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On the Local Theory of Profinite Groups

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by

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Let G be a finite group, and H be a subgroup of G. The transfer homomorphism emerges from the natural action of G on the cosets of H. The transfer was first introduced by Schur in 1902 [22] as a construction in group theory, which produce a homomorphism from a finite group G into H/H' an abelian group where H is a subgroup of G and H' is the derived group of H. One important first application is Burnside's normal p-complement theorem [5] in 1911, although he did not use the transfer homomorphism explicitly to prove it.

Burnside Theorem. *Let G be a finite group, and let P be a sylow p-subgroup that is contained in the center of its normalizer, then G has a normal subgroup H which has elements of P as its coset representatives.*

Emil Artin in 1929 [1] Extended the definition to the situation G is infinite and H is a subgroup of G of finite index. The first place the transfer homomorphism appeared in a textbook was in 1937, Lehrbuch der Gruppentheorie, by Hans Zassenhaus [28]. In 1959 the transfer was popularized in the mathematical community in America by Marshall Hall in the textbook The Theory of Groups [14] where the transfer "in German *Die Verlagerung*" as a homomorphism from a group G into a subgroup H of finite index was introduced.

In an effort to define the transfer homomorphism for profinite groups, Oliver Schirokauer in 1996 [21] published a paper in which he presented a new definition for the standard cohomological transfer as an integral.

In this study we will give a definition of the transfer homomorphism for profinite groups which is an analog of Marshall Hall's definition of the transfer homomorphism for finite groups. Our definition depends on the group action and structure, in relation to axiom of choice and ordinal numbers, using the permanent map.

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Chapter 1

Statement of Main Results

In Chapter 3, we give some examples of pro-2 groups like the profinite dihedral group:

Example 1.0.1. The profinite dihedral group D^* is the inverse limit of the inverse system of finite dihedral groups, D_n, $n \in \mathbb{N}$.

Example 1.0.2. The inverse limit of the dihedral 2-groups D_{2^n} is isomorphic to D_{2^∞}.

and we give an extension of the generalized quaternion group as a profinite group:

Example 1.0.3. The profinite generalized quaternion group Q^* is the inverse limit of finite generalized quaternion groups.

We characterize the profinite Hamiltonian groups as an inverse limit of finite Hamiltonian groups.

Theorem 1.0.4. *Let H be a pro-Hamiltonian group. Then H is a profinite abelian group, or $H = Q^* \times E^{2*} \times A^*$ where E^{2*} is an elementary abelian pro-2 group, Q^* is the profinite generalized quaternion group and A^* is a profinite abelian group in which every element is of odd Steintz prime order.*

We also extend some results of sylow intersections to profinite groups:

Lemma 1.0.5. *Let G be a profinite group, and let $K = \bigcap(S \mid S \in Syl_p(G))$. Then:*

(a) $K \trianglelefteq G$*, and* K *is a pro-*p *subgroup of* G*.*

(b) K *contains every normal pro-*p *subgroup of* G*.*

Theorem 1.0.6. *Let* G *be a profinite group, let* $D = P \cap Q$ *be a maximal sylow* p*-intersection of* $P, Q \in Syl_p(G)$ *with* $P \neq Q$*. Then:*

(a) D *is a proper subgroup of each sylow* p*-subgroup of* $N_G(D)$*.*

(b) Any two distinct sylow p*-subgroups of* $N_G(D)$ *have intersection* D*.*

(c) There exists a one-to-one correspondence between sylow p*-subgroups of* G *containing* D *and the sylow* p*-subgroups of* $N_G(D)$*.*

The main result of this study is in chapter 4 in which we introduce the transfer for profinite groups using the permanent map. Let G be a profinite group, H be a closed subgroup of G of not necessarily finite index, and K be a normal subgroup of H such that H/K is abelian. Let Y be a transversal of H in G and $|Y| = n \in \mathbb{SN}$, where $\mathbb{SN}$ is the set of all Steinitz numbers. For an infinite Steinitz number n we say that a monomial $(n \times n)$-matrix A over a commutative ring with identity R is *known* if the limit

$$\lim_{i \to n} \prod_{m \leq i} a_m$$

exists and converges in R regardless of the order of the indices, where $a_m \in R$ is the nonzero entry in the m-th row of A.

Let $M_n(H/K)$ be the group of all $(n \times n)$ known monomial matrices with entries from H/K. We define the transfer homomorphism of profinite groups as a composition of the monomial representation

$$\mu : G \to M_n(H/K)$$

and the permanent map

$$perm_n : M_n(H/K) \to H/K.$$

This definition is an analog for the one of the transfer for finite groups in [14]. In an effort to define the transfer homomorphism for profinite groups, Oliver Schirokauer in 1996 [21] published a paper in which he presented a new definition for the standard cohomological transfer as an integral, our aim is to define the transfer using the group structure and the permanent map. In [10] the authors proved Burnside Normal p-Complement Theorem for Profinite Groups

Theorem 1.0.7 (Burnside Normal p-Complement Theorem for Profinite Groups)**.** *Let G be a profinite group, and let P be a sylow pro-p subgroup that is contained in the center of its normalizer, then G has a closed normal subgroup H which has elements of P as its coset representatives.*

without using the transfer, instead they use the following lemma

Lemma 1.0.8 ([10])**.** *Let G be an inverse limit of finite groups G_i where i is in a directed set I. Then G has a normal sylow p-complement if and only if G_i has a normal sylow p-complement for all $i \in I$.*

We give a proof of this theorem using the transfer following the work in [14] chapter 14, also we used the transfer to prove

Theorem 1.0.9 (Focal Subgroup Theorem for Profinite Groups)**.** *Let G be a profinite group, P a pro-p sylow subgroup of G and G' be the derived group of G. Then*

$$V_P^G(G) \cong P/P \cap G'$$

which was also been proved in [10] for profinite groups by extending from the finite version without using the transfer.

Chapter 2

Introduction

Profinite groups arise naturally in mathematics as finite groups, Galois groups, the p-adic integers, $p \in \mathbb{P}$, G/N where G is any compact Hausdorff group and N is the connected component of the identity of G. Profinite groups have a Sylow theory. The Local Theory of finite groups is based on the Sylow theory of finite groups. It asks what global properties of a group G can be deduced from the knowledge of P, a Sylow p-subgroup of G, and $N_G(P)$. The collection of all subgroups of G, H, such that $H = N_G(Q)$ where Q is a p-subgroup of G for some prime p which divides the order of G are called the p-local subgroups of G. Burnside's p-complement theorem, as an application of the Transfer homomorphism, is an example of such a result. This may also be viewed as a first application of fusion in finite groups. This chapter will summarize the necessary background for this dissertation.

2.1 Finite Groups

In this section we present the basic theory which is important to our intent to extend results related to finite groups to profinite groups. Following chapter 14 in Marshal Hall [14] for the convenience of the reader we give the following.

Definition 2.1.1. Let G be a group, and $H \leq G$ of finite index, take $S = \{u_1, \cdots, u_n\}$, be the set of left coset representatives of H in G. A monomial permutation M is a mapping $u_i \to h_{ij}u_j$ where $i = 1, \cdots, n$, and $j = i(j)$.

Remark 2.1.2. If $\mathbb{M}$ is the set of all monomial permutations, we can make it into a group via the multiplication: for $M_1, M_2 \in \mathbb{M}$, with $M_1 : u_i \to h_{ij}u_j$ and $M_2 : u_j \to h_{jk}u_k$, we write $M_1M_2 : u_i \to (h_{ij}h_{jk})u_k$, and the identity of $\mathbb{M}$ is $u_i \to u_i$.

Remark 2.1.3. let R be a monomial representation of a finite group G with multipliers from $H \leq G$. set

$$\pi(g) : u_i \to h_{ij}u_j \quad i, j = 1, ..., n$$

then the mapping

$$g \to \left[h_{ij}\, H'\right]_{n\times n}$$

is a homomorphism of G into $M_n(H/H')$.

Definition 2.1.4. Let G be a finite group, $H \leq G$, and let $S = \{u_1, \cdots, u_n\}$ to be the set of left coset representatives of H in G. Set $\phi(z) = x_j$ for $z = hx_j$. The transfer homomorphism is defined from G into H/H' via

$$V_{G\to H}(G) = \prod_{i=1}^{n} x_i g\phi(x_i g)^{-1} \mod H'$$

Remark 2.1.5. We note that the map ϕ can be considered as a choice function, which we will call ν in chapter 4.

Theorem 2.1.6 ([14])**.** *(a) The mapping $g \to V_{G\to K}(g)$ is a homomorphism of G into K/K'.*

(b) The transfer $V_{G\to K}$ is well defined; i.e independent of the choice of representatives.

(c) if $T \leq K \leq G$, then $V_{G\to T}(g) = V_{K\to T}[V_{G\to K}(g)]$.

The next result is known as Burnside normal p-complement theorem for finite groups.

Theorem 2.1.7 (Burnside's Theorem [14]). *Let G be a finite group, and let P be a sylow p-subgroup that is contained in the center of its normalizer, then G has a normal subgroup H which has elements of P as its coset representatives.*

The following lemma is used to show the above theorem.

Lemma 2.1.8. *Let G be a finite group, if two subsets K_1 and K_2 of G are normal in a sylow p-subgroup P of G and are conjugate in G then K_1 and K_2 are conjugate in $N_G(P)$.*

This leads to a well known results for finite groups, such as theorems of P. Hall, Grün, and Wielandt. The importance of these results arise from the relationship between the sylow p-subgroup of a group G and the factor group G/K of G which are p-groups. Now, Let $H' \leq H_0 \leq H \leq G$ such that $|G:H| = n$ so that

$$G = \bigcup_{i=1}^{n} Hx_i$$

and

$$V_{G\to H}(g) = \prod_{i=1}^{n} x_i g[\phi(x_i g)]^{-1} H_0.$$

For $g \in G$ and $i = 1, 2, \cdots, n$ we define $ig \in \{1, 2, \cdots, n\}$ such that $x_i g x_{ig}^{-1} \in H$. Then for a fixed $g \in G$ we have the permutation $\pi(g) : i \to ig$ of the transitive permutation representation of G on the left cosets of H in G. Thus we can write

$$V_{G\to H}(g) = \prod_{i} x_i g x_{ig}^{-1} H_0$$

in the permutation $\pi(g)$ we can write it as t disjoint cycles

$$\pi(g) = \prod_{j=1}^{t} \sigma_j$$

choose the least element in each σ_j and form the set $C_H(g)$. For $i \in C_H(g)$ let $r_i = |\sigma_j|$

where i appears in σ_j. Then

$$\sum_{i\in C_H(g)} r_i = n$$

Lemma 2.1.9 ([14])**.**

$$V_{G\to H}(g) = \prod_{i\in C_H(g)} x_i g^{r_i} x_i^{-1}\, H_0$$

Where $x_i g^{r_i} x_i^{-1}$ is the first power of $x_i g x_i^{-1}$ which lies in H.

The following is needed for the proof of P. Hall theorem and Hall-Wielandt theorem in [14].

Definition 2.1.10. The *diagonal contribution*, $d(g)$ is the contribution of cycles of length one in $\pi(g)$ to $V_{G\to H}(g)$ is given by

$$d(g) = \prod_{i=ig} x_i g x_i^{-1}\, H_0$$

Remark 2.1.11. Note that $d(g) = \prod_{i=1}^{n} h_{ii}$ with $\pi(g) = [h_{ij}\, H_0]_{n\times n}$.

