5-2021

From Multi-Prime to Subset Labelings of Graphs

Bethel I. McGrew
Western Michigan University, bethelmcgrew@gmail.com

FROM MULTI-PRIME TO SUBSET LABELINGS OF GRAPHS

by

Bethel I. McGrew

A dissertation submitted to the Graduate College
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
Mathematics
Western Michigan University
May 2021

Doctoral Committee:

Ping Zhang, Ph.D., Chair
Gary Chartrand, Ph.D.
Patrick Bennett, Ph.D.
Ebrahim Salehi, Ph.D.

FROM MULTI-PRIME TO SUBSET LABELINGS OF GRAPHS

Bethel I. McGrew, Ph.D.

Western Michigan University, 2021

A graph labeling is an assignment of labels (elements of some set) to the vertices or edges (or both) of a graph G. If only the vertices of G are labeled, then the resulting graph is a vertex-labeled graph. If only the edges are labeled, the resulting graph is an edge-labeled graph. The concept was first introduced in the 19th century when Arthur Cayley established Cayley's Tree Formula, which proved that there are n^{n-2} distinct labeled trees of order n. Since then, it has grown into a popular research area.

In this study, we first review several types of labelings, then turn to the particular problem of multi-prime labelings, where products of distinct primes are assigned as labels that are disjoint for adjacent vertices and intersecting for non-adjacent vertices. We express the problem in the equivalent language of *subset labelings*, denoting elements in a label by their indices 1, 2, … , k. A graph's *subset index* is the smallest number of elements k from which we can assign a subset labeling f, considered as a function with domain $V(G)$ and range $\mathcal{P}^*([k])$ (i.e., the power set of $[k]$ with the empty set omitted).

It turns out that the problem of determining the subset index for graph classes such as paths and cycles is nontrivial. For paths of order n, we determine the index up to $n = 24$, and for cycles of order n, we determine it up to $n = 18$. We also describe the connection between the problem of determining the subset index of a graph and a combinatorics problem related to the so-called Erdős-Ko-Rado Theorem, namely the problem of determining the largest possible family of sets such that every set is disjoint from at most some fixed number of sets in the collection. Our work on subset labelings of cycles in particular has resulted in the correction of a significant research result in this area, reopening the problem for further research.

We also consider the problem of determining the subset index for other graph classes, including prisms and grids, for which we present upper bounds in terms of the subset indices of paths and cycles. We conclude by studying the problem for select graph unions.

ACKNOWLEDGMENTS

I would like to begin by thanking two professors who were a particular inspiration when I first sat under them as a young undergraduate at Western Michigan University and whose friendship has been a gift throughout my studies. For his gentle encouragement and spirit of wonder, and for showing me that mathematics could be beautiful, I thank Dr. Arthur White. For her relentless rigor and contagious passion in the classroom, I thank Dr. Niloufer Mackey. I can only aspire to be to my own students what these teachers were to me.

Secondly, I thank my advisor, Dr. Ping Zhang, for her dedication and countless hours spent reviewing this work. I additionally thank my first and second readers, Dr. Gary Chartrand and Dr. Patrick Bennett, for taking the time to discuss the manuscript with me, point out errors, and suggest improvements in key sections.

Lastly, I thank my parents, Drs. Timothy and Lydia McGrew, who first taught me the meaning of academic integrity and without whose steadfast support and love none of my work would be possible.

Bethel I. McGrew

TABLE OF CONTENTS

ACKNOWLEDGMENTS ii
LIST OF FIGURES v
CHAPTER
1. INTRODUCTION 1
1.1 Graceful Labelings 1
1.2 Harmonious Labelings 3
1.3 Prime Labelings 5
1.4 Graph Colorings 6
1.5 Kneser Graphs 7

2. MULTI-PRIME LABELINGS OF GRAPHS 10
2.1 Introduction 10
2.2 A Different View of Multi-Prime Labelings 13
2.3 Preliminary Results 14

3. SUBSET LABELINGS OF PATHS 21
3.1 Introduction 21
3.2 Preliminary Results on Paths 22
3.3 Paths of Order 12 through 22 24
3.4 The Path of Order 23 27

4. SUBSET LABELINGS OF CYCLES 48
4.1 Introduction 48
4.2 Preliminary Results on Cycles 50
4.3 Cycles of Order 9 through 12 52

Table of Contents—Continued

CHAPTER

4.4 Cycles of Order 13 through 18 60

4.5 Subset Labelings and Almost Intersecting Set Families 69

5. SUBSET LABELINGS OF PRISMS AND GRIDS 74

5.1 Introduction 74

5.2 Subset Labelings of Prisms 76

5.3 Subset Labelings of Grids 86

6. SUBSET LABELINGS OF GRAPH UNIONS 96

6.1 Preliminary Results on Graph Unions 96

6.2 Subset Index of kK_2 98

6.3 Subset Labelings of Unions of Cycles 99

BIBLIOGRAPHY 109

LIST OF FIGURES

1.1: A graceful graph 2

1.2: Three graphs that are not graceful 3

1.3: Harmonious labelings of K_3 and K_4 3

1.4: Harmonious labelings of P_n for $n = 6, 7, 8, 9$ and $K_{1,6}$ 4

1.5: Trees T for which $h(T) = |E(T)|$ 4

1.6: A prime graph Q_3 6

2.1: Multi-prime labelings of K_3, $C_4 = K_{2,2}$ and C_5 11

2.2: A 5-subset labeling of the Petersen graph P 13

2.3: Showing that the converse of Proposition 2.3.2 is false 16

2.4: A subset labeling of Q_3 17

2.5: Showing that the converse of the Subset Lemma is false 18

3.1: Subset labelings of P_4, P_5 and P_6 21

3.2: Subset labelings of P_n for $n = 7, 8, 9$ 24

4.1: Subset labelings of C_5, C_6 and C_7 49

5.1: A subset labeling f of $C_3 \square K_2$ 77

5.2: The labeling f in Case 0 78

5.3: The labeling f for $C_6 \square K_2$ in the proof of Theorem 5.2.2 83

5.4: A subset labeling of $P_3 \square K_2$ 87

5.5: Illustrating the subset labelings of $P_n \square K_2$ for $n = 7, 8, 9$ described in the proof of Theorem 5.3.5 91

5.6: Constructing a subset labeling of L_{n+1} as described in the proof of Theorem 5.3.7 95

Chapter 1

Introduction

A graph labeling is an assignment of labels (elements of some set) to the vertices or the edges (or both) of a graph G. If only the vertices of G are labeled, then the resulting graph is a vertex-labeled graph. In the case of edges, the resulting graph is edge-labeled. Graph labelings have been traced back to the 19th century, when the famous British mathematician Arthur Cayley proved that there are n^{n-2} distinct labeled trees of order n. This resulted in what is commonly called *Cayley's Tree Formula* (see [2]). Over the past few decades, the subject of graph labelings has grown into one of the most popular research areas of graph theory. Gallian [7] has compiled a periodically updated survey of many kinds of labelings and numerous results, obtained from well over a thousand referenced research articles. As an introduction, we will briefly review three of the most-studied graph labelings, namely graceful labelings, harmonious labelings, and prime labelings (see [2, 5, 18, 19] for more details). In particular, prime labelings are closely related to the research topic of this work. We refer to the books [2, 5] for graph theory notation and terminology not described in this work.

1.1 Graceful Labelings

In 1966, Alexander Rosa presented a talk titled "On certain valuations of the vertices of a graph" at the Theory of Graphs International Symposium in Rome. One such valuation was the so-called β-valuation. For a graph G of size m, Rosa defined this vertex labeling as an injective function

$$f : V(G) \to \{0, 1, \ldots, m\}$$

for which the induced edge labeling

$$f' : E(G) \to \{1, 2, \ldots, m\}$$

defined by $f'(uv) = |f(u) - f(v)|$ is bijective. In 1972, Golomb [9] referred to a β-valuation as a *graceful labeling* and a graph possessing a graceful labeling as a *graceful graph*. It is this terminology that has become standard. The graph G of order 5 and size 6 in Figure 1.1 is an example of a graceful graph, shown with one possible graceful labeling.

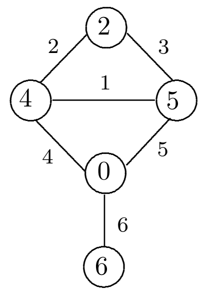

Figure 1.1: A graceful graph

The determination of which graphs are graceful is the major problem in this area. Among the results obtained on graceful graphs are the following (see [2, 7]):

- ⋆ The cycle C_n is graceful if and only if $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$.
- ⋆ The path P_n is graceful for all positive integers n
- ⋆ The complete graph K_n is graceful if and only if $n \leq 4$.
- ⋆ The graph $K_{s,t}$ is graceful for all positive integers s and t.
- ⋆ The n-cube Q_n is graceful for all positive integers n.
- ⋆ Every caterpillar is graceful.
- ⋆ Every tree of order at most 27 is graceful.

The three graphs shown in Figure 1.2 are the only connected graphs of order 5 that are not graceful. In fact, it has been shown that almost all graphs are not graceful [6].

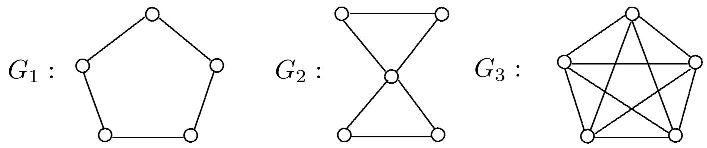

Figure 1.2: Three graphs that are not graceful

One of the best known conjectures dealing with graceful graphs involves trees and is due to Kotzig and Ringel (see [7]).

The Graceful Tree Conjecture *Every nontrivial tree is graceful.*

1.2 Harmonious Labelings

In 1980, Ronald Graham and Neil Sloane [10] introduced another graph labeling called a harmonious labeling. Let G be a connected graph of order n and size m. Then $m \geq n-1$. For $m \geq n$, a labeling ℓ of G is *harmonious* if $\ell : V(G) \to \mathbb{Z}_m$ is injective and the edge labeling $\ell' : E(G) \to \mathbb{Z}_m$ defined by $\ell'(uv) = \ell(u) + \ell(v)$ is bijective. A graph that admits a harmonious labeling is called a *harmonious graph.* For example, the complete graphs K_3 and K_4 are harmonious. A harmonious labeling for each of K_3 and K_4 is shown in Figure 1.3.

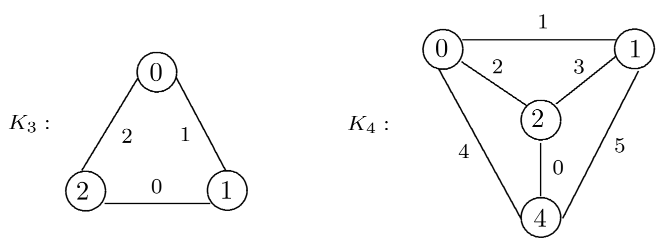

Figure 1.3: Harmonious labelings of K_3 and K_4

For $m = n-1$, the graph G is a tree and such a labeling is impossible. In this case, some element of $\mathbb{Z}_m$ is assigned to two vertices of G, while all other elements of $\mathbb{Z}_m$ are used exactly once. Here, the labeling ℓ is not injective. Harmonious labelings of P_n for $n = 6, 7, 8, 9$ and the star $K_{1,6}$ are shown in Figure 1.4.

Many classes of trees have been shown to be harmonious (see [2, 7]). For example, all paths and stars are harmonious. In fact, Graham and Sloane made the following conjecture.

The Harmonious Tree Conjecture *Every nontrivial tree is harmonious.*

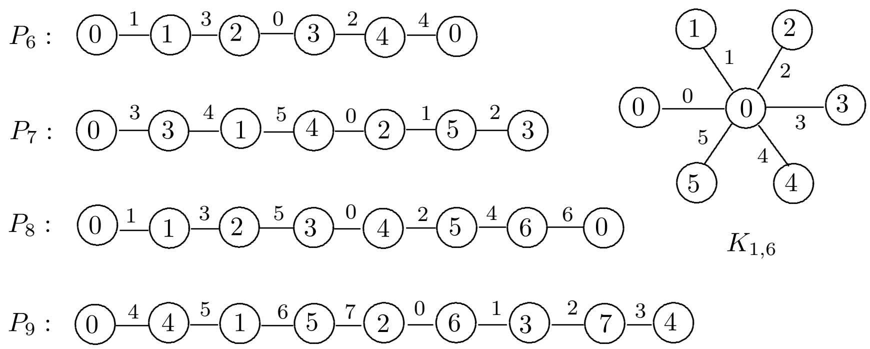

Figure 1.4: Harmonious labelings of P_n for $n = 6, 7, 8, 9$ and $K_{1,6}$

In 2018, a related harmonious concept of trees was introduced in [2]. For a nontrivial tree T of size m, the *harmonious number* $h(T)$ of T is the number of elements of $\mathbb{Z}_m$ that can be repeated in some harmonious labeling of T. Therefore, $0 \le h(T) \le m$ for every tree T of size m. If the Harmonious Tree Conjecture is true, then $1 \le h(T) \le m$. It was observed in [2] that if T is a tree of even size m and the Harmonious Tree Conjecture is true, then $h(T)$ is even and $2 \le h(T) \le m$. Figure 1.5 shows three trees $T \in \{P_3, P_4, K_{1,3}\}$ of size $m \ge 2$ such that $h(T) = m$.

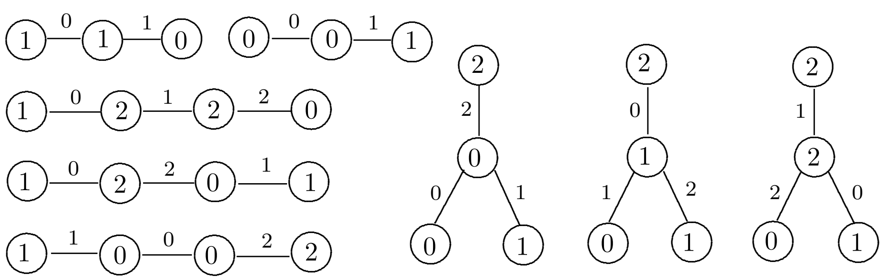

Figure 1.5: Trees T for which $h(T) = |E(T)|$

The following question appeared in [2]:

Does there exist a class of trees T for which $h(T) = 1$?

We show that in fact, no such tree of size 2 or more exists. Indeed, more can be said.

Theorem 1.2.1 *If T is a nontrivial harmonious tree of size m, then $h(T) = m$.*

Proof. Since T is harmonious, there is a harmonious labeling $\ell_0 : V(T) \to \mathbb{Z}_m$ of T such that some element $a \in \mathbb{Z}_m$ is assigned to exactly two vertices of T and all other elements of $\mathbb{Z}_m$ are used exactly once. We show that the labeling $\ell_1 : V(T) \to \mathbb{Z}_m$ of T defined by

$$\ell_1(v) = \ell_0(v) + 1 \text{ for each vertex } v \text{ of } T$$

is also a harmonious labeling of T. First, we show that ℓ_1 assigns $a+1$ to exactly two vertices of T and each of all other elements of $\mathbb{Z}_m$ to exactly one vertex of T. If $\ell_1(u) = \ell_1(v)$ in $\mathbb{Z}_m$ where $u, v \in V(T)$, then $\ell_0(u) + 1 = \ell_0(v) + 1$ in $\mathbb{Z}_m$ and so $\ell_0(u) = \ell_0(v)$ in $\mathbb{Z}_m$. Since ℓ_0 is harmonious, it follows that either

$$\text{(i) } u = v \text{ or (ii) } u \neq v \text{ and } \ell_0(u) = \ell_0(v) = a.$$

This implies that if $\ell_1(u) = \ell_1(v)$ in $\mathbb{Z}_m$, then either

$$u = v \text{ or } \ell_1(u) = \ell_1(v) = a + 1.$$

Next, we show that $\ell_1' : E(T) \to \mathbb{Z}_m$, defined by $\ell_1'(uv) = \ell_1(u) + \ell_1(v)$, is bijective. First, observe that if $e = uv \in E(T)$, then

$$\ell_1'(e) = \ell_1(u) + \ell_1(v) = (\ell_0(u) + 1) + (\ell_0(v) + 1) = \ell_0'(e) + 2.$$

If $\ell_1'(e) = \ell_1'(f)$ in $\mathbb{Z}_m$ where $e, f \in E(T)$, then $\ell_0'(e) + 2 = \ell_0'(f) + 2$ in $\mathbb{Z}_m$, and so $\ell_0'(e) = \ell_0'(f)$ in $\mathbb{Z}_m$. Since ℓ_0' is injective, $e = f$ and ℓ_1' is injective. Furthermore, $|E(T)| = |\mathbb{Z}_m| = m$, and so ℓ_1' is bijective. Therefore, ℓ_1 is a harmonious labeling of T. In ℓ_1, the element $a + 1$ is assigned to exactly two vertices of T and all other elements of $\mathbb{Z}_m$ are used exactly once. By a finite induction, it follows that for each element $x \in \mathbb{Z}_m$ there is a harmonious labeling of T in which x is repeated. Therefore, $h(T) = m$. ■

1.3 Prime Labelings

Around 1980, Roger Entringer (see [7]) introduced another graph labeling called a prime labeling. For a graph G of order n, a *prime labeling* of G is a vertex labeling of G with the distinct integers in the set $[n] = \{1, 2, \ldots, n\}$ such that the labels of every two adjacent vertices of G are relatively prime. That is, a prime labeling is a bijective function $f : V(G) \to [n]$ such that $\gcd(f(u), f(v)) = 1$ for each pair u, v of adjacent vertices of G. If there exists a prime labeling of G, then G is called a *prime graph*. For example, the graph Q_3 of the cube is a prime graph. A prime labeling of this graph is shown in Figure 1.6. While adjacent vertices must have relatively prime labels, nonadjacent vertices may have relatively prime labels as well.

Determining which graphs are prime is a major research problem. Results obtained include the following (see [2, 7]):

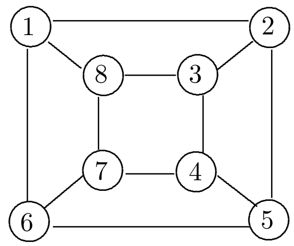

Figure 1.6: A prime graph Q_3

- ⋆ Every cycle is a prime graph.
- ⋆ For each integer $t \geq 2$, the complete bipartite graph $K_{2,t}$ is prime.
- ⋆ For each integer $r \geq 3$, the graph $K_{r,r}$ is not prime.
- ⋆ While $K_{3,4}$, $K_{3,5}$, and $K_{3,6}$ are prime, $K_{3,t}$ is not prime for $t \geq 7$.
- ⋆ All paths and stars are prime.
- ⋆ All trees of order at most 50 are prime.
- ⋆ All caterpillars with maximum degree at most 5 are prime.

In fact, Entringer (see [7, 16]) made the following conjecture in the 1980s.

The Prime Tree Conjecture *Every nontrivial tree is prime.*

In 2011, Haxell, Pikhurko, and Taraz [12] proved that all trees having a sufficiently large order are prime. The Prime Tree Conjecture remains open, however.

Over the past few decades, numerous types of graph labelings have been introduced and studied. Our work here is closely related to prime labelings of graphs.

1.4 Graph Colorings

The topic of graph colorings is an area of graph theory that is closely related to graph labelings. This section reviews some basic terms and definitions.

A *vertex k-coloring* of a graph G is a function $c : V(G) \to [k] = \{1, 2, \ldots, k\}$ where k is a positive integer. The minimum k for which a proper (vertex) k-coloring of a graph G exists is the *chromatic number* of G and is denoted by $\chi(G)$. The following are some well-known results about the chromatic number of a graph.

Proposition 1.4.1 *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

The *clique number* $\omega(G)$ of a graph G is the maximum order of a complete subgraph of G. In particular, $\omega(K_n) = n$ and $\omega(G) = 2$ for every nonempty bipartite graph G. Let $\Delta(G)$ denote the maximum degree of G.

Theorem 1.4.2 *For every graph G, $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.*

For each odd integer $n \geq 3$, the connected graphs C_n and K_n have the property that $\chi(C_n) = 3 = \Delta(C_n) + 1$ and $\chi(K_n) = n = \Delta(K_n) + 1$. Brooks [1] showed that these two classes of graphs are the only connected graphs with this property.

Theorem 1.4.3 *If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.*

A *k-edge coloring* of a graph G is a function $c : E(G) \to [k]$ where k is a positive integer. If no two adjacent edges in G are colored the same, then c is referred to as a *proper edge coloring*. The minimum positive integer k for which G has a proper k-edge coloring is its *chromatic index*, denoted by $\chi'(G)$. By the definition of the chromatic index $\chi'(G)$ of a nonempty graph G, it follows that $\chi'(G) \geq \Delta(G)$. The most famous theorem dealing with chromatic index was obtained by Vizing in [17].

Theorem 1.4.4 *For every nonempty graph G, $\chi'(G) \leq \Delta(G) + 1$.*

As a result of Vizing's theorem, the chromatic index of a nonempty graph G is one of two numbers, namely either $\Delta(G)$ or $\Delta(G) + 1$. A graph G with $\chi'(G) = \Delta(G)$ is called a *class one graph* while a graph G with $\chi'(G) = \Delta(G) + 1$ is called a *class two graph*.

1.5 Kneser Graphs

Listed below is a sequence of eight nonempty subsets of the set $[6] = \{1, 2, \ldots, 6\}$:

$$\{1,2,3\}, \{4,5\}, \{1,6\}, \{2,5\}, \{3,4,6\}, \{1,5\}, \{2,4,6\}, \{1,3,5\}$$

For simplicity, we will refer to the subsets as 123, 45, 16, and so on.

There are two properties of this sequence that are of interest. First, every two consecutive terms in the sequence are disjoint. Second, no two non-consecutive terms are disjoint.

There is nothing especially unique about these two properties as far as the set [6] is concerned. For example, if we were to add the integer 7 to any of the subsets that are not consecutive, then the resulting sequence still possesses the two properties mentioned above. We can continue doing this with the integers 8, 9, and so on. Consequently, the sequence

$$1237, 458, 1679, 25, 3468, 159, 246, 1359$$

consisting of eight nonempty subsets of the set [9] also has the properties. While we can construct such a sequence of nonempty subsets of $[n]$ for every integer $n \geq 6$, can this be accomplished if $n < 6$? We'll return to this question soon.

This sequence of eight nonempty subsets of the set [6] has the same two properties mentioned above, but the first and last subsets in the sequence are disjoint:

$$12,\ 456,\ 13,\ 256,\ 14,\ 236,\ 15,\ 346$$

As before, we can accomplish the same two properties by replacing [6] by any set $[n]$, when $n \geq 6$. But could we accomplish these properties by replacing the set [6] by some set $[n]$ where $n < 6$?

These sequences and the two properties they possess suggest a problem that can be expressed in terms of graphs. A graph G_1 can be associated with the first sequence when $V(G_1) = \{v_1, v_2, \ldots, v_8\}$ such that the vertex v_i corresponds to the ith term in the sequence and $v_i v_j \in E(G_1)$ if the ith and jth subsets are disjoint. Hence, $G_1 = P_8$. Defining G_2 from the second sequence in a similar manner results in $G_2 = C_8$. Consequently, the vertices of P_8 can be labeled with nonempty subsets of [6] such that only adjacent vertices have disjoint labels. The vertices of C_8 can be so labeled as well. For P_8 and C_8, is [6] the smallest set with these properties? This question need not be restricted to P_8 and C_8. A similar question can be asked of any graph, which suggests a labeling problem for graphs.

These questions are reminiscent of a famous class of graphs, namely the Kneser graphs. Let k and n be the two positive integers with $n > 2k$. It is possible to partition the $\binom{n}{k}$ k-element subsets of $[n]$ into $n - 2k + 2$ classes so that no pair of disjoint k-element subsets belong to the same class. The Kneser graph $KG_{n,k}$ has its vertex set consisting of the k-element subsets $[n]$, where two vertices are adjacent if these subsets are disjoint. The graph $KG_{5,2}$ is the Petersen graph. While the observation above states that $\chi(KG_{n,k}) \leq n-2k+2$, Kneser believed that $\chi(KG_{n,k}) = n - 2k + 2$. This was verified by Lovász. In the case of Kneser graphs, one begins with a collection of subsets of the same size, and from this, a graph is constructed. In our case, we begin with a graph G and assign each vertex of G a nonempty subset of a fixed set where the subsets can have any size, but any two vertices are assigned disjoint labels if and only if they are adjacent. While this can always be done, we are primarily concerned to determine the size of the smallest universal set that can accomplish this.

Chapter 2

Multi-Prime Labelings of Graphs

2.1 Introduction

As mentioned in Chapter 1, a *prime labeling* of a graph G of order n is a vertex labeling of G with distinct integers in the set $[n] = \{1, 2, \ldots, n\}$ such that the labels of every two adjacent vertices of G are relatively prime. That is, a prime labeling is a bijective function $f : V(G) \to [n]$ such that $\gcd(f(u), f(v)) = 1$ for each pair u, v of adjacent vertices of G. If there exists a prime labeling of G, then G is called a *prime graph*. Clearly, some graphs do not have prime labelings. For example, the complete graph K_4 of order 4 is not prime. Regardless of how the vertices of K_4 are labeled with distinct elements of [4], the two vertices labeled 2 and 4 cannot be be adjacent. Similarly, for each integer $n \geq 4$, the complete graph K_n is not prime. This concept has been studied extensively by many since Entringer introduced it in 1980 (see [2, 7], for example).

A related vertex labeling concept was introduced and studied in [2]. A vertex labeling f of a nontrivial connected graph G is a *multi-prime labeling* if $f : V(G) \to [2, \infty)$ has the property that $\gcd(f(u), f(v)) = 1$ if and only if $uv \in E(G)$. By the definition, we can assume that the label $f(v)$ of each vertex v of G is the product of *distinct* primes.

For example, a multi-prime labeling is shown in Figure 2.1 for each of the three graphs K_3, $C_4 = K_{2,2}$ and C_5. In general, a multi-prime labeling of the complete graph K_n of order $n \geq 3$ can be obtained by assigning distinct primes to distinct vertices, while a multi-prime labeling of a complete multipartite graph G can be obtained by assigning the same prime to two vertices of G if and only if these two vertices belong to the same partite set.

We list some of the major differences between a multi-prime labeling and a prime labeling.

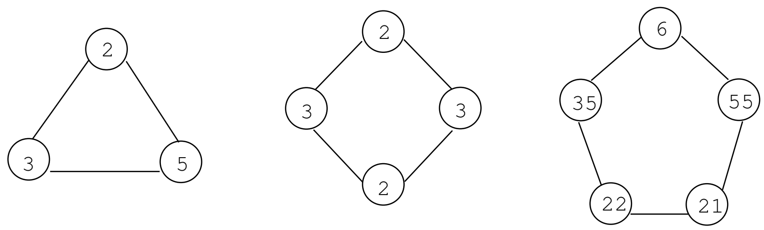

Figure 2.1: Multi-prime labelings of K_3, $C_4 = K_{2,2}$ and C_5

(1) In a prime labeling, the vertex labels come from the set $[n]$ for some positive integer n; while in a multi-prime labeling, the vertex labels come from the set $[2, \infty)$.

(2) A prime labeling is bijective; while a multi-prime labeling is not required to be injective. Thus, distinct vertices may be labeled the same in a multi-prime labeling.

(3) In a prime labeling f, it is required that $\gcd(f(u), f(v)) = 1$ for each pair u, v of adjacent vertices of G. However, it is also possible that $\gcd(f(x), f(y)) = 1$ when x and y are *nonadjacent* vertices of G. In a multi-prime labeling, adjacent vertices are the *only* pairs of vertices whose labels are relatively prime.

Consequently, in a multi-prime labeling, 1 is not the label of any vertex, and so every vertex label is divisible by at least one prime. Furthermore, if u and v are two nonadjacent vertices, then their labels are not relatively prime, thus there is at least one prime that divides both labels. There is no requirement in a multi-prime labeling of a graph that distinct vertices must be assigned distinct labels. If two vertices are labeled the same, however, then these vertices cannot be adjacent.

A nontrivial connected graph G is a *multi-prime graph* if G has a multi-prime labeling. It was shown in [2] that every nontrivial connected graph has a multi-prime labeling. For completeness, we include a proof of this result here (see [2]).

Theorem 2.1.1 *Every nontrivial connected graph has a multi-prime labeling.*

Proof. We proceed by induction on the order n of a connected graph. The result is immediate for small values of n, say $n = 2, 3, 4$. Assume that the statement is true for all connected graphs of order n for an integer $n \geq 4$. Let G be a connected graph of order $n+1$, let v be a non-cut-vertex of G, and let $G' = G - v$. Since G' is a connected graph of order n, it follows by the induction hypothesis that G' has a multi-prime labeling. Let such a labeling f' of G' be given using k primes, say $p_1, p_2, \ldots, p_k$. Suppose that $V(G') = \{u_1, u_2, \ldots, u_n\}$, where $N(v) = \{u_1, u_2, \ldots, u_r\}$, $1 \le r \le n$. Define a vertex labeling f of G by

$$f(x) = \begin{cases} p_{i+k} f'(u_i) & \text{if } x = u_i \text{ for } 1 \le i \le n \\ p_{n+k+1} \prod_{i=k+r+1}^{k+n} p_i & \text{if } x = v. \end{cases}$$

Since $xy \in E(G)$ if and only if $\gcd(f(x), f(y)) = 1$, it follows that f is a multi-prime labeling of G. ■

The following observation is therefore a consequence of Theorem 2.1.1.

Observation 2.1.2 *Every nontrivial connected graph is a multi-prime graph.*

Observation 2.1.2 suggests a parameter dealing with multi-prime labelings, which was introduced and studied in [2]. Let G be a nontrivial connected graph. For a multi-prime labeling f of G, the *multi-prime index* $\rho(f)$ of f is the number of primes p such that $p \mid f(x)$ for *at least* one vertex x of G. The *multi-prime index* $\rho(G)$ of the graph G itself is the minimum multi-prime index among all multi-prime labelings of G. The multi-prime indices of the cycles C_n, $3 \le n \le 5$, are stated next (see [2]).

Example 2.1.3 $\rho(C_3) = 3$, $\rho(C_4) = 2$, *and* $\rho(C_5) = 5$.

As an illustration, we include an argument which shows that $\rho(C_5) = 5$. In the multi-prime labeling f_0 of C_5 in Figure 2.1, each label is divisible by at least one of the five primes $2, 3, 5, 7, 11$ and so $\rho(f_0) = 5$. Hence, $\rho(C_5) \le 5$. It remains to show that $\rho(C_5) \ge 5$. That is, we show that $\rho(f) \ge 5$ for *every* multi-prime labeling f of C_5. First, we observe that no label in a multi-prime labeling of C_5 can be a single prime q, for otherwise, the labels of the two vertices not adjacent to a vertex having q as a label are both divisible by q. However, these two vertices are adjacent in C_5 and their labels are not relatively prime, which is impossible.

Thus, every label must be divisible by at least two distinct primes, and the labels of two adjacent vertices must be divisible by a total of at least four distinct primes, say $2, 3, 5, 7$. No vertex label can be divisible by three of these primes, for otherwise, the label of each neighbor of this vertex is divisible by a single prime, which we have seen is impossible. Therefore, in the cycle $C_5 = (v_1, v_2, v_3, v_4, v_5, v_1)$, each vertex label is divisible by exactly two primes, say $f(v_1) = 2 \cdot 3$ and $f(v_2) = 5 \cdot 7$. We may assume that $2 \mid f(v_3)$ and $3 \mid f(v_4)$. Also, we may assume that $5 \mid f(v_4)$ and $7 \mid f(v_5)$. However, then, $2 \mid f(v_5)$, which is impossible since v_1 and v_5 are adjacent and $2 \mid f(v_1)$. Therefore, every multi-prime labeling of C_5 requires the use of at least five primes. Thus, five primes is the smallest number of primes that can be used in a multi-prime labeling of C_5. Hence, $\rho(C_5) = 5$, as desired.

The following two observations will be useful to us (see [2]).

Observation 2.1.4 *If G is a nontrivial connected graph, then $\rho(G) \geq 2$.*

Observation 2.1.5 *If H is an induced subgraph of a graph G, then $\rho(H) \leq \rho(G)$.*

2.2 A Different View of Multi-Prime Labelings

As described in [2], multi-prime labelings of a nontrivial connected graph can be looked at in another way and can be considered as another type of vertex labeling of a graph. In order to describe this, we first introduce some additional definitions and notation.

For a positive integer r, let $\mathcal{P}([r])$ denote the power set of $[r]$, while $\mathcal{P}^*([r]) = \mathcal{P}([r]) - \{\emptyset\}$ denotes the set of nonempty subsets of $[r]$. Consequently, $|\mathcal{P}^*([r])| = 2^r - 1$. Let G be a nontrivial connected graph. For a positive integer r, a vertex labeling $f : V(G) \to \mathcal{P}^*([r])$ of G is a *subset labeling* provided $f(u) \cap f(v) = \emptyset$ if and only if $uv \in E(G)$. The minimum positive integer r for which G has such a subset labeling is the *subset index* of G. To illustrate these concepts, we consider the famous Petersen graph P of order 10. Figure 2.2 shows a 5-subset labeling of P. (For simplicity, we write the set $\{a\}$ as a, $\{a, b\}$ as ab, $\{a, b, c\}$ as abc, and so on.) In fact, the subset index of the Petersen graph P is 5, as we will soon see.

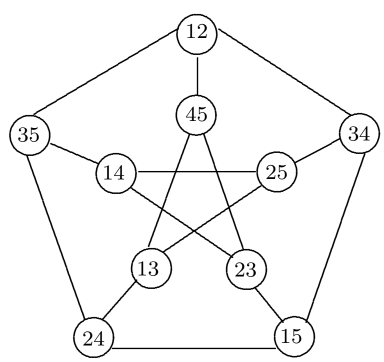

Figure 2.2: A 5-subset labeling of the Petersen graph P

We now return to multi-prime labelings of nontrivial connected graphs. Let f be a multi-prime labeling of a nontrivial connected graph G. As before, let

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ p_5 = 11,\ p_6 = 13, \ldots$$

be the sequence of all primes such that $p_i < p_{i+1}$ for each positive integer i. If u and v are two vertices of G, then u and v are assigned labels

$$f(u) = p_{i_1}p_{i_2}\cdots p_{i_k} \text{ and } f(v) = p_{j_1}p_{j_2}\cdots p_{j_\ell},$$

where $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ are distinct primes and $p_{j_1}, p_{j_2}, \ldots, p_{j_\ell}$ are distinct primes.

- ⋆ If $uv \in E(G)$, then all $k + \ell$ primes $p_{i_1}, p_{i_2}, \ldots, p_{i_k}, p_{j_1}, p_{j_2}, \ldots, p_{j_\ell}$ are distinct and so $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, j_2, \ldots, j_\ell\} = \emptyset$.
- ⋆ If $uv \notin E(G)$, then at least one of the primes $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ is the same as one of the primes $p_{j_1}, p_{j_2}, \ldots, p_{j_\ell}$, that is, $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, j_2, \ldots, j_\ell\} \neq \emptyset$.

This observation gives rise to a different view of multi-prime labelings of graphs. That is, multi-prime labelings can be considered as subset labelings. To see this, let f be a multi-prime labeling of G. Then f can be considered as a function

$$f : V(G) \to \mathcal{P}^*([r]) \text{ for some integer } r \geq 2$$

such that $f(u) \cap f(v) = \emptyset$ if and only if $uv \in E(G)$. Thus, rather than assigning the integer $f(u) = p_{i_1}p_{i_2}\cdots p_{i_k}$ to a vertex u of G, we can assign the subset $f(u) = \{i_1, i_2, \ldots, i_k\}$ of $[r]$ to u. Hence, the concepts of multi-prime labeling and subset labeling are essentially the same concept. Consequently, the subset index of G and the multi-prime index $\rho(G)$ of G are the same parameter. For this reason, we use $\rho(G)$ for the subset index of G as well.

For a positive integer k, an integer is called a *k-prime* integer if it is the product of k distinct primes, and a k-element set is called a *k-set*. Thus, in a multi-prime labeling of a graph, the label of each vertex is a k-prime integer for some positive integer k; while in a subset labeling, the label of each vertex is a k-set for some positive integer k.

To illustrate these concepts, we return to the Petersen graph P. Since P has the subset labeling shown in Figure 2.2, it follows that $\rho(P) \leq 5$. Since P contains the 5-cycle as an induced subgraph and $\rho(C_5) = 5$ by Example 2.1.3, it follows that $\rho(P) \geq 5$ by Observation 2.1.5. Hence, $\rho(P) = 5$, as we mentioned earlier.

2.3 Preliminary Results

In this section, we present several preliminary results dealing with multi-prime labelings of graphs using the terminology of subset labelings. In fact, henceforth we will only be discussing this topic in terms of subset labelings. Again, for simplicity, we write the set $\{a\}$ as a, $\{a, b\}$ as ab, $\{a, b, c\}$ as abc, and so on.

The following observation is useful in determining the subset index of a graph.

Observation 2.3.1 *If a graph G has a subset labeling $f : V(G) \to \mathcal{P}^*([k])$ for some integer $k \geq 2$, then G has a subset labeling $g : V(G) \to \mathcal{P}^*([\ell])$ for each integer $\ell \geq k$.*

According to Observation 2.3.1, establishing $\rho(G) = k \geq 3$ for a graph G requires showing that there exists a subset labeling of G using integers from the set $\mathcal{P}^*([k])$ but no subset labeling of G using integers from the set $\mathcal{P}^*([k-1])$.

For a vertex v of a graph G, let $N_G(v)$ or $N(v)$ (if the graph G is understood) denote the neighborhood of v.

Proposition 2.3.2 *Let f be a subset labeling of a nontrivial connected graph G. If $u, v \in V(G)$ such that $N(u) \neq N(v)$, then $f(u) \neq f(v)$.*

Proof. Assume, to the contrary, that there are vertices x and y in G such that $N(x) \neq N(y)$ but $f(x) = f(y)$. Since $f(x) = f(y)$, it follows that $xy \notin E(G)$. Since $N(x) \neq N(y)$, we may assume that there exists $z \in N(x) - N(y)$. Thus, $xz \in E(G)$ and $yz \notin E(G)$. This implies that $f(x) \cap f(z) = \emptyset$ and $f(y) \cap f(z) \neq \emptyset$. However, then, $\emptyset = f(x) \cap f(z) = f(y) \cap f(z) \neq \emptyset$, which is impossible. ∎

The converse of Proposition 2.3.2 is false. If G is a nontrivial connected graph with $\rho(G) = k$ and $f : V(G) \to \mathcal{P}^*([k])$ is a subset labeling of G, it is possible that there exist vertices u and v of G such that $f(u) \neq f(v)$ but $N(u) = N(v)$. For example, consider the three graphs G_1, G_2, G_3 of Figure 2.3, where a subset labeling is given for each of these graphs. Thus, $\rho(G_1) \leq 5$ and $\rho(G_i) \leq 6$ for $i = 2, 3$. In [2], it was shown that $\rho(P_6) = 5$ and $\rho(P_8) = 6$. Since P_6 is an induced subgraph of G_1, it follows that $\rho(G_1) = 5$. For $i = 2, 3$, since P_8 is an induced subgraph of G_i, it follows that $\rho(G_i) = 6$. In the subset labeling of G_i ($i = 1, 2, 3$) shown in Figure 2.3, the two solid vertices have different labels, but they have the same neighborhood.

