Characterizing Undergraduate Students’ Proving Processes around “Stuck Points”

Yaomingxin Lu

Western Michigan University, luyaomingxin@gmail.com

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CHARACTERIZING UNDERGRADUATE STUDENTS’ PROVING PROCESSES
AROUND “STUCK POINTS”

by

Yaomingxin Lu

A dissertation submitted to the Graduate College
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
Mathematics
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Doctoral Committee:

Mariana Levin, Ph.D., Chair
Tabitha Y. Mingus, Ph.D.
Laura R. Van Zoest, Ph.D.
Shiv S. Karunakaran, Ph.D.
Learning to prove mathematical propositions is a cornerstone of mathematics as a discipline (de Villiers, 1990). However, since proving is a different mathematical activity as compared to students’ prior experience, research has also shown that many undergraduate students struggle to learn to prove, including those who major in mathematics (Moore, 1994; Selden, 2012). While the field has generated research that has analyzed the final products of proof (Selden & Selden, 2009) and there are frameworks for analyzing problem-solving processes (e.g., Carlson & Bloom, 2005; Schoenfeld, 1985, 2010), much remains to be known about analyzing undergraduate students’ proving processes. With a focus on impasses in the proving processes, this dissertation study provides a more fine-grained account by characterizing both students’ overall proving process and their navigating actions. This study explores (a) undergraduates’ proving processes, (b) where students get stuck during the process of constructing proofs, and (c) how students navigate out of their stuck points. In particular, the results of this study can be interpreted as providing information about how undergraduate students engage in productive struggle as they attempt to prove mathematical statements (Hiebert & Grouws, 2007). Given the difficulty undergraduate students face in higher-level math courses, understanding the ways in which they struggle is important for building more inclusive classroom environments.
The data for this study consisted of semi-structured task-based interviews with 10 undergraduates enrolled in a transition-to-proof course. Interviews were video-recorded and students’ real-time proof work was captured using a Livescribe™ pen. Through my data analysis, I created an analytical framework for classifying students’ stuck points and their navigating actions, grounded in the proving processes maps that I generated for each participant. The results of this study indicated that undergraduate students do not engage in strictly linear or sequential proving processes, and they often encounter multiple stuck points when engaging in proving activities. Undergraduate students’ proving processes around stuck points can be categorized into three main types: Type I (no related outcome produced), Type II (related outcome produced but not linked to the main argument), and Type III (at least one related outcome produced and linked with the main argument). Although navigating actions or attempts were observed in all cases analyzed, not all actions led to making productive progress or led students to successfully complete proof tasks. With a deeper look at the navigating actions in two comparative cases, I observed that certain actions, such as setting a goal for their work, producing related results, and linking these results back to the main argument, occurred only in productive progress.

This study provides both theoretical and pedagogical tools for unpacking specific moments of students’ struggles, which is important for understanding and supporting students’ growth with respect to the discipline of mathematics. The different proving processes and navigation actions characterized in this study can help instructors to have a better understanding about productive struggle and to support students in engaging in more productive struggle during their proving practices.
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CHAPTER 1
INTRODUCTION

In the discipline of mathematics, very little is regarded as highly as proof. It is proof that mathematicians work toward and from and that sets mathematics apart from the empirical sciences. Some may argue that there is no need for every student to study proof, since very few students will become mathematicians. However, the processes of thought that are cultivated by the learning of proof make unique and important contributions to a student’s educational experience.

Therefore, learning to prove mathematical propositions is a cornerstone of the discipline of mathematics (de Villiers, 1990). Research has examined the purposes of proof (e.g., de Villiers, 1990), and the role of proof in the mathematics classroom (e.g., Hanna, 1990). Since proving is typically a different kind of mathematical activity compared with students’ previous experience, research has also shown that many undergraduate students struggle to learn to prove, including those who major in mathematics (Moore, 1994; Selden, 2012).

Within the proof literature that focuses on undergraduate students, much work has concerned students’ final proof products (such as written work or exams). However, such products cannot reveal the sequence of mental and/or physical actions that resulted in the final proof that students present in their written work (Selden & Selden, 2009). In particular, the proving process might entail several places where students get stuck, only some of which ultimately contribute to the final proof. Thus, an emphasis on process, particularly on “stuck points,” affords many things that other kinds of data collection obscure. Despite the importance
and interest in students’ proving processes, there has not been much research that focuses on characterizing the actual proving processes for undergraduate students.

In order to characterize students’ proving processes, this study focuses on students’ stuck points. This is for several reasons. Hiebert and Grouws (2007) define productive struggle as an intellectual effort that students expend to make sense of mathematical concepts and tasks that are challenging but fall within students’ reasonable capabilities. However, there has been limited investigation in the mathematics education literature about students’ struggles. While the idea of struggle has been examined in problem solving in the middle grades (e.g., Warshauer, 2011), there has been sparse research on this idea in proving at the collegiate level. This study contributes to filling this gap. This knowledge is critical since if we know how students engage in proving processes, then we can ask, “Where is the place for us to intervene as instructors and how should we intervene?”
CHAPTER 2
LITERATURE REVIEW

To study undergraduate students’ proving processes, it is necessary to first examine the research literature and to unpack relevant phenomena in proof research, including existing perspectives on the meaning of proof, products of proof, and processes of proving. With this in hand, I will turn next to presenting existing theoretical constructs, such as problem-solving processes and cognitive processes in proving. Lastly, I will review existing research on stuck points and productive struggle that directly inform my study of undergraduate students’ proving processes and navigating actions.

**Proof and Proving in Mathematics**

**The Meaning of Proof**

The term *proof* has been used in a number of different ways in the field of mathematics education (Stylianides, Stylianides, & Weber, 2017). Some researchers have defined proof from a mathematical standpoint, associating it with logical deductions that link premises with conclusions (Griffiths, 2000). Other researchers have defined proof from a cognitive perspective or a social perspective, focusing on arguments that help an individual gain conviction in a mathematical claim (Harel & Sowder, 2007) or on how members of a mathematical community approve an argument as a proof (Balacheff, 1991). Stylianides (2007) developed an influential and widely used definition of proof that was informed by the literature on proof from mathematics education and by analysis of data of student argumentation and consensus forming in a elementary classroom. According to his definition, *proof* is a *mathematical argument*, a
connected sequence of assertions for or against a mathematical claim. This definition, together with the three components of an argument (set of accepted statements, modes of argumentation, modes of argument representation), not only merges different mathematical, social, cognitive, and pedagogical points of view, but it is explicit enough to describe proof across all school grades. It acknowledged that proof should be connected to the norms of the community and representational practices that are available to them. As Stylianides, Stylianides, and Weber (2017) claim, a proof can be thought of as an argument with certain norms of expression based on this definition.

These different definitions illustrate that there is a lack of common language about what proof means in mathematics education research. Thus, there is a need for developing a more explicit definition of proof in order to interpret and synthesize research findings and to contribute to the coherence of the body of knowledge produced by studies in this area. In particular, Stylianides et al. (2017) call for the study of proving processes, since each of the definitions discussed above seem to discuss only the final proof product; the proving process itself is not the focus of those definitions.

**Products Versus Processes of Proving**

Reiss and Renkl (2002) pointed out that some instructional approaches and student beliefs are implicitly guided by the false idea that proof is always a linear and systematic process. This has major implications for the teaching and learning of proof. As mentioned in the previous chapter, much existing research in the proof literature focused on the final product of proof rather than the proving process. However, as Boero (1999) pointed out, mathematicians do not engage in strictly linear or sequential proving processes. He distinguished between proof product and proving process by how proof may be generated. Other research suggests that the
teaching of proving to students by presenting only the final product of proof may encourage the memorization of the proofs and does not help students to understand how to produce proofs themselves (Raman, 2003). Thus, it is important to study the process of proof, which can afford us many things compared to studying only the final proof product itself.

Several researchers (Alcock, 2010; Raman, 2003; Weber, 2001; Weber & Alcock, 2004) provided insights into some aspects of the proving process. Raman (2003) suggested that there are three essentially different kinds of ideas involved in the production and evaluation of proof: a heuristic idea, procedural idea, and key idea. Key ideas are central for students’ proof production. Others (e.g., Weber & Alcock, 2004) have described proof construction as involving both semantic and syntactic modes. Semantic modes involve thinking about the mathematical objects to which a statement refers, and syntactic modes involve thinking about and manipulating a statement based on its form. Weber (2001) claimed that some of the undergraduates in his study failed to construct a proof because they could not use the syntactic knowledge that they had. Alcock (2010), in her interview study of mathematicians’ views on proof production and student difficulties, discerned four modes of thinking involved in proof construction: instantiation, structural thinking, creative thinking, and critical thinking. Besides structural thinking, which is syntactic, all other modes are semantic (Alcock, 2010). She argued that each of the four modes of thinking was important in the construction of proofs and should be taught in introductory proof courses. Looking across these studies, one can notice that the categories of students’ thinking that are described are all at a very coarse grain size and thus do not immediately apply to studying the moment-by-moment dynamics of proving processes.

In contrast, Selden and Selden (2009) took a lead in focusing on processes of proof construction and proposed two aspects (or parts) of a student’s written proof: the formal-
rhetorical part and the problem-centered part. The formal-rhetorical part of a proof is the part of
a proof that depends only on unpacking and using the logical structure of the statement of the
theorem, associated definitions, and earlier results. In general, this part does not depend on a
deep understanding of, or intuition about, the concepts involved or on genuine problem solving
in the sense of Schoenfeld (1985). Selden and Selden called the remaining part of constructing a
proof the problem-centered part. It is the part that does depend on genuine mathematical problem
solving, intuition, and a deeper understanding of the concepts involved. Thus, Selden and Selden
separated the conceptual parts of proof activity from the selecting and setting up a proof
framework parts of proof activity, which is procedural. However, Stylianides et al. (2017)
challenged the dichotomy between formal-rhetorical and problem-centered activities in proof
writing and called for a more nuanced characterization between these two activities. Levin’s
(2018) work on strategy systems takes a more fine-grained look at the interrelation between
concepts and strategies and may be useful for unpacking the relationships between formal-
rhetorical and problem-solving activities in proof construction.

From the review of the proof literature, although students’ and mathematicians’ proof
practices and activities have been examined in several ways, there are still several aspects of
their processes of proving that have not been examined. In the next section, I turn to discussing
theoretical constructs that will be needed for engaging in the study of undergraduate students’
proving processes.

The Problem-Solving Process and Model of Arguments

In this section, I first describe research around the problem-solving process, since much
of the literature has noted the close links between proving and problem solving. I then turn to
describing Toulmin’s (1993) model of the structure of arguments as a way to highlight the specific nature of mathematical reasoning processes involved in proving.

**The Problem-Solving Process**

A number of authors have remarked on the close relationship between problem solving and proving (e.g., Furinghetti & Morselli, 2009; Moore, 1994). Even though there is no existing proving-process framework, multiple problem-solving frameworks have been developed.

In the early 1980s, many researchers interested in understanding human problem solving were informed by newly developed cognitive perspectives, such as information processing (Newell & Simon, 1972). Whereas many other researchers focused solely on information-processing models of cognition, Schoenfeld’s foundational work on problem solving revealed phenomena not accounted for in the previous accounts. For example, Schoenfeld (1983) discussed a series of methodological and psychological issues related to the use of qualitative methods to study problem solving (i.e., clinical interviews and talk-aloud protocol analyses). His work revealed the need to take into account broader socio-cognitive and metacognitive perspectives, such as the environment and individuals’ beliefs that could affect the generalization and interpretation of verbal protocol data of problem solving (Schoenfeld, 1983). He raised questions about the interpretation of verbal data through only the dimensions of task variable and cognitive structures. Initially, Schoenfeld added two more dimensions besides cognitive structures: belief systems and metacognition (i.e., self-regulation, including conscious knowledge about knowledge). The knowledge generated from this research encouraged the field to attend to multiple dimensions in the process of mathematical problem solving and learning.

To elaborate, Schoenfeld (1983) proposed a framework that includes three qualitative categories of knowledge and behaviors that researchers need to consider when analyzing and
interpreting verbal data: resources, control, and belief systems. The first category, resources, is an individual’s foundation of basic mathematical knowledge, including facts, algorithms, and understanding possessed by the individual. The second category, control, refers to control over individuals’ resources, such as planning, monitoring, and metacognitive acts with regard to selecting and using resources. The third category, belief systems, refers to the beliefs students have about mathematics and themselves as mathematics learners that individuals bring to bear on the problem situation. All three of these categories must be taken into account when explaining a student’s behavior in mathematics.

Later, Schoenfeld (1985) refined this framework by adding heuristics as a fourth category. Schoenfeld refined the theoretical framework that could be used for investigating problem solving and, more broadly, for investigating mathematical thinking. This framework contains four domains that he claimed must necessarily be addressed by any work intending to investigate mathematical problem-solving performance: resources, heuristics, control, and belief systems. I will now discuss the refined version of each of these dimensions in turn.

Resources

When students work on mathematics problems, they select strategies, adapt them in response to feedback, spend time, and make many other decisions to optimize their performance (Schoenfeld, 1985). Schoenfeld concluded that a knowledge base is available for more prescriptive characterization of problem-solving strategies. However, with the detailed heuristics to learn, higher-order thinking and beliefs would need to be introduced to specify when, why, and how to use the lower-order prescriptive processes. Thus, he emphasized the need for research on metacognition and beliefs.
Heuristic Strategies

Throughout the 1970s and into the 1980s, research in problem solving was dominated by heuristic strategies—a set of broad problem-solving techniques and strategies that can help individuals get started in problem solving but by themselves are not meant to provide optimized or final solutions to tasks. Schoenfeld (1985) documented that general heuristic strategies, such as those proposed by Polya (1957), were too general to learn to implement directly. Through his observations and subsequent analyses, Schoenfeld recognized that, for students to be able to solve problems effectively, strategies needed to be instantiated in a set of sub-strategies that are specific to the type of problem being solved and that are developed from experience solving problems. These sub-strategies needed to become part of the solver’s knowledge base, which must be developed from this perspective, taking Polya’s question “Have you seen a problem like this before?” to a deeper level and a broader level.

Metacognition Control

With respect to problem solving, Schoenfeld (1985) referred to metacognitive control as the ability to break up more complex problems into a few sub-problems—do the sub-problems first, then sequence the sub-problems, and finally complete the whole problem. But this work is complex. These metacognitive control decisions involve strategic planning, self-monitoring, and intentionally adapting problem-solving paths to achieve a specific goal.

Schoenfeld, reflecting on courses he taught in problem solving, described a Vygotskian approach for supporting students in developing metacognitive control. He described circulating from group to group and prompting them with questions such as one would ask themselves to try to make progress and get unstuck. At first, these questions did not appear to be the questions students were asking themselves, but they learned to ask similar questions of themselves in
Thus, Schoenfeld demonstrated the ways in which metacognitive control could be appropriated and internalized by students—first operating at an external social level, and then moving to the individual level.

Thus, Schoenfeld’s studies have demonstrated the crucial role of metacognitive control during mathematical problem solving of novel, ill-structured problems. This is particularly important because it fills an explanatory gap in previous accounts. For instance, when students have sufficient content knowledge to solve a problem, they may still fail to do so because they lack suitable metacognitive control to select, continue, or abandon a specific strategy. For novices, a metacognitive control failure often involves quickly selecting a strategy without evaluating their prior knowledge, making a plan, or understanding the complexities of the problem (Schoenfeld, 1985). Schoenfeld (1992) describes a novice’s “wild goose chase” problem-solving approach as “read the task, make a decision about direction quickly, and pursue that direction come hell or high water” (p. 356). While the majority of the novice undergraduate mathematics majors in his studies focused on specific formulas and equations, contrasting analysis of expert mathematicians revealed that they sought to understand the goal structure of a problem and to identify which mathematical tools might help.

The enhanced metacognitive awareness and self-scaffolding provided by metacognitive control can improve problem solving, satisfaction, and mathematics achievement. Later studies show that students exerting greater metacognitive control typically identify more metacognitive knowledge that is relevant, acquire more relevant resources through self-scaffolding, and spend more sustained, meaningful time on a problem (Chiu & Kuo, 2009).
Belief Systems

According to Schoenfeld (1985), beliefs are an individual’s understanding that shapes the way they conceptualize and engage in mathematical behavior, including student beliefs and teacher beliefs, about the nature of mathematics and processes of doing mathematics. In particular, beliefs are influenced by one’s own prior experiences and culture and can shape one’s mathematical behavior. Schoenfeld pointed out that it is often the case that resources are assumed to be the primary determinant of success in problem solving. That is, if the requisite mathematical content for a particular problem is known, then the problem should be solvable. Schoenfeld uncovered the inappropriateness of this assumption. For instance, mathematicians with powerful heuristics and control are likely to be able to solve problems even when their resources are lacking, and students who possess the necessary resources may be unable to solve problems because their belief systems impede activating and using knowledge they “have.”

Schoenfeld’s framework of four attributes of knowledge and behaviors researchers need to consider when analyzing problem-solving behaviors has been used for later research in problem solving (Lesh & Zawojewski, 2007). Inspired from both Polya’s (1957) four steps and Schoenfeld’s (1985) four attributes, Carlson and Bloom (2005) created a Multidimensional Problem-Solving Framework that describes problem-solving phases and attributes through investigating how 12 mathematicians solved a number of different problems. The Multidimensional Problem-Solving Framework has four phases, each with the same four associated problem-solving attributes. The four phases are Orienting, Planning, Executing, and Checking. Carlson and Bloom’s problem-solving framework serves as one of the core conceptual frameworks for this study. I will discuss this framework in detail in the framework chapter.
**Toulmin’s Model of Arguments**

In addition to Carlson and Bloom’s (2005) framework, described in detail in the previous section, I also considered an existing framework for understanding the structure of arguments, attributable to Toulmin (1993). As has been discussed, there are two aspects of a written proof, the formal-rhetorical part and the problem-centered part based on Selden and Selden (2013). Even though Stylianides et al. (2017) challenged the dichotomy between formal-rhetorical and problem-centered activities in proof writing and called for a more nuanced characterization between these two activities, we can still regard the formal-rhetorical part of the proving processes as a unique characteristic to distinguish proving processes from problem-solving processes. We have already talked about the problem-centered part in the previous section as the process of problem solving. The formal-rhetorical part of a proof was defined as “unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results” (Selden & Selden, 2009, pp. 308-309), in other words, the “argumentation.”

Toulmin’s (1993) model has been proposed as a methodological tool in educational literatures (Pedemonte, 2007) to study arguments. Toulmin’s model, also called “Toulmin’s model of arguments,” focuses on the structure of a proof, which is the logical connection between the statements. In Toulmin’s basic model, every argument begins with three fundamental parts: the claim, the data, and the warrant (as shown in Figure 1).

\[
\begin{align*}
D & : \text{Data} & C & : \text{Claim} \\
W & : \text{Warrant}
\end{align*}
\]

*Figure 1. Toulmin’s Basic Model*
Toulmin’s (1993) model deconstructs arguments into six parts: Claim, Evidence (support for claim), Warrant (connection between claim and evidence), Backing (legitimacy of assertions), Counterargument (potential objections to the claim), and Qualifier (modifications of the claim that add specificity to assumptions). In an argument, the first step is expressed by an assertion, which is called the claim. The second step is to produce or use data to support the claim. The last step is to provide justifications, which serve as a bridge between the claim and the data. In addition, a qualifier, a rebuttal, and backing are the three auxiliary elements that may be used to describe an argument (Toulmin, 1993). As can be seen from the description, Toulmin’s model focuses on the structure of the argument, but not the actual actions of the actor.

Recently, researchers started to be interested in the gap between arguments and proofs. Pedemonte (2007) analyzed the types of arguments that are easier to translate into proofs. It brings useful insights about why students have difficulties transferring their informal arguments to proofs. However, these types of research do not seem to separate the product of proof and process of proving. The analysis of the “content” of the proof is not sufficient to analyze all cognitive aspects in relationship between argumentation and proof (Pedemonte, 2007).

Previous research focuses only on the structure of the argument but not the actual actions and cognitions of individuals. To address this, notably Karunakaran (2014) focused on comparing the bundles observed in the proving processes and the intentions behind the bundles observed within the proving processes between the expert and novice provers to describe the use of an individual’s mathematical knowledge in the process of proving. Through this research, I aim to bridge the gap from argument structure to student actions in proving processes.
Stuck Points and Productive Struggle

During the course of problem solving and proving, places where individuals, both students and experts, get stuck are natural and expected occurrences. Despite this, there has been limited investigation in mathematics education literature about students’ stuck points and how they navigate out of being stuck as they work. Although the idea of the importance of struggle in learning has been examined in problem solving (e.g., Warshauer, 2011) in the middle grades, there has been sparse research on this idea in the context of proving at the college level. In the following paragraphs, I review the literature as it pertains to stuck points in proving processes.

Stuck Points in Proving Processes

Savic (2012) conducted a preliminary study of how mathematicians recover from proving impasses. Savic gave mathematicians statements about a domain that they may not necessarily have been familiar with (semigroups). He discovered that the majority of mathematicians encountered impasses in their proving process. In his study, all of the mathematicians were able to recover from the impasses. The data Savic was able to collect on impasses and recovery are interesting because previous literature in mathematics education and educational psychology comparing experts and novices had tended to compare performance on tasks that were familiar to the experts but that were unfamiliar to the novices. Methodologically, Savic used Livescribe pens that were given to the mathematicians for days to gather real-time data. Even though one can see what a mathematician does during the proving process based on the writing, in these data, one cannot see each mathematician’s real action, facial expression, and body language. Further, even though Savic used an exit interview to gain a better understanding of mathematicians’ thinking, no real-time think-aloud data were available to understand the actions in a moment-by-moment manner. Savic’s results included a qualitative description of a number
of ways that mathematicians get “unstuck,” including actions directly related to proving (using prior knowledge, using different methods, using different techniques, generating examples) and also actions that were not directly related (doing other problems, walking around, doing other things, eating and sleeping). Although a mathematician may have had a key insight while walking around (for example) that allowed them to get unstuck, from this study, we still know little about the actual cognitive process of getting unstuck.

Further, proving experts are different from novices. For example, they know what to do when getting stuck from their previous experience. For novices, like undergraduate students who are not yet experienced with constructing proofs, recovering from impasses might be difficult and of a different character from the experts in this study.

Later, Satyam (2018) described students’ process of proving, focusing on the interplay between cognitive processes and student affect, such as attitudes, beliefs, and emotions. Her study focused on the interplay between proving processes and managing emotional responses. Satyam focused on learning about how students managed “negative” emotions during proving, and, in addition, on characterizing students’ perceptions of satisfying moments during proving processes. Methodologically, in addition to video records of students’ proving processes, she asked participants to construct emotion graphs immediately after a proof attempt. Her findings suggested that students’ satisfying moments were largely about accomplishments with and without struggle, understanding, external validation, and interacting with peers. Satyam’s work has demonstrated the importance of examining students’ stuck points when trying to characterize their proving processes. However, in addition to the role of affect in cognition, there is more we need to investigate about students’ “stuck points” in proving.
As mentioned above, when students are engaged in novel tasks, whether problem solving or proving tasks, it is natural for them to encounter impasses. When studying students’ proving process around stuck points, it is important to be aware of what contributes to the success or failure of overcoming such stuck points. The aim of this study called for a construct that can characterize students’ actions through the lens of stuck points. The widely used notion of productive struggle emerged.

**Productive Struggle**

Hiebert and Grouws (2007) defined productive struggle as an intellectual effort that students expend to make sense of mathematical concepts and tasks that are challenging but fall within the students’ reasonable capabilities. Struggle often conveys a negative meaning and may be viewed as a problem in mathematics classrooms (Hiebert & Wearne, 2003). Researchers, however, suggest that struggling to make sense of mathematics should be a necessary component of learning mathematics with understanding (Hiebert & Grouws, 2007). Hiebert and Wearne (1993) stated that “all students need to struggle with challenging problems if they are to learn mathematics deeply” (p. 6). According to Hiebert and Grouws, struggle and its connection to learning are central to the issue of how to improve student learning and understanding of mathematics. A description of what a student’s productive struggle looks like can provide insight into what teachers can do to engage students in a more productive process in doing mathematics (Hiebert & Grouws, 2007). Previous research on student struggles has been limited and has primarily focused on examining whether struggle occurred without examining in detail the nature of students’ struggles (e.g., Inagaki, Hatano, & Morita, 1998; Santagata, 2005).

In the Common Core Standards for Mathematical Practice, the first standard states that students should “make sense of problems and persevere in solving them” (National Governors
Perseverance is really important in problem solving since the initial approach may not produce a solution. In the recent NCTM (2014) publication *Principles to Actions: Ensuring Mathematical Success for All*, the writing team identified the support of productive struggle in learning mathematics as one of eight important teaching practices. Warshauer (2014) argued that struggle can be observable in most classrooms even if it is perceived as a phenomenon. She argues teachers need strategies to support student struggle so it is meaningful for their learning.

To fill in this gap, Warshauer (2011, 2014) investigated the nature of productive struggle in middle-grade classrooms in detail. However, the actions students take when encountering struggle in settings other than middle-grade classrooms and how it could be productive needs further exploration (Warshauer, 2014). This study aims to address this need by investigating what struggles look like for undergraduate students when they are engaging in proving processes and what to do to make their struggles more productive.

**Summary of the Literature Review**

This chapter presented a review of the research literature that focused on three major areas. In the first section, several major phenomena in proof research were discussed in order to build the foundation for this study. Much research in the proof construction literature focuses on the difficulties that students face when proving a theorem (e.g., Weber, 2001), rather than characterizing the proving process itself. There is some research that provides us some insights into students’ understanding in proof construction; however, the big categories of students’ thinking are at too coarse a grain size to apply in a moment-by-moment manner to proving processes (e.g., Alcock, 2010; Raman, 2003; Weber & Alcock, 2004). To help characterize undergraduate students’ proving processes, three theoretical constructs were presented in the
second section. There are frameworks for analyzing problem-solving processes (e.g., Carlson & Bloom, 2005; Schoenfeld, 1985, 2010) and argumentation (Toulmin, 1993). However, not much research has yet attended to characterizing students’ moment-by-moment proving process around stuck points. The last section of this chapter presented the focus of a stuck point and productive struggle as a potential construct to study undergraduate students’ proving processes and navigating actions.

**Research Questions**

As described previously, the purpose of this study was to examine the ways that researchers can study undergraduate students’ processes of constructing proofs. Specifically, the study focused on the following overarching research questions:

1. What kinds of stuck points do students encounter as they engage in proving?
2. In proving attempts that generate stuck points, what characterizes the overall proving process?
3. When students encounter stuck points in proving processes and are actively involved in navigating out of a stuck point, which actions appear to contribute to the success or failure of their attempts?