Lemma 2.1.12 ([14])**.** *If u and v are conjugate in G, then $d(u) = d(v)$, and $d(u^{-1}) = [d(u)]^{-1}$.*

Now, for $h \in H$ we define $d^*(h) \equiv h^{-1}d(h)$. Using lemma 2.1.12, we have

Lemma 2.1.13 ([14])**.** *If $h \in H$, then*

$$V_{G\to H}(h) \equiv h^n \prod_{i\in C_H(h)} d^*(h^{r_i})d^*(x_i h^{-r_i} x_i^{-1})\, H_0$$

Corollary 2.1.14. *If $d^*(h) \in H_0$ for all $h \in H$, then for any $h \in H$ we have $V_{G\to H}(h) = h^n$.*

We set up the stage for the main theorems in this section from chapter 14 in [14]. Let $p \in \mathbb{P}$, and G any finite group, define

$$\mathscr{O}^p(G) = \langle x \in G \mid (|x|, p) = 1\rangle$$

thus $G/\mathcal{O}^p(G)$ is the maximal p-factor group of G. Let P be a sylow p-subgroup of G, and define $N = N_G(P)$, and let H be any subgroup of G containing N. Now set $P^* = P \cap \mathcal{O}^p(G)$, $N^* = N \cap \mathcal{O}^p(G)$, and $H^* = H \cap \mathcal{O}^p(G)$, so that $G = \mathcal{O}^p(G)P = \mathcal{O}^p(G)N = \mathcal{O}^p(G)H$ and $|P/P^*| = |N/N^*| = |H/H^*| = |G/\mathcal{O}^p(G)|$. Here $\mathcal{O}^p(G)$ is fully invariant subgroup of G, note that P^* is a sylow p-subgroup of $\mathcal{O}^p(G)$, and N^* normalizes both $\mathcal{O}^p(G)$ and P, and so N^* normalizes $\mathcal{O}^p(G) \cap P = P^*$. Now $\mathcal{O}^p(\mathcal{O}^p(G)) = \mathcal{O}^p(G)$ but is may happen that $\mathcal{O}^p(H^*) \subsetneqq H^*$. If We consider the case when $\mathcal{O}^p(H^*) \subsetneqq H^*$. Note that $\mathcal{O}^p(H^*)$ is fully invariant subgroup of H^*, and H^* is normal in H. Since H/H^* is a p-group we have $\mathcal{O}^p(H) = \mathcal{O}^p(H_1)$. Now we define

$$H_0 = H^{*p}[H^*, H]\mathcal{O}^p(H^*)$$

where

$$H^{*p} = \langle x^p \mid x \in H^* \rangle$$

and

$$[H^*, H] = \langle [h^*, h] \mid h^* \in H^*, h \in H \rangle$$

Since $H/\mathcal{O}^p(H^*)$ is a finite p-group and so it is also nilpotent. This means that $[H^*, H]/\mathcal{O}^p(H^*) \lneqq H^*/\mathcal{O}^p(H^*)$. Now, by definition, $H^{*p} \subset T$ where $T \lneqq H^*$ with $|H^* : T| = p$ such that $[H^*, H]\mathcal{O}^p(H^*) \subsetneqq T \subsetneqq H^*$. So, if $\mathcal{O}^p(H^*) \subsetneqq H^*$ we have $H_0 \lhd H^*$ and H^*/H_0 is a p-group.

Theorem 2.1.15 (P. Hall [14]). *Let G be any finite group, P a sylow p-subgroup, $N = N_G(P)$, and H a subgroup containing N. Let $\mathcal{O}^p(G)$ be defined as before, and set $P^* = P \cap \mathcal{O}^p(G)$, $N^* = N \cap \mathcal{O}^p(G)$, and $H^* = H \cap \mathcal{O}^p(G)$. Then $\mathcal{O}^p(H^*) = \mathcal{O}^p(H)$, and if $\mathcal{O}^p(H^*) \neq H^*$, then $H_0 = H^{*p}[H^*, H]\mathcal{O}^p(H^*)$ is a proper subgroup of H^*, and H^* can be obtained by adjoining to H_0 certain conjugates lying in H^* of elements*

$$e_p(u, z_i) = [u, \overbrace{z_i, \cdots, z_i}^{(p\text{-}1)}]$$

where $u \in P^*$ *and* $z_i \in P$ *for* $i = 1, \cdots, s$. *If*

$$G = \bigcup_{i=1}^{s} H t_i P$$

is a decomposition of of G *into double cosets of* H *and* P, *let* π_i *for* $i = 1, \cdots, s$ *be the transitive representation of* P *on the cosets of* H *in* Ht_iP. *Then* π_i *is not of degree one, and we choose* z_i *so that* $\pi_i(z_i)$ *is of order* p *in the center of* $\pi(P)$.

Corollary 2.1.16 ([14]). *If* $e_p(u,z) = 1$ *for all* $u, z \in P_1$, *then*

$$\mathscr{O}^p(N_1) = N = \mathscr{O}^p(G_1) \cap N$$

and

$$G_1/\mathscr{O}^p(G_1) = N_1/\mathscr{O}^p(N_1)$$

This will happen in particular if the class of P_1 *is less than* p.

Theorem 2.1.17 (Hall-Wielandt [14]). *Let* P_1 *be a sylow* p*-subgroup of* G_1 *and* Q_1 *be a weakly closed subgroup of* P_1. *Let* $N_1 = N_{G_1}(P_1)$ *and* $H_1 = N_{G_1}(Q_1)$. *Then any one of the following conditions will ensure* $\mathscr{O}^p(H_1) = H = \mathscr{O}^p(G_1) \cap H_1$, *and so* $G_1/\mathscr{O}^p(G_1) = H_1/\mathscr{O}^p(H_1)$.

1. $e_p(u,z) = 1$ *for all* $u \in P_1$ *and all* $z \in Q_1$.

2. $e_p(u,z) = 1$ *for all* $u, z \in Q_1$.

3. $Q_1 \subseteq Z_{p-1}(P_1)$, *where* $Z_{p-1}(P_1)$ *is the* $(p-1)$*st term of ascending central series for* P_1.

Corollary 2.1.18 ([14]). *Let* Q_1 *be a characteristic subgroup of* P_1. *If* Q_1 *is not weakly closed in* P_1, *then there is another sylow* p*-subgroup* P_2 *which contains* Q_1 *but in which* Q_1 *is not normal. This must be the case if* Q_1 *satisfies either one of the conditions in Hall-Wielandt theorem, but* $G_1/\mathscr{O}^p(G_1) \ncong H_1/\mathscr{O}^p(H_1)$.

Theorem 2.1.19 ([14])**.** *Let P be a sylow p-subgroup of G. Then $V_{G\to P}(G) \cong P/(P\cap G')$.*

Theorem 2.1.20 (First Theorem of Grün [14])**.** *Let P be a sylow p-subgroup of G. Then $V_{G\to P}(G) \cong P/P^*$, where*

$$P^* = [P\cap N_{G'}(P)] \bigcup_{z\in G} (P\cap z^{-1}P'z).$$

Theorem 2.1.21 (Second Theorem of Grün [14])**.** *If G is p-normal, then*

$$G/\mathscr{O}^p(G)G' \cong N(Z(P))/\mathscr{O}^p(N(Z(P)))(N(Z(P)))'$$

where $P \in Syl_p(G)$.

Theorem 2.1.22 ([14])**.** *[Hall-Grün] If G is p-normal, then the greatest factor group of G which is a p-group is isomorphic to that for the normalizer of the center of a sylow p-subgroup.*

Our aim is to give proves of the above for profinite groups.

2.2 Topological Groups

In this section we follow [25].

Definition 2.2.1. A group G is said to be a topological group if it satisfies the following conditions:

1. G is a topological space, and

2. The map $\phi : G\times G \to G$, given by $\phi(x,y) = xy^{-1}$ is continuous.

Theorem 2.2.2 ([25])**.** *Let G be a topological group.*

(a) *the map* $(x,y) \mapsto xy$ *from* $G \times G$ *to* G *is continuous. and the map* $x \mapsto x^{-1}$ *from* G *to* G *is a homeomorphism. For* $g \in G$ *the maps* $x \mapsto xg$ *and* $x \mapsto gx$ *from* G *to* G *are homeomorphisms.*

(b) *Every right or left coset of an open (closed) subgroup of* G *is open (closed).*

(c) *Any subgroup of* G *is a closed subgroup, and any closed subgroup of* G *of finite index is open. Moreover if* G *compact then any open subgroup is of finite index.*

(d) *If* H *is a subgroup containing a non-empty open subset* U *of* G *then* H *is open in* G.

(e) *If* H *is a subgroup of* G *and* $K \trianglelefteq G$ *then* H *is a topological group with respect to the subgroup topology and* G/K *is a topological group with respect to the quotient topology, and the quotient map* $q : G \to G/K$ *takes open sets to open sets.*

Lemma 2.2.3 ([25])**.** *Let* G *be a compact topological group. If* C *is a subset which is both closed and open such that* $1 \in C$*, then* C *contains an open normal subgroup.*

Proposition 2.2.4 ([25])**.** *Let* G *be a compact totally disconnected topological group.*

(a) *Every open set in* G *is a union of cosets of open normal subgroups.*

(b) *A subset of* G *is both closed and open if and only if it is a union of finitely many cosets of open normal subgroups.*

(c) *If* X *is a subset of* G *then its closure* $\overline{X}$ *satisfies*

$$\overline{X} = \cap\{NX \mid N \text{ an open normal subgroup of } G\}$$

In particular

$$C = \cap\{NC \mid N \text{ an open normal subgroup of } G\}$$

for each closed subset C, and the intersection of the open normal subgroups of G is the trivial subgroup.

Proposition 2.2.5 ([25])**.** *Let G be a compact totally disconnected topological group.*

(a) Every open set in G is a union of cosets of open normal subgroups.

(b) A subset of G is both closed and open if and only if it is a union of finitely many cosets of open normal subgroups.

(c) If X is a subset of G then its closure $\overline{X}$ satisfies

$$\overline{X} = \bigcap \{NX \mid N \triangleleft_o G\}$$

In particular

$$C = \bigcap \{NC \mid N \triangleleft_o G\}$$

for each closed subset C, and $\bigcap_{N \triangleleft_o G} N = id_G$.

Lemma 2.2.6 ([25])**.** *Let $\{G_i \mid i \in I\}$ be a family of topological groups, let*

$$C = \bigtimes_{i \in I} G_i$$

We can make C into a topological group by taking the product topology and defining multiplication pointwise.

2.3 Profinite Groups

In this section we follow [25].

Definition 2.3.1. A *directed* set is a partially ordered set (or a poset), I, such that for all $i_1, i_2 \in I$ there is a $j \in I$ with $i_1 \leq j$ and $i_2 \leq j$.

Definition 2.3.2. An *inverse System* (G_i, ϕ_{ij}, I) consists of finite groups G_i with continuous homomorphisms $\phi_{ij} : G_j \to G_i$ with the property: Whenever $i, j, k \in I$ such that $i \le j \le k$ we have $\psi_{ii} = id_{G_i}$ and $\phi_{ij}\phi_{jk} = \phi_{ik}$. Where I is a directed poset.

Example 2.3.3. Let $I = \mathbb{N}$, and consider the system $(\mathbb{Z}/p^i\mathbb{Z}, \phi_{ij}, I)$, and for $i \le j$ in I define $\phi_{ij} : \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$. Indeed $(\mathbb{Z}/p^i\mathbb{Z}, \phi_{ij}, I)$ is an inverse system.

Definition 2.3.4. Let (G_i, ϕ_{ij}, I) be an inverse system of finite groups and let Y be a group, the family $(\psi_i : Y \to G_i)_{i \in I}$ of continuous maps is called *compatible* if $\phi_{ij}\psi_j = \psi_i$ whenever $i \le j$ in I.

Definition 2.3.5. An *inverse limit* (G, ϕ_i, I) of an inverse system (G_i, ϕ_{ij}, I) of finite groups is the profinite group G with compatible continuous homomorphisms $\phi_i : G \to G_i$ with the property: Whenever $(\psi_i : Y \to G_i)_{i \in I}$ is a compatible family of continuous homomorphisms from a group Y there is a unique homomorphism $\psi : Y \to G$ such that $\phi_i\psi = \psi_i$ for all $i \in I$.

Example 2.3.6. The inverse limit of of the inverse system in example 2.3.3 is the p-adic integers $\widehat{\mathbb{Z}}_p$.

Definition 2.3.7. The inverse limit of an inverse system (G_i, ϕ_{ij}, I) of finite groups is called a profinite group G denoted by

$$G = \varprojlim_{i \in I} G_i$$

The following proposition gives a description of profinite groups.

Proposition 2.3.8 ([25])**.** *Let (G_i, ϕ_{ij}, I) be an inverse system*

(a) The inverse limit G is unique up to isomorphism.

(b) If $C = \bigtimes\{G_i \mid i \in I\}$ and for each i let π_i be the projection map from C to G_i. Define

$$X = \{c \in C \mid \phi_{ij}\pi_i(c) = \pi_j(c) \ \textit{ for all } \ i, j \textit{ with } j \ge i\}$$

and $\psi_i = \pi_i|_X$ for each i. Then (X, ψ_i) is an inverse limit of (G_i, ϕ_{ij}, I)

(c) If (G_i, ϕ_{ij}, I) is an inverse system of finite groups and continuous homomorphisms, then X is called the special inverse limit group denoted by $s\varprojlim G_i$, and the maps ψ_i are continuous homomorphisms.

Proposition 2.3.9 ([25]). *Let (G_i, ϕ_{ij}, I) be an inverse system, and set $G = \varprojlim G_i$ the profinite group. Then*

(a) *If each G_i is Hausdorff, so is G.*

(b) *If each G_i is totally disconnected, so is G.*

(c) *If each G_i is Hausdorff, then $s\varprojlim G_i$ is closed in the cartesian product*

$$C = \bigtimes_{i \in I} G_i$$

(d) *If each G_i is compact and Hausdorff, so is G.*

(e) *If each G_i is non-empty compact and Hausdorff group, then G is non-empty.*

Proposition 2.3.10 ([25]). *Let (G, ψ_i) be the inverse limit of the inverse system (G_i, ϕ_{ij}, I) of non-empty compact Hausdorff groups. Then*

(a) $\psi_i(G) = \bigcap_{j \geq i} \phi_{ij}(G_j)$ *for each $i \in I$.*

(b) *The set $\psi_i^{-1}(U)$ with $i \in I$ and U open in G_i form a base for the topology on G.*

(c) *If Y is a subset of G such that $\psi_i(Y) = G_i$ for each $i \in I$ then Y is dense in G.*

(d) *If θ is a map from a group Y to G, then θ is continuous if and only if each map $\psi_i\theta$ is continuous.*

(e) *If $f : G \to A$ is a continuous map to a discrete space, then f factors through G_i for some $i \in I$; i.e for some i there is a continuous map $g : G_i \to A$ such that $f = g\phi_i$.*

The next proposition gives an insight on the characterization of profinite groups.