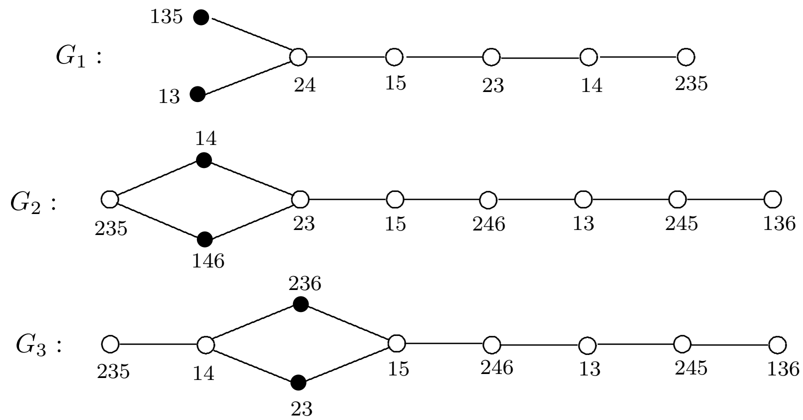

Figure 2.3: Showing that the converse of Proposition 2.3.2 is false

The *distance* $d(u, v)$ between vertices u and v in a nontrivial connected graph G is the minimum length of a $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is a $u - v$ *geodesic* in G. The *eccentricity* $e(v) = \max\{d(v, w) : w \in V(G)\}$ of a vertex v of G is the distance between v and a vertex farthest from v in G. The *diameter*

$$\text{diam}(G) = \max\{e(v) : v \in V(G)\}$$

of G is thus the largest eccentricity among the vertices of G, and the *radius*

$$\text{rad}(G) = \min\{e(v) : v \in V(G)\}$$

is the smallest eccentricity among the vertices of G. Therefore, the diameter of G is the greatest distance between any two vertices of G. A vertex v with $e(v) = \text{rad}(G)$ is called a *central vertex* of G and a vertex v with $e(v) = \text{diam}(G)$ is called a *peripheral vertex* of G. Two vertices u and v of G with $d(u, v) = \text{diam}(G)$ are *antipodal vertices* of G. Necessarily, if u and v are antipodal vertices in G, then both u and v are peripheral vertices.

Proposition 2.3.3 *Let f be a subset labeling of a nontrivial connected graph G. If $u \in V(G)$ such that $e(u) \geq 3$, then $|f(u)| \geq 2$. Consequently, if* $\text{rad}(G) \geq 3$, *then $|f(u)| \geq 2$ for every vertex u of G.*

Proof. Assume, to the contrary, that there exists $x \in V(G)$ such that $e(x) \geq 3$ but $f(x) = \{a\}$ is a singleton subset. Since $e(x) \geq 3$, there exists a vertex y in G such that $d(x, y) = 3$. Let (x, u, v, y) be an $x - y$ geodesic. Then $v, y \notin N(x)$, which implies that $f(x) \cap f(v) \neq \emptyset$ and $f(x) \cap f(y) \neq \emptyset$. Since $f(x) = \{a\}$, it follows that $a \in f(v) \cap f(y)$. But this is impossible, since $vy \in E(G)$. Furthermore, if $\text{rad}(G) \geq 3$, then $e(u) \geq 3$ for every vertex u of G, and the result follows. ■

We now illustrate these results by determining the subset index of the 3-cube Q_3.

Example 2.3.4 $\rho(Q_3) = 6$.

Proof. Let $Q_3 = C_4 \,\square\, K_2$, where $(u_1, u_2, u_3, u_4, u_1)$ and $(v_1, v_2, v_3, v_4, v_1)$ are the two 4-cycles of Q_3 and $u_iv_i \in E(Q_3)$ for $1 \leq i \leq 4$. The subset labeling $f : V(Q_3) \to \mathcal{P}^*([6])$ is defined by

$$f(u_1) = 23,\ f(u_2) = 16,\ f(u_3) = 24,\ f(u_4) = 15,$$
$$f(v_1) = 14,\ f(v_2) = 25,\ f(v_3) = 13,\ f(v_4) = 26,$$

as shown in Figure 2.4. Thus, $\rho(Q_3) \leq 6$. It remains to show that $\rho(Q_3) \geq 6$.

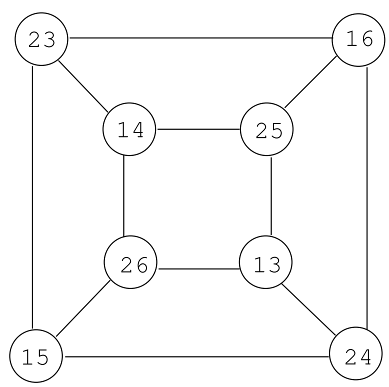

Figure 2.4: A subset labeling of Q_3

By Observation 2.3.1, it suffices to show that there is no subset labeling of Q_3 using the nonempty subsets of [5]. Assume, to the contrary, that there is a subset labeling $f : V(G) \to \mathcal{P}^*([5])$. Since $N(u) \neq N(v)$ for every pair u, v of distinct vertices of Q_3, it follows that $f(u) \neq f(v)$ by Proposition 2.3.2. Furthermore, because $e(v) = 3$ for every vertex v of Q_3, it follows by Proposition 2.3.3 that $|f(v)| \geq 2$. If $|f(x)| = 3$ for some vertex x of Q_3, say $f(x) = 123$, then $f(y) = 45$ for each of the three neighbors y of x, which is impossible. Thus, the label of every vertex of Q_3 is a 2-set label. We may assume that $f(u_1) = 12$. This implies that each of $f(u_2)$, $f(v_1)$, and $f(u_4)$ is a 2-element subset of $\{3, 4, 5\}$. We may assume that $f(u_2) = 34$ and $f(v_1) = 35$. However, then, $f(v_2) = 12 = f(u_1)$, which is a contradiction. Therefore, $\rho(Q_3) = 6$. ■

We saw for the subset labeling f of C_5 in Example 2.1.3 that $|f(v)| \geq 2$ for every vertex v of C_5 even though $\text{rad}(C_5) = 2$. Therefore, the converse of Proposition 2.3.3 is false. However, for each vertex x of C_5, there is an induced path P_4 with initial vertex x in C_5. In fact, the proof of Proposition 2.3.3 gives the following observation.

> *Let f be a subset labeling of a nontrivial connected graph G. If $u \in V(G)$ such that u is the initial vertex of an induced path of length 3, then $|f(u)| \geq 2$.*

The following observation will be used repeatedly in subsequent sections.

Proposition 2.3.5 ***The Subset Lemma.*** *Let f be a subset labeling of a nontrivial connected graph G. If $u, v \in V(G)$ such that $N(u) - N(v) \neq \emptyset$, then $f(v) \not\subseteq f(u)$.*

Proof. Assume, to the contrary, that there exist $x, y \in V(G)$ such that $N(x) - N(y) \neq \emptyset$ but $f(y) \subseteq f(x)$. Let $z \in N(x) - N(y)$. Then $xz \in E(G)$ but $yz \notin E(G)$. Since $f(y) \subseteq f(x)$ and $f(z) \cap f(y) \neq \emptyset$, it follows that $f(z) \cap f(y) \subseteq f(z) \cap f(x)$, and so $f(z) \cap f(x) \neq \emptyset$, which is impossible since $xz \in E(G)$. ■

The contrapositive of the Subset Lemma can be stated as follows.

> *Let f be a subset labeling of a nontrivial connected graph G. If $u, v \in V(G)$ such that $f(v) \subseteq f(u)$, then $N(u) \subseteq N(v)$.*

The converse of the Subset Lemma is false. For example, consider the subset labeling $f : V(P_8) \to \mathcal{P}^*([6])$ of P_8 shown in Figure 2.5. We have seen that $\rho(P_8) = 6$. For the two vertices u and v of P_8, it follows that $N(u) \subseteq N(v)$, but $f(v) = \{2, 3, 6\} \not\subseteq \{2, 3, 5\} = f(u)$.

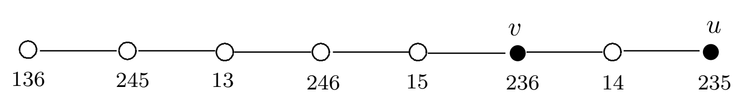

Figure 2.5: Showing that the converse of the Subset Lemma is false

The following observation is a useful consequence of the Subset Lemma.

Proposition 2.3.6 *If $\rho(G) = r$, then for any two adjacent vertices $u, v \in V(G)$ such that $N(u) - N(v) \neq \emptyset$ and $N(v) - N(u) \neq \emptyset$, it is impossible to have $|f(u) \cup f(v)| = r$.*

Proof. Suppose, to the contrary, that there exist adjacent vertices u and v with either $N(u) - N(v) \neq \emptyset$ or $N(v) - N(u) \neq \emptyset$ such that $|f(u) \cup f(v)| = r$. Say that there exists some vertex $w \in N(v) - N(u)$. If $|f(w)| = |f(u)|$, then $f(w) = f(u)$. If $|f(w)| < |f(u)|$, then $f(w) \subset f(u)$. Either way, we contradict the Subset Lemma. ■

The *clique number* $\omega(G)$ of a graph G is the maximum order of a complete subgraph of G. A graph without triangles is called *triangle-free*. The *girth* $g(G)$ of a graph G with cycles is the length of a smallest cycle in G. Consequently, if $g(G) \geq 4$, then G is triangle-free. If G is a connected graph of order at least 3 with $\omega(G) = \omega$, $g(G) = g \geq 3$, and $\text{diam}(G) = d$, then G contains an induced complete graph K_ω of order ω, an induced cycle C_g of order g, and an induced path P_d of order $d + 1$. It was shown in [2] that $\rho(K_\omega) = \omega$. Thus, as an immediate consequence of Proposition 2.1.5, we have a lower bound for the subset index of a connected graph that is related to its clique number, its girth and its diameter.

Proposition 2.3.7 *If G is a connected graph of order at least 3 having clique number $\omega(G) = \omega$, girth $g(G) = g \geq 3$, and diameter $\text{diam}(G) = d$, then*

$$\rho(G) \geq \max\{\omega(G),\ \rho(C_g),\ \rho(P_{d+1})\}.$$

There is another related distance concept involving induced paths in a connected graph that provides an improved lower bound for the subset index of the graph. For two vertices u and v in a connected graph G, the *induced detour distance* between u and v is the length of a longest induced $u - v$ path in G, which is denoted by $d^*(u, v)$. An induced $u - v$ path of length $d^*(u, v)$ is called an *induced $u - v$ detour*. The *induced detour diameter* of G, denoted by $\text{diam}^*(G)$, is the length of a longest induced path in G. This concept was introduced by Chartrand, Johns, and Tian in [4]. Thus, for every connected graph G and every two vertices u and v of G, it follows that

$$d^*(u, v) \geq d(u, v) \text{ and } \text{diam}^*(G) \geq \text{diam}(G).$$

A graph G is called a *detour graph* if $d(u, v) = d^*(u, v)$ for every two vertices u and v of G. Hence, if G is a detour graph, then $\text{diam}(G) = \text{diam}^*(G)$. For example, every tree is a detour graph. A characterization of detour graphs has been established in [4]. Again, the following lower bound for the subset index of a connected graph is an immediate consequence of Proposition 2.1.5.

Proposition 2.3.8 *If G is a connected graph having induced detour diameter d^*, then*

$$\rho(G) \geq \rho(P_{d^*+1}).$$

Although there are many connected graphs G having induced detour diameter d^* such that $\rho(G) = \rho(P_{d^*+1})$, the inequalities in Propositions 2.3.7 and 2.3.8 can be strict, which we will discuss in Chapters 3 and 4. By Propositions 2.3.7 and 2.3.8, it will be useful to study the subset indices of paths and cycles, which are the topics of Chapters 3 and 4.

Chapter 3

Subset Labelings of Paths

3.1 Introduction

In Proposition 2.3.7, we saw that if G is a connected graph of order at least 3 having girth $g(G) = g$ and diameter d, then

$$\rho(G) \geq \rho(C_g) \quad \text{and} \quad \rho(G) \geq \rho(P_{d+1}).$$

Therefore, it is of interest to study the subset indices of paths and cycles. Figure 3.1 shows subset labelings of P_n for $n = 4, 5, 6$. In fact, $\rho(P_n) = n - 1$ if $n = 4, 5, 6$.

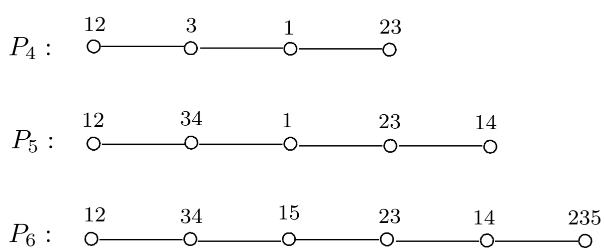

Figure 3.1: Subset labelings of P_4, P_5 and P_6

The subset indices (multi-prime indices) of paths was a topic discussed in [2], where the following result was stated.

Theorem 3.1.1 *If $n \geq 3$, then $\rho(P_n) \leq \rho(P_{n+1}) \leq \rho(P_n) + 1$. Furthermore,*

$$\lim_{n\to\infty} \rho(P_n) = \infty.$$

Consequently, for each integer $r \geq 3$, there exists an integer n_r (possibly many such integers) for which $\rho(P_{n_r}) = r$. However, the problem of determining the exact value of the subset index of a given path in general remains open. In fact, the values of $\rho(P_n)$ were only stated for $3 \leq n \leq 11$ in [2], namely:

- $\star$ if $3 \leq n \leq 6$, then $\rho(P_n) = n - 1$;
- $\star$ $\rho(P_7) = 5$;
- $\star$ if $8 \leq n \leq 11$, then $\rho(P_n) = 6$.

In particular, the subset index of P_{12} was not given. We now investigate $\rho(P_n)$ for some integers $n \geq 12$. It is convenient to introduce some additional terminology and notation before proceeding further. Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 3$ and let f be a vertex labeling of P_n. The *label sequence* of f is defined as

$$\mathcal{S}_f(P_n) = (f(v_1), f(v_2), \ldots, f(v_n)).$$

3.2 Preliminary Results on Paths

We first present several preliminary results on the subset labelings of paths in general. If u and v are two distinct vertices of the path P_n of order $n \geq 3$, then $N(u) \neq N(v)$. Hence, the following result is an immediate consequence of Proposition 2.3.2.

Proposition 3.2.1 *If f is a subset labeling of the path P_n of order $n \geq 3$, then $f(u) \neq f(v)$ for every two distinct vertices u and v of P_n.*

Consequently, if $\rho(P_n) = k$ for an integer $n \geq 3$, then k is the minimum positive integer for which there exists a sequence S of n distinct nonempty subsets of $[k]$ such that every two consecutive terms in S are disjoint and every two nonconsecutive terms in S are not disjoint.

For each positive integer n, the radius of P_n is $\text{rad}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$; therefore, if $n \geq 6$, then $e(u) \geq 3$ for every vertex u of P_n. Thus, the following result is an immediate consequence of Proposition 2.3.3.

Proposition 3.2.2 *If f is a subset labeling of P_n of order $n \geq 6$, then $|f(u)| \geq 2$ for every vertex u of P_n.*

We also have the following consequence of the Subset Lemma.

Proposition 3.2.3 *Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 4$ and let f be a subset labeling of P_n. Suppose that v_i and v_j are two vertices of P_n where $1 \leq i, j \leq n$. If $\deg v_j = 2$ or $d(v_i, v_j) \geq 3$, then $f(v_i) \not\subseteq f(v_j)$.*

Proof. Suppose that v_i and v_j are two vertices of P_n, $1 \leq i, j \leq n$, such that $\deg v_j = 2$ or $d(v_i, v_j) \geq 3$. First, suppose that $\deg v_j = 2$. Then either $v_{j-1} \in N(v_j) - N(v_i)$ or $v_{j+1} \in N(v_j) - N(v_i)$. In either case, $N(v_j) - N(v_i) \neq \emptyset$. Next, suppose that $d(v_i, v_j) \geq 3$. If $j < i$, then $v_{j+1} \in N(v_j) - N(v_i)$; while if $i < j$, then $v_{j-1} \in N(v_j) - N(v_i)$. In either case, $N(v_j) - N(v_i) \neq \emptyset$. Thus, $f(v_i) \not\subseteq f(v_j)$ by the Subset Lemma. ■

Proposition 3.2.4 *Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 4$ and let f be a subset labeling of P_n. Suppose that $v_i \in V(P_n)$ where $1 \leq i \leq n$ such that $f(v_i) = \{a, b\}$.*

- ⋆ *If $i + 2k + 1 \in [n]$ for some positive integer k, then*

$$\{f(v_{i+2k}) \cap f(v_i), f(v_{i+2k+1}) \cap f(v_i)\} = \{\{a\}, \{b\}\}.$$

Furthermore, if $f(v_{i+2}) \cap f(v_i) = \{a\}$, then

$$f(v_{i+2k}) \cap f(v_i) = \{a\} \text{ and } f(v_{i+2k+1}) \cap f(v_i) = \{b\} \text{ for each positive integer } k.$$

- ⋆ *If $i - 2k - 1 \in [n]$ for some positive integer k, then*

$$\{f(v_{i-2k}) \cap f(v_i), f(v_{i-2k-1}) \cap f(v_i)\} = \{\{a\}, \{b\}\}.$$

Furthermore, if $f(v_{i-2}) \cap f(v_i) = \{a\}$, then

$$f(v_{i-2k}) \cap f(v_i) = \{a\} \text{ and } f(v_{i-2k-1}) \cap f(v_i) = \{b\} \text{ for each positive integer } k.$$

Proof. We may assume, without loss of generality, that $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $i \neq 3$. Since $f(v_{i+2}) \cap f(v_i) \neq \emptyset$, $f(v_{i+3}) \cap f(v_i) \neq \emptyset$, and $f(v_{i+2}) \cap f(v_{i+3}) = \emptyset$, we may assume, without loss of generality, that $f(v_{i+2}) \cap f(v_i) = \{a\}$ and $f(v_{i+3}) \cap f(v_i) = \{b\}$. Similarly, $f(v_{i+2k}) \cap f(v_i) \neq \emptyset$, $f(v_{i+2k+1}) \cap f(v_i) \neq \emptyset$, and $f(v_{i+2k}) \cap f(v_{i+2k+1}) = \emptyset$. Because $f(v_{i+2}) \cap f(v_i) = \{a\}$ and $f(v_{i+3}) \cap f(v_i) = \{b\}$, it follows that $f(v_{i+2k}) \cap f(v_i) = \{a\}$ and $f(v_{i+2k+1}) \cap f(v_i) = \{b\}$ for each positive integer k when $i + 2k + 1 \in [n]$.

Using a similar argument to the above, we see that if $f(v_{i-2}) \cap f(v_i) = \{a\}$ and $f(v_{i-3}) \cap f(v_i) = \{b\}$ (if $i - 3 \in [n]$), then $f(v_{i-2k}) \cap f(v_i) = \{a\}$ and $f(v_{i-2k-1}) \cap f(v_i) = \{b\}$ for each positive integer k when $i - 2k - 1 \in [n]$. ■

Proposition 3.2.5 *Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 4$ and let f be a subset labeling of the path P_n. If v_i and v_j are vertices of P_n such that $d(v_i, v_j) \geq 5$ and $d(v_i, v_j)$ is odd, then either $|f(v_i)| \neq 2$ or $|f(v_j)| \neq 2$.*

Proof. Assume, to the contrary, that there are vertices v_i and v_j of P_n such that $d(v_i, v_j) \geq 5$ is odd but $|f(v_i)| = |f(v_j)| = 2$. We may assume, without loss of generality, that $i < j$. Since $d(v_i, v_j) = j - i \geq 5$ is odd, it follows that i and j are of opposite parity and v_i and v_j are not adjacent. Hence, $f(v_i) \cap f(v_j) \neq \emptyset$. We may assume that $f(v_i) = ab$ and $f(v_j) = ac$, where then $b \neq c$ by Proposition 3.2.1.

First, we apply Proposition 3.2.4 to $f(v_i) = ab$. Since $f(v_j) = ac$, it follows that $a \notin f(v_{j-1})$. By Proposition 3.2.4 then, $f(v_i) \cap f(v_t) = \{b\}$ for each integer $t \in \{j-1, j-3, \ldots, i+2\}$. In particular, $a \notin f(v_{i+2})$. Next, we apply Proposition 3.2.4 to $f(v_j) = ac$. Since $f(v_i) = ab$, it follows that $a \notin f(v_{i+1})$. By Proposition 3.2.4 then, $f(v_j) \cap f(v_t) = \{c\}$ for each integer $t \in \{i+1, i+3, \ldots, j-2\}$. In particular, $c \in f(v_{i+1})$, and so $c \notin f(v_{i+2})$. Consequently, $f(v_{i+2}) \cap f(v_j) = \emptyset$.

On the other hand, $d(v_i, v_j) = j - i \geq 5$, and so v_{i+2} is not adjacent to v_j. Hence, $f(v_{i+2}) \cap f(v_j) \neq \emptyset$, producing a contradiction. ∎

To illustrate these properties, Figure 3.2 shows a subset labeling of P_n for each integer $n = 7, 8, 9$.

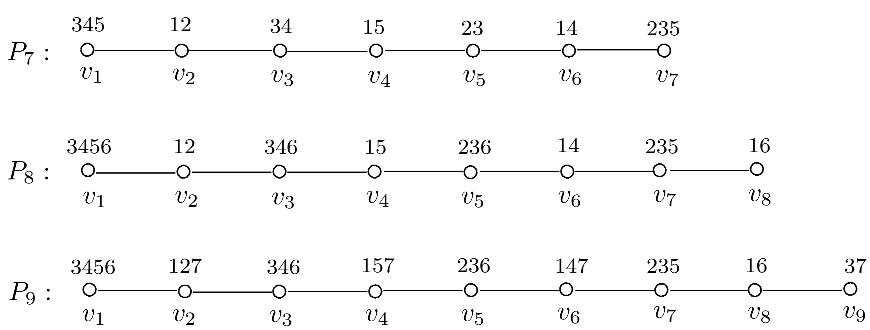

Figure 3.2: Subset labelings of P_n for $n = 7, 8, 9$

3.3 Paths of Order 12 through 22

For each integer n with $12 \leq n \leq 22$, we present a subset labeling $f : V(P_n) \to \mathcal{P}^*([7])$ by means of the label sequence of f as follows (where, as before, a set $\{a, b, c\}$ is written as abc

and $\{a,b,c,d\}$ as $abcd$):

$$\begin{aligned}
\mathcal{S}_f(P_{12}) &= (1234, 567, 123, 457, 126, 347, 156, 237, 146, 235, 167, 2345)\\
\mathcal{S}_f(P_{13}) &= (1234, 567, 123, 457, 126, 347, 156, 237, 146, 235, 167, 245, 1367)\\
\mathcal{S}_f(P_{14}) &= (1234, 567, 123, 457, 126, 347, 156, 237, 146, 235, 167, 245, 137, 2456)\\
\mathcal{S}_f(P_{15}) &= (1234, 567, 123, 457, 126, 347, 156, 237, 146, 235, 167, 245, 137, 246, 1357)\\
\mathcal{S}_f(P_{16}) &= (1234, 567, 123, 457, 126, 347, 156, 237, 146, 235, 167, 245, 137, 246, 135, 2467)\\
\mathcal{S}_f(P_{17}) &= (1234, 567, 123, 457, 126, 347, 156, 237, 146, 235, 167, 245, 137, 246, 135, 267, 1345)\\
\mathcal{S}_f(P_{18}) &= (4567, 123, 567, 124, 367, 125, 347, 126, 345, 267, 135, 247, 136, 257, 146,\\
&\quad 235, 147, 2356)\\
\mathcal{S}_f(P_{19}) &= (4567, 123, 567, 124, 367, 125, 347, 126, 345, 267, 135, 247, 136, 257, 146,\\
&\quad 235, 147, 236, 1457)\\
\mathcal{S}_f(P_{20}) &= (4567, 123, 567, 124, 367, 125, 347, 126, 345, 267, 135, 247, 136, 257, 146,\\
&\quad 235, 147, 236, 157, 2346)\\
\mathcal{S}_f(P_{21}) &= (2457, 136, 245, 367, 125, 347, 156, 234, 567, 123, 456, 127, 345, 267,\\
&\quad 135, 246, 357, 146, 235, 147, 236),\\
\mathcal{S}_f(P_{22}) &= (2457, 136, 245, 367, 125, 347, 156, 234, 567, 123, 456, 127, 345, 267,\\
&\quad 135, 246, 357, 146, 235, 147, 236, 1457)
\end{aligned}$$

In fact, for $12 \le n \le 21$, a subset labeling $f : V(P_n) \to \mathcal{P}^*([7])$ of P_n can be obtained from a subset labeling $g : V(P_{n+1}) \to \mathcal{P}^*([7])$ of P_{n+1} by deleting $g(v_{n+1})$ and replacing $g(v_n)$ by $[7] - g(v_{n-1})$. That is, $f(v_i) = g(v_i)$ for $1 \le i \le n-1$ and $f(v_n) = [7] - g(v_{n-1})$. Since each of these labelings was done with seven integers, it follows that $\rho(P_n) \le 7$ for $12 \le n \le 22$. In fact, $\rho(P_n) = 7$ for $12 \le n \le 22$, as we show next.

Theorem 3.3.1 *If $12 \le n \le 22$, then $\rho(P_n) = 7$.*

Proof. Since $\rho(P_{12}) \le \rho(P_{13}) \le \cdots \le \rho(P_{22})$ by Theorem 3.1.1, it suffices to show that $\rho(P_{12}) \ge 7$ and $\rho(P_{22}) \le 7$. We have shown that there is a subset labeling $f : V(P_{22}) \to \mathcal{P}^*([7])$ defined by

$$\begin{aligned}
\mathcal{S}_f(P_{22}) &= (2457, 136, 245, 367, 125, 347, 156, 234, 567, 123, 456, 127, 345, 267,\\
&\quad 135, 246, 357, 146, 235, 147, 236, 1457),
\end{aligned}$$

and so $\rho(P_{22}) \le 7$.

Next, we show that $\rho(P_{12}) \geq 7$. Since $\rho(P_{12}) \geq \rho(P_{11}) = 6$, we need only show that $\rho(P_{12}) \neq 6$. Assume, to the contrary, that $\rho(P_{12}) = 6$. Then there exists a subset labeling $f : V(P_{12}) \to \mathcal{P}^*([6])$ of P_{12}. By Proposition 3.2.2, no vertex of P_{12} can have a 1-set label. First, we verify the following four facts about the interior vertices of P_{12}:

Fact 1: No interior vertex of P_{12} can have a 4-set label.

Fact 2: No two adjacent interior vertices can have 3-set labels.

Fact 3: Of every two adjacent interior vertices, at least one has a 2-set label.

Fact 4: No element in [6] can belong to five 2-set labels of interior vertices.

Proof of Fact 1: Assume, to the contrary, that there is an interior vertex v_i such that $f(v_i)$ of P_{12} is a 4-set. We may assume, without loss of generality, that $2 \leq i \leq 6$ and that $f(v_i) = \{1, 2, 3, 4\}$. Since $f(v_{i-1}) \cap f(v_i) = f(v_i) \cap f(v_{i+1}) = \emptyset$, it follows that $f(v_{i-1}) \subseteq \{5, 6\}$ and $f(v_{i+1}) \subseteq \{5, 6\}$. Since, by Proposition 3.2.2, $|f(v)| \geq 2$ for every vertex v of P_{12}, it follows that $f(v_{i-1}) = \{5, 6\} = f(v_{i+1})$. By Proposition 3.2.1, this is impossible. Therefore, **Fact 1** is true.

Proof of Fact 2: Assume, to the contrary, that there are adjacent interior vertices v_i and v_{i+1} such that $f(v_i)$ and $f(v_{i+1})$ are both 3-set labels. Then $|f(v_i) \cup f(v_{i+1})| = 6$. We may assume, without loss of generality, that $2 \leq i \leq 6$. Then, since $N(v_i) - N(v_{i+1}) \neq \emptyset$ and $N(v_{i+1}) - N(v_i) \neq \emptyset$, by Proposition 2.3.6 we cannot have $|f(v_i) \cup f(v_{i+1})| = 6$. This is a contradiction. Therefore, **Fact 2** is true.

Fact 3 is then a consequence of **Facts 1** and **2**.

Proof of Fact 4: Assume, to the contrary, there is an element in [6], say 1, that belongs to five 2-set labels of interior vertices of P_{12}. Since P_{12} has ten interior vertices, it follows by **Fact 3** that at least five of these ten interior vertices of P_{12} have 2-set labels. Of course, no element of [6] can belong to two adjacent vertices of P_{12}. Thus, we may assume, without loss of generality, that 1 belongs to the labels of five interior vertices of P_{12} no two of which are adjacent. Since no two vertices have identical labels by Proposition 3.2.1, and since at least one of every two adjacent interior vertices has a 2-set label, we may assume that $f(v_3) = 12$, $f(v_5) = 13$, $f(v_7) = 14$, $f(v_9) = 15$, $f(v_{11}) = 16$. Then, since $f(v_2) \cap f(v_j) \neq \emptyset$ for $j \in \{5, 7, 9, 11\}$ but $\{1, 2\} \not\subseteq f(v_2)$, it follows that $f(v_2) = 3456$, which contradicts Fact 1. Therefore, **Fact 4** is true.

By **Fact 3**, at least one of every two adjacent interior vertices has a 2-set label. Thus, we may assume that (1) one of $f(v_2)$ and $f(v_{11})$ is a 2-set label or (2) one of $f(v_3)$ and $f(v_{10})$ is a 2-set label. We consider these two cases.

Case 1. *Either $f(v_2)$ or $f(v_{11})$ is a 2-set label, say $f(v_2) = 12$ is a 2-set label.* Since $d(v_2, v_{11}) = 9$ is odd, it follows by Proposition 3.2.5 that $f(v_{11})$ is a 3-set label. We may assume that $1 \in f(v_i)$ for each $i \in \{4, 6, 8, 10, 12\}$ and $2 \in f(v_i)$ for each $i \in \{5, 7, 9, 11\}$. Since $d(v_2, v_7) = 5$, $d(v_2, v_9) = 7$, and $d(v_2, v_{11}) = 9$ are odd and $|f(v_2)| = 2$, it follows by Proposition 3.2.5 that $|f(v_7)| = |f(v_9)| = |f(v_{11})| = 3$. It then follows by **Fact 3** that $|f(v_6)| = |f(v_8)| = |f(v_{10})| = 2$. Since $d(v_3, v_{10}) = 7$ and $d(v_5, v_{10}) = 5$ are odd, it follows by Proposition 3.2.5 that $|f(v_3)| = |f(v_5)| = 3$. This implies that $|f(v_4)| = 2$. Thus, 1 belongs to five 2-set labels of interior vertices of P_{12}, namely $1 \in f(v_i)$ for $i = 2, 4, 6, 8, 10$, which is impossible by **Fact 4**.

Case 2. *$f(v_3)$ or $f(v_{10})$ is a 2-set label, say $f(v_3) = 12$ is a 2-set label.* Since $d(v_3, v_{10}) = 7$ is odd, it follows by Proposition 3.2.5 that $f(v_{10})$ is a 3-set label, which implies that $f(v_{11})$ is a 2-set label. Then the argument in Case 1 (replacing v_2 by v_{11}) produces a contradiction.

Hence, there is no subset labeling $f : V(P_{12}) \to \mathcal{P}^*([6])$ of P_{12}, and so we conclude that $\rho(P_{12}) \geq 7$. Therefore, $\rho(P_{12}) = 7$. ■

There are many subset labelings $f : V(P_{20}) \to \mathcal{P}^*([7])$ in which $|f(v)| = 3$ for every vertex v of P_{20}. Three of these labelings are listed below.

$$\begin{aligned}
\mathcal{S}_f(P_{20}) &= (136, 245, 367, 125, 347, 156, 234, 567, 123, 456, 127, 345, 267,\\
&\quad 135, 246, 357, 146, 235, 147, 236)\\
\mathcal{S}_f(P_{20}) &= (247, 136, 257, 146, 235, 147, 236, 157, 246, 137, 456, 123, 567,\\
&\quad 124, 367, 125, 347, 126, 345, 167)\\
\mathcal{S}_f(P_{20}) &= (237, 456, 123, 567, 124, 367, 125, 347, 126, 345, 167, 235, 147,\\
&\quad 236, 157, 246, 135, 247, 136, 257).
\end{aligned}$$

3.4 The Path of Order 23

We saw in Theorem 3.3.1 that $\rho(P_n) = 7$ for $12 \leq n \leq 22$. The path P_{22} is the path of largest order whose subset index is 7. In this section, we show that $\rho(P_{23}) = 8$. To establish this fact, we first present several preliminary results.

Proposition 3.4.1 *Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n \geq 4$ and let f be a subset labeling of P_n. Suppose that v_i and v_j are two distinct vertices of P_n such that $|f(v_i)| = |f(v_j)| = 2$ and i and j are of the same parity.*

- ⋆ *If i and j are both odd, then $f(v_i) \cap f(v_j) \subseteq f(v_p)$ for each odd integer p.*
- ⋆ *If i and j are both even, then $f(v_i) \cap f(v_j) \subseteq f(v_p)$ for each even integer p.*

Proof. Since $d(v_i, v_j) = |i-j| \geq 2$, it follows that v_i and v_j are not adjacent in P_n. Thus, $f(v_i) \cap f(v_j) \neq \emptyset$. Assume, without loss of generality, that $i < j$ and $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Let $f(v_i) = ab$ and $f(v_j) = ac$. Then $b \neq c$ by Proposition 3.2.1 and so $f(v_i) \cap f(v_j) = \{a\}$. Since i and j are of the same parity, it follows that $j = i + 2t$ for some positive integer t. Because $a \in f(v_i)$ and $a \in f(v_{i+2t})$, the argument used in the proof of Proposition 3.2.4 shows that $a \in f(v_{i+2k})$ for each integer k for which $i + 2k \in [n]$. Since i and $i + 2k$ are of the same parity, the result follows. ■

To simplify the terminology in the proofs, we refer to the subscript of a vertex as the *index* of the vertex. We say that a vertex v_i is *even-indexed* if its subscript i is even and is *odd-indexed* otherwise.

Proposition 3.4.2 *There is no subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23} such that three or more vertices of P_{23} have 2-set labels.*

Proof. Assume, to the contrary, that there exists a subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23} such that at least three vertices of P_{23} have a 2-set label. Let v_i, v_j, v_k be the first three vertices of P_{23} such that $|f(v_i)| = |f(v_j)| = |f(v_k)| = 2$ (that is, i, j, k are the smallest distinct subscripts such that $|f(v_i)| = |f(v_j)| = |f(v_k)| = 2$). We may assume, without loss of generality, that $1 \leq i < j < k \leq 23$ and $j \leq 12$. We consider three cases, according to the locations of the vertices v_i, v_j, and v_k on P_{23}.

Case 1. *The three vertices v_i, v_j, v_k are consecutive on P_{23}.* Thus, $\{i, j, k\} = \{i, i+1, i+2\}$. We may assume, without loss of generality, that $f(v_i) = 12, f(v_{i+1}) = 45$, and $f(v_{i+2}) = 13$. Here, i can either be odd or even. We will only consider the situation when i is odd, since the argument is similar when i is even. By Proposition 3.4.1, it follows that $f(v_i) \cap f(v_\ell) = \{1\}$ for all odd integers ℓ and $\{2, 3\} \subset f(v_\ell)$ for all even integers ℓ, $\ell \neq i-1, i+3$. By Proposition 3.2.4, we may further assume, without loss of generality, that $f(v_{i+1}) \cap f(v_\ell) = \{4\}$ for all even integers $\ell \geq i+3$. Thus, $\{2, 3, 4\} \subseteq f(v_\ell)$ for all even integers $\ell \geq i+3$. In particular, since $j = i+1 \leq 12$, it follows that $\{2, 3, 4\} \subseteq f(v_\ell)$

for each $\ell \in \{14, 16, 18, 20, 22\}$. However, there are only three distinct labels (namely $\{2,3,4\}, \{2,3,4,6\}, \{2,3,4,7\}$) that are available for these vertices, which is a contradiction by Proposition 3.2.1.

Case 2. *Exactly two of v_i, v_j, v_k are consecutive on P_{23}.* We may assume, without loss of generality, that v_i and v_j are adjacent while v_j and v_k are not adjacent. Thus, $\{i, j, k\} = \{i, i+1, k\}$, where $k \geq i+3$. We may further assume that $f(v_i) = 12, f(v_{i+1}) = 34$, and $f(v_k) = 13$. Here, i can be odd or even. We will only consider the situation when i is odd, since the argument is similar when i is even. Since either $d(v_i, v_k)$ is odd or $d(v_{i+1}, v_k)$ is odd, it follows by Proposition 3.2.5 that $k = i+3$ or $k = i+4$. We consider these two subcases.

Subcase 2.1. $k = i+3$. In this subcase, $i+1$ and $i+3$ are both even. By Proposition 3.4.1, this forces $f(v_{i+1}) \cap f(v_\ell) = \{3\}$ for all v_ℓ, ℓ even and $\{1,4\} \subseteq f(v_\ell)$ for all v_ℓ, ℓ odd such that $v_\ell \notin N(v_{i+1}) \cup N(v_{i+3})$. Furthermore, we observe by Proposition 3.2.4 that for each even integer $\ell \in [i+5, 22]$, it follows that $f(v_i) \cap f(v_\ell) = \{1\}$, and for each odd integer $\ell \in [i+6, 21]$, $f(v_i) \cap f(v_\ell) = \{2\}$. So for all odd integers $\ell \in [i+6, 21]$, $\{2,4\} \subseteq f(v_\ell)$, and for all even integers $\ell \in [i+5, 22]$, $\{1,3\} \subseteq f(v_\ell)$. But, by Propositions 3.2.1 and 3.2.3, this is impossible since $f(v_k) = 13$ and there is at least one interior vertex $v_\ell \notin N(v_k)$. Indeed, since $j \leq 12$, we have that v_{k+2} is an interior even-indexed vertex. Hence, we have $f(v_k) \subseteq f(v_{k+2})$, which is impossible.