This study contributes to research and practice about mathematics education teaching and learning in several ways. First, this work attempts to characterize the proving processes of novice provers (undergraduate students), a population whose proving processes have not been significantly examined yet. Second, this work examines undergraduate students’ stuck points in the proving processes, which have not been examined largely in previous research, with the particular focus on their navigating actions.
CHAPTER 3
CONCEPTUAL FRAMEWORK

In this chapter, I present the frameworks that undergird my research design. First, I will discuss proving as problem solving (Moore, 1994). I will then discuss my conceptual framework that was built on the problem-solving framework of Carlson and Bloom (2005) and will provide an initial guide for my analysis. Lastly, I discuss the operationalization of productive struggle given by Warshauer (2011) that informed the development of my coding scheme for understanding what students do when they encounter impasses in mathematical work.

Conceptualizing Proving as Problem Solving

As discussed in the literature review, a number of authors have remarked on the close relationship between problem solving and proving (e.g., Furinghetti & Morselli, 2009; Moore, 1994). In fact, Weber (2005) suggested focusing on problem-solving aspects of proof because this “allows insight into some important themes that other perspectives on proving do not address, including the heuristics that mathematicians use to construct proofs” (p. 352). In this study, I conceptualized proving as a kind of problem-solving process. Thus, I will explain Carlson and Bloom’s (2005) problem-solving framework, which has informed my analytical approach.

Carlson and Bloom’s Problem-Solving Framework

As discussed in Chapter 1, Carlson and Bloom’s (2005) Multidimensional Problem-Solving Framework (Figure 2) expands Schoenfeld’s (1985) framework. They developed four phases for problem-solving processes, each phase with the same four associated problem-solving
attributes from Schoenfeld. The four associated problem-solving attributes are resources, heuristics, affect, and monitoring, which were developed from Schoenfeld. The four phases in Carlson and Bloom’s framework are Orienting, Planning, Executing, and Checking. I will discuss each phase in turn.

Figure 2. Multidimensional Problem-Solving Framework by Carlson and Bloom (2005)

**Orienting:** The effort put forth to read and understand the problem through constructing a table, graph, or text. Goals and givens are established.

**Planning:** A conjecture is formulated by considering different solution approaches and relevant mathematical concepts or knowledge. An approach is decided.
Executing: Selecting and implementing various procedures and heuristics through constructing logical statements, algebraic computations.

Checking: The results or validity of the answer is tested or verified.

If the solution was determined not to be viable by the solver, the process will cycle back or forward from the planning phases. Inside of the planning phase, there is the conjecture sub-cycle (conjecture, imagine, and evaluate). This sub-cycle can be repeated until a solution path is found.

The original context of the work of Carlson and Bloom (2005) considered only how experts (like mathematicians) behave to successfully solve problems. Thus, their Multidimensional Problem-Solving Framework does not characterize where a novice may seem to get stuck and how they may behave if they get stuck. My previous pilot work (described in Appendices C, D, F, G, & H) illustrated both the affordances and the constraints of this framework for analyzing data of novice provers’ processes. In my dissertation study, I thus began by characterizing the major approaches or actions that were evident in students’ proof construction through “stuck points” (as shown in Figure 3) in their proving processes.
To characterize students’ stuck points, I was informed by the classification of different types of student struggles given in Warshauer (2011) (see Table 1). Warshauer argued student struggle needs to be encouraged but also supported, so that students’ struggle is productive and meaningful for learning.

**Warshauer’s Classification of Different Types of Struggles**

In the context of middle-grades students’ reasoning around impasses, Warshauer (2011) identified four types of struggles that students encounter as they work on challenging tasks in a class session. These four are (1) encountering difficulty in figuring out how to get started, (2) carrying out their task, (3) expressing uncertainty about their chosen strategy, or (4) expressing an error or misconception in problem solving (Warshauer, 2014).
Table 1

*Warshauer’s Classification of Kinds of Student Struggles*

<table>
<thead>
<tr>
<th>Kind of struggles</th>
<th>Example descriptions</th>
</tr>
</thead>
</table>
| 1. Getting started                      | • Confusion about what the task is asking  
• Claim forgetting type of problem    
• Gesture uncertainty and resignation  
• No work on paper                      |
| 2. Carrying out a process               | • Encounter an impasse                                                               
• Unable to implement a process from a formulated representation                    
• Unable to implement a process due to its algebraic nature                          
• Unable to carry out an algorithm                                                   
• Forget facts or formula                                                            |
| 3. Uncertainty in explaining and sense-making | • Difficulty in explaining their work                                              
• Express uncertainty                                                               
• Unclear about reasons for their choice of strategy                                 
• Unable to make sense of their work                                                 |
| 4. Expressing misconceptions and errors | • Misconception related to mathematical content (e.g., multiplication always makes bigger) |

Given my interest in students’ actions to navigate impasses in their mathematical work, Warshauer’s framework provided a concrete starting point. In order to investigate the possible connection between struggle and proving, I examined students’ productive struggle as students worked on proof-related tasks. In particular, I investigated episodes where students made mistakes, expressed misconceptions, or claimed to be lost or confused. A close examination of students’ stuck points helped to reveal the nature of the struggle students were having in making
sense of mathematics. Thus, these struggle types served as a conceptual framework to analyze the interview episode where undergraduate students were engaged in a proving process to struggle.

**Productive Struggle Versus Unproductive Struggle**

Once I had classified the different types of struggles, the next question I needed to consider was what could be considered to be productive about so-called “productive struggle.” Warshauer (2011) claimed that while struggles are situated with the task and student, struggles can be directed more productively by teachers’ support. She identified a struggle to be productive if the cognitive demand of the task is maintained, student thinking is supported, and student actions were enabled to move forward by teachers’ responses and questions. On the other hand, the struggle is categorized as unproductive if students are not making progress, they stop trying, or teachers’ support had reduced the cognitive demand of the task. Thus, whether or not the struggle is productive depends mostly on teachers’ practice and student-teacher interaction, in Warshauer’s definition.

Different from Warshauer (2011), my study focuses more on the students’ own actions and autonomy in the proving process. Since previous research on productive struggle mainly examined the K-12 problem-solving environment with teachers’ interaction instead of undergraduate students’ proving processes and their own navigating actions, I aim to extend the understanding of these constructs and ideas in this study.

In the next chapter, I will describe in detail my process of appropriating and further developing my conceptual framework in preparation for the analysis in this dissertation. This process involved a negotiation between the existing frameworks with my pilot data of specific proving processes.
CHAPTER 4
DESIGN AND DATA COLLECTION

In this chapter, I will outline and justify the methodology used to answer the research questions. I will describe the study context of the transition-to-proof course, the recruitment plan for participants, the research design and instruments, and the methods of data collection. Lastly, I will also describe the results of my pilot study to illustrate aspects of my data analysis process.

Context of the Study

Transition-to-Proof Course

My study of undergraduate students (novice provers) drew participants from the Introduction to Proof course\(^1\) at a large research university in the Midwest. The Introduction to Proof (ITP) course is required for undergraduates majoring in STEM and mathematics education. The course focuses on developing mathematical argumentation techniques and proof strategies. In addition, students explore diverse problems and demonstrate relationships among several areas of mathematics in order to prepare them for the proof-intensive work required of them in higher-level mathematics courses. Some basic concepts of real analysis, linear algebra, and number theory are introduced in this course.

\(^1\) Some of the participants were interviewed during Winter or Summer term due to their schedule conflicts during the Fall or Spring semester. Because of this, at the time of the interview, some of the participants had already completed the Introduction to Proof course.
Participants

I chose to work with novice students with basic knowledge in proof because I hypothesized that by observing individuals with a limited knowledge base and limited problem-solving experience, I would learn more about proving processes that hadn’t been characterized in the problem-solving processes framework of Carlson and Bloom (2005). I define *novice provers* in this study to be undergraduate students who have had only some initial experience with proof and proving (less than a semester). I chose to work with students at the end of their ITP course (or immediately after its completion) so that they would have enough basic knowledge about both proofs and the content of the tasks to engage effectively with the designed proof tasks in my protocol.

Recruitment of Participants

Students who had completed or who were in the final stage of completing transition-to-proof courses were recruited for task-based interviews. I visited the ITP course (transition-to-proof course) in the beginning of the semester to recruit participants. Students indicated their willingness to participate by signing an IRB-approved informed consent form. Of the 12 students who volunteered for this study, 10 interviews were conducted in Spring 2020, before the lockdown due to the COVID-19 pandemic began. In the end, two students who had agreed to participate in the interview could not make it to the in-person interview because of distance or personal issues.

Description of the Participants

Table 2 presents the names (pseudonyms) of all 10 participants, with information on gender, ethnic group, and major.
Table 2

*Descriptions of the Participants*

<table>
<thead>
<tr>
<th>Name</th>
<th>Gender</th>
<th>Ethnic group</th>
<th>Major</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matt</td>
<td>M</td>
<td>White</td>
<td>Computer Science</td>
</tr>
<tr>
<td>Jeff</td>
<td>M</td>
<td>White</td>
<td>Statistics</td>
</tr>
<tr>
<td>Porter</td>
<td>M</td>
<td>Hispanic</td>
<td>Economics, Math</td>
</tr>
<tr>
<td>Eric</td>
<td>M</td>
<td>White</td>
<td>Physics, Math</td>
</tr>
<tr>
<td>Devin</td>
<td>M</td>
<td>White</td>
<td>Physics, Math</td>
</tr>
<tr>
<td>Nikki</td>
<td>F</td>
<td>White</td>
<td>Secondary Education, Math</td>
</tr>
<tr>
<td><em>Lucy</em></td>
<td>F</td>
<td>White</td>
<td>Secondary Education, Math</td>
</tr>
<tr>
<td>Rachel</td>
<td>F</td>
<td>White</td>
<td>Applied Math</td>
</tr>
<tr>
<td>Rina</td>
<td>F</td>
<td>White</td>
<td>Secondary Education, Math</td>
</tr>
<tr>
<td>Soni</td>
<td>M</td>
<td>Asian</td>
<td>Computer Science</td>
</tr>
</tbody>
</table>

Lucy is listed here, but she completed only the initial 30 minutes of the interview. She requested to withdraw from the study, concerned about her performance on the tasks. Accordingly, her interview data was withdrawn from the analysis.

**Research Design and Instruments**

To answer the research questions, I conducted semi-structured, task-based interviews with nine undergraduates enrolled in the transition-to-proof course. Participants were asked to work on two introductory number theory tasks in real-time using a Livescribe pen.
Task-Based Interview

I conducted interviews to characterize relative novices’ proof construction processes. The interviews took place on campus at a mutually convenient place. All interviews were audio- and video-recorded for analysis.

Each participant was asked to prove or disprove two mathematical statements. The majority of the interviews took around 70 minutes, which provided enough time for the participants to have at least 30 minutes to engage with each task. In order to capture students’ proving processes with their specific strategies and steps, a think-aloud protocol was used (Schoenfeld, 1985). Participants were asked to verbalize their thoughts aloud while doing a task, both in the moment and shortly after the task. My role as an interviewer was to encourage the participants to think aloud about their proving process and pay attention to their “stuck points” and what they were doing to overcome them. I chose to minimize interviewer intervention during the proving processes. This was because my phenomenon of interest was the proving process and what students do when they get stuck, so there was a high chance that talking to them would be regarded as a “hint” or an interruption that could fundamentally shape their own process. To avoid that, I asked only questions like “I haven’t heard you say much in the past couple of minutes. What were you thinking?” in order to support them in thinking aloud.

Since recalling definitions and theorems is not the focus for this study, I made available to the participants a set of definitions/theorems/lemmas that were relevant to the statements that students needed to prove.

Task Selection

According to Dawkins and Karunakaran (2016), it is not really possible to remove content from a measure of proof competence because competence with proof is multi-faceted
and content dependent. With this in mind, I chose the content of number theory for several reasons that served my research purposes. First, number theory is suitable for students across different levels as it does not require previous knowledge of particular notations for the student to understand the task statements. My goal was for the tasks to neither be heavily dependent on content knowledge nor rely solely upon a singular specific proof technique, such as mathematical induction. Second, the tasks were designed so as to permit the use of several different approaches. For example, statements that could only be proved easily using mathematical induction were excluded; however, statements that could be proven by either contrapositive or contradiction were viable because of the technique choices available.

Initially, in my first round of the pilot study, I started with four tasks. With four tasks to complete, each student had only around 20 minutes to work on each task. I realized the time given for each task was not enough, since students didn’t have enough time to fully engage with proof-task and overcome their stuck points. After the first round of the pilot study, I decided to reduce the number of tasks to two. With a similar amount of time for the entire interview, students would have more time to work on each task. The analysis of my pilot study showed that after reducing the number of tasks, students did have a much longer time to engage with each task and, importantly, after a long period of being stuck, some of the students did overcome their “stuck points.”

The tasks can be found in Figure 4, and the interview protocol, along with expected responses, can be found in Appendix E.
Task 1
Suppose $a, b$ and $c$ are integers such that $a^2 + b^2 = c^2$. Is it true that at least one of $a$ or $b$ is even?

Task 2
Prove or disprove: An integer is divisible by 9 if the sum of its digits is divisible by 9.

*Figure 4. Interview Tasks*

**Data Collection**

The data collected for the study consisted of video-recordings and audio-recordings of the task interviews, the written work produced by the participants, the real-time “Pencast” by the Livescribe pen (including audio and writing in action movie) and interviewer notes. To capture participants’ facial expressions and body language (which may indicate their affective response to stuck points), I wore camera glasses in the interview. To provide background and context for our discussions in the interviews, I observed selected course activities and collected the syllabus for the transition-to-proof course to get a sense of their mathematical knowledge and proof background. Protocols that appeared to yield rich data as students progressed toward perceiving sets of stuck points were transcribed for further analysis. All students’ names were removed from the data and replaced with a code-name from a master list (destroyed after all data were collected).

**Camera Glasses**

To capture students’ moment-by-moment facial expressions in order to better indicate their “stuck points,” I wore a pair of camera glasses during the interview. The hidden camera glasses look like normal glasses from the outside but can record video and take pictures. The
camera glasses recorded the interviewee’s perspective without adding another bulky camera set-up into the interview setting. The videos were captured with time stamps in order to better reference my records.

**Livescribe Echo Pen Data**

In order to capture the real-time proving process, I asked the student in the interview to use the Livescribe Echo Pen (and to write on special paper). The Livescribe pen captured both audio and real-time writing using a camera near end of the ballpoint pen. When one presses on the “record” square at the bottom of the special paper with the pen, the pen goes into audio record mode, which then allows for the real-time capturing of the writing and speaking. The pen can be stopped by a “stop” button, and all proving episodes are time-and-date stamped. One can click or point on any of the sentences to replay the writing and speaking in that specific moment. The pen data can be uploaded to computer software called Echo Desktop, where I could export each students’ proving session together in one PDF file called a “Pencast.” I transcribed each of the Pencasts into a timeline coordinated with their real-time writing and talking. I then analyzed each timeline based on my analytical framework (discussed in the next section). One Pencast transcription is provided in Appendix G as an example.

**Pilot Studies**

In order to test out the research design and predict the findings, I conducted three cycles of pilot studies in Spring 2019, Summer 2019, and Fall 2019. The results of these three cycles of pilot studies can be categorized into two levels: logistical and analytical.

**Logistical Results**

In Spring 2019, I did my first round of the pilot study. I had four students participate in the task-based interview, all from the Introduction to Proof course, after their midterms. I started
using four tasks and asked all students to complete all the tasks on paper. Each student had around 20 minutes to work on each problem. However, from the interviews, I realized the time given for each task was not enough. Students did not have sufficient time to fully engage with proof task and overcome their stuck points. After this round of the pilot study, I decided to reduce the number of tasks to two. In this round of pilot work, I also realized that if students all completed the tasks on the paper, I could not capture their facial expressions or body language. Thus, it would be hard for me to identify “stuck points” later in my analysis. As a result, I decided to ask students to write on the blackboard hanging on the wall, so the camera would be better able to capture their facial expressions and body language when they got stuck.

In Summer 2019, with the refined two tasks, I interviewed five more students, all from a proof-based number theory course. Students in this class had all recently taken the Introduction to Proof class. The aim for this round of the pilot study was to test out the refinement of the research design and also determine the final set of the two tasks. In this iteration, I also tried both paired interviews and individual interviews. My analysis in this round revealed several difficulties of interview pairs to characterize each individual’s proving processes. I faced many challenges when interviewing pairs of students. For example, since some of them didn’t know each other before the interview, they sometimes didn’t want to work together. In cases where they did know each other, one seemed to talk more than the other. Also, if one student agreed with the other student’s idea, it was hard to determine whether that student had the same understanding or just indicated that they understood. Thus, for analytical clarity with respect to my dissertation study on proving processes, I decided to interview only individual students.

These interviews helped to finalize the content area for the chosen tasks. I used four different tasks from different content areas for the first round of the pilot study and realized that
relative novice students could easily get stuck in just understanding the task itself. For instance, some students pointed out in the first round of the interviews that it was difficult for them to understand some of the language or symbols used in the analysis tasks. This was not helpful for my purposes: I did not want unfamiliarity with language and symbols to be the obstacle to making progress in proving a statement. The two number theory tasks in this round of the pilot study all worked out well. However, there was still some problem with the existing two tasks in the second round of the pilot study. One of the questions was a typical theorem question that the student may have seen before in their proof class. To prevent a student simply recalling the proving process from what they did before, I changed the question in the last round of the pilot study.

In Fall 2019, with the final two tasks, I interviewed three more students, all from the transition-to-proof course (after their midterm). The aim of this final round of the pilot study was to test out the finalized tasks and implement the new technology of the Livescribe pen. In order to capture the real-time proving process closer than would be possible with the video record, I asked the student in the interview to use the Livescribe Echo Pen and special paper. Both the finalized tasks and Livescribe pen worked well in this pilot study and I decided to use them in the dissertation study.

**Analytical Results**

For the last two rounds of the pilot study, both tasks I chose to use were in the domain of number theory. I retained this content focus in my dissertation study design. Initially, I used Carlson and Bloom’s (2005) Multidimensional Problem-Solving Framework to try to analyze these data. I then decided to use “stuck points” as important points to devote analytic attention to in characterizing students’ proving processes.
A new tool implemented in this study is the use of Livescribe Echo Pen. Besides audio and video transcripts, I also transcribed Livescribe Pencasts based on their time stamps. The resulting transcript is divided into three columns: the time stamps, written words/sentences and spoken words/sentences. The advantage of also having the Livescribe Pencast transcript was to have a better record of students’ written work and how it unfolded moment by moment, especially if their written work was not captured fully in the video. The Livescribe transcript served as a backup and reference for the audio and video transcript and interview notes.

Here is a short example of the Livescribe Pencast transcript and coding. A more detailed coding example can be found in Appendix G. In Figure 5 below, we see two colors of the Pencast: the dark green text indicates the ongoing writing, while the lighter green text indicates the part that hasn’t been written yet at a given time stamp.

<table>
<thead>
<tr>
<th>Time</th>
<th>Writing</th>
<th>Speaking</th>
<th>Coding</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:53-2:20</td>
<td>None</td>
<td>“I am thinking the best way to go about it is to do the contrapositive. It seems easier to determine the contrapositive of a or b is even and work backwards to prove $a^2 + b^2 \neq c^2$.”</td>
<td>Planning</td>
</tr>
<tr>
<td>2:23-3:30</td>
<td>Proof; [contrapositive]</td>
<td>Same as writing</td>
<td>Executing</td>
</tr>
<tr>
<td></td>
<td>Suppose a and b are odd.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Then let $a = 2x + 1$ and $b = 2y + 1$ for $x, y \in \mathbb{Z}$. Then we get $a^2 + b^2 = (2x + 1)^2 + (2y + 1)^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 5. Livescribe Pencast Transcript and Coding Example
Two cases were chosen from the pilot: one student who got stuck and succeeded in navigating out of their stuck point, and another pair of students who got stuck but were not able to navigate out of their stuck point. The details of the two cases are provided in Appendix G.

Comparing their actions, I realized that working in pairs or individually might not be the over-riding factor for whether a student can overcome their stuck point. Thus, for the dissertation study, I decided to focus only on interviews with individual students and to specifically capture individual students’ proving processes.

Based on the pilot cases analysis, Carlson and Bloom’s (2005) framework had the following problems when analyzing undergraduate students’ proving process.

*Different cycling back pattern:* Carlson and Bloom (2005) claimed that the process of problem-solving should always be captured as Orienting-Planning-Executing-Checking with possible cycling back from Planning to Executing and Checking. However, from what I saw, this process is not always followed in this way. For my cases, the cycling back was mostly between only Planning and Executing. Checking was included only once for the pair I analyzed in the beginning, even though they cycled back and forth multiple times between Planning and Executing.

*Not completing the full cycle of Orienting-Planning-Executing-Checking:* The major difference I noticed was the lack of an Orienting phase in my cases. This might be due to one of the differences between proving and problem solving: it is sometimes hard to construct a table, graph, or diagram to make sense of a proof task because of the abstract nature of many proof problems. Also, the pair I analyzed stopped their proving process with the last stuck point, instead of ending at another Checking.
The need to extend the Planning and Executing phases: Based on my pilot analysis I concluded that there should be more actions and there should be some sub-cycles included inside the Planning and Executing phases. A specific sub-cycle that occurs multiple times in the process of proving is encountering multiple “stuck points.”

Multiple stuck points not identified: As can be seen from my analysis, the majority of the students’ stuck points seemed to occur in the phase of Executing. Students seemed to encounter several stuck points during the Executing stage.

Thus, there is a need to develop a framework that captures the different cycling and students’ navigating actions around stuck points. I will describe in detail in the next chapter how I developed such a framework. One thing I also noted from the pilot results was that not all navigating actions led students to successfully prove the tasks. Thus, I was especially interested in whether there might be some possible indicators that some actions would be more productive than others.
In this chapter, I will describe in detail how I analyzed my data. As I have discussed in the previous chapter, my pilot study showed that, in order to account for the proving processes of undergraduate students, the existing problem-solving frameworks need to be expanded. In particular, the role of students’ stuck points in their process is important, as is understanding more about what students do when they get stuck. In the next sections, I will detail my process of developing, testing, and refining my coding scheme, leading to the development of my analytical framework.

Analytical Tools and Processes

I seek to characterize relative novices’ proof construction processes and understand what they do in “stuck points.” The focus of this study lent itself to the adoption of certain grounded theory methods to guide data collection and data analysis (Glaser & Strauss, 1967; Shkedi, 2005; Yin, 2014), specifically, the data analysis strategies of open coding, axial coding, selective coding, and constant comparative analysis. The data analysis was carried out through the negotiation of the existing conceptual framework with empirical data at hand, to appropriate and adapt this framework to my research context.

Operationalizing and Identifying Stuck Points

I began by segmenting transcripts of students’ proving processes into episodes of different phases, drawing from Carlson and Bloom’s (2005) problem-solving phases: Orienting, Planning, Executing, and Checking. I then identified episodes in which students got “stuck.”
Identifying a “stuck point” can be hard since it requires finding observable behaviors to serve as indicators of a person’s internal mental state. In this sense, body and facial language become more important in telling whether a person has gotten stuck. For my purposes, *stuck* meant a period of time during the proving process when a prover felt or recognized that their argument had not been progressing fruitfully and that they had no new ideas. Whether the participant discovered an error was not important; the prover’s awareness that the argument had not been progressing but they were hesitant about what to do next was the key focus for me. For the purposes of my study, an impasse (an episode of “stuckness”) consisted of interruption in the proving processes that was initiated by a student struggle that was in some way visible, whether voiced, gestured, or written. Specifically, I considered a potential “stuck point” to be a moment of silence through audio or some facial or body language through video.

An impasse ended when the student overcame a stuck point and continued attempting or finishing the task, or the student gave a sign of no resolution and indicated wanting to switch to a different task or end the interview. In other cases, the episode ended when the student continued to struggle but the interviewer had to move on because of time. In total, there were 41 episodes of such “stuck points.”

I used a process of iteration between constructing descriptions and explanations of students’ actions or behaviors and the subsequent adjustments based on further evidence that is consistent with the constant comparative method of grounded theory (Glaser & Strauss, 1967). To find out whether there were some common actions for a student in both tasks, or some common actions among the students for each task, a cross-cases analysis was also used to make comparison between the cases.
Table 3 summarizes the unit of analysis for each research question, which will be discussed in more detail in the next chapters.

Table 3

*Unit of Analysis for Each Research Question*

<table>
<thead>
<tr>
<th>Research Question</th>
<th>Unit of Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Kinds of Stuck Points</td>
<td>Instances of Identified Stuck Points ($N = 91$)</td>
</tr>
<tr>
<td>2. Overall Proving Processes</td>
<td>Episodes around Stuck Points ($N = 41$)</td>
</tr>
<tr>
<td>3. Navigating Actions</td>
<td>Episodes around Stuck Points ($N = 41$), Instances of Navigating Actions, Monitoring and Persisting ($N = 230$)</td>
</tr>
</tbody>
</table>

**Creating Initial Coding Scheme and Developing the Proving Process Map**

To capture each student’s stuck points for each individual task, I used a flow chart to map out each student’s proving processes. These argument flow charts were inspired by Lew and Zazkis’s (2019) flow chart (Figure 6) to map out students’ arguments in proof.

*Figure 6. Students’ Arguments Flow Chart by Lew and Zazkis (2019)*

The different categories for this study around stuck points on the process map were developed from the data. I coded all 41 episodes related to students’ stuck points and navigating
actions using grounded theory methods to create my initial coding scheme. The general process of generating the bottom-up coding scheme can be explained in Figure 7.

![Diagram of the process of creating the initial coding scheme]

Figure 7. Process of Creating the Initial Coding Scheme

Through this process, several major categories were identified. Each student’s proving processes were captured in three major categories initially: their arguments, their stuck point, and their actions. After identifying those bigger categories, smaller categories were then identified inside of each major category. Specific examples of the categories will be described in the sections below.

In students’ arguments, there were initial arguments and related results from their navigating actions. In their actions, there are students’ initial explorations and their navigating actions once they face a stuck point. Since I was primarily interested in students’ arguments and actions around their stuck points, I refined categories related to proving activity to the following four: initial argument, stuck points, navigating actions, and related outcomes.
Similar to Lew and Zazkis (2019), I then expanded the proving process map into two dimensions. Vertically, the process map captured the four categories as described. Horizontally, the map captured the order of each individual argument and action, which were the main focuses of the proof literature, as discussed in the literature review. The arrows in between indicate the directions of the movements and the relationship between the arguments and the actions. Instead of capturing only arguments, I also added a category that captures students’ actions to help bridge the gap of the relationship between the argument and the actions, as described above.

In Figure 8 below, I provide an example of one student’s proving process map for Task 2. From this process map, one can clearly see the student’s arguments, stuck points, navigating actions, and outcomes related to those actions, as well as how one moved from one to the other. This proving process map provides a bird’s-eye view of the proving process for each task for each individual student.

*Figure 8. Example of Student’s Proving Process Map for Task 2*
Testing and Refining the Coding Scheme

As I have discussed, my pilot study showed that in order to account for the proving processes of undergraduate students, the existing frameworks need to be expanded. Thus, I needed to develop a framework or coding scheme to capture students’ actions around those stuck points. In this section, I will detail my process of developing, testing, and refining my coding scheme, leading to the development of my analytical framework.