Proposition 2.3.11 ([25]). *Let G be compact Hausdorff totally disconnected group, then G is the inverse limit of its discrete quotient groups.*

We introduce some notations and disclaimers: for a profinite group G we will write $H \leq G$ to mean that H is closed subgroup of G, and when it is open we will write $H \leq_o G$.

Definition 2.3.12. A *filter base* in G is a family I of open normal subgroups of G such that for all $K_1, K_2 \in I$ there is a subgroup $K_3 \in I$ with $K_3 \subset K_1 \cap K_2$.

The next two results set up the stage for the characterization of profinite groups.

Proposition 2.3.13 ([25]). *Let (G, ψ_i) be an inverse limit of an inverse system (G_i, ϕ_{ij}, I) of compact Hausdorff topological groups and let $L \lhd_o G$. Then $\ker \psi_i \leq L$ for some i. Consequently G/L is isomorphic as a topological group to a quotient group of a subgroup of some G_i, and if each map ψ_i is onto then G/L is isomorphic to a quotient group of some G_i.*

Proposition 2.3.14 ([25]). *Let G be a topological group and I be a filter base of closed normal subgroups, and for $K, L \in I$ define $K \leq' L$ if and only if $L \leq K$. Thus I is a directed set with respect to $\leq'$ and the surjective homomorphism $q_{KL} : G/L \to G/K$ defined for $K \leq' L$, make the groups G/K into an inverse system. Let $(\widehat{G}, \psi_K) = \varprojlim G/K$, there is a continuous homomorphism $\theta : G \to \widehat{G}$ with kernel $\bigcap_{K \in I} K$, with image a dense subgroup of $\widehat{G}$, and such that $\psi_K \theta$ is the quotient map from G to G/K for each $K \in I$. If G is compact then θ is onto; if G is compact and $\bigcap_{K \in I} K = 1$ then θ is an isomorphism of topological groups.*

The term *class* of finite groups is a class in the usual sense which also closed with respect to isomorphic images; i.e if $\mathscr{C}$ is a class and if $F_1 \in \mathscr{C}$ and $F_1 \cong F_2$ then $F_2 \in \mathscr{C}$. Let $\mathscr{C}$ be a class of finite groups, a group F in $\mathscr{C}$ is called a $\mathscr{C}$-group. If G is the inverse limit of $\mathscr{C}$-groups then we call G a pro-$\mathscr{C}$ group. Note that $\mathscr{C}$-groups are

pro-$\mathscr{C}$ groups. We say that $\mathscr{C}$ is closed for subgroups if every subgroup of a $\mathscr{C}$-group is a $\mathscr{C}$-group, same terminology is used for quotient groups. In addition, we say that $\mathscr{C}$ is closed for direct products if $F_1 \times F_2 \in \mathscr{C}$ whenever $F_1 \in \mathscr{C}$ and $F_2 \in \mathscr{C}$.
One of the important classes are the class of all finite groups, the class of finite p-groups with $p \in \mathbb{P}$ is fixed, and the class of all finite cyclic groups.

Definition 2.3.15. A *pro-p* group is the inverse limit of finite p-groups, and a *procyclic* group is the inverse limit of finite cyclic groups.

Theorem 2.3.16 ([25])**.** *Let $\mathscr{C}$ be a class of finite groups, which is closed for subgroups and direct products, and let G be a profinite group. The following are equivalent:*

(a) *G is a pro-$\mathscr{C}$ group.*

(b) *G is isomorphic as a topological group to a closed subgroup of a Cartesian product of $\mathscr{C}$-groups.*

(c) *G is compact and $\bigcap\{N \mid N \lhd_o G, G/N \in \mathscr{C}\} = 1$.*

(d) *G is compact and totally disconnected, and for $L \lhd_o G$ there is a subgroup $N \lhd_o G$ with $N \leq L$ and $G/N \in \mathscr{C}$. If in addition $\mathscr{C}$ is closed for quotients then (d) can be replaced by*

(e) *G is compact and totally disconnected, and $G/L \in \mathscr{C}$ for every $L \lhd_o G$.*

Taking $\mathscr{C} = \mathscr{C}_f$ in the above theorem we obtain an important characterization of profinite groups:

Corollary 2.3.17 ([25])**.** *Let G be a profinite group. The following are equivalent:*

(a) *G is a profinite group.*

(b) *G is isomorphic as a topological group to a closed subgroup of a Cartesian product of finite groups.*

(c) *G is compact and $\bigcap\{N \mid N \lhd_o G\} = 1$.*

(d) G is compact and totally disconnected.

Then next important result gives a description of how profinite groups its subgroups and its factor groups can be viewed as inverse limits.

Theorem 2.3.18 ([25])**.** *Let G be a profinite group. If I is a filter base of closed normal subgroups of G such that $\bigcap_{N\in I} N = 1$ then:*

$$G \cong \varprojlim_{N\in I} G/N$$

moreover

$$H \cong \varprojlim_{N\in I} H/(H\cap N)$$

for each closed subgroup H and

$$G/K \cong \varprojlim_{N\in I} G/KN$$

for each closed normal subgroup K.

The above result also shows that subgroups and factor groups of profinite groups are profinite.

Remark 2.3.19.

1. In the context of profinite groups, any subgroup is closed in G, we will write " $\leq$ " to denote closed subgroups, and " $\leq_o$ " for open subgroups.

2. If G is profinite, then

$$G \cong \varprojlim_{N \triangleleft_o G} G/N$$

3. If G is profinite and $K \leq G$, then

$$K \cong \varprojlim_{N \triangleleft_o G} K/(K\cap N) \cong \varprojlim_{N \triangleleft_o G} KN/N$$

2.4 Sylow Theory For Profinite Groups

In this section we follow Ribes [20] and Wilson [25]. The following are essential for this section.

Proposition 2.4.1 ([20]). *Let $C_1, C_2, \cdots$ be a countably infinite set of nonempty closed subsets of a profinite group G, such that each C_i have empty interior. Then*

$$G \neq \bigcup_{n=1}^{\infty} C_i$$

and so the cardinality of G is either finite or uncountable.

Definition 2.4.2. A *Steinitz number*, n, is a formal infinite product

$$n = \prod_{p \in \mathbb{P}} p^{n(p)}$$

where $n(p)$ is non-negative natural number or ∞. By $\mathbb{SN}$ we mean the set of all Steinitz numbers.

Now we give some properties of Steinitz numbers following Ribes and Zalesskii. [20].

Definition 2.4.3. Let

$$\left\{ n_i = \prod_{p \in \mathbb{P}} p^{n(p,i)} \mid i \in I \right\}$$

be a collection of Steinitz numbers. Then

1. $n_i \mid n_j$ if $n(p,i) \leq n(p,j)$ for all $p \in \mathbb{P}$
2. $\prod_J n_i = \prod_{p \in \mathbb{P}} p^{n(p)}$ where $n(p) = \sum_J n(p,i)$ with $J \subseteq I$.
3. $gcd\{n_i | i \in J\} = \prod_{p \in \mathbb{P}} p^{n(p)}$ where $n(p) = \min_i\{n(p,i) | i \in J\}$ with $J \subseteq I$.
4. $lcm\{n_i | i \in J\} = \prod_{p \in \mathbb{P}} p^{n(p)}$ where $n(p) = \max_i\{n(p,i) | i \in J\}$ with $J \subseteq I$.

With this we can define the order and the index in profinite groups.

Definition 2.4.4. Let G be a profinite group, and $H \leq G$. Let $\mathscr{U}$ denote the set of all open normal subgroups of G. The index of H in G is defined as

$$|G:H| = lcm\{|G/U : HU/U| \mid U \in \mathscr{U}\} \in \mathbb{SN}$$

the order of G, $|G|$, is defined as

$$|G| = |G:1| = lcm\{|G/U| \mid U \in \mathscr{U}\} \in \mathbb{SN}$$

We proceed with some important properties and extensions from well known classical results.

Theorem 2.4.5 ([20])**.** *Let G be a profinite group.*

(a) If $H \leq G$ then $|G:H|$ is a natural number if and only if H is an open subgroup of G.

(b) If $H \leq G$ then

$$|G:H| = lcm\{|G:U| \mid H \leq U \leq_o G\}$$

(c) If $H \leq G$ and $\mathscr{U}'$ be a filter in G consisting of open normal subgroups, then

$$|G:H| = lcm\{|G/U : HU/U| \mid U \in \mathscr{U}'\}$$

(d) If $K \leq H \leq G$ then $|G:K| = |G:H||H:K|$. This is Lagrange's theorem.

(e) Let $\{H_i \mid i \in I\}$ be a family of closed subgroups of G filtered from below. Assume that $H = \bigcap_{i \in I} H_i$. Then

$$|G:H| = lcm\{|G:H_i| \mid i \in I\}$$

(f) Let (G_i, ϕ_{ij}) be a surjective inverse system of profinite groups over a directed poset I. Let $G = \varprojlim_{i \in I} G_i$. Then

$$|G| = lcm\{|G_i| \mid i \in I\}$$

(g) For any collection $\{G_i \mid i \in I\}$ of profinite groups,

$$|\prod_{i \in I} G_i| = \prod_{i \in I} |G_i|$$

With these results in hand, we can begin the sylow theory.

Definition 2.4.6. Let G be a profinite group, and $p \in \mathbb{P}$. A sylow p-subgroup of G is a subgroup P such that $|P|$ possibly infinite power of p and $(|G : P|, p) = 1$.

The following results are an extension of the classical sylow theorems for finite groups.

Theorem 2.4.7 ([25])**.** *Let G be a profinite group, and $p \in \mathbb{P}$*

(a) G has a sylow p-subgroup.

(b) If P is a sylow p-subgroup of G and T is a pro-p subgroup of G then $T^g \leq P$ for some $g \in G$.

(c) Every pro-p subgroup of G is contained in a sylow p-subgroup of G.

(d) If P_1 and P_2 are sylow p-subgroups of G then $P_1^g = P_2$ for some $g \in G$.

Theorem 2.4.8 ([20])**.** *Let G be a profinite group, K a normal subgroup and P a sylow p-subgroup of G, $p \in \mathbb{P}$. Then*

(a) $K \cap P$ is a sylow p-subgroup of K.

(b) KP/K is a sylow p-subgroup of G/K.

(c) $G = N_G(Q)K$ *for each sylow p-subgroup Q of K.*

(d) $H = N_G(H)$ *whenever H is a closed subgroup which contains* $N_G(Q)$ *for some sylow p-subgroup Q of K.*

Chapter 3

Pro-p Groups

3.1 Preliminaries

With section 4 in chapter 2 in mind, we present some known results on pro-p groups.

Theorem 3.1.1 ([20]). *A pro-p group is always pronilpotent.*

Lemma 3.1.2 ([20]). *Let P be a pro-p group, then all maximal subgroups of P are normal.*

Lemma 3.1.3 ([20]). *Let P be a pro-p group. If H is an open normal proper subgroup of P, then*

$$H < N_P(H)$$

Theorem 3.1.4 ([25]). *Let G be a profinite group, and let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$.*

1. *The map $\delta : \widehat{\mathbb{Z}} \times G \to G$ with $\delta(n,g) = g^n$ for $n \in \widehat{\mathbb{Z}}$ exists and is unique. If $g \in G$ and $z \in \widehat{\mathbb{Z}}$ then g^z is defined.*

2. *For $g \in G$ and $z_1, z_2 \in \widehat{\mathbb{Z}}$ then $g^{z_1+z_2} = g^{z_1}g^{z_2}$ and $(g^{z_1})^{z_2} = g^{z_1z_2}$.*

3. *If $g_1, g_2 \in G$ commute and $z \in \widehat{\mathbb{Z}}$ then $(g_1g_2)^z = g_1^z g_2^z$.*

Lemma 3.1.5 ([20]). *Any procyclic pro-p group is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ for some $n \geq 0$ or to $\widehat{\mathbb{Z}}_p$. Any procyclic group is isomorphic to a quotient group of $\widehat{\mathbb{Z}}$.*

For the proof of last result see [20]. The next result is a characterization of finite p-groups [14].