Subcase 2.2. $k = i+4$. In this subcase, i and $i+4$ are both odd. Again, by Proposition 3.4.1, this forces $f(v_i) \cap f(v_\ell) = 1$ for all odd integers ℓ and $\{2,3\} \subseteq f(v_\ell)$, while $1 \notin f(v_\ell)$ for all $v_\ell \notin N(v_i) \cup N(v_{i+4})$ such that ℓ is even. On the other hand, we observe by Proposition 3.2.4 that for odd integers $\ell \in [i+6, 22]$, $f(v_{i+1}) \cap f(v_\ell) = \{3\}$. So $3 \in f(v_\ell)$ for all odd $\ell \in [i+6, 22]$. But this is impossible, since we just showed that $3 \in f(v_\ell)$ for all even $\ell \in [i+6, 22]$, which is a contradiction.

Case 3. *None of v_i, v_j, v_k are consecutive on P_{23}.* We consider two subcases.

Subcase 3.1. *All three vertex indices have the same parity.* By Proposition 3.4.1, all of $f(v_i), f(v_j), f(v_k)$ must share the same element. Without loss of generality, let i, j, k be odd, and let $f(v_i) = 12, f(v_j) = 13$, and $f(v_k) = 14$. Then all even-indexed vertices not in $N(v_i) \cup N(v_j) \cup N(v_k)$ must include the subset $\{2,3,4\}$ and exclude the element $\{1\}$. Since $|N(v_i) \cup N(v_j) \cup N(v_k)| \leq 6$, there remain at least five even-indexed vertices to label. No vertex label can have cardinality at least 5, and only three 4-sets and a triple satisfy this condition, namely 2345, 2346, 2347, and 234. By Proposition 3.2.1, no label can be duplicated. Thus, there are more even-indexed vertices than possible labels, which is a contradiction.

Subcase 3.2. *Two vertex indices are even and one vertex index is odd or two vertex indices are odd and one vertex index is even.* Without loss of generality, we may assume that i and k are both odd and j is even. By Proposition 3.2.5, we cannot have $d(v_i, v_j) \geq 5$ or $d(v_j, v_k) \geq 5$. Hence, we must have $d(v_i, v_j) = 3 = d(v_j, v_k)$, so that $j = i + 3$ and $k = i + 6$. Since $i + 3$ and $i + 6$ are of opposite parity, it follows by Proposition 3.2.4 that if $f(v_i) = ab$ and $f(v_i) \cap f(v_{i+3}) = \{a\}$, then we must have $f(v_i) \cap f(v_{i+6}) = \{b\}$. Thus, it's impossible that all three vertex labels should share the same element. So if $f(v_{i+3}) = ac$, then we must have $f(v_{i+6}) = bc$. Without loss of generality, let $f(v_i) = 12, f(v_{i+3}) = 13$, and $f(v_{i+6}) = 23$.

Since we assumed that $j \leq 12$, it follows that $i + 8 \leq j + 8 \leq 22$, and so v_{i+8} is an interior vertex. By Proposition 3.2.3, we cannot have $f(v_{i+6}) \subseteq f(v_{i+8})$. Since $f(v_{i+3}) \cap f(v_{i+6}) = \{3\}$, it follows that $1 \in f(v_{i+7})$. Therefore, since $f(v_i) \cap f(v_{i+8}) \neq \emptyset$ and $f(v_{i+3}) \cap f(v_{i+8}) \neq \emptyset$, it follows that $\{2, 3\} \subseteq f(v_{i+8})$. However, then $f(v_{i+6}) \subseteq f(v_{i+8})$, which, as we just noted, is impossible.

By Cases 1–3, the proof is complete. ■

Proposition 3.4.3 *There is no subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23} such that for some integer i with $3 \leq i \leq 21$, the vertex v_i has a 4-set label.*

Proof. Assume, to the contrary, that $f(v_i) = 4567$ for some integer i with $3 \leq i \leq 21$. We may further assume that i is odd since the argument is similar when i is even. Because the complement of 4567 is 123, it follows by Propositions 3.2.1 and 3.2.3 that $|f(v_{i-1})| = 2 = |f(v_{i+1})|$. Without loss of generality, suppose that $f(v_{i-1}) = 12$ and $f(v_{i+1}) = 13$. By Proposition 3.2.4, all labels of even-indexed vertices contain the element 1, while all labels of odd-indexed vertices that are not adjacent to v_{i-1} or v_{i+1} include the subset $\{2, 3\}$ and exclude the element 1. By Proposition 3.4.2, we must have $|f(v_j)| \geq 3$ for all integers j with $j \neq i - 1, i + 1$. Furthermore, no other vertex can have a 4-set label, for otherwise, this would result in at least three 2-set labels. Hence, we conclude that $|f(v_j)| = 3$ for all integers j with $j \neq i - 1, i, i + 1$. However, there are only four triples containing the subset $\{2, 3\}$ (namely 234, 235, 236, 237), but there are at least eight odd-indexed vertices whose vertex labels contain {2, 3}, which is a contradiction. ■

Proposition 3.4.4 *There is no subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23} such that exactly two vertices of P_{23} have 2-set labels.*

Proof. Suppose, to the contrary, that there exist exactly two vertices v_i and v_j such that $|f(v_i)| = |f(v_j)| = 2$. We may assume, without loss of generality, that $1 \le i < j \le 23$ and $i \le 11$. Thus, $|f(v_k)| \ge 3$ for each integer $k \in [23] - \{i, j\}$. By Proposition 3.4.3, at most two interior vertices, namely v_2 and v_{22}, can have 4-set labels. Hence, at least seventeen interior vertices have 3-set labels. We consider two cases, depending on the adjacency of v_i and v_j.

Case 1. *The two vertices v_i and v_j are adjacent.* Thus, $j = i + 1$. Suppose, without loss of generality, that $f(v_i) = 12$ and $f(v_{i+1}) = 34$. By Proposition 3.2.3, we may further assume that if $k \ne i-1, i+2$, then $f(v_k)$ must include exactly one element from $f(v_i)$ and exactly one element from $f(v_{i+1})$. There are twelve total triples satisfying these conditions, namely

$$\{135, 136, 137, 235, 236, 237, 145, 146, 147, 245, 246, 247\}.$$

However, by the reasoning above, at least seventeen interior vertices of P_{23} must be labeled with triples, which is a contradiction.

Case 2. *The two vertices v_i and v_j are not adjacent.* Suppose, without loss of generality, that $f(v_i) = 12$ and $f(v_j) = 13$. By Proposition 3.2.5, either $d(v_i, v_j) = 3$, or $d(v_i, v_j) \ge 2$ is even. We consider these two subcases. We consider only the case when i is odd, since the proof is similar when i is even.

Subcase 2.1. *$d(v_i, v_j) = 3$ and so $j = i+3$.* By Proposition 3.2.4, if $k \in [j+2, 22]$ and k is even, then $f(v_i) \cap f(v_k) = \{1\}$; while if $k \in [j+3, 21]$ and k is odd, then $f(v_i) \cap f(v_k) = \{2\}$. This implies that $f(v_{i+3}) \cap f(v_k) = \{3\}$ for all odd integers $k \in [j+3, 21]$. Thus, $\{2, 3\} \subseteq f(v_k)$ for all odd integers $k \in [j+3, 21]$ but $1 \notin f(v_k)$ for all odd integers $k \in [j+3, 21]$. Similarly, if $k \in [3, i-2]$ is odd, then $f(v_{i+3}) \cap f(v_k) = \{1\}$; while if $k \in [2, i-1]$ is even, then $f(v_{i+3}) \cap f(v_k) = \{3\}$. This implies that $f(v_i) \cap f(v_k) = \{2\}$ for all even integers $k \in [2, i-3]$. Thus, $\{2, 3\} \subseteq f(v_k)$ for all even integers $k \in [2, i-3]$ but $1 \notin f(v_k)$ for all even integers $k \in [2, i-3]$. Thus, there are at least five integers $k \in [2, i-3] \cup [j+3, 21]$ such that $|f(v_k)| = 3$ and $\{2, 3\} \subseteq f(v_k)$. On the other hand, there are only four triples containing the subset $\{2, 3\}$, namely $234, 235, 236, 237$. This is a contradiction.

Subcase 2.2. *$d(v_i, v_j)$ is even.* Since i and j are odd in this case, it follows by Proposition 3.4.1 that $1 \in f(v_k)$ for all odd integers k and $\{2, 3\} \subseteq f(v_k)$ for all even integers k such that $v_k \notin N(v_i) \cup N(v_j)$. Then there are at least fifteen interior vertices $v_k \notin N(v_i) \cup N(v_j)$ to label in the case that v_i and v_j have disjoint neighborhoods contained in the interior of P_{23}. By Proposition 3.4.3, at least thirteen of these vertices must be labeled with triples. Since we have shown that $|f(v_i)| < 4$ for $i \notin \{2, 22\}$, it follows that we must label at least eight odd-indexed vertices and five even-indexed vertices with triples. But there are only four triples containing $\{2, 3\}$ without the element 1, namely 234, 235, 236, and 237. Thus, the five vertices with even index whose labels must be triples containing $\{2, 3\}$ but not 1 cannot all be labeled with triples. This is a contradiction. (Note that had we chosen i and j to be even integers instead, we would have needed to label at least seven even-indexed and six odd-indexed interior vertices with triples.)

By Cases 1 and 2, we conclude that $|f(v_i)| = 2$ for at most one vertex in a subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23}. ∎

Proposition 3.4.5 *There is no subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23} such that v_2 or v_{22} has a 4-set label.*

Proof. Assume, to the contrary, that $f(v_2) = 4567$ say. Since the complement of 4567 is the triple 123, and since duplicate labels are disallowed, this forces $|f(v_3)| = 2$ and either $f(v_3) = 12$ or $f(v_3) = 13$. We now claim that $f(v_1) = 123$ is forced. If this were not the case, then either

(1) $|f(v_1)| = 2 = |f(v_3)|$, which is impossible by Proposition 3.4.4, or

(2) $f(v_3) = 123$ and $f(v_1) \subset f(v_3)$, which is impossible by Proposition 3.2.3.

From this, it follows that $|f(v_{22})| < 4$, for otherwise, it would force $|f(v_{21})| = 2$. Thus, $|f(v_k)| = 3$ for all $4 \le k \le 22$. Without loss of generality, let $f(v_4) = 345$. Suppose further, without loss of generality, that $1 \in f(v_k)$ for all odd integers $k \in [5, 21]$ and $2 \in f(v_k)$ for all even integers $k \in [6, 22]$. This then forces $f(v_5) = 167$. For all remaining odd integers $k \in [7, 21]$, there are $\binom{5}{2} - 1 = 9$ choices of triples for $f(v_k)$, namely

$$\{134, 135, 136, 137, 145, 146, 147, 156, 157\}.$$

Since the triple 267 was eliminated by choosing $f(v_4) = 345$ and $f(v_5) = 167$, it follows for even integers $k \in [6, 22]$ that there are also $\binom{5}{2} - 1 = 9$ choices of triples for $f(v_k)$, namely

$$\{234, 235, 236, 237, 245, 246, 247, 256, 257\}.$$

In particular, $f(v_6) \in \{234, 235, 245\}$. Whichever triple is chosen, the other two will be eliminated from consideration for the labels of $f(v_k)$ where $k \in [8, 22]$. Then there are sixteen remaining interior vertices to label, including eight even-indexed vertices. But the set $\{267, 234, 235, 245\}$ of triples has been eliminated, leaving only six choices. This is a contradiction.

This completes the proof that $|f(v_i)| \neq 4$ for $i = 2, 22$ in a subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23}. ■

We are now prepared to present the following result.

Theorem 3.4.6 *If $f : V(P_{23}) \to \mathcal{P}^*([7])$ is a subset labeling of P_{23}, then $|f(v)| \neq 2$ for every vertex v of P_{23}.*

Proof. Assume, to the contrary, that there exists a subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ such that $|f(v_i)| = 2$ for some vertex v_i of P_{23}. Thus, v_i is the only vertex of P_{23} having a 2-set label by Propositions 3.4.2 and 3.4.4. We may assume that $f(v_i) = 12$ is a 2-set label for exactly one integer i with $1 \leq i \leq 12$. Observe that there are precisely 20 triples from $\mathcal{P}^*([7])$ containing exactly one of 1 and 2, namely

$$\{134,\ 135,\ 136,\ 137,\ 145,\ 146,\ 147,\ 156,\ 157,\ 167,\ 234,\ 235,\ 236,\ 237,\ 245,\ 246,\ 247,\ 256,\ 257,\ 267\}.$$

We consider three cases, according to whether $i = 1$, $i = 2$, or $3 \leq i \leq 12$.

Case 1. $i = 1$. Without loss of generality, let $f(v_1) = 12$. Then by Proposition 3.2.4, for $i = 3, 4, \ldots, 23$, we may assume that $f(v_1) \cap f(v_j) = \{1\}$ for all odd integers $j \geq 3$, while $f(v_1) \cap f(v_j) = \{2\}$ for all even integers $j \geq 4$. Now, as we have shown in Propositions 3.4.3, 3.4.4, and 3.4.5, it must be the case that $|f(v_j)| = 3$ for $j \in [3, 22]$. There are precisely ten triples from [7] containing 1 and excluding 2 and vice versa, all of which must be used for the twenty interior vertices. Without loss of generality, assume that $f(v_3) = 134$. Then $f(v_5) \in \{135, 136, 137, 145, 146, 147\}$. The choice of $f(v_5)$ will then force the choice of $f(v_4)$, which will eliminate a third additional unused triple consisting of elements from $f(v_3) \cup f(v_5)$. Thus, sixteen possible label choices will remain to label seventeen interior vertices. This is a contradiction.

Case 2. $i = 2$. Without loss of generality, let $f(v_2) = 12$. Again, by Proposition 3.2.4, assume that $f(v_2) \cap f(v_j) = \{1\}$ for all odd integers $j \notin N(v_i)$, while $f(v_2) \cap f(v_j) = \{2\}$ for all even integers j. Then we must label the nineteen vertices $v_4, v_5, \ldots, v_{22}$

with triples. Similarly to Case 1, we may label $f(v_4) = 134$ and note that $f(v_6) \in \{135, 136, 137, 145, 146, 147\}$. Again, we note that any of these choices will force $f(v_5)$, which in turn will eliminate yet a third unused triple of elements containing 1 from $f(v_4) \cup f(v_6)$. Furthermore, $f(v_6)$ will eliminate all triples of elements in its complement from the choices for $f(v_k), k > 8$, including the other two triples whose intersection with $f(v_2)$ equals 2. Then exactly three triples whose intersection with $f(v_2)$ equals 1 and three triples whose intersection with $f(v_2)$ equals 2 have been eliminated, or six triples in total. Thus, only fourteen triples remain to label the remaining fifteen interior vertices. This is a contradiction.

Case 3. $i \in [3, 12]$. Recall that there are two lists of triples whose intersection with $f(v_i)$ equals 1 or 2, respectively, namely

$$\{134,\ 135,\ 136,\ 137,\ 145,\ 146,\ 147,\ 156,\ 157,\ 167\}$$

and

$$\{234,\ 235,\ 236,\ 237,\ 245,\ 246,\ 247,\ 256,\ 257,\ 267\}.$$

If i is even, then it remains to label ten even-indexed interior vertices with all ten triples from one list and eight odd-indexed interior vertices with eight triples from the other list. If i is odd, it remains to label nine even-indexed interior vertices and nine odd-indexed interior vertices with nine triples from each of the two lists above. Either way, one of 1 or 2 is in $f(v_j)$ for at least nine interior vertices v_j. Since the proof is similar if i is even, we will assume that i is odd and that $f(v_i) \cap f(v_j) = \{1\}$ for all odd-indexed vertices v_j. Now, notice for any two odd-indexed vertex labels that share a pair that there is exactly one triple from the list of ten containing the element 2 that may be drawn from the complements for both labels. Indeed, given a set $\{a_1, a_2, \ldots, a_7\}$, if the elements of $f(v_j)$ are $\{a_1, a_2, a_3\}$ and the elements of $f(v_k)$ are $\{a_1, a_2, a_4\}$, then the complements are $\{a_4, a_5, a_6, a_7\}$ and $\{a_3, a_5, a_6, a_7\}$, respectively. There are four possible triples of elements from each complementary set. Exactly one triple is shared, namely $\{a_5, a_6, a_7\}$. For example, if $i \in [5, 19]$ and $f(v_{i-2}) = 137$ while $f(v_{i+2}) = 157$, then the complement of 137 is 2456 and includes triples 245, 246, and 256. The complement of 157 is 2346 and includes the triples 234, 236 and 246. Precisely one triple, namely 246, is shared.

Notice further that for any two odd-indexed vertex labels that share only one element, no triples from the list of ten containing 2 are shared between their complements. Again, given the set $\{a_1, a_2, \ldots, a_7\}$, if the elements of $f(v_j)$ are $\{a_1, a_2, a_3\}$ and the elements of $f(v_k)$ are $\{a_1, a_4, a_5\}$, then the complements are $\{a_4, a_5, a_6, a_7\}$ and $\{a_2, a_3, a_6, a_7\}$, respectively. Each complementary set yields a list of four possible triples. No triples are shared. For example, if $f(v_{i+2}) = 146$ rather than $f(v_{i+2}) = 157$, then its complement includes the triples 257, 235, and 237, which has no overlap with the list of triples 245, 246, and 256 from the complement of $f(v_{i-2}) = 137$.

Since more triples are thus eliminated from consideration by two vertices v_j and v_k such that $|f(v_j) \cap f(v_k)| = 1$ than by two vertices such that $|f(v_j) \cap f(v_k)| = 2$, it suffices to obtain a contradiction in the case that every two odd-indexed vertex labels share a pair of elements. Without loss of generality, consider two vertices v_j and v_k where j, k are odd integers in [5, 19]. We may assume that $\{f(v_j), f(v_k)\} = \{137, 157\}$. Then, whichever triple is selected for $f(v_{j-1})$ and $f(v_{k+1})$ respectively, the other two triples in $\{245, 246, 256\}$ and $\{234, 236, 246\}$ will be eliminated. Thus, a total of at most thirteen triples from our original list of twenty remain to label fourteen interior vertices, which is a contradiction.

By Cases 1–3, we conclude that $|f(v_i)| > 2$ for all integers i in any subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23}. ∎

Theorem 3.4.7 *There is no subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23} such that at least one of the two end-vertices of P_{23} has a 4-set label and all interior vertices of P_{23} have 3-set labels.*

Proof. Let $P_{23} = (v_1, v_2, \ldots, v_{23})$. Assume, to the contrary, that there exists a subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23} such that one of the end-vertices of P_{23} has a 4-set label and every interior vertex has a 3-set label. We may assume that $|f(v_1)| = 4$ and $|f(v_i)| = 3$ for $2 \le i \le 22$. Since there are 35 3-element subsets of [7] and P_{23} has 21 interior vertices, it follows that exactly 21 of these 3-element subsets are used by f for the interior vertices of P_{23} and exactly 14 of them are not used by f for the interior vertices of P_{23}. To simplify the notation, we once again refer to a 3-set $\{a, b, c\}$ as the triple *abc*. We show that at least 15 triples in [7] cannot be used as the labels of the interior vertices of P_{23} by f, producing a contradiction.

We may assume that $f(v_1) = 4567$, $f(v_2) = 123$, and $f(v_3) = 456$. Hence, $f(v) \notin \{457, 467, 567\}$ for every vertex v of P_{23} (and so 457, 467, 567 are eliminated). We may assume that $f(v_4) = 127$, eliminating 137 and 237. We may assume that $f(v_5) = 345$, eliminating 346 and 356. So far, the seven triples in the set

$$\{457, 467, 567, 137, 237, 346, 356\}$$

are eliminated. Thus, $f(v_6) \in \{126, 167, 267\}$. Since each of these three triples contains 6, and 167 and 267 are equivalent, we may assume that $f(v_6) = 126$ or $f(v_6) = 167$. We consider these two cases.

Case 1. $f(v_6) = 126$ *while* $167, 267$ *are eliminated as possible labels.* Thus, the nine triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 167,\ 267\}$$

are eliminated. Now, $f(v_7) \in \{347, 357, 457\}$. Since 457 is eliminated while 347 and 357 are equivalent, we may assume that $f(v_7) = 347$ and eliminate 357. Next, $f(v_8) \in \{125, 156, 256\}$. Since 156 and 256 are equivalent, we may assume that $f(v_8) = 125$ or $f(v_8) = 156$. There are two subcases.

Subcase 1.1. $f(v_8) = 125$ *and* $156, 256$ *are eliminated as possible labels.* So far, the 12 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 167,\ 267,\ 357,\ 156,\ 256\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 8$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ f(v_8) = 125.$$

Since $f(v_9) \in \{346, 367, 467\}$ and 346, 347 are eliminated, $f(v_9) = 367$. Thus, $f(v_{10}) \in \{124, 145, 245\}$. Since 145 and 245 are equivalent, we may assume that $f(v_{10}) = 124$ or $f(v_{10}) = 145$.

Subcase 1.1.1. $f(v_{10}) = 124$ *and* $145, 245$ *are eliminated as possible labels.* Now $f(v_i)$ is defined for $1 \le i \le 10$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ 125,\ 367,\ 124 = f(v_{10}).$$

However, then, there is no triple for $f(v_{11})$. Thus, Subcase 1.1.1 cannot occur.

Subcase 1.1.2. $f(v_{10}) = 145$ *and* $124, 245$ *are eliminated as possible labels.* So far, the 14 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 167,\ 267,\ 357,\ 156,\ 256,\ 124,\ 245\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 10$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ 125,\ 367,\ 145 = f(v_{10}).$$

The only choice for $f(v_{11})$ is 236. Thus, we have

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ 125,\ 367,\ 145,\ 236 = f(v_{11}).$$

Hence, $f(v_{12}) \in \{147, 157\}$. Whichever of 147 and 157 is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total, which is a contradiction. Thus, Subcase 1.1.2 cannot occur. Consequently, Subcase 1.1 cannot occur.

Subcase 1.2. $f(v_8) = 156$ *and* $125, 256$ *are eliminated as possible labels.* So far, the 12 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 167,\ 267,\ 357,\ 125,\ 256\}$$

are eliminated and the labels $f(v_i)$ are defined for $1 \le i \le 8$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ 156 = f(v_8).$$

Since 237 is eliminated, it follows that $f(v_9) = 234$ or $f(v_9) = 247$. We consider these two subcases.

Subcase 1.2.1. $f(v_9) = 234$ *while* 247 *is eliminated as a possible label.* So far, the 13 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 167,\ 267,\ 357,\ 125,\ 256,\ 247\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 9$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ 156,\ 234 = f(v_9).$$

Thus, 157 is the only choice for $f(v_{10})$. Thus, the labels $f(v_i)$ are defined for $1 \le i \le 10$:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ 156,\ 234,\ 157 = f(v_{10}).$$

Hence, $f(v_{11}) = 236$ or $f(v_{11}) = 246$.

- ⋆ If $f(v_{11}) = 236$, then 246 is eliminated and so 14 triples are eliminated in total. Now, $f(v_i)$ is defined for $1 \le i \le 11$:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ 156,\ 234,\ 157,\ 236 = f(v_{11}).$$

Hence, $f(v_{12}) \in \{145, 147\}$. Whichever of 145 and 147 is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.

⋆ If $f(v_{11}) = 246$, then 236 is eliminated and so 14 triples are eliminated in total. Now, $f(v_i)$ is defined for $1 \leq i \leq 11$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ 156,\ 234,\ 157,\ 246 = f(v_{11}).$$

The only choice for $f(v_{12})$ is 135. However, then, there is no triple available for $f(v_{13})$.

Thus, Subcase 1.2.1 cannot occur.

Subcase 1.2.2. $f(v_9) = 247$ *while* 234 *is eliminated as a possible label.* So far, the 13 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 167,\ 267,\ 357,\ 125,\ 256,\ 234\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 9$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 126,\ 347,\ 156,\ 247 = f(v_9).$$

Hence, $f(v_{10}) \in \{135, 136\}$.

⋆ If $f(v_{10}) = 135$, then 136 is eliminated and so 14 triples are eliminated in total. This forces $f(v_{11}) = 246$, $f(v_{12}) = 157$, $f(v_{13}) = 236$, and $f(v_{14}) \in \{145, 147\}$. Whichever of 145 and 147 is selected for $f(v_{14})$, the other is eliminated, giving 15 triples eliminated in total.

⋆ If $f(v_{10}) = 136$, then 135 is eliminated and so 14 triples are eliminated in total. Then $f(v_{11}) \in \{245, 257\}$. Whichever of 245 and 257 is selected for $f(v_{11})$, the other is eliminated, giving 15 triples eliminated in total.

Thus, Subcase 1.2.2. (and so Subcase 1.2) cannot occur, completing the proof of Case 1.

Case 2. $f(v_6) = 167$ *while* $126, 267$ *are eliminated as possible labels.* Thus, the 9 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 6$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167 = f(v_6).$$

Now, $f(v_7) \in \{234, 235, 245\}$. We consider three subcases.

Subcase 2.1. $f(v_7) = 234$ *while* 235, 245 *are eliminated as possible labels.* Thus, the 11 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 235,\ 245\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 7$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 234 = f(v_7).$$

Hence, $f(v_8) = 156$ or $f(v_8) = 157$. We consider two subcases.

Subcase 2.1.1. $f(v_8) = 156$ *while* 157 *is eliminated as a possible label.* Thus, the 12 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 235,\ 245,\ 157\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 8$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 234,\ 156 = f(v_8).$$

Hence, $f(v_9) \in \{247, 347\}$.

$\star$ If $f(v_9) = 247$, then 347 is eliminated. Thus, the 13 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 235,\ 245,\ 157,\ 347\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 9$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 234,\ 156,\ 247 = f(v_9).$$

Hence, $f(v_{10}) = 135$ or $f(v_{10}) = 136$.

- If $f(v_{10}) = 135$, then 136 is eliminated and so the 14 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 235,\ 245,\ 157,\ 347,\ 136\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 10$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 234,\ 156,\ 247,\ 135 = f(v_{10}).$$

This forces $f(v_{11}) = 246$, $f(v_{12}) = 357$, and $f(v_{13}) \in \{124, 146\}$. Whichever of 124 and 146 is selected for $f(v_{13})$, the other is eliminated, giving 15 triples eliminated in total.

- If $f(v_{10}) = 136$, then 135 is eliminated and so the 14 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 235,\ 245,\ 157,\ 347,\ 135\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 10$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 234,\ 156,\ 247,\ 136 = f(v_{10}).$$

This forces $f(v_{11}) = 257$ and $f(v_{12}) \in \{134, 146, 346\}$. Whichever of $134, 146, 346$ is selected for $f(v_{12})$, the other two are eliminated, giving 16 triples eliminated in total.

⋆ If $f(v_9) = 347$, then 247 is eliminated. Thus, the 13 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 235,\ 245,\ 157,\ 247\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 9$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 234,\ 156,\ 347 = f(v_9).$$

Hence, $f(v_{10}) = 125$ or $f(v_{10}) = 256$.

- If $f(v_{10}) = 125$, then 256 is eliminated and so the 14 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 235,\ 245,\ 157,\ 247,\ 256\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 10$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 234,\ 156,\ 347,\ 125 = f(v_{10}).$$

Then $f(v_{11}) \in \{346, 367\}$. Whichever of $346, 367$ is selected for $f(v_{11})$, the other is eliminated, giving 15 triples eliminated in total.

- If $f(v_{10}) = 256$, then 125 is eliminated and so the 14 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 235,\ 245,\ 157,\ 247,\ 125\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 10$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 234,\ 156,\ 347,\ 256 = f(v_{10}).$$

Then $f(v_{11}) \in \{134, 147\}$. Whichever of $134, 147$ is selected for $f(v_{11})$, the other is eliminated, giving 15 triples eliminated in total.

Thus, Subcase 2.1.1 cannot occur.

Subcase 2.1.2. $f(v_8) = 157$ *and* 156 *is eliminated.* Thus, the 12 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 235,\ 245,\ 156\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 8$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 234,\ 157 = f(v_8).$$

Hence, $f(v_9) \in \{236, 246, 346\}$. Whichever of $236, 246, 346$ is selected for $f(v_9)$, the other two are eliminated, giving 14 triples eliminated in total.

- ⋆ If $f(v_9) = 236$, then $246, 346$ are eliminated and $f(v_{10}) \in \{145, 147\}$. Whichever of $145, 147$ is selected for $f(v_{10})$, the other is eliminated, giving 15 triples eliminated in total.
- ⋆ If $f(v_9) = 246$, then $236, 346$ are eliminated and $f(v_{10}) \in \{135, 357\}$. Whichever of $135, 357$ is selected for $f(v_{10})$, the other is eliminated, giving 15 triples eliminated in total.
- ⋆ If $f(v_9) = 346$, then $236, 246$ are eliminated and $f(v_{10}) \in \{125, 127, 257\}$. Whichever of $125, 127, 257$ is selected for $f(v_{10})$, the other two are eliminated, giving 16 triples eliminated in total.

Thus, Subcase 2.1 cannot occur.

Subcase 2.2. $f(v_7) = 235$ *and* $234, 245$ *are eliminated.* Thus, the 11 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 234,\ 245\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 7$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 235 = f(v_7).$$

Hence, $f(v_8) = 146$ or $f(v_8) = 147$. We consider these two subcases.

Subcase 2.2.1. $f(v_8) = 146$ *and* 147 *is eliminated.* Thus, the 12 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 234,\ 245,\ 147\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 8$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 235,\ 146 = f(v_8).$$

Hence, $f(v_9) \in \{257, 357\}$.

⋆ If $f(v_9) = 257$, then 357 is eliminated, giving 13 triples eliminated. Then $f(v_{10}) \in \{134, 136\}$.

- If $f(v_{10}) = 134$, then 136 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 256$, $f(v_{12}) = 347$, and $f(v_{13}) \in \{125, 156\}$. Whichever of $125, 156$ is selected for $f(v_{13})$, the other is eliminated, giving 15 triples eliminated in total.

- If $f(v_{10}) = 136$, then 134 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 247$ and $f(v_{12}) \in \{135, 156\}$. Whichever of $135, 156$ is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.

⋆ If $f(v_9) = 357$, then 257 is eliminated, giving 13 triples eliminated. Then $f(v_{10}) \in \{124, 246\}$.

- If $f(v_{10}) = 124$, then 246 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 367$ and $f(v_{12}) \in \{125, 145\}$. Whichever of $125, 145$ is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.

- If $f(v_{10}) = 246$, then 124 is eliminated, giving 14 triples eliminated. Then $f(v_{11}) \in \{135, 157\}$. Whichever of $135, 157$ is selected for $f(v_{11})$, the other is eliminated, giving 15 triples eliminated in total.

Thus, Subcase 2.2.1 is impossible.

Subcase 2.2.2. $f(v_8) = 147$ *and* 146 *is eliminated.* Thus, the 12 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 234,\ 245,\ 146\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 8$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 235,\ 147 = f(v_8).$$

Hence, $f(v_9) \in \{236, 256\}$.

- ⋆ If $f(v_9) = 236$, then 256 is eliminated, giving 13 triples eliminated. Then $f(v_{10}) \in \{145, 157\}$.
  - If $f(v_{10}) = 145$, then 157 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 367$ and $f(v_{12}) \in \{124, 125\}$. Whichever of $124, 125$ is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.
  - If $f(v_{10}) = 157$, then 145 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 246$ and $f(v_{12}) \in \{135, 357\}$. Whichever of $135, 357$ is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.
- ⋆ If $f(v_9) = 256$, then 236 is eliminated, giving 13 triples eliminated. Then $f(v_{10}) \in \{134, 347\}$.
  - If $f(v_{10}) = 134$, then 347 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 257$ and $f(v_{12}) \in \{136, 146\}$. Whichever of $136, 146$ is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.
  - If $f(v_{10}) = 347$, then 134 is eliminated, giving 14 triples eliminated. Then $f(v_{11}) \in \{125, 156\}$. Whichever of $125, 156$ is selected for $f(v_{11})$, the other is eliminated, giving 15 triples eliminated in total.

Thus, Subcase 2.2.2 is impossible. Consequently, Subcase 2.2 cannot occur.

Subcase 2.3. *$f(v_7) = 245$ and $234, 235$ are eliminated.* Thus, the 11 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 234,\ 235\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 7$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 245 = f(v_7).$$

Hence, $f(v_8) = 136$ or $f(v_8) = 367$. We consider these two subcases.

Subcase 2.3.1. *$f(v_8) = 136$ and 367 is eliminated.* Thus, the 12 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 234,\ 235,\ 367\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 8$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 245,\ 136 = f(v_8).$$

Hence, $f(v_9) \in \{247, 257\}$.

⋆ If $f(v_9) = 247$, then 257 is eliminated, giving 13 triples eliminated. Then $f(v_{10}) \in \{135, 156\}$.

- If $f(v_{10}) = 135$, then 156 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 246$ and $f(v_{12}) \in \{157, 357\}$. Whichever of $157, 357$ is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.
- If $f(v_{10}) = 156$, then 135 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 347$ and $f(v_{12}) \in \{125, 256\}$. Whichever of $125, 256$ is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.

⋆ If $f(v_9) = 257$, then 247 is eliminated, giving 13 triples eliminated. Then $f(v_{10}) \in \{134, 146\}$.

- If $f(v_{10}) = 134$, then 146 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 256$ and $f(v_{12}) \in \{147, 347\}$. Whichever of $147, 347$ is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.
- If $f(v_{10}) = 146$, then 134 is eliminated, giving 14 triples eliminated. This forces $f(v_{11}) = 357$ and $f(v_{12}) \in \{124, 246\}$. Whichever of $124, 246$ is selected for $f(v_{12})$, the other is eliminated, giving 15 triples eliminated in total.

Thus, Subcase 2.3.1 cannot occur.

Subcase 2.3.2. *$f(v_8) = 367$ and 136 is eliminated.* Thus, the 12 triples in the set

$$\{457, 467, 567, 137, 237, 346, 356, 126, 267, 234, 235, 136\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 8$ as follows:

$$f(v_1) = 4567, 123, 456, 127, 345, 167, 245, 367 = f(v_8).$$

Hence, $f(v_9) \in \{124, 125, 145\}$.

⋆ If $f(v_9) = 124$, then $125, 145$ are eliminated. Thus, the 14 triples in the set

$$\{457, 467, 567, 137, 237, 346, 356, 126, 267, 234, 235, 136, 125, 145\}$$

are eliminated and $f(v_i)$ is defined for $1 \leq i \leq 9$ as follows:

$$f(v_1) = 4567, 123, 456, 127, 345, 167, 245, 367, 124 = f(v_9).$$

This forces $f(v_{10}) = 357$ and $f(v_{11}) \in \{146, 246\}$. Whichever of $146, 246$ is selected for $f(v_{11})$, the other is eliminated, giving 15 triples eliminated in total.

⋆ If $f(v_9) = 125$, then $124, 145$ are eliminated. Thus, the 14 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 234,\ 235,\ 136,\ 124,\ 145\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 9$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 245,\ 367,\ 125 = f(v_9).$$

This forces $f(v_{10}) = 347$ and $f(v_{11}) \in \{156, 256\}$. Whichever of $156, 256$ is selected for $f(v_{11})$, the other is eliminated, giving 15 triples eliminated in total.

⋆ If $f(v_9) = 145$, then $124, 125$ are eliminated. Thus, the 14 triples in the set

$$\{457,\ 467,\ 567,\ 137,\ 237,\ 346,\ 356,\ 126,\ 267,\ 234,\ 235,\ 136,\ 124,\ 125\}$$

are eliminated and $f(v_i)$ is defined for $1 \le i \le 9$ as follows:

$$f(v_1) = 4567,\ 123,\ 456,\ 127,\ 345,\ 167,\ 245,\ 367,\ 145 = f(v_9).$$

This forces $f(v_{10}) = 236$ and $f(v_{11}) \in \{147, 157\}$. Whichever of $147, 157$ is selected for $f(v_{11})$, the other is eliminated, giving 15 triples eliminated in total.

Thus, Subcase 2.3.2 (and so Subcase 2.3) cannot occur. Consequently, the proof of Case 2 is completed. ■

We are now prepared to present the following result.

Theorem 3.4.8 $\rho(P_{23}) = 8$.

Proof. Let $P_{23} = (v_1, v_2, \ldots, v_{23})$. Since there is a subset labeling of P_{23} using labels in the set $\mathcal{P}^*([8])$ defined by

$$\begin{aligned}\mathcal{S}_f(P_{23}) &= (2457, 136, 245, 367, 125, 347, 156, 234, 567, 123, 456, 127, 345, 267,\\ &\quad 135, 246, 357, 146, 235, 1478, 236, 1457, 2368),\end{aligned}$$

it follows that $\rho(P_{23}) \le 8$. Since $\rho(P_{23}) \ge 7$, it remains to show that $\rho(P_{23}) \ne 7$. By Proposition 3.2.2, no vertex of P_{23} can have a 1-set label, and by Proposition 3.2.1, we have $f(u) \ne f(v)$ for any subset labeling f for every two distinct vertices of P_{23}. We now assume, to the contrary, that there exists a subset labeling $f : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23}. First, we present the following facts:

Fact 1: No vertex of P_{23} can have a 2-set label by Theorem 3.4.6.

Fact 2: No vertex of P_{23} can have a 5-set label.

Fact 3: No interior vertex of P_{23} can have a 4-set label.

Fact 4: It is impossible that (1) $|f(v_1)| = 4$ or (2) $|f(v_{23})| = 4$ and $|f(v_i)| = 3$ for $2 \leq i \leq 22$ by Theorem 3.4.7.

Proof of Fact 2: Suppose that $|f(v_i)| = 5$ for some integer i with $1 \leq i \leq n$. We may assume that $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $f(v_i) = [5]$. However, then, $f(v_{i+1}) = 67$, which is impossible by **Fact 1**.

Proof of Fact 3: Suppose that $|f(v_i)| = 4$ for some integer i with $2 \leq i \leq n-1$. We may assume that $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $f(v_i) = [4]$. Since every two 4-element subsets of [7] have at least one element in common, it follows that $|f(v_{i-1})| \neq 4$ and $|f(v_{i+1})| \neq 4$. Furthermore, by **Fact 1**, $|f(v_{i-1})| \neq 2$ and $|f(v_{i+1})| \neq 2$. Hence, $|f(v_{i-1})| = |f(v_{i+1})| = 3$. However, then, $f(v_{i-1}) = 567 = f(v_{i+1})$, which is a contradiction by Proposition 3.2.1.