Testing the Initial Coding Scheme With the Literature

As explained in the previous sections, using thematic analysis, I developed four main categories around stuck points as my initial coding scheme: initial argument, stuck points, navigating actions, and related outcomes. I will detail the process of using the existing literature and frameworks to enhance my initial coding scheme in the next few paragraphs. Specifically, once I had the overall categories for general proving processes, it was necessary to develop sub-categories to capture students’ actions.

Warshauer (2011) identified four types of struggles that students encounter as they work on challenging tasks in a classroom session (as shown in Table 1). While useful for understanding the nature of struggles, this framework does not capture clearly where in the problem-solving or proving processes students encounter those struggles, nor does it focus on the actions they take to try to navigate out of those struggles. In addition, based on the nature of the two proof tasks (prove or disprove a given statement), students also needed to determine whether to prove or disprove the statement in the initial stages.

I then worked from Warshauer’s categories and tried to find some themes. For example, Warshauer’s categories of being unable to implement a process from a formulated representation, being unable to implement a process due to its algebraic nature, and being unable
to carry out an algorithm are all related to computational problems; forgetting type of problems and expressing a misconception all related to lack of content knowledge. To capture where in the proving processes students would encounter these struggles, I combined my reorganization of Warshauer’s framework with Carlson and Bloom’s (2005) four phases (see Table 4).

Table 4

*Initial Coding Scheme*

<table>
<thead>
<tr>
<th>Phases (Carlson and Bloom’s phases)</th>
<th>Codes</th>
</tr>
</thead>
</table>
| Initial exploration (Orienting and Planning) | • Difficulty in understanding the statement  
• Difficulties in determining whether the statement is valid |
| Stuck points and navigating actions (Executing) | • Difficulty in explaining their work  
• Lack of content knowledge  
• Computational problems  
• Difficulties in using and understanding a strategy  
• Responding to impasses |
| Validating (Checking) | • Difficulties in validation |

Once the initial coding scheme with those themes was developed, I completed the first round of data analysis to test and refine those themes. Specifically, I used the coding scheme to classify the different types of “stuck points” I had observed in my data to see if there was anything missing in my initial coding scheme. As I was analyzing the transcripts using the initial coding scheme, I also classified the actions and approaches that students used to try to navigate out of the “stuck points” they encountered. The initial coding scheme was elaborated as new instances, categories, or themes were found.
One of the major differences I found between my analysis and the initial coding scheme was the unique argumentation properties that proving processes have compared to the problem-solving processes that Warshauer (2011) and Carlson and Bloom (2005) were concerned with. Thus, I added some categories related to argumentation or generalization from my data into my emergent coding scheme: (1) connecting to previous arguments, (2) difficulties in generalizing, (3) difficulties in linking the arguments, and (4) difficulties in converting informal ideas and arguments into mathematical language.

Secondly, since my work focused on “stuck points,” I reorganized my coding scheme to highlight students’ responses to “stuck points” or impasses. These often occur in Carlson and Bloom’s (2005) Planning and Executing phases. Based on the first round of my data analysis, my participants’ proving processes could be divided into three major phases: (1) initial exploration of the task, (2) navigating out of stuck points, and (3) validating the proof. Thus, I refined my coding scheme to center around those three phases instead of Carlson and Bloom’s four phases.

To sum up, in the phase of initial exploration of the task, students explore the validity of the statement and make a conjecture based on their previous knowledge or arguments. These actions correspond to the Orienting and Planning phases Carlson and Bloom (2005) identified. When trying to execute their initial ideas, undergraduate students often run into several stuck points. They sometimes identify their stuck points and then take some actions to try to resolve the impasses. The section of “identifying stuck points” was developed from my initial coding scheme and Warshauer’s (2011) classification of different type of stuck points.

Refining the Coding Scheme

Besides different types of stuck points that were identified by the students, their actions for attempting to resolve those stuck points, monitoring, and persistent actions were also
classified and captured in the refined coding scheme. Based on my second round of inductive thematic analysis, students’ navigating actions could be classified as (1) trying examples, (2) trying different representations, (3) trying different strategies, and (4) trying different resources (material, social, and mental). In addition, I also observed that students behaved differently when they encountered different stuck points and that, interestingly, similar navigating behaviors didn’t necessarily lead to similar outcomes. Thus, other factors used in solving the problem, such as their monitoring process and persistence, also needed to be taken into account.

Table 5 shows the refined coding scheme.

In this study, because of my research questions, most of my analytic attention focused on the second proving process phase concerning navigating out of stuck points. Figure 9 summarizes the process of how I develop my analytical framework in trying to answer my research questions.
Table 5

**Refined Coding Scheme**

<table>
<thead>
<tr>
<th>Proving process phases</th>
<th>Codes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial exploration of the task</td>
<td>• Exploring the validity of the statement</td>
</tr>
<tr>
<td>(occurs in the Orienting and Planning phases)</td>
<td>• Connecting to existing argument</td>
</tr>
<tr>
<td>Stuck Points and navigating the stuck points</td>
<td>• Identifying stuck points</td>
</tr>
<tr>
<td>(occurs mostly in the Planning and Executing phases)</td>
<td>◦ Difficulty in understanding the statement</td>
</tr>
<tr>
<td></td>
<td>◦ Lack of content knowledge</td>
</tr>
<tr>
<td></td>
<td>◦ Computational problem</td>
</tr>
<tr>
<td></td>
<td>◦ Difficulties in generalization</td>
</tr>
<tr>
<td></td>
<td>◦ Difficulties in linking the arguments</td>
</tr>
<tr>
<td></td>
<td>◦ Difficulties in convert into mathematical language</td>
</tr>
<tr>
<td></td>
<td>◦ Difficulties in strategy</td>
</tr>
<tr>
<td></td>
<td>◦ Difficulties in validation</td>
</tr>
<tr>
<td></td>
<td>• Attempting to resolve the stuck points</td>
</tr>
<tr>
<td></td>
<td>◦ Try examples</td>
</tr>
<tr>
<td></td>
<td>◦ Try different representations</td>
</tr>
<tr>
<td></td>
<td>◦ Use/Switch proof strategy</td>
</tr>
<tr>
<td></td>
<td>◦ Use of resources</td>
</tr>
<tr>
<td></td>
<td>▪ Material resources: definitions, formulas, notes</td>
</tr>
<tr>
<td></td>
<td>▪ Mental resources: relevant ideas</td>
</tr>
<tr>
<td></td>
<td>▪ Social resources: asking for help</td>
</tr>
<tr>
<td></td>
<td>• Monitoring process</td>
</tr>
<tr>
<td></td>
<td>• Persisting in the proving process</td>
</tr>
<tr>
<td>Validating the proof (Checking)</td>
<td>• Difficulties in validation</td>
</tr>
</tbody>
</table>
In this section, my finalized coding scheme (see Table 6), which serves as the primary analytical framework for this study, will be explained in detail with description and examples. Some of the definitions grew out of and were informed by my work analyzing the proving processes of a different set of undergraduate students as part of a different research project.²

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² T2P (Transitions to Proof Research Group at Michigan State University and Western Michigan University; Co-PIs: Dr. Mariana Levin, Dr. Shiv Karunakaran, and Dr. Jack Smith) centers on Transition-to-Proof courses and tracking the longitudinal development of STEM majors’ autonomy and agency in mathematical proof and proving. The research group had paid close attention to students’ impasses as an expression of mathematical autonomy. The T2P approach to analyzing autonomy in mathematical autonomy was influenced from my research of students’ stuck points and at the same time my involvement in this research process has also helped me in extending my thinking and constructs around analyzing students’ impasses.
Table 6

Analytical Framework for Characterizing Students’ Actions Around Stuck Points

<table>
<thead>
<tr>
<th>Actions</th>
<th>Description or sub-codes</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identifying Stuck Points</td>
<td>Engages in analyzing their argument up to the point of impasse, including determining where argument is solid as well as where it is weak.</td>
<td>“I know that to use contrapositive I need to say that a squared plus b squared is not equals to c squared. But I don’t have like, information on c squared. Like, I don't know if it's, oh, no, odd. If it's odd, then the square is odd. Right?” (Nikki, Task 1)</td>
</tr>
<tr>
<td>Attempting to resolve stuck points</td>
<td>Trying examples</td>
<td>“So, so. So for example, we have 18 by nine and one plus eight, nine by nine (writing on the paper)…” (Rachel, Task 2)</td>
</tr>
<tr>
<td></td>
<td>Trying different representations</td>
<td>“And I'm not, I'm not saying that 18 is the 1 multiply by 8 together. That's an integer……So I could represent them it two different ways, 6 plus 2. Well, there's different things…….(silence)” (Porter, Task 2)</td>
</tr>
<tr>
<td></td>
<td>Using/Switching Strategy</td>
<td>“Can I use Induction? This type of problem looks like an induction problem.” (Porter, Task 2)</td>
</tr>
<tr>
<td>Use of Resources</td>
<td>Materials Resources (Notes, textbook, internet, etc.)</td>
<td>Eric started to look into the formula sheet and found the general formula after a few minutes. (Eric, Task 2)</td>
</tr>
<tr>
<td></td>
<td>Mental Resources (Mental process of ideas)</td>
<td>“Yeah, yeah, I just take cases in my head……so even on an even……so do I want if it's at most one of them is odd that would mean they have a case one wants odd ones even when the two of them are even……okay let me write…….” (Devin, Task 1)</td>
</tr>
<tr>
<td></td>
<td>Social Resources (Ask for help at tutor lab, learning centers, office hours, etc.)</td>
<td>“If it's a question like this, because like with the other one, I knew that I could do it. But with this one, I really think I can't. So I would ask the teacher, ‘How can I start this?’ Yeah.” (Porter, Task 2)</td>
</tr>
<tr>
<td>Monitoring Progress</td>
<td>An understanding, evaluation or recognition of where they are in the proving process when faced with an impasse.</td>
<td>“Am I right? (talking to himself) Yeah, yeah of course I'm right cuz it's nine time so it didn't it didn't it didn't mean anything this stuff like…Oh, wait, wait, maybe…maybe a minute something…I just feel like I kind of know the thing… (silence)” (Soni, Task 2)</td>
</tr>
<tr>
<td>Persisting in Solving</td>
<td>Allowing time in proving</td>
<td>“Yeah, so see, it's gonna be square root of this stuff for square root of this stuff is k plus k plus 2n. Wait, no, ah, ah, wait, I'm almost there! So k squared…….(silence)” (Soni, Task 2)</td>
</tr>
</tbody>
</table>
Identifying Stuck Points

Identifying stuck points refers to when students engage in analyzing their argument up to the point of impasses, including determining where their argument is solid as well as where it is weak. For instance, when solving Task 1, one of my participants, Porter, indicated he was stuck. He identified his stuck point clearly:

I know that to use contrapositive I need to say that a squared plus b squared is not equal to c squared. But I don’t have like, information on c^2. Like, I don't know if it’s, oh, no, odd. If it’s odd, then the square is odd. Right? And then adding them to odd numbers. (Porter, Task 1)

This is a good example of a student identifying what they knew and where they got stuck.

Attempting to Resolve the Stuck Points

This refers to all the attempts or actions taken to try to resolve the identified source of struggle. Inside this category, there are specific attempts or navigating actions, such as trying examples, trying different representations, using or switching strategies, and using different resources. In terms of trying examples, there were many instances in which the students tried examples to make sense of the argument. For example, Rachel tried a few examples of the multiple of 9s to try to find the connection between the digits and the sum of the digits. In terms of trying different representations, a good example is provided by Soni, who tried to rewrite all the 10s as 9+1 to try to find the differences between counting by 10s and counting by 9s. For switching strategies, both Porter and Soni thought about applying induction instead of the direct proof that they attempted initially. Participants also looked into material resources such as formula sheets to try to help themselves move forward.

Monitoring Progress

Monitoring refers to a self-understanding, evaluation, or recognition of where the student is in the problem-solving process when faced with an impasse. Monitoring one’s process was
observed in many students. To exemplify, Soni kept checking on himself when making decisions about how to move forward:

Am I right? (talking to himself). Yeah, yeah of course I’m right cuz it’s nine times so it didn’t it didn’t it didn’t mean anything this stuff like…Oh, wait, wait, maybe…maybe a minute something…I just feel like I kind of know the thing… (silence). (Soni, Task 2)

In my later analysis, I will show that monitoring progress is a really important step in trying to overcome stuck points.

Persisting in Solving

Persistence refers to when the students allow themselves time. This action may be more obvious in some students compared to others. For instance, after around 20 minutes of trying Task 2 without much progress, Nikki told me directly, “I have enough for this problem; I want to move on” (Nikki, Task 2). While persistence alone would not guarantee that Nikki would have made progress, it is necessary for progress.

After finalizing the analytical framework, I then went back and uniformly applied the coding scheme to the dataset. From the first several rounds of data analysis, different kinds of undergraduates’ stuck points had emerged. In the next chapter, I will discuss the characteristics of each of the kinds.
CHAPTER 6

THE NATURE OF STUDENTS’ STUCK POINTS

The development of the analytical framework helps me to answer my first research question, “What kinds of stuck points do students encounter as they engage in proving?” After the first several rounds of data analysis, different kinds of undergraduates’ stuck points also emerged. In this chapter, I will discuss the characteristics of each of the kinds, providing prototypical examples from the dataset as needed for the purpose of illustrating the nature of each kind.

Understanding Patterns of Stuck Points

After finalizing the analytical framework, I then went back and uniformly applied it to the dataset. In this analysis, I compiled the types of navigating actions by participant. Table 7 shows the breakdown for each participant of their different major actions in responding to stuck points.

Based on Table 7, we can create a bird’s-eye view of each participant’s overall actions in responding to stuck points. With respect to the four major categories, “persisting in proving the statement” was the least observed among the four major categories, while “monitoring” was the most observed. With regard to each participant’s actions in navigating stuck points, Soni, Eric, and Porter each had a strong profile of engagement around stuck points, while the rest of the participants had similar, but lower, numbers of instances of navigating actions (around 20). Thus, I considered Soni, Eric, and Porter to be prime candidates for selection for a deeper case analysis of moment-by-moment proving processes and actions around stuck points.
Table 7  
Actions in Responding to Stuck Points by Participants

<table>
<thead>
<tr>
<th>Participants/Codes</th>
<th>Identifying stuck points</th>
<th>Attempting to resolve</th>
<th>Monitoring</th>
<th>Persisting</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Devin</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>0</td>
<td>24</td>
</tr>
<tr>
<td>Eric</td>
<td>12</td>
<td>21</td>
<td>30</td>
<td>7</td>
<td>70</td>
</tr>
<tr>
<td>Jeff</td>
<td>4</td>
<td>8</td>
<td>4</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>Matt</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>Nikki</td>
<td>12</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>Porter</td>
<td>16</td>
<td>11</td>
<td>13</td>
<td>3</td>
<td>43</td>
</tr>
<tr>
<td>Rachel</td>
<td>6</td>
<td>10</td>
<td>9</td>
<td>2</td>
<td>27</td>
</tr>
<tr>
<td>Rina</td>
<td>11</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>22</td>
</tr>
<tr>
<td>Soni</td>
<td>13</td>
<td>24</td>
<td>29</td>
<td>8</td>
<td>74</td>
</tr>
<tr>
<td>Total</td>
<td>91</td>
<td>99</td>
<td>107</td>
<td>24</td>
<td>321</td>
</tr>
</tbody>
</table>

Besides navigating actions, across the interviews there were 91 instances in which participants’ stuck points were identified by themselves in different formats (questions, think-alouds, and in their written work). I paid close attention to those instances and pulled them out to look at them in particular to try to answer my first research question about the patterns and kinds of students’ stuck points in proving. Based on the memo of my description for each instance, I first categorized those instances into several small categories of kinds of stuck points:

- Lack of content knowledge that is needed to pursue further
- Difficulty in computation or algebraic manipulation
• Difficulty in generating examples
• Difficulty in generalizing formulas/equations based on their examples
• Difficulty in linking the arguments together
• Difficulty in explaining their work in mathematical terms
• Difficulty in using any kind of strategy or proof technique
• Difficulty in switching to a different strategy
• Unclear about the reasons for the use of certain strategy
• Difficulty in validating whether the previous steps are correct
• Uncertain about whether they have proved the statement

I then tried to put similar small categories together to try to generate themes. There are content-related stuck points, such as when the student indicated that they needed certain content knowledge like “modular arithmetic” to pursue further completing the proof. There are computation-related stuck points, when students are having difficulties in algebraic manipulation. There are strategy-related stuck points, when student is having difficulties in using any kind of strategy or proof technique, or having difficulties in switching to a different strategy, or is unclear about the reason for the use of a certain strategy. There are also validation-related stuck points, when students have difficulty in validating whether the previous steps are correct, or when they are uncertain about whether they have proved the statement. There are some stuck points around generalizing, when students have difficulties in generalizing from examples or have difficulties in generalizing formulas/equations based on their examples. Finally, there are stuck points around their argumentation, when students have difficulties in linking the different arguments together or explaining their work in mathematical language. Table 8 shows some examples of how I turned those initial smaller categories (codes) into the themes.
Table 8

*Examples of Assigning Codes with Themes*

<table>
<thead>
<tr>
<th>Codes</th>
<th>Theme</th>
</tr>
</thead>
<tbody>
<tr>
<td>Difficulty in generalizing from examples</td>
<td>Generalization</td>
</tr>
<tr>
<td>Difficulty in generalizing formulas/equations</td>
<td></td>
</tr>
<tr>
<td>based on their examples</td>
<td></td>
</tr>
<tr>
<td>Difficulty in linking the arguments together</td>
<td>Argumentation</td>
</tr>
<tr>
<td>Difficulty in explaining their argument in</td>
<td></td>
</tr>
<tr>
<td>mathematical terms</td>
<td></td>
</tr>
</tbody>
</table>

Once I made sure the themes were accurate representation of the data, I then returned to the data set and compared my themes against it to see if anything was missing. The finalized six themes are explained below with examples and also in Table 9. I want to note that not all codes of stuck points fit into all of these categories; there were still a few instances that were unique to a given context. In addition, some instances belong to more than one theme, thus the total percentage does not add up to 100%. The six kinds described below account for 97% of the total instances.

**Generalization**

Stuck points around generalization are defined as when students have difficulties in generalizing examples based on their argument or in generating general representations/formulas based on their examples. I put those codes under the theme of generalization because they both involve a certain level of generalizing that students are doing from their examples, either to generalize an argument or a formula/equation. There are many instances when students questioned, “How do I show it works for all cases?” or “How can I represent, in general, all numbers with its digits?” after successfully having generated several examples. In most cases,
students had fewer difficulties in generalizing examples to help make sense of their argument.

For example, for Task 2, after trying multiple examples and realizing each of those examples is divisible by 9 if its digits are divisible by 9, many students like Porter had difficulties in trying to generalize this to show it works for all integers. Porter tried to represent the integers using different strategies, but those attempts still didn’t help him to generate a formula or equation that shows it works for all integers.

**Argumentation**

Stuck points around argumentation refer to when students have difficulties in going beyond their intuition or informal explorations to linking the argument and expressing their argument in mathematical terms. Those codes are grouped together under the theme of argumentation because they are all difficulties students have related to argument, whether producing the argument, linking two arguments together, or writing their informal explorations into formal proofs. Many instances have been observed that belong to this theme. During their proving processes, many students constantly asked questions about their argumentation, such as “How can these ideas help me to prove the statement?” or “How do all the results tell me about the statement?” or “How do I link those arguments together?” This is also one of the major differences between proving and problem-solving processes. For examples, for Task 1, Nikki successfully showed the result of \( a^2 + b^2 \) is even when both \( a \) and \( b \) are odd. But how to use this related argument to link back and show the parity for \( c \) became the problem for her. There are also students that mentioned they tested the statement that an integer is divisible by 9 if its digits are divisible by 9, but they had difficulties in transforming their informal exploration in a formal proof.
Table 9

*Kinds of Students’ Stuck Points in Proving the Interview Tasks*

<table>
<thead>
<tr>
<th>Kinds of Stuck Points/Codes</th>
<th>Sub-Categories</th>
<th>Example of Students’ Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Content Knowledge</td>
<td>Difficulty in understanding the task statement</td>
<td>“What does mod means?”</td>
</tr>
<tr>
<td></td>
<td>Lack of content knowledge that is needed to carry on (next step is not identified)</td>
<td>“What is proof by contradiction look like?”</td>
</tr>
<tr>
<td>Computation</td>
<td>Difficulty in implementing a process due to its algebraic nature</td>
<td>“How to expand ((9 + 1)^3)?”</td>
</tr>
<tr>
<td></td>
<td>Errors related to computation</td>
<td></td>
</tr>
<tr>
<td>Generalization</td>
<td>Difficulty in generating examples</td>
<td>“How to represent all numbers with its digits in general?”</td>
</tr>
<tr>
<td></td>
<td>Difficulty in generalizing formulas/equations based on their examples</td>
<td>“How to write these into a general formula?”</td>
</tr>
<tr>
<td>Argumentation</td>
<td>Difficulty in going beyond common sense</td>
<td>“How can these ideas help me to prove the statement?”</td>
</tr>
<tr>
<td></td>
<td>Difficulty in linking arguments</td>
<td>“How does all the results tell me about the statement?”</td>
</tr>
<tr>
<td></td>
<td>Difficulty in explaining their argument in mathematical terms</td>
<td></td>
</tr>
<tr>
<td>Strategy</td>
<td>Difficulty in using a particular kind of strategy or proof technique (i.e., proof by induction, proof by contradiction, contrapositive, direct proof, counterexample)</td>
<td>“Can I use induction here?”</td>
</tr>
<tr>
<td></td>
<td>Difficulty in switching to a different strategy</td>
<td>“What strategy should I use to prove or disprove this statement?”</td>
</tr>
<tr>
<td></td>
<td>Unclear about the reasons for the use of certain strategy</td>
<td></td>
</tr>
<tr>
<td>Validation</td>
<td>Difficulty in validating whether the previous steps are correct</td>
<td>“Have I proved the statement?”</td>
</tr>
<tr>
<td></td>
<td>Uncertain about whether they have proved the statement</td>
<td>“Is my assumption correct?”</td>
</tr>
</tbody>
</table>

**Strategy**

Stuck points around strategy usage refer to when students have difficulty in using a strategy or switching to different kind of strategy or are unclear about their reasons for choosing
a particular strategy. For some of the students, it was difficult for them to distinguish between strategies such as contradiction and contrapositive, induction and strong induction. They asked, “What strategy should I use to prove or disprove this statement?” There are also instances in which a student chose a strategy without a reason particularly connected to the task at hand. For example, their choice appeared to be based on having learned it in class recently. For example, Porter indicated the reason he chose induction for Task 2 was because he had just learned induction recently in class. Thus, deciding upon, using, and switching strategies is also one of the major challenges many students faced during the interview.

**Computation**

Stuck points around computation refer to when students have difficulties in implementing a process due to its algebraic nature or they make other errors related to computation. Computational difficulties may also be mentioned multiple times in the instance. For example, for Task 2, many students found it difficult to deal with the $k^{th}$ term and questioned, “How can I expand $(9 + 1)^3$?” Also, even though students might not realize it or find out by themselves during the interview, some of their computational errors were also noted. For instance, Matt assumed his calculation of $\sqrt{4(k + l)^2 + 4k + 4l + 1}$ reduced to $2\sqrt{(k + l)^2 + k + l + 1}$, but in fact, his calculation was not correct since he broke out the square root.

**Content Knowledge**

Stuck points around content knowledge refer to when students have difficulties in understanding the task statement or have challenges in accessing the content knowledge needed to carry on in their proof attempt. This happened especially for Task 2. For example, students like Devin recalled that the statement could be proved using modular arithmetic. However, since he did not understand the definition of mod, he raised the question “How can I apply the
definition of mod here? What actually does mod mean?” With a lack of understanding of modular arithmetic, many students took other routes to grapple with Task 2, which was originally designed as a modular arithmetic problem.

**Validation**

Stuck points around validation refer to when students have difficulties in validating whether the previous steps in their argument are correct or they are uncertain about whether they have proved their argument. Some students, when asked about whether they think they have proved the statement, indicated that they are not really sure about what they did and still question, “Have I succeeded in proving the statement?”

**Overlap**

As I have mentioned, there are instances of stuck points that belong to more than one theme. For example, when Nikki was stuck with using proof by contraction, she had difficulties both in understanding what exactly is proof by contraction (content knowledge), but also whether proof by contraction would work here (strategy). Similarly, when Porter is asking “how to show it work for all cases,” he is concerned not only with how to make a generalization based on his examples but also how to make the argument from his exploration. Thus, out of the 91 total instances of stuck points, 109 total themes were identified, including the overlaps.

**Other**

As mentioned above, the first six categories account for 97% of all stuck points observed in these data. All the other stuck points that didn’t clearly belong to the above categories and were not similar in some way to each other were categorized as “other.” For instance, Rina identified that the source of one of her stuck points was that she thought there was a lack of information given in the task. Thus, the source of her stuck point was not from her internally.
Based on those bigger categories, each instance of the different kinds of stuck point was then counted, as shown in Table 10, from most to least frequently occurring. Note that a few instances contained multiple kinds, so each instance can have more than one theme associated with it. Thus, the percentages do not add up to 100%.

Table 10

Frequency and Percentage of the Stuck Points by Kind

<table>
<thead>
<tr>
<th>Kind</th>
<th>Counts</th>
<th>Percent (of N = 91)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argumentation</td>
<td>47</td>
<td>51.6%</td>
</tr>
<tr>
<td>Generalization</td>
<td>33</td>
<td>36.2%</td>
</tr>
<tr>
<td>Strategy</td>
<td>11</td>
<td>12.1%</td>
</tr>
<tr>
<td>Content Knowledge</td>
<td>8</td>
<td>8.8%</td>
</tr>
<tr>
<td>Computation</td>
<td>4</td>
<td>4.4%</td>
</tr>
<tr>
<td>Validation</td>
<td>3</td>
<td>3.3%</td>
</tr>
<tr>
<td>Other</td>
<td>3</td>
<td>3.3%</td>
</tr>
<tr>
<td>Total</td>
<td>109</td>
<td>120%</td>
</tr>
</tbody>
</table>

Based on the percentages across the full data set, the most common types of stuck points are around argumentation and generalization. These two categories accounted for more than 87% of the data. Specifically, for the two proof-based interview tasks in my study, the majority of the participants’ stuck points involved generating a general formula based on their examples, linking the different arguments they were generating, and explaining their work in mathematical terms. All of these aspects are closely associated with the nature of proof tasks (with argumentation in particular, as opposed to problem solving, more generally). Thus, the development of my
analytical framework for analyzing the students’ stuck points helped me in answering my first research question about the kinds of struggles students encounter in their proving processes.