Theorem 3.1.6 ([14]). *The p-groups of order p^n which contain a cyclic subgroup of index p are of the following type:*

1. *Abelian*

   (a) *If cyclic then it is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$.*

   (b) *For $n \geq 2$, we have $x^{p^{n-1}} = 1$, $y^p = 1$, and $xy = yx$.*

2. *Non abelian*

   (a) *p is odd and $n \geq 3$ we have $x^{p^{n-1}} = 1$, $y^p = 1$, and $yx = x^{1+pn-2}y$.*

   (b) *$p = 2$, $n \geq 3$ we have the generalized quaternion group: $x^{2^{n-1}} = 1$, $y^2 = x^{2^{n-2}}$, and $yx = x^{-1}y$.*

   (c) *$p = 2$, $n \geq 3$ we have the dihedral group: $x^{2^{n-1}} = 1$, $y^2 = 1$, and $yx = x^{-1}y$.*

   (d) *$p = 2$, $n \geq 4$ we have : $x^{2^{n-1}} = 1$, $y^2 = 1$, and $yx = x^{1+2^{n-2}}y$.*

   (e) *$p = 2$, $n \geq 4$ we have : $x^{2^{n-1}} = 1$, $y^2 = 1$, and $yx = x^{-1+2^{n-2}}y$.*

Note that for a system of abelian cyclic p-group G, we can take the inverse system and have $G \cong \widehat{\mathbb{Z}}_p$. The next two sections will provide some generalizations of this results to pro-p groups.

3.2 Example: The Profinite Dihedral 2-Group

We consider the collection of all finite dihedral groups, D_{2n} with $n \in \mathbb{N}$, represented as

$$D_{2n} = \langle r, s \mid r^n = 1 = s^2,\ srs = r^{-1} \rangle$$

Example 3.2.1. The profinite dihedral group D^* is the inverse limit of the inverse system of finite dihedral groups, D_n, $n \in \mathbb{N}$.

Proof. First we identify

$$D_{2n} = \langle r, s \mid r^n = 1 = s^2,\ srs = r^{-1} \rangle$$

$$D_{2k} = \langle r_1, s_1 \mid r_1^k = 1 = s_1^2,\ s_1 r_1 s_1 = r_1^{-1} \rangle$$

$$D_{2m} = \langle r_2, s_2 \mid r_2^m = 1 = s_2^2,\ s_2 r_2 s_2 = r_1^{-1} \rangle$$

and let D^* be the profinite dihedral group, its generators are given explicitly as

$$D^* = \overline{\langle x, y \mid y^2 = (xy)^2 = 1 \rangle}$$

where x has Steinitz number order. To construct an inverse system of dihedral groups, let $I = \mathbb{N}$ be the index set, defining the relation $\leq$ on I by $n \leq m$ if and only if $n|m$, we have I as an indexed set. Define

$$\phi_{kn} : D_{2n} \to D_{2k}$$

for any $n, k \in I$, such that $k|n$, via $\phi_{kn}(r^i s^j) = r_1^i s_1^j$, this is well defined since $k|n$ and the exponent on the right are taken modulo k and 2. It is a group homomorphism, to see this let $r^i s^j, r^l s^t \in D_{2n}$ and take

$$\phi_{kn}(r^i s^j \cdot r^l s^t) = \phi_{kn}(r^{i-l} s^{j+t}) = r_1^{i-l} s_1^{j+t}$$

on the other hand

$$\phi_{kn}(r^i s^j) \cdot \phi_{kn}(r^l s^t) = r_1^i s_1^j \cdot r_1^l s_1^t = r_1^{i-l} s_1^{j+t}$$

where $i - j$ is modulo k and $j + t$ is modulo 2. Consider the system (D_{2n}, ϕ_{kn}), we will show that this is an inverse system of finite dihedral groups. First note that $\phi_{kk} : D_{2k} \to$

D_{2k}, by our definition this is the identity map. Now, let $m|k|n$ in I, and consider the maps

$$\phi_{mk} : D_{2k} \to D_{2m}$$

and

$$\phi_{kn} : D_{2n} \to D_{2k}$$

take $\phi_{mk} \circ \phi_{kn} : D_{2n} \to D_{2m}$, note that

$$\begin{aligned}(\phi_{mk} \circ \phi_{kn})(r^i s^j) &= \phi_{mk}(\phi_{kn}(r^i s^j)) \\ &= \phi_{mk}(r_1^i s_1^j) \\ &= r_2^i s_2^j\end{aligned}$$

$$\phi_{mn}(r^i s^j) = r_2^i s_2^j$$

This shows $\phi_{mk} \circ \phi_{kn} = \phi_{mn}$, and hence the system (D_{2n}, ϕ_{kn}) is an inverse system. Consider the profinite dihedral group, D^*, as a subgroup of $GL_2(\widehat{\mathbb{Z}})$ as follows:

set

$$x = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

and

$$y = \begin{bmatrix} -1 & z \\ 0 & 1 \end{bmatrix}$$

where $z \in \widehat{\mathbb{Z}}$ such that $z \neq 0$, it is easy to see that x has an infinite Steinitz number order and y is of order 2. Indeed $yxy = x^{-1}$. So

$$D^* \cong \overline{\langle x, y \rangle}$$

set $H = \overline{\langle x \rangle}$ and $K = \overline{\langle y \rangle}$, and define $\Phi : K \to Aut(H)$ by sending $k \in K$ to the inversion automorphism on H, the associated action is $k \cdot h = h^{-1}$. so we have $D^* \cong H \rtimes_{\Phi} K$. From the semidirect product we have the short exact sequence

$$1 \longrightarrow \widehat{\mathbb{Z}} \longrightarrow D^* \longrightarrow C_2 \longrightarrow 1$$

we also have

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow D_n \longrightarrow C_2 \longrightarrow 1$$

applying the inverse limit functor to the above short exact sequence, we obtain

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varprojlim \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \varprojlim D_n & \longrightarrow & \varprojlim C_2 & \longrightarrow & 1 \\ & & \downarrow{\cong} & & \downarrow{\gamma} & & \downarrow{\cong} & & \\ 1 & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & D^* & \longrightarrow & C_2 & \longrightarrow & 1 \end{array}$$

by the five lemma for short exact sequences [9], we deduce that γ is an isomorphism. $\square$

Example 3.2.2. The inverse limit of the dihedral 2-groups D_{2^n} is isomorphic to D_{2^∞}.

Proof. We follow the proof of lemma 2.2.1. First note that (D_{2^n}, ϕ_{mn}, I) is an inverse system with $I = \mathbb{N}$ and ϕ_{mn} is as in lemma 2.2.1 by setting the relation $\leq$ on I as the usual inequality. We have $D_{2^n} \cong \mathbb{Z}/2^n\mathbb{Z} \rtimes C_2$, this translates to the short exact sequence:

$$1 \longrightarrow \mathbb{Z}/2^n\mathbb{Z} \longrightarrow D_n \longrightarrow C_2 \longrightarrow 1$$

now consider D_{2^∞} as subgroup of $GL_2(\widehat{\mathbb{Z}})$, using

$$x = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

and

$$y = \begin{bmatrix} -1 & z \\ 0 & 1 \end{bmatrix}$$

where z is a nonzero element in $\widehat{\mathbb{Z}}$, it is easy to see that x has an infinite Steinitz number order and y is of order 2. Indeed $yxy = x^{-1}$. So

$$D_{2^\infty} \cong \overline{\langle x, y \rangle}$$

set $H = \overline{\langle x \rangle}$ and $K = \overline{\langle y \rangle}$, and define $\Phi : K \to Aut(H)$ by sending $k \in K$ to the inversion automorphism on H, the associated action is $k \cdot h = h^{-1}$. so we have $D_{2^\infty} \cong H \rtimes_\Phi K$. From the semidirect product we have the short exact sequence

$$1 \longrightarrow \widehat{\mathbb{Z}}_2 \longrightarrow D_{2^\infty} \longrightarrow C_2 \longrightarrow 1$$

then

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varprojlim \mathbb{Z}/2^n\mathbb{Z} & \longrightarrow & \varprojlim D_{2^n} & \longrightarrow & \varprojlim C_2 & \longrightarrow & 1 \\ & & \downarrow \cong & & \downarrow \lambda & & \downarrow \cong & & \\ 1 & \longrightarrow & \widehat{\mathbb{Z}}_2 & \longrightarrow & D_{2^\infty} & \longrightarrow & C_2 & \longrightarrow & 1 \end{array}$$

and by the five lemma for short exact sequences, we deduce that λ is an isomorphism.

□

3.3 Example: The Profinite Generalized Quaternion Group

The generalized quaternion group is given by:

$$Q_i = \langle x, y \mid x^{2^i} = y^4 = 1,\, x^{2^{i-1}} = y^2,\, x^y = x^{-1} \rangle$$

It is a non-abelian 2-group of order 2^{i+2}. Any element in Q_i can be written as $x^a y^b$ where a is modulo 2^i and b is modulo 4, and for any $x^a y^b$ and $x^c y^d$ in Q_i the product $x^a y^b \cdot x^c y^d$ is $x^{a+(-1)^b c} y^{b+d}$, to see this we need to establish the following relation:

$$y^b x y^{-b} = x^{(-1)^b}$$

since b is modulo 2, it it either even or odd; if b is even then

$$y^b x y^{-b} = x \tag{3.1}$$

since even powers of y can be absorbed into the powers of x using the relation $x^{2^{i-1}} = y^2$. In the case b to be odd, note that any odd number can be written as $2s+1$ for $s \in \mathbb{N}$, so we can rewrite $y^b = y^{2s}y$, and form relation 3.1 and the relation $x^y = x^{-1}$ we have

$$y^b x y^{-b} = x^{-1} \tag{3.2}$$

the relations 3.1 and 3.2 together gives

$$y^b x y^{-b} = x^{(-1)^b}$$

using induction, we can prove that

$$y^b x^c y^{-b} = (y^b x y^{-b})^c$$

so

$$x^a y^b \cdot x^c y^d = x^a (y^b x^c y^{-b}) y^{b+d} = x^a x^{(-1)^b c} y^{b+d} = x^{a+(-1)^b c} y^{b+d}.$$

Regarded as finite 2-groups we can construct the pro-2 group for a system of generalized quaternion groups. We begin with the construction of the inverse system. To construct an inverse system of generalized quaternion groups, let $I = \mathbb{N}$ be the index set, defining the relation on I by $n \leq m$ as the usual inequality, we have I as an indexed set. Define

$$\phi_{kn} : Q_n \to Q_k$$

for any $n, k \in I$, such that $k < n$, via $\phi_{kn}(x^i y^j) = x^i y^j$, where i is modulo 2^n and j is modulo 4. This map is well defined since $k < n$, it is also a homomorphism, to see this let $x^i y^j, x^l y^t \in Q_n$ with i, l are modulo 2^n and j, t are modulo 4. Take

$$\phi_{kn}(x^i y^j \cdot x^l y^t) = \phi_{kn}(x^{i+(-1)^j l} y^{j+t}) = x^{i+(-1)^j l} y^{j+t}$$

on the other hand

$$\phi_{kn}(x^i y^j) \cdot \phi_{kn}(x^l y^t) = x^i y^j \cdot x^l y^t = x^{i+(-1)^j l} y^{j+t}$$

so ϕ_{kn} is indeed a homomorphism.

Consider the system (Q_n, ϕ_{kn}), we will show that this is an inverse system of finite generalized quaternion groups. First note that $\phi_{kk} : Q_k \to Q_k$, by our definition this is the identity map. Now, let $m < k < n$ in I, and consider the maps

$$\phi_{mk} : Q_k \to Q_m$$

and

$$\phi_{kn} : Q_n \to Q_k$$

take $\phi_{mk} \circ \phi_{kn} : Q_n \to Q_m$, note that

$$\begin{aligned}(\phi_{mk} \circ \phi_{kn})(x^i y^j) &= \phi_{mk}(\phi_{kn}(x^i y^j)) \\ &= \phi_{mk}(x^i y^j) \\ &= x^i y^j\end{aligned}$$

on the other hand

$$\phi_{mn}(x^i y^j) = x^i y^j$$

this shows $\phi_{mk} \circ \phi_{kn} = \phi_{mn}$, and hence (Q_n, ϕ_{kn}) is an inverse system. Define

$$\mathscr{C} = \prod_{n \in I} Q_n$$

and for each $n \in I$ let π_n be the projection map

$$\pi_n : \mathscr{C} \to Q_n$$

define

$$s \varprojlim_{n \in I} Q_n = \left\{ \mathbf{c} \in \mathscr{C} \;\middle|\; \phi_{mk}\pi_k(\mathbf{c}) = \pi_m(\mathbf{c}), \text{ for all } m \leq k \right\}$$

and

$$Q^* = \overline{\langle \bar{x}, \bar{y} \rangle}$$

where $\bar{x} = (x_i)_{i\in I}$ and $\bar{y} = (y_i)_{i\in I}$, we have

$$|\bar{x}| = LCM(2^i)_{i\in I} = 2^\infty \in \mathbb{SN} \qquad \text{and} \qquad |\bar{y}| = LCM(4)_{i\in I} = 4$$