By Proposition 3.2.2 and **Facts 1** and **2**, it follows that $|f(v)| \notin \{1, 2, 5\}$ for every vertex v of P_{23}. By Facts 3 and 4, it follows that $|f(v)| = 3$ for each vertex v of P_{23}. We may assume that $f(v_1) = 123$ and $f(v_2) = 456$. Since f is a subset labeling of P_{23}, it follows that $f(v_1) \cap f(v_i) \neq \emptyset$ for $3 \leq i \leq 23$. However, then, the subset labeling $g : V(P_{23}) \to \mathcal{P}^*([7])$ of P_{23} defined by $g(v_1) = f(v_1) \cup \{7\}$ and $g(v_i) = f(v_i)$ for $2 \leq i \leq 23$ is also a subset labeling, which contradicts **Fact 4**.

We conclude that $\rho(P_{23}) \geq 8$. Consequently, $\rho(P_{23}) = 8$. ■

Finally, we present a subset labeling $f : V(P_{24}) \to \mathcal{P}^*([8])$:

$$\begin{aligned}\mathcal{S}_f(P_{24}) = (&1234, 567, 123, 458, 2367, 158, 2467, 358, 1467, 235, 1678, 245,\\ &1378, 256, 3478, 125, 3468, 257, 1346, 278, 1345, 268, 1357, 2468)\end{aligned}$$

We now summarize the values of $\rho(P_n)$ for $3 \leq n \leq 23$ as follows:

$$\rho(P_n) = \begin{cases} n-1 & \text{if } 3 \leq n \leq 6 \\ 5 & \text{if } n = 7 \\ 6 & \text{if } 8 \leq n \leq 11 \\ 7 & \text{if } 12 \leq n \leq 22 \\ 8 & \text{if } n = 23, 24. \end{cases}$$

We have seen that $\rho(P_{n+1}) \leq \rho(P_n)+1$ for $n \geq 3$ by Theorem 3.1.1. Therefore, the following is a consequence of Theorems 3.1.1 and 3.4.8.

Corollary 3.4.9 *If* $n \geq 23$, *then* $8 \leq \rho(P_n) \leq n - 15$.

It remains to establish sharp lower and upper bounds for the subset index of paths.

Chapter 4

Subset Labelings of Cycles

4.1 Introduction

In this chapter, we turn our attention to subset labelings of cycles. For each integer $n \geq 3$, the path P_{n-1} of order $n-1$ is an induced subgraph of the cycle C_n of order n. By Observation 2.1.5, it follows that

$$\rho(C_n) \geq \rho(P_{n-1}) \text{ for every integer } n \geq 3.$$

Since $\rho(P_n) \leq \rho(P_{n+1}) \leq \rho(P_n) + 1$ by Theorem 3.1.1, it follows that

$$\rho(C_n) \geq \rho(P_n) - 1 \text{ for every integer } n \geq 3.$$

Consequently, there is a (probably expected) connection between subset labelings of cycles and paths. The following is a consequence of Theorem 3.1.1 (see [2]).

Proposition 4.1.1 $\lim_{n\to\infty} \rho(C_n) = \infty$.

Since $C_3 = K_3$ is the complete graph of order 3, $C_4 = K_{2,2}$ is the complete bipartite graph of order 4, and the subset indices of these two classes of graphs are known, we consider cycles of order 5 or more. Exact values of $\rho(C_n)$ have been determined in [2] for $n = 5, 6, 7$.

Proposition 4.1.2 $\rho(C_5) = \rho(C_6) = 5$ *and* $\rho(C_7) = 7$.

Figure 4.1 shows a subset labeling of C_n of order n for each integer $n = 5, 6, 7$.

While $\lim_{n\to\infty} \rho(C_n) = \infty$ by Proposition 4.1.1, it follows by Proposition 4.1.2 that there is no result for cycles corresponding to Theorem 3.1.1 for paths, namely,

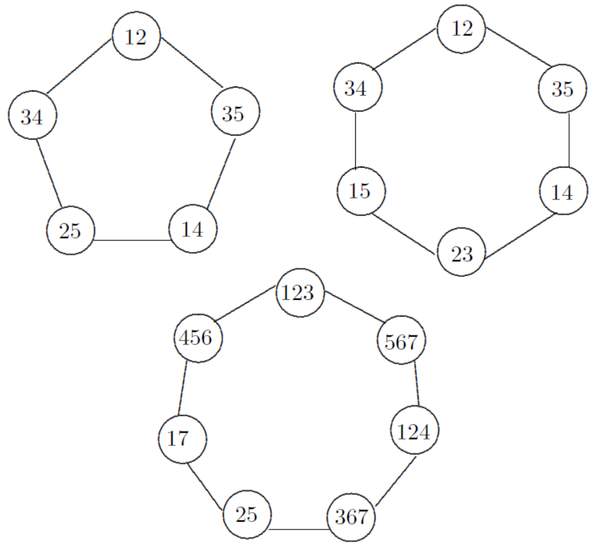

Figure 4.1: Subset labelings of C_5, C_6 and C_7

$$\rho(P_n) \le \rho(P_{n+1}) \le \rho(P_n) + 1 \text{ for each integer } n \ge 3.$$

Perhaps surprisingly, $\{\rho(C_n)\}_{i=3}^{\infty}$ is not even a monotone sequence. For example, even though $5 = \rho(C_6) < \rho(C_7) = 7$ by Proposition 4.1.2, it turns out that $\rho(C_8) = 6$, which we will show. This implies that $\{\rho(C_n)\}_{i=3}^{\infty}$ fails to be a monotone sequence. In general, when defining our labels we continue to refer to subset $\{a\}$ as a, subset $\{a, b\}$ as ab, subset $\{a, b, c\}$ as abc, and so forth. Also, as a preliminary observation, we note that since the eccentricity of every vertex of C_n is at least 4 for $n \ge 8$, by Proposition 2.3.3 we have $|f(v)| \ge 2$ for every $v \in C(n)$, $n \ge 8$.

Proposition 4.1.3 $\rho(C_8) = 6$.

Proof. Let $C_8 = (v_1, v_2, \ldots, v_8, v_1)$. Since the labeling $f : V(C_8) \to \mathcal{P}^*([6])$ defined by

$$(f(v_1), f(v_2), \ldots, f(v_8)) = (12, 456, 13, 256, 14, 236, 15, 346)$$

is a subset labeling of C_8, it follows that $\rho(C_8) \le 6$. It remains to show that $\rho(C_8) \ge 6$. Assume, to the contrary, that there is a subset labeling $f : V(C_8) \to \mathcal{P}^*([5])$. Since $N(u) \ne N(v)$ for every two distinct vertices u and v and $\text{diam}(C_8) = 4 > 3$, it follows that $f(u) \ne f(v)$ and $|f(x)| \in \{2, 3\}$ for every vertex x of C_8. If $|f(x)| = 3$ for some vertex x of C_8, then the two neighbors of x have the same label, which is impossible. Thus, $|f(x)| = 2$ for every vertex x of C_8. We may assume, without loss of generality, that $f(v_8) = 35$,

$f(v_1) = 12$, $f(v_2) = 34$, and $f(v_3) = 15$. Since $f(v_4) \cap f(v_3) = \emptyset$ and $f(v_4) \cap f(v_8) \neq \emptyset$, this forces $f(v_4) = 23$. Similarly, since $f(v_5) \cap f(v_4) = \emptyset$ and $f(v_5) \cap f(v_i) \neq \emptyset$ for $i = 1, 8$, this forces $f(v_5) = 15 = f(v_3)$, which is impossible. Thus, $\rho(C_8) \geq 6$. We conclude $\rho(C_8) = 6$. ■

We now proceed to investigate $\rho(C_n)$ for integers $n \geq 9$. As with P_n, we introduce the notation of a *label sequence* for cycle C_n. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ be a cycle of order $n \geq 3$, and let f be a vertex labeling of C_n. Then the label sequence of f is defined as

$$\mathcal{S}_f(C_n) = (f(v_1), f(v_2), \ldots, f(v_n)).$$

4.2 Preliminary Results

We begin with some preliminary observations on subset labelings of cycles in general. First, we note that for all integers $n \geq 5$ and for any two vertices u and v in $V(C_n)$, we have both $N(u) - N(v) \neq \emptyset$ and $N(v) - N(u) \neq \emptyset$. Thus, if f is a subset labeling of a cycle C_n of order $n \geq 5$, it follows by the Subset Lemma that neither $f(v) \subseteq f(u)$ nor $f(u) \subseteq f(v)$.

The following three results discuss some properties that a subset labeling f of a cycle C_n may possess for n sufficiently large when $f(v)$ is a 2-element set for some $v \in V(C_n)$.

Proposition 4.2.1 *Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ be an odd cycle of order $n \geq 5$ and let f be a subset labeling of C_n. If $f(v_i) = ab$ for some integer i with $1 \leq i \leq n$, then either*

- $\star$ $(f(v_\ell) \cap f(v_i))_{\ell=i+2}^{i-2} = (a, b, a, b, \ldots, a, b)$ *or*
- $\star$ $(f(v_\ell) \cap f(v_i))_{\ell=i+2}^{i-2} = (b, a, b, a, \ldots, b, a)$,

where the subscript of each vertex of C_n is expressed as one of $1, 2, \ldots, n$.

Proof. Suppose, without loss of generality, that $f(v_1) = ab$. By the Subset Lemma, no vertex label can be duplicated and no other vertex label can include the subset $\{a, b\}$. Furthermore, $f(v_1) \cap f(v_3) = \{a\}$ or $f(v_1) \cap f(v_3) = \{b\}$. We will only consider the situation when $f(v_1) \cap f(v_3) = \{a\}$, since the argument is similar if $f(v_1) \cap f(v_3) = \{b\}$. Since $f(v_1) \cap f(v_3) = \{a\}$, this forces $f(v_1) \cap f(v_4) = \{b\}$, which in turn forces $f(v_1) \cap f(v_5) = \{a\}$, and so on. If n is even, then we will eventually obtain $f(v_1) \cap f(v_{n-2}) = \{b\}$ and $f(v_1) \cap f(v_{n-1}) = \{a\}$. If n is odd, then we will eventually obtain $f(v_1) \cap f(v_{n-2}) = \{a\}$ and $f(v_1) \cap f(v_{n-1}) = \{b\}$. ■

Proposition 4.2.2 *Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ be an even cycle of order $n \geq 6$ and let f be a subset labeling of C_n. If $f(v) = ab$ for some vertex v of C_n, then either*

- ⋆ *$f(v_i) \cap f(v) = \{a\}$ for every vertex v_i of C_n such that $d(v_i, v) \geq 2$ and i is odd and $f(v) \cap f(w) = \{b\}$ for every neighbor w of v_i that is not a neighbor of v, or*
- ⋆ *$f(v_i) \cap f(v) = \{b\}$ for every vertex v_i of C_n such that $d(v_i, v) \geq 2$ and i is odd and $f(v) \cap f(w) = \{a\}$ for every neighbor w of v_i that is not a neighbor of v.*

Proof. Suppose, without loss of generality, that $f(v_1) = ab$. By the Subset Lemma, no vertex label can be duplicated and no other vertex label can include the subset $\{a, b\}$. Furthermore, $f(v_1) \cap f(v_3) = \{a\}$ or $f(v_1) \cap f(v_3) = \{b\}$. First, suppose that $f(v_1) \cap f(v_3) = \{a\}$. This forces $f(v_1) \cap f(v_4) = \{b\}$, which in turn forces $f(v_1) \cap f(v_5) = \{a\}$, and so on. Since n is even, we will eventually obtain $f(v_1) \cap f(v_{n-2}) = \{b\}$ and $f(v_1) \cap f(v_{n-1}) = \{a\}$. Thus, we obtain $f(v_1) \cap f(v_i) = \{a\}$ for all odd integers i and $f(v_1) \cap f(v_i) = \{b\}$ for all even integers $i \neq 2$. Next, suppose that $f(v_1) \cap f(v_3) = \{b\}$. Using a similar argument, we obtain that $f(v_1) \cap f(v_i) = \{b\}$ for all even integers $i \neq 2$, and $f(v_1) \cap f(v_i) = \{a\}$ for all odd integers i. ■

Proposition 4.2.3 *Let C_n be a cycle of order $n \geq 8$ and let f be a subset labeling of C_n.*

(i) If $n \geq 9$ is odd, then there do not exist two vertices u and v with $d(u, v) = 2$ or $d(u, v) \geq 4$ such that $|f(u)| = 2 = |f(v)|$.

(ii) If $n \geq 8$ is even, then there do not exist two vertices u and v for which $d(u, v)$ is odd and $|f(u)| = 2 = |f(v)|$.

Proof. We first verify (i). Let $n \geq 9$ be an odd integer. Suppose, to the contrary, that there exist two vertices u and v with $d(u, v) = 2$ or $d(u, v) \geq 4$ such that $|f(u)| = 2 = |f(v)|$. Thus, $f(u) \cap f(v) \neq \emptyset$. Let $f(u) = ab$ and $f(v) = ac$. Since n is odd, there is a $u - v$ path P of odd length and a $u - v$ path P' of even length on C_n. By assumption, the length of P is at least 5. We may assume that $P = (u = v_1, v_2, v_3, \ldots, v_{k-1}, v_k = v)$ for some even integer $k \geq 6$. Since $f(v_1) = ab$ and $f(v_2) \cap f(v_k) \neq \emptyset$, it follows that $c \in f(v_2)$. Similarly, since $f(v_k) = ac$ and $f(v_{k-1}) \cap f(v_1) \neq \emptyset$, it follows that $b \in f(v_{k-1})$. This implies that $c \in f(v_i)$ for each even integer i with $2 \leq i \leq k-2$ and $b \in f(v_i)$ for each odd integer i with $3 \leq i \leq k-1$. Since $f(v_4) \cap f(v_1) \neq \emptyset$, but $b \in f(v_3) \cap f(v_5)$, we have $a \in f(v_4)$ forced, so that $\{a, c\} \subseteq f(v_4)$. However, then, $a, c \notin f(v_3)$, and so $f(v_3) \cap f(v_k) = \emptyset$, which is a contradiction.

Next, we verify (ii). Let $n \geq 8$ be an even integer. Suppose, to the contrary, that there exist two vertices u and v for which $d(u, v)$ is odd and $|f(u)| = 2 = |f(v)|$. Since $d(u, v)$ is odd, the two $u - v$ paths on C_n are both odd, and at least one of these two $u - v$ paths has length 5 or more. An argument similar to that employed in (i) produces a contradiction here as well. ■

4.3 Cycles of Order 9 through 12

In this section, we determine the subset indices of the cycles C_n for $9 \leq n \leq 12$. Let

$$C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$$

be a cycle of order $n \geq 3$.

We first investigate $\rho(C_n)$ for $n \in \{9, 10, 12\}$. We will see soon that $\rho(C_{11})$ is a special case.

Proposition 4.3.1 *$\rho(C_9) = 7$, $\rho(C_{10}) = 6$, and $6 \leq \rho(C_{12}) \leq 7$.*

Proof. Since $\rho(P_n) = 6$ for $8 \leq n \leq 11$, it follows that $\rho(C_n) \geq \rho(P_8) = 6$ for $n = 9, 10, 12$. For $n = 9, 10, 12$, there is a subset labeling $f : V(C_n) \to \mathcal{P}^*([k])$, where $k = 7$ if $n = 9, 12$ and $k = 6$ if $n = 10$, defined by

$$\begin{aligned}
\mathcal{S}_f(C_9) &= (123, 456, 137, 246, 135, 247, 356, 124, 567) \\
\mathcal{S}_f(C_{10}) &= (12, 345, 16, 234, 15, 246, 13, 256, 14, 356) \\
\mathcal{S}_f(C_{12}) &= (123, 456, 237, 156, 347, 125, 367, 245, 136, 257, 134, 567).
\end{aligned}$$

Thus, $\rho(C_{10}) = 6$ and $6 \leq \rho(C_n) \leq 7$ for $n = 9, 12$.

Next, we show that $\rho(C_9) = 7$. Since $\rho(C_9) \in \{6, 7\}$, it remains to show that $\rho(C_9) \neq 6$. Let $C_9 = (v_1, v_2, \ldots, v_9, v_1)$. Assume, to the contrary, that there exists a subset labeling $f : V(C_9) \to \mathcal{P}^*([6])$ of C_9. Then $|f(v)| \geq 2$ for every vertex v of C_9. First, we make three observations.

(1) No vertex label can be a 4-set.

If (1) is false, then we may assume that $f(v_1) = 1234$. However, then, $f(v_2) = 56 = f(v_9)$, which is impossible by the Subset Lemma.

(2) No two consecutive vertex labels can be 3-sets.

This is an immediate consequence of Proposition 2.3.6.

(3) No two consecutive vertex labels can be 2-sets.

If (3) is false, then we may assume that $f(v_1) = 12$ and $f(v_2) = 34$. By Proposition 4.2.1, we may further assume that $1 \in f(v_i)$ for each (odd) integer $i \in \{3, 5, 7\}$ and $2 \in f(v_i)$ for each (even) integer $i \in \{4, 6, 8\}$. Similarly, we may assume that $4 \in f(v_i)$ for each (odd) integer $i \in \{5, 7, 9\}$ and $3 \in f(v_i)$ for each (even) integer $i \in \{4, 6, 8\}$. This implies that $\{2, 3\} \subseteq f(v_i)$ for $i \in \{4, 6, 8\}$ and $\{1, 4\} \subseteq f(v_i)$ for $i \in \{5, 7\}$. By the Subset Lemma, no two vertices can have the same label, and no vertex label can be a subset of another vertex label. Thus, each of $f(v_4)$, $f(v_6)$, and $f(v_8)$ is a 3-set label containing the subset $\{2, 3\}$. Since $\{1, 4\}$ is a subset of $f(v_5)$ and $f(v_7)$, it follows that 5 and 6 are the only possibilities for the third element of $f(v_4)$, $f(v_6)$, and $f(v_8)$. However, then, at least two of $f(v_4)$, $f(v_6)$, and $f(v_8)$ are the same, which, again, is impossible by the Subset Lemma.

By (1)-(3), we conclude that the vertex labels of C_9 alternate between 2-sets and 3-sets along the cycle C_9. Since C_9 is an odd cycle, this is impossible. Therefore, $\rho(C_9) = 7$. ■

By Proposition 4.3.1, it follows that $\rho(C_9) = 7$, $\rho(C_{10}) = 6$, and $6 \le \rho(C_{12}) \le 7$. We will return to $\rho(C_{12})$ later. Before this, however, we show that $\rho(C_{11}) = 8$, which again illustrates that $\{\rho(C_n)\}_{i=3}^{\infty}$ is not a monotone sequence. In order to show that $\rho(C_{11}) = 8$, we first present some preliminary results on subset labelings of C_{11}.

Proposition 4.3.2 *If $f : V(C_{11}) \to \mathcal{P}^*([7])$ is a subset labeling of C_{11}, then $|f(v)| \ne 4$ for every vertex v of C_{11}.*

Proof. Let $C_{11} = (v_1, v_2, \ldots, v_{11}, v_1)$ and suppose, to the contrary, that $|f(v)| = 4$ for some vertex v of C_{11}. We may assume that $f(v_1) = 1234$. Thus, each of $f(v_2)$ and $f(v_{11})$ is a subset of $\{5, 6, 7\}$. By the Subset Lemma, $f(v_2) \not\subseteq f(v_{11})$ and $f(v_{11}) \not\subseteq f(v_2)$. Thus, each of $f(v_2)$ and $f(v_{11})$ is a 2-set label. However, since $n = 11$ is odd and $d(v_2, v_{11}) = 2$, this is impossible by Proposition 4.2.3. ■

Theorem 4.3.3 *If $f : V(C_{11}) \to \mathcal{P}^*([7])$ is a subset labeling of C_{11}, then at most one vertex of C_{11} has a 2-set label.*

Proof. Let $C_{11} = (v_1, v_2, \ldots, v_{11}, v_1)$. Suppose, to the contrary, that there are at least two vertices u and v such that $|f(u)| = |f(v)| = 2$. By Proposition 4.2.3, either $d(u,v) = 1$ or $d(u,v) = 3$, and u and v are the only two vertices of C_n with 2-set labels. We consider two cases, according to whether $d(u,v) = 1$ or $d(u,v) = 3$.

Case 1. $d(u,v) = 1$. Without loss of generality, let $f(v_1) = 12$ and $f(v_2) = 34$. We may further assume that $1 \in f(v_3)$, which forces $2 \in f(v_4)$. Similarly, we may assume that $3 \in f(v_4)$. Thus, $\{2,3\} \subseteq f(v_4)$, and so $\{1,4\} \subseteq f(v_5)$. Furthermore, $\{2,3\} \subseteq f(v_i)$ for $i = 6, 8, 10$. There are two subcases, depending on whether $|f(v_4)| = 3$ or $|f(v_4)| = 2$.

Subcase 1.1. $|f(v_4)| = 3$. By Proposition 4.3.2, $|f(v_i)| = 3$ for $i \in [5, 11]$. We thus have a total of four 3-set labels containing the subset $\{2,3\}$, but since the third element must be drawn from $\{5,6,7\}$, there are only three possible labels. This is a contradiction.

Subcase 1.2. $|f(v_4)| = 2$. In this case, $f(v_4) = 23$, which excludes the triples $145, 146$ from consideration for $f(v_7)$ and $f(v_9)$, leaving only the possibility of 147. This is a contradiction.

Case 2. $d(u,v) = 3$. Without loss of generality, let $f(v_1) = 12$ and $f(v_4) = 13$. This implies that $3 \in f(v_2)$ and $2 \in f(v_3)$. We may assume that $f(v_2) = 345$ and $f(v_3) = 267$. By Proposition 4.3.2, $|f(v_i)| = 3$ for $i \in [5, 11]$. We have $(f(v_i) \cap (f(v_1) \cup f(v_4))_{i=5}^{11} = (\{2\}, \{1\}, \{2,3\}, \{1\}, \{2,3\}, \{1\}, \{3\})$. The triple 167 is excluded because $f(v_2) = 345$, and the triple 145 is excluded because $f(v_3) = 267$. This means that we cannot assign the labels 234 and 235 to the vertices v_7 and v_9 (or vice versa), for otherwise this would force $f(v_8) = 167$. Similarly, we cannot assign the labels 236 and 237 to v_7 and v_9 (or vice versa), for otherwise this would force $f(v_8) = 145$. This leaves four possible subcases.

Subcase 2.1. $f(v_7) = 236$ *and* $f(v_9) = 234$. This forces $f(v_8) = 157$. Then $|f(v_8) \cap f(v_{10})| = 2$ and 167 is excluded, so $f(v_{10}) = 156$ is forced. Then $f(v_{11}) = 347$ is forced. Similarly, $|f(v_8) \cap f(v_6)| = 2$ and 145 is excluded, so 147 is forced for $f(v_6)$. Then $f(v_5) = 256$ is forced. However, then, $f(v_5) \cap f(v_{11}) = \emptyset$. This is a contradiction.

Subcase 2.2. $f(v_7) = 237$ *and* $f(v_9) = 234$. This forces $f(v_8) = 156$. Then $|f(v_8) \cap f(v_{10})| = 2$, but 167 is excluded, so $f(v_{10}) = 157$ is forced. Then $f(v_{11}) = 346$ is forced. Similarly, $|f(v_8) \cap f(v_6)| = 2$, but 145 is excluded, so 146 is forced for $f(v_6)$. Then $f(v_5) = 257$ is forced. However, then, $f(v_5) \cap f(v_{11}) = \emptyset$. This is a contradiction.

Subcase 2.3. $f(v_7) = 237$ *and* $f(v_9) = 235$. This forces $f(v_8) = 146$. Then $|f(v_8) \cap f(v_{10})| = 2$, but 167 is excluded, so $f(v_{10}) = 147$ is forced. Then $f(v_{11}) = 356$ is forced.

Similarly, $|f(v_8) \cap f(v_6)| = 2$, but 145 is excluded, so 156 is forced for $f(v_6)$. Then $f(v_5) =$ 247 is forced. However, then, $f(v_5) \cap f(v_{11}) = \emptyset$. This is a contradiction.

Subcase 2.4. $f(v_7) = 236$ *and* $f(v_9) = 235$. This forces $f(v_8) = 147$. Then $|f(v_8) \cap f(v_{10})| = 2$, but 167 is excluded, so $f(v_{10}) = 146$ is forced. Then $f(v_{11}) = 357$ is forced. Similarly, $|f(v_8) \cap f(v_6)| = 2$, but 145 is excluded, so 157 is forced for $f(v_6)$. Then $f(v_5) =$ 246 is forced. However, then, $f(v_5) \cap f(v_{11}) = \emptyset$. This is a contradiction.

Consequently, at most one vertex of C_{11} can have a 2-set label. ■

Proposition 4.3.4 *If* $f : V(C_{11}) \to \mathcal{P}^*([7])$ *is a subset labeling of* C_{11}*, then* $|f(v)| \neq 2$ *for every vertex* v *of* C_{11}*.*

Proof. Let $C_{11} = (v_1, v_2, \ldots, v_{11}, v_1)$. Assume, to the contrary, that there is a vertex v with a 2-set label, say $f(v_1) = 12$. By Proposition 4.3.3, we have $|f(v_i)| = 3$ for all $i \neq 1$, say

$$f(v_2) = 345,\ f(v_3) = 167,\ f(v_4) = 234,\ f(v_5) = 156, \text{ and } f(v_6) = 247.$$

This implies that $1 \in f(v_i)$ for $i \in \{7, 9\}$ and $2 \in f(v_i)$ for $i \in \{8, 10\}$. Among the triples containing either 1 or 2, notice that in addition to the labels used, we have excluded the set

$$\{127,\ 157,\ 267,\ 235,\ 245,\ 237\}.$$

There are two possibilities for $f(v_7)$, that is $f(v_7) = 135$ or $f(v_7) = 136$. We consider these two cases.

Case 1. $f(v_7) = 135$. This forces $f(v_8) = 246$, since the elements of $f(v_8)$ must come from $\{2, 4, 6, 7\}$, but 267 and 247 are excluded. Then the elements of $f(v_9)$ must come from $\{1, 3, 5, 7\}$, but 157 and 135 have been excluded, so 137 is forced. Then the elements of $f(v_{10})$ must come from $\{2, 4, 5, 6\}$, but 245 and 246 have been excluded, so 256 is forced. This then forces $f(v_{11}) = 347$. However, then, $f(v_{11}) \cap f(v_5) = \emptyset$, which is a contradiction.

Case 2. $f(v_7) = 136$. This forces $f(v_8) = 257$, since the elements of $f(v_8)$ must come from $\{2, 4, 5, 7\}$, but 245 and 247 are excluded. Then the elements of $f(v_9)$ must come from $\{1, 3, 4, 6\}$. There are two possible subcases.

Subcase 2.1. $f(v_9) = 134$. This forces $f(v_{10}) = 256$, since the elements must come from $\{2, 5, 6, 7\}$, but 257 and 267 are excluded. This then forces $f(v_{11}) = 347$. However, then, $f(v_{11}) \cap f(v_5) = \emptyset$, which is a contradiction.

Subcase 2.2. $f(v_9) = 146$. The elements of $f(v_{10})$ must then be drawn from $\{2, 3, 5, 7\}$; yet all three triples 235, 237, and 257 have been excluded, which is a contradiction.

By Cases 1 and 2, we conclude that $|f(v)| \neq 2$ for every vertex v of C_n. ∎

We are now prepared to show that $\rho(C_{11}) = 8$.

Theorem 4.3.5 $\rho(C_{11}) = 8$

Proof. The subset labeling $g : V(C_{11}) \to \mathcal{P}^*([8])$ defined by

$$\mathcal{S}_g(C_{11}) = (123, 456, 1278, 345, 167, 245, 367, 145, 236, 147, 568)$$

shows that $\rho(C_{11}) \leq 8$. Next, we show that $\rho(C_{11}) \geq 8$. Assume, to the contrary, that there is a subset labeling $f : V(C_{11}) \to \mathcal{P}^*([7])$ of C_{11}. First, we present the following three facts:

> **Fact 1**: No vertex of C_{11} can have a 6-set label or a 5-set label. The neighboring vertices of a vertex with a 6-set label would be labeled with a singleton (indeed, the same singleton), which is impossible. The neighboring vertices of a vertex with a 5-set label would be labeled with duplicate 2-sets, which is also impossible.
>
> **Fact 2**: No vertex of C_{11} can have a 2-set label. This follows from Propositions 4.3.3 and 4.3.4.
>
> **Fact 3**: No vertex of C_{11} can have a 4-set label. This follows from Proposition 4.3.2.

By **Facts 1–3**, every vertex has a 3-set label. We make the useful observation that this forces $|f(v_i) \cap f(v_{i+2})| = 2$ for all i, else if we had $|f(v_i) \cap f(v_{i+2})| = 1$ for some i, this would mean $|f(v_i) \cup f(v_{i+2})| = 5$, forcing $|f(v_{i+1})| = 2$ which contradicts **Fact 2**. It follows immediately that we cannot have $|f(v_i) \cap f(v_{i+3})| = 2$, else then we would have $|f(v_{i+1}) \cap f(v_{i+3})| = 1$ forced. Now, say

$$f(v_1) = 123,\ f(v_2) = 567,\ f(v_3) = 124 \text{ and } f(v_4) = 356.$$

This implies that $4 \in f(v_{11})$ and $7 \in f(v_5)$. Then $f(v_5) \in \{127, 147, 247\}$. Since 147 and 247 are equivalent, we consider only the cases 127 and 147.

Case 1. $f(v_5) = 127$. This forces $\{4, 7\} \subseteq f(v_{11})$ and $\{3, 4\} \subseteq f(v_6)$. Since 345 and 346 are equivalent triples, say $f(v_6) = 345$. This forces $6 \in f(v_7)$. Then $f(v_7) \in \{126, 267, 167\}$.

Since 167 and 267 are equivalent, it suffices to consider the two cases $f(v_7) = 126$ and $f(v_7) = 267$.

Subcase 1.1. $f(v_7) = 126$. This forces $f(v_8) = 347$, which forces $5 \in f(v_9)$. We then have $f(v_9) \in \{125, 156, 256\}$. But this would force either $f(v_{10}) = 36$, $f(v_{10}) = 23$, or $f(v_{10}) = 13$, respectively, which contradicts **Fact 2** in all cases. Thus, this case cannot occur.

Subcase 1.2. $f(v_7) = 267$. This forces $\{1, 5\} \subseteq f(v_8)$. Then $f(v_8) \in \{135, 145\}$.

Subcase 1.2.1. $f(v_8) = 135$. This forces $f(v_9) = 246$, which in turn forces $f(v_{10}) = 135$. This contradicts the Subset Lemma.

Subcase 1.2.2. $f(v_8) = 145$. This forces $\{2, 3\} \subseteq f(v_9)$, which in turn forces $f(v_{10}) = 156$, which in turn forces $f(v_{11}) = 47$. Then $f(v_{11}) \cap f(v_4) = \emptyset$, since $f(v_4) = 356$. This is a contradiction.

By Subcases 1.2.1-1.2.2, Case 1.2 cannot occur. By Cases 1.1-1.2, Case 1 cannot occur.

Case 2. $f(v_5) = 147$. Observing further that $f(v_{11}) \in \{456, 457, 467\}$, and observing that 457 and 467 are equivalent, we consider the two subcases $f(v_{11}) = 456$ and $f(v_{11}) = 457$.

Subcase 2.1. $f(v_{11}) = 456$. Recall that $f(v_2) = 567$, $f(v_3) = 124$, and $f(v_4) = 356$. Since $5, 6 \notin f(v_{10})$ and $f(v_2) \cap f(v_{10}) \neq \emptyset$, this forces $7 \in f(v_{10})$. Similarly, since $f(v_4) \cap f(v_{10}) \neq \emptyset$, this forces $3 \in f(v_{10})$. Thus, $\{3, 7\} \subseteq f(v_{10})$. Further, since $1, 4 \notin f(v_6)$ and $f(v_6) \cap f(v_3) \neq \emptyset$, this forces $2 \in f(v_6)$. Since $f(v_5) \cap f(v_7) \in \{\{4, 7\}, \{1, 7\}, \{1, 4\}\}$, we consider these three subcases.

Subcase 2.1.1. $f(v_5) \cap f(v_7) = \{4, 7\}$. Since vertex labels cannot be duplicated, $f(v_7) \neq 147$. Because $2 \in f(v_6)$, it follows that $f(v_7) \neq 247$. Since $f(v_1) = 123$ and $f(v_7) \cap f(v_1) \neq \emptyset$, this forces $f(v_7) = 347$. But then $|f(v_{10}) \cap f(v_7)| = 2$, which contradicts the observation made in the paragraph following **Fact 3**.

Subcase 2.1.2. $f(v_5) \cap f(v_7) = \{1, 7\}$. Since $f(v_5) = 147$ and $f(v_8) \cap f(v_5) \neq \emptyset$, this forces $4 \in f(v_8)$. This in turn forces $\{5, 6\} \subseteq f(v_9)$, since $|f(v_{11}) \cap f(v_9)| = 2$. However, then, this forces $f(v_8) = 234$, so that $f(v_8) \cap f(v_2) = \emptyset$. This is a contradiction.

Subcase 2.1.3. $f(v_5) \cap f(v_7) = \{1, 4\}$. Since $f(v_7) \cap f(v_8) = \emptyset$, we know $1, 4 \notin f(v_8)$. But since $f(v_5) = 147$ and $f(v_8) \cap f(v_5) \neq \emptyset$, we have $7 \in f(v_8)$. Further, since $f(v_3) = 124$ and $f(v_8) \cap f(v_3) \neq \emptyset$, we have $2 \in f(v_8)$. Now, since $f(v_{11}) = 456$ and $f(v_{11}) \cap f(v_8) \neq \emptyset$, we have $f(v_8) \in \{257, 267\}$. Since these are equivalent, let $f(v_8) = 257$. Then this forces

$f(v_9) = 146 = f(v_7)$, which is a contradiction.

By Subcases 2.1.1-2.1.3, it is impossible that $f(v_{11}) = 456$, and so Subcase 2.1 cannot occur.

Subcase 2.2. $f(v_{11}) = 457$. Recalling that $f(v_2) = 567$, this forces $6 \in f(v_{10})$. Further, since $f(v_5) = 147$, this forces $2 \in f(v_6) \cap f(v_3)$ and $5 \in f(v_6) \cap f(v_{11})$, so that $\{2, 5\} \subseteq f(v_6)$. Further, this forces $1 \in f(v_{10})$. Thus, $\{1, 6\} \subseteq f(v_{10})$. We now consider three subcases for $f(v_5) \cap f(v_7)$.

Subcase 2.2.1. $f(v_5) \cap f(v_7) = \{4, 7\}$. This forces $f(v_8) \cap f(v_{11}) = \{5\}$ and $f(v_8) \cap f(v_5) = \{1\}$, so $\{1, 5\} \subseteq f(v_8)$. Since $|f(v_{11}) \cap f(v_9)| = 2$, this in turn forces $\{4, 7\} \subset f(v_9)$. Recalling that $f(v_4) = 356$, this further forces $3 \in f(v_9)$. Thus, $f(v_9) = 347$, which in turn forces $3 \in f(v_6)$, so $f(v_6) = 235$. Recalling that $f(v_4) = 356$, we then have $6 \in f(v_7)$ forced, so $f(v_7) = 467$. But then, since $f(v_1) = 123$, we have $f(v_7) \cap f(v_1) = \emptyset$, which is a contradiction.

Subcase 2.2.2. $f(v_5) \cap f(v_7) = \{1, 7\}$. This forces $4 \in f(v_8)$. Then, since $|f(v_{11}) \cap f(v_9)| = 2$, we must have $\{5, 7\} \subseteq f(v_9)$. Recalling $f(v_3) = 124$, since $f(v_9) \cap f(v_3) \neq \emptyset$, we have $2 \in f(v_9)$ forced, so $f(v_9) = 257$. This in turn forces $f(v_8) = 346$. But then this together with $\{2, 5\} \subseteq f(v_6)$ forces $f(v_7) = 17$, which contradicts **Fact 2**.

Subcase 2.2.3. $f(v_5) \cap f(v_7) = \{1, 4\}$. This forces $\{2, 7\} \subseteq f(v_8)$, which in turn forces $f(v_9) = 345$. But then this forces $f(v_8) = 267$, which together with $\{2, 5\} \subseteq f(v_6)$ forces $f(v_7) = 134$. But then recalling that $f(v_2) = 567$, we have $f(v_7) \cap f(v_2) = \emptyset$, which is a contradiction.

By Subcases 2.2.1-2.2.3, we conclude it is impossible to have $f(v_{11}) = 457$, and so Case 2.2 cannot occur.

By Cases 2.1–2.2, we conclude that $f(v_{11}) \notin \{456, 457\}$, and so Case 2 cannot occur. Thus, by Cases 1-2, there is no subset labeling $f : V(C_{11}) \to \mathcal{P}^*([7])$ such that all vertex labels are triples. Consequently, $\rho(C_{11}) = 8$. ∎

We now determine the value of $\rho(C_{12})$.

Proposition 4.3.6 $\rho(C_{12}) = 7$.

Proof. We have seen that $\rho(C_{12}) \in \{6, 7\}$. Thus, it remains to show that $\rho(C_{12}) \neq 6$. Let $C_{12} = (v_1, v_2, \ldots, v_{12}, v_1)$. Assume, to the contrary, that there exists a subset labeling

$f : V(C_{12}) \to \mathcal{P}^*([6])$ of C_{12}. Then $|f(v)| \geq 2$ for every vertex v of C_{12}. First, we make three observations.

(1) No vertex label can be a 4-set.

If (1) is false, and $|f(v)| = 4$ for some v, then we have $|f(u)| = 2$ forced for $u \in N(v)$, which contradicts Proposition 2.3.6.

(2) No two consecutive vertex labels can be 3-sets.

This is also an immediate consequence of Proposition 2.3.6.

(3) No two consecutive vertex labels can be 2-sets.