If we compare my findings with the findings of Warshauer (2011), there are major differences for each category. Even though the two studies categorize the nature of struggles differently—Warshauer in terms of different proving phases and this study in terms of specific instances—for Warshauer, only around 30% of the students’ stuck points were attributed to argumentation and validation. Instead, the majority of the source of struggle for the students in her study revolved around getting started and carrying out a process (57%). Interestingly, all of the participants in this study had no difficulties in getting started. This could be due to the two different populations of the study (middle school students vs. undergraduates) and the context of the study (classroom vs. interview). I will discuss more about the differences between these studies in the discussion chapter.

In responding to those different kinds of stuck points, the students in my study used multiple different kinds of actions to try and navigate out of their stuck points. Out of the 99 instances of attempts to resolve stuck points, trying examples, trying different representations, using/switching strategies, and using different resources were some of the most common actions. To move beyond tabulating the frequency and percentage of those different actions, I argue it is important to consider each action in context and then to characterize the overall proving process. In the next chapter, I will describe the analysis of students’ navigating actions when they encountered stuck points and then will go deeper with two illustrative contrasting case analyses to respond to my second and third research questions.
Chapter Summary

The focus of my first research question was identifying the kinds of stuck points students encounter in their proving processes. These kinds were identified using grounded theory techniques and thematically grouping the coding for places where students struggled in proving into six categories: (1) argumentation, (2) generalization, (3) strategy usage, (4) content knowledge, (5) computational difficulties, and (6) validation. Even though there were also stuck points that didn’t fit in those six broad themes, these six accounted for 97% of the total instances (120%) when students experienced an impasse. Based on those themes, each instance of the different kind of stuck point was then tabulated from most to least frequently occurring. It is also worth noting that all of the instances occurred after the initial exploration stage, so the students had already had an opportunity to get started in thinking about the task. Interestingly, all of the participants in this study had no difficulties in getting started.

My findings indicated that the majority of the undergraduate students in my study had difficulties around argumentation and generalization when proving number theory tasks. The finding of the kinds of stuck points can help to better inform teachers about their students’ difficulties as they consider appropriate instructional support or guidance. In addition, students can also self-evaluate their own stuck points by noting the aspects that they are unable to address or progress (Warshauer, 2011). The results show that students may not get stuck entirely for the whole task but might struggle on certain aspects during their proving processes. With the idea that the focus should not be on getting the correct result or answer, each stuck point will seem more manageable to students.

Further, although the majority of students got stuck around argumentation and generalization, their proving processes and the actions they took to try to navigate out of their
stuck points were different. Those differences turned out to be the major factors of whether or not a student would be successful in navigating out of their stuck points. Thus, it was important to specifically look into students’ process and actions around their stuck points—the focus of research questions 2 and 3.
CHAPTER 7

CHARACTERIZING STUDENTS’ OVERALL PROVING PROCESSES

The overarching goal of my study is to characterize undergraduate students’ stuck points and navigating actions during their proving processes. The previous chapter addressed my first research question concerning the nature of the stuck points students in my study encountered in their proving processes. To answer my second research question, “In proving attempts that generate stuck points, what characterizes the overall proving process?” I will first describe three general types of proving processes around stuck points (generated from an analysis across all the study data).

Types of Overall Proving Processes

In the previous chapters, the data collection and the data analysis process were described in detail, and an analysis of the kinds of stuck points students in the study encountered was also presented. In this section, I turn to focusing on the qualities of navigating actions students used, situated in the context of their entire proving attempts. My analysis resulted in three major types of navigating actions that were generated based on each participant’s proving process map and the major categories of my analytical framework. As I discussed in the data analysis chapter, for each task, each participant’s proving process was characterized by their process map based on shifts between (1) initial argumentation (Argument), (2) major stuck points (Stuck Points), (3) actions they took to try to navigate out of being stuck (Navigating Actions), and (4) related results (Related Outcome). If they are able to use the related outcomes (new insight, useful calculation, new connection) that they generated in a given cycle, then the proving process
progresses, either to completion or until they reach a new impasse. In the next sections, I will describe the three major types of proving process cycles around stuck points based on these shifts and illustrate with several examples for each type. I will present the processes flow types in the order of increasing complexity and going from no related outcome produced, to related outcome produced but not a link to the argument, to at least one related outcome linked with the argument.

**Type 1: No Related Outcome Produced**

*Trajectory: Arguments → Stuck Points ← Navigating Actions*

For this process type (as in all the types I consider), the student started with some sort of argument and got stuck when trying to construct a proof. The student then took several actions or attempts to try to navigate out of the stuck point but did not produce any related results, thus going back to their navigating actions. In summary, this type can be characterized as going back and forth between stuck points and navigating actions without any related outcome to help one move forward (see Figure 10).

![Figure 10. Process Flow for Type 1 Process Around Stuck Point](image)

*Figure 10. Process Flow for Type 1 Process Around Stuck Point*
Only two of the participants’ proving processes can be characterized as this type: Nikki (Task 2) and Porter (Task 2). I will be discussing the case of Porter in detail in the next section; thus, I will use Nikki’s process to explain this type of proving process here (see Figure 11).

Figure 11. Nikki’s Work for Task 2

As soon as Nikki saw the task, she remembered that she had done a similar problem before. She quickly wrote down the problem and figured out what were the “P” and “Q” in this statement to set up for a proof by contrapositive. She then rewrote the statement as “If the sum $x_1 + x_2 + \cdots + x_k$ equals 9, then the number $n$ is divisible by 9” (Nikki, Task 2). But Nikki got stuck after she had decided that was the argument that she wanted to prove. “How are they related?” she questioned. At this point she went back to the beginning of the problem and double-checked which statement should be thought of as “P” and which statement as “Q,” then told me that she didn’t know where to go next. She then started trying some examples such as 27,
then thought about what happened if the sum was 9 or the sum of a multiple of 9 (like 18). But these navigating actions didn’t help her to go forward. She asked aloud, “How can I represent this example in a more general form?” and she got stuck again with her attempts to try examples. Nikki then tried to look for the general equation for expressing the sum of the digits. She found the expression on the formula sheet and tried to understand the expression again by computing some examples and by rewriting 27 as $2 \times 10^1 + 7$. But she was soon stuck again with the $k^{th}$ term: “What does $k$ here mean in the expression of $10^k$? Is $k$ the number of digits or something else?” (Nikki, Task 2). She told me this general form she found (on the provided formula sheet) didn’t really help her, because she didn’t even understand the expression itself. After several attempts, Nikki didn’t produce any related outcome from her navigating actions and she remained stuck in the end of the episode.

As we have noticed from Nikki’s example, this type of process doesn’t involve the production of related outcomes from navigating actions. Students with this process tend to go back and forth between stuck points and navigating actions since no significant related outcome is produced to help them move forward. Thus, in the end of the process, students don’t overcome the stuck point and they are unable to make progress on their argument.

**Type 2: Related Outcome Produced but Not Link to the Argument**

*Trajectory: Arguments $\rightarrow$ Stuck Point 1 $\rightarrow$ Navigating Actions $\rightarrow$ Related Outcomes $\rightarrow$ Stuck Point 2 ($\leftrightarrow$ Navigating Actions)*

The majority of the participants’ proving processes can be characterized as this type. For this process type, like before, a student starts with some sort of argument and gets stuck when trying to prove the argument. The student then takes several actions or attempts to try to navigate out of the stuck point (so far, as in Type 1). However, the student successfully overcomes one stuck point by producing a new idea or connection (a “related outcome”). However, they soon
run into another stuck point when they try to expand the related outcome into an argument. This process may also include some navigating actions to try to overcome the second stuck point, but the student is unsuccessful in doing so (see Figure 12).

For the majority of participants in my study, Task 1 seemed to be more approachable than Task 2. This result actually matches the intention of the two interview tasks to reveal a broader range of proving processes. Thus, many of the participants’ Task 1 processes are like the Type 3 process that will be discussed below. Here I will mainly focus on Task 2, since several participants were able to generate some related outcomes via their navigating actions.

Several participants’ proving processes fit in this type: Devin (Task 2), Jeff (Task 2), Rachel (Task 1), Rina (Task 1), Matt (Task 1 & Task 2), Nikki (Task 1), and Porter (Task 1). I will now discuss one example for each task.

In Task 2, Devin conjectured the statement works for all multiples of 9 after trying examples like 9, 18, 27, 36. He wanted to prove the statement, but asked, “How can I represent all the multiples of 9 in general?” For him, this was a stuck point. He tried different examples and rewrote 27 as 20 + 7 to try to navigate out. Devin realized the tens digit is a multiple of 10
when trying different examples, so he generalized that any two-digit number can be written as
$10 \cdot a_1 + a_0$ where $a_1$ is the tens place and $a_0$ is the ones place. With this generalized form, he
moved forward. The next question that arose for him was then, “How can this idea of
representing digits relate to divisibility by 9?” He thus ran into another stuck point. Based on the
definition of divisibility, Devin now assumed the number is divisible by 9 and tried to see if he
could find any general patterns. If the number is divisible by 9, then $9n = 10 \cdot a_1 + a_0$. He used
different computation by rearranging the equation as $10 \cdot a_1 - 9n = -a_0, 9 \cdot a_1 + a_1 - 9n =$
$-a_0, 9(a_1 - n) = a_0 + a_1$. He concluded the statement works for two-digit numbers. However,
he then became stuck again on how to represent the multiples of 9 that have more than two
digits. Following his example with two digits, he tried to use the general formula for multiple
digits and tried to arrange the equation in the same way that he had before. But he wasn’t able to
rearrange the equation and didn’t go any further from this point (see Figure 13).

*Figure 13. Devin’s Work for Task 2*
Similar to Devin, Rina also had generated the conjecture that the statement worked for all multiples of 9 by trying examples like 18, 54, and 81. But she then got stuck with how to show the statement in general. She tried to represent a given number in terms of the sum of its digits in a general form like Devin and assumed $9/N$ where $N$ is a number such that $\text{ones} + \text{tens} + \cdots = \text{multiple of 9}$ and $m_1, m_2, m_3$ are the face values of each of the digits. Thus, any number can be represented as $N = 1 \times m_1 + 10 \times m_2 + 100 \times m_3 + \cdots$ But with this generalized formula in hand, how to use this to prove the statement became her new problem. She then thought about whether a number that is a multiple of 9 could be factored differently. Rina took the examples of 36 and 135, then rewrote them as $36 = 4 \times 9$, $135 = 1 \times 100 + 3 \times 10 + 5 \times 1 = 100 + 30 + 5 = 2 \times 5 \times 5 + 6 \times 5 + 5 = 5 \times 27 = 5 \times 3 \times 9$. But “What do these factorizations tell me?” became her new question, a question she still had in the end (see Figure 14).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{rinas-work-for-task-2.png}
\caption{Rina’s Work for Task 2}
\end{figure}
The examples of Devin and Rina illustrate what Type 2 processes look like and how they differ from Type 1 processes in terms of the role of related outcomes. Both Devin and Rina had several related outcomes produced by overcoming some of their stuck points. This helped them to move forward and, in fact, they were just a few steps away from fully overcoming all the stuck points and successfully proving the statement. Thus, at this point, it will be insightful to compare their processes with at least one related outcome linked with argument.

**Type 3: At Least One Related Outcome Linked with Argument**

*Trajectory: Arguments → Stuck Points → Navigating Actions → Related Outcomes → Arguments*

Similar to the previous two types, for this type of process, a student also starts out with an initial argument and they get stuck. The student then takes several actions or attempts to try to navigate out of the stuck point (as in Types 1 and 2 process flows). However, the Type 3 process advances to proving the statement using the related outcomes generated in the process and ultimately results in having at least one related outcome linked with the argument. Since there could be multiple stuck points resulting in multiple processes, the student will be categorized in this type as long as at least one productive cycle is complete. The cycle can repeat if a second or third stuck point is encountered. To summarize, this type of process always involves a related outcome each time and the participant is able to leverage these related outcomes to move forward with developing their argument (see Figure 15).
Several participants’ proving processes fit in this type: Devin (Task 1), Jeff (Task 1), Rachel (Task 1), Rina (Task 1), Eric (Task 1 & Task 2), and Soni (Task 1 & Task 2). I will now discuss one full cycle example for each task.

For Task 1, Rachel successfully overcame several stuck points and proved the statement. After reading the statement, she immediately thought about an initial approach based on proof by contradiction. She divided the task into three cases, $a$ and $b$ both even, $a$ and $b$ both odd, one of $a$ and $b$ even and one odd. She started with $a$ is even and $b$ is odd, set $a = 2k$ and $b = 2l + 1$, then rewrote $c$ in terms of $a$ and $b$. She then moved to case 2, assuming that $a$ and $b$ are both odd. Set $a = 2k + 1$ and $b = 2l + 1$. Then $c^2 = a^2 + b^2 = (2k + 1)^2 + (2l + 1)^2 = 4k^2 + 4k + 4l^2 + 4l + 2$. At this point, she ran into her first stuck point, “What does $4k^2 + 4k + 4l^2 + 4l + 2$ mean?” she questioned. She tried to finish the last case for both $a$ and $b$ is even, but that still didn’t help her to make any further conclusions. She was silent and thinking for a while, then a sudden “a-ha” moment came up. “What if I go from the other direction? Do I know about $c$?” She then went from the other direction and realized that $c$ can be even or odd so can be written as $2m$ or $2m + 1$. Then $a^2 + b^2 = (2m)^2 = 4m^2$ or $a^2 + b^2 = (2m + 1)^2 = 4m^2 +
4m + 1. But how to interpret and link the two results together became her new problem. She realized 4m^2 + 4m + 1 are odd, but 4k^2 + 4k + 4l^2 + 4l + 2 is even, thus proving a^2 + b^2 ≠ c^2 by contradiction. Thus, at least one of a or b is even (see Figure 16).

<table>
<thead>
<tr>
<th>(a, b, c \in \mathbb{Z}) s.t. (a^2 + b^2 = c^2)</th>
<th>(c) is an integer</th>
</tr>
</thead>
<tbody>
<tr>
<td>at least (a) or (b) is even</td>
<td>(c) is odd</td>
</tr>
<tr>
<td>(c = 2m + 1) for some integer (m)</td>
<td>(c = 2m) for some integer (m)</td>
</tr>
<tr>
<td>(a = 2k) for some integer (k) and (b = 2l) for some integer (l)</td>
<td>(4m^2 + 4m + 1 = a^2 + b^2)</td>
</tr>
<tr>
<td>(c = \sqrt{a^2 + b^2})</td>
<td>(m = 1, a = 3, b = 0)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case 2: (a) is even, (b) is odd</th>
<th>(c) is even</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c = 2m) for some integer (m)</td>
<td>(c^2 = 4m^2 = a^2 + b^2)</td>
</tr>
<tr>
<td>if (m = 0) then (a) and (b) are even</td>
<td>(m = 4, a^2 + b^2 = 4, c^2 = 16)</td>
</tr>
<tr>
<td>(a^2 + b^2 = (2k + 1)^2 + (2l + 1)^2 = 4k^2 + 4k + 4l^2 + 4l + 2)</td>
<td>(c^2 = 4k^2 + 4k + 4l^2 + 4l + 2)</td>
</tr>
</tbody>
</table>

**Figure 16. Rachel’s Work for Task 1**

Soni (who will also be discussed in more detail in the case analysis in the next chapter) successfully overcame several stuck points in both Task 1 and Task 2. Since I will be describing his proving process in detail in the next section, I will just summarize his general process for Task 2 here. Soni started with outlining some of his ideas for finding the differences between counting by 9s and counting by 10s and rewrote 10 as 9 + 1. Having this general goal in mind about what he would do, he felt the need for a generalized formula to represent any integer with its sum of the digits. Soni looked into the formula sheet and, like Porter, he found the formula that any integer can be represented as \(a_k(10)^k + a_{k-1}(10)^{k-1} + \cdots + a_0(10)^0\). But then how to use the general formula to prove the statement became his problem. Soni then thought about induction to help him achieve his goal. However, he soon realized that it didn’t work here. He decided to go back to his initial thinking about representing 10 as 9 + 1 and substituting all 10s
in the formula as $9 + 1$. However, he got stuck again on how to expand $a_k(9 + 1)^k + a_{k-1}(9 + 1)^{k-1} + \cdots + a_0(9 + 1)^0$. Soni then decided to try a specific example first, then follow the same way to show the generalized cases. Following this way of thinking, he successfully navigated out of his stuck points and proved the statement in the end.

From both Rachel and Soni, we see what could be described as a more productive struggle engaged in their proving processes since they completed at least one cycle of link back to the argument. Looking across both Rachel’s and Soni’s cases, we would notice both of them were clear about the goals for each of their navigating actions and had a clear understanding about where they got stuck. They also successfully produced several related outcomes from their navigating actions to help them move forward.

Based on those three types, I have categorized my participants’ work for Task 1 and Task 2 based on their proving process map, as shown in Table 11. Note here that students are considered as making productive progress (Type 3) as long as at least one productive cycle is complete, which means they don’t need to successfully prove the task in order to be considered as making productive progress. I also want to point out that even though the focus of the analysis is about the phenomena of stuck points rather than people, those cases of different students help to understand and illustrate the different processes better.
Table 11

*Proving Process Type by Task and Participants*

<table>
<thead>
<tr>
<th>Name</th>
<th>Task</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Devin</td>
<td>1</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Eric</td>
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<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
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<td>X</td>
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<tr>
<td>Jeff</td>
<td>1</td>
<td></td>
<td></td>
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<td></td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Matt</td>
<td>1</td>
<td></td>
<td>X</td>
<td></td>
</tr>
<tr>
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<td></td>
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</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Porter</td>
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<td>Rachel</td>
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<td></td>
<td>2</td>
<td></td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

My results indicated that the majority of my participants’ proving processes were more productive in their actions for Task 1 compared to Task 2. Looking vertically at the table, the majority of the participants’ work was considered either Type 2 or 3, and only Porter and Nikki’s work on Task 2 was considered Type 1. Looking horizontally in the table, participants can be grouped into three major categories: (1) two Type 3 processes (Soni and Eric), one Type 3 and one Type 2 (Devin, Jeff, Rachel, Rina), two Type 2 (Matt), and one Type 2 and one Type 1
(Porter and Nikki). This overview analysis supports my decision to focus on Porter and Soni in the case analysis I will present in Chapter 8 since I wanted to analyze specific actions associated with each type of process.

**Chapter Summary**

My second research question concerned characterizing students’ overall proving processes around stuck points. Three types of processes were generated based on the participants’ proving process maps for each task. Those three types are no related outcome produced, related outcome produced but not a link to the argument, and at least one related outcome linked with the argument. I will summarize each below.

*No Related Outcome Produced.* For this process type, a student starts with an initial argument and then gets stuck when trying to prove the argument. The student then takes several actions or attempts to try to navigate out of the stuck point they have encountered but cannot produce any related results; thus, they go back to their navigating actions. In summary, this type can be characterized as oscillating back and forth between stuck points and navigating actions without a related outcome that could help them move forward. Only a few of the participants’ proving processes were characterized as this type.

*Related Outcome Produced but Not Link to the Argument.* The majority of the participants’ proving processes were characterized as this type. For this process type, a student starts with an initial argument and then gets stuck when trying to prove it. The student then takes several actions or attempts to try to navigate out of the stuck point, as in Type 1. However, the difference in this type is that the student does generate related outcomes (new ideas or connections). In this case, they soon run into another stuck point when trying to expand the related outcome and link back to the argument. This process may also include some navigating
actions to try to overcome the second stuck point they encountered, but in this case they do not succeed in navigating out of their stuck point.

*At Least One Related Outcome Linked With Argument.* Similar to the previous two types, for this type of process the student also starts with an initial argument and then gets stuck on the way to a proof. As in the first two cases, the student takes several actions to navigate out of a stuck point. However, this process advances to a successful argument because the student is able to leverage related outcomes and results. The cycle continues when a second or third stuck point is encountered. To summarize, this type of process results in at least one related outcome when trying to overcome stuck points and the student is considered to have made productive progress.

The finding of the three proving process types shows that students’ proving processes are not linear; there could be multiple stuck points, thus resulting in multiple sub-processes. The student is considered to have engaged in productive struggle if *at least one* productive cycle is complete.

Given the differences among the three types, it is worth comparing those three types of processes in more detail. To do that, I will describe and discuss two illustrative cases in detail in the next chapter, including the proving processes I have discussed for all of these three types. The aim of the discussion of the two cases is to explore what supports we can provide for students who are in the Type 1 or Type 2 proving process to help them move to Type 3 (and thus help them to engage in more productive struggle when they encounter impasses).
CHAPTER 8

CHARACTERIZING STUDENTS’ NAVIGATING ACTIONS

The overarching goal of my study is to characterize undergraduate students’ stuck points and navigating actions during their proving processes. The previous two chapters addressed my first research question concerning the nature of the stuck points students in my study encountered in their proving processes and my second research question examining students’ overall proving processes. In this chapter, I will explore my third research question: “When students encounter stuck points in proving processes and are actively involved in navigating out of a stuck point, which actions appear to contribute to the success or failure of their attempts?” To answer this research question, I will present a detailed analysis of two contrasting cases. The purpose of discussing these two cases in depth is to provide a clearer and deeper view into each of the three general types of the proving process I described, as well as characterizing actions the students in the two cases took to try to overcome their stuck points. The analysis lays the foundation for discussing the relative productivity of different possible actions students may take as they are engaging in proving.

The Choice of Porter and Soni

The goal of my last research question was to discuss some specific actions students take to try to navigate out of their stuck points and how those actions play a role in their overall proving processes. In the analytical framework described in Chapter 5, I generated four major categories of actions to focus attention on: identifying stuck points, attempting to resolve stuck points, monitoring proving processes, and persistence. Based on my analysis, there are more
instances observed in the first three categories (91, 99, 107, respectively) compared with persistence (24). I discussed the different kinds of stuck points students identified in relation to my first research question. Here, I will summarize and expand upon the other three types of actions.

Within the 99 instances of different attempts to resolve stuck points, I observed specific attempts or navigating actions, such as trying examples, trying different representations, using or switching strategies, and using different resources. Those types of actions appeared in all three types of the proving processes. Given the similar navigating actions, why are some students successfully able to navigate out of their stuck points while others are not? Thus, there is a need to look deeper into specific cases to understand the differences.

Soni and Porter were chosen in particular for several reasons. First, they had a similar proof background. Both Soni and Porter were in the same ITP class, and for both of them, it was their first time taking a proof-based course. Secondly, both Porter and Soni made adequate progress in proving or disproving both tasks, so there is rich evidence to analyze. From my analysis in Chapter 6, Soni, Eric, and Porter had the most instances with their actions in responding to stuck points. However, from my findings in Chapter 7, Eric’s process was very similar to Soni’s (productive progress for both tasks and he had a similar number of instances for each category of navigating actions). Thirdly, there are interesting patterns for both Porter’s and Soni’s proving processes, which will be discussed later in the chapter. Lastly, there is an important dimension of contrast between the two cases: Soni successfully overcame his stuck points multiple times, while Porter still remained stuck at the end of the session. That said, even though Porter might not have been able to overcome all of his stuck points in the end, he was pretty close and may just have needed some support to be successful. Interestingly, we can see
both Type 2 and Type 3 actions involved in Porter’s proving processes for Task 1 and Task 2. Thus, it is worth spending the time to have a closer look at their proving processes for each task and compare the navigating actions around stuck points of the two.

**Porter’s Proving Process and Navigating Actions**

In this section, I will describe Porter’s proving process for the two interview tasks. For each task, Porter’s process involved several shifts from (1) his initial argument, (2) major stuck points related to his different arguments, (3) navigating actions he tried, and (4) related outcomes produced if the stuck point was overcome. Given the multiple shifts in his proving processes, I created proving process maps (Figures 17 and 20) to show a top-level view of the different shifts. This visualization also helped to keep track of how stuck points relate to the proving processes as a whole. I then will analyze each of his major stuck points and the actions he took to try to navigate out of his stuck point.

Porter started quickly in deciding the strategy to use for this problem would be contrapositive. He indicated it is easier to check the conclusion by using contrapositive.

P: So, if I use contrapositive, I know that it would be, instead of at least one of a and b is even though, it would be both a and b are odd. And that way, I only have like, one thing to check like I can put that and…… Then it makes it easier to like check the conclusion. Because if it was the other way around, then we have two conclusions to check or like two different things.
**Task 1**

Suppose $a, b$ and $c$ are integers such that $a^2 + b^2 = c^2$; is it true that at least one of $a$ or $b$ is even?

---

**Figure 17. Porter’s Proving Processes Map for Task 1**

**Task 1 Stuck Point 1**

"I am not sure what I have to prove right now."

However, right after deciding he was going to use contrapositive, Porter ran into his first stuck point. He indicated that he was not really clear about what he needed to show right now, and he felt he was missing the conclusion.

P: So I’m thinking that I’m not sure what I have to like, prove right now. Because like I want I want to say that. If one is one of your P’s, even if it’s contrapositive and I feel like I’m missing the conclusion. So the $a$ squared plus $b$ squared is not equal to $c$ squared... (silence)
P: Yeah, I’m stuck. I know that to use contrapositive I need to say that \( a \) squared plus \( b \) squared is not equals to \( c \) squared. But I don’t have like, information on \( c^2 \). Like, I don’t know if it’s, oh, no, odd. If it’s odd, then the square is odd. Right? And the adding them to odd numbers. That’s not normal. Now, not necessarily.

Porter tried to overcome this impasse by writing down the negated statement based on the definition of contrapositive; then he supposed that \( a \) and \( b \) were both odd and set \( a = 2k + 1 \) and \( b = 2l + 1 \),

\[
c^2 = a^2 + b^2 = (2k + 1)^2 + (2l + 1)^2 = 4k^2 + 4k + 1 + 4l^2 + 4l + 1
\]

\[
= 2(2k^2 + 2k + 2l + 2l^2) = \text{even}
\]

He concluded that \( c^2 \) is even, which meant \( a^2 + b^2 \) is even.

P: So, I am thinking about if I have to use odd and then I squares on here, yeah, I mean…So that mean that \( A \) squared could be just this part it can disappear I got to…say so odd…No, just give me the same thing…… (silence)

P: I’m still…like what am I? I don’t know what to think right now but like this part so you need to show…(silence) \( a \) squared plus \( b \) squared now equal to \( c \) squared right? (talking to himself) Yeah. Can’t be squared……This is an even number……

Once Porter successfully showed \( a^2 + b^2 \) is even, he had a hard time moving forward (see Figure 18).

---

**Figure 18.** Porter’s Work for Showing \( a^2 + b^2 \) Is Even
Task 1 Stuck Point 2

“So how do you go back to prove it's not equal to $c^2$ to use contrapositive?”

Porter indicated that he needed more information about $c$ to move forward and make this contradiction.