Also

$$\bar{y}\bar{x}\bar{y}^{-1} = (y_i)_{i\in I}(x_i)_{i\in I}(y_i^{-1})_{i\in I} = (y_i x_i y_i^{-1})_{i\in I} = (x_i^{-1})_{i\in I} = \bar{x}^{-1}$$

since $x_i^{2^{n-2}} = y_i^2$ for all $i \in I$, we can write any element in Q^* as $\bar{x}^a\bar{y}^b$ where b is modulo 4. Now we show that $Q^* = \varprojlim_{n\in I} Q_n$, let

$$\phi_n : Q^* \to Q_n \qquad \text{via} \qquad \phi_n(\bar{x}^a\bar{y}^b) = x_n^a y_n^b$$

and

$$\theta : Q^* \to s\varprojlim_{n\in I} Q_n \qquad \text{via} \qquad \theta(\bar{x}^a\bar{y}^b) = (\phi_n(\bar{x}^a\bar{y}^b))_{n\in I}$$

First we show that θ is injective: Let $\bar{x}^{a_1}\bar{y}^{b_1}$ and $\bar{x}^{a_2}\bar{y}^{b_2}$ be in Q^*, if $\theta(\bar{x}^{a_1}\bar{y}^{b_1}) = \theta(\bar{x}^{a_2}\bar{y}^{b_2})$ then

$$(\phi_n(\bar{x}^{a_1}\bar{y}^{b_1}))_{n\in I} = (\phi_n(\bar{x}^{a_2}\bar{y}^{b_2}))_{n\in I}$$

and so

$$x_n^{a_1} y_n^{b_1} = x_n^{a_2} y_n^{b_2} \qquad \text{for all} \qquad n \in I$$

this gives $\bar{x}^{a_1}\bar{y}^{b_1} = \bar{x}^{a_2}\bar{y}^{b_2}$. Now we show that θ is subjective, let $\mathbf{c} = (x_n^{a_n} y_n^b)_{n\in I}$, where a_n is modulo 2^n and b is modulo 4. Take $\bar{a} = \prod_{n\in I} a_n$, and consider $\bar{x}^{\bar{a}}\bar{y} \in Q^*$, we have

$$\theta(\bar{x}^{\bar{a}}\bar{y}^b) = (\phi_n(\bar{x}^{\bar{a}}\bar{y}^b))_{n\in I} = (x_n^{a_n} y_n^b)_{n\in I} = \mathbf{c}$$

this shows that θ is a surjection. Therefore θ is a bijection from Q^* to $\varprojlim_{n\in I} Q_n$, thus

$Q^* = \varprojlim_{n \in I} Q_n$. Thus we have the following

Example 3.3.1. The profinite generalized quaternion group Q^* is the inverse limit of finite generalized quaternion groups.

3.4 Profinite Hamiltonian Groups

Definition 3.4.1. A finite group G is an elementary abelian p-group if every non-trivial element has order p.

We can write any elementary abelian group as

$$(\mathbb{Z}/p\mathbb{Z})^n$$

for $n \in \mathbb{Z}_{\geq 0}$. We note that not every elementary abelian group is cyclic, for example

$$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

is elementary abelian but not cyclic.

Remark 3.4.2. An elementary abelian pro-2 group is the inverse limit of finite elementary abelian 2-groups.

Theorem 3.4.3 (Goursat's Lemma [3])**.** *There is a bijective correspondence between subgroups G of $A \times B$ and quintuples $\{\bar{G}_1, G_1, \bar{G}_2, G_2, f\}$, where $G_1 \trianglelefteq \bar{G}_1 \leq A$, $G_2 \trianglelefteq \bar{G}_2 \leq B$, and $f : \bar{G}_1/G_1 \to \bar{G}_2/G_2$ is an isomorphism.*

Definition 3.4.4. A finite group is said to be *Hamiltonian* if all of its subgroups are normal.

Let $I = \mathbb{N}$, and let $\mathscr{H} = \{H_i\}_{i \in I}$ be a collection of hamiltonian groups, first we see that any subgroup of H_i is hamiltonian, since subgroups which are normal in the whole group are also normal in any subgroup that contain them. Let $N_i \triangleleft H_i$ and consider

H_i/N_i since subgroups are of the form K_iN_i/N_i where $K_i \leq H_i$ which is also normal, but then $K_iN_i \lhd H_i$ and so $K_iN_i/N_i \lhd H_i/N_i$ and so $H_i/N_i \in \mathscr{H}$. Now let $K_i, L_i \in \mathscr{H}$ and take $K_i \times L_i$ by Goursat's lemma we can see that any subgroup of the product is normal and hence $K_i \times L_i \in \mathscr{H}$.

Let $H^* = \varprojlim_{i \in I} H_i$, any closed subgroup K^* of H^* can be viewed as

$$K^* \cong \varprojlim_{N \trianglelefteq_o H^*} K^*/(K^* \cap N)$$

where $K^*/(K^* \cap N)$ is isomorphic to a normal subgroup in H^*/N for some open normal subgroup N of H^*. So

$$(K^*)^{h^*} \cong \varprojlim_{N \trianglelefteq_o H^*} (K^*/(K^* \cap N))^{h_N} \cong \varprojlim_{N \trianglelefteq_o H^*} K^*/(K^* \cap N) = K^*$$

and so any subgroup of H^* is normal.

Define $\mathscr{E}$ to be the set of all finite elementary abelian 2-groups, and $\mathscr{A}_{odd}$ to be the set of all abelian groups in which every element of is of odd prime order. Note that both $\mathscr{E}$ and $\mathscr{A}_{odd}$ are closed under subgroups, quotients, and products. For $i \in \mathbb{N}$, let $E_i \in \mathscr{E}$ and set

$$E^* = \varprojlim_{i \in \mathbb{N}} E_i$$

is an elementary abelian pro-2 group. Also, for $A_i \in \mathscr{A}_{odd}$ we have

$$A^* = \varprojlim_{i \in \mathbb{N}} A_i$$

is a profinite abelian group in which every element is of odd Steinitz prime order.

Theorem 3.4.5. *Let H be a pro-Hamiltonian group. Then H is a profinite abelian group, or $H = Q^* \times E^* \times A^*$ where E^* is an elementary abelian pro-2 group, Q^* is a profinite quaternion group and A^* is a profinite abelian group in which every element is of odd Steinitz prime order.*

Proof. Any finite Hamiltonian group is either abelian or can be written as

$$H_i = Q_i \times E_i \times A_i$$

where E_i is elementary abelian 2-group, and A is abelian group in which every element is of odd prime order. Taking the inverse limit we have

$$\varprojlim_{i \in \mathbb{N}} H_i = \varprojlim_{i \in \mathbb{N}} (Q_i \times E_i \times A_i)$$

since

$$\varprojlim_{i \in \mathbb{N}} (R_i \times S_i) = \varprojlim_{i \in \mathbb{N}} R_i \times \varprojlim_{i \in \mathbb{N}} S_i$$

for R_i and S_i are finite groups, we have

$$\varprojlim_{i \in \mathbb{N}} H_i = \varprojlim_{i \in \mathbb{N}} Q_i \times \varprojlim_{i \in \mathbb{N}} E_i \times \varprojlim_{i \in \mathbb{N}} A_i = Q^* \times E^* \times A^*$$

so

$$H^* \cong Q^* \times E^* \times A^*$$

□

3.5 Sylow Intersections in Profinite Groups

Definition 3.5.1. Let G be a profinite group, and P and Q be sylow p-subgroups of G such that $P \neq Q$. The subgroup $P \cap Q$ of G is called a maximal sylow p-intersection if whenever $P \cap Q \leq R \cap S$ for sylow p-subgroups R and S of G, we have $P \cap Q = R \cap S$.

Lemma 3.5.2. *Let G be a profinite group, and let $K = \bigcap(S \mid S \in Syl_p(G))$. Then:*

(a) $K \trianglelefteq G$, and K is a pro-p subgroup of G.

(b) K contains every normal pro-p subgroup of G.

Proof. Let $g \in G$ be arbitrary, since $K \leq S$ for all $S \in Syl_p(G)$ we have $K^g \leq S^g$ for all $S \in Syl_p(G)$, by proposition 2.4.7(d) we have $K^g \leq K$ and so $K \trianglelefteq G$.

To show that K is a pro-p subgroup, Let $I = (N \mid N \triangleleft_o G)$ be a filter base such that $\bigcap(N \mid N \triangleleft_o G) = 1$. Then by theorem 2.3.18 we have

$$K \cong \varprojlim_{N \in I} K/(K \cap N)$$

here clearly $K \cap N \triangleleft K$ since $K \trianglelefteq G$. and since $K/(K \cap N) \leq G/(K \cap N)$ is a p-subgroup for all $N \in I$, this proves (a). Now, Let H be normal pro-p subgroup of G, since sylow p-subgroups are maximal pro-p subgroups we have $H \leq K$. So there exists $P \in syl_p(G)$ such that $H \leq P$. By normality $H^g = H$ for all $g \in G$. we have

$$H = H^g \leq P^g \quad \text{for all } g \in G$$

Thus

$$H \leq \bigcap_{g \in G} P^g = \bigcap(P \mid P \in Syl_p(G)) = K$$

this gives (b). □

Theorem 3.5.3. *Let G be a profinite group, let $D = P \cap Q$ be a maximal sylow p-intersection of $P, Q \in Syl_p(G)$ with $P \neq Q$. Then:*

(a) D is a proper subgroup of each sylow p-subgroup of $N_G(D)$.

(b) Any two distinct sylow p-subgroups of $N_G(D)$ have intersection D.

(c) There exists a one-to-one correspondence between sylow p-subgroups of G containing D and the sylow p-subgroups of $N_G(D)$.

Proof. Since $D = P \cap Q$, where $P, Q \in Syl_p(G)$, note that $D \trianglelefteq N_G(D)$. So D is a pro-p subgroup of $N_G(D)$ contained in every sylow p-subgroup of $N_G(D)$. For strict inequality: since $D = P \cap Q$, we have $D < P$ and this gives $D < N_P(D) \leq N_G(D)$, and we have (a). For part (b) let R_1 and S_1 be sylow p-subgroups of $N_G(D)$ such that $R_1 \neq S_1$, we want to show that $D = R_1 \cap S_1$. As $D \lhd N_G(D)$ we have $D \leq R_1$ and $D \leq S_1$, so $D \leq R_1 \cap S_1$. Let R and S be sylow p-subgroups of G such that $R_1 \leq R$ and $S_1 \leq S$. then $D = R \cap S$, and

$$D \leq R_1 \cap S_1 \leq R \cap S = D.$$

Hence

$$D = R_1 \cap S_1$$

For part (c), let $\mathscr{D} = \left\{ P \in Syl_p(G) \;\middle|\; D \leq P \right\}$ and define $\Phi : \mathscr{D} \to Syl_p(N_G(D))$ by

$$\Phi(P) = P \cap N_G(D).$$

Let $S \in Syl_p(N_G(D))$, then there is an $S^* \in \mathscr{D}$ such that $S \leq S^*$ and

$$\Phi(S^*) = S^* \cap N_G(D)$$

but $S \leq S^* \cap N_G(D)$ so $\Phi(S^*) = S$ $\square$

Chapter 4

The Transfer Homomorphism for Profinite Groups

4.1 Necessary Background in Set Theory

Let X be a well ordered set, a segment of X is a subset S such that for any $a,b \in X$ with $a \leq b$ then $a \in S$ implies that $b \in S$. If S is proper and a is the first element of $X - S$ then

$$S_a = \{b \in X \mid b < a\}$$

called the initial segment. For two well-ordered sets X and Y we say that X is order isomorphic to Y if there is a bijective function $f : X \to Y$ such that if $a \leq b$ in X then $f(a) \leq f(b)$ in Y, we write $X \approx Y$. An ordinal number α is a well-ordered set if $a = S_a$ for all $a \in \alpha$. If X is a well-ordered set such that $X \approx \alpha$ then we say that X has ordinal number α and we write $ord(X) = \alpha$. If X is finite then α is a finite ordinal otherwise α is an infinite ordinal see [4].

Example 4.1.1 ([4]). Take $\mathbb{N} = \{0,1,2,\cdots\}$ with the usual order, denote the ordinal number of $\mathbb{N}$ by ω. Let $X = \{a,b,c,d,\cdots\}$ be a well ordered set with $a < b < c < d < \cdots$, then $S_a \approx 0$ so a occupies the first position, likewise b occupies the second position,

c the third and so on, see [3]. In general:

Theorem 4.1.2 ([2]). *Every well-ordered set has a unique ordinal number.*

Ordinal numbers can be added, multiplied and exponentiated. The class of ordinal numbers **Ord** forms a chain

$$0 < 1 < 2 < \cdots < \omega < \omega + 1 < \omega + 2 < \cdots < \omega 2 < \omega 2 + 1 < \cdots$$

For an ordinal α, we call the ordinal number $\alpha + 1$ a *successor* of α, and we call α a *predecessor* of the ordinal number $\alpha + 1$. An ordinal number is called a *limit ordinal* number if and only if it does not have a predecessor.