If (3) is false, then we may assume that $f(v_1) = 12$ and $f(v_2) = 34$. By Proposition 4.2.1, we may further assume that $1 \in f(v_i)$ for each (odd) integer $i \in \{3, 5, 7, 9, 11\}$ and $2 \in f(v_i)$ for each (even) integer $i \in \{4, 6, 8, 10\}$. Similarly, we may assume that $4 \in f(v_i)$ for each (odd) integer $i \in \{5, 7, 9, 11\}$ and $3 \in f(v_i)$ for each (even) integer $i \in \{4, 6, 8, 10, 12\}$. This implies that $\{2, 3\} \subseteq f(v_i)$ for $i \in \{4, 6, 8, 10\}$ and $\{1, 4\} \subseteq f(v_i)$ for $i \in \{5, 7, 9, 11\}$. By the Subset Lemma, we must have $|f(v_i)| = 3$ for $i \in \{4, 6, 8, 10\}$ with no duplicates. Now, since $\{1, 4\}$ is a subset of $f(v_5)$, $f(v_7)$, $f(v_9)$, and $f(v_{11})$, it follows that 5 and 6 are the only possibilities for the third element of the four labels $f(v_i)$ for $i \in \{4, 6, 8, 10\}$. However, then, at least two of $f(v_4)$, $f(v_6)$, $f(v_8)$, and $f(v_{10})$ are duplicates, which is a contradiction.

By (1)-(3), we conclude that the vertex labels of C_{12} alternate between 2-sets and 3-sets along the cycle C_{12}. Thus, we may assume that

$$\begin{aligned} |f(v_i)| &= 2 \text{ for each (odd) integer } i \in \{1, 3, 5, 7, 9, 11\} \\ |f(v_i)| &= 3 \text{ for each (even) integer } i \in \{2, 4, 6, 8, 10, 12\}. \end{aligned}$$

We may further assume that $f(v_1) = 12$, $f(v_2) = 345$, and $f(v_{12}) = 346$. By Proposition 4.2.1, we may also assume that $1 \in f(v_i)$ for each (odd) integer $i \in \{3, 5, 7, 9, 11\}$ and $2 \in f(v_i)$ for each (even) integer $i \in \{4, 6, 8, 10\}$. This forces $f(v_3) = 16$ and $f(v_{11}) = 15$. Since $f(v_3) = 16$, we have $6 \in f(v_i)$ for $i = \{6, 8, 10\}$ forced. Since $f(v_{11}) = 15$, we have $5 \in f(v_i)$ for $i = \{4, 6, 8\}$ forced. However, then, $f(v_6) = 256 = f(v_8)$, which contradicts the Subset Lemma. We conclude that $\rho(C_{12}) = 7$. ■

In summary, the values of $\rho(C_n)$ for $n \in \{9, 10, 11, 12\}$ are listed below.

$$\rho(C_n) = \begin{cases} 7 & \text{if } n = 9 \\ 6 & \text{if } n = 10 \\ 8 & \text{if } n = 11 \\ 7 & \text{if } n = 12 \end{cases}$$

4.4 Cycles of Order 13 through 18

In this section, we now determine the exact value of $\rho(C_n)$ for $13 \le n \le 18$. We begin with $\rho(C_n)$ for $n \in \{13, 15, 16, 17, 18\}$. The number $\rho(C_{14})$ will be determined afterward.

Proposition 4.4.1 *If $n \in \{13, 15, 16, 17, 18\}$, then $\rho(C_n) = 7$.*

Proof. If $n \ge 13$, then P_{12} is an induced subgraph of C_n, and so $\rho(C_n) \ge \rho(P_{12}) = 7$. Thus, if $n \in \{13, 14, 15, 16, 17, 18\}$, then $\rho(C_n) \ge \rho(P_{12}) = 7$. For $n = 13, 15, 16, 17, 18$, there is a subset labeling $f : V(C_n) \to \mathcal{P}^*([7])$ defined by

$$\begin{aligned}
\mathcal{S}_f(C_{13}) &= (123, 456, 127, 345, 167, 245, 367, 145, 236, 147, 256, 134, 567) \\
\mathcal{S}_f(C_{15}) &= (123, 456, 127, 345, 167, 234, 156, 247, 136, 257, 146, 357, 246, \\
&= 135, 467) \\
\mathcal{S}_f(C_{16}) &= (123, 456, 137, 245, 167, 235, 146, 257, 346, 125, 347, 156, 247, \\
&\quad 356, 124, 567) \\
\mathcal{S}_f(C_{17}) &= (123, 456, 127, 345, 167, 245, 136, 247, 135, 246, 157, 236, 147, \\
&\quad 256, 347, 125, 467) \\
\mathcal{S}_f(C_{18}) &= (456, 127, 345, 126, 347, 125, 367, 145, 236, 147, 235, 146, 257, \\
&\quad 136, 247, 135, 246, 137).
\end{aligned}$$

Hence, $\rho(C_n) = 7$ for $n = 13, 15, 16, 17, 18$. ■

Note that $\rho(C_{14}) \ge \rho(P_{12}) = 7$. However, it turns out that $\rho(C_{14}) = 8$. In order to establish this fact, we first present some preliminary results.

Theorem 4.4.2 *If $f : V(C_{14}) \to \mathcal{P}^*([7])$ is a subset labeling of C_{14}, then at most one vertex of C_{14} has a 2-set label.*

Proof. Let $C_{14} = (v_1, v_2, \ldots, v_{14}, v_1)$. Suppose, to the contrary, that there are at least two vertices u and v such that $|f(u)| = |f(v)| = 2$. By Proposition 4.2.3, it follows that $d(u, v) = 2k$ for some positive integer k. We first prove that u and v are the only two vertices of C_n with 2-set labels. Suppose, to the contrary, that there is a third vertex w such that $f(w)$ is a 2-set label. Note that since any two of these vertices are separated by an even distance, all three of their indices will have the same parity. We consider three cases:

Case 1. $d(u, v) = 2 = d(v, w)$. Without loss of generality, say $f(v_1) = 12, f(v_3) = 13$, and $f(v_5) = 14$. Then we have $1 \in f(v_7) \cap f(v_9) \cap f(v_{11}) \cap f(v_{13})$ and $\{2, 3, 4\} \subseteq f(v_8) \cap f(v_{10}) \cap f(v_{12})$. By the Subset Lemma, each of $f(v_8), f(v_{10})$ and $f(v_{12})$ must have cardinality at least 4. No label can be a 5-set label, since then its neighbors would have duplicate 2-set labels, which would contradict the Subset Lemma. Without loss of generality, suppose $f(v_8) = 2345, f(v_{10}) = 2346$, and $f(v_{12}) = 2347$. As a consequence of Proposition 2.3.6, we must have $|f(v_i)| \leq 2$ for any v_i in the neighborhood of a vertex with a 4-set label. Since singletons are not allowed, we conclude $|f(v_i)| = 2$ for $i \in \{7, 9, 11, 13\}$. But there are only six possible 2-set labels from [7] containing the element 1. Since $1 \in f(v_i)$ for all seven odd-indexed vertices v_i, this produces a contradiction.

Case 2. $d(u, v) = 4, d(v, w) = 2$. Without loss of generality, suppose $f(v_1) = 12, f(v_5) = 13$, and $f(v_7) = 14$. Then we have $1 \in f(v_i)$ for all odd i, $\{2, 3, 4\} \subseteq f(v_{10}) \cap f(v_{12})$, $\{2, 3\} \subseteq f(v_8)$, and $\{3, 4\} \subseteq f(v_2) \cap f(v_{14})$. By the Subset Lemma, we must have $|f(v_{10})| = 4 = |f(v_{12})|$. Without loss of generality, say $f(v_{10}) = 2345, f(v_{12}) = 2346$. This then forces $f(v_{11}) = 17$ and $f(v_{13}) = 15$, which in turn forces $f(v_8) = 2357$, which in turn forces $f(v_9) = 16$. But then we have $\{3, 4, 6, 7\} \subseteq f(v_2) \cap f(v_{14})$ forced, which leads to a contradiction of the Subset Lemma.

Case 3. $d(u, v) = 4 = d(v, w)$. Without loss of generality, suppose that $f(v_1) = 12, f(v_5) = 13$, and $f(v_9) = 14$. Then we have $1 \in f(v_i)$ for all i odd, $\{2, 3, 4\} \subseteq f(v_{12})$, $\{2, 3\} \subseteq f(v_8) \cap f(v_{10})$, $\{2, 4\} \subseteq f(v_4) \cap f(v_6)$, and $\{3, 4\} \subseteq f(v_2) \cap f(v_{14})$. By the Subset Lemma, we must have both $|f(v_2)|$ and $|f(v_{14})|$ at least 3. Suppose we have $5 \in f(v_2)$ and $6 \in f(v_{14})$. Then this forces $6 \in f(v_3)$ and $5 \in f(v_{13})$. We now consider two subcases.

Subcase 3.1. $f(v_3) = 16$. This forces $f(v_{11}) = 2346$, which in turn forces $f(v_{14}) = 3467$ by the Subset Lemma. This then forces $f(v_{13}) = 15$, which forces $\{2, 3, 5, 6\} \subseteq f(v_8) \cap f(v_{10})$. Then, similarly to Case 2, this leads to a contradiction.

Subcase 3.2. $f(v_3) = 167$. We consider two subcases for $f(v_{13})$.

Subcase 3.2.1. $f(v_{13}) = 15$. This forces $\{2, 3, 5\} \subseteq f(v_8) \cap f(v_{10})$. By the Subset Lemma, without loss of generality this then forces $f(v_8) = 2356, f(v_{10}) = 2357$. This in turn forces $f(v_7) = 17$. But then we have $f(v_7) \subseteq f(v_3)$, which contradicts the Subset Lemma.

Subcase 3.2.2. $f(v_{13}) = 157$. Notice now that since we have $\{2,4\} \subseteq f(v_6)$ and $\{2,3\} \subseteq f(v_8)$, and since we must have $f(v_7) \cap f(v_{12}) \neq \emptyset$, this forces $|f(v_{12})| = 4$. But this contradicts Proposition 2.3.6, since then we have $|f(v_{13}) \cup f(v_{14})| = 7$.

By Subcases 3.2.1 and 3.2.2, Subcase 3.2 cannot occur. By Subcases 3.1 and 3.2, Case 3 cannot occur.

Hence, we conclude that at most two vertices u and v can have 2-set labels. We now consider three cases for $d(u,v)$:

Case 1. $d(u,v) = 2$. Say $f(v_1) = 12$ and $f(v_3) = 13$. Then we have $1 \in f(v_i)$ for all i odd and $\{2,3\} \subset f(v_i)$ for $i \in \{6,8,10,12\}$. Without loss of generality, suppose $f(v_6) = 234, f(v_8) = 235, f(v_{10}) = 236, f(v_{12}) = 237$. This forces $f(v_5) = 1567$, which in turn forces $f(v_4) = 24$ and $f(v_6) = 23$, which contradicts Proposition 4.2.3.

Case 2. $d(u,v) = 4$. Say $f(v_1) = 12$ and $f(v_5) = 13$. Then we have $1 \in f(v_i)$ for all i odd and $\{2,3\} \subset f(v_i)$ for all i even such that v_i is not adjacent to v_1 or v_5. Without loss of generality, suppose $f(v_8) = 234, f(v_{10}) = 235, f(v_{12}) = 236$. This would force $f(v_3) = 1456$, which would then force $f(v_2) = 37$ and $f(v_4) = 27$, which contradicts Proposition 4.2.3.

Case 3. $d(u,v) = 6$. Say $f(v_1) = 12$ and $f(v_7) = 13$. Then we have $1 \in f(v_i)$ for all i odd and $\{2,3\} \subset f(v_i)$ for all i even such that v_i is not adjacent to v_1 or v_7. Without loss of generality, say $f(v_4) = 234, f(v_{10}) = 235, f(v_{12}) = 236$. Then this forces $\{1,5,6\} \subseteq f(v_3) \cap f(v_5)$. Since we are unable to assign distinct 4-set labels to $f(v_3)$ and $f(v_5)$, this contradicts the Subset Lemma.

By Cases 1-3, we conclude that no more than one vertex can have a 2-set label. This concludes the proof. ∎

Proposition 4.4.3 *If $f : V(C_{14}) \to \mathcal{P}^*([7])$ is a subset labeling of C_{14}, then $|f(v)| \neq 4$ for every vertex v of C_{14}.*

Proof. Let $C_{14} = (v_1, v_2, \ldots, v_{14}, v_1)$ and suppose, to the contrary, that $|f(v)| = 4$ for some vertex v of C_{14}. We may assume that $f(v_1) = 1234$. Thus, each of $f(v_2)$ and $f(v_{14})$ is a subset of $\{5,6,7\}$. Since we have $f(v_2) \not\subseteq f(v_{14})$ and $f(v_{14}) \not\subseteq f(v_2)$ by the Subset Lemma, it follows that each of $f(v_2)$ and $f(v_{14})$ is a 2-set label. This is impossible by Theorem 4.4.2. ∎

Proposition 4.4.4 *If* $f : V(C_{14}) \to \mathcal{P}^*([7])$ *is a subset labeling of* C_{14}*, then* $|f(v)| \neq 2$ *for every vertex* v *of* C_{14}*.*

Proof. Let $C_{14} = (v_1, v_2, \ldots, v_{14}, v_1)$. Assume, to the contrary, that there is a vertex v with a 2-set label, say $f(v_1) = 12$. By Proposition 4.4.3 and Theorem 4.4.2, every other vertex label is a 3-set label. We may assume that

$$f(v_2) = 345,\ f(v_3) = 167,\ f(v_4) = 234,\ f(v_5) = 156, \text{ and } f(v_6) = 247.$$

This implies that $1 \in f(v_i)$ for $i \in \{7, 9, 11, 13\}$ and $2 \in f(v_i)$ for $i \in \{8, 10, 12\}$. Among the triples containing either 1 or 2, notice that in addition to the labels used, we have excluded the set

$$\{127,\ 157,\ 267,\ 235,\ 245,\ 237,\ 347\}$$

We now have either $f(v_7) = 135$ or $f(v_7) = 136$ forced. We consider these two cases.

Case 1. $f(v_7) = 135$. This forces $(f(v_8), f(v_9), f(v_{10}), f(v_{11})) = (246, 137, 256, 347)$. Since 347 has been eliminated, this is a contradiction.

Case 2. $f(v_7) = 136$. This forces $f(v_8) = 257$, since the elements of $f(v_8)$ must come from $\{2, 4, 5, 7\}$, but 245 and 247 are excluded. Then the elements of $f(v_9)$ must come from $\{1, 3, 4, 6\}$. There are two possible subcases.

Subcase 2.1. $f(v_9) = 134$. This forces $f(v_{10}) = 256$, since the elements must come from $\{2, 5, 6, 7\}$, but 257 and 267 are excluded. This then forces $f(v_{11}) = 347$. However, then, $f(v_{11}) \cap f(v_5) = \emptyset$, which is a contradiction.

Subcase 2.2. $f(v_9) = 146$. The elements of $f(v_{10})$ must then be drawn from $\{2, 3, 5, 7\}$, yet all three triples 235, 237, and 257 have been excluded, which is a contradiction.

By Cases 1 and 2, we conclude that $|f(v)| \neq 2$ for every vertex v of C_n. ■

We will now show that $\rho(C_{14}) = 8$.

Theorem 4.4.5 $\rho(C_{14}) = 8$

Proof. The subset labeling $g : V(C_{14}) \to \mathcal{P}^*([8])$ defined by

$$\mathcal{S}_g(C_{14}) = (123, 456, 1278, 345, 167, 245, 367, 148, 256, 347, 156, 247, 135, 468)$$

shows that $\rho(C_{14}) \leq 8$. Next, we show that $\rho(C_{14}) \geq 8$. Assume, to the contrary, that there is a subset labeling $f : V(C_{14}) \to \mathcal{P}^*([7])$. First, we observe the following three facts:

Fact 1: No vertex of C_{14} can have a 5-set label or a 6-set label.

Fact 2: No vertex of C_{14} can have a 4-set label. This follows from Proposition 4.4.3.

Fact 3: No vertex of C_{14} can have a 2-set label. This follows from Proposition 4.4.4.

By **Facts 1-3**, every vertex has a 3-set label. We may assume

$$f(v_1) = 123, f(v_2) = 456, f(v_3) = 127, f(v_4) = 345.$$

This further eliminates the triples $137, 237, 346, 356$ and forces $7 \in f(v_{14})$ and $6 \in f(v_5)$. We note that $f(v_5) \in \{267, 167, 126\}$, but 167 and 267 are equivalent. Thus, without loss of generality, we may consider only the cases $f(v_5) = 267$ and $f(v_5) = 126$.

Case 1. $f(v_5) = 267$. Then $f(v_i)$ is defined for $1 \leq i \leq 5$ as follows:

$$f(v_1) = 123, 456, 127, 345, 267 = f(v_5).$$

with eliminated triples $137, 237, 346, 356, 126, 167$. We note that $f(v_6) \in \{134, 135, 145\}$. Since the cases $f(v_6) = 134$ and $f(v_6) = 135$ are equivalent, it suffices to consider the two non-equivalent subcases $f(v_6) = 134$ and $f(v_6) = 145$.

Subcase 1.1. $f(v_6) = 134$. The label 567 has been eliminated for $f(v_7)$, so $f(v_7) \in \{256, 257\}$. Since the labels 256 and 257 are equivalent, we may consider only the case $f(v_7) = 257$. Since 136 and 346 have been eliminated for $f(v_8)$, we see that $f(v_8) = 146$. Then $f(v_i)$ is defined for $1 \leq i \leq 8$ as follows:

$$f(v_1) = 123, 456, 127, 345, 267, 134, 257, 146 = f(v_8).$$

The following triples have been eliminated thus far for $(f(v_9), f(v_{10}), \ldots, f(v_{14}))$:

$$\{137, 237, 346, 356, 126, 167, 135, 145, 256, 567, 136, 346\}$$

We have $\{3, 5\} \subset f(v_9)$ forced, which in turn forces $4 \in f(v_{10})$. Further, notice that $f(v_{14}) \in \{457, 467\}$. We consider these two subcases.

Subcase 1.1.1. $f(v_{14}) = 457$. This forces $f(v_{13}) = 236$, which forces $\{1, 7\} \subset f(v_{12})$. The third element must be either 5 or 4. If $f(v_{12}) = 157$, then this forces $f(v_{11}) = 236$, which has already been used for $f(v_{13})$. This is a contradiction. Hence, we must have $f(v_{12}) = 147$, which forces $f(v_{11}) = 235$. But this has been eliminated by $f(v_8) = 146$, so this also leads to a contradiction.

Subcase 1.1.2. $f(v_{14}) = 467$. This forces $7 \in f(v_9)$, so that $f(v_9) = 357$. This then forces $\{2, 4\} \subset f(v_{10})$, where the third element must be either 1 or 6. If we choose $f(v_{10}) = 124$, this forces $f(v_{11}) = 356$, which has been eliminated. This is a contradiction. If we choose $f(v_{10}) = 246$, this forces $f(v_{11}) = 135$, which has also been eliminated. This is also a contradiction.

By Subcases 1.1.1-1.1.2, we conclude that $f(v_6) = 134$ is not possible.

Subcase 1.2. $f(v_6) = 145$. Then $f(v_i)$ is defined for $1 \leq i \leq 6$:

$$f(v_1) = 123, 456, 127, 345, 267, 145 = f(v_6)$$

and the triples eliminated for $f(v_7), f(v_8), \ldots, f(v_{14})$ are

$$\{137, 237, 346, 356, 126, 167, 134, 135\}.$$

We consider the two possible cases $f(v_7) \in \{236, 367\}$.

Subcase 1.2.1. $f(v_7) = 236$. This choice further eliminates the triple 457 from consideration for $f(v_{14})$, leaving only the choices 467 and 567. Since these choices are equivalent, we will consider only the case $f(v_{14}) = 467$. There are two non-equivalent choices for $f(v_8)$, namely 147 and 157.

Subcase 1.2.1.1. $f(v_8) = 147$. We now have $f(v_i)$ defined as follows for $i = 14, 1, 2, \ldots, 8$:

$$f(v_{14}) = 467, 123, 456, 127, 345, 267, 145, 236, 147 = f(v_8).$$

Since $f(v_{13}) = 125$ is forced, as is $f(v_9) = 256$, this forces $f(v_{12}) = 467$. Since this label has already been used, a contradiction is produced.

Subcase 1.2.1.2. $f(v_8) = 157$. We now have $f(v_i)$ defined as follows for $i = 14, 1, 2, \ldots, 8$:

$$f(v_{14}) = 467, 123, 456, 127, 345, 267, 145, 236, 147 = f(v_8).$$

This forces $\{2,4\} \subset f(v_9)$. We now consider the two options for $f(v_{13})$, namely 125 and 235. If we choose $f(v_{13}) = 125$, this forces $f(v_{12}) = 347$, which forces $\{5,6\} \subset f(v_{11})$, which, in turn, forces $\{5,6\} \subset f(v_{10})$. Together with $\{2,4\} \subset f(v_9)$, this forces $f(v_{10}) = 137$, which has been eliminated. This is a contradiction. If, instead, $f(v_{13}) = 235$, this forces $f(v_{12}) = 146$, which forces $f(v_{11}) = 257$. Together with $\{2,4\} \subset f(v_9)$, this forces $f(v_{10}) = 136$. But now there is no possibility for the third element of $f(v_9)$. This is a contradiction.

Subcase 1.2.2. $f(v_7) = 367$. We consider the two non-equivalent cases for $f(v_{14})$, namely 457 and 467.

Subcase 1.2.2.1. $f(v_{14}) = 457$. We now have $f(v_i)$ defined as follows for $i = 14, 1, 2, \ldots, 7$:

$$f(v_{14}) = 457, 123, 456, 127, 345, 267, 145, 367 = f(v_7).$$

If $f(v_8) = 245$, then $f(v_9) = 1367$, which is impossible since no label can be a 4-set. Since the case when $f(v_8) = 124$ is equivalent to that of $f(v_8) = 125$, we consider only $f(v_8) = 124$. This choice forces $(f(v_9), f(v_{10}), f(v_{11})) = (357, 246, 157)$, which forces $3 \in f(v_{12})$. But $f(v_{14}) = 457$ forces $3 \in f(v_{13})$. This is a contradiction.

Subcase 1.2.2.2. $f(v_{14}) = 467$. This forces $\{2,4\} \subset f(v_8)$. We now have $f(v_i)$ defined as follows for $i = 14, 1, 2, \ldots, 7$:

$$f(v_{14}) = 467, 123, 456, 127, 345, 267, 145, 367 = f(v_7).$$

This forces $f(v_{13}) = 235$, $\{1,4\} \subset f(v_{12})$, and $5 \in f(v_{11})$. We consider the two choices 147, 146 for $f(v_{12}$. If $f(v_{12}) = 147$, this forces $f(v_{11}) = 256$, which forces $f(v_{10}) = 347$. Together with $\{2,4\} \subset f(v_8)$, this forces $f(v_9) = 156$. But now no choices remain for the third element of $f(v_8)$, which is a contradiction. If, instead, $f(v_{12}) = 146$, then this forces $\{5,7\} \subset f(v_{11})$. The two options for $f(v_{11})$ are 257 and 357. If $f(v_{11}) = 257$, this forces $f(v_{10}) = 136$, which together with $\{2,4\} \subset f(v_8)$ makes it impossible to label $f(v_9)$. This is a contradiction. If $f(v_{11}) = 357$, this forces $f(v_8) = 245$, which forces $f(v_9) = 136$. But this makes it impossible to label $f(v_{10})$. This is a contradiction.

Hence, we have $f(v_{12}) \neq 147, 146$, which implies that $f(v_{14}) \neq 467$.

By Subcases 1.2.1-1.2.2, we conclude that $f(v_6) \neq 145$ and by Subcases 1.1-1.2, we conclude that $f(v_5) \neq 267$.

Case 2. $f(v_5) = 126$. We note that the label 457 has been eliminated for $f(v_6)$, so $f(v_6) \in \{347, 357\}$. These two subcases are equivalent. Hence, without loss of generality, we consider only the case $f(v_6) = 347$. We now have $f(v_i)$ defined for $1 \le i \le 6$ as follows:

$$f(v_1) = 123, 456, 127, 345, 126, 347 = f(v_6).$$

We note that $f(v_7) \in \{125, 156, 256\}$. The cases 156 and 256 are equivalent. Thus, it suffices to consider only the non-equivalent cases $f(v_7) = 156$ and $f(v_7) = 125$.

Subcase 2.1. $f(v_7) = 156$. We now have $f(v_i)$ defined as follows for $1 \le i \le 7$:

$$f(v_1) = 123, 456, 127, 345, 126, 347, 156 = f(v_7).$$

We now consider each of the non-equivalent subcases $f(v_{14}) \in \{457, 467, 567\}$.

Subcase 2.1.1. $f(v_{14}) = 457$. Now, the only choices for $f(v_8)$ are 234 and 247, which are equivalent. Hence, we consider $f(v_8) = 234$ only. We now have $f(v_i)$ defined as follows for $i = 14, 1, 2, \ldots, 8$:

$$f(v_{14}) = 457, 123, 456, 127, 345, 126, 347, 156, 234 = f(v_8).$$

This labeling forces $(f(v_9), f(v_{10}), f(v_{11}), f(v_{12})) = (157, 246, 135, 247)$. Since 247 has been eliminated, this is a contradiction.

Subcase 2.1.2. $f(v_{14}) = 467$. This forces $\{3, 5\} \subset f(v_{13})$. If we choose $f(v_{13}) = 135$, this forces $f(v_{12}) = 246$, which has been eliminated. Hence, $f(v_{13}) = 235$ is forced, which forces $\{1, 4\} \subset f(v_{12})$. In this subcase, the choices 234 and 247 for $f(v_8)$ are not equivalent. If $f(v_8) = 247$, this forces $f(v_9) = 136$, which forces $\{2, 5\} \subset f(v_{10})$. The label $f(v_{11}) = 367$ is now forced, but this eliminates all choices for $f(v_{12})$, which is a contradiction. If instead $f(v_8) = 234$, this forces $f(v_9) = 157$, which forces $\{2, 6\} \subset f(v_{10})$, which, in turn, forces $1 \in f(v_{11})$. But this is a contradiction since $1 \in f(v_{12})$. We conclude that $f(v_{14}) \neq 467$.

Subcase 2.1.3. $f(v_{14}) = 567$. This forces $f(v_8) = 147$, which forces $\{2, 3\} \subset f(v_9)$, which forces $1 \in f(v_{10})$. This choice of $f(v_{14})$ also forces $\{1, 4\} \subset f(v_{13})$. The two choices for $f(v_{13})$ are 124 and 134. If $f(v_{13}) = 124$, this forces $f(v_{12}) = 367$, which forces $\{4, 5\} \subset f(v_{11})$. We now have $f(v_{10}) = 167$ forced, which has been eliminated. This is a contradiction. If instead $f(v_{13}) = 134$, this forces $f(v_{12}) = 257$. Together with $1 \in f(v_{10})$, this forces $f(v_{11}) = 346$, which has been eliminated. This is a contradiction.

By Subcases 2.1.1-2.1.3, we conclude that $f(v_7) \neq 156$.

Subcase 2.2. $f(v_7) = 125$. As the choices 457 and 467 have been eliminated for $f(v_{14})$, the remaining case is $f(v_{14}) = 567$. Further, since the choices 346 and 467 have been eliminated for $f(v_8)$, the only remaining case is $f(v_8) = 367$. Thus, we have defined $f(v_i)$ for $i = 14, 1, 2, \ldots, 8$ as follows:

$$f(v_{14}) = 567, 123, 456, 127, 345, 126, 347, 125, 367 = f(v_8)$$

Beginning from $f(v_8)$, we have $\{4,5\} \subset f(v_9), \{3,6\} \subset f(v_{10})$, and $7 \in f(v_{11})$. Beginning from $f(v_{14})$, we have $\{3,4\} \subset f(v_{13})$ and $\{5,7\} \subset f(v_{12})$. But this is a contradiction, since we cannot have $7 \in f(v_{11}) \cap f(v_{12})$.

By Subcases 2.1-2.2, we conclude that $f(v_6) \neq 347$. Hence, $f(v_5) \neq 126$.

By Cases 1-2, we conclude that $\rho(C_{14}) \geq 8$. Thus, $\rho(C_{14}) = 8$. ■

In summary:

$$\rho(C_n) = \begin{cases} 7 & \text{if } n = 13 \\ 8 & \text{if } n = 14 \\ 7 & \text{if } 15 \leq n \leq 18. \end{cases}$$

We summarize the values of $\rho(C_n)$ for $3 \leq n \leq 18$ below:

$$\rho(C_n) = \begin{cases} 3 & \text{if } n = 3 \\ 2 & \text{if } n = 4 \\ 5 & \text{if } n = 5, 6 \\ 7 & \text{if } n = 7 \\ 6 & \text{if } n = 8 \\ 7 & \text{if } n = 9 \\ 6 & \text{if } n = 10 \\ 8 & \text{if } n = 11 \\ 7 & \text{if } n = 12, 13 \\ 8 & \text{if } n = 14 \\ 7 & \text{if } 15 \leq n \leq 18. \end{cases}$$

We have seen that $\rho(P_n) = 7$ for $12 \leq n \leq 22$, thus $\rho(C_n) \geq \rho(P_{12}) = 7$ for $n \geq 13$. By Theorem 3.4.8, it follows that if $n \geq 24$, then $\rho(C_n) \geq \rho(P_{23}) = 8$. In fact, we have the following conjecture:

Conjecture 4.4.6 *If $n \geq 19$, then $\rho(C_n) \geq 8$.*

We conclude this chapter with a connection between the subset labeling problem and a closely related problem in combinatorics, showing how our work on cycles in particular has led to the reopening of a highly nontrivial question for further research.

4.5 Subset Labelings and Almost Intersecting Set Families

Given any positive integer r, an *intersecting family* $\mathcal{F}$ of sets in $\mathcal{P}([r])$ is a collection of sets such that for any two sets $A, B \in \mathcal{F}$, A and B have nonempty intersection. In 1961, Erdős, Ko, and Rado proved the following theorem on intersecting set families:

The Erdős-Ko-Rado Theorem. Let n and k be positive integers with $n \geq 2k$. If $\mathcal{F}$ is a family of distinct k-element subset of $[n]$ such that each pair of subsets has nonempty intersection, then the number of sets in $\mathcal{F}$ is at most $\binom{n-1}{k-1}$.

In [8], co-authors Dániel Gerbner, Nathan Lemons, Cory Palmer, Balázs Patkós, and Vajk Szécsi introduced the related concept of an "almost" intersecting set family. A family $\mathcal{F}$ is defined as *ℓ-almost intersecting* if for any set $F \in \mathcal{F}$, there are exactly ℓ other sets in $\mathcal{F}$ disjoint from F. A family is defined as $\leq \ell$-almost intersecting if for any set $F \in \mathcal{F}$, there are at most ℓ other sets in $\mathcal{F}$ disjoint from F. They then introduced the problem of establishing analogous results to the Erdős-Ko-Rado Theorem for almost intersecting set families.

We can relate this problem to the subset labeling problem by noting that if $\ell = \Delta(G)$ for a graph G, then the list of labels in any subset labeling of the graph will constitute a $\leq \ell$-almost intersecting set family (since $\deg v \leq \Delta(G)$ for any v, and the label assigned to v must be disjoint from the labels assigned to the neighbors of v). In particular, for any path P_n, the list of labels in a subset labeling is a ≤ 2-almost intersecting set family, since each end-vertex label is disjoint from exactly one other vertex label, while each interior vertex label is disjoint from exactly two other vertex labels. Further, for a cycle C_n, the list of labels in a subset labeling constitutes a 2-almost intersecting set family, since each vertex label is disjoint from exactly two other vertex labels. Since the empty set cannot be included in a nontrivial ℓ-almost intersecting family, we will use the notation $\mathcal{P}^*([r])$ for a positive integer r.

For each positive integer r, a "tight" upper bound (i.e., upper bound with attainment) for the size of a 1-almost intersecting set family from $\mathcal{P}^*([r])$ was established in [8]. The authors also claimed to have done the same for the size of a 2-almost intersecting family from $\mathcal{P}^*([r])$, presenting what was believed to be the best (largest) possible construction for such a family. It turns out that their calculations were in error, as we will show. The problem of establishing a best possible construction thus remains open for further research and is connected to the problem of establishing a general formula for the subset index of a cycle C_n as a function of n.

We say a set family $\mathcal{F}$ is *Sperner* if no set $A \in \mathcal{F}$ contains another set $B \in \mathcal{F}$. For example, $\mathcal{F} = \{12, 34, 25, 13, 45\}$ is Sperner. Observe that when the sets in this particular collection $\mathcal{F}$ are assigned as vertex labels to the cycle C_5 in the order above, they constitute a subset labeling $f : V(C_5) \to \mathcal{P}^*([5])$. Indeed, by the Subset Lemma, we can assert in general that for $n \geq 5$, the list of labels in a subset labeling of a cycle C_n is a 2-almost intersecting Sperner family. As we will see, the construction for a 2-almost intersecting family in $\mathcal{P}^*([r])$ described in [8] is non-Sperner. But our counterexamples in the case $r = 7$ were obtained while generating subset labelings $f : V(C_n) \to \mathcal{P}^*([7])$ for $n = 17, 18$. As observed, these constituted necessarily Sperner families. This suggests that the true best possible construction for a 2-almost intersecting family may be Sperner, although a systematic process for producing such a family in general is still unknown.

It was claimed in [8] that for any integer $r \geq 3$, the best possible upper bound for the size $|\mathcal{F}|$ of a 2-almost intersecting family $\mathcal{F}$ in $\mathcal{P}^*([r])$ is given by:

$$|\mathcal{F}| \leq \begin{cases} 2\binom{r-2}{\frac{r-2}{2}} & \text{if } r \text{ is even} \\ 4\binom{r-3}{\lfloor r/2 \rfloor - 2} & \text{if } r \text{ is odd} \end{cases}$$

In [8], a family attaining this bound was constructed in the following way: If $r = 2k+2$, then let $\mathcal{F}_1, \mathcal{F}_2$ be a partition of the $\binom{2k}{k}$ k-element sets in $[2k]$ such that $A \in \mathcal{F}_1$ if and only if $\overline{A} \in \mathcal{F}_2$. Then the following family possesses the required properties and has size $2\binom{2k}{k}$:

$$\mathcal{F} = \mathcal{F}_1 \cup \{A \cup \{2k+1\} : A \in \mathcal{F}_1\} \cup \mathcal{F}_2 \cup \{A \cup \{2k+2\} : A \in \mathcal{F}_2\}$$

Similarly, if $r = 2k+1$, the following family possesses the required properties and has size $4\binom{2k-2}{k-2}$: Let $\mathcal{F}_1 = \{A \in \binom{[2k-1]}{k-1} : a \in A\}$, $\mathcal{F}_2 = \{A \in \binom{[2k-1]}{k} : a \notin A\}$ for some fixed $a \in [r]$, and define

$$\mathcal{F} = \mathcal{F}_1 \cup \{A \cup \{2k\} : A \in \mathcal{F}_1\} \cup \mathcal{F}_2 \cup \{A \cup \{2k+1\} : A \in \mathcal{F}_2\}.$$

To illustrate this for an even r, say $r = 6$. Then $k = 2$, and the following family of size 12 is generated by this construction:

$$\mathcal{F}_1 = \{12, 13, 23\}, \mathcal{F}_2 = \{34, 24, 14\}$$
$$\mathcal{F} = \mathcal{F}_1 \cup \{125, 135, 235\} \bigcup \mathcal{F}_2 \cup \{346, 246, 146\}$$

To illustrate this for an odd r, suppose $r = 7$. Then $k = 3$, and the following family of size 16 is generated by this construction, where we have chosen to fix $a = 1$:

$$\mathcal{F}_1 = \{12, 13, 14, 15\}, \mathcal{F}_2 = \{234, 235, 245, 345\}$$
$$\mathcal{F} = \mathcal{F}_1 \cup \{126, 136, 146, 156\} \bigcup \mathcal{F}_2 \cup \{2347, 2357, 2457, 3457\}$$

In justifying their claim that this construction is best possible, the authors establish the following additional notation: Let $\mathcal{F} \subset \mathcal{P}^*([r])$ be a 2-almost intersecting family. Let $\mathcal{F}_2^U$ denote the subfamily of sets in $\mathcal{F}$ which contain some other set in $\mathcal{F}$, let $\mathcal{F}_2^L$ denote the subfamily of sets in $\mathcal{F}$ which are contained in some other set in $\mathcal{F}$, and let $\mathcal{F}_1 = \mathcal{F} - (\mathcal{F}_2^U \cup \mathcal{F}_2^L)$. Then we may write $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2^U \cup \mathcal{F}_2^L$. The authors then define a *good* 2-almost intersecting set family $\mathcal{F}$ as a family satisfying the following two properties:

- ⋆ $A \in \mathcal{F}_2^U$ implies $\overline{A} \in \mathcal{F}_2^U$.
- ⋆ For any $A_1, A_2 \in \mathcal{F}$ with $A_1 \subset A_2$, we have $|A_2 - A_1| = 1$.

The families constructed above satisfy these conditions. When $r = 6$, we have

$$\mathcal{F}_2^U = \{125, 135, 235, 346, 246, 146\}$$
$$\mathcal{F}_2^L = \{12, 13, 23, 34, 24, 14\}$$

When $r = 7$, we have

$$\mathcal{F}_2^U = \{126, 136, 146, 156, 2347, 2357, 2457, 3457\}$$
$$\mathcal{F}_2^L = \{12, 13, 14, 15, 234, 235, 245, 345\}$$

However, in the case of a Sperner family $\mathcal{F}$, we have $\mathcal{F}_1 = \mathcal{F}$, since by definition there are no subset relations among the sets in the collection. Yet any nontrivial Sperner family $\mathcal{F}$ is trivially "good," since the statement "If $A \in \mathcal{F}_2^U$, then $\overline{A} \in \mathcal{F}_2^U$" is not false. Similarly, even though there are no two sets $A_1, A_2 \in \mathcal{F}$ such that $A_1 \subset A_2$, it is not false to say that *if* $\mathcal{F}$ contained two such sets, then we would have $|A_2 - A_1| = 1$.

We now consider the upper bound the authors obtain in the course of their proof attempt, which, it turns out, is not equivalent to the bound in the theorem statement. The bound in the proof is established in terms of the following quantities a_k and b_k, which are derived from an inequality the authors separately prove for "good" intersecting families. We omit the full details of the derivation here:

$$a_k = \frac{2k!(r-k)! - 2k!(r-k-1)!}{r!}$$

$$b_k = \frac{2k!(r-k)!+2(k+1)!(r-k-1)!-2k!(r-k-1)!-k!(r-k-2)!-(k+1)!(r-k-2)!-k!(r-k-1)!+k!(r-k-2)!}{r!}$$

The authors claim that if $m = \min\{a_k, \frac{b_k}{2} : 1 \le k \le n-1\}$, then $|\mathcal{F}| \le \frac{1}{m}$. They then prove separately that as k varies from 1 to $n-1$, the quantities a_k and b_k are each minimized when $k = \lfloor r/2 \rfloor$, then obtain by substitution that $a_{\lfloor r/2 \rfloor} > \frac{b_{\lfloor r/2 \rfloor}}{2}$. From this, they conclude that $\frac{1}{m} = \frac{2}{b_{\lfloor r/2 \rfloor}}$.