P: So……So I know that $a$ squared plus $b$ squared right……I just don’t have information about $c$. $c^2$ is supposed to be even that many of you know it’s just an integer……then we know that the contrapositive says square has to be odd……can’t repeat that again that because I got this answer that $a$ squared plus $b$ squared is equal to event to an even integer then that means that using contrapositive we know that $c^2$ has to be odd to make this contradiction make the conversion possible.

P: But I just like looking at this with this, one identifier chosen, then $c^2$ has to be odd when both $a$ and $b$ are odd, so if I put it back to normal, at least one of them is even, not odd. If at least one of them is even then we know that $c^2$ would be right here. If I was one of them is even though we know that $c^2$ beat by both of them……(silence)

Since the assumption that $a$ and $b$ are both odd didn’t help him to overcome how to prove it’s not equal to $c^2$, Porter then considered two other cases: $a$ is even and $b$ is odd, and both $a$ and $b$ are even.

P: But if both of them are even, then we get this question even integer, because if it’s at least, then one case could be that both $a$ and $b$ are even. But if both $a$ and $b$ are even that would be……[inaudible] Okay. So and even an odd integer when you add them, that gives you an odd integer and that would make this equation true. And also if the two of them are even that would have been $c^2$ also has to be even. So that wouldn’t mean the equation holds in both cases. But I already wrote all this. That was my……How can I write…….(silence) Okay, yeah……(silence)……But I’m assuming that they’re equal.

However, the two cases didn’t seem to help him showing the contrapositive. “That’s my question. So how do you go back to proving it’s not equal to $c^2$?” (see Figure 19).
Figure 19. Porter’s Attempts on Two Other Cases

Since the above general cases didn’t lead him far and didn’t seem to make much sense to him, Porter then switched to try specific examples and see whether that could help him. He tried to add two odd numbers and he confirmed the sum is an even number.

P: This is not equal to $c^2$. Can I do that? $c^2$ equals the square root...square. If I add two numbers, can I get an even number? (talking to himself) I mean yes, 9 plus 3 can get you on your number ha okay...

Thus far, we see that Porter was stuck on how to use contrapositive to relate back to $c$. He had tried both other general cases and specific examples to help him overcome. However, both approaches didn’t lead him far. When trying the specific examples, he was actually going back what he had -already proved in the general cases. Porter started to go back and forth in circles.

Task 1 Stuck Point 3

“Did I choose the strategy of contrapositive wrong?”
After all the previous attempts didn’t help him to use the results to relate back to the contrapositive statement, Porter started to question his initial choice of contrapositive. “Did I choose the contrapositive wrong? Still don’t know how to relate the result to the statement.” He worked through everything that he had done in the beginning. He then tried to negate the statement again.

Y: So why do you go back and reconsider your contrapositive?

P: Because I’m stuck so when I get stuck is working through everything I’ve done from the beginning. Even like the method I thought about it. So maybe putting the negative question that I’m getting wrong. They said most of $a, b$ desired. That’s my accent why I should know how to put this a negative…….But at the end that is going to walk us……I know that the two even numbers, even number and have an even and an odd number…….(silence)

He also checked a few more cases and checked his logic. He claimed again he was missing the contrapositive to show $c$ is not equal.

P: Yeah, yeah, I just take cases in my head…….so even on an even…….so do I want if it’s at most one of them is odd that would mean they have a case one wants odd ones even when the two of them are even…….okay let me write…….Even and odd number…….Actually, I should get just what I got. So, it could be some number…….

Y: So, so what did you say?

P: So I said that in this case because the even embodies odd then if I add them that I should get an odd number it might be logic is correct. So I did get that number, but I was trying to use the logic that this was even plus odd, then $c$ has to be odd. And I got a $c$ is odd, because I’m missing contrapositive they shouldn’t be that $c$ is not equal.

After trying several examples and different cases still didn’t help him, Porter started to question whether contrapositive may not work for this problem since he couldn’t make the connections between the $c^2$ and $a^2 + b^2$. He continued to make subsequent attempts to work through and check what he had done, but was still stuck on relating the results back to $c$ itself.

Y: So what is preventing you in this part?
P: Back to the relationship to $c^2$ and making connections in between. Because I mean, I know it’s like could be by theorem. A lot of things I just don’t know how to relate them.

To summarize Porter’s proving process on Task 1, he started immediately with a specific strategy of contrapositive, but he soon ran into Stuck Point 1 on how to show it by contrapositive. He then tried to navigate out by assuming both $a$ and $b$ are odd and based on that $a^2 + b^2$ is even. However, he then had difficulties relating this outcome back to $c^2$ by using contrapositive. After trying the other two cases, which still didn’t help him to move forward, Porter started to question his strategy of contrapositive. He double checked everything and started to go back and forth in circles. I will discuss several important points about Porter’s proving processes and navigating actions on Task 1 in the discussion section, after discussing his proving processes and navigating actions on Task 2 (see Figure 20).

Porter’s initial reaction after reading Task 2 was, “I have never had something like that…….” Since he didn’t know where to start, Porter decided to disprove it in the beginning because he thought attempting to disprove the statement would help him to think until he could prove it.

Y: So why did you decide to disprove it?

S: Okay, cuz maybe trying to disprove it will help me think until I can prove it. Because right now when I say like an integer, it’s divisible by nine if the sum of its digits, like I don’t know, what would be the connection between the digits. Just like if I see A and B, I’m gonna think they’re multiplying but they’re not by themselves. Right now. Um, they’ll be confused……actually. So the disprove part will probably help me to think yeah……

But this thought didn’t remain for too long. After he tried the example of 18 and realized they learned a similar rule in middle school, Porter then decided to switch his strategy and prove it instead.

P: So, so. So for example, we have 18 by 9 and 1 + 8, 9 by 9 (writing on the paper)…….Actually, I remember in middle school, they told us that if you add the digits,
it’s, it’s a multiple of three, then you can divide it by three. If it was the same with nine……(silence) Yeah, there’s no way to disprove it. So it’s like a pattern that I’m thinking in my head, that you’re always gonna get 9 or 18 or 27. When you like, going onwards when you add them. So I have to prove it.

But Porter soon ran into a stuck point once he decided that he was going to prove it instead. He told me that “this one’s harder than the other one.”

P: When I go to like 99, it’s the same thing. So I don’t I don’t think there’s a way to disprove it. So you go back to four, seven. Now, I think technically I should use the divisibility I just don’t know how……(silence)

---

**Task 2**

Prove or disprove: An integer is divisible by 9 if the sum of its digits is divisible by 9.

---

**Figure 20. Porter’s Proving Processes Map for Task 2**
Task 2 Stuck Point 1

“How to link that it is divisible by 9 with the digits?”

To help himself overcome his stuck point, Porter thought about the divisibility theorem they learned in class. “The divisibility rule should play a role here in the question since I saw similar questions before.” But he realized he had difficulties representing the sum of its digits. So he decided to use the example of 18 again to make sense of the divisible by 9 part. He thought about 1 multiplied by 8 but that was not related to 9.

P: So I do thinking about the divisibility. I mean, cuz I know I need to, like prove that this is divisible by nine. And these other things also divisible by nine. I think I’m just having problems with when I see like the sum of its digits. Like I don’t know how to connect them to the digits with like the, like, for example, if I think 18 I know I tend to avoid 9 and I don’t want to say it’s divisible by 9 is another connection between 1 + 8 and 2.

(See Figure 21.)

Figure 21. Porter’s Initial Approach and His Example of 18

Task 2 Stuck Point 2

“I am having difficulties of representing the relationship between 18 and 1+8”

Porter indicated that he was having difficulties representing the relationship between 18 and 1 + 8. Then he tried to represent each digit by separating 8 as 2 + 6. But that still didn’t lead him anywhere.

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P: I am having difficulties representing 18, and the sum of 1 and 8. Mm hmm. What I was thinking first A, B, but that’s the thing that’s like the multiplication. And I'm not, I’m not saying that 18 is the 1 multiply by 8 together. That’s an integer……So I could represent them it two different ways, 6 + 2. Well, there’s different things……(silence)

P: So I’m thinking about maybe there are some ways you can do that will show 1 and 8…Same as comes in like that……I have a plus……I’m seeing some type of pattern, I think (silence).

After a few minutes of silence, Porter suddenly decided he was going to switch strategies to use mathematical induction to prove since the generalized form of the sum of the digits looked like an induction type of problem he had done: “Can I use Induction? This type of problem looks like an induction problem.” He assumed the base case was 18 and proved his base case. But Porter soon got stuck again with the inductive steps.

P: First one would be more than two digits. The first one that would work would be 18 [Porter’s base case was wrong]. I think I have an equation to like to put k + 1……I don’t need to create like some kind of equation right. I’ll do…Yeah…So get them to see……no this is growing like plus one plus one minus seeing this……(silence)

Y: So what are you doing right now?

P: I was just trying to remind myself the divisibility. I’d like to change forms. To see if I can see like a connection between a and b.

Y: Do you find any connections here?

P: I mean, I know that if I have that 18 equals to 1 + 8 that I know this is divisible by 2. I know that a next like if I tried to find I can mix I know that the x can be the digital 18. I just don’t know how to continue for the inductive cases.

(See Figure 22.)

<table>
<thead>
<tr>
<th>9b = a1·1 + a0·10 + a3·100+…</th>
</tr>
</thead>
<tbody>
<tr>
<td>If the integer is 18, 911+84, and 9156, 10+6</td>
</tr>
</tbody>
</table>

Figure 22. Porter’s Sketch of Using Induction for the Base Case [Porter’s base case was wrong.]
Task 2 Stuck Point 3

“How to prove the inductive case for the $k+1$ term?”

Porter claimed that this inductive case was hard to show since it’s hard to show the relationship between the sum of the digits and the digits itself. He decided to try a few examples to see whether two digits or three digits would make sense for him. “I didn’t know what to do so just trying random ideas.” He tried examples to remind himself how the divisibility worked out. He tried the number 18 and saw 18 is divisible by 2 but still didn’t go far.

P: Yeah, let’s try. So for instance, if I have any of the integers like a team, right? I can represent a term using its first that digits, which is the ones digits. And the 10s digits, know what? The ones digits and the 10s digits. Okay. So then this one will be represented as my 10 digits, it’s 10. And my one stitches, it’s eight which is 18.

He then tried the different representation of digits, then represented 27 as $20 + 7$. But this still didn’t help him much.

P: Mm hmm. So if I have 27, and I want to represent it as the 10s and the ones so I have $20 + 7$. So this is my ones digits, and every of my ones digit is times one. And every of my 10 steps can be represented as that number times 10.

P: Yeah, but I still don’t know how I could make like, I don’t know, maybe an equation. Like a general equation, but this time around just I see the pattern, you see the pattern, but I don’t know how to write the pattern. [silence and started look for general equations].

He decided to look for some general formula to represent the sum of the digits in general. He found the formula in the formula sheet that any integer can be represented as $a_k(10)^k + a_{k-1}(10)^{k-1} + \cdots + a_0(10)^0$. He was trying to make sense of the formula by trying out examples like 333 and 1121. But instead of helping him, Porter felt this formula frustrated him.

P: So if I have a 333 for instance, then this first one can be represented as three times 100. And this one is three times 10. And this one is three times one. If I add them together, they will become three. Okay, so that’s representing the sum of any integer. Turning to check that if the number is divisible by nine……Then the sum of the digits on the number is also the result……(silence) Yeah.
P: Because I don’t see like, if I put like a 1 plus 1 a 2 plus 1, like if I put plus one then it’s not gonna…..[He made a mistake in calculation] Sure……that that other numbers also like divisible by nine. And it’d be fine for me to prove anything. I cannot use as an assumption, a number that’s not divisible by nine. It’s the sum of digits is frustrating…..(silence)

Porter then started to question his strategies again. He said he was having difficulties with representing the multiple digits using the induction, so maybe induction was not the right strategy to use. Porter summarized his overall switches of strategies for this task in the following:

P: I wanted to prove it in the beginning, but I don’t know how. And I thought it was gonna be direct proof, but not work. I got stuck. So I tried to change into math induction, but then I didn’t know how to use this formula for example. Because I mean, with math induction, I could do have to take the smallest number. So it would be in this case 18. And I always told that case works like a math induction with like, the formula. Like with a plus one. Like one plus timing we have like time. If I had a plus one, that would just be like 11 + 9. I don’t know that. I don’t know how to use it here.

Porter continued stuck at the end. When asked to identify his major stuck points, Porter said “Understanding what to do. I just have never had a problem where it’s like the digits, the digits itself, they have like a meaning.”

To summarize Porter’s proving process on Task 2, similar to Task 1, Porter started by choosing a specific strategy to use, which is disprove. But after he tried several examples and found it all worked out, Porter then switched his strategy to direct proof. He tried a few examples with different representations, but it didn’t help. He suddenly decided to switch his strategy again to induction since they did a similar problem in class. Based on the definition of induction, Porter assumed the base case to be 18 (18 is not actually the base case; the base case is 9) and showed it works. However, he then had difficulty proving the inductive steps as general cases. Porter then looked at the formula sheet and found a generalized form representing any integer as the sum of its digits. He used a few examples like 333 to make sense of the formula but didn’t go far. He concluded that the formula didn’t help and induction might not work here.
I will discuss several important points about Porter’s proving processes and navigating actions on Task 2 in the discussion section.

**Soni’s Proving Process and Navigating Actions**

In this section, I will describe Soni’s proving process for the two interview tasks. Similar to Porter, for each task, Soni’s process involved several shifts from (1) his initial argument, (2) major stuck points related to his different arguments, (3) actions he took to try to navigate out of his stuck point, and (4) related outcomes if the stuck point is overcome. Given the multiple shifts in his proving processes, I used a proving process map (Figures 23 and 26) to show a top-level view of the different shifts. This visualization also helped to keep track of how stuck points relate to the proving processes as a whole. I will then analyze in detail each of his major stuck points and the actions he took to try to navigate out of his stuck point.

Soni started solving Task 1 by trying the example of a Pythagorean triples 3, 4, 5 and it worked. He immediately thought about contradiction and assumed both $a$ and $b$ are odd.

S: Oh, so here’s the thing if it’s even……We’ll assume P is the statement because this time is true, right? So P is [inaudible] So this stuff is true, but we’re proving Q, there’s one $a$ or $b$ has to be, have to be even. So there’s there is a thing called contradiction. Proof by contradiction means like if I’m proving something is not so……(silence) So the $c$ could be $\sqrt{a^2 + b^2}$. So we’re assume so I’m going to prove by contradiction……so one of the so what is not Q now Q is going to be the both are not even. They’re odd.

To check the validity of the statement and confirm his thinking, Soni tried a generalized form of $k$ and $2k$ to represent even and odd numbers. He made some calculations in his head and concluded the result would be an even number if assuming both $a$ and $b$ are odd.

S: So we’ll just take so……I have to reorganize this stuff. $k$ is odd number and $2k$ is even……(silence) so, I mean still there….so both are odd…The square of odd number it’s gonna be odd. So, so to our number plus together equals an even number is maybe two, right? (talking to himself) Yes, they’re both just the square. So, so we assume $a$ and $b$ are both odd and……So to our number plus together equals even number so the even numbers……(silence)
But he then got stuck about how the result of adding two odd numbers could even help to prove the statement.

**Task 1**

Suppose $a, b$ and $c$ are integers such that $a^2 + b^2 = c^2$; is it true that at least one of $a$ or $b$ is even?

![Soni’s Proving Processes Map for Task 1](image)

*Figure 23. Soni’s Proving Processes Map for Task 1*

**Task 1 Stuck Point 1**

“How can this help to prove the statement?”

After he concluded that the sum of $a^2$ and $b^2$ is even, Soni questioned how this result would help to prove the statement using contradiction. He was monitoring his thinking while trying to navigate out of his stuck point.
S: Is there any statement saying like if one even number subtract an even number equals an even number? (talking to himself) No, it can be just……wait I need to I need to prove this……so even number……Oh…….(silence) They are relative like the greatest common divider their relative prime. So, does that mean that there’s no, there’s no way you can separate to another square? (talking to himself) Did I just make myself confused? (talking to himself) (silence)

After several rounds of monitoring, he then questioned his strategy of proving by contradiction (see Figure 24).

S: By the way, like, I’m just trying to find a way like no wrong way to prove this stuff. I mean, prove by contradiction. Like I can’t, I can’t assume that not P not Q. Because the Q is what we find to prove. I cannot like assume because if I’m assumed the non-Q is true. That means like, like, assume that stuff is true. Because I can assume this stuff is not. We’re trying to prove this. So I have to assume this, like $a^2 + b^2$ now equal to $c^2$ when Q is not even, it’s odd. But I’m stuck here like, I tried to use some numbers to instead of $a$, $b$, and $c$, and trying to find they’re not equivalent when it’s odd. But the thing is, if I’m using $k^2$, just $a^2 + b^2$, it’s our two. Now it’s an even number. So an even number. An even number. This one it’s an even number. So if it’s even number I guess square root of a given number, square root of a number you can see…(silence)

Figure 24. Soni’s Initial Approach and Scratch of the Contradiction
Once he double checked his strategy of contradiction and realized it might still work, he started to generate some examples of the sum of the two odds squared like $1^2 + 3^2 = 10$, to see whether a specific example could give him some ideas. Soni then realized 10 can’t be a square root of an integer. But he indicated that it’s hard for him to use this specific example to show the generalized cases.

S: I just know like, like the $c$ like $a$, $a^2 + b^2$ they both even. So the number comes out the sum of $a^2 + b^2$ when those even the number cannot be like, like can that cannot be square root because it’s not integer it’s a real number. It’s really just because you never, it’s never going to like, see, I have the example one, like 1 + 9 equals 10. And they’re never going to be square root like you can separate by entity, it becomes an integer and you cannot separate like by the square root. So how am I going to show? (talking to himself) (silence)

Soni was thinking and silent for a few minutes, and there was a sudden “a-ha” moment:

“Oh, maybe I can use this. So $c$ is equal to square root of something.” He then wrote on the paper: since $odd^2 = odd$ and $odd + odd = even$, then $a^2 + b^2$ is even. So $c = \sqrt{a^2 + b^2} = \sqrt{even}$ which lead him to a related outcome after assuming A and B are both odd. “But is $\sqrt{even}$ also an even number?” He continued to question.

Task 1 Stuck Point 2

“Is $\sqrt{even}$ also an even number?”

After he produced his first related outcome that showed $c^2$ is even, Soni got stuck again and wondered about whether the $\sqrt{even}$ is also an even number. He was thinking and silent again. But I could see he was monitoring throughout his thinking process.

S: But the thing is I’m stuck here like how am I going to show I’m going to show this number cannot be square root two. Is there any like, I guess there’s some way to show number cannot be square. Like, it’s not so, okay……(silence) Oh, I got it. I got it. So……no, I’m not I’m not wait, wait……it could be……(silence)

S: I just can’t show that like, so key is even now I can just like $k$ plus $k$ equals two $k$ and square root of two $k$. Like square root two. No I can’t I can’t I can’t show that because
square root two is irrational number (talking to himself). If I can show somehow if I can show……all right.

During this long time of his thinking process, Soni encouraged himself multiple times in solving the problem and did not give up when some of his attempts didn’t work out. He also tried different representations of $a$ and $b$, but it seemed that also didn’t help him much.

S: So we’re assuming $a$ equals to $k$ and $b$ is equals to, we’re assume, assume, assume $a$ is equal to $k$, $b$ is equal to $k$ plus two times a number and, the $n$ can be any integer and belongs to integer. So, two times a number plus an odd it’s going to be odd. It’s always going to be odd. So, now we can solve like, plus $k$ plus two $n$ square equals to something. So……(silence) Yeah, so see, it’s gonna be square root of this stuff for square root of this stuff is $k$ plus $k$ plus two $n$. Wait, no, ah, ah, wait, I’m almost there! So $k$ squared….(silence)

S: So, so you will so this part because, because it’s even something……Yeah, so screwed over even now while I’m trying to see it…….I’m almost there.

After all the previous attempts didn’t work out, Soni started to think in the other direction of $c$,

“Do we know if $c$ is even?” He talked about how it won’t work in this case since $\sqrt{2}$ is not an integer.

S: So, square root of two it’s a rational number, it’s not an integer. So that’s not means like, like two or number plus together in the square root of a tool like the $\sqrt{2(a^2 + b^2)}$. It’s going to be always going to be there’s like additional two inside of it. Oh, I had to clarify. So $4k$ is our number. So there is like a never going to be to……Okay, it’s our number. So I’m showing here, B it’s going to be, because they’re not saying it could be the same or not the same. So it depends on $n$, $n$ could be any number two times any number equals an even number, even number plus an odd number equals an odd number.

So, so that’s how I know making sure $a$ and $b$ they both are odd number. And when our number plus together, I’m trying to do the algebra here and, and I find out there’s always going to be a $\sqrt{2}$ includes in the whole result. So the $\sqrt{2}$ is a irrational number. Because $\sqrt{2}$ is an irrational number, it means it means……See, never like, see, it’s not it’s never going to be an integer.

Following this way of thinking, Soni then tried the examples of 3 and 5 and wrote $3^2 = 9, 5^2 = 25, 9 + 25 = 34$, then realized $\sqrt{34}$ is also not an integer. Then, based on this he concluded it is not always true, since $c$ can also be a non-integer, which is a contradiction (see Figure 25).
When I asked him whether he thought he had proved the statement, he said,

Not now, but I can go from here. By contradiction in these two are number plus together. Like two or number plus together, it’s never gonna be like, have a result was this like, see it? No, never gonna be an integer. So that means like, some of them or both has to be even number…… To prove meant above when a square plus b squared equals even number c never, never to be an integer. That’s a contradiction. Right?

He was still thinking about the problem while taking to me, and I could see that he was not really satisfied with his argument and questioned whether he had proved the statement even in the end when the interview had finished.

To summarize Soni’s proving process on Task 1, he didn’t decide his strategy immediately; instead, he tried several examples and mapped out some ideas both in his head and also on the paper. Once he had some ideas to start with, he then indicated he was going to use contradiction to prove it. But he also had difficulty in using contradiction specifically to prove this statement. He also tried to navigate out by assuming both $a$ and $b$ are odds and based on that
Once he realized that $a^2 + b^2$ is even, he moved a bit forward by concluding $c^2$ is even. Soni then stuck with showing whether $\sqrt{\text{even}}$ is also even. He tried different representations of $a$ and $b$ but that still didn’t help. After a thinking and monitoring process, Soni had an “a-ha” moment of thinking in the other direction of $c$, since $c$ can be even or odd. He then proved $\sqrt{\text{even}}$ is not always an integer, which means $c$ is not always an integer, a contradiction. I will discuss several important points about Soni’s proving processes and navigating actions on Task 1 in the discussion section, after discussing his proving processes and navigating actions on Task 2 (Figure 26).

**Task 2**

Prove or disprove: An integer is divisible by 9 if the sum of its digits is divisible by 9.

[Diagram: Soni’s Proving Processes Map for Task 2]

*Figure 26. Soni’s Proving Processes Map for Task 2*
After reading the problem, Soni expressed his interest in solving the problem, “Really? That’s that’s……That’s fun. Let me think about this……” He immediately thought about 10 could be represented as 9 + 1. To test the validity of this statement and decide whether he was going to prove or disprove, Soni tried several multiple of 9s (e.g., 81, 108) and realized they all satisfied the condition. He conjectured that all numbers divisible by 9 should satisfy this condition.

S: I never think about that stuff. Huh? Why? Oh, Oh, cuz 10 — 9, like 1. Yeah, yeah. Yeah. [Talking to himself] Guess I have a thought like that all the stuff is like 10 minus maybe there’s a number, the integer, we’ll call it n, the sum of its digit divided by nine. So I score plus 990 plus 999 plus nine. Wow, eight plus nine. Whoa. Oh, it’s not it’s digit like, like it’s not divided by nine. It’s gonna be always the nine right? Even our 2000 there’s a number there’s a digit should be zero. So what if I have 945? Is it? Yeah, it does. Oh, okay (silence)……

But he realized it might be hard to prove:

I mean, why? I can see it’s true, but why is it true. So how do you prove this kind of a question? It’s a new like, never saw this, uh, never I never think about this but it’s true. Never think about this. I know it’s true, but how can I use mathematical stuff prove this stuff is true.

He wondered whether proof by contradiction would still work here.

S: Oh, the sum of digit……So that’s if the sum…… so we’re assuming this one is true. So then we’re proving integers divided by nine. I can I can prove backwards right? What is called can I prove backwards? (talking to himself) What is that call? contradiction? contradiction? Yeah……there are different so I can prove contrapositive I can prove this one and some integer divided by nine.

But he soon ran into another problem once he decided to use proof by contrapositive: “What’s the relation between the digit and between the digit and nine? (silence).” He found it hard to link or make connections between the digits and digits sum. “I don’t know this this stuff. Like I know it’s true. I never think about this.”

Task 2 Stuck Point 1

“How to represent all numbers in general?”
Soni started by trying examples to see whether it could lead him in any direction. He clearly stated the goals of his attempts: “So I want to know the relation between……the relationship between this digit and……I’m trying to find a way to represent a digit that having a relationship with 9.” To achieve this goal, he tried the example of 90, tried multiple representations, and then compared it to 100, then wrote it as $90 = 100 - 10$.

S: Oh, I think like, like I want to show. I want to know, like, the 10s the digit. Yeah, the digit. So I want to know the digital stuff. And maybe, maybe $110 - 11$. It’s gonna be the same. So I want to know the relation between the relationship between this digit and….wait what wait…….During the same, always gonna be the same. Yeah the same is digit before the digit like if it’s two digit is one always going to be the same as this two digit…..I’m going to show this……(silence) Am I on the right track? (talking to himself)…yes…I am…I want to find the differences between counting by 9s and counting by 10s.

Soni stated that he was trying to find a way to represent the digits and the sum. After he confirmed he was on the right track, Soni started to look at the formula sheet and he found the general formula to represent multiple digits after a few minutes. However, how to use the general formula to prove the statement became his new problem.

S: [read the formula] Recall the numerical value of any integer $n$ can be present as $n$ equals what is the sake oh…I have a question. So, how does this stuff help you to find $n$? That’s just going to help you to represent the digit now for the $99 = 100 - 1$ right? [talking to himself] plug in $k$ times plus one, plus a $k$ minus one times 9 plus a 0. So now we can separate out…Can we? I think we can. $k$ times 9, $k$ plus 1…2…Okay. Oh wait, what am I doing? What am I doing? (talking to himself) I can’t do this. No, no, no, that’s not a plus two times two times nine…….Oh wait, wait. I think this represent digit. So digit, they pass together……How can I separate this stuff?

Soni then encountered a long period of thinking and silence. He had a hard time deciding how to use the digit representation. Similar to Porter, Soni also thought about using induction to prove. But he soon realized that it wouldn’t really help him.

S: Oh, oh no…I have the thought. So strong induction. Base case. Can you go to one? That’s my divider now. This one when $a$ $k$ equals 1, what is $a$? oh one $k$ equals 1. That’s 10 half of standard stuff. What is the it’s the last stages. So if this doesn’t make sense, maybe we can start from that and instead let’s start with the biggest one. What is the 0? A
0... I... So that’s the ones place... I can’t link them together. They’re not there......(silence) Also this part I’m so confused, how to prove the stuff using mathematical language (silence)?