Theorem 4.1.3 (The Principle of Transfinite recursion over **Ord** [4]). *Let $\phi(\alpha)$ be a statement that is either true or false for each $\alpha \in$ **Ord**. If $\phi(\beta)$ is true for each $\beta < \alpha$ implies that $\phi(\alpha)$ is true, then $\phi(\alpha)$ is true for each $\alpha \in$ **Ord** i.e.*

1. *$\phi(0)$ is true and if $\phi(\alpha)$ is true, then $\phi(\alpha + 1)$ is true for each non-limit ordinal α.*

2. *If β is a limit ordinal and $\phi(\alpha)$ is true for each $\alpha < \beta$ then $\phi(\beta)$ is true.*

Note that ordinal numbers capture the position of an element in a set. While similar in construction the cardinal numbers capture the cardinality of a set; i.e the number of elements in this set. The class of cardinal numbers **Card** form a chain

$$0 < 1 < 2 < \cdots < \aleph_0 < \aleph_1 < \aleph_2 < \cdots$$

where $\aleph_0 = card(\mathbb{N})$, $\aleph_1 = card(\mathscr{P}(\mathbb{N}))$ and $\aleph_1 = card(\mathscr{P}(\mathscr{P}(\mathbb{N})))$ and so on. The following theorems are equivalent.

Theorem 4.1.4 ([2]). *Every set has a unique cardinal number.*

Theorem 4.1.5 (Theorem AC1 [2]). *Any collection of nonempty sets has a choice function, i.e Let C be a collection of nonempty sets. Then we can choose a member from each set in that collection. In other words, there exists a function f defined on C with the property that, for each set S in the collection, f(S) is a member of S.*

Theorem 4.1.6 (Theorem CAC [2]). *Any collection of mutually disjoint nonempty sets has a transversal.*

Theorem 4.1.7 (Zermelo's Theorem [2]). *Every set can be well ordered.*

Theorem 4.1.8 ([2]). *Every well-ordered set is isomorphic to a unique ordinal number.*

4.2 The Permanent Mapping

Definition 4.2.1. Let $A = [a_{i,j}]$ be an $(n \times n)$-matrix and S_n be the symmetric group on $\{1, 2, \cdots, n\}$. The *permanent* of A is given as

$$perm(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

Definition 4.2.2. A *monomial* matrix A over a commutative ring R with unity, is an $(n \times n)-$matrix in which each row and each column has exactly one nonzero entry from R. It can be written as a product of a permutation matrix P and a diagonal matrix D

$$A = DP$$

since

$$M_n(R) = D_n \rtimes \Sigma_{n \times n}$$

where D_n is the group of all diagonal $(n \times n)$-matrices with entries from R, and $\Sigma_{n \times n}$ is the group of all permutation $(n \times n)$-matrices. Then

$$perm(A) = \prod_{i=1}^{n} a_i \quad \text{where } a_i \text{ is the non-zero entry in row } i.$$

Remark 4.2.3. Let A and B be two monomial matrices over a commutative ring with unity, Then $perm(AB) = perm(A)perm(B)$. To see this note that AB is another monomial matrix in which each nonzero entry has the form $(ab)_i$,so

$$perm(AB) = \prod_{i=1}^{n} (ab)_i$$

by commutativity and associativity

$$perm(AB) = \prod_{i=1}^{n} a_i \prod_{i=1}^{n} b_i = perm(A)perm(B)$$

4.3 The Transfer of Profinite Groups

Let G be a profinite group with H as a closed subgroup. Consider the set

$$H \setminus G = \{Hg \mid g \in G\}$$

of all the right cosets of H in G, we have

$$G = \bigsqcup_{g \in G} Hg$$

By theorem 4.2.3, the cardinality of $H \setminus G$ is $|G : H| = n \in \mathbb{SN}$. Considered as an uncountable set we can define a well-ordering using Zermelo's theorem. Since it is a collection of mutually disjoint nonempty sets, by theorem 4.2.5 it has a transversal Y which has the same cardinality as $H \setminus G$. Well order Y by the choice function

$$\nu : H \setminus G \to Y \quad \text{by } \nu(Hx) = y_{Hx} \text{ for } x \in G \text{ and } y_{Hx} \in Hx$$

it follows that this function ν is an order preserving bijection, using theorem 4.1.7, we can take $\Delta \in \mathbb{SN}$ to index Y

$$Y = \{y_\alpha\}_{\alpha=1}^{\Delta}$$

as a first element in Y we take id_G which corresponds to $\alpha = 1$. Now, Let (G_i, ϕ_{ij}, I) be an inverse system of finite groups, set

$$G = \varprojlim_{i \in I} G_i.$$

The set of right cosets $H \setminus G$, which is a topological space with respect to the quotient topology, is the profinite topological space [25]

$$H \setminus G = \varprojlim_{i \in I} H_i \setminus G_i$$

The natural group action of G on $H \setminus G$ is the continuous map

$$\phi : G \times H \setminus G \to H \setminus G, \quad \text{with } \phi(g, Hy_\alpha) = (Hy_\alpha)g \text{ where } \alpha \in \Delta$$

this action will give the permutation representation

$$\rho : G \to S_{H \setminus G}$$

given by

$$\rho(g)(Hy_\alpha) = (Hy_\alpha)g = Hy_\beta$$

where $\alpha, \beta \in \Delta$, note that if $g \in Hy_\alpha$ then $\alpha = \beta$ otherwise $\alpha \neq \beta$. This permutation representation will permute the elements in $H \setminus G$ likewise the elements in Y. If $g \in G$ then

$$(Hy_\alpha)g = Hy_\beta$$

we have

$$h_{\alpha\beta} = y_\alpha \, g \, y_\beta^{-1} \in H$$

Definition 4.3.1. For a Steinitz number n we say that a monomial $(n \times n)$-matrix A over a commutative ring with identity R is *known* if the limit in transfinite induction always exist. i.e

$$\lim_{i \to n} \prod_{m \leq i} a_m \quad \text{converges in } R$$

and is independent of the order of the indices, where $a_j \in R$ is the only nonzero entry in the j-th row of A,

Fix an ordering Δ of Y and let $K \lhd H$ so that H/K is abelian. Define $M_\Delta(H/K)$ to be the set of all known monomial matrices with entries from H/K and for any $A \in M_\Delta(H/K)$ we have exactly one nonzero element in each row and column from H/K. For any $z \in H/K$ we define $0 * z = z * 0 = 0$ and $z + 0 = 0 + z = z$.

Definition 4.3.2. Let G be a profinite group, H be a closed subgroup of G, and K be a closed normal subgroup of H such that H/K is abelian. Let $Y = \{y_\alpha\}_{\alpha=1}^{\Delta}$ be a set of coset representatives of H in G. The *monomial representation* of a profinite group G over the abelian group H/K is given by

$$\mu : G \to M_\Delta(H/K) \quad \text{via } \mu(g) = A \in M_\Delta(H/K)$$

where $A = [a_{\alpha,\beta}]$ with

$$a_{\alpha\beta} = \begin{cases} y_\alpha \, g \, y_\beta^{-1} K & \text{if } y_\alpha \cdot g = y_\beta \\ 0 & \text{otherwise} \end{cases}$$

Theorem 4.3.3. *The monomial representation in the above definition is a well defined continuous homomorphism.*

Proof. To show that it is well defined, let $Z = \{z_\alpha\}_{\alpha\in\Delta}$ be another set of coset representatives of H in G with relation $z_\alpha = u_\alpha y_\alpha$. For any $g \in G$ we have

$$z_\alpha \, g \, z_\alpha^{-1} K = u_\alpha y_\alpha \, g \, (u_\alpha y_\alpha)^{-1} K = u_\alpha (y_\alpha \, g \, y_\alpha^{-1}) u_\alpha^{-1} K$$

then

$$z_\alpha \, g \, z_\alpha^{-1} K = \left(u_\alpha K\right)\left(y_\alpha \, g \, y_\alpha^{-1} K\right)\left(u_\alpha^{-1} K\right) = y_\alpha \, g \, y_\alpha^{-1} K.$$

To show that it is a homomorphism, let g_1, g_2 be in G, and call $\mu(g_1) = A_1$ and $\mu(g_2) = A_2$. Then $\mu(g_1)\mu(g_2) = A_1 A_2 \in M_\Delta(H/K)$ is another monomial matrix in which each nonzero entry is given as $y_\alpha \, g_1 g_2 \, y_\gamma^{-1} K$, to see this take $A_1 = [a_{\alpha,\beta}]$ and $A_2 = [b_{\delta,\gamma}]$ where

$$a_{\alpha\beta} = \begin{cases} y_\alpha \, g_1 \, y_\beta^{-1} K & \text{if } y_\alpha \cdot g = y_\beta \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_{\delta\gamma} = \begin{cases} y_\delta \, g_2 \, y_\gamma^{-1} K & \text{if } y_\delta \cdot g = y_\gamma \\ 0 & \text{otherwise} \end{cases}$$

for g_1 and g_2 in G and $\alpha, \beta, \delta, \gamma \in \Delta$. Since A_1 and A_2 have exactly one nonzero entry in each row and each column we have for $A_1A_2 = [a_{\alpha,\beta} b_{\delta,\gamma}]$ where $\beta = \delta$, so the nonzero entries of A_1A_2 look like

$$\left(y_\alpha \, g_1 \, y_\beta^{-1} K\right)\left(y_\beta \, g_2 \, y_\gamma^{-1} K\right) = \left(y_\alpha \, g_1 \, y_\beta^{-1} y_\beta \, g_2 \, y_\gamma^{-1}\right) K = y_\alpha \, g_1 g_2 \, y_\gamma^{-1} K$$

then $\mu(g_1g_2) = \mu(g_1)\mu(g_2)$. Finally, multiplication in H/K is continuous since H/K is a profinite group. □

Definition 4.3.4. Let G be a profinite group and H be a closed subgroup of G with index Δ in $\mathbb{SN}$. Let K be a normal subgroup of H such that H/K is abelian. The *permanent* of a monomial matrix $A \in M_\Delta(H/K)$ the known monomial matrices is the map

$$perm_\Delta : M_\Delta(H/K) \to H/K$$

where

$$M_\Delta(H/K) = D_\Delta \rtimes \Sigma_{\Delta\times\Delta}$$

and D_Δ is the set of all $\Delta \times \Delta$ diagonal matrices, and $\Sigma_{\Delta\times\Delta}$ is the set of all $\Delta \times \Delta$ permutation matrices. Which satisfies

$$perm_\Delta(A) = \prod_{\alpha\in\Delta} a_\alpha \in H/K \tag{4.1}$$

and

$$perm_\Delta(A) = \prod_{\substack{(\alpha,\beta)\in\Delta\times\Delta \\ a_{\alpha\beta}\neq 0}} a_{\alpha\beta} \in H/K \tag{4.2}$$

where $a_{\alpha\beta} = a_\alpha \in H/K$, and the products are commutative and associative.

Theorem 4.3.5. *Let G be a profinite group, H a closed subgroup and K be a closed normal in H such that H/K is abelian. Then $perm_\Delta$ is a a continuous homomorphism into H/K.*

Proof. To show that $perm_\Delta$ is a homomorphism, let A and B be in $M_\Delta(H/K)$, using equation (4.1) we have

$$perm_\Delta(A) = \prod_{\alpha\in\Delta} a_\alpha = \prod_{\substack{(\alpha,\beta)\in\Delta\times\Delta \\ a_{\alpha,\beta}\neq 0}} a_{\alpha\beta}$$

and

$$perm_\Delta(B) = \prod_{\delta\in\Delta} b_\delta = \prod_{\substack{(\delta,\gamma)\in\Delta\times\Delta \\ b_{\delta\gamma}\neq 0}} b_{\delta\gamma}$$

then

$$perm_\Delta(A)perm_\Delta(B) = \prod_{\alpha\in\Delta} a_\alpha \prod_{\delta\in\Delta} b_\delta = \prod_{\substack{(\alpha,\beta)\in\Delta\times\Delta \\ a_{\alpha\beta}\neq 0}} a_{\alpha\beta} \prod_{\substack{(\delta,\gamma)\in\Delta\times\Delta \\ b_{\delta\gamma}\neq 0}} b_{\delta\gamma}$$

hence

$$perm_\Delta(A)perm_\Delta(B) = \prod_{\substack{(\alpha,\beta),(\beta,\gamma)\in\Delta\times\Delta \\ a_{\alpha\beta},b_{\beta\gamma}\neq 0}} a_{\alpha\beta}b_{\beta\gamma}$$

on the other hand for AB we have

$$perm_\Delta(AB) = \prod_{\substack{(\alpha,\beta),(\beta,\gamma)\in\Delta\times\Delta \\ a_{\alpha\beta},b_{\beta\gamma}\neq 0}} a_{\alpha\beta}b_{\beta\gamma}$$

hence

$$perm_\Delta(AB) = perm_\Delta(A)perm_\Delta(B)$$

Finally, since multiplication is continuous, then $perm_\Delta$ is continuous. □

Definition 4.3.6. Let G be a profinite group and H be a closed subgroup of G such that $|G:H| = n$, with $n \in \mathbb{SN}$, let $H' = [H,H]$. Let Y be a transversal of H in G with