Here, the authors make the erroneous final conclusion that the size of the families in their proposed best possible construction is exactly $\frac{2}{b_{\lfloor r/2 \rfloor}}$, which would make the statement $|\mathcal{F}| \le \frac{1}{m}$ equivalent to the statement of the theorem. In fact, these expressions are not equivalent in general for r odd, as we show by example in the case $r = 7$. Substitution into a_3 yields $\frac{1}{23\frac{1}{3}}$, and substitution into b_3 yields $\frac{1}{12}$. Then we have $\frac{b_3}{2} = \frac{1}{24}$, so $m = \frac{b_3}{2}$, and we have $|\mathcal{F}| \le \frac{1}{m} = 24$. On the other hand, substitution into $4\binom{r-3}{\lfloor r/2 \rfloor - 2}$ yields 16.

In fact, it is false that the proposed construction is best possible in general, as we show by counterexample for $r = 7$. Indeed, we have already produced 2-almost intersecting Sperner families of size 18 and 17 in $\mathcal{P}^*([7])$, namely, the lists of labels in the subset labelings $f : V(C_{18}) \to \mathcal{P}^*([7])$ and $f : V(C_{17}) \to \mathcal{P}^*([7])$:

$$\mathcal{S}_f(C_{18}) = (456, 127, 345, 126, 347, 125, 367, 145, 236, 147, 235, 146, 257, 136, 247, 135, 246, 137)$$

$$\mathcal{S}_f(C_{17}) = (123, 456, 127, 345, 167, 245, 136, 247, 135, 246, 157, 236, 147, 256, 347, 125, 467)$$

In general, it is clear that the largest n such that $\rho(C_n) = r$ is bounded above by the size of the largest possible 2-almost intersecting Sperner family in $\mathcal{P}^*([r])$. These problems thus suggest themselves for further research:

Problem 4.5.1 *Given any integer r, is the best possible construction for a 2-almost intersecting family of sets in $\mathcal{P}^*([r])$ a Sperner family?*

Problem 4.5.2 *Given any integer r, is the size of the best possible construction for a 2-almost intersecting Sperner family of sets in $\mathcal{P}^*([r])$ equal to or greater than the largest n such that $\rho(C_n) = r$?*

Chapter 5

Subset Labelings of Prisms and Grids

5.1 Introduction

The *Cartesian product* $G \square H$ of two graphs G and H has vertex set $V(G \square H) = V(G) \times V(H)$. Two distinct vertices (u, v) and (x, y) of $G \square H$ are adjacent if either (1) $u = x$ and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(G)$. In this chapter, we explore the subset indices of the *prism* $C_n \square K_2$ and the *grid* $P_n \square P_q$ in the special case where $q = 2$. We establish bounds for these indices in terms of the subset indices of cycles and paths, respectively.

We begin with a couple of preliminary results.

Proposition 5.1.1 *Let G be a graph containing a clique K_r and let f be a subset labeling of G. If $v \in V(G)$ such that v is not adjacent to any vertex of K_r, then $|f(v)| \geq r$.*

Proof. Suppose, to the contrary, that there exists some vertex $v \in V(G)$ such that v is not adjacent to any vertex of K_r and $|f(v)| < r$. Then, by the Pigeonhole Principle, there must be some element $a \in f(v)$ such that $a \in f(u_i) \cap f(u_j)$ for some two vertices $u_i, u_j \in V(K_r)$. Yet, since K_r is a clique, we have $f(u_i) \cap f(u_j) = \emptyset$ for any two vertices $u_i, u_j \in V(K_r)$. This is a contradiction. ■

Theorem 5.1.2 *Let $G = K_n \square K_2$. For $n \geq 2$, $\rho(G) = n(n-1)$.*

Proof. To prove $\rho(G) \geq n(n-1)$, suppose, to the contrary, that there exists a subset labeling $f : V(G) \to \mathcal{P}^*([n(n-1)-1])$. Denote the two copies of K_n in the product as K_n and $K_n^{'}$. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_n^{'}) = \{v_1, v_2, \ldots, v_n\}$, and let $E(G) -$

$E(K_n) - E(K_n^{'}) = \{u_i v_i\}_{i=1}^{n}$. By the Pigeonhole Principle, without loss of generality, there exists some vertex $x \in V(K_n)$ such that $|f(x)| < n-1$. Yet $|V(G) - N[x]| = n-1$, and the vertices of $V(G) - N[x]$ form the sub-clique $K_{n-1}^{'}$. Hence $|f(x)| \geq n-1$. By Proposition 5.1.1, this is a contradiction. Thus, $\rho(K_n \,\square\, K_2) \geq n(n-1)$.

To prove $\rho(G) \leq n(n-1)$, we present the following labeling $f : V(G) \to \mathcal{P}^*([n(n-1)])$, displayed here in the form of an $n \times n-1$ matrix A. For $1 \leq i \leq n$, $f(u_i)$ is the set of elements in the ith row of A, consisting of the integers in the set $[(i-1)(n-1)+1, i(n-1)]$. For $1 \leq j \leq n-1$, $f(v_j)$ is the set of elements in the jth column of A except for that element in the jth row. Finally, $f(v_n) = [n(n-1)] - \bigcup_{i=1}^{n-1} f(v_i)$.

$$A = \begin{bmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 \\ n & n+1 & n+2 & \dots & 2n-3 & 2(n-1) \\ 2(n-1)+1 & 2(n-1)+2 & 2(n-1)+3 & \dots & 2(n-1)+(n-2) & 3(n-1) \\ 3(n-1)+1 & 3(n-1)+2 & 3(n-1)+3 & \dots & 3(n-1)+(n-2) & 4(n-1) \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ (n-1)(n-1)+1 & (n-1)(n-1)+2 & (n-1)(n-1)+3 & \dots & (n-1)(n-1)+(n-2) & n(n-1) \end{bmatrix}$$

We claim this labeling is a subset labeling. To begin with, it is clear by construction that $f(u_i) \cap f(u_j) = \emptyset$ for any two i, j, and similarly $f(v_i) \cap f(v_j) = \emptyset$ for any two i, j. In particular, note that $f(v_n)$ consists precisely of the $n-1$ elements deleted from the lists for each of $f(v_1), f(v_2), \dots, f(v_{n-1})$. Further, for $1 \leq i \leq n-1$, each $f(v_i)$ selects exactly the jth element from each of $f(u_j)$ except for $f(u_i)$, thus ensuring both that $f(v_i) \cap f(u_i) = \emptyset$ and that $f(v_i) \cap f(u_j) \neq \emptyset$ for all $j \neq i$. Finally, note that $f(v_n)$ selects the first element from $f(u_1)$, the second element from $f(u_2)$, the third element from $f(u_3)$, and so on up to element number $n-1$ from $f(u_{n-1})$. Hence, by construction, it clearly has nonempty intersection with each of $f(u_i)$ for $i \neq n$. This concludes the proof. ∎

For $n = 5$, we give the matrix A together with the corresponding labeling.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \end{bmatrix}$$

$$f(u_1) = \{1,2,3,4\}, f(u_2) = \{5,6,7,8\}, f(u_3) = \{9,10,11,12\}$$
$$f(u_4) = \{13,14,15,16\}, f(u_5) = \{17,18,19,20\}$$

$$f(v_1) = \{5,9,13,17\}, f(v_2) = \{2,10,14,18\}, f(v_3) = \{3,7,15,19\}$$
$$f(v_4) = \{4,8,12,20\}, f(v_5) = \{1,6,11,16\}$$

5.2 Subset Labelings of Prisms

We now turn to prisms in general, beginning with $\rho(C_n \,\square\, K_2)$ for integers $n \geq 3$. In the following results, we will consistently return to the labeling notation of $ab = \{a, b\}, abc = \{a, b, c\}$, etc. Since $C_3 \,\square\, K_2 = K_3 \,\square\, K_2$, it follows by Theorem 5.1.2 that $\rho(C_3 \,\square\, K_2) = 6$. As an illustration, we present a subset labeling $f : V(C_3 \,\square\, K_2) \to \mathcal{P}^*([6])$ of $C_3 \,\square\, K_2$, shown in Figure 5.1. Thus, for smallest values of n where $n = 3, 4, 5$,

$$\rho(C_n \,\square\, K_2) = \begin{cases} 6 & \text{if } n = 3, 4 \\ 9 & \text{if } n = 5 \end{cases} \tag{5.1}$$

Since $\rho(C_3) = 3$, $\rho(C_4) = 2$, and $\rho(C_5) = 5$, it follows that $\rho(C_3 \,\square\, K_2) = \rho(C_3) + 3$ and $\rho(C_n \,\square\, K_2) = \rho(C_n) + 4$ for $n = 4, 5$. The 5-cycle C_5 is not an induced subgraph of $C_3 \,\square\, K_2$, while C_{n+2} is an induced subgraph of $C_n \,\square\, K_2$ for each integer $n \geq 4$.

Next, we present upper and lower bounds for the subset index of $\rho(C_n \,\square\, K_2)$ in terms of $\rho(C_{n+2})$ and $\rho(C_n)$ for each integer $n \geq 4$.

Theorem 5.2.1 *For each integer $n \geq 4$,*

$$\rho(C_{n+2}) \leq \rho(C_n \,\square\, K_2) \leq \rho(C_n) + 4.$$

Proof. For an integer $n \geq 4$, let $G = C_n \,\square\, K_2$. Since C_{n+2} is an induced subgraph of G, it follows that $\rho(C_{n+2}) \leq \rho(G)$. Therefore, it remains to show that $\rho(G) \leq \rho(C_n) + 4$.

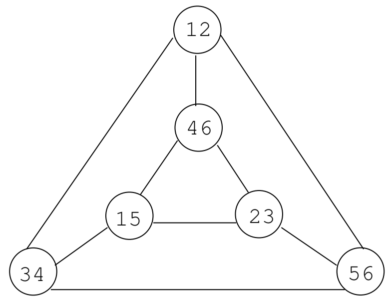

Figure 5.1: A subset labeling f of $C_3 \,\square\, K_2$

Suppose that $\rho(C_n) = k$. Then C_n has a subset labeling $g : V(C_n) \to \mathcal{P}^*([k])$. The subset labeling g gives rise to another subset labeling

$$h : V(C_n) \to \mathcal{P}^*([5, k+4])$$

of C_n defined by

$$h(x) = \{i + 4 : \ i \in g(x)\} \text{ for each vertex } x \text{ of } C_n.$$

Write $n = 4q + r$ for some integers q and r, where $q \geq 1$ and $r \in \{0, 1, 2, 3\}$. Next, we define a subset labeling $f : V(G) \to \mathcal{P}^*([k+4])$ of G by considering four cases, according to the value of r. In what follows, let G be constructed from two copies of the n-cycle

$$C = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1) \text{ and } C' = (v_1, v_2 \ldots, v_n, v_{n+1} = v_1)$$

by adding the edges $u_i v_{i+1}$ for $1 \leq i \leq n$, where the subscript of a vertex is expressed as a positive integer modulo n.

Case 0. $r = 0$ *and* $n = 4q$. We define the labeling $f : V(G) \to \mathcal{P}^*([k+4])$ of G by

$$f(u_i) = \begin{cases} h(u_i) \cup \{4\} & \text{if } i \equiv 0 \pmod 4 \\ h(u_i) \cup \{a\} & \text{if } i \equiv a \pmod 4 \text{ where } a = 1, 2, 3 \text{ and } 1 \leq i \leq n \end{cases}$$

$$f(v_i) = \begin{cases} h(u_i) \cup \{4\} & \text{if } i \equiv 3 \pmod 4 \text{ and } 1 \leq i \leq n. \\ h(u_i) \cup \{a+1\} & \text{if } i \equiv a \pmod 4 \text{ where } a = 0, 1, 2 \text{ and } 1 \leq i \leq n. \end{cases}$$

This is illustrated in Figure 5.2 for $n = 4$. It remains to show that f is a subset labeling of G. Let x and y be two vertices of G. If $x, y \in V(C)$ or $x, y \in V(C')$, say the former, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. Hence, $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(C)$. Thus, we may assume that $x \in V(C)$ and $y \in V(C')$. By the symmetry of G

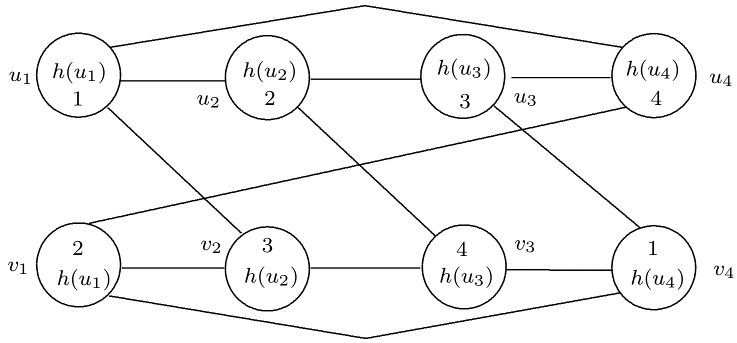

Figure 5.2: The labeling f in Case 0

and the definition of f, we may further assume that $x = u_1$ and $y = v_i$ for some integer i with $1 \leq i \leq n$.

- ⋆ If $y = v_2$, then $xy = u_1v_2 \in E(G)$. Since $f(u_1) = h(u_1) \cup \{1\}$ and $f(v_2) = h(u_2) \cup \{3\}$ and $h(u_1) \cap h(u_2) = \emptyset$, it follows that $f(u_1) \cap f(v_2) = \emptyset$.

- ⋆ If $y = v_n$, then $xy \notin E(G)$. Since $1 \in f(u_1) \cap f(v_n)$, it follows that $f(u_1) \cap f(v_n) \neq \emptyset$.

- ⋆ If $y = v_i$ where $i \in [n] - \{2, n\}$, then $xy \notin E(G)$. Since $h(u_1) \cap h(u_i) \neq \emptyset$ for $i \in [n] - \{2, n\}$ and $h(u_1) \subseteq f(u_1)$, it follows that $f(u_1) \cap f(v_i) \neq \emptyset$ for $i \in [n] - \{2, n\}$.

Therefore, f is a subset labeling of G, and so $\rho(G) \leq k + 4$ when $n \equiv 0 \pmod 4$.

Case 1. *$r = 1$, so $n = 4q + 1$.* Since $\rho(C_5 \,\square\, K_2) = 9$ and $\rho(C_5) = 5$, it follows that $\rho(C_5 \,\square\, K_2) = \rho(C_5) + 4$. Thus, we may assume that $n \geq 9$. We define a labeling $f : V(G) \to \mathcal{P}^*([k + 4])$ of G as follows. For each $i \in [4q] = [n - 1]$, we define $f(u_i)$ as in Case 0, and for each $i \in [4q - 1] \cup \{n\} = [n - 2] \cup \{n\}$, we define $f(v_i)$ as in Case 0.

For the remaining ten vertices of G (namely u_i for $i \in [n-4, n]$ and v_i for $i \in [n-5, n-1]$), we define

$$\begin{aligned} f(u_{n-4}) &= h(u_{n-4}) \cup \{2\} \\ f(u_{n-3}) &= h(u_{n-3}) \cup \{1\} \\ f(u_{n-2}) &= h(u_{n-2}) \cup \{3\} \\ f(u_{n-1}) &= h(u_{n-1}) \cup \{2\} \\ f(u_n) &= h(u_n) \cup \{4\} \end{aligned}$$

and

$$\begin{aligned} f(v_{n-5}) &= h(u_{n-5}) \cup \{2\} \\ f(v_{n-4}) &= h(u_{n-4}) \cup \{1\} \\ f(v_{n-3}) &= h(u_{n-3}) \cup \{3\} \\ f(v_{n-2}) &= h(u_{n-2}) \cup \{2\} \\ f(v_{n-1}) &= h(u_{n-1}) \cup \{4\} \end{aligned}$$

It remains to show that f is a subset labeling of G. Let x and y be two vertices of G. If $x, y \in V(C)$ or $x, y \in V(C')$, say the former, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. Hence, $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(C)$. Thus, we may assume that $x \in V(C)$ and $y \in V(C')$. If $x = u_i$ for $i \in [n-7]$ and $y = v_i$ for $i \in [n]$, then the proof is similar to that in Case 0. Hence, we may assume that

$$x = u_i \text{ where } i \in [n-6, n] = \{n-6, n-5, n-4, n-3, n-2, n-1, n\}$$

and $y = v_i$ for $1 \leq i \leq n$.

- ⋆ If $x = u_{n-6}$ and $y = v_{n-5}$, then $xy = u_{n-6}v_{n-5} \in E(G)$. Since $f(u_{n-6}) = h(u_{n-6}) \cup \{3\}$, $f(v_{n-5}) = h(u_{n-5}) \cup \{2\}$, and $h(u_{n-6}) \cap h(u_{n-5}) = \emptyset$, it follows that $f(u_{n-6}) \cap f(v_{n-5}) = \emptyset$.

- ⋆ If $x = u_{n-5}$ and $y = v_{n-4}$, then $xy = u_{n-5}v_{n-4} \in E(G)$. Since $f(u_{n-5}) = h(u_{n-5}) \cup \{4\}$, $f(v_{n-4}) = h(u_{n-4}) \cup \{1\}$, and $h(u_{n-5}) \cap h(u_{n-4}) = \emptyset$, it follows that $f(u_{n-5}) \cap f(v_{n-4}) = \emptyset$.

- ⋆ If $x = u_i$ where $i \in \{n-4, n-3, n-2\}$ and $y = v_{i+1}$, then $xy = u_i v_{i+1} \in E(G)$. By the definition of f, we have $f(u_i) = h(u_i) \cup \{a\}$ while $f(v_{i+1}) = h(u_{i+1}) \cup \{a+1\}$. Since $h(u_i) \cap h(u_{i+1}) = \emptyset$, it follows that $f(u_i) \cap f(u_{i+1}) = \emptyset$.

- ⋆ If $x = u_{n-1}$ and $y = v_n$, then $xy = u_{n-1}v_n \in E(G)$. Since $f(u_{n-1}) = h(u_{n-1}) \cup \{2\}$, $f(v_n) = h(u_n) \cup \{1\}$, and $h(u_{n-1}) \cap h(u_n) = \emptyset$, it follows that $f(u_{n-1}) \cap f(v_n) = \emptyset$.

- ⋆ If $x = u_n$ and $y = v_1$, then $xy = u_n v_1 \in E(G)$. Since $f(u_n) = h(u_n) \cup \{4\}$, $f(v_1) = h(u_1) \cup \{2\}$, and $h(u_n) \cap h(u_1) = \emptyset$, it follows that $f(u_n) \cap f(v_1) = \emptyset$.

- ⋆ If $x = u_i$ where $i \in [n-6, n]$ and $y = v_{i-1}$, then $xy \notin E(G)$. By the definition of f, we have

  $$(f(u_i) \cap f(v_{i-1}))_{i=n-6}^{n} = (\{3\}, \{4\}, \{2\}, \{1\}, \{3\}, \{2\}, \{4\}).$$

  Hence, $f(u_i) \cap f(v_{i-1}) \neq \emptyset$.

- ⋆ If $x = u_i$ where $i \in [n-6, n]$ and $y = v_j$ where $j \neq i-1, i+1$, then $xy \notin E(G)$. Since $h(u_i) \subseteq f(u_i)$, $h(u_j) \subseteq f(u_j)$, and $h(u_i) \cap h(u_j) \neq \emptyset$, it follows that $f(u_i) \cap f(u_j) \neq \emptyset$.

Therefore, f is a subset labeling of G, and so $\rho(G) \leq k+4$ when $n \equiv 1 \pmod 4$.

Case 2. $r = 2$ *and* $n = 4q+2$. We define the labeling $f : V(G) \to \mathcal{P}^*([k+4])$ of G as follows. For each $i \in [4q] = [n-2]$, we define $f(u_i)$ as in Case 0, and for each $i \in [4q-1] \cup \{n\} = [n-3] \cup \{n\}$, we define $f(v_i)$ as in Case 0. For the remaining four vertices of G (namely, u_i for $i \in \{n-1, n\}$ and v_i for $i \in \{n-2, n-1\}$), we define

$$\begin{aligned} f(u_{n-1}) &= h(u_{n-1}) \cup \{2\} \\ f(u_n) &= h(u_n) \cup \{3\} \\ f(v_{n-2}) &= h(u_{n-2}) \cup \{2\} \\ f(v_{n-1}) &= h(u_{n-1}) \cup \{3\}. \end{aligned}$$

It remains to show that f is a subset labeling of G. Let x and y be two vertices of G. If $x, y \in V(C)$ or $x, y \in V(C')$, say the former, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. Hence, $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(C)$. Thus, we may assume that $x \in V(C)$ and $y \in V(C')$. If $x = u_i$ for $1 \leq i \leq n-4$ and $y = v_i$ for $1 \leq i \leq n$, then the proof is similar to that in Case 0. Thus, we may assume that

$$x = u_i \text{ where } i \in [n-3, n] = \{n-3, n-2, n-1, n\}$$

and $y = v_i$ for some integer i with $1 \leq i \leq n$. By the definition of f, we may further assume that $u_i = u_{n-3}$.

- ⋆ If $y = v_{n-2}$, then $xy = u_{n-3}v_{n-2} \in E(G)$. Since $f(u_{n-3}) = h(u_{n-3}) \cup \{3\}$, $f(v_{n-2}) = h(u_{n-2}) \cup \{2\}$, and $h(u_{n-3}) \cap h(u_{n-2}) = \emptyset$, it follows that $f(u_{n-3}) \cap f(v_{n-2}) = \emptyset$.
- ⋆ If $y = v_{n-4}$, then $xy \notin E(G)$. Since $4 \in f(u_{n-3}) \cap f(v_{n-4})$, it follows that $f(u_{n-3}) \cap f(v_{n-4}) \neq \emptyset$.
- ⋆ If $y = v_i$ where $i \in [n] - \{n-2, n-4\}$, then $xy \notin E(G)$. Since $h(u_{n-3}) \cap h(u_i) \neq \emptyset$ for $i \in [n] - \{n-2, n-4\}$ and $h(u_{n-3}) \subseteq f(u_{n-3})$, it follows that $f(u_{n-3}) \cap f(v_i) \neq \emptyset$.

Therefore, f is a subset labeling of G, and so $\rho(G) \leq k+4$ when $n \equiv 2 \pmod 4$.

Case 3. $r = 3$ *and* $n = 4q+3$. A labeling $f : V(G) \to \mathcal{P}^*([k+4])$ of G is defined as follows. For each $i \in [n]$, we define $f(u_i)$ as in Case 0, and for each $i \in [4q-1] \cup \{n\} =$

$[n-4] \cup \{n\}$, we define $f(v_i)$ as in Case 0. For the remaining three vertices of G (namely, v_i for $i \in \{n-3, n-2, n-1\}$), we define

$$\begin{aligned} f(v_{n-3}) &= h(u_{n-3}) \cup \{1\} \\ f(v_{n-2}) &= h(u_{n-2}) \cup \{2\} \\ f(v_{n-1}) &= h(u_{n-1}) \cup \{3\}. \end{aligned}$$

It remains to show that f is a subset labeling of G. Let x and y be two vertices of G. If $x, y \in V(C)$ or $x, y \in V(C')$, say the former, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. Hence, $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(C)$. Thus, we may assume that $x \in V(C)$ and $y \in V(C')$. By the symmetry of G and the definition of f, if $x = u_i$ for $1 \le i \le n-5$, then the proof is similar to the one in Case 0. Therefore, we assume that $x = u_i$ for $i \in \{n-4, n-3, n-2, n-1, n\}$ and $y = v_i$ for some integer i with $1 \le i \le n$.

- ⋆ If $x = u_{n-4}$ and $y = v_{n-3}$, then $xy = u_{n-4}v_{n-3} \in E(G)$. Since $f(u_{n-4}) = h(u_{n-4}) \cup \{3\}$ and $f(v_{n-3}) = h(u_{n-3}) \cup \{2\}$ and $h(u_{n-4}) \cap h(u_{n-3}) = \emptyset$, it follows that $f(u_{n-4}) \cap f(v_{n-3}) = \emptyset$.

- ⋆ If $x = u_{n-3}$ and $y = v_{n-2}$, then $xy = u_{n-3}v_{n-2} \in E(G)$. Since $f(u_{n-3}) = h(u_{n-3}) \cup \{4\}$ and $f(v_{n-2}) = h(u_{n-2}) \cup \{2\}$, and $h(u_{n-3}) \cap h(u_{n-2}) = \emptyset$, it follows that $f(u_{n-3}) \cap f(v_{n-2}) = \emptyset$.

- ⋆ If $x = u_{n-2}$ and $y = v_{n-1}$, then $xy = u_{n-2}v_{n-1} \in E(G)$. Since $f(u_{n-2}) = h(u_{n-2}) \cup \{1\}$ and $f(v_{n-1}) = h(u_{n-1}) \cup \{3\}$, and $h(u_{n-2}) \cap h(u_{n-1}) = \emptyset$, we have $f(u_{n-2}) \cap f(v_{n-1}) = \emptyset$.

- ⋆ If $x = u_{n-1}$ and $y = v_n$, then $xy = u_{n-1}v_n \in E(G)$. Since $f(u_{n-1}) = h(u_{n-1}) \cup \{2\}$ and $f(v_n) = h(u_n) \cup \{1\}$, and $h(u_{n-1}) \cap h(u_n) = \emptyset$, we have $f(u_{n-1}) \cap f(u_n) = \emptyset$.

- ⋆ If $x = u_n$ and $y = v_1$, then $xy = u_n v_1 \in E(G)$. Because $f(u_n) = h(u_n) \cup \{3\}$ and $f(v_1) = h(u_1) \cup \{2\}$, and $h(u_n) \cap h(u_1) = \emptyset$, it follows that $f(u_n) \cap f(v_1) = \emptyset$.

- ⋆ If $x = u_i$ where $i \in [n-4, n]$ and $y = v_{i-1}$, then $xy \notin E(G)$. By the definition of f, we have

  $$(f(u_i) \cap f(v_{i-1}))_{i=n-4}^{n} = (\{3\}, \{4\}, \{1\}, \{2\}, \{3\}, \{1\}).$$

  Hence, $f(u_i) \cap f(v_{i-1}) \neq \emptyset$.

- ⋆ If $x = u_i$ for $i \in [n-4, n]$ and $y = v_j$ where $j \neq i-1, i+1$, then $xy \notin E(G)$. Because $h(u_i) \subseteq f(u_i)$, $h(u_j) \subseteq f(u_j)$, and $h(u_i) \cap h(u_j) \neq \emptyset$, it follows that $f(u_i) \cap f(u_j) \neq \emptyset$.

Therefore, f is a subset labeling of G, and so $\rho(G) \le k+4$ when $n \equiv 3 \pmod 4$. ■

There are integers $n \ge 6$ such that $\rho(C_{n+2}) < \rho(C_n \,\square\, K_2) < \rho(C_n)+4$. In fact, if $n \ge 3$ and $n \equiv 0 \pmod 3$, then the upper bound for $\rho(C_n \,\square\, K_2)$ in Theorem 5.2.1 can be improved, as we show next.

Theorem 5.2.2 *For each integer $n \ge 3$ and $n \equiv 0 \pmod 3$,*

$$\rho(C_n \,\square\, K_2) \le \rho(C_n)+3.$$

Proof. Since $\rho(C_3 \,\square\, K_2) = 6 = \rho(C_3)+3$, we may assume that $n \ge 6$. Let $G = C_n \,\square\, K_2$ be constructed from two copies of the n-cycle

$$C = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1) \text{ and } C' = (v_1, v_2 \ldots, v_n, v_{n+1} = v_1)$$

by adding the edges $u_i v_{i+1}$ for $1 \le i \le n$, where the subscript of a vertex is expressed as a positive integer modulo n. Suppose that $\rho(C_n) = k$. Then C_n has a subset labeling $g : V(C_n) \to \mathcal{P}^*([k])$. The subset labeling g gives rise to another subset labeling

$$h : V(C_n) \to \mathcal{P}^*([4, k+3])$$

of C_n defined by $h(x) = \{i+3 : \ i \in g(x)\}$ for each vertex x of C_n. With the aid of the labeling h, we now define the labeling $f : V(G) \to \mathcal{P}^*([k+3])$ of G by

$$f(u_i) = \begin{cases} h(u_i) \cup \{3\} & \text{if } i \equiv 0 \pmod 3 \\ h(u_i) \cup \{a\} & \text{if } i \equiv a \pmod 3 \text{ where } a = 1,2 \text{ and } 1 \le i \le n \end{cases}$$

$$f(v_i) = \begin{cases} h(u_i) \cup \{3\} & \text{if } i \equiv 2 \pmod 3 \text{ and } 1 \le i \le n. \\ h(u_i) \cup \{a+1\} & \text{if } i \equiv a \pmod 3 \text{ where } a = 0,1 \text{ and } 1 \le i \le n. \end{cases}$$

This labeling is shown in Figure 5.3 for $n = 6$.

It remains to show that f is a subset labeling of G. Let x and y be two vertices of G. If $x, y \in V(C)$ or $x, y \in V(C')$, say the former, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. Hence, $f(x) \cap f(y) = \emptyset$ if and only if $xy \in E(C)$. Thus, we may assume that $x \in V(C)$ and $y \in V(C')$. By the symmetry of G and the definition of f, we may further assume that $x = u_1$ and $y = v_i$ for some integer i with $1 \le i \le n$.

- ⋆ If $y = v_2$, then $xy = u_1 v_2 \in E(G)$. Since $f(u_1) = h(u_1) \cup \{1\}$ and $f(v_2) = h(u_2) \cup \{3\}$ and $h(u_1) \cap h(u_2) = \emptyset$, it follows that $f(u_1) \cap f(v_2) = \emptyset$.

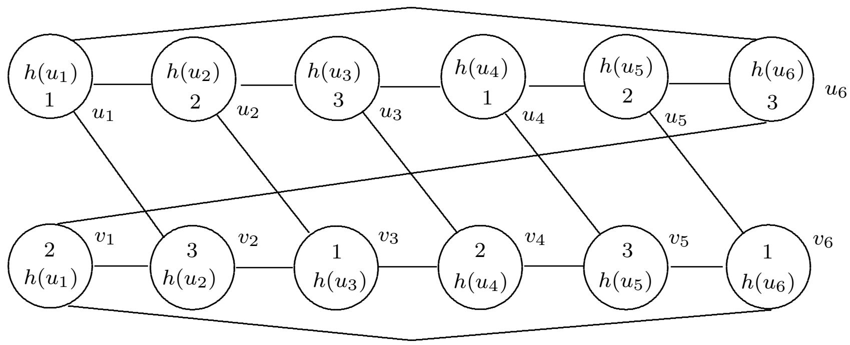

Figure 5.3: The labeling f for $C_6 \,\square\, K_2$ in the proof of Theorem 5.2.2

- ⋆ If $y = v_n$, then $xy \notin E(G)$. Since $1 \in f(u_1) \cap f(v_n)$, it follows that $f(u_1) \cap f(v_n) \neq \emptyset$.
- ⋆ If $y = v_i$ where $i \in [n] - \{2, n\}$, then $xy \notin E(G)$. Since $h(u_1) \cap h(u_i) \neq \emptyset$ for $i \in [n] - \{2, n\}$ and $h(u_1) \subseteq f(u_1)$, it follows that $f(u_1) \cap f(v_i) \neq \emptyset$ for $i \in [n] - \{2, n\}$.

Therefore, f is a subset labeling of G, and so $\rho(G) \leq k + 4$ when $n \equiv 0 \pmod 3$. ■

We saw that $\rho(C_3 \,\square\, K_2) = 6 = \rho(C_3) + 3$. As another illustration, we show that $\rho(C_6 \,\square\, K_2) = \rho(C_6) + 3 = 8$.

Proposition 5.2.3 $\rho(C_6 \,\square\, K_2) = 8$.

Proof. Since $\rho(C_6) = 5$ and $\rho(C_6 \,\square\, K_2) \leq \rho(C_6) + 3 = 8$ by Theorem 5.2.2, it suffices to show that $\rho(C_6 \,\square\, K_2) \geq 8$. Let $G = C_6 \,\square\, K_2$. Assume, to the contrary, that G has a subset labeling $f : V(G) \to \mathcal{P}^*([7])$. Suppose that G is constructed from two copies of the 6-cycle

$$C = (u_1, u_2, \ldots, u_6, u_7 = u_1) \text{ and } C' = (v_1, v_2 \ldots, v_6, v_7 = v_1)$$

by adding the edges $u_i v_i$ for $1 \leq i \leq 6$. Since the eccentricity of every vertex of G is 4, it follows that $|f(u)| \geq 2$ for every vertex u of G. If $u, v \in V(G)$ and $u \neq v$, then $N(u) - N(v) \neq \emptyset$, and so $f(v) \nsubseteq f(u)$ by the Subset Lemma. Consequently, $|f(u)| \leq 5$ for every vertex u of G and $f(u) \neq f(v)$ for every two distinct vertices u and v of G.

First, we verify the following three facts:

Fact 1: There exists some vertex $u \in V(G)$ such that $|f(u)| \geq 3$.

Fact 2: No vertex of G can have a 4-set label.

Fact 3: No two adjacent vertices of G can have 3-set labels.

Proof of Fact 1: Suppose, to the contrary, that $|f(u)| = 2$ for every vertex u of G. We may assume, without loss of generality, that $f(u_1) = 12$, $f(u_2) = 34$, $f(u_3) = 15$, $f(u_4) = 23$, $f(u_5) = 14$, and $f(u_6) = 35$. However, then, this forces $\{3, 4, 5\} \subseteq f(v_1)$, which is a contradiction. Thus, Fact 1 holds.

Proof of Fact 2: Suppose, to the contrary, some vertex label of G is a 4-set. We may assume, without loss of generality, that $f(u_1) = 1234$. We claim that $|f(v)| = 2$ for each $v \in N(u_1) = \{u_2, v_1, u_6\}$; for otherwise, we may assume that $f(u_6) = 567$ or $f(v_1) = 567$, say $f(u_6) = 567$. However, then, $f(v_1) \subseteq f(u_6)$ and $f(u_2) \subseteq f(u_6)$, which is impossible by the Subset Lemma. Thus, as claimed, $|f(v)| = 2$ for each $v \in N(u_1) = \{u_2, v_1, u_6\}$. Suppose, without loss of generality, that $f(u_2) = 56$, $f(u_6) = 57$, and $f(v_1) = 67$. This again forces $f(v_2) \subseteq f(u_1)$, which again is impossible by the Subset Lemma. Thus, **Fact 2** holds.

Proof of Fact 3: Suppose, to the contrary, that there are two adjacent vertices u and v such that $|f(u)| = |f(v)| = 3$. We may assume that (1) $u = u_1$ and $f(u_1) = 123$ and (2) $v \in \{u_2, v_1\}$ and $f(v) = 456$. First, suppose that $f(u_2) = 456$. Since $f(v_2) \not\subseteq f(u_1)$ by the Subset Lemma, this forces $7 \in f(v_2)$. Because $f(u_1) = 123$ and $7 \in f(v_2)$, this forces $f(v_1) \subseteq f(u_2)$, which again is impossible by the Subset Lemma. Next, suppose that $f(v_1) = 456$. Similarly, this forces $7 \in f(v_2)$. The fact that $f(u_1) = 123$ and $7 \in f(v_2)$ forces $f(u_2) \subseteq f(v_1)$, which is impossible. Thus, **Fact 3** holds.

By **Facts 1** and **2**, we may assume that $f(u_1) = 123$. It then follows by **Facts 2** and **3** that $|f(v)| = 2$ for each $v \in N(u_1) = \{u_2, v_1, u_6\}$. Suppose, without loss of generality, that $f(u_2) = 57$ and $f(u_6) = 56$. Then $6 \notin f(v_1)$ and $7 \notin f(v_1)$; for otherwise, say $6 \in f(v_1)$. However, then, $f(v_2) \cap f(u_6) = \emptyset$, which is impossible. This forces $f(v_1) = 45$, which in turn forces $4 \in f(u_3) \cap f(v_4)$ and $5 \in f(u_4)$. Since $f(v_4) \cap f(u_2) \neq \emptyset$ and $f(v_4) \cap f(u_6) \neq \emptyset$, it follows that $6, 7 \in f(v_4)$. Thus, $f(v_4) = 467$. However, then, $f(v_4) \cap f(u_1) = \emptyset$, which is a contradiction.

Therefore, $\rho(G) \geq 8$, and so $\rho(G) = 8$. ■

We now investigate the lower bound $\rho(C_{n+2})$ for the subset index $\rho(C_n \square K_2)$ of $C_n \square K_2$ in Theorem 5.2.1. We have seen for each integer n with $3 \leq n \leq 6$ that $\rho(C_n \square K_2) \neq \rho(C_{n+2})$. In fact, this is also true for $n = 7, 8$, as we show next. Since

$$\rho(C_7) = \rho(C_9) = 7 \text{ and } \rho(C_8) = \rho(C_{10}) = 6,$$

it follows by Theorem 5.2.1 that

$$7 \leq (C_7 \,\Box\, K_2) \leq 11 \text{ and } 6 \leq (C_8 \,\Box\, K_2) \leq 10.$$

We now show that $\rho(C_n \,\Box\, K_2) \geq 8$ for $n = 7, 8$.

Proposition 5.2.4 *If* $n = 7, 8$, *then* $\rho(C_n \,\Box\, K_2) \geq 8$.

Proof. We will only consider the case when $n = 7$, since the proof for the case when $n = 8$ is similar. Let $G = C_7 \,\Box\, K_2$. Assume, to the contrary, that G has a subset labeling $f : V(G) \to \mathcal{P}^*([7])$. Suppose that G is constructed from two copies of the 7-cycle

$$C = (u_1, u_2, \ldots, u_7, u_8 = u_1) \text{ and } C' = (v_1, v_2 \ldots, v_7, v_8 = v_1)$$

by adding the edges $u_i v_i$ for $1 \leq i \leq 7$. By the Subset Lemma and 5.1.1, $|f(u)| \geq 2$ for every vertex u of G and $f(u) \not\subseteq f(v)$ for every two distinct vertices u and v of G. Furthermore, $|f(u)| \leq 5$ for every vertex u of G. First, we verify the following three facts:

Fact 1: There exists some vertex $u \in V(G)$ such that $|f(u)| \geq 3$.

Fact 2: No vertex of G can have a 4-set label.

Fact 3: No two adjacent vertices of G can have 3-set labels.