Y: Um, so how did you decide to use induction?

S: I was thinking about like I can prove using this but that doesn’t work. That doesn’t work well because I don’t know because this one I don’t know......so I can’t use the induction.

Once induction didn’t work out, Soni tried to go back to his initial thought about the difference of representing 10s and 9s. Since 9 can be written as 10 – 1, 10 can be rewritten as 9 + 1; then he substituted 10 = 9 + 1 into the general formula and he got any integer can be represented as $a_k(9 + 1)^k + a_{k-1}(9 + 1)^{k-1} + \cdots + a_0(9 + 1)^0$ (see Figure 27).

![Figure 27. Soni’s Work of Exploring the Difference Between Counting by Nines and Counting by Tens](image)

Y: How did you decide to go back to your thought about representing 10 as 9 + 1?

S: I’m thinking, cuz I was thinking about this. They’re always gonna be some, like, they’re always gonna be 10 times something for this, this part, and always gonna be 10 times even, it’s not even just only 9, it’s 10 times 10 to the one. Right? So 10 to the zero, it’s gonna be one. So just having that thought, like, I know this type of question, I have to
just by experience is really just the experience, because I did some question like, like, kind of similar. So I know like, in this kind of a type of question, I can do some like aid. If it’s aid. I just use 10 minutes, just, I don’t know. I just go through a whole my math education experience of the whole my math education I have this kind of like experience like magic thought.

Right after he came up with this “magic thought,” he got stuck again with the $k^{th}$ power for $(9 + 1)^k$. “I have this or something like that. The $9 + 1$, how am I going to separate this stuff with the power of $k$? (silence).”

**Task 2 Stuck Point 2**

“How to deal with $(9 + 1)^k$? “

Soni got stuck again on how to expand the expression of $(9 + 1)^k$ after substituting $10 = 9 + 1$ in the equation. He indicated the difficulty he was having was using mathematical language to link back to divisibility.

S: Oh, wait, think about this, this whole stuff divided by nine. So it means like a sum of $k$ divided by 2, like 9 divides this stuff too. That’s the expression of like that’s the sum of the digit. That’s the sum of the digit. Because from here, I see this one is the digit. So that’s the digit….

Y: So what you’re trying to do?

S: I’m trying to prove the digit. So if this 1 divided by 9 and just prove this one divided by nine too…..Am I gonna prove this whole thing like not a sum?

He went into his silence and thinking mode again. Then he suddenly had an “a-ha” moment and thought he could try the example, follow this representation first and see how it works, then try to use the same way of showing this example to show the general $k^{th}$ digits.

S: And basically it’s sum like 945…..So I can see the connection! I just have some thought…..(writing on the paper)

$$[945 = 9(10)^2 + 4(10)^1 + 5(10)^0 = 9(9 + 1)^2 + 4(9 + 1)^1 + 5(9 + 1)^0$$

$$= 9(9^2 + 2 * 9 + 1^2) + (4 * 9 + 4 * 1) + (5 * 1)$$

$$= 9^3 + 2 * 9^2 + 4 * 9 + (9 + 4 + 5)]$$
It works for 945 by factor out the 9. Here are the digits 9, 4, 5 (pointing at the last bracket), I found the relationship! Now I can follow the similar ways to show the general forms…….Follow this way of proving 945, the general formula works by changing the specific 9, 4, 5 to the digits $a_1, a_2, \ldots, a_n$.

Y: Excellent!

Following this way of proving 945, Soni showed the general formula works based on this example (see Figure 28).

Figure 28. Soni’s Work of Representing 945 Using Counting by 9s

To summarize Soni’s proving process on Task 2, similar to Task 1, Soni started with outlining some of his ideas of finding the differences between counting by 9s and counting by 10s and rewriting 10 as $9 + 1$. He had this general goal in mind; he felt that he needed a generalized formula to represent any integer with the sum of its digits. Soni looked at the formula sheet, and like Porter, he found the formula that any integer can be represented as $a_k(10)^k + a_{k-1}(10)^{k-1} + \cdots + a_0(10)^0$. But how to use the general formula to prove the statement became his problem. Like Porter, Soni also thought about induction to help him achieve his goal. But he soon realized that it didn’t work here. Soni then went back to his initial thinking about representing 10 as $9 + 1$ and substituted $9 + 1$ for all the 10s in the formula. However, he got stuck again on how to expand $a_k(9 + 1)^k + a_{k-1}(9 + 1)^{k-1} + \cdots + \sum_{i=0}^{\infty} (a_i(10)^i)$.
Sonu had an “a-ha” moment about trying a specific example first then following the same way to show the generalized cases. Following this way of thinking, Sonu successfully navigated out of his stuck point and proved the statement in the end. I will discuss several important points about Sonu’s proving processes and navigating actions on Task 2 in the next section.

**Discussion**

In the previous section, we paid close attention to Porter’s and Sonu’s proving process maps and specific navigating actions around each of their stuck points. If we map out just those four major shifts for both Porter and Sonu, we will see that Sonu completed the full cycles of arguments → stuck points → navigating actions → related outcomes → arguments for both tasks. He always had a related outcome each time when he overcame a stuck point to help him move forward. On the other hand, even though Porter’s process map looks a bit different for Task 1 and Task 2, he didn’t complete full cycles for either task. For Task 1, he went from arguments → stuck points → navigating actions → related outcomes → stuck points → navigating actions. Similar to Task 1, he also went back and forth between stuck points and navigating actions for Task 2 (arguments → stuck points ↔ navigating actions) without any related outcome to help him move forward. Table 12 shows the comparison of their major proving processes cycles around stuck points.

In this section, I will summarize the general proving processes of Porter and Sonu around stuck points based on the developed analytical framework (discussed in Chapter 5), identify several themes and claims that emerged from their stuck points and navigating actions, and finally discuss why some actions Sonu took led him to make more productive progress compared to Porter.
Both Porter and Soni had some thoughts right after they read the statement of Task 1. However, Task 2 seemed to give them more struggles to start with. Both of them claimed that they “never had something like that.” But their reaction to this never-seen task was a bit different.

Porter chose to disprove it first since he said, “I couldn’t just prove it so let’s try to disprove it first.” He also tried to recall from his memory: “I remember in middle school, they told us that if you add the digits, it’s, it’s a multiple of three, then you can divide it by three. If it was the same with nine……” However, he couldn’t remember clearly how they accomplished the division by three previously and after trying several examples, his approach of attempting to disprove the statement also did not seem to work. He then ran into a stuck point about how to link the divisibility by 9 condition with the digits.
Even though Soni also claimed that he never thought about or saw this task, his immediate reaction was “Really? That’s fun. Let me think about this…” (as opposed to going back to recall from his memory). He tried several multiples of 9 and realized they all satisfied the condition. He conjectured that all numbers divisible by 9 should satisfy this condition. But he then questioned, “What about the other numbers? What are their connections?”

The initial exploration is helpful to set up the overall plan for the later proving processes. As we can see, Porter’s plan was mainly from recalling his previous experience or learning, whereas Soni focused on outlining some different ideas that he thought about.

**Identifying Stuck Points**

Both Porter and Soni encountered several stuck points while doing Task 1 and Task 2. I have clearly described each of their stuck points and their processes of navigation in the previous sections. Here, we discuss more about how they themselves identified their stuck points.

There are more instances of Porter’s attempts to identify his stuck points compared with Soni. If we look at all of the instances in which Porter tried to identify his stuck points, we would see several general descriptions, such as “I am not sure what I have to prove right now”; “I think technically I should use the divisibility rule but I just don’t know how”; “A lot of things—I just don’t know how to relate them”; and “I don’t know how to write the pattern.” Those general descriptions didn’t really help him to make decisions about his navigating actions since they are too broad in nature. But other ones, such as “So I know that \(a^2 + b^2\) right……I just don’t have information about \(c\)”; “So how do you go back to proving it’s not equal to \(c^2\)”; “I am having difficulties representing 18 and the sum of 1 and 8”; and “I just don’t know how to continue for the inductive case” gave him clearer direction to later identify his goals for each navigating action and attempt.
Even though Soni also had some general descriptions like “how to prove the stuff using mathematical language?” the majority of his descriptions of his stuck points were pretty specific, such as, “But I’m stuck here like, I tried to use some numbers instead of $a$, $b$, and $c$, and trying to find they’re not equivalent when it’s odd”; “But the thing is I’m stuck here. Like how am I going to show I’m going to show this number cannot be $\sqrt{2}$”; and “I have this or something like that. The $9 + 1$, how am I going to separate this stuff with the power of $k$.” Those clear directions did help him later when setting up the goals of the attempts and deciding the different navigating actions.

**Identifying Goals for Attempts**

As we have discussed for their identification of stuck points, a clear goal was really helpful for Soni and Porter for their navigating actions and attempts.

For Task 1, Porter clearly indicated that he had showed $a^2 + b^2$ is even, but he just had no information about $c$. So all of the later actions he took were around this main goal of finding more information about $c$. Regardless of whether or not Porter succeeded in the end, he remained very clear on what he needed to prove for this task. However, for Task 2, even though Porter tried multiple examples, he didn’t really have a clear goal for those examples in mind. Thus, his analysis of the example only remained on the surface level of rewriting the number into different forms, like 333 as $300 + 33 + 3$, but with no idea about how this would help to show the general cases.

Compared to Porter, Soni had a specific goal in mind for Task 2 when trying the different examples. His goal was to “find a way to represent the digits that have a relationship with 9 by writing 10 as $9 + 1$.” With this goal in mind, his example of 945 became $9(10)^2 + 4(10)^1 + 5(10)^0 = 9(9 + 1)^2 + 4(9 + 1)^1 + 5(9 + 1)^0$. He expanded the equation and successfully
found the relationship between its digits 9, 4, 5. Thus, his attempt of trying the example of 945 helped him prove the argument compared with Porter’s example of 333.

**Navigating Actions**

Since the participants were all in an interview setting instead of a classroom setting, some types of navigating actions that they had used in classroom settings might not have occurred during their proving processes in the interview (e.g., interaction with peers or with instructors). Thus, I mainly focus on each student’s autonomous actions to try to navigate out without considering some of their social actions.

Porter’s navigating action was mostly “strategy” driven. He usually thought about what strategy to use first, then switched to a different strategy if he got stuck. For instance, for Task 2, he initially thought about disproving the statement since he didn’t know how to prove it. After trying several examples and realizing that disproving didn’t work out, he switched to a direct proof approach. But that still didn’t lead him far. He then brought out the idea of induction, since he saw a “similar problem in class.” However, he still couldn’t figure out how to show the inductive case for integers with multiple digits and concluded that induction also didn’t help. Even though we can see that he had a good understanding of different strategies, the frequent switch of strategies also indicated he didn’t have a clear goal in mind for each of his attempts.

Interestingly, opposite to Porter, Soni had some difficulty remembering the name of some strategies, “What is that called? Contradiction? Or contrapositive?” He usually started with mapping and writing out his ideas then deciding the strategy. For Task 2, Soni started with no specific strategy but an idea or overall goal of finding the differences between counting by 9s and counting by 10s; he represented 10 as 9 + 1. Thus, all the actions he took to try navigating
out of his stuck point were centered around this goal, including trying different examples, trying induction and realizing it didn’t help, and trying different representations.

One other thing to note is that both Porter and Soni looked for and found a generalized formula for representing multiple digit numbers (any integer can be represented as $a_k(10)^k + a_{k-1}(10)^{k-1} + \cdots + a_0(10)^0$) when they got stuck. However, Porter used the example of rewriting 333 as 300 + 30 + 3 to make sense of the formula, but this still didn’t help him think further. On the other hand, Soni, with the central goal of representing 10 as $9 + 1$, immediately substituted $9 + 1$ into the formula and moved forward with the example of 945. Thus, similar navigating actions may not lead to the same outcome in the end. Besides different goals of navigating actions, monitoring and persistence are also important factors to consider when analyzing the students’ actions.

**Monitoring Throughout**

As we have defined in Chapter 4, monitoring is an understanding, evaluation, or recognition of where you are in the problem-solving/proving process when faced with an impasse. It’s important for students to provide recognition of work that is completed and of work that is yet to be completed.

Both Porter and Soni had some level of monitoring throughout their navigating actions. However, the difference was whether they could recognize where they were in their overall progress. For Porter, this recognition was not always easy. He indicated that he felt a bit lost during the process, “I am still……like where am I?” “I don’t need to create like some kind of equation, right? I do……yeah……no.” But he also made several monitoring moves to ensure that he was on the right track: “Turning to check that if the number is divisible by nine……Then
the sum of the digits on the number is also the result……(silence) Yeah. Not working.” “$a^2 + b^2$ now equal to $c^2$ right? Yeah.”

Even though Soni also encountered some difficulties with recognizing his overall progress, like “Did I just make myself confused?” he was constantly checking and monitoring on each of his steps: “Am I on the right track? Yes…I am…I want to find the differences between counting by 9s and counting by 10s”; “Oh wait, wait. I think this representative digit, so digit, they together……How can I separate this stuff?”; “No, I can’t. I can’t. I can’t show that because square root two is irrational number (talking to himself).” There are also more instances of monitoring for Soni than for Porter, which might be another factor contributing to Soni’s success for those attempts.

**Self-Motivation and Perseverance**

As we saw from both Porter’s and Soni’s proving processes, both of them spent a long time trying to navigate out of their stuck points. Thus, it is important to persevere in proving the problem and to keep oneself motivated throughout the long navigating process.

Porter’s self-motivation and perseverance in doing the two tasks was not much observed during his navigating processes. Even though he constantly gave some signs that he might give up soon, like “This one’s harder than the other one,” “It’s the sum of digits that is frustrating…,” and “I don’t know that. I don’t know how to use it here” with a frustrated facial expression, he seemed to at least have tried everything he could think of. At the end of the Task 2, after trying around 20 minutes, he finally gave up by indicating, “I can’t go further.” So in general, Porter didn’t simply give up when he encountered stuck points.

Soni’s self-motivation and perseverance were a bit easier to observe. He constantly motivated himself when his attempts didn’t work out by saying, “Wait I need to I need to prove
this…”; “Wait, no, ah, ah, wait, I’m almost there! So k square…(silence)”; and “Oh, wait, wait, maybe….maybe a minute something…I just feel like I kind of know the thing…I am almost there” as a way to persist. Even after the task-based interview ended, Soni still couldn’t help himself from continuing to think about Task 2.

To close this section, I will now summarize in Table 13 Porter’s and Soni’s overall proving processes and some themes we have observed based on the analytical framework.

Table 13 summarizes the key points as discussed in the previous sections. In general, we can see that even though some of Porter’s and Soni’s navigating actions might look similar (e.g., look for a generalized formula to represent digits), with different identification of stuck points, different goals for the attempts, different monitoring and persistence levels, Soni seemed to make more productive progress in general compared to Porter. I will discuss in the last chapter how instructors can support students to engage in making more productive progress.
Table 13

Comparison of Soni’s and Porter’s Navigating Actions

<table>
<thead>
<tr>
<th></th>
<th>Porter</th>
<th>Soni</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use of resources (material, social or mental)</td>
<td>Makes progress on task, but searched for memory of task or resources when impasse is encountered (remember did similar task in middle school, did similar task using induction)</td>
<td>Displays use of resources, but it is done in the service of constructing an argument in the moment and not for remembering/reusing from resources. (look for a general formula to substitute 10 as $9 + 1$)</td>
</tr>
<tr>
<td>Identify Stuck Points</td>
<td>Superficial attempt to identify source of struggle (try random ideas, re-stating ideas)</td>
<td>Engages in analyzing the argument up to the point of impasse, including determining where argument is solid.</td>
</tr>
<tr>
<td>Attempt to navigating out</td>
<td>Having identified the source of struggle, there is an acknowledgement that alternative strategies (i.e. induction), representations, etc. may be useful to resolve the impasse. But these are not pursued fully.</td>
<td>Expressly attempted to resolve the identified source of struggle. This may involve the use of alternative strategies, representations, etc. (Represent 10 as $9 + 1$ and substitute into general equations)</td>
</tr>
<tr>
<td>Monitoring</td>
<td>Offers some evidence of evaluating the progress made up to the point of impasse or provides some recognition of the work yet to be completed. May not present any evidence of recognizing the overall progress has been made in the process.</td>
<td>Displays an understanding of where they are in the proving/problem-solving process when faced with an impasse. Provides recognition of work that is completed and of the work that is yet to be completed.</td>
</tr>
<tr>
<td>Persistence</td>
<td>Gives up relatively quickly; does not try to extend the attempt (takes about 28 minutes)</td>
<td>Uses the allotted time; still tries to finish the attempt when the interview time is end (takes about 46 minutes)</td>
</tr>
</tbody>
</table>

Chapter Summary

From the analysis of Porter and Soni, as well as considering the processes of the other participants, I observed that specific navigating actions students took to try to navigate out of their stuck points could vary. However, additional actions that were not initially identified in the analytical framework appeared only in Type 2 or Type 3 processes. Those unidentified actions fit
in two major categories: (1) setting up a goal for the attempts, and (2) producing related outcomes.

*Setting Up a Goal for Their Attempts.* The majority of the students were able to identify their overall goals during their initial exploration. Some of the goals they formulated were broad, such as prove or disprove the statement, but others were more specific in terms of what they wanted to do for each of their attempts or steps. For instance, for Task 2, even though some of the participants (like Porter) tried multiple examples, they didn’t really have a clear goal for what those examples were supposed to do for them in the scope of the proof. Thus, their analysis of the examples they generated remained at a surface level of rewriting a number into different forms, but without an idea for how this would help to demonstrate the general case. In contrast, Soni had a clear goal in mind for Task 2 when trying the different examples he generated. His goal was to “find a way to represent the digits that have a relationship with 9 by writing 10 as 9 + 1.” With this goal in mind, his example of 945 didn’t just remain on the surface level and he successfully found the relationship between its digits 9, 4, 5. This type of action is mainly found in students’ productive or partially productive attempts.

*Producing Related Outcomes and Linking Them Back to the Argument.* Actions that produced related outcomes and linked those results back to the argument or goal were also observed. For some students, they were not aware of the production of the related results, thus no further action was taken. On the other hand, those who were aware of the intermediate results they produced and were able to link these sub-results back to their main argument did succeed in overcoming their stuck points in the end (like Soni). This type of action is mainly found in students’ Type 3 process when students are making productive progress.
In terms of actions related to monitoring and persistence, the majority of those actions were aligned with actions in the analytical framework. However, in addition to those types of actions, a related class of action, “self-motivation,” also emerged. For instance, Soni constantly motivated himself when his previous attempts didn’t work. In my study, actions related to self-motivation were observed only in productive proving processes.

Even though there were 107 total instances of students’ monitoring actions, not all of the monitoring actions led them to achieve their goals. Recognizing where one is in one’s overall process and reflecting back on goals appeared in Type 2 and Type 3 processes; this was an important indicator for the productivity of their actions. In the case of Porter, he indicated that he felt a bit lost during the process: “I am still……like where am I?” Soni, on the other hand, was able to recognize his overall progress and constantly checked each of his steps with respect to his goals: “Am I on the right track?…yes…I am…I want to find the differences between counting by 9s and counting by 10s,” which appeared strongly instrumental in helping him to move forward.

Although I discussed Porter’s and Soni’s navigating actions in detail, different navigating actions were taken by all of the participants in the study, representing different ways of trying to overcome a stuck point. However, not all of those navigating actions could be considered to be making productive progress, since not all participants were successful in producing at least one related outcome that linked back to the argument.

I will close this chapter by proposing a model of what students’ processes and actions around stuck points might look like. This model came out of my analytical framework, together with the results from my analysis. Many were documented in the analytical framework, but some of the influencing factors, such as setting a clear goal for the attempts, producing related results,
and linking the related results back to the goal, came from the analysis. Figure 29 illustrates how the different actions may relate to each other in the proving processes around stuck points.

Figure 29. Proposed Model for How the Different Actions May Related to Each Other in a Productive Proving Process Around Stuck Points

There are two components of the framework: the overall proving process from an argument, to stuck point, to navigating actions, to related results and back to the argument; and the specific actions one takes to move along the process, including identifying stuck points, attempting to resolve, monitoring, and persisting in solving. Some indicating factors have also been noted, such as setting up a goal for each attempt, producing related results, and linking back to the goal or argument.

In the next chapter, I will present a discussion of my results, including implications and directions for future research.
CHAPTER 9
CONCLUSIONS, IMPLICATIONS AND FUTURE RESEARCH

Having presented and explained (1) the kinds of stuck points that occur when students are engaging in proving, (2) characteristics of the different types of students’ proving processes around stuck points, and (3) specific actions students take to try to navigate out of their stuck points, in this chapter I will present a discussion of the results of this study and possible implications of the results. The discussion includes relating the results concerning productive stuck point navigating actions to the literature on productive struggle more generally. The concluding section of this chapter examines limitations of this study and directions for future research.

Discussion and Conclusions

As I discussed in the literature review, although students’ and mathematicians’ proof practices have been examined in several ways, there are still important aspects that have not been examined in the existing literature. The findings that I presented in previous chapters raise some questions to consider: What are some differences with the result of this study and with previous literature? How will this study contribute to the development of a proving process framework? What can be considered to be more productive actions, and what can instructors do to support students to engage in a more productive struggle? In the following paragraphs, I will discuss some possible answers to these questions based on the existing literature and the results of this study.
Differences in Results Compared to the Existing Literature

One of the major differences I found between the framework I developed and previous research is the unique argumentation properties that the proving process has compared to the problem-solving processes that Warshauer (2011) and Carlson and Bloom (2005) were trying to capture. Based on the percentages across the full data set, the most common types of stuck points were around generalization and argumentation. These two categories accounted for more than an 87% share of my data.

Even though Warshauer (2011) and this study categorize student struggles differently (Warshauer’s study in terms of different proving phases and this study in terms of specific instances of identified stuck points), one can observe notable differences when comparing the findings of the two studies about kinds of student struggle in proving (Lu) and problem solving (Warshauer). For Warshauer, the majority of the students’ struggles were around getting started and carrying out a process (57%), which is not necessarily associated with forming generalizations or connecting arguments together logically. Instead, in my study, the majority of students’ stuck points occurred in the executing phase after students’ initial explorations and when students already had set up an initial plan.

These differences in results might be due to important dimensions of difference between the two studies: the setting, the population, and the task content. Previous studies on students’ productive struggle, like Warshauer (2011), have mainly been focused on middle school students’ problem-solving activities in classroom settings where students are working together with their peers. Because of these differences in setting and population, those studies tend to focus more on teachers’ support in order to help students overcome their struggles. However, this study focused on undergraduate students’ individual proving processes in a task-based interview
setting. The role of the interviewer in this context was not to provide support for the student but rather to observe students’ actions and process, and at the same time get enough in-the-moment thinking of the students.

The high frequency of students’ stuck points around generalization and argumentation in the executing stage might also be due to the nature of the two interview tasks. As discussed in the methods chapter, I purposely selected these two interview tasks on the same content of number theory for multiple reasons. One of the reasons was that both questions would be easy to understand with accessible entry points and multiple solution paths. Thus, it would be less likely for students to get stuck at the initial stage of the proving process. The number theory content itself might also contribute to the high frequency of stuck points around generalization and argumentation. Indeed, according to Dawkins and Karunakaran (2016), it is not really possible to remove content from a measure of proof competence because competence with proof is multi-faceted.

**Contribution to the Development of a Proving Process Framework**

Thus far in the mathematics education literature, there does not exist a framework that has specifically focused on students’ proving processes. While no proving frameworks yet exist for analyzing students’ proving processes, there are frameworks for analyzing problem-solving processes (cf. Carlson & Bloom, 2005; Schoenfeld, 1985, 2010). Previous work by Carlson and Bloom (2005) had characterized how experts behave to successfully solve problems. However, this problem-solving framework does not characterize how an undergraduate student may behave if they “get stuck” in solving problems and where they seem to “get stuck.”

To address those issues and develop a novel proving process framework, based on Warshauer’s (2011) classification and my data analysis, through the use of thematic analysis
(identifying common themes), I classified different types of stuck points and the processes and actions that students used to try to navigate out of those stuck points. At this point, the analytical framework I developed aimed to examine whether students were identifying the source of their struggle, making attempts to resolve the stuck point, monitoring their process that led to the stuck point, and allowing time/persisting.

In addition to my analytical framework, in order to capture each student’s overall proving processes around stuck points and bridging the gap of the relationship between the argument and the actions, I also used a process map to give a fine-grained description of students’ proving processes for each task (as described in Chapter 5). Previous proof research, building largely upon the work of Toulmin (1993), focused only on the structure of the argument but not the actual action and the actor (e.g., Pedemonte, 2007). The process map captured both students’ arguments and actions and also the relationship between the two for each task and for each individual student.

Besides, three types of processes were generated based on participants’ proving process maps for each task. Those three types were termed. For each type, students’ movements between each argument and actions were mapped out. My results indicate that the majority of my participants were more in Type 3 in their actions for Task 1 compared with Task 2. This may be due to the different purpose of the two tasks, since Task 1 was designed so that they would use different approaches without requiring certain concepts (odd and even numbers), and Task 2 required the knowledge of certain number theory concepts (modular arithmetic). In any case, the majority of the students had process maps and actions that were categorized as Type 2 and Type 3. From the analysis of the actions, even though the specific navigating actions students took to try to navigate out of their stuck points did vary, there were similar influencing factors for some
of their actions. Some of their actions were influenced by their class content and what they had been doing recently. For instance, many of the participants mentioned the use of induction for Task 2, which was the strategy they had just learned recently. The finding of those three types shows that students’ proving processes are not linear; there could be multiple stuck points thus resulting in multiple sub-processes. But the student is considered to be making productive progress if at least one productive cycle is complete.

The finalized framework, as presented in Chapter 6, proposed a model of what students’ process and actions around stuck points look like. This framework summarized the finding of the research questions of this study and filled in the gap of the existing literature. Thus, the three tools—different themes to examine students’ kinds of stuck points, proving process map to capture students’ overall proving processes, and developed analytical framework to capture specific navigating actions—could be used as a methodological tool for future research.

**Guidelines to Support Students Making More Productive Progress**

Data from this study suggest that some students did spend a significant amount of time, with multiple navigating actions, to try to overcome their stuck points. Thus, presenting such lengthy examples of students’ proving processes and actions with proof-related tasks could be illuminating for both students engaging in proof for the first time and also for instructors and teachers.

Although I focused primarily upon Porter’s and Soni’s navigating actions in detail, different navigating actions were taken by different participants in the study, which represented different ways of trying to overcome a stuck point. However, not all of those navigating actions would be considered as making productive progress (completing at least one cycle, linking one related outcome back to the main argument). As we have observed from the results chapter,
certain actions, such as having a goal for each of their attempts, being aware of the produced results, and being able to link those results back to the goal, occurred in only some of the partially productive processes and all of the productive processes (but not in any of the unproductive processes). Thus, in thinking about what teachers or instructors can do to support students to engage in more productive struggle in proving processes, three ideas emerged.