$|Y| = \Delta \in \mathbb{SN}$. The transfer map from G into H/H' is the composition

$$V_H^G(g) = perm_\Delta(\mu(g)) = \prod_{\alpha,\beta\in\Delta} y_\alpha \, g \, y_\beta^{-1} \; H'$$

Theorem 4.3.7. *The transfer map from G into H/H' is a well defined continuous homomorphism.*

Proof. We have seen that $perm_\Delta$ is a continuous homomorphism and μ is a well defined continuous homomorphism, the result follows. □

Let G be a profinite group, H is a closed subgroup with $|G:H| \in \mathbb{SN}$ and let $Y = \{y_\alpha\}_{\alpha\in\Delta}$ to be a right transversal of H in G with $|Y| = \Delta$, let H' be the characteristic group of H. For $g \in G$ let αg to be in Δ such that $y_\alpha \, g \, y_{\alpha g}^{-1} \in H$, we can write

$$V_H^G(g) = perm_\Delta(\mu(g)) = \prod_{y_\alpha, y_{\alpha g}\in Y} y_\alpha \, g \, y_{\alpha g}^{-1} \; H' = \prod_{\alpha\in\Delta} y_\alpha \, g \, y_{\alpha g}^{-1} \; H'$$

If we fix g we get the permutation $\pi(g) : \alpha \to \alpha g$, suppose that $\pi(g)$ decomposes into Γ disjoint cycles $C_\gamma(g)$ where $\gamma \in \Gamma \leq \Delta$, each cycle consists of members in Δ, define

$$C_\gamma^{set}(g) = \left\{\varepsilon \in \Delta \;\middle|\; \varepsilon \text{ appears in } C_\gamma(g)\right\}$$

these sets partition Δ i.e

$$\Delta = \bigsqcup_{\gamma\in\Gamma} C_\gamma^{set}(g).$$

Each cycle has a length $r_\gamma \in \mathbb{SN}$ which also can be thought as an ordinal number, and for $\gamma \in \Gamma$ take $\left\{y_\gamma g^\kappa \;\middle|\; 0 \leq \kappa < r_\gamma\right\}$ to be the g orbits (or cycles) where $y_\gamma g^{r_\gamma} \in H y_\gamma$ and $y_\gamma g^\kappa \notin H y_\gamma$ for any $\kappa < r_\gamma$. Then

$$G = \bigsqcup_{\gamma\in\Gamma} \bigsqcup_{\kappa<r_\gamma} H y_\gamma g^\kappa$$

then $\left\{y_\gamma g^\kappa \;\middle|\; 0 \leq \kappa < r_\gamma\right\}$ can be used as a transversal of H in G.

Lemma 4.3.8. *From the construction above we have*

$$V_H^G(g) = \prod_{\gamma\in\Gamma} y_\gamma\, g^{r_\gamma}\, y_\gamma^{-1}\ H'$$

where $y_\gamma\, g^{r_\gamma}\, y_\gamma^{-1}$ is the first power of $y_\gamma\, g\, y_\gamma^{-1}$ that lies in H.

Proof. For $g \in G$ we have a disjoint coset decomposition

$$G = \bigsqcup_{\gamma\in\Gamma} \bigsqcup_{\kappa<r_\gamma} H y_\gamma g^\kappa$$

then $y_\gamma, y_\gamma g, \cdots, y_\gamma g^{r_\gamma-1}$ is a transversal of H in G. Now fix $\gamma \in \Gamma$ and consider

$$y_\gamma g^{r_\gamma-1} g = y_\gamma g^{r_\gamma} \in H y_\gamma$$

and for any $\kappa < r_\gamma - 1$ we have

$$y_\gamma g^\kappa g = y_\gamma g^{\kappa+1} \in H y_\gamma g^{\kappa+1}$$

Consider the transfer $V_H^G(g)$ we have

$$V_H^G(g) = \prod_{y_\alpha, y_\beta \in Y} y_\alpha\, g\, y_\beta^{-1}\ H'$$

where $y_\beta = \nu(y_\alpha g)$. In calculating the transfer we have

$$y_\gamma g^\kappa g \left(y_\gamma g^{\kappa+1}\right)^{-1} = y_\gamma g^{\kappa+1} \left(g^{\kappa+1}\right)^{-1} y_\gamma^{-1} = 1$$

for any $\kappa < r_\gamma - 1$, and

$$y_\gamma g^{r_\gamma-1} g \left(y_\gamma g^{r_\gamma}\right)^{-1} = y_\gamma g^{r_\gamma} y_\gamma^{-1}$$

then

$$V_H^G(g) = \prod_{\gamma\in\Gamma} y_\gamma\, g^{r_\gamma}\, y_\gamma^{-1}\ H'$$

$\square$

Theorem 4.3.9 (Transitivity of the Transfer)**.** *Let G be a profinite group, and let H and K be closed subgroups such that $K \leq H \leq G$ with each index is in $\mathbb{SN}$. Then for all $g \in G$ we have*

$$V_K^G(g) = V_K^H\left(V_H^G(g)\right)$$

Proof. Let Y be a right transversal of H in G of cardinality Δ and Z be a right transversal of K in H of cardinality Γ, then

$$G = \bigsqcup_{\alpha \in \Delta} H y_\alpha \qquad \text{and} \qquad H = \bigsqcup_{\delta \in \Gamma} K z_\delta$$

then

$$G = \bigsqcup_{\alpha \in \Delta} \bigsqcup_{\delta \in \Gamma} K z_\delta y_\alpha.$$

To finish off the proof we need to show that for any $z \in Z$ and $y \in Y$ that $(z_\gamma\, y_\alpha)\, g\, (z_\delta\, y_\beta)^{-1} \in K$. This is the same as showing that ZY is a right transversal of K in G; where $ZY = \{zy \mid z \in Z,\, y \in Y\}$, let $g \in G$ then $Hg = Hy$ for some $y \in Y$, so $gy^{-1} \in H$. Then $K(gy^{-1}) = Kz$ for some $z \in Z$, we have $Kg = K(zy)$ this shows that $zy \in Kg$. We claim that this element zy is the unique element of ZY in Kg; since otherwise if $Kz_\gamma\, y_\alpha = Kz_\delta\, y_\beta$, where $z_\gamma, z_\delta \in Z$ and $y_\alpha, y_\beta \in Y$, but then $Ky_\alpha = Ky_\beta$ since $Kz_\gamma, Kz_\delta \subset H$, so we have $y_\alpha = y_\beta$ since Y is a right transversal of H is G. Thus $Kz_\gamma y_\alpha = Kz_\delta y_\alpha$ therefore $Kz_\gamma = Kz_\delta$, but Z is a right transversal of K in H we have $z_\gamma = z_\delta$. Hence ZY is a right transversal of K in G. Now we show that

$$(z_\gamma\, y_\alpha)\, g\, (z_\delta\, y_\beta)^{-1} \in K.$$

From the above claim, we have $K(z_\gamma\, y_\alpha)g = Kz_\delta\, y_\beta$, and since $Kz_\gamma, Kz_\delta \subset H$ we have $Hy_\alpha\, g = Hy_\beta$, so $y_\alpha \cdot g = y_\beta$ and there is some $h \in H$ such that $h = y_\alpha\, g\, y_\beta^{-1}$ So

$$Kz_\gamma\, k = Kz_\gamma\, y_\alpha\, g\, y_\beta^{-1} = Kz_\delta\, y_\beta\, y_\beta^{-1} = Kz_\delta$$

so $z_\gamma \cdot k = z_\delta$ and there is some $kinK$ such that $k = z_\gamma k z_\delta^{-1}$. Finally

$$(z_\gamma y_\beta) g (z_\delta y_\beta)^{-1} = z_\gamma y_\beta g y_\beta^{-1} z_\delta^{-1} = z_\gamma k z_\delta^{-1} = h$$

so for any $g \in G$ we have $y_\alpha g = h_{\alpha\beta} y_\beta$, but $h_{\alpha\beta} \in H$ then $z_\delta g = k_{\alpha\beta\delta\gamma} z_\gamma$, then

$$z_\delta y_\alpha = k_{\alpha\beta\delta\gamma} z_\gamma y_\beta$$

where $k_{\alpha\beta\delta\gamma} \in K$. Hence for $g \in G$ we have

$$V_K^G(g) = \prod_{\alpha\in\Delta,\delta\in\Gamma} k_{\alpha\beta\delta\gamma} K'$$

and

$$V_H^G(g) = \prod_{\alpha\in\Delta} h_{\alpha\beta} H'$$

but

$$V_K^H(h_{\alpha\beta}) = \prod_{\delta\in\Gamma} k_{\alpha\beta\delta\gamma} K'$$

then

$$V_K^G(g) = \prod_{\alpha\in\Delta} V_K^H(h_{\alpha\beta}) K' = V_K^H\left(\prod_{\alpha\in\Delta} h_{\alpha\beta} K'\right) = V_K^H\left(V_H^G(g)\right)$$

4.4 Applications of the Profinite Transfer

The first application is Burnside's theorem for profinite groups, which is an analog of Burnside's theorem in [14]. we will prove it using the transfer homomorphism.

Theorem 4.4.1 (Burnside's Normal p-Complement Theorem for Profinite Groups [14])**.** *Let G be a profinite group and P is a sylow p-subgroup of G be such that $P \leq Z(NG(P))$. Then G has a closed normal subgroup H which has the elements of P as its coset representatives.*

We note that this theorem was proved for profinite groups in the context of fusion in [10]. To prove the theorem we need the following lemma, which is proved for finite groups in [14].

Lemma 4.4.2. *Let G be a profinite group, if two subsets K_1 and K_2 of G are normal in a sylow pro-p subgroup P of G and are conjugate in G then K_1 and K_2 are conjugate in $N_G(P)$.*

Proof. Suppose that $K_1^x = K_2$ for some $x \in G$. Since $K_1 \trianglelefteq P$ then $K_2 = K_1^x$ is normal in $P^x = Q$. So both P and Q are contained in the normalizer of K_2. As sylow pro-p subgroups are conjugate in $N_G(K_2)$. So $Q^y = P$ for some $y \in G$ with $K_2^y = K_2$. Now , let $z = xy$ we have $P^z = P$, $K_1^z = K_2$. This ends the proof of the lemma. □

here is a proof given by an axiomitized transfer.

Proof. First note that P is abelian so $P' = 1$, as a closed subgroup we have

$$G = \bigsqcup_{\alpha \in \Delta} P y_\alpha$$

where Y is a transversal of P in G with cardinality Δ. Let $u \in P$ and consider the orbits of u in G given by $\{y_\gamma u^\kappa \mid 0 \le \kappa < r_\gamma\}$ where $\gamma \in \Gamma \le \Delta$ and $y_\gamma u^{r_\gamma} \in P y_\gamma$ and $y_\gamma u^\kappa \notin P y_\gamma$ for any $\kappa < r_\gamma - 1$. Consider the transfer homomorphism from G to P, we have

$$V_P^G(u) = \prod_{\alpha,\beta \in \Delta} y_\alpha \, u \, y_\beta^{-1} \; P' = \prod_{\alpha,\beta \in \Delta} y_\alpha \, u \, y_\beta^{-1}$$

as $P' = 1$. View the permutation u on P in terms of the cycle structure, in calculating the transfer we have

$$y_\gamma u^\kappa u \big(y_\gamma u^{\kappa+1}\big)^{-1} = y_\gamma u^{\kappa+1} \big(u^{\kappa+1}\big)^{-1} y_\gamma^{-1} = 1$$

for any $\kappa < r_\gamma - 1$, and

$$y_\gamma u^{r_\gamma - 1} u \big(y_\gamma u^{r_\gamma}\big)^{-1} = y_\gamma u^{r_\gamma} y_\gamma^{-1}$$

then

$$V_P^G(u) = \prod_{\gamma\in\Gamma} y_\gamma\, u^{r_\gamma}\, y_\gamma^{-1}$$

Now, $y_\gamma\, u^{r_\gamma}\, y_\gamma^{-1} \in P$ is conjugate to u^{r_γ} in G, let us consider $K_1 = \{y_\gamma\, u^{r_\gamma}\, y_\gamma^{-1}\}$ and $K_2 = \{u^{r_\gamma}\}$, since P is abelian K_1 and K_2 are normal in P, using the last lemma we have $y_\gamma\, u^{r_\gamma}\, y_\gamma^{-1} = z\, u^{r_\gamma}\, z^{-1}$ with $z \in N_G(P)$. By hypothesis P in in the center of its normalizer, this means that $zu^{r_\gamma}z^{-1} = u^{r_\gamma}$. Hence

$$V_P^G(u) = \prod_{\gamma\in\Gamma} u^{r_\gamma}$$

and since r_γ where $\gamma \in \Gamma$ are the lengths of the cycles that partitioned Δ we have

$$V_P^G(u) = u^\Delta = u^{|G:P|}$$

Since P is a pro-p sylow subgroup of order $p^{m(p)}$, with $m(p) \leq \infty$, we have $p \nmid |G:P|$. Thus in the transfer of G onto P, $V_P^G(P) = P$ because trivially the transfer cannot be larger than P we have $V_P^G(G) = P$. The kernel of this homomorphism must be a subgroup, H, of index $p^{m(p)}$ in G and of order $|G:P|$. Hence H is a normal subgroup of index $p^{m(p)}$ in G and so the elements of P can be used as a transversal of H in G. $\square$

The next result is the focal subgroup theorem for profinite groups, again we will use the transfer in the proof.

