيلو غرام يجب أن تكون قادرًا على الشراء مقابل ريال واحد؟

- إذا كان لديك 3/4 كيلو غرام من الحلوى وقمت بتقسيم الحلوى إلى ثلاث أجزاء، فكم من الحلوى سيكون لديك في كل جزء؟
الجزء الثاني:
اختر التعبير المناسب الذي يمثل النموذج التالي

- أ

\[
4 \div \frac{1}{3} = \frac{4}{1} \times \frac{3}{1} = 12
\]

- ب

\[
\frac{1}{3} \div 4 = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}
\]

- ج

\[
\frac{2}{3} \times \frac{1}{2} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}
\]
Appendix D

Interview Protocol
Set 1

Part 1:

1) People seem to have different approaches to solving problems involving division with fractions. How do you solve a problem like this one?

\[ \frac{3}{4} \div \frac{1}{2} = \]

2) Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story for

\[ \frac{3}{4} \div \frac{1}{2} = ? \]

Part 2:

3) Using the representation to solve the following story problem:

If \( \frac{3}{5} \) of a bag of candy weighs \( 6 \frac{3}{4} \) pounds how much does 1 bag of candy weigh?

4) Imagine that you are giving this task for your students, and this is the answer for one of your students, Reem.

\[ 6 \frac{3}{4} \div 3 = 2 \frac{1}{4} \]

\[ 6 \div 3 = 2 \]

\[ \frac{3}{4} \div 3 = \frac{1}{4} \]

\[ \frac{1}{4} \times 5 = 11 \frac{1}{4} \]

a. Do you think Reem’s strategy is reasonable? Explain.

b. Do you think Reem’s strategy could apply to other problem? Explain.

c. Do you think Reem’s strategy is efficient and could be taught for students? Explain.
Set 2:

Part 1:

1) People seem to have different approaches to solving problems involving multiplication with fractions. How do you solve a problem like this one?

\[ 2\frac{1}{2} \times \frac{3}{4} = ? \]

2) Imagine that you are teaching multiplication with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story for \( 2\frac{1}{2} \times \frac{3}{4} = ? \)?

Part 2:

3) Using the representation to solve the following story problem:

Mary has \( 6 \frac{3}{4} \) boxes of candy. One box of candy weighs \( \frac{2}{3} \) pounds, how many pounds of candy do she has?

4) Imagine that you are giving this task for your students, and this is the answer for one of your students, Dana.

\[
\frac{3}{4} \times \frac{2}{3} = ?
\]

\[
6 \times \frac{1}{3} \times 2 = 2 \times 2 = 4
\]

\[
\frac{3}{4} \times \frac{1}{3} \times 2 = \frac{1}{4} \times 2 = \frac{1}{2}
\]

\[
6 \frac{3}{4} \times \frac{2}{3} = 4 \frac{1}{2}
\]

a. Do you think Dana’s strategy is reasonable? Explain.

b. Do you think Dana’s strategy could apply to other problem? Explain.

c. Do you think Dana’s strategy is efficient and could be taught for students? Explain.
المملحق د
المقابلة

المجموعة الأولى:

الجزء الأول

أ - لدى الناس أساليب مختلفة لحل المسائل الرياضية التي تتضمن قسمة الكسور. كيف تحل مسألة كهذا؟

$$1 \frac{3}{4} \div 1 \frac{1}{2} =$$

ب - تخيلي أنك تدرس موضوع قسمة الكسور، ولجعل هذا الموضوع ذو معنى لطالباتك وجدت أن الكثير من المعلمات يقومون بربط الرياضيات بمواضيع أخرى، أحيانا يربطون المواضيع المجردة بالحياة اليومية أو المسائل اللطيفة لتوسيع المحتوى. ما هي القصة أو المسأله اللطيفة المناسبة لقسمة الكسور؟

$$1 \frac{3}{4} \div 1 \frac{1}{2} =$$

الجزء الثاني

أ - استخدمي النماذج لحل المسألة اللطيفة التالية:

إذا كان وزن $\frac{3}{5}$ من كيس الحلوى هو $\frac{3}{4}$ كيلوغرام، كم وزن كيس 1 من الحلوى؟