First, having a clear goal seems to be helpful both in making decisions and also in monitoring. This idea helps to answer the question, “How do we understand the situation where students who should have all the resources needed to prove statements do not succeed?” As Schoenfeld (1985) might claim, when students have sufficient content knowledge to solve a problem, they may still fail to do so because they lack suitable metacognitive control to select, continue, or abandon a specific idea or strategy. For instance, for Task 2, even though Porter tried multiple examples, he didn’t really have a clear goal of those examples in mind. Thus, his analysis of the example remained only on the surface level of rewriting the number into different forms, like 333 as 300 + 33 + 3, but he had no idea about how this would help to show the general cases. Compared to Porter, Soni had a really clear goal in mind for Task 2 when trying the different examples. His goal was to “find a way to represent the digits that have a relationship with 9 by writing 10 as 9 + 1.” With this goal in mind, he expanded the equation and successfully found the relationship between its digits 9, 4, 5. Thus, his attempt of trying the example of 945 became more productive compared with Porter’s example of 333, and it helped him to prove the statement in the end. Practically, teachers could help students to set up overall goals for the problems and several sub-goals for each of students’ attempts.

Second, producing related outcomes or arguments from the navigating actions and linking those back to the argument or goal is a necessary step to overcome the stuck points. This
idea aligns with what Schoenfeld (1985) claimed as metacognition control, which is the ability to break up more complex problems into a few sub-problems, do the sub-problem first, then sequence the sub-problems, then complete the whole problem. Thus, students need to be able to reduce the task complexity and see how the sub-arguments relate to the main argument in order to make productive progress. For example, both Porter and Soni looked for and found a generalized formula for representing multiple digits numbers but they got stuck. However, Porter used the example of rewriting 333 as 300 + 30 + 3 to make sense of the formula, but this still didn’t help him think further. While Soni, with the central goal of representing 10 as 9 + 1, immediately substituted 9 + 1 into the formula, used the example of 945 to produce a related result, then used the result to link back to the general formula. Therefore, teachers could help students to reduce the complexity of the problem in some way by approaching the problem through examples or changing the problem slightly and then get back to the original problem.

Lastly, self-motivation and persistence also played a role in productive actions. A lot of this idea shares much in common with previous studies on productive struggle (e.g., Warshauer, 2015; Lynch, Hunt, & Lewis, 2018). For example, Soni constantly motivated himself when his previous attempt didn’t work out by saying, “Wait, no, ah, ah, wait, I’m almost there! So $k^2$ …..(silence)”and “Oh, wait, wait, maybe…Maybe a minus something…I just feel like I kind of know the thing…I am almost there” to help him keep persisting. He also couldn’t help himself from continuing to think and validate his proof for Task 2 even after the interview ended. Thus, teachers could allow time for students to progress and keep them motivated throughout the long process. From my findings of the three types, allowing time alone is not satisfactory; students need to be informed about how to use the time.
From these three ideas, I will now discuss what this study can tell us about whether a navigating action will be considered making productive progress. In general, students need to engage in a way that they are able to make some progress in order to be productive. To be more detailed, and based on our data analysis, productive progress occurs when the student can understand/identify their stuck points, have a clear goal for each of their attempts, engage in different attempts to navigate out, produce related results and link those results back to the argument, monitor throughout, and persist in solving the problem. Students will be considered making productive progress if at least one cycle is complete. Thus, it will be helpful to evaluate students’ kinds of stuck points, overall proving process, and navigating actions to identify where and in which steps the student needs help in the proving process.

Implications

This study characterized undergraduate students’ proving processes and navigating actions around their stuck points. Given the importance of proving processes in the proof literature, this study has implications theoretically, methodologically, and pedagogically for researchers of proof and for instructors or curriculum developers of proof-related courses.

Theoretically, this work gives a more fine-grained account of what students do when they get stuck through mapping out their proving processes around stuck points. To examine both students’ overall proving process and actions in proving attempts that generate stuck points, two analytical tools were used. First, a process map was used to capture students’ overall process. Then, my analytical framework, based on previous research and my initial data analysis, was applied to examine specific actions. My results indicated that undergraduates’ proving processes around stuck points can be characterized into three main types: unproductive, partially productive, and productive. With a deeper look at Porter’s and Soni’s navigation actions, I
observed that certain actions, such as having a goal for each of their attempts, being aware of the intermediate results they had produced, and being able to link those results back to the goal, only occur in some of the partially productive processes and all of the productive processes. Thus, the result of the study provides theoretical tools for unpacking the moments for students’ struggle, which is important for understanding students’ mathematical growth.

Methodologically, to capture undergraduate students’ proving processes, this study made use of Livescribe pens to examine students’ proving processes with number theory tasks, with a particular focus on their stuck points. The use of the Livescribe pen to capture students’ proving processes is relatively novel in mathematics education research (e.g., Lew & Zazkis, 2019; Savic, 2015) and, to my knowledge, has not yet been applied to examine undergraduate students’ stuck points. Future research on processes can also adapt Livescribe to capture students’ moment-by-moment actions. In addition, compared with other studies in the proof literature, this research used only two interview tasks to make sure participants had enough time engaging with the task and were able to explore several navigating actions. Allowing time for students to struggle with the tasks is important for research on students’ stuck points or impasses.

Pedagogically, the evaluation of undergraduate processes offers evidence that students do not engage in strictly linear or sequential proving processes. As indicated in the results chapter, undergraduate students often encountered multiple stuck points when engaging in proving activities. As stated in the literature review, right now this nonlinearity of the proving process is not represented in textbooks or instructions or students’ beliefs. Thus, through this study, I want to send the message, or have our curriculum send the message, that being stuck and getting unstuck is part of the process. To do that, instruction or the textbook could provide some
examples of the map of students’ proving processes to acknowledge that being stuck is acceptable and expected.

In addition, different proving processes and navigation actions characterized in this study can help instructors to have a better understanding of productive struggle and to be more explicit in their own modeling of their own actions for their students. Thus, this study suggests strategies for proof-based course instructors in supporting students engaging in more productive struggle during their proving practices through reflecting on their practice and considering actions that they can take in facilitating productive struggles for their students. These guidelines might include (1) setting up classroom norms that acknowledge that struggle is an important part of learning math, (2) designing tasks that engage students in productive struggle, (3) asking questions that help students identify their goal for each attempt and the source of their struggle, (4) encouraging students to reflect on their work (monitoring), and (5) allowing time for students to persevere.

**Limitations and Future Research**

As with all qualitative research studies, this study is limited in the extent of the generalizability of its results. The research design of this study is not meant to produce results for a large-scale characterization of all undergraduate students’ proving processes around stuck points. The purpose of this study was to generate analytic generalization (Firestone, 1993), rather than sample-to-population or case-to-case generalizations that arise from the data. That is, I do not claim that the characteristics of the participants’ overall proving processes and Porter’s and Soni’s navigating actions presented here are necessarily applicable to the larger population of undergraduate provers. The findings from the current study focused on better understanding
undergraduate students’ overall proving process and actions when they encounter stuck points and providing a grounding for future research.

Since this study has a particular focus on students’ actions and processes, the next question is to look into students’ understanding and knowledge. It could be possible that a student might have the knowledge that is needed to prove the task, but they might not be able to use their knowledge. Thus, it’s important to also understand why students are doing what they are doing in their actions. Levin’s (2018) work on strategy systems takes a more fine-grained look at the interrelation between concepts and strategies and may be useful for unpacking the relationships.

Another limitation of this study was my choice of the two tasks from only number theory. This was a conscious choice due to the goal of the study: to look at students’ stuck points after they had generated an initial approach. The limitation with using tasks exclusively from number theory is that it is possible that the number theory tasks rely more on generalizing from examples. The actions noted in this study might be similar or different given a different content. Future research can look into other content, such as abstract algebra, where students’ use of examples or other processes may be different.

Lastly, the sample from this study was chosen from students who are in the transition-to-proof course or the course immediately after. Thus, the actions or processes may reflect only novice provers and might be differ for a different student population. An important question for future research could be to examine proving processes or actions for students who have taken multiple proof-related courses and are considered to be more expert-like in proving.

This study captured only undergraduate individual proving processes and actions in a task-interview setting, in which I was trying to do non-interventions during the interview. Future
research could involve designed interventions to provide students with controlled hints and see how far they can go with the hint. Other possible avenues for future research include investigations of students’ proving processes in a classroom setting, especially when students work in groups. In that setting, more social interactions among different students and between students and the instructor will necessarily occur. It will be interesting to see how those interactions may result in similarities and differences to the current study. Another possible direction for future work includes sharing my framework and process maps with proof-based course instructors and studying how they use them. Alternatively, another direction could be to understand more about how students might use the framework to evaluate their own proving processes when they reach stuck points.

To conclude, this study provides theoretical, methodological, and pedagogical tools for unpacking the moments of students’ struggles, which is important for understanding and supporting students’ growth in the discipline. The different proving processes and navigation actions characterized in this study can help instructors to have a better understanding about productive struggle and to support students engaging in more productive struggle during their proving practices.
REFERENCES


Appendix A

Human Subjects Institutional Review Board
Approval Letter
Date: February 7, 2019

To: Mariana Levin, Principal Investigator
    Yaomingxin Lu, Student Investigator for dissertation

From: Amy Naugle, Ph.D., Chair

Re: IRB Project Number 19-01-24

This letter will serve as confirmation that your research project titled “Characterizing Undergraduates’ Competence with Proof Construction” has been approved under the expedited category of review by the Western Michigan University Institutional Review Board (IRB). The conditions and duration of this approval are specified in the policies of Western Michigan University. You may now begin to implement the research as described in the application.

Please note: This research may only be conducted exactly in the form it was approved. You must seek specific board approval for any changes to this project (e.g., add an investigator, increase number of subjects beyond the number stated in your application, etc.). Failure to obtain approval for changes will result in a protocol deviation.

In addition, if there are any unanticipated adverse reactions or unanticipated events associated with the conduct of this research, you should immediately suspend the project and contact the Chair of the IRB for consultation.

The Board wishes you success in the pursuit of your research goals.

A status report is required on or prior to (no more than 30 days) February 6, 2020 and each year thereafter until closing of the study. The IRB will send a request.

When this study closes, submit the required Final Report found at https://wmich.edu/research/forms.

Note: All research data must be kept in a secure location on the WMU campus for at least three (3) years after the study closes.
Appendix B

Participant Consent Form
Principal Investigator: Dr. Mariana Levin
Student Investigator: Yaomingxin Lu
Title of Study: Characterizing Undergraduates’ Competence with Proof Construction

STUDY SUMMARY: This consent form is part of an informed consent process for a research study and it will provide information that will help you decide whether you want to take part in this study. Participation in this study is completely voluntary. The purpose of the research is to investigate how undergraduate students develop their competence with proof construction and will serve as Yaomingxin Lu’s dissertation for the requirements of the Ph.D. of Mathematics Education. If you choose to participate in the first level of the study, data collected will come from course tasks that are part of the normal classroom practice, therefore you are not asked to do anything beyond what is expected of any student enrolled in a M3140 or Math 3400 or Math 5800, other than granting permission for the researcher to copy your work. If you choose to participate in the interview component of the study, you will be asked to participate in an interview outside the classroom. The interview will last no longer than 45 minutes, use your code name to ensure confidentiality, take place on campus at mutually convenient place between you and the interviewer, and would occur during the semester. There are no additional risks or costs to participating in this study, other than the time commitment given if you choose to participate in the interview component. What is learned from this study may benefit the future course instructors of proof-related upper-division courses at Western Michigan University, future proof-intensive course students and, through published results, inform researchers and course instructors at other institutions of ways to provide the best possible preparation for undergraduate student to go into more advanced-level mathematics and develop their proof construction competence. Your alternative to taking part in the research study is not to take part in it.

You are invited to participate in a research study titled “Developing Undergraduates’ Proof Competence with Proof Construction” in connection with your proof-intensive course. This consent document will explain the purpose of this research project and will go over all of the time commitments, the procedures used in the study, and the risks and benefits of participating in this research project. Please read this consent form carefully and completely and please ask any questions if you need more clarification.

What are we trying to find out in this study?
The primary purpose of this study is to investigate how undergraduate students develop their competence with proof construction.

Who can participate in this study?
All students enrolled in the Math 3140 or Math 3400 or Math 5800 classes are encouraged to participate in the study, “Characterizing Undergraduates’ Competence with Proof Construction.”
Students who self-select to not participate in this study will be excluded without any negative impact on them. Among all of the students who choose to participate in this study, only around 10 students will be selected randomly by the researcher to be the focus of this study. There are no consequences for not participating in the study.

**Where will this study take place?**
Part of this study will take place in the classroom of Math 3140, Math 3400 and Math 5800 on the main campus of Western Michigan University. If you choose to participate in the second level of the study, you will be video recorded and interviewed while working on a proof-related task outside the classroom for around 30 minutes. The interviews will take place on campus at a mutually convenient time and place and would occur during the semester.

**What is the time commitment for participating in this study?**
The time commitment for participating in this study is the length of the current semester since the data collected consists of the tasks you complete during the term. If you choose to participate in the interview component of the study, those will also take place during the term. The interview itself will last no longer than 45 minutes.

**What will you be asked to do if you choose to participate in this study?**
If you choose to participate in the first level of the study, data collected will come from course tasks that are part of the normal classroom practice, therefore you are not asked to do anything beyond what is expected of any student enrolled in Math3140, Math 3400 or Math 5800, other than granting permission for the researcher to copy your work. Your name will be removed prior to any copying of your work to ensure anonymity in the data analysis. Code names will be created and used to permit examining your (anonymous) mathematical proof thinking across the semester.

If you choose to participate in the interview component of the study, you will be asked to participate in an interview outside the classroom. The interview will last no longer than 45 minutes, use your code name to ensure confidentiality, take place on campus at mutually convenient place between you and the interviewer, and would occur during the semester. The interviewer will be Yaomingxin Lu, a Ph.D. student in the department of mathematics and not the course instructor.

**What information is being measured during the study?**
This study is examining your proof construction competence through exploring some proof-related task. To this end, your work from course tasks and your responses during the interviews will be analyzed to examine your proof construction competence and to assess the impact of the tasks and problem-solving strategies you used while solving the problems.

**What are the risks of participating in this study and how will these risks be minimized?**
There are no known additional risks to participating in this study, other than those that may occur as part of your daily routine as an undergraduate student. If you choose to participate in the interview component of the study, you may experience some minor discomfort that can occur during interviews (e.g. mild stress owing to the one-to-one setting).
When the interviewer observes such discomfort, s/he will provide wait time for re-composure and remind you that you are able to halt the interview at any point in time without consequence. You may choose to remove yourself from the study or the interview at any time with no course grade penalty.

**What are the benefits of participating in this study?**
There are no direct benefits to you for participating in this study. What is learned from this study may benefit the future course instructors of proof-related upper-division courses at Western Michigan University, future proof-intensive courses’ students and, through published results, inform researchers and course instructors at other institutions of ways to provide the best possible preparation for undergraduate student to go into more advanced-level mathematics and develop their proof construction competence.

**Are there any costs associated with participating in this study?**
There are no costs associated with participating in this study other than the time commitment given if you choose to participate in the interview component. Those who participate in the study are completing the work expected of any student enrolled in Math 3140, Math 3400 or Math 5800 class.

**Is there any compensation for participating in this study?**
Those participating in the interviews will get $20 gift card and may gain more insight into their own thinking and learning processes as they might benefit from the interview tasks which are very similar to their classroom tasks and homework.

**Who will have access to the information collected during this study?**
Only Dr. Mariana Levin and Yaomingxin Lu, will have access to the information collected during this study. All data will have names removed prior to copying and code names assigned. Video data collected will use code names when necessary to insure confidentiality. If any collected information is used during a public presentation, your identity will be kept confidential through the use of these code names or other pseudonyms.

**What will happen to my information collected for this research after the study is over?**
The information collected about you for this research will not be used by or distributed to investigators for other research.

**What if you want to stop participating in this study?**
You can choose to stop participating in the study at any time for any reason. You will not suffer any prejudice or penalty by your decision to stop your participation. You will experience NO consequences either academically or personally if you choose to withdraw from this study.

To stop participating email Dr. Mariana Levin at mariana.levin@wmich.edu or Ms. Yaomingxin Lu at yaomingxin.lu@wmich.edu and indicate that you no longer want to participate in the study.
Should you have any questions prior to or during the study, you can contact the primary investigator, Mariana Levin at mariana.levin@wmich.edu or 269-387-4592 or Yaomingxin Lu at yaomingxin.lu@wmich.edu. You may also contact the Chair, Human Subjects Institutional Review Board at 269-387-8293 or the Vice President for Research at 269-387-8298 if questions arise during the course of the study.

This consent document has been approved for use for one year by the Western Michigan University Institutional Review Board (WMU IRB) as indicated by the stamped date and signature of the board chair in the upper right corner. Do not participate in this study if the stamped date is older than one year.

Please continue to provide an indication of your participation levels.

---------------------------------------------------------------------

IF YOU AGREE TO PARTICIPATE, PLEASE CHECK ONLY ONE OPTION

☐ I have read this informed consent document. The risks and benefits have been explained to me. I agree to take part in this study and I am willing to be interviewed.

☐ I have read this informed consent document. The risks and benefits have been explained to me. I agree to take part in this study, but I do not want to be interviewed.

☐ I have read this informed consent document. The risks and benefits have been explained to me. I do not want to be part of the study at all.

---------------------------------------------------------------------

Please Print Your Name

---------------------------------------------------------

Participant’s signature                           Date

---------------------------------------------------------------------

Email address for interview contact
Appendix C

Pilot Study Interview Protocol
Research title: Characterizing Undergraduates’ Competence with Proof Construction

Interviewer: Yaomingxin Lu (Ph.D. student at WMU)

Interviewee:

Time and place:

Thank you for agreeing to be interviewed. This interview is designed to examine your proof construction thinking. We can use your responses to better understand how students can best learn to construct proof. Your answers will remain confidential and will not be used for grading purposes by the instructor. If during the interview, you decide to stop the questioning and not continue, there is no penalty. Thank you for your willingness to participate in the interview.

(Sign consent form, ask about video tape)

Your background:

Tell me about yourself (5min),

1. What’s your major, what year are you?

2. When was the first time that you learn proof?

3. What is your view of proofs? Did that view ever change?

4. How successful do you think you are with proofs?

I am going to ask you to prove or disprove different mathematical statements. Please try to describe as completely as possible how you decide on an approach to making your decision on the validity of the statement. I’m trying to understand students’ thought process, in how you approach proofs; it’s not about the final answer. So I’m going to give you some questions I’d like you to prove. It would be helpful to me if you share your thinking with me: what you’re trying to do, why you’re trying to do that, etc. I want you to vocalize everything you’re thinking about the
problem. Pretend you’re at home and you’re just talking to yourself out loud. I can’t answer any questions about the math or what to do next if you get stuck.

Any questions I ask or notes I take means that I’m interested in what you’re doing, it’s not a sign that I’m judging your work or that your thinking is incorrect. I may ask questions every so often like, “What do you mean by...?” or “Why did you decide...” I may ask you “What are you thinking right now?” if you’re been quiet for some time.

There are two sets of tools I will provide to you when you are solving the task. The first tool is the set of strategies you have learned in class, including proof by induction, contrapositive, contradiction, direct proof and so on. This might help you to choose a strategy you will use for proving the task or change to another strategy if you get stuck. The second is the set of definitions/theorems/lemma that might be useful to proof or disprove the statement.

Do you have any questions before we begin?

After tasks are complete: (debriefing)

1. Were there any places where you got stuck?

2. How did you overcome that in the end? (Do you make small sub-proofs or expand steps?)

   Or if didn’t overcome, what was preventing you?
Appendix D

Pilot Study Round 1 and 2 Interview Tasks
**First round Pilot study**

Task 1:
Prove or disprove: If \( n \) is a nonnegative integer, then 5 divides \( 2 \cdot 4^n + 3 \cdot 9^n \).

Task 2:
Prove or disprove: The set \( A = \left\{ \frac{1}{1+x} : x \in \mathbb{R} \text{ and } |x| > 1 \right\} \) is unbounded.

Task 3:
Prove or disprove: If \( p \) is prime, then \( \sqrt{p} \) is irrational.

Task 4:
Prove or disprove: Let \( A, B, \) and \( C \) be sets. If \( A \times C = B \times C, \) then \( A = B \).

**Second round Pilot study**

Task 1:
Prove or disprove: An integer is divisible by 9 if the sum of its digits is divisible by 9.

Task 2:
Prove or disprove: If \( p \) is prime, then \( \sqrt{p} \) is irrational.
Appendix E

Task-Based Interview Protocol
Protocol

**Interviewer:** Yaomingxin Lu

**Goal of the Interview:** to characterize relative novices’ (undergraduate students) proof construction processes on designed task that is difficult to access based on the final proof product alone through examining of their “stuck points” during this process.

**Interviewee:**

**Time and place:**

Thank you for agreeing to be interviewed. This interview is designed to examine your proof construction thinking. We can use your responses to better understand how students can best learn to construct proof. Your answers will remain confidential and will not be used for grading purposes by the instructor. If during the interview, you decide to stop the questioning and not continue, there is no penalty. Thank you for your willingness to participate in the interview.

(Sign consent form, ask about video recording)

**Your background:**

Tell me about yourself (5 min):

1. What’s your major, what year are you?
2. When was the first time that you remember learning proof?
3. What do you think is the purpose of proof? What counts as proof to you?

**Interview Tasks:**

I am going to ask you to do the following two proof-related tasks. Each of the tasks was either a mathematical statement or situation, and you will be asked to prove or disprove each of these statements or situation. Please try to describe as completely as possible how you decide on an approach to making your decision on the validity of the statement. The interview will take
around an hour, you will have at least 60 min to work on both tasks. You will not be stopped for doing any task until you indicate that you have tried everything that you can. You will be asked to verbal your thoughts aloud while doing a task, both in the moment and shortly after the task.

My role as an interviewer was to encourage you to think aloud about your proving process instead of answering your questions or interact with you during the proving processes. I will only ask questions like: "I haven't heard you say much in the past couple of minutes. What were you thinking?" to keep you think-aloud to minimize our interactions during the proving processes. Also, I am more interested in your proving processes rather than whether you will get the “correct” answer.

I will provide you with a set of definitions/theorems/lemma that you might need to proof or disprove the statement. Feel free to use the ones you think are relevant/useful for the task. If there are any other definitions/theorems/lemma that you couldn’t not recall and are not in the paper, you can always ask, and I will provide that to you.

Do you have any questions before we begin?

To this end, I will follow the following guidelines:

1. Make sure the student is thinking aloud. Do not allow for any major portion of time (more than 2-3 minutes) to pass where the student is not saying anything.

2. Make sure to listen for statements that indicate that the student is deciding on a new strategy or path to work on the problem.

3. Make sure to listen for statements that indicate that the student is deciding that a particular path or strategy is not working.

4. Make sure to listen for statements that indicate that the subject has determined the validity of the statement.
An understanding of even and odd is necessary for this proof.

(By contradiction) Suppose that both $a$ and $b$ are odd. Then we can write $a = 2m + 1$ and $b = 2n + 1$ for integers $m$ and $n$, and therefore $a^2 + b^2 = (2m + 1)^2 + (2n + 1)^2 = 4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 4(m^2 + m + n^2 + n) + 2$.

We divide into two cases: $c$ is even, or $c$ is odd.

If $c$ is odd, then so is $c^2$. However, the calculation above showed that $a^2 + b^2$ is even, and this is a contradiction. If $c$ is even, then it is divisible by 2, and so $c^2$ is divisible by 4. However, $a^2 + b^2$ is equal to a multiple of 4 plus 2, and so it is not divisible by 4. In either case we have a contradiction. Therefore, at least one of $a$ and $b$ is even.

Task 2

Prove or disprove: An integer is divisible by 9 if the sum of its digits is divisible by 9.

(Recall that the numerical value of any integer $N$ can be represent as $N = a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots + a_1 \cdot 10^1 + a_0 \cdot 10^0$ where $a_k, a_{k-1} ... a_2, a_1, a_0$ are the digits)

Expected solution:

An understanding of basic modular arithmetic is necessary for this proof.

(Direct proof) Suppose that the base-ten representation of $N$ is

$N = a_k a_{k-1} ... a_2 a_1 a_0$,

where $a_i$ is a digit for each $i$. Then the numerical value of $N$ is given by

$N = a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots + a_1 \cdot 10^1 + a_0 \cdot 10^0$

Now we know that, since $10 - 1 = 9$, we have $10 \equiv 1 \pmod{9}$ and so we have $10^j \equiv 1^j \equiv 1 \pmod{9}$ for every $j$.

Therefore, we can write

$a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \cdots + a_1 \cdot 10^1 + a_0 \cdot 10^0$

$= a_k \cdot 1 + a_{k-1} \cdot 1 + \cdots + a_1 \cdot 1 + a_0 \cdot 1 \pmod{9}$
Therefore, we have $N \equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \pmod{9}$.

That is, $N$ differs from the sum of its digits by a multiple of 9. It follows, then, that $N$ is a multiple of 9 if and only if the sum of its digits is a multiple of 9.