Proof of Fact 1: Suppose, to the contrary, that $|f(u)| = 2$ for every vertex u of G. We may assume, without loss of generality, that $f(u_1) = 12$, $f(u_2) = 34$, $f(u_3) = 15$, $f(u_4) = 23$, and $f(u_5) = 14$. However, then, this forces $\{2, 3, 5\} \subseteq f(u_6)$, which is a contradiction. Thus, **Fact 1** holds.

Proof of Fact 2: Suppose, to the contrary, some vertex label of G is a 4-set. We may assume, without loss of generality, that $f(u_1) = 1234$. By the Subset Lemma, it follows that $|f(v)| = 2$ for each $v \in N(u_1) = \{u_2, v_1, u_7\}$. Suppose, without loss of generality, that $f(u_2) = 56$, $f(u_7) = 57$, and $f(v_1) = 67$. This again forces $f(v_2) \subseteq f(u_1)$, which is impossible. Thus, **Fact 2** holds.

Proof of Fact 3: Suppose, to the contrary, that there are two adjacent vertices u and v such that $|f(u)| = |f(v)| = 3$. We may assume that (1) $u = u_1$ and $f(u_1) = 123$ and (2) $v \in \{u_2, v_1\}$ and $f(v) = 456$. First, suppose that $f(u_2) = 456$. Since $f(v_2) \not\subseteq f(u_1)$ by the Subset Lemma, this forces $7 \in f(v_2)$. Because $f(u_1) = 123$ and $7 \in f(v_2)$, this forces

$f(v_1) \subseteq f(u_2)$, which is impossible. Next, suppose that $f(v_1) = 456$. Similarly, this forces $7 \in f(v_2)$. The fact that $f(u_1) = 123$ and $7 \in f(v_2)$ forces $f(u_2) \subseteq f(v_1)$, which again is a contradiction. Thus, **Fact 3** holds.

By **Facts 1** and **2**, we may assume that $f(u_1) = 123$. It then follows by **Facts 2** and **3** that $|f(v)| = 2$ for each $v \in N(u_1) = \{u_2, v_1, u_7\}$. Suppose, without loss of generality, that $f(u_2) = 57$ and $f(u_7) = 56$. Then $6 \notin f(v_1)$ and $7 \notin f(v_1)$; for otherwise, say $6 \in f(v_1)$. However, then, $f(v_2) \cap f(u_7) = \emptyset$, which is impossible. This forces $f(v_1) = 45$, which in turn forces $4 \in f(u_3) \cap f(v_4)$ and $5 \in f(u_4)$. Since $f(v_4) \cap f(u_2) \neq \emptyset$ and $f(v_4) \cap f(u_7) \neq \emptyset$, it follows that $6, 7 \in f(v_4)$. Thus, $f(v_4) = 467$. However, then, $f(v_4) \cap f(u_1) = \emptyset$, which is a contradiction. Therefore, $\rho(G) \geq 8$. ■

By (5.1) and Propositions 5.2.3 and 5.2.4, we have the following:

If $3 \leq n \leq 8$, *then* $\rho(C_n \,\square\, K_2) \neq \rho(C_{n+2})$.

The following question remains open.

Problem 5.2.5 *Does there exist* $n \geq 9$ *such that* $\rho(C_n \,\square\, K_2) = \rho(C_{n+2})$*?*

5.3 Subset Labelings of Grids

We now investigate the subset labelings of grids $P_n \,\square\, K_2$ for integers $n \geq 2$. Since

$$\rho(P_2 \,\square\, K_2) = \rho(C_4) = 2,$$

we may assume that $n \geq 3$. Since P_{n+2} and $P_{n-1} \,\square\, K_2$ are induced subgraphs of $P_n \,\square\, K_2$ and $P_n \,\square\, K_2$ is an induced subgraph of $P_{n+1} \,\square\, K_2$, we have the following useful observation.

Observation 5.3.1 *For each integer* $n \geq 3$,

$$\max\{\rho(P_{n+2}), \rho(P_{n-1} \,\square\, K_2)\} \leq \rho(P_n \,\square\, K_2) \leq \rho(P_{n+1} \,\square\, K_2).$$

First, we determine $\rho(P_3 \,\square\, K_2)$ for $3 \leq n \leq 6$. By Observation 5.3.1, it is useful to recall the following known values of $\rho(P_n)$:

$$\rho(P_n) = \begin{cases} n-1 & \text{if } 3 \leq n \leq 6 \\ 5 & \text{if } n = 7 \\ 6 & \text{if } 8 \leq n \leq 11 \\ 7 & \text{if } 12 \leq n \leq 22 \\ 8 & \text{if } n = 23. \end{cases}$$

Example 5.3.2 $\rho(P_3 \,\Box\, K_2) = 4$.

Proof. Let $G = P_3 \,\Box\, K_2$. By Observation 5.3.1, $\rho(G) \geq \rho(P_5) = 4$. Figure 5.4 shows a subset labeling $f : V(G) \to \mathcal{P}^*([4])$ of G, and so $\rho(G) \leq 4$. Thus, $\rho(G) = 4$. ■

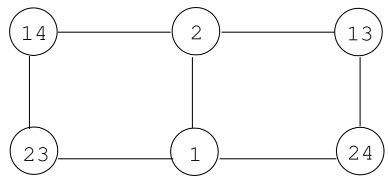

Figure 5.4: A subset labeling of $P_3 \,\Box\, K_2$

Example 5.3.3 $\rho(P_4 \,\Box\, K_2) = \rho(P_5 \,\Box\, K_2) = 6$.

Proof. By Observation 5.3.1, it suffices to show that

$$\rho(P_4 \,\Box\, K_2) \geq 6 \text{ and } \rho(P_5 \,\Box\, K_2) \leq 6.$$

Let $G = P_4 \,\Box\, K_2$ be constructed from two copies (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4) of the 4-path P_4 by adding the edges $u_i v_i$ for $1 \leq i \leq 4$. Assume, to the contrary, that G has a subset labeling $f : V(G) \to \mathcal{P}^*([5])$. By the Subset Lemma and 5.1.1, it follows that $|f(u)| \geq 2$ for every vertex u of G and $f(u) \neq f(v)$ for every two distinct vertices u and v of G. This implies that $|f(u)| = 2$ for every vertex u of G. Thus, we may assume that $f(u_1) = 12$, $f(u_2) = 34$, and $f(v_1) = 35$. We may further assume that $1 \in f(x)$ where $x \in \{u_3, v_4, v_2\}$ and $2 \in f(x)$ where $x \in \{u_4, v_3\}$. This forces $5 \in f(x)$ where $x \in \{u_3, v_4\}$. However, then, $f(u_3) = f(v_4) = 15$, which is impossible. Thus, $\rho(P_4 \,\Box\, K_2) \geq 6$.

Next, we show that $\rho(P_5 \,\Box\, K_2) \leq 6$. Let $P_5 \,\Box\, K_2$ be constructed from two copies (u_1, u_2, u_3, u_5) and (v_1, v_2, v_3, v_5) of the 5-path by adding the edges $u_i v_i$ for $1 \leq i \leq 5$. We define a subset labeling $f : V(P_5 \,\Box\, K_2) \to \mathcal{P}^*([6])$ by

$$f(u_1) = 356,\ f(u_2) = 12,\ f(u_3) = 34,\ f(u_4) = 15,\ f(u_5) = 236,$$

$$f(v_1) = 124,\ f(v_2) = 35,\ f(v_3) = 16,\ f(v_4) = 23,\ f(v_5) = 145.$$

Thus, $6 \leq \rho(P_4 \,\Box\, K_2) \leq \rho(P_5 \,\Box\, K_2) \leq 6$, giving the desired result. ■

Example 5.3.4 $\rho(P_6 \,\Box\, K_2) = 7$.

Proof. First, we show that $\rho(P_6 \,\square\, K_2) \geq 7$. Let $G = P_6 \,\square\, K_2$ be constructed from two copies $(u_1, u_2, \ldots, u_6)$ and $(v_1, v_2, \ldots, v_6)$ of the 6-path by adding the edges u_iv_i for $1 \leq i \leq 6$. Assume, to the contrary, that G has a subset labeling $f : V(G) \to \mathcal{P}^*([6])$. By the Subset Lemma and 5.1.1, it follows that $2 \leq |f(u)| \leq 3$ for every vertex u of G and $f(u) \neq f(v)$ for every two distinct vertices u and v of G. We claim that $|f(u_i)| = |f(v_i)| = 2$ for $2 \leq i \leq 5$. If this is not the case, then we may assume $f(u_2) = 123$ or $f(u_3) = 123$. If $f(u_i) = 123$ for $i = 2, 3$, then $f(u_{i+1}) \cup f(v_i) = 456$. This forces $f(v_{i+1}) \subseteq \{1, 2, 3\} = f(u_i)$, which is impossible by the Subset Lemma. Thus, $|f(u_i)| = |f(v_i)| = 2$ for $2 \leq i \leq 5$. We may assume that $f(u_2) = 12$, $f(u_3) = 34$, and $f(v_2) = 35$. We may further assume that $1 \in f(x)$ where $x \in \{u_4, u_6, v_3, v_5\}$ and $2 \in f(x)$ where $x \in \{u_5, v_6, v_4\}$. This forces $5 \in f(x)$ where $x \in \{u_4, u_6, v_5\}$. However, then, $f(u_4) = f(v_5) = 15$, which is impossible. Thus, $\rho(G) \geq 7$.

To show that $\rho(G) \leq 7$, we define a subset labeling $f : VG) \to \mathcal{P}^*([7])$ by

$$f(u_1) = 126,\ f(u_2) = 347,\ f(u_3) = 15,\ f(u_4) = 236,\ f(u_5) = 14,\ f(u_6) = 2357$$
$$f(v_1) = 2457,\ f(v_2) = 12,\ f(v_3) = 346,\ f(v_4) = 17,\ f(v_5) = 235,\ f(v_6) = 146.$$

Thus, $\rho(P_6 \,\square\, K_2) \leq 7$ and so $\rho(P_6 \,\square\, K_2) = 7$. ■

Next, we establish lower and upper bounds for $\rho(P_n \,\square\, K_2)$ for $n \geq 3$ in terms of $\rho(P_{n+2})$ and $\rho(P_n)$ in general.

Theorem 5.3.5 *For each integer $n \geq 3$,*

$$\rho(P_{n+2}) \leq \rho(P_n \,\square\, K_2) \leq \rho(P_n) + 3.$$

Proof. Let $G = P_n \,\square\, K_2$. By Observation 5.3.1, it remains to verify that $\rho(G) \leq \rho(P_n)+3$. Since this is true for $3 \leq n \leq 6$ by Examples 5.3.2, 5.3.3 and 5.3.4, we may assume that $n \geq 7$. Let $G = P_n \,\square\, K_2$ be constructed from two copies of the path of order n, namely

$$P = (u_1, u_2, \ldots, u_n) \text{ and } P' = (v_1, v_2, \ldots, v_n),$$

by adding the edges u_iv_i for $1 \leq i \leq n$. Suppose that $\rho(P_n) = k$. Then P_n has a subset labeling $g : V(P_n) \to \mathcal{P}^*([k])$. This subset labeling g gives rise to a subset labeling $h : V(P_n) \to \mathcal{P}^*([4, k+3])$ of P_n defined by

$$h(x) = \{i + 3 :\ i \in g(x)\} \text{ for each vertex } x \text{ of } P_n.$$

We define the labeling $f : V(G) \to \mathcal{P}^*([k+3])$ of G by

$$f(u_i) = \begin{cases} h(u_i) & \text{if } i \in \{1,2\} \\ h(u_i) \cup \{3\} & \text{if } i \equiv 2 \pmod 3 \\ h(u_i) \cup \{a+1\} & \text{if } i \equiv a \pmod 3 \text{ where } a = 0,1 \text{ and } 1 \le i \le n \end{cases}$$

$$f(v_i) = \begin{cases} h(u_{i+1}) \cup \{3\} & \text{if } i \equiv 0 \pmod 3 \text{ and } 1 \le i \le n-2. \\ h(u_{i+1}) \cup \{a\} & \text{if } i \equiv a \pmod 3 \text{ where } a = 1,2 \text{ and } 1 \le i \le n-2. \\ h(u_{i+1}) & \text{if } i = n-1. \\ h(u_{i-1}) \cup \{2,3\} & \text{if } i = n \text{ and } n \equiv 0 \pmod 3). \\ h(u_{i-1}) \cup \{1,3\} & \text{if } i = n \text{ and } n \equiv 1 \pmod 3). \\ h(u_{i-1}) \cup \{1,2\} & \text{if } i = n \text{ and } n \equiv 2 \pmod 3). \end{cases}$$

It remains to show that f is a subset labeling of G. Let x and y be two vertices of G. If $x, y \in V(P)$ or $x, y \in V(P')$ and $\{x,y\} \neq \{v_{n-3}, v_n\}$, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. Next, assume that $\{x,y\} = \{v_{n-3}, v_n\}$.

- If $n \equiv 0 \pmod 3$, then $3 \in f(v_{n-3})$ and $3 \in f(v_n)$.
- If $n \equiv 1 \pmod 3$, then $1 \in f(v_{n-3})$ and $1 \in f(v_n)$.
- If $n \equiv 2 \pmod 3$, then $2 \in f(v_{n-3})$ and $2 \in f(v_n)$.

Hence, $f(v_{n-3}) \cap f(v_n) \neq \emptyset$. Thus, we may assume that $x \in V(P)$ and $y \in V(P')$.

⋆ If $x = u_1$ and $y = v_1$, then $xy = u_1v_1 \in E(G)$. Since $f(u_1) = h(u_1)$ and $f(v_1) = h(u_2) \cup \{1\}$ and $h(u_1) \cap h(u_2) = \emptyset$, it follows that $f(u_1) \cap f(v_1) = \emptyset$.

⋆ If $x = u_2$ and $y = v_2$, then $x = u_2v_2 \in E(G)$. Since $f(u_2) = h(u_2)$ and $f(v_2) = h(u_3) \cup \{2\}$ and $h(u_2) \cap h(u_3) = \emptyset$, it follows that $f(u_2) \cap f(v_2) = \emptyset$.

⋆ If $x = u_1$ and $y = v_i$ for $2 \le i \le n$, then $xy \notin E(G)$. Since $h(u_1) \cap h(u_i) \neq \emptyset$ for $2 \le i \le n$, it follows that $f(u_1) \cap f(v_i) \neq \emptyset$ for $2 \le i \le n$.

⋆ If $x = u_2$ and $y = v_i$ for $3 \le i \le n$, then $xy \notin E(G)$. Since $h(u_2) \cap h(u_i) \neq \emptyset$ for $3 \le i \le n$, it follows that $f(u_2) \cap f(v_i) \neq \emptyset$ for $3 \le i \le n$.

⋆ If $x = u_3$ and $y = v_3$, then $xy = u_3v_3 \in E(G)$. Since $f(u_3) = h(u_3) \cup \{1\}$ and $f(v_3) = h(u_4) \cup \{3\}$ and $h(u_3) \cap h(u_4) = \emptyset$, it follows that $f(u_3) \cap f(v_3) = \emptyset$. By the symmetry of the graph structure and the definition of the labeling f, the same reasoning holds in general for the adjacent vertices u_i and v_i when $3 \le i \le n-1$.

- ⋆ If $x = u_3$ and $y = v_1$, then $xy = u_3v_1 \notin E(G)$. Since $f(u_3) = h(u_3) \cup \{1\}$ and $f(v_1) = h(u_2) \cup \{1\}$, it follows that $f(u_3) \cap f(v_1) \neq \emptyset$. By the symmetry of the graph structure and the definition of the labeling f, the same reasoning holds in general for the nonadjacent vertices u_i and v_{i-2} when $3 \leq i \leq n$.

- ⋆ If $x = u_3$ and $y = v_i$ for $i \notin \{1, 3\}$, then $u_3v_i \notin E(G)$. Since $h(u_3) \subseteq f(u_3)$ and $h(u_{i+1}) \subseteq f(v_i)$, and $h(u_3) \cap h(u_{i+1}) \neq \emptyset$, it follows that $f(u_3) \cap f(v_i) \neq \emptyset$. By the symmetry of the graph structure and the definition of the labeling f, the same reasoning holds in general for the nonadjacent vertices u_i and v_j where $j \notin \{i-2, i\}$.

- ⋆ If $x = u_n$ and $y = v_n$, then $xy \in E(G)$. We consider three possibilities.

  - If $n \equiv 0 \pmod 3$, then $f(u_n) = h(u_n) \cup \{1\}$ and $f(v_n) = h(u_{n-1}) \cup \{2, 3\}$.
  - If $n \equiv 1 \pmod 3$, then $f(u_n) = h(u_n) \cup \{2\}$ and $f(v_n) = h(u_{n-1}) \cup \{1, 3\}$.
  - If $n \equiv 2 \pmod 3$, then $f(u_n) = h(u_n) \cup \{3\}$ and $f(v_n) = h(u_{n-1}) \cup \{1, 2\}$.

  Since $h(u_n) \cap h(u_{n-1}) = \emptyset$ in all cases, it follows that $f(u_n) \cap f(v_n) = \emptyset$.

- ⋆ If $x = u_{n-2}$ and $y = v_n$, then $u_{n-2}v_n \notin E(G)$. We consider three possibilities.

  - If $n \equiv 0 \pmod 3$ or $n - 2 \equiv 1 \pmod 3$, then $f(u_{n-2}) = h(u_{n-2}) \cup \{2\}$ and $f(v_n) = h(u_{n-1}) \cup \{2, 3\}$.
  - If $n \equiv 1 \pmod 3$ or $n - 2 \equiv 2 \pmod 3$, then $f(u_{n-2}) = h(u_{n-2}) \cup \{3\}$ and $f(v_n) = h(u_{n-1}) \cup \{3, 1\}$.
  - If $n \equiv 2 \pmod 3$ or $n - 2 \equiv 0 \pmod 3$, then $f(u_{n-2}) = h(u_{n-2}) \cup \{1\}$ and $f(v_n) = h(u_{n-1}) \cup \{1, 2\}$.

  Hence, $f(u_{n-2}) \cap f(v_n) \neq \emptyset$ in all cases.

Therefore, f is a subset labeling of G, and so $\rho(G) \leq k + 3$. ■

The subset labelings of $P_n \square K_2$ are shown in Figure 5.5 for $n = 7, 8, 9$.

Both the lower and upper bounds for $\rho(P_n \square K_2)$ described in Theorem 5.3.5 can be attained. Furthermore, strict inequalities in Theorem 5.3.5 are also possible. For $3 \leq n \leq 6$, the values of $\rho(P_n \square K_2)$ are listed as follows:

$$\rho(P_n \square K_2) = \begin{cases} 4 & n = 3 \\ 6 & n = 4, 5 \\ 7 & n = 6. \end{cases}$$

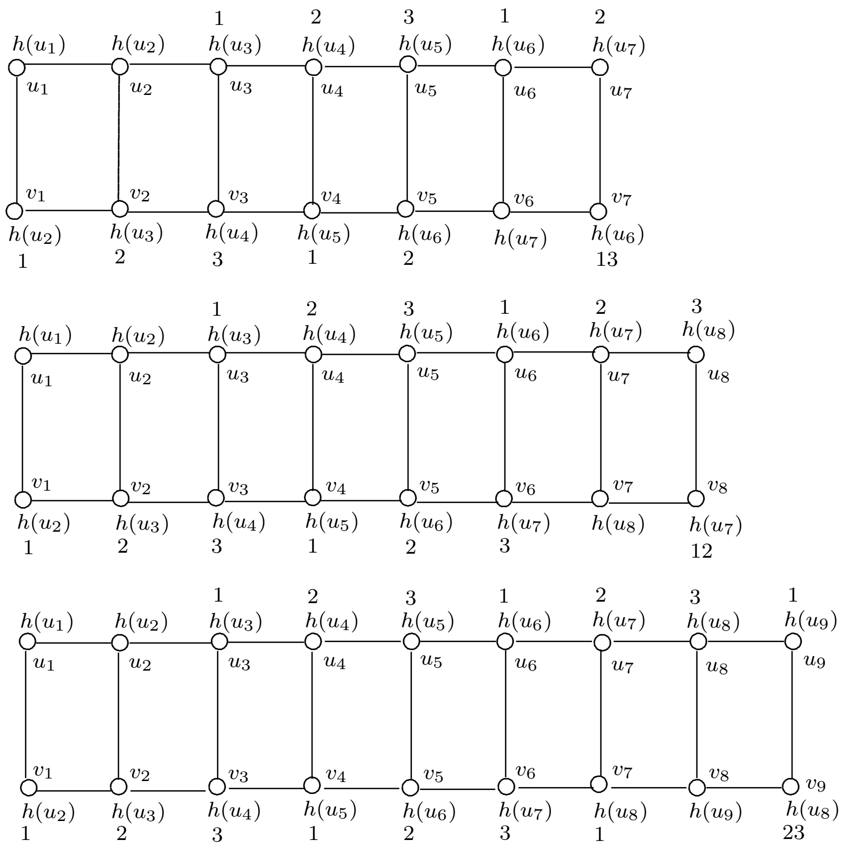

Figure 5.5: Illustrating the subset labelings of $P_n \,\square\, K_2$ for $n = 7, 8, 9$ described in the proof of Theorem 5.3.5

Consequently,

$$\rho(P_3 \,\square\, K_2) = \rho(P_5) = 4,$$

$$\rho(P_4 \,\square\, K_2) = \rho(P_4) + 3 = 6,$$

$$5 = \rho(P_7) < \rho(P_5 \,\square\, K_2) = 6 < \rho(P_5) + 3 = 7, \text{ and}$$

$$6 = \rho(P_8) < \rho(P_6 \,\square\, K_2) = 7 < \rho(P_6) + 3 = 8.$$

With the aid of Observation 5.3.1, Example 5.3.4 and Theorem 5.3.5, we next show that $\rho(P_7 \,\square\, K_2) = 8$, and so $\rho(P_7 \,\square\, K_2) = \rho(P_7) + 3$.

Proposition 5.3.6 $\rho(P_7 \,\square\, K_2) = 8$.

Proof. By Observation 5.3.1, Example 5.3.4 and Theorem 5.3.5, it follows that $\rho(P_7 \,\square\, K_2) = 7$ or $\rho(P_7 \,\square\, K_2) = 8$. Thus, it suffices to show that $\rho(P_7 \,\square\, K_2) \neq 7$. Let $G = P_7 \,\square\, K_2$ be constructed from two copies $(u_1, u_2, \ldots, u_7)$ and $(v_1, v_2, \ldots, v_7)$ of the 7-path by adding the edges u_iv_i for $1 \leq i \leq 7$. Assume, to the contrary, that G has a subset labeling $f : V(G) \to \mathcal{P}^*([7])$. By the Subset Lemma and 5.1.1, it follows that $2 \leq |f(u)| \leq 4$ for every vertex u of G and $f(u) \neq f(v)$ for every two distinct vertices u and v of G. First, we verify the following three facts:

Fact 1: No vertex of degree 3 in G can have a 4-set label.

Fact 2: There exists an integer i with $3 \leq i \leq 5$ such that at least one of u_i and v_i has a 3-set label.

Fact 3: No two adjacent vertices of degree 3 in G can have 3-set labels.

Proof of Fact 1: Assume, to the contrary, that $|f(u)| = 4$ for some vertex u of degree 3 in G. We may assume that $u = u_i$ where $2 \leq i \leq n-1$. We will consider only the case when $u = u_2$ and $f(u_2) = 1234$, since the arguments for other cases are similar. By the Subset Lemma, we have $|f(x)| = 2$ for each $x \in N(u_2) = \{u_1, v_2, u_3\}$. Without loss of generality, we may assume that $f(u_1) = 56$, $f(u_3) = 57$, and $f(v_2) = 67$. Since $f(u_3) \cup f(v_2) = \{5, 6, 7\}$, it follows that $f(v_3) \subseteq f(u_2)$, which contradicts the Subset Lemma. Thus, Fact 1 holds.

Proof of Fact 2: By Fact 1, if $\deg_G u = 3$, then $2 \leq |f(u)| \leq 3$. Assume, to the contrary, that $|f(x)| = 2$ if $x \in \{u_i, v_i\}$ for $3 \leq i \leq 5$. Without loss of generality, we may assume that $f(u_3) = 12$, $f(u_4) = 34$, $f(u_5) = 15$, $f(v_5) = 23$, $f(v_4) = 16$, and $f(v_3) = 35$. This, however, forces $\{2, 3\} \subseteq f(u_6)$, and so $f(v_5) \subseteq f(u_6)$, which contradicts the Subset Lemma. Thus, Fact 2 holds.

Proof of Fact 3: Assume, to the contrary, that there are two adjacent vertices u and v of degree 3 such that $|f(u)| = |f(v)| = 3$. Then either $\{u, v\} = \{u_i, u_{i+1}\}$ for $2 \leq i \leq n-2$ or $\{u, v\} = \{u_i, v_i\}$ for $2 \leq i \leq n-1$. We will consider the situations when $\{u, v\} = \{u_2, u_3\}$ and $\{u, v\} = \{u_2, v_2\}$ since the arguments for other cases are similar. First, suppose that $\{u, v\} = \{u_2, u_3\}$ where say $f(u_2) = 123$ and $f(u_3) = 456$. This forces $7 \in f(v_3)$, for otherwise, $f(v_3) \subseteq f(u_2)$, which is impossible by the Subset Lemma. However, then, $f(u_2) \cup f(v_3) = \{1, 2, 3, 7\}$, which forces $f(v_2) \subseteq f(u_3)$. Again, this is impossible by the Subset Lemma. Next, suppose that $\{u, v\} = \{u_2, v_2\}$, where say $f(u_2) = 123$ and $f(v_2) = 456$. This forces $7 \in f(v_3)$, for otherwise, $f(v_3) \subseteq f(u_2)$, which is impossible by the Subset Lemma. Here again, $f(u_2) \cup f(v_3) = \{1, 2, 3, 7\}$, which forces $f(u_3) \subseteq f(v_2)$, a contradiction by the Subset Lemma.

By **Fact 2**, we may assume that $f(u_3) = 123$ or $f(u_4) = 123$. We consider two cases.

Case 1. $f(u_3) = 123$. By **Fact 3**, $|f(x)| = 2$ for each $x \in \{u_2, v_3, v_4\}$. We may suppose, without loss of generality, that $f(u_4) = 45$ and $f(v_3) = 46$. Since $f(u_2) \cap f(v_4) \neq \emptyset$, it follows that $f(u_2) \neq 56$. Furthermore, $f(u_2) \cup f(v_3) \neq \{4, 5, 6, 7\}$; for otherwise, $f(v_2) \subseteq f(u_3)$, which is impossible by the Subset Lemma. Therefore, $f(u_2)$ cannot be any of $45, 46, 56, 57, 67$, which forces $f(u_2) = 47$. This, in turn, forces $6 \in f(u_5)$ and $4 \in f(u_6)$, which forces $\{5, 6, 7\} \subseteq f(v_6)$. Since $|f(v_6)| \leq 3$ by **Fact 1**, it follows that $f(v_6) = 567$. However, then, $f(v_6) \cap f(u_3) = \emptyset$, a contradiction.

Case 2. $f(u_4) = 123$. By the argument used in Case 1, we may assume, without loss of generality, that $f(u_5) = 45, f(v_4) = 46$, and $f(u_3) = 47$ and conclude that $\{5, 6, 7\} \subseteq f(v_7)$ and $\{5, 6, 7\} \subseteq f(v_1)$. Since $f(v_7) \cap f(u_4)$ and $f(v_1) \cap f(u_4) \neq \emptyset$, it follows that $f(v_1) = t_1 567$ and $f(v_1) = t_2 567$, where $t_1 \neq t_2$ and $t_1, t_2 \in \{1, 2, 3\}$. We can further assume that $f(v_1) = 1567$ and $f(v_7) = 2567$. This forces $2 \in f(u_1) \cap f(v_2)$ and $1 \in f(u_7) \cap f(v_6)$. If $x \in \{u_1, v_2, u_7, v_6\}$, then $7 \notin f(x)$ and $f(x) \cap f(u_3) \neq \emptyset$, which forces $4 \in f(x)$. Consequently, $\{2, 4\} \subseteq f(u_1) \cap f(v_2)$ and $\{1, 4\} \subseteq f(u_7) \cap f(v_6)$. Since $f(u_1) \not\subseteq f(v_2)$ by the Subset Lemma, this forces $f(u_1) = 234$. Thus, $f(u_1) \cup f(u_3) = \{2, 3, 4, 7\}$, and so $f(u_2) \subseteq \{1, 5, 6\}$. On the other hand, $f(v_4) \cup f(u_5) \cup f(v_6) \subseteq \{1, 4, 5, 6\}$, and so $f(v_5) \subseteq \{2, 3, 7\}$. However, then, $u_2 v_5 \notin E(G)$ and $f(u_2) \cap f(v_5) = \emptyset$, a contradiction.

Therefore, we conclude that $\rho(G) = 8$. ■

We have seen that if $n \geq 3$, then $\rho(P_n) \leq \rho(P_{n+1}) \leq \rho(P_n) + 1$. Consequently,

$$0 \leq \rho(P_{n+1}) - \rho(P_n) \leq 1 \text{ for each integer } n \geq 3.$$

This, however, is not the case for $\rho(P_n \square K_2)$. For example, we saw that $\rho(P_3 \square K_2) = 4$ in Example 5.3.2 and $\rho(P_4 \square K_2) = 6$ in Example 5.3.3. Thus, $\rho(P_4 \square K_2) - \rho(P_3 \square K_2) = 2$. In fact, $\rho(P_{n+1} \square K_2)$ cannot exceed $\rho(P_n \square K_2)$ by more than 2, as we show next.

Theorem 5.3.7 *For each integer $n \geq 3$,*

$$\rho(P_n \square K_2) \leq \rho(P_{n+1} \square K_2) \leq \rho(P_n \square K_2) + 2.$$

Proof. Since $\rho(P_n \square K_2) \leq \rho(P_{n+1} \square K_2)$ by Observation 5.3.1, it remains to verify the upper bound. Let $F = P_n \square K_2$ be constructed from the two n-paths $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ by adding the edges $u_i v_i$ for $1 \leq i \leq n$, and let $G = P_{n+1} \square K_2$ be constructed from the two $(n+1)$-paths $(u_1, u_2, \ldots, u_n, u_{n+1})$ and $(v_1, v_2, \ldots, v_n, u_{n+1})$ by adding the edges $u_i v_i$ for $1 \leq i \leq n + 1$. Suppose that $\rho(P_n \square K_2) = k$. Let $g : V(F) \to \mathcal{P}^*([k])$ be a subset labeling of F. This labeling gives rise to a subset labeling $h : V(F) \to \mathcal{P}^*([3, k+2])$ defined by

$$h(x) = \{i + 2 : \ i \in g(x)\} \text{ for each vertex } x \text{ of } F.$$

We now define the labeling $f : V(G) \to \mathcal{P}^*([k+2])$ of G by

$$f(u_i) = \begin{cases} h(u_i) & \text{if } 1 \le i \le n-3 \text{ and } i = n \\ h(u_i) \cup \{1\} & \text{if } i = n-2 \\ h(u_i) \cup \{2\} & \text{if } i = n-1 \\ h(u_{n-1}) \cup \{1\} & \text{if } i = n+1 \end{cases}$$

$$f(v_i) = \begin{cases} h(v_i) & \text{if } 1 \le i \le n-3 \text{ and } i = n \\ h(v_i) \cup \{2\} & \text{if } i = n-2 \\ h(v_i) \cup \{1\} & \text{if } i = n-1 \\ h(v_{n-1}) \cup \{2\} & \text{if } i = n+1. \end{cases}$$

It remains to show that f is a subset labeling of G. Let x and y be two vertices of G. If $x, y \in V(F)$, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. If $\{x, y\} = \{u_{n+1}, v_{n+1}\}$, say $x = u_{n+1}$ and $y = v_{n+1}$, then $f(x) = h(u_{n-1}) \cup \{1\}$ and $f(y) = h(v_{n-1}) \cup \{2\}$. Since $h(u_{n-1}) \cap h(v_{n-1}) = \emptyset$, it follows that $f(x) \cap f(y) = \emptyset$. Next, we assume that $x \in \{u_{n+1}, v_{n+1}\}$ and $y \in V(F)$. By the definition of f, we may assume that $x = u_{n+1}$. Hence, $f(x) = f(u_{n+1}) = h(u_{n-1}) \cup \{1\}$.

- ⋆ If $y = u_i$ where $1 \le i \le n-3$, then $xy \notin E(G)$. Since $h(u_i) \cap h(u_{n-1}) \ne \emptyset$, $f(x) = h(u_{n-1}) \cup \{1\}$ and $f(u_i) = h(u_i)$, it follows that $f(x) \cap f(y) \ne \emptyset$.
- ⋆ If $y = u_{n-2}$, then $1 \in f(x) \cap f(y)$. Thus, $xy \notin E(G)$ and $f(x) \cap f(y) \ne \emptyset$.
- ⋆ If $y = u_{n-1}$, then $h(u_{n-1}) \subseteq f(x) \cap f(y)$. Hence, $xy \notin E(G)$ and $f(x) \cap f(y) \ne \emptyset$.
- ⋆ If $y = u_n$, then $xy \in E(G)$. Since $h(u_{n-1}) \cap h(u_n) = \emptyset$ and $1 \notin h(u_n)$, it follows that $f(x) \cap f(y) = \emptyset$.
- ⋆ If $y = v_i$ where $1 \le i \le n-2$ or $i = n$, then $xy \notin E(G)$. Since $h(v_i) \cap h(u_{n-1}) \ne \emptyset$, $f(x) = h(u_{n-1}) \cup \{1\}$, and $f(v_i) = h(v_i)$, it follows that $f(x) \cap f(y) \ne \emptyset$.
- ⋆ If $y = v_{n-1}$, then $1 \in f(x) \cap f(y)$. Thus, $xy \notin E(G)$ and $f(x) \cap f(y) \ne \emptyset$.

Hence, f is a subset labeling of G, and so $\rho(G) \le k + 2 = \rho(F) + 2$. ■

For an integer $n \geq 2$, the graph $P_n \square K_2$ is often referred to as *the ladder graph* and therefore is denoted by L_n. With this notation, the following corollary is a consequence of Theorems 5.3.5 and 5.3.7:

Corollary 5.3.8 *For each integer* $n \geq 2$,

$$\max\{\rho(P_{n+3}),\ \rho(L_n)\} \leq \rho(L_{n+1}) \leq 2 + \min\{\rho(P_{n+1}) + 1,\ \rho(L_n)\}.$$

For $n = 8$ and $k = \rho(L_8)$, Figure 5.6 illustrates the subset labeling

$$f : V(L_9) \to \mathcal{P}^*([k+2]) \text{ of } L_9$$

that is constructed from a subset labeling $g : V(L_8) \to \mathcal{P}^*([k])$ of L_8, as described in the proof of Theorem 5.3.7.

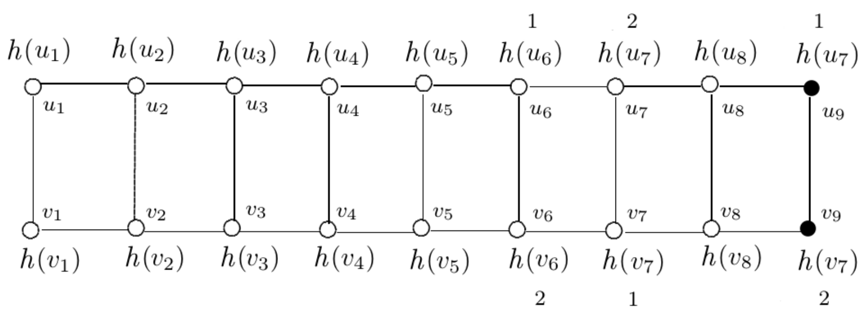

Figure 5.6: Constructing a subset labeling of L_{n+1}
as described in the proof of Theorem 5.3.7

Chapter 6

Subset Labelings of Graph Unions

A *union* of graphs $G_1, G_2, \ldots, G_n$, also called the *sum* $G_1 + G_2 + \cdots + G_n$, is the graph whose vertex set is the disjoint union of the vertex sets of $G_1, G_2, \ldots, G_n$ and whose edge set is the disjoint union of the edge sets of $G_1, G_2, \ldots, G_n$. We give a few preliminary results, then focus on the subset index problem for unions of complete graphs and unions of cycles.

6.1 Preliminary Results on Graph Unions

We first present an upper bound for the subset index of the union of two graphs.

Theorem 6.1.1 *For two graphs G_1 and G_2, $\rho(G_1 + G_2) \leq \rho(G_1)\rho(G_2)$.*

Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_p\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_q\}$. Let $\rho(G_1) = r_1, \rho(G_2) = r_2$. Define subset labelings $f_1 : V(G_1) \to \mathcal{P}^*([r_1])$ and $f_2 : V(G_2) \to \mathcal{P}^*([r_2])$. We define a subset labeling $f : V(G_1 + G_2) \to \mathcal{P}^*([r_1 r_2])$ by the $r_1 \times r_2$ matrix A below (denoting the ith row as R_i, $1 \leq i \leq r_1$ and the jth column as C_j, $1 \leq j \leq r_2$) in the following way: Let $R_i \subseteq f(u)$ if and only if $i \in f_1(u)$ for some vertex $u \in V(G_1)$, and let $C_j \subseteq f(v)$ if and only if $j \in f_2(v)$ for some vertex $v \in V(G_2)$.

$$A = \begin{bmatrix} 1 & 2 & 3 & \ldots & r_2 \\ r_2+1 & r_2+2 & r_2+3 & \ldots & 2r_2 \\ 2r_2+1 & 2r_2+2+2 & 2r_2+3 & \ldots & 3r_2 \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ (r_1-1)r_2+1 & (r_1-1)r_2+2 & (r_1-1)r_2+3 & \ldots & r_1 r_2 \end{bmatrix}$$

By construction, for $x, y \in V(G_1)$ or $x, y \in V(G_2)$, we have $f(x) \cap f(y) = \emptyset$ if and only if $f_1(x) \cap f_1(y) = \emptyset = f_2(x) \cap f_2(y)$, while for $x \in V(G_1)$ and $y \in V(G_2)$, we have $f(x) \cap f(y) \neq \emptyset$. Hence, $\rho(G_1 + G_2) \leq \rho(G_1)\rho(G_2)$. ■

We illustrate this with the example $P_5 + C_7$, recalling that $\rho(P_5) = 4$ and $\rho(C_7) = 7$. Let $P_5 = (u_1, u_2, u_3, u_4)$ and $C_7 = (v_1, v_2, v_3, \ldots, v_7, v_1)$, and define $f_1 : V(P_5) \to \mathcal{P}^*([4])$ and $f_2 : V(C_7) \to \mathcal{P}^*([7])$ as follows:

$$\mathcal{S}_{f_1}(P_5) = (12, 34, 1, 23, 14)$$
$$\mathcal{S}_{f_2}(C_7) = (123, 456, 127, 345, 167, 234, 567)$$

Then, referring to the matrix A below, we may define $f : V(P_5 + C_7) \to \mathcal{P}^*([28])$ as follows, where for convenience we write the union of the collections of elements in rows R_{i_1} and R_{i_2} as $R_{i_1}R_{i_2}$. Similarly, we denote the union of the collections of elements in columns C_{j_1} and C_{j_2} as $C_{j_1}C_{j_2}$, and so on for unions of three or more rows/columns.

$$(f(u_1), f(u_2), \ldots, f(u_5)) =$$
$$(R_1R_2, R_3R_4, R_1, R_2R_3, R_1R_4)$$

$$(f(v_1), f(v_2), f(v_3), \ldots, f(v_7)) =$$
$$(C_1C_2C_3, C_4C_5C_6, C_1C_2C_7, C_3C_4C_5, C_1C_6C_7, C_2C_3C_4, C_5C_6C_7)$$

$$A = \begin{bmatrix} 1 & 2 & 3 & \ldots & 7 \\ 8 & 9 & 10 & \ldots & 14 \\ 15 & 16 & 17 & \ldots & 21 \\ 22 & 23 & 24 & \ldots & 28 \end{bmatrix}$$

This bound is sharp, attained by the class of complete graph unions $K_p + K_q$.