ب - تخيلي أنك تعطي هذه المسأله لطالباتك، وهذه هي إجابة الطالبة، ريم

$$6 \frac{3}{4} \div 3 = 2 \frac{1}{4}$$

$$6 \div 3 = 2$$

$$3 \div 3 = 1 \frac{1}{4}$$

$$2 \frac{1}{4} \times 5 = 11 \frac{1}{4}$$

هل تعتقد أن إستراتيجية ريم منطقية؟ أشرح إجابتك

هل تعتقد أن إستراتيجية ريم يمكن استخدامها لحل مسألة أخرى؟ أشرح إجابتك

هل تعتقد أن إستراتيجية ريم فعالة وكافية ويمكن تدريسها للطلاب؟ أشرح إجابتك
المجموعة الثانية

الجزء الأول

أ - لدى الناس أساليب مختلفة لحل المسائل الرياضية التي تتضمن ضرب الكسور. كيف تحل مسألة كهذه؟

ب - تخيلي أنك تدرس موضوع ضرب الكسور، ولجلع هذا الموضوع ذو معنى لطالباتك وجدت أن الكثير من المعلمات يقومون بربط الرياضيات بمواضيع أخرى، أحيانا يربطون المواضيع المجردة بالحياة اليومية أو المسائل اللفظية لتوضيح المحتوى. ما هي القصة أو المسألة اللفظية المناسبة لـ

الجزء الثاني

أ - استخدمي النماذج لحل المسألة اللفظية التالية:

مرحباً لديها 6 صندوق من الحلوى. وزن صندوق الحلوى الواحد هو $\frac{2}{3}$ كيلوغرام، كم كيلوغرام من الحلويات لديها؟

ب - تخيلي أنك تعطي هذه المسألة لطالباتك، وهذه هي إجابة الطالبة، دانه

هل تعتقد أن استراتيجية دانه منطقية؟ اشرح إجابتك

هل تعتقد أن استراتيجية دانه يمكن استخدامها لحل مسألة أخرى؟ اشرح إجابتك

هل تعتقد أن استراتيجية دانه فعالة وكافية؟ ويمكن تدريسها للطالبات؟ اشرح إجابتك

\[
\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}
\]

\[
\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}
\]
Appendix E

Summary of Teacher Responses to Interview Tasks
<table>
<thead>
<tr>
<th>FRACTION DIVISION</th>
<th>LA</th>
<th>SA</th>
<th>MA</th>
<th>SH</th>
<th>ZH</th>
<th>MN</th>
<th>AZ</th>
<th>AA</th>
<th>MR</th>
<th>RA</th>
<th>HU</th>
<th>ZM</th>
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</thead>
<tbody>
<tr>
<td>Approaches to solving division with fractions</td>
<td>SM</td>
<td>SM</td>
<td>SM</td>
<td>SM</td>
<td>SM</td>
<td>SM</td>
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<td>R</td>
<td>RS</td>
<td>S</td>
<td>IC</td>
<td></td>
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<tr>
<td>Interpretation of students’ alternative strategy on division with fractions</td>
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<td>V</td>
<td>V</td>
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<td>T</td>
<td>E</td>
<td>N</td>
<td>N</td>
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</tbody>
</table>
### Fraction Multiplication

#### Approaches to solving a multiplication with fractions
- Standard Method (SM)
- Distributive Law (D)
- Incomplete (IN)
- Incorrect (IC)
- Equal Group (EG)
- Measurement Division (MD)
- Appropriate (A)
- Inappropriate (IA)
- Solving Problem by Using Representation (R)
- Solving Problem Symbolically (S)
- Solving Problem by Representation and Symbolically, or Using Representation with some use of Symbolic (RS)
- Valid (V)
- Invalid (IV)
- Generalizable (G)
- Efficient (E)
- Teachable (T)
- Not generalizable, Not efficient, or Not teachable (N)