Theorem 4.4.3. *Let G be a profinite group, P a pro-p sylow subgroup of G and $G' = [G,G]$. Then*

$$V_P^G(G) \cong P/P\cap G'$$

Proof. Let $Y = \{y_\alpha\}_{\alpha=1}^{\Delta}$ where $\Delta = |G:P|$ be a transversal of P in G. Consider the transfer homomorphism $V_P^G : G \to P/P'$ and note that since P/P' is a p-group then for any $x \in G$ with order prime to p we have $V_P^G(x) = 1$. Since G is generated by P and the other sylow subgroups for other primes we have $V_P^G(G) = V_P^G(P)$. Let $u \in P$ by lemma

4.3.9 we have

$$V_P^G(u) = \prod_{\gamma \in \Gamma} y_\gamma\, u^{r_\gamma}\, y_\gamma^{-1}\ P'$$

with $\gamma \in \Gamma \leq \Delta$. Note that

$$y_\gamma\, u^{r_\gamma}\, y_\gamma^{-1} = u^{r_\gamma}[u^{r_\gamma}, y_\gamma^{-1}]$$

since $[u^{r_\gamma}, y_\gamma^{-1}]$ is in G' we have

$$V_P^G(u) = \prod_{\gamma \in \Gamma} u^{r_\gamma}\ G' = u^{|G:P|}\ G'.$$

Since $\big(p, |G:P|\big) = 1$ then $V_P^G(u) \not\equiv 1$ for $u \in P$ and $u \notin G'$. We also have $V_P^G(G') \equiv 1$ since $V_P^G(G)$ is abelian. Hence the kernel of $P \to V_P^G(P)$ is $P \cap G'$, then

$$V_P^G(G) \cong P/P \cap G'$$

□

We present the following as a beginning effort to prove M. Hall Theorem in [14].

Definition 4.4.4. The *diagonal contribution*, $d(g)$ is the contribution of cycles of length one in $\pi(g)$ to $V_H^G(g)$, and is given by

$$d(g) = \prod_{\alpha = \alpha g} y_\alpha\, g\, y_\alpha^{-1} H'$$

Remark 4.4.5. Note that $d(g) = \prod_{\alpha \in \Delta} h_{\alpha\alpha} H'$ where $\pi(g) = \big[h_{\alpha\beta} H'\big]_{(\alpha,\beta) \in \Delta \times \Delta}$.

Lemma 4.4.6. *Let G be a profinite group, if u and v are conjugate in G, then $d(u) = d(v)$, and $d(u^{-1}) = [d(u)]^{-1}$.*

Proof. Let $v = t^{-1}ut$ for some $t \in G$. From $\alpha = \alpha u$ we have $\alpha = \alpha t v t^{-1}$ which gives $\alpha t = \alpha t v$, this means that $\alpha = \alpha u$ is equivalent to $\alpha t = \alpha t v$. By definition

$$d(v) = \prod_{\alpha t = \alpha t v} y_{\alpha t} v y_{\alpha t}^{-1} H' = \prod_{\alpha = \alpha u} y_{\alpha t} v y_{\alpha t}^{-1} H'$$

since $v = t^{-1}ut$ we have $v = t^{-1}y_\alpha^{-1}y_\alpha u x_\alpha^{-1}x_\alpha t$ we have

$$d(v) = \prod_{\alpha=\alpha u} \left(y_{\alpha t}t^{-1}y_\alpha^{-1}\right)\left(y_\alpha u y_\alpha^{-1}\right)\left(y_\alpha t y_{\alpha t}^{-1}\right)H'$$

but $y_{\alpha t}t^{-1}y_\alpha^{-1}$ and $y_\alpha t y_{\alpha t}^{-1}$ are in H we obtain

$$d(v) = \prod_{\alpha=\alpha u} y_\alpha u y_\alpha^{-1}H' = d(u)$$

since $\alpha = \alpha u$ is equivalent to $\alpha = \alpha u^{-1}$ we have

$$d(u^{-1}) = \prod_{\alpha=\alpha u} y_\alpha u^{-1}y_\alpha^{-1}H' = \prod_{\alpha=\alpha u} \left(y_\alpha u y_\alpha^{-1}\right)^{-1}H' = [d(u)]^{-1}$$

□

Definition 4.4.7. Let G be a profinite group and H be a closed subgroup of G. For $h \in H$ define

$$d^*(h) = h^{-1}d(h)$$

Lemma 4.4.8. *Let G be a profinite group and H be a closed subgroup of G. If $h \in H$ then*

$$V_H^G(h) = h^{|G:H|}\prod_{\gamma\in\Gamma} d^*(h^{r_\gamma})\left[d^*(y_\gamma\, h^{r_\gamma}\, y_\gamma)\right]^{-1}$$

Proof. From $d^*(h) = h^{-1}d(h)$ we have $h = d(h)\left[d^*(h)\right]^{-1}$ and by lemma 4.4.5 $h = d(h)d^*(h^{-1})$. Again by lemma 4.4.5 we have

$$d(h^{r_\gamma}) = d(y_\gamma\, h^{r_\gamma}\, y_\gamma)$$

hence if $y_\gamma\, h^{r_\gamma}\, y_\gamma$ is in H we have

$$y_\gamma\, h^{r_\gamma}\, y_\gamma = d(h^{r_\gamma})\left[d^*(y_\gamma\, h^{r_\gamma}\, y_\gamma)\right]^{-1} = h^{r_\gamma}d^*(h^{r_\gamma})\left[d^*(y_\gamma\, h^{r_\gamma}\, y_\gamma)\right]^{-1}$$

by lemma 4.3.9 we have for $h \in H$

$$V_H^G(h) = \prod_{\gamma \in \Gamma} y_\gamma \, g^{r_\gamma} \, y_\gamma^{-1} \; H'$$

but

$$y_\gamma \, h^{r_\gamma} \, y_\gamma = d(h^{r_\gamma}) \big[d^*(y_\gamma \, h^{r_\gamma} \, y_\gamma) \big]^{-1} = h^{r_\gamma} d^*(h^{r_\gamma}) \big[d^*(y_\gamma \, h^{r_\gamma} \, y_\gamma) \big]^{-1}.$$

then

$$V_H^G(h) = \prod_{\gamma \in \Gamma} h^{r_\gamma} d^*(h^{r_\gamma}) \big[d^*(y_\gamma \, h^{r_\gamma} \, y_\gamma) \big]^{-1} H' = h^{|G:H|} \prod_{\gamma \in \Gamma} d^*(h^{r_\gamma}) \big[d^*(y_\gamma \, h^{r_\gamma} \, y_\gamma) \big]^{-1}$$

□

Corollary 4.4.9. *If $d^*(h)$ is in H' for any $h \in H$ then $V_H^G(h) = h^{|G:H|}$.*

Proof. This is an immediate consequence of the last lemma. □

Remark 4.4.10. All results on the diagonal contribution is work towards the proof of Philip Hall Theorem in [14]

Chapter 5

Future Work

First, can we define the transfer without axiomatizing? That is: Is the set of known $(n \times n)$-monomial matrices the set of $(n \times n)$-monomial matrices?

In addition to many well known results in [14], [27], [11] and [24] as an application to the transfer for finite groups, we will list some of these results for future work. In chapter 14 of [14], Marshall Hall used the transfer as a tool to provide and prove some important results, some of which were extended to profinite groups in chapter 4. Here we list other results of chapter 14 in [14] which we like to extend to profinite groups, these results were listed in chapter 2.

We would like to provide an analog of the results by Yoshida [26] in section 4 we wish to extend to profinite groups. let G be a profinite group, and let H, K be closed subgroups of G, by $Rep(H/G\backslash K)$ we mean a complete set of double coset representatives in G.

Definition 5.0.1. Let G be a profinite group, S a closed subgroup of S, and W closed normal subgroup of S.

- W is a weakly closed subgroup of S with respect to G if for any $g \in G$ with $W^g \leq S$ we have $W^g = W$.

- W is strongly closed subgroup of S with respect to G if for any $x \in G$ we have $W^g \cap S \leq W$.

Definition 5.0.2. Let G be a profinite group, and H be closed subgroups of G, and S be closed subgroups of H. We say that S is of *sylow-type* in H with respect to G if any G-conjugate of S contained in H is also an H-conjugate to S.

Example 5.0.3. If S is weakly closed subgroup of a sylow subgroup of H, then S is of sylow type in H with respect to G.

Theorem 5.0.4. *Let G be a profinite group, and P a sylow p-subgroup of G that is weakly regular. Then $P \cap G' = P \cap N_G(P)'$.*

Theorem 5.0.5. *Let G be a profinite group, and P a sylow p-subgroup of G. Let $Q \lhd_c P \leq_c H \leq_c G$ such that Q is of sylow type in H with respect to G and that $[x, y; p-1] \in \Phi^*(Q)$ for each $x \in P$ and $y \in Q$. Then $P \cap G' = \big(P \cap H'\big)\big(P \cap N_G(Q)'\big)$.*

Bibliography

[1] Artin, E. (1929). Idealklassen in oberkörpern und allgemeines reziprozitätsgesetz. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 7, pages 46–51. Springer.

[2] Bach, C. N. (1995). *Zermelo's Well-ordering Theorem and the Axiom of Choice*. University of California, Berkeley.

[3] Bauer, K., Sen, D., and Zvengrowski, P. (2011). A generalized goursat lemma. *arXiv preprint arXiv:1109.0024*.

[4] Bland, P. E. (2011). *Rings and their modules*. Walter de Gruyter.

[5] Burnside, W. (1911). *Theory of groups of finite order*. University.

[6] Charatonik, J. J. and Charatonik, W. J. (2001). Inverse limits and openness of the induced mappings. *Topology and its Applications*, 114(3):235–260.

[7] Cohn, P. M. (2012). *Basic algebra: groups, rings and fields*. Springer Science & Business Media.

[8] Craven, D. A. (2008). Fusion systems: Group theory, representation theory, and topology.

[9] Dummit, D. S. and Foote, R. M. (2004). *Abstract algebra*, volume 3. Wiley Hoboken.

[10] Gilotti, A., Ribes, L., and Serena, L. (1999). Fusion in profinite groups. *Annali di Matematica pura ed applicata*, 177(1):349–362.

[11] Glauberman, G. (1968). A characteristic subgroup of a p-stable group. *Canadian Journal of Mathematics*, 20:1101–1135.

[12] Gorenstein, D. (2007). *Finite groups*, volume 301. American Mathematical Soc.

[13] Halbeisen, L. J. (2012). *Combinatorial set theory*. Springer.

[14] Hall, M. *The Theory of Groups,(1959).*

[15] Halmos, P. R. (1974). Relations. In *Naive Set Theory*, pages 26–29. Springer.

[16] Higgins, P. J. (1975). *An introduction to topological groups*. Cambridge Univ. Press.

[17] Huppert, B. (1967). *Endliche gruppen I*, volume 134. Springer-verlag.

[18] Kaufmann, F. (2012). *The infinite in mathematics: logico-mathematical writings*, volume 9. Springer Science & Business Media.

[19] Powell, M. B. and Higman, G. (1971). *Finite simple groups: proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute)*. Academic Press.

[20] Ribes, L. and Zalesskii, P. (2000). *Profinite groups*. Springer.

[21] Schirokauer, O. (1997). A cohomological transfer map for profinite groups. *Journal of Algebra*, 195(1):74–92.

[22] Schur, I. (1902). *Neuer Beweis eines Satzes über endliche Gruppen*.

[23] Stancu, R. and Symonds, P. (2014). Fusion systems for profinite groups. *Journal of the London Mathematical Society*, 89(2):461–481.

[24] Suzuki, M. (1978). A transfer theorem. *Journal of Algebra*, 51(2):608–618.

[25] Wilson, J. S. (1998). Profinite groups. *Oxford*, 87:90.

[26] Yoshida, T. (1978). Character-theoretic transfer. *Journal of Algebra*, 52(1):1–38.

[27] Yoshida, T. (1988). Character-theoretic transfer (ii). *Journal of Algebra*, 118(2):498–527.

[28] Zassenhaus, H. (1937). Lehrbuch der gruppentheorie. *Leipzig und Berlin*, 2.

[29] Zassenhaus, H. (1958). *The theory of groups, Chelsea Publ.*