Corollary 6.1.2 $\rho(K_p + K_q) = \rho(K_p)\rho(K_q)$

Proof. By Proposition 6.1.1, it suffices to show that $\rho(K_p + K_q) \geq \rho(K_p)\rho(K_q)$. Observe that for any subset labeling f of $K_p + K_q$, we must have $|f(x)| \geq q$ for all $x \in V(K_p)$, since every two vertices of K_p are adjacent, while $|f(y)| \geq p$ for all $y \in V(K_q)$, since every two vertices of K_q are adjacent. We conclude that $\rho(K_p + K_q) \geq pq$. ■

6.2 Subset Index of kK_2

We now consider the special case of kK_2, $k \geq 2$. The following conjecture for the index of such graphs was made independently by Chartrand and McGrew, first formulated by Chartrand as follows:

Conjecture 6.2.1 *Let k and s be integers where $k \geq 2$ and $s \geq 3$.*

(i) If $\binom{2s-1}{s} \leq k \leq \binom{2s}{s-1}$, then $\rho(kK_2) = 2s+1$.

(ii) If $\binom{2s}{s-1} + 1 \leq k \leq \binom{2s+1}{s}$, then $\rho(kK_2) = 2s+2$.

That is, if $((k))$ is the minimum integer t such that $k \leq \binom{t-1}{\lfloor \frac{t-2}{2} \rfloor}$, then $\rho(kK_2) = ((k))$.

It turns out that this conjecture is true. In fact, the problem is equivalent to one of the almost intersecting family problems introduced in [8]. We observe that the labels in a subset labeling $f : V(kK_2) \to \mathcal{P}^*([r])$ form a 1-almost intersecting family of size $2k$. That is, each label is disjoint from exactly one other label in the collection. Since there are no other constraints imposed by the structure of the graph, we observe that the size of the largest 1-almost intersecting family $\mathcal{F}$ with sets from $\mathcal{P}^*([r])$ for a given r equals twice the largest k such that $\rho(kK_2) = r$ for a given r. In [8], the authors state and prove the following, using theorems of Sperner and Bollobás:

$$|\mathcal{F}| \leq \begin{cases} \binom{r}{r/2} & \text{if } r \text{ is even} \\ 2\binom{r-1}{\lfloor r/2 \rfloor - 1} & \text{if } r \text{ is odd} \end{cases}$$

For example, if $r = 6$, then $|\mathcal{F}| \leq \binom{6}{6/2} = 20$. In terms of the notation in our conjecture, $6 = ((10))$. If $r = 7$, then $|\mathcal{F}| \leq 2\binom{6}{2} = 30$, or in our notation, $7 = ((15))$.

We give subset labelings $f : V(10K_2) \to \mathcal{P}^*([6])$ and $f : V(15K_2) \to \mathcal{P}^*([7])$, which also constitute best-possible constructions for $|\mathcal{F}|$ as proven in the result from [8]. In general, we denote the set of vertices from one side of the k copies of K_2 as $u_1, u_2, ..., u_k$ and the set from the other side as $v_1, v_2, \ldots, v_k$. We define a subset labeling $f : V(10K_2) \to \mathcal{P}^*([6])$ as follows:

$$(f(u_1), f(u_2), \ldots, f(u_{10})) = (123, 124, 125, 126, 134, 135, 136, 145, 146, 156)$$
$$(f(v_1), f(v_2), \ldots, f(v_{10})) = (456, 356, 346, 345, 256, 246, 245, 236, 235, 234)$$

We define a subset labeling $f : V(15K_2) \to \mathcal{P}^*([7])$ as follows:

$$(f(u_1), f(u_2), \ldots, f(u_{15})) =$$
$$(123, 124, 125, 126, 127, 134, 135, 136, 137, 145, 146, 147, 156, 157, 167)$$
$$(f(v_1), f(v_2), \ldots, f(v_{15})) =$$
$$(4567, 3567, 3467, 3457, 3456, 2567, 2467, 2457, 2456, 2367, 2357, 2356, 2347, 2346, 2345)$$

We can make a further connection to the index of the *friendship graph* F_k, which is the graph of order $2k+1$ and size $3k$ constructed by identifying k copies of C_3 with a common vertex. We show that if $\rho(kK_2) = r$, then $\rho(F_k) = r+1$. To show $\rho(F_k) \geq r+1$, suppose, to the contrary, that there exists a subset labeling $f : V(F_k) \to \mathcal{P}^*([r])$. Then this would force the label for the common vertex u to have nonempty intersection with the label for some vertex in kK_2, a contradiction. To show $\rho(F_k) \leq r+1$, suppose we have a subset labeling $g : V(kK_2) \to \mathcal{P}^*([r])$. Define a subset labeling $f : V(F_k) \to \mathcal{P}^*([r+1])$ by letting $f(v) = g(v)$ for each vertex v of kK_2 and letting $f(u) = r+1$ for the common vertex u. We conclude that $\rho(F_k) = r+1$.

The problem of finding the index of kK_n for $n \geq 3$ remains unsolved.

6.3 Subset Labelings of Cycle Unions

We begin with the following observation for the union $kC_4 = k(K_2 \,\square\, K_2)$:

Proposition 6.3.1 *Let $k \geq 1$. Then $\rho(kC_4) = \rho(kK_2)$.*

Proof. Let $r = \rho(kK_2)$. Since kK_2 is an induced subgraph of kC_4, we have $\rho(kC_4) \geq r$. To show that $\rho(kC_4) \leq r$, let $g : V(kK_2) \to \mathcal{P}^*([r])$ be a subset labeling of kK_2. For $1 \leq i \leq k$, let $C_4^{(i)}$ denote the ith of the k 4-cycles, where $C_4^{(i)} = (v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, v_4^{(i)}, v_1^{(i)})$. Then we may define $f : V(kC_4) \to \mathcal{P}^*([r])$ in the following way: For each $C_4^{(i)}$, let the k subgraphs K_2 formed by $(v_1^{(i)}, v_4^{(i)})$ be labeled according to g, then let $f(v_3^{(i)}) = f(v_1^{(i)})$ and $f(v_4^{(i)}) = f(v_2^{(i)})$. Then we have $(f(v_1^{(i)}) \cup f(v_3^{(i)})) \cap (f(v_2^{(i)}) \cup f(v_4^{(i)})) = \emptyset$ for each i. Since g is a subset labeling, $f(x) \cap f(y) \neq \emptyset$ for $x \in C_4^{(i)}$ and $y \in C_4^{(j)}$, $i \neq j$. Hence, $\rho(kC_4) \leq r$. And so, $\rho(kC_4) = r$. ■

From Proposition 6.1.1, $\rho(G_1 + G_2) \leq \rho(G_1)\rho(G_2)$ for every two graphs G_1 and G_2. However, when G_1 and G_2 are cycles C_p and C_q, $p \leq q$, we can establish much closer upper bounds for $\rho(C_p + C_q)$ in terms of the larger cycle. We begin with the special case $p = q$.

Theorem 6.3.2 *Let $G = 2C_p$, $p \geq 5$.*

- ⋆ *If p is odd, $\rho(G) \leq \rho(C_p) + 3$.*
- ⋆ *If p is even, $\rho(G) \leq \rho(C_p) + 2$.*

Proof. Denote the two copies of C_p as C_p and $C_p^{'}$. Let $C_p = (u_1, u_2, \ldots, u_p, u_1)$ and $C_p^{'} = (v_1, v_2, \ldots, v_p, v_1)$, and let $\rho(C_p) = r$. Suppose $h : V(C_p) \to \mathcal{P}^*([r])$ is a subset labeling of C_p.

To prove the first statement, consider the labeling $f : V(G) \to \mathcal{P}^*([r+3])$ defined by:

$$f(u_i) = \begin{cases} h(u_i) \cup \{r+3\} & \text{if } i = 1 \\ h(u_i) \cup \{r+1\} & \text{if } i \text{ is even, } i \neq p-1 \\ h(u_i) \cup \{r+2\} & \text{if } i \text{ is odd, } i \notin \{1,3\} \\ h(u_i) \cup \{r+2, r+3\} & \text{if } i = 3 \\ h(u_i) \cup \{r+1, r+3\} & \text{if } i = p-1 \end{cases}$$

$$f(v_i) = \begin{cases} h(u_{i+1}) \cup \{r+3\} & \text{if } i \in \{1, p-1\} \\ h(u_{i+1}) \cup \{r+1\} & \text{if } i \text{ even, } i \neq p-1 \\ h(u_{i+1}) \cup \{r+2\} & \text{if } i \text{ odd, } i \notin \{1,p\} \\ h(u_{i+1}) \cup \{r+1, r+2\} & \text{if } i = p \end{cases}$$

We show that f is a subset labeling. If vertices $x = u_j$ and $y = u_k$ are both in $V(C_p)$, then we have $f(x) \cap f(y) = \emptyset$ if and only if $h(u_{j-1}) \cap h(u_{k-1}) = \emptyset$. If vertices $x = v_j$ and $y = v_k$ are both in $V(C_p^{'})$, then $f(x) \cap f(y) = \emptyset$ if and only if $h(u_j) \cap h(u_k) = \emptyset$. Furthermore, if $x = u_i \in V(C_p)$ and $y = v_j \in V(C_p^{'})$, where $j \notin \{i-2, i\}$, then $f(x) \cap f(y) \neq \emptyset$, since $h(u_i) \cap h(u_j) \neq \emptyset$. It remains to consider the following cases:

- ⋆ If $x = u_i$, $y = v_i$, i even and $i \neq p-1$, then $xy \notin E(G)$. Since $r+1 \in f(u_i) \cap f(v_i)$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_{p-1}$ and $y = v_{p-1}$, then $xy \notin E(G)$. Since $r+3 \in f(u_{p-1}) \cap f(v_{p-1})$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_i$, $y = v_i$, i odd, $i \neq 1$, then $xy \notin E(G)$. Since $r+2 \in f(u_i) \cap f(v_i)$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_1$, $y = v_1$, then $xy \notin E(G)$. Since $r+3 \in f(u_1) \cap f(v_1)$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_i$, $y = v_{i-2}$, i odd, $i \notin \{1, 3\}$, then $xy \notin E(G)$. Since $r + 2 \in f(u_i) \cap f(v_{i-1})$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_1$, $y = v_{p-1}$, then $xy \notin E(G)$. Since $r + 3 \in f(u_1) \cap f(v_{p-1})$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_3$, $y = v_1$, then $xy \notin E(G)$. Since $r+3 \in f(u_3) \cap f(v_1)$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_i$, $y = v_{i-2}$, i even, then $xy \notin E(G)$. Since $r + 1 \in f(u_i) \cap f(v_{i-2})$, we have $f(x) \cap f(y) \neq \emptyset$.

We conclude that f is a subset labeling. Hence, $\rho(2C_p) \leq \rho(C_p) + 3$ for p odd.

To prove the second statement, consider the labeling $f : V(G) \to \mathcal{P}^*([r+2])$ defined by:

$$f(u_i) = \begin{cases} h(u_i) \cup \{r+1\} & \text{if } i \text{ is odd} \\ h(u_i) \cup \{r+2\} & \text{if } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} h(u_{i+1}) \cup \{r+1\} & \text{if } i \text{ is odd} \\ h(u_{i+1}) \cup \{r+2\} & \text{if } i \text{ is even} \end{cases}$$

We show that f is a subset labeling. If vertices $x = u_j$ and $y = u_k$ are both in $V(C_p)$, then we have $f(x) \cap f(y) = \emptyset$ if and only if $h(u_j) \cap h(u_k) = \emptyset$. If vertices $x = v_j$ and $y = v_k$ are both in $V(C_p^{'})$, then $f(x) \cap f(y) = \emptyset$ if and only if $h(u_{j+1}) \cap h(u_{k+1}) = \emptyset$. Furthermore, if $x = u_i \in V(C_p)$ and $y = v_j \in V(C_p^{'})$, $j \notin \{i-2, i\}$, then $f(x) \cap f(y) \neq \emptyset$, since $h(u_i) \cap h(u_j) \neq \emptyset$. It remains to consider the following cases:

- ⋆ If $x = u_i, y = v_j$, $j \in \{i-2, i\}$, i odd, then $xy \notin E(G)$. Since $r + 1 \in f(u_i) \cap f(v_j)$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_i, y = v_j$, $j \in \{i-2, i\}$, i even, then $xy \notin E(G)$. Since $r + 2 \in f(u_i) \cap f(v_j)$, we have $f(x) \cap f(y) \neq \emptyset$.

We conclude that f is a subset labeling. Hence, $\rho(2C_p) \leq \rho(C_p) + 2$ for p even. ∎

These upper bounds are attained for $p = 5$ and $p = 6$.

Observation 6.3.3 *Let $G = C_5 + C_q$, where $q \geq 5$ is odd. Then $\rho(G) \geq 8$.*

Proof. Let $C_5 = (u_1, u_2, u_3, u_4, u_5, u_1)$. Assume, to the contrary, that there exists a subset labeling $f : V(G) \to \mathcal{P}^*([7])$. First, we observe that $|f(u_i)| \geq 3$ for all $u_i \in V(C_5)$. Otherwise, if say $f(u_2) = 12$, then, without loss of generality, we must have $1 \in f(v_i)$ for i odd and $2 \in f(v_i)$ for i even. But then this would mean $1 \in f(v_1) \cap f(v_q)$, which is a contradiction. Thus, without loss of generality, say $f(u_1) = 123, f(u_2) = 456, f(u_3) = 127, f(u_4) = 345$. This forces $f(u_5) = 67$, which is a contradiction. Hence, $\rho(G) \geq 8$. ■

In particular, $\rho(2C_5) \geq 8$. We give a subset labeling $f : V(2C_5) \to \mathcal{P}^*([8])$:

$$(f(u_1), f(u_2), f(u_3), f(u_4), f(u_5)) = (123, 456, 178, 236, 458)$$
$$(f(v_1), f(v_2), f(v_3), f(v_4), f(v_5)) = (125, 348, 156, 247, 368)$$

We conclude that $\rho(2C_5) = \rho(C_5) + 3 = 8$.

Proposition 6.3.4 *Let $G = 2C_6$. Then $\rho(G) = 7$.*

Proof. Let $C_6 = (u_1, u_2, u_3, u_4, u_5, u_6, u_1)$ and $C_6' = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$. To show that $\rho(G) \leq 7$, we give a subset labeling $f : V(G) \to \mathcal{P}^*([7])$:

$$(f(u_1), f(u_2), f(u_3), f(u_4), f(u_5), f(u_6)) = (135, 246, 157, 234, 156, 247)$$
$$(f(v_1), f(v_2), f(v_3), f(v_4), f(v_5), f(v_6)) = (123, 456, 127, 345, 126, 457)$$

It remains to show that $\rho(G) \geq 7$. Assume, to the contrary, that there exists a subset labeling $f : V(G) \to \mathcal{P}^*([6])$. We must have $|f(x)| < 4$ for all $x \in V(G)$, otherwise we would have $f(y) \subseteq f(z)$ for $y, z \in N(x)$, which contradicts the Subset Lemma. Further, by Proposition 2.3.6, we cannot have $|f(x)| = 3 = |f(y)|$ for any two adjacent vertices $x, y \in V(G)$. Hence, we must have $|f(x)| = 2$ for at least three vertices $x \in V(C_6)$ and at least three vertices $y \in V(C_6')$. For our vertices from $V(C_6')$, suppose we choose v_i where either all i are odd or all i are even, say without loss of generality $f(v_1) = 12, f(v_3) = 14, f(v_5) = 16$. Then this forces $f(u_j) = 246 = f(u_k)$ for some two vertices $u_j, u_k \in V(C_6)$, which contradicts the Subset Lemma. Hence, without loss of generality, we must choose two vertices v_i even and one odd. Say $f(v_1) = 12, f(v_2) = 34$, and $f(v_4) = 13$. Then, without loss of generality, we have $\{1, 4\} \subseteq f(u_i)$ for all i odd and $\{2, 3\} \subseteq f(u_i)$ for all i even. Since at least three of these labels must be 2-sets, we have at least two equal, which contradicts the Subset Lemma. Thus, $\rho(G) \geq 7$. We conclude that $\rho(G) = 7$. ■

Since C_p is an induced subgraph of G, we have $\rho(C_p) \leq \rho(G)$ in general. This lower bound is attained in the case $p = 7$. Letting $C_7 = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_1), C_7' = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$, we give a subset labeling $f : V(2C_7) \to \mathcal{P}^*([7])$:

$$(f(u_1), f(u_2), f(u_3), f(u_4), f(u_5), f(u_6), f(u_7)) = (135, 246, 357, 146, 257, 136, 247)$$
$$(f(v_1), f(v_2), f(v_3), f(v_4), f(v_5), f(v_6), f(v_7)) = (123, 456, 127, 345, 167, 234, 567)$$

For even p, we make a conjecture for the index of the general union kC_p.

Conjecture 6.3.5 *$\rho(kC_p) \leq \rho(C_p) + k$ for even $p \geq 4$.*

We now present an upper bound for $\rho(C_p + C_q)$ when $p < q$.

Theorem 6.3.6 *Let $G = C_p + C_q$, $p < q$. Then $\rho(G) \leq \rho(C_q) + 3$.*

Proof. Let $C_p = (u_1, u_2, \ldots, u_p, u_1)$ and $C_q = (v_1, v_2, \ldots, v_q, v_1)$. We consider four cases.

Case 1. p odd, q even. Let $\rho(C_q) = r$, and let $h : V(C_q) \to \mathcal{P}^*([r])$ be a subset labeling of C_q. Let $a \in h(v_3) \cap h(v_q)$. We claim the following labeling $f : V(G) \to \mathcal{P}^*([r+3])$ is a subset labeling:

$$f(u_i) = \begin{cases} \{r+1, r+2\} & \text{if } i = 1 \\ h(v_{i-1}) \cup \{a, r+3\} & \text{if } i = 2 \\ h(v_{i-1}) \cup \{r+1\} & \text{if } i \text{ odd, } i \notin \{1, p\} \\ h(v_{i-1}) \cup \{r+2\} & \text{if } i \text{ even, } i \neq 2 \\ h(v_{i-1}) \cup \{r+3\} & \text{if } i = p \end{cases}$$

$$f(v_i) = \begin{cases} h(v_i) \cup \{r+1\} & \text{if } i \text{ odd, } i \notin \{p-2, p, q-1\} \\ h(v_i) \cup \{r+2\} & \text{if } i \text{ even, } i \neq 2 \\ h(v_i) \cup \{r+2, r+3\} & \text{if } i = 2 \\ h(v_i) \cup \{r+1, r+3\} & \text{if } i \in \{p-2, p, q-1\} \end{cases}$$

If vertices x and y are both in $V(C_q)$, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. If $x = u_j$ and $y = u_k$ are both in $V(C_p)$, where $j, k \neq 1$, then we have $f(x) \cap f(y) = \emptyset$ if and only if $h(v_{j-1}) \cap h(v_{k-1}) = \emptyset$. Furthermore, if $x = u_i$ and $y = v_j$, $i \neq 1$, $j \notin \{i-2, i\}$, then $f(x) \cap f(y) \neq \emptyset$, since $h(v_{i-1}) \cap h(v_j) \neq \emptyset$. It remains to consider the following possibilities for x and y:

- ⋆ If $x = u_1$ and $y = u_2$, then $xy \in E(G)$. Since $r+1, r+2 \notin f(u_2)$ and $r+3 \notin f(u_1)$, we have $f(x) \cap f(y) = \emptyset$.

- ⋆ If $x = u_1$ and $y = u_p$, then $xy \in E(G)$. Since $r+1, r+2 \notin f(u_p)$ and $r+3 \notin f(u_1)$, we have $f(x) \cap f(y) = \emptyset$.
- ⋆ If $x = u_1$ and $y = u_i$, $i \notin \{2, p\}$, then $xy \notin E(G)$. Since $r+1 \in f(u_i)$ for i odd and $r+2 \in f(u_i)$ for i even, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_1$ and $y = v_i$, then $xy \notin E(G)$. Since $r+1 \in f(v_i)$ for i odd and $r+2 \in f(v_i)$ for i even, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_2$ and $y = v_2$, then $xy \notin E(G)$. Since $r+3 \in f(u_2) \cap f(v_2)$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_2$ and $y = v_q$, then $xy \notin E(G)$. Since $a \in f(u_2) \cap f(v_q)$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_i$ and $y \in \{v_i, v_{i-2}\}$, i even, $i \neq 2$, then $xy \notin E(G)$. Since $r+2 \in f(u_i) \cap f(v_i) \cap f(v_{i-2})$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_i$ and $y \in \{v_i, v_{i-2}\}$, i odd, $i \notin \{1, p\}$, then $xy \notin E(G)$. Since $r+1 \in f(u_i) \cap f(v_i) \cap f(v_{i-2})$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_p$ and $y \in \{v_{p-2}, v_p\}$, then $xy \notin E(G)$. Since $r+3 \in f(u_p) \cap f(v_{p-2}) \cap f(v_p)$, we have $f(x) \cap f(y) \neq \emptyset$.

We conclude that f is a subset labeling.

Case 2. p even, q odd. Let $\rho(C_q) = r$, and suppose $h : V(C_q) \to \mathcal{P}^*([r])$ is a subset labeling of C_q. Since $q \geq p+1$, we observe that there exists some a such that $a \in h(v_{p-2}) \cap h(v_q)$. We claim the following labeling $f : V(G) \to \mathcal{P}^*([r+3])$ is a subset labeling:

$$f(u_i) = \begin{cases} h(v_i) \cup \{r+3\} & \text{if } i \in \{1, p-1\} \\ h(v_i) \cup \{r+1\} & \text{if } i \text{ even, } i \neq p \\ h(v_i) \cup \{r+2\} & \text{if } i \text{ odd, } i \notin \{1, p-1\} \\ \{r+1, r+2, a\} & \text{if } i = p \end{cases}$$

$$f(v_i) = \begin{cases} h(v_i) \cup \{r+1\} & \text{if } i \text{ odd, } i \neq q \\ h(v_i) \cup \{r+2, r+3\} & \text{if } i \in \{2, p-2, p\} \\ h(v_i) \cup \{r+2\} & \text{if } i \text{ is even, } i \notin \{2, p-2, p\} \\ h(v_i) \cup \{r+3\} & \text{if } i = q \end{cases}$$

If vertices x and y are both in $V(C_q)$, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. If x and y are both in $V(C_p)$, where $x, y \neq u_p$, then if $h(v_j) \subseteq f(x)$ and $h(v_k) \subseteq f(y)$, we have $f(x) \cap f(y) = \emptyset$ if and only if $h(v_j) \cap h(v_k) = \emptyset$. Furthermore, if $x = u_i$ and $y = v_j$, $i \neq p$, $j \notin \{i-1, i+1\}$, then $f(x) \cap f(y) \neq \emptyset$, since $h(v_i) \cap h(v_j) \neq \emptyset$. It remains to consider the following possibilities for x and y:

- ⋆ If $x = u_p$ and $y \in \{u_1, u_{p-1}\}$, then $xy \in E(G)$. We observe $r+3 \notin f(u_p)$ and $r+1, r+2 \notin f(u_1) \cup f(u_{p-1})$. Further, since $a \in h(v_{p-2}) \cap h(v_q)$, it follows that $a \notin h(v_1) \cup h(v_{p-1})$. Thus, we have $f(x) \cap f(y) = \emptyset$.

- ⋆ If $x = u_p$ and $y \in u_i, i \notin \{1, p-1\}$, then $xy \notin E(G)$. Since $r+1 \in f(u_i)$ for all i even, while $r+2 \in f(u_i)$ for all i odd, $i \notin \{1, p-1\}$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_p$ and $y \in v_i$, $i \neq q$, then $xy \notin E(G)$. Since $r+1 \in f(v_i)$ for all i odd, $i \neq q$, while $r+2 \in f(v_i)$ for all i even, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_p$ and $y = v_q$, then $xy \notin E(G)$. Since $a \in f(u_p) \cap f(v_q)$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_1$ and $y \in \{v_2, v_q\}$, then $xy \notin E(G)$. Since $r+3 \in f(u_1) \cap f(v_2) \cap f(v_q)$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_{p-1}$ and $y \in \{v_{p-2}, v_p\}$, then $xy \notin E(G)$. Since $r+3 \in f(u_{p-1}) \cap f(v_{p-2}) \cap f(v_p)$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_i$ and $y \in \{v_{i-1}, v_{i+1}\}$, i odd, $i \notin \{1, p-1\}$, then $xy \notin E(G)$. Since $r+2 \in f(u_i) \cap f(v_{i-1}) \cap f(v_{i+1})$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_i$ and $y \in \{v_{i-1}, v_{i+1}\}$, i even, $i \neq p$, then $xy \notin E(G)$. Since $r+1 \in f(u_i) \cap f(v_{i-1}) \cap f(v_{i+1})$, we have $f(x) \cap f(y) \neq \emptyset$.

We conclude that f is a subset labeling.

Case 3. p and q odd. Let $\rho(C_q) = r$, and suppose $h : V(C_q) \to \mathcal{P}^*([r])$ is a subset labeling of C_q. Since $q \geq p+2$, we observe that there exists some a such that $a \in h(v_p) \cap h(v_q)$. We claim the following labeling $f : V(G) \to \mathcal{P}^*([r+3])$ is a subset labeling:

$$f(u_i) = \begin{cases} h(v_i) \cup \{r+3\} & \text{if } i \in \{1, p-1\} \\ h(v_i) \cup \{r+1\} & \text{if } i \text{ even, } i \neq p-1 \\ h(v_i) \cup \{r+2\} & \text{if } i \text{ odd, } i \notin \{1, p\} \\ \{r+1, r+2, a\} & \text{if } i = p \end{cases}$$

$$f(v_i) = \begin{cases} h(v_i) \cup \{r+1\} & \text{if } i \text{ odd, } i \notin \{p-2, p, q\} \\ h(v_i) \cup \{r+1, r+3\} & \text{if } i \in \{p-2, p\} \\ h(v_i) \cup \{r+3\} & \text{if } i = q \\ h(v_i) \cup \{r+2, r+3\} & \text{if } i = 2 \\ h(v_i) \cup \{r+2\} & \text{if } i \text{ even, } i \neq 2 \end{cases}$$

If vertices x and y are both in $V(C_q)$, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. If x and y are both in $V(C_p)$, where $x, y \neq u_p$, then if $h(v_j) \subseteq f(x)$ and $h(v_k) \subseteq f(y)$, we have $f(x) \cap f(y) = \emptyset$ if and only if $h(v_j) \cap h(v_k) = \emptyset$. Furthermore, if $x = u_i$ and $y = v_j$, $i \neq p$, $j \notin \{i-1, i+1\}$, then $f(x) \cap f(y) \neq \emptyset$, since $h(v_i) \cap h(v_j) \neq \emptyset$. It remains to consider the following possibilities for x and y:

- ⋆ If $x = u_p$ and $y = u_1$, then $xy \in E(G)$. Since $r+1, r+2 \notin f(u_1)$, and since $a \in h(v_q)$ implies $a \notin h(v_1)$, we have $f(x) \cap f(y) = \emptyset$.
- ⋆ If $x = u_p$ and $y = u_{p-1}$, then $xy \in E(G)$. Since $r+1, r+2 \notin f(u_{p-1})$, and since $a \in h(v_p)$ implies $a \notin h(v_{p-1})$, we have $f(x) \cap f(y) = \emptyset$.
- ⋆ If $x = u_p$ and $y = v_q$, then $xy \notin E(G)$. Since $a \in f(u_p) \cap f(v_q)$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_p$ and $y = v_i, i \neq q$, then $xy \notin E(G)$. Since $r+1 \in f(v_i)$ for i odd, $i \neq q$ and $r+2 \in f(v_i)$ for i even, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_1$ and $y = v_q$, then $xy \notin E(G)$. Since $r+3 \in f(u_1) \cap f(v_q)$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_1$ and $y = v_2$, then $xy \notin E(G)$. Since $r+3 \in f(u_1) \cap f(v_2)$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_{p-1}$ and $y \in \{v_{p-2}, v_p\}$, then $xy \notin E(G)$. Since $r+3 \in f(u_{p-1}) \cap f(v_{p-2}) \cap f(v_p)$, we have $f(x) \cap f(y) \neq \emptyset$.
- ⋆ If $x = u_i$ and $y \in \{v_{i-1}, v_{i+1}\}$ for even $i \neq p-1$, then $xy \notin E(G)$. Since $r+1 \in f(u_i) \cap f(v_{i-1}) \cap f(v_{i+1})$, we have $f(x) \cap f(y) \neq \emptyset$.

⋆ If $x = u_i$ and $y \in \{v_{i-1}, v_{i+1}\}$ for odd $i \notin \{1, p\}$, then $xy \notin E(G)$. Since $r+2 \in f(u_i) \cap f(v_{i-1}) \cap f(v_{i+1})$, we have $f(x) \cap f(y) \neq \emptyset$.

We conclude that f is a subset labeling.

Case 4. p and q even. Let $\rho(C_q) = r$, and suppose $h : V(C_q) \to \mathcal{P}^*([r])$ is a subset labeling of C_q. Since $p > 3$, we observe that there exists some a such that $a \in h(v_1) \cap h(v_{p-1})$. We claim the following labeling $f : V(G) \to \mathcal{P}^*([r+3])$ is a subset labeling:

$$f(u_i) = \begin{cases} h(v_i) \cup \{r+1\} & \text{if } i \text{ is odd, } i \neq p-1 \\ \{r+1, r+2, a\} & \text{if } i = p-1 \\ h(v_i) \cup \{r+2\} & \text{if } i \text{ is even, } i \notin \{2, p-2, p\} \\ h(v_i) \cup \{r+2, r+3\} & \text{if } i = 2 \\ h(v_i) \cup \{r+3\} & \text{if } i = p-2 \\ h(v_q) \cup \{r+3\} & \text{if } i = p \end{cases}$$

$$f(v_i) = \begin{cases} h(v_i) \cup \{r+3\} & \text{if } i = 1 \\ h(v_i) \cup \{r+1\} & \text{if } i \text{ is even} \\ h(v_i) \cup \{r+2, r+3\} & \text{if } i \in \{p-3, p-1, q-1\} \\ h(v_i) \cup \{r+2\} & \text{if } i \text{ is odd, } i \notin \{1, p-3, p-1, q-1\} \end{cases}$$

If vertices x and y are both in $V(C_q)$, then $f(x) \cap f(y) = \emptyset$ if and only if $h(x) \cap h(y) = \emptyset$. If x and y are both in $V(C_p)$, where $x, y \neq u_{p-1}$, then if $h(v_i) \subseteq f(x)$ and $h(v_j) \subseteq f(y)$, we have $f(x) \cap f(y) = \emptyset$ if and only if $h(v_i) \cap h(v_j) = \emptyset$. Furthermore, if $x = u_i$ and $y = v_j$, $i \neq p-1$, $j \notin \{i-1, i+1\}$, then $f(x) \cap f(y) \neq \emptyset$, since $h(v_i) \cap h(v_j) \neq \emptyset$. It remains to consider the following possibilities for x and y:

⋆ If $x = u_{p-1}$ and $y \in \{u_{p-2}, u_p\}$, then $xy \in E(G)$. We observe that $a \in h(v_1) \cap h(v_{p-1})$ implies $a \notin h(v_q) \cup h(v_{p-2})$. Further, $r+3 \notin f(u_{p-1})$, and $r+1, r+2 \notin f(u_{p-2}) \cup f(u_p)$. Hence, we have $f(x) \cap f(y) = \emptyset$.

⋆ If $x = u_{p-1}$ and $y = u_i$, $i \notin \{p-2, p\}$, then $xy \notin E(G)$. Since $r+1 \in f(u_i)$ for all odd i and $r+2 \in f(u_i)$ for all even $i \notin \{p-2, p\}$, we have $f(x) \cap f(y) \neq \emptyset$.

⋆ If $x = u_{p-1}$ and $y = v_i$, then $xy \notin E(G)$. Since $a \in h(v_1)$, $r+1 \in f(v_i)$ for all even i, and $r+2 \in f(v_i)$ for all odd $i \neq 1$, we have $f(x) \cap f(y) \neq \emptyset$.

⋆ If $x = u_2$ and $y = v_1$, then $xy \notin E(G)$. Since $r+3 \in f(u_2) \cap f(v_1)$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_{p-2}$ and $y \in \{v_{p-3}, v_{p-1}\}$, then $xy \notin E(G)$. Since $r+3 \in f(u_{p-2}) \cap f(v_{p-3}) \cap f(v_{p-1})$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_p$ and $y \in \{v_1, v_{q-1}\}$, then $xy \notin E(G)$. Since $r+3 \in f(u_p) \cap f(v_1) \cap f(v_{q-1})$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_i$ and $y \in \{v_{i-1}, v_{i+1}\}$, i odd, $i \neq p-1$, then $xy \notin E(G)$. Since $r+1 \in f(u_i) \cap f(v_{i-1}) \cap f(v_{i+1})$, we have $f(x) \cap f(y) \neq \emptyset$.

- ⋆ If $x = u_i$ and $y \in \{v_{i-1}, v_{i+1}\}$, i even, $i \notin \{p-2, p\}$, then $xy \notin E(G)$. Since $r+2 \in f(u_i) \cap f(v_{i-1}) \cap f(v_{i+1})$, we have $f(x) \cap f(y) \neq \emptyset$.

We conclude that f is a subset labeling. By the four cases above, we conclude that $\rho(C_p + C_q) \leq \rho(C_q) + 3$ for any two integers p, q with $p < q$. ∎

It remains an open question whether there exist integers p and q such that $5 \leq p < q$ and $\rho(C_p + C_q) = \rho(C_q) + 3$. In some cases, the index may attain its lower bound of $\rho(C_q)$. For example, recalling that $\rho(C_{11}) = 8$, we give a subset labeling $f : V(C_7 + C_{11}) \to \mathcal{P}^*([8])$, letting $C_7 = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_1)$ and $C_{11} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_1)$:

$$(f(u_1), f(u_2), f(u_3), f(u_4), f(u_5), f(u_6), f(u_7)) = (136, 247, 3568, 147, 268, 134, 2578)$$

$$(f(v_1), f(v_2), f(v_3), f(v_4), f(v_5), f(v_6), f(v_7), f(v_8), f(v_9), f(v_{10}), f(v_{11})) = \\ (123, 4567, 128, 3456, 178, 2456, 378, 1256, 348, 1257, 468)$$

The general problem of finding bounds for a union $C_{p_1} + C_{p_2} + C_{p_3} + \cdots + C_{p_n}$ of cycles, $n \geq 3$, remains open for further research.

Bibliography

[1] R. L. Brooks, On coloring the nodes of a network. *Proc. Cambridge Philos. Soc.* **37** (1941), 194–197.

[2] G. Chartrand, C. Egan, and P. Zhang, *How to Label a Graph.* Springer, New York (2019).

[3] G. Chartrand and F. Harary, Planar permutation graphs. *Ann. Inst. H. Poincaré* (*Sect. B*) **3** (1967) 433-438.

[4] G. Chartrand, G. L. Johns, and S. L. Tian, Detour distance in graphs. *Quo vadis, graph theory? Ann. Discrete Math.* **55** North-Holland, Amsterdam (1993) 127-136,

[5] G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Second Edition, Chapman & Hall/CRC Press, Boca Raton (2020).

[6] P. Erdős and R. J. Wilson, On the chromatic index of almost all graphs. *J. Combin. Theory Ser. B* **23** (1977) 255-257.

[7] J. A. Gallian, A dynamic survey of graph labeling. *Electron. J. Combin.* (2017), #DS6.

[8] D. Gerbner, N. Lemons, C. Palmer, B. Patkós, V. Szécsi, Almost intersecting families of sets. *SIAM Journal on Discrete Mathematics.* **26(4)** (2012)

[9] S. W. Golomb, How to number a graph. *Graph Theory and Computing.* Academic Press, New York (1972), 23–37.

[10] R. L. Graham and N. J. A. Sloane, On additive bases and harmonious graphs. *SIAM J. Alg. Disc. Math.* **1** (1980), 382–404.

[11] F. Harary, *Graph Theory.* Addison-Wesley, Reading, MA (1969).

[12] P. Haxell, O. Pikhurko, and A. Taraz, Primality of trees. *J. Combin.* **2** (2011), 481–500.

[13] T. A. McKee and F. R. McMorris, *Topics in Intersection Graph Theory*. SIAM Monographs on Discrete Mathematics and Applications, Philadelphia, PA (1999).

[14] J. Mycielski, Sur le coloriage des graphes. *Colloq. Math.* **3** (1955) 161-162.

[15] E. A. Nordhaus and J. W. Gaddum, On complementary graphs. *Amer. Math. Monthly* **63** (1956) 175-177.

[16] R. Tout, A. N. Dabboucy, and K. Howalla, Prime labeling of graphs. *Nat. Acad. Sci. Lett.* **5** (1982), 365–368.

[17] V. G. Vizing, On an estimate of the chromatic class of a p-graph. (Russian) *Diskret. Analiz.* **3** (1964) 25-30.

[18] P. Zhang, *Color-Induced Graph Colorings*. Springer, New York (2015).

[19] P. Zhang, *A Kaleidoscopic View of Graph Colorings*. Springer, New York (2016).