#### Generating a story problem for fraction multiplication

#### Representation of a story problem for fraction multiplication

#### Interpretation of students’ alternative strategy on multiplication with fractions

<table>
<thead>
<tr>
<th>FRACTION MULTIPLICATION</th>
<th>LA</th>
<th>SA</th>
<th>MA</th>
<th>SH</th>
<th>ZH</th>
<th>MN</th>
<th>AZ</th>
<th>AA</th>
<th>MR</th>
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<th>ZM</th>
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<tr>
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<tr>
<td>Generating a story problem for fraction multiplication</td>
<td>V</td>
<td>IV</td>
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<tr>
<td>Representation of a story problem for fraction multiplication</td>
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<td>A</td>
<td>IA</td>
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<tr>
<td>Interpretation of students’ alternative strategy on multiplication with fractions</td>
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</tbody>
</table>

Standard Method (SM); Distributive Law (D); Incomplete (IN); Incorrect (IC); Equal Group (EG); Measurement Division (MD); Appropriate (A); Inappropriate (IA); Solving Problem by Using Representation (R); Solving Problem Symbolically (S); Solving Problem by Representation and Symbolically, or Using Representation with some use of Symbolic (RS); Valid (V); Invalid (IV); Generalizable (G); Efficient (E); Teachable (T); Not generalizable, Not efficient, or Not teachable (N).

The three colors refer to the teachers’ levels of MKT that are relative among the participating teachers in this analysis. Teachers with a high level of MKT had IRT scores between 0 and 0.5 (purple). Teachers with a middle level of MKT had IRT scores between –0.5 and 0 (blue). Teachers with a low level of MKT had IRT scores below –0.5 (orange).
Appendix F

Human Subjects Institutional Review Board Approval Letter
Date: August 26, 2019

To: Ok-Kyeong Kim, Principal Investigator
    Mona Aladil, Student Investigator for dissertation

From: Amy Naugle, Ph.D., Chair

Re: IRB Project Number 19-08-08

This letter will serve as confirmation that your research project titled “Saudi Elementary Mathematics Teachers’ Knowledge for Teaching Fractions” has been approved under the expedited category of review by the Western Michigan University Institutional Review Board (IRB). The conditions and duration of this approval are specified in the policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note: This research may only be conducted exactly in the form it was approved. You must seek specific board approval for any changes to this project (e.g., add an investigator, increase number of subjects beyond the number stated in your application, etc.). Failure to obtain approval for changes will result in a protocol deviation.

In addition, if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the IRB for consultation.

The Board wishes you success in the pursuit of your research goals.

A status report is required on or prior to (no more than 30 days) August 25, 2020 and each year thereafter until closing of the study.

When this study closes, submit the required Final Report found at https://wmich.edu/research/forms.

Note: All research data must be kept in a secure location on the WMU campus for at least three (3) years after the study closes.
Appendix G

The Ministry of Education Approval Letter
الملكية العربية السعودية
وزارة التعليم
(20)
الإدارة العامة للتعليم بمحافظة الأحساء
إدارة التخطيط والتطوير-بحث سياسات التعليم

إلى من يهم الأمر

<table>
<thead>
<tr>
<th>الاسم</th>
<th>منتى بنت خليفة عبد اللطيف العديل</th>
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<td>رقم السجل المدني</td>
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<td>الجامعة التي تدرس بها</td>
<td>جامعة غرب متنقلا بapphirea</td>
</tr>
<tr>
<td>المدرسة الدراسية لعلم الرياضيات</td>
<td>المدرسة الدراسية</td>
</tr>
<tr>
<td>زيارة ميدانية</td>
<td>من 10 سبتمبر 2019 إلى 30 أكتوبر 2019</td>
</tr>
</tbody>
</table>

نُقِدْكم علماً بأن الباحثة قد تقدمت بطلب تسهيل مهامها. في جميع معلومات ولا مانع لدى إدارة التخطيط والتطوير بالإدارة العامة للتعليم بمحافظة الأحساء من تسهيل مهامها البحثية وفقاً للإجراءات التنظيمية واللوائح المعمول بها.

والسلام عليكم ورحمة الله وبركاته...

مدير إدارة التخطيط والتطوير

حمد بن سعود العمير