**Debriefing**

1. Was there anything that was particularly difficult or took you long?
2. (When there were delays) What were you thinking of at this point in time?
3. Were there any places where you got stuck? Did you overcome it in the end?
4. If you did overcome being stuck, how did you overcome that in the end? If not, what did you feel was preventing you?
Appendix F

Pilot Study Round 2 Data Transcript Coding Example
<table>
<thead>
<tr>
<th>Time</th>
<th>Transcript</th>
<th>Coding (Codes: Orienting Planning Executing Checking Monitoring Stuck Points)</th>
</tr>
</thead>
<tbody>
<tr>
<td>29:07-29:54 (Video 1)</td>
<td>My initial thought with this one is….eh….is that….the square root 2 proof, but now when I thinking about it more….eh….yeah… So...I guess the first thought is like (start writing on the board)....my initial thought about this statement is that it is not true, because square root of anything besides perfect squares aren’t irrational….p is not a perfect square for any p, so….my initial thought is that is not true…</td>
<td>Planning</td>
</tr>
<tr>
<td>29:55-30:35</td>
<td>So from what I know from the square root of 2 proof, you assume that is rational, so you assume that root p is rational (writing on the board)....what it means that is rational…. That means the square root of p can be written as some a over b, for a and b in Z. so that means that p is equal to some a square over b square by algebra (actually write $a^2/b$ on the board)</td>
<td>Executing</td>
</tr>
<tr>
<td>30:40-31:00</td>
<td>And that’s where I need to think more…(realize it is $b^2$ and changed on the board)</td>
<td>Stuck point 1</td>
</tr>
<tr>
<td>31:01-31:25</td>
<td>this implies that $pb^2 = a^2$ ok, so where I am now is that I am going to start throwing stuff at it and see that if I can get something that I want… (start writing random things on the board) that means p divides $a^2$</td>
<td>Executing</td>
</tr>
<tr>
<td>31:26-31:50</td>
<td>(Silence)</td>
<td>Stuck point 2</td>
</tr>
<tr>
<td>31:51-32:00</td>
<td>is $a/b$ as reduced as possible?</td>
<td>Planning</td>
</tr>
<tr>
<td>32:01-02:40 (Video 2)</td>
<td>Does that help? (looking at $p/a^2$) doesn’t help…. (Silence)</td>
<td>Stuck point 3 (major)</td>
</tr>
<tr>
<td>02:40-02:54</td>
<td>cause I am doing different things…. $a^2$ is congruence to 0 mod p, what does that mean….is there anything that I can backtrack and say about root p….or a for that matter…</td>
<td>Executing</td>
</tr>
<tr>
<td>02:55-04:33</td>
<td>Or I have issues to go from here…. (Silence)</td>
<td>Stuck point 4 (major)</td>
</tr>
<tr>
<td>Time</td>
<td>Activity</td>
<td>Notes</td>
</tr>
<tr>
<td>--------</td>
<td>----------------------------------------------------------------------------------------------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>04:33-05:49</td>
<td>(start writing on the board on the left side, ( pb^2 = a^2, \sqrt{pb} = a )) I am just trying to throw things on the board</td>
<td>Executing</td>
</tr>
<tr>
<td>05:50-07:51</td>
<td>No, that’s where I get started…(erases the whole thing that was just written) (Silence and looking at the board) eh…. I feel… I guess I am not quite sure what I am… trying… to do… right now…. I am just trying to do some operations on the stuff that I know to see if they give me any information… but I am not sure whether I am working towards a clear contradiction. Y: so what do you want to prove in the beginning?</td>
<td>Stuck point 5 (major)</td>
</tr>
<tr>
<td>07:58-08:29</td>
<td>I want to find some contradiction…. I guess…. my initial thought is that you assume a/b is the lowest term, but then you want to find some smaller b or some common divisors… so… eh…. Okay, wait… so for that…. I want to find something that a and b have in common….</td>
<td>Planning</td>
</tr>
<tr>
<td>08:54-09:14</td>
<td>p divides a, if p divides b then it will be done…. or p divides ( a^2 )</td>
<td>Executing</td>
</tr>
<tr>
<td>09:18-09:51</td>
<td>Oh, wait!…..no….. (Silence)</td>
<td>Stuck point 6</td>
</tr>
<tr>
<td>10:22-11:19</td>
<td>(talking to himself)</td>
<td>Monitoring</td>
</tr>
<tr>
<td>11:19-11:36</td>
<td>(Writes a doesn’t divide p, implies a divides ( b^2 )) Do I know anything about p?</td>
<td>Executing</td>
</tr>
<tr>
<td>11:39-12:22</td>
<td>I don’t…. (Silence)</td>
<td>Stuck point 7</td>
</tr>
<tr>
<td>12:51-15:32</td>
<td>(Writes on the board a divides p or a divides ( b^2 )) if p equals to 1, that doesn’t work…. (cross out a=1) (start writing ( pb^2 = a^2 ) on the top of the board, implies ( b = \pm 1 ), implies ( a = \sqrt{p} ) (cross implies ( b = \pm 1 ), implies ( a = \sqrt{p} ) which can’t be…. (cross out a divides p) (write a divides ( b^2 ), implies a doesn’t divide b)</td>
<td>Executing</td>
</tr>
<tr>
<td>16:12-19:31</td>
<td>this is shocking…. but…. Y: you got it?</td>
<td>Checking</td>
</tr>
</tbody>
</table>
I don’t think so, but I am making some assumptions…. 
$b^2 = p$, that’s where I am not sure….that means $b = \pm \sqrt{p}$? 
But b is an integer, we gonna assume $\sqrt{p}$ is not an integer 
because then p will not be prime….then implies b as 
an integer equals to some non integer, so a can not equal p. 
So that means the case divides p is out. So we are in the case 
of a divides $p^2$, if a divides $b^2$, that means some elements of 
the prime factorization of $b^2$...I guess what i think about is 
you see a is prime factorization of $b^2$. But you can’t because 
a and b are assumed to be relatively prime in the beginning, 
so a can not divide $b^2$. So that’s now a contradiction, 
yes, so that’s the contradiction!

Emma and Bob’s Transcript of Task 2 (Pilot Study Round 2 Task 2)

<table>
<thead>
<tr>
<th>Time</th>
<th>Transcript</th>
<th>Coding (Codes: Orienting Planning Executing Checking Monitoring Stuck Points)</th>
</tr>
</thead>
</table>
| 00:02-01:04| B: There is a rational definition and we can put that in our conversation. 
E: I think that we may want to use quadratic formula definition too. 
Y: Why do you think that? 
E: I just feel it...coz we get some square root in there...I am 
browsing all the definitions.... 
B: we know that...we don’t know what we have....what we 
learned before in class...in previous class, we don’t have a 
formal expression for irrational....but we have an expression 
for rational, because rational is some p over some q, and q 
can not be 0. So if we use a counterexample, or 
contradiction, we could say that, suppose square root p is 
rational then prove it is irrational by contradiction. That’s 
what I am thinking 
E: en.... Okay, so that’s the strategy we want to use....you 
wanna try that one first? 
B: Sure. | Planning                                                                                                                                   |
| 01:15-02:25| E: (Writing on the board while talking) So suppose square root p is rational....then as this form....then we can write 
$\sqrt{p}$ equal to some x over y where y can not equal to 0, and x 
and y are in the reals? 
B: Yeah.                                                                                                                                  | Executing                                                      |
B: Alright, so that then.
E: Let’s square.
B: No, square root.
E: Let’s square the square root….to make square root of p looks like p….so then this is what happening (point to the writing on the board)
B: So you get \( p^2 = x^2/y^2 \) …yeah….
E: And this equals \( x^2 \cdot (1/y^2) \)…..en….

03:06-03:50
E: But that doesn’t (pointing to p is prime)...
B: Wait…something is wrong…
E: I feel that we want to show that…like here (point to \( p^2 = x^2/y^2 \)), we have this factorization of p…we know \( x^2 \) is going to be….no, we don’t know that…they are integers…that would be nice they would…
(Silence)

03:52-04:27
E: I was thinking that if we could show there is another factorization of p that has p and 1, and that will means p is not prime which will be our contradiction….but we want these (pointing to \( x^2 \cdot (1/y^2) \)) to be integer values… but not guarantee that…
B: right…
B: But we know that p is prime which we don’t need to show…. 
E: I thought that’s what you want to contradict….
B: (Pointing to sqr(p) is irrational) I want to contradict to this…
E: Yes…If you suppose p is rational, then you come to something that doesn’t work, that’s a contradiction…
B: right…
E: So we have this p that has factorization that is not p and 1 (explaining to Bob)….then our assumption is wrong….

04:57-05:50
B: But like you said, we don’t know if this is integers ((pointing to \( x^2 \cdot (1/y^2) \))...
E: So i think there are something we need to think a bit differently…. (looking at definitions again)

05:51-06:43
B: I am thinking….we want to show that it is p time something else right….no never mind….I was thinking maybe we can multiply \( y^2 \) on both sides (write on the board)…..
E: so \( py^2 = x^2 \)…..but we want…

06:49-11:31
(Silence)
<table>
<thead>
<tr>
<th>Time</th>
<th>Event</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>11:43-12:11</td>
<td>E: Do we have a statement that we can write ( p ) is rational....not really....we can have 3, ( \frac{1}{3} ) and 9....3 is prime so it is not a factorization of this....this is my only issue with that....we want to find a integer...</td>
<td>Executing</td>
</tr>
<tr>
<td>12:12-14:05</td>
<td>(Silence)</td>
<td>Stuck Point 4</td>
</tr>
<tr>
<td>14:25-14:52</td>
<td>E: contradiction....contrapositive....(looking at different strategies)....assume ( \sqrt{p} ) is rational, then show ( p ) is not prime. You want to try that? (erase everything out)</td>
<td>Planning (Heuristic)</td>
</tr>
<tr>
<td>15:23-17:30</td>
<td>B: so we can do the same thing.....where.....E: (write everything same as before until ( p^2 = x^2 / y^2 )....it is the same issue....)</td>
<td>Executing</td>
</tr>
<tr>
<td>17:31-23:25</td>
<td>(Silence)</td>
<td>Stuck Point 5</td>
</tr>
</tbody>
</table>
Appendix G

Livescribe Data Transcript and Coding Example
Nikki’s Transcript of Task 1

**Proposition:** Suppose $a, b, c \in \mathbb{Z}$ s.t. $a^2 + b^2 = c^2$. Prove that either $a$ or $b$ is even.

**Proof:** Suppose $a$ AND $b$ are odd. Then let $a = 2x+1$ and $b = 2y+1$ for $x, y \in \mathbb{Z}$.

Then we get

$$a^2 + b^2 = (2x+1)^2 + (2y+1)^2$$

$$= 4x^2 + 4x + 1 + 4y^2 + 4y + 1$$

$$= 2(2x^2 + 2x + 2y^2 + 2y) + 2$$

Thus, $a^2 + b^2$ is even.

**Writing Time:** 0:00-1:50

Suppose $a, b, c \in \mathbb{Z}$ s.t. $a^2 + b^2 = c^2$. Prove that either $a$ or $b$ is even.

**Speaking Time:** 1:53-2:20

“I am thinking the best way to go about it is to do the contrapositive. It seems easier to determine the contrapositive of $a$ or $b$ is even and work backwards to prove $a^2 + b^2 \neq c^2$.”

**Coding:** Orienting

**Planning:** (Heuristic)
<table>
<thead>
<tr>
<th>Time</th>
<th>Writing</th>
<th>Speaking</th>
<th>Coding</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:23-3:30</td>
<td>Proof: [contrapositive] Suppose $a$ and $b$ are odd. Then let $a = 2x + 1$ and $b = 2y + 1$ for $x, y \in \mathbb{Z}$. Then we get $a^2 + b^2 = (2x + 1)^2 + (2y + 1)^2$</td>
<td>Same as writing</td>
<td>Executing</td>
</tr>
<tr>
<td>3:31-3:41</td>
<td>$(2x + 1) \cdot (2x + 1)$</td>
<td>Same as writing</td>
<td>Executing</td>
</tr>
<tr>
<td>3:42-4:26</td>
<td>$= 4x^2 + 4x + 1 + 4y^2 + 4y + 1$</td>
<td>&quot;</td>
<td>Executing</td>
</tr>
<tr>
<td></td>
<td>$= 4x^2 + 4x + 4y^2 + 4y + 2$</td>
<td>= $4x^2 + 4x + 1 + 4y^2 + 4y + 2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$= 2(2x^2 + 2x + 2y^2 + 2y)$</td>
<td>and from there we can factor out the 2... = $2(2x^2 + 2x + 2y^2 + 2y)$</td>
<td></td>
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<tr>
<td>Time</td>
<td>Writing</td>
<td>Speaking</td>
<td>Coding</td>
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<td>-------------------------------------------------------------------------</td>
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</tr>
<tr>
<td>4:27-4:58</td>
<td>None</td>
<td>“so $c^2$ would have been even, but…I don’t know how that play into the question overall…”</td>
<td>Stuck Point</td>
</tr>
<tr>
<td>4:59-5:10</td>
<td>Since $c^2$ is even (then cross the whole sentence out)</td>
<td>Same as writing</td>
<td>Executing (Monitoring)</td>
</tr>
<tr>
<td>5:11-5:40</td>
<td>Since $2x^2 + 2x + 2y^2 + 2y \in \mathbb{Z}$, it follows that $c^2$ is even when $a^2$ and $b^2$ are odd.</td>
<td>Same as writing</td>
<td>Executing (Heuristic)</td>
</tr>
<tr>
<td>5:42-6:22</td>
<td>None</td>
<td>“would that just be the end of the proof then…I have no idea”</td>
<td>Stuck Point</td>
</tr>
<tr>
<td>6:23-7:12</td>
<td>None</td>
<td>None</td>
<td>Stuck Point</td>
</tr>
</tbody>
</table>

Since $2x^2 + 2x + 2y^2 + 2y \in \mathbb{Z}$, it follows that $c^2$ is even when $a^2$ and $b^2$ are odd.

Thus, in the original proof $c$ would be odd, and either $a$ or $b$ (but not both) must be even ($2$ odds make an even, $2$ evens make an even, and $1$ odd and $1$ even make an odd).

$2$ odds make even: Let $n \in \mathbb{Z}$ such that $n = 2x + 1$, and let $p \in \mathbb{Z}$ such that $p = 2y + 1$.

Contradiction: Suppose $a, b, c \in \mathbb{Z}$.

$a^2 + b^2 = c^2$. We must show that $a$ and $b$ are odd.

$2^2 + 2^2 = 2^2$. We must show that $2x^2 + 2y^2 + 2y$ is even.

Since $x + y \in \mathbb{Z}$, it follows that $n + p$ is even.
<table>
<thead>
<tr>
<th>Time</th>
<th>Writing</th>
<th>Speaking</th>
<th>Coding</th>
</tr>
</thead>
<tbody>
<tr>
<td>7:13-8:10</td>
<td>Thus, in the original proof $c^2$ would be odd, and wither a or b (but not both) must be even (2 odds make an even, 2 evens make an even, and 1 odd and 1 even make an odd).</td>
<td>“Ok, so from there we could say… Thus, in the original proof $c^2$ would be odd, and wither a or b (but not both) must be even. Because two odds make an even, 2 evens make an even, and 1 odd and 1 even make an odd.”</td>
<td>Executing</td>
</tr>
<tr>
<td>8:12-9:25</td>
<td>2 odds make even: let $n$ exist in integers such that $n = 2x + 1$, and let $p \in \mathbb{Z}$ such that $p = 2y + 1$. $n + p = 2x + 1 + 2y + 1 = 2x + 2y + 2 = 2(x + y)$, since $x, y \in \mathbb{Z}$, it follows that $n + p$ is even. Write 2 even (then crossed out)</td>
<td>Same as writing</td>
<td>Planning</td>
</tr>
<tr>
<td>9:25-11:29</td>
<td>None</td>
<td>“If you see the contrapositive of the contrapositive, you put it back…you get an odd $c^2$, which implies that a or b must be even because in the contrapositive we assume both of them must be odd…”</td>
<td>Executing</td>
</tr>
<tr>
<td>11:32-11:55</td>
<td>Write a square next to $a^2$ and $b^2$ are odd.</td>
<td>“this is where I gonna end the proof”</td>
<td>Executing</td>
</tr>
<tr>
<td>Time</td>
<td>Writing</td>
<td>Speaking</td>
<td>Coding</td>
</tr>
<tr>
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<td>--------------------------------------------------------------------------------------------</td>
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<tr>
<td>11:55-12:02</td>
<td></td>
<td>Y: “Can you clarify your thinking for me, how is the $c^2$ is even help you to make the conclusion that you have done the proof?”</td>
<td></td>
</tr>
<tr>
<td>12:05-13:14</td>
<td>None</td>
<td>None</td>
<td>Stuck Point</td>
</tr>
<tr>
<td>13:15-13:20</td>
<td>None</td>
<td>“I don’t think it does… I still don’t see the relation of $c^2$ been even or odd and…”</td>
<td>Stuck Point</td>
</tr>
<tr>
<td>13:21-14:29</td>
<td>None</td>
<td>None</td>
<td>Stuck Point</td>
</tr>
<tr>
<td>14:30-15:17</td>
<td>None</td>
<td>“I might have just misunderstand the question…I don’t know how you go about proving a or b is even when you have $a, b, c \in \mathbb{Z} s.t. a^2 + b^2 = c^2$, I am not seeing the relation between one statement of the other…”</td>
<td>Stuck Point</td>
</tr>
<tr>
<td>15:18-16:50</td>
<td>None</td>
<td>“I don’t think I can do it…”</td>
<td>Stuck Point</td>
</tr>
<tr>
<td>Time</td>
<td>Writing</td>
<td>Speaking</td>
<td>Coding</td>
</tr>
<tr>
<td>------------</td>
<td>---------</td>
<td>--------------------------------------------------------------------------</td>
<td>--------------</td>
</tr>
<tr>
<td>16:52-18:08</td>
<td>None</td>
<td>“I probably did this question way wrong…possibly should use contradiction…”</td>
<td>Planning</td>
</tr>
<tr>
<td>18:09-18:18</td>
<td>None</td>
<td>“I think it might be easier to go about using contradiction…”</td>
<td>Planning</td>
</tr>
<tr>
<td>18:19-18:28</td>
<td>None</td>
<td>“Well, let’s see. If we say it is true then…we must show a and b are odd…but”</td>
<td>Planning</td>
</tr>
<tr>
<td>18:29-20:24</td>
<td>None</td>
<td>None</td>
<td>Stuck Point</td>
</tr>
<tr>
<td>20:25-21:06</td>
<td>None</td>
<td>“I still don’t think I know how I will go about it even if I am using contradiction…I don’t see the relation that I can contradict…still the same problem when I was using contrapositive.”</td>
<td>Stuck Point</td>
</tr>
<tr>
<td>21:14-21:54</td>
<td>Suppose</td>
<td>Suppose $a, b, c \in \mathbb{Z}$ s.t. $a^2 + b^2 = c^2$. We must show that a and b are odd.</td>
<td>Planning</td>
</tr>
<tr>
<td></td>
<td></td>
<td>“If I take the contradiction, then we get: suppose $a, b, c \in \mathbb{Z}$ s.t. $a^2 + b^2 = c^2$. We must show that a and b are odd.”</td>
<td></td>
</tr>
<tr>
<td>21:55-22:54</td>
<td>None</td>
<td>“I don’t know how I will show a and b are odd when the only known relation is $a^2 + b^2 = c^2$.”</td>
<td>Stuck Point</td>
</tr>
</tbody>
</table>
Appendix H

Pilot Study Case Analysis Results
For the last two rounds of the pilot study, both tasks I chose to use were in the domain of number theory. I retained this focus on my dissertation study design. Initially, I used Carlson and Bloom (2005)’s multidimensional problem-solving framework to try to analyze these data. I then decided to use “stuck points” as important points to devote analytic attention to in characterizing students’ proving processes. Below, I will discuss two cases from the pilot, selected intentionally to be one interview in which the student got stuck but did succeed in navigating out and one interview in which the students got stuck, but were not able to navigate out of the stuck point.

The case of navigating out

The student in the first case is called Paul. Paul is a senior, mathematics and computer science major. Paul has taken Math 3140 in his sophomore year. Paul has successfully solved both tasks in the end. The first task went pretty smooth for him, but he got stuck multiple times when doing the second task when asking to prove or disprove the square root of any prime is irrational. Based on Carlson and Bloom (2005)’s problem solving phases, Paul got stuck at the executing stage when he tried to find a way to show “$p = \frac{a^2}{b^2}$”. After several tries without progressing, Paul started to cycle back to the planning stage; he wondered he might have done something wrong in the beginning which might be the possible reason for his stuckness. He went back and did the planning stage again without changing the initial strategy he used, prove by contradiction, but soon got stuck again at the same place. He spent around 10 mins writing out everything that could be related to “$p = \frac{a^2}{b^2}$” including trying to change it to different expression through algebraic manipulation, but none of his work got him anywhere. He decided to go back to the planning phase again, but this time started from the beginning by reading the task again.
He told me that he was looking for the goal of his proof while reading. He reminded himself that he should prove “\( p \neq \frac{a^2}{b^2} \)” instead of “\( p = \frac{a^2}{b^2} \)” because he is trying to use proof by contradiction and he already knows \( p \) is prime. From here, he found a clear path to go and with some minor stuck points trying to show \( p \) doesn’t divide \( a^2 \), he successfully overcame after several stuck points.

Using Carlson and Bloom (2005)’s four problem solving phases and adding the stuck point stages, Paul’s proving process for the second task can be captured as follows (detailed coding can be found in Appendix F):

<table>
<thead>
<tr>
<th>Transcript</th>
<th>Proving Phases</th>
</tr>
</thead>
<tbody>
<tr>
<td>29:07-29:54</td>
<td>Planning</td>
</tr>
<tr>
<td>29:55-30:35</td>
<td>Executing</td>
</tr>
<tr>
<td>30:40-31:00</td>
<td>Stuck point 1 (minor)</td>
</tr>
<tr>
<td>31:01-31:25</td>
<td>Executing</td>
</tr>
<tr>
<td>31:26-31:50</td>
<td>Stuck point 2 (minor)</td>
</tr>
<tr>
<td>31:51-32:00</td>
<td>Planning</td>
</tr>
<tr>
<td>32:01-02:40 (end of first video)</td>
<td>Stuck point 3 (major)</td>
</tr>
<tr>
<td>02:40-02:54</td>
<td>Executing</td>
</tr>
<tr>
<td>02:55-04:33</td>
<td>Stuck point 4 (major)</td>
</tr>
<tr>
<td>04:33-05:49</td>
<td>Executing</td>
</tr>
<tr>
<td>05:50-07:51</td>
<td>Stuck point 5 (major)</td>
</tr>
<tr>
<td>07:58-08:29</td>
<td>Planning</td>
</tr>
<tr>
<td>08:54-09:14</td>
<td>Executing</td>
</tr>
<tr>
<td>09:18-09:51</td>
<td>Stuck point 6 (minor)</td>
</tr>
<tr>
<td>10:22-11:19</td>
<td>Monitoring</td>
</tr>
</tbody>
</table>
I claim a stuck point happens when a student is silent or not taking any actions for at least 20 seconds. However, when coding the data, I realize that there are differences between different “stuck points.” There are ones where students got stuck for a short period of time and soon have some idea in mind, so during those short periods of less than a minute or two, I code as minor “stuck points.” For the ones that students took more than few minutes and still didn’t take any actions or still didn’t talk in that long period of time, I code those as major “stuck points.”

From what I saw of his proving process, I saw several problems when analyzing Paul’s proving process using Carlson and Bloom (2005)’s framework, which we will summarize later after the discussion of the second case.

*The case that didn’t navigate out*

The pair of students in the second case are Emma and Bob. Emma and Bob are both seniors, Secondary Math Education Majors, they became friends after taking many classes together. Both of them had taken Math 3140 last year when they were Juniors. They were willing to be interviewed together and work together for the two tasks. Emma and Bob also successfully solved the first task, however, they also got stuck multiple times when doing the second task. Emma and Bob, same as Paul, also got stuck at the executing stage when they tried to prove “$p = \frac{a^2}{b^2}$”, which is the key to lead to the next steps. They tried different ways of changing the equation algebraically but couldn’t go far. Then Emma went back to the statements they wrote in
the beginning and double checked those statements to make sure they were correct so far while Bob was still trying different algebraic manipulations. Once Emma told Bob that everything they did so far seemed to be correct, they then ran into a major stuck point. During those 5 minutes of stuck points, Emma was looking at all of the possible definitions and theorems that they could apply to this situation while Bob was looking at the board the whole time. Then, Emma decided to erase everything they did before and told Bob that they might have used the wrong strategy or approach. So they decided to start from the beginning and use contrapositive instead of contradiction this time. After a discussion about what they need to prove for contrapositive and going over the same steps as before again, they still stopped at the place when they got “$p = \frac{a^2}{b^2}$.” They then saw that they went back to the same place where they were stuck for the first time and decided they couldn’t prove this task.

Using Carlson and Bloom (2005)’s four problem solving phases and adding to the stuck point stages, Emma and Bob’s proving process for the second task can be captures as follow (detailed coding can be found in Appendix F):

<table>
<thead>
<tr>
<th>Transcript</th>
<th>Proving Phases</th>
</tr>
</thead>
<tbody>
<tr>
<td>00:02-01:04</td>
<td>Planning</td>
</tr>
<tr>
<td>01:15-02:25</td>
<td>Executing</td>
</tr>
<tr>
<td>03:06-03:50</td>
<td>Stuck Point 1 (minor)</td>
</tr>
<tr>
<td>03:52-04:27</td>
<td>Executing</td>
</tr>
<tr>
<td>04:57-05:50</td>
<td>Stuck Point 2 (minor)</td>
</tr>
<tr>
<td>05:51-06:43</td>
<td>Executing</td>
</tr>
<tr>
<td>06:49-11:31</td>
<td>Stuck Point 3 (major)</td>
</tr>
<tr>
<td>11:43-12:11</td>
<td>Executing</td>
</tr>
</tbody>
</table>
We see Paul ended with the normal checking phases as described by Carlson and Bloom (2005), but Emma and Bob ended at a “stuck point” phase. There also some other problems I run into when trying to characterize students’ proving processes using the existing Carlson and Bloom (2005)’s framework.

“Productive” actions vs “Unproductive” actions

To have a “productive” stuck point, there are two characteristics that I claim. First, the students’ previous argument should be based on correct mathematical ideas. If the student got stuck because of the incorrect arguments previously, the stuck point is has become unproductive. Second, the stuck point is considered to be productive if the student navigates it out. Thus, even if the students’ previous arguments are all based on correct mathematical ideas, if that student never navigates it out it won’t be considered to be “productive”.

Thus, I will say some of Paul’s stuck points are “productive” since he navigated it out in the end after some of stuck points. What might be possible incidences that lead to this “productiveness”? What are some of the actions/approaches that Paul took are “productive”? On the other hand, we will say some of Emma and Bob’s stuck points are not productive since they didn’t navigate it out in the end. What might be possible incidences that lead to this

<table>
<thead>
<tr>
<th>Time</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>12:12-14:05</td>
<td>Stuck Point 4 (major)</td>
</tr>
<tr>
<td>14:25-14:52</td>
<td>Planning</td>
</tr>
<tr>
<td>15:23-17:30</td>
<td>Executing</td>
</tr>
<tr>
<td>17:31-23:25</td>
<td>Stuck Point 5 (major)</td>
</tr>
</tbody>
</table>

Table 3.3 Emma and Bob’s Proving Processes
“unproductiveness”? Or were Emma and Bob very close to be “productive”? Let look at what
did each student do when they get stuck.

When Paul got stuck, he first tried different algebraic manipulations to see whether that
can help to get him further based on what he already has. When that didn’t work out, he then
decided to go back to verify whether he did all of the previous steps correct. Based on two
criteria of “productive” the first one is that all of the previous mathematical statements are
correct. Once he verified that he was correct so far, he wrote down all of the relevant algebraic
manipulations and concepts, but it still didn’t have him go any further. So he went back to the
planning stage again, but this time, he re-read the question and asked himself what he was trying
to prove. By asking this question to himself, he realized that he was actually doing the proof by
contradiction, so instead of proving $p = \frac{a^2}{b^2}$, if he can show “$p \neq \frac{a^2}{b^2}$”, then it would contradict
the assumption he made in the beginning. From here, he found a clear path to go and with some
minor stuck points trying to show p doesn’t divides $a^2$, he successfully navigated out after
several stuck points.

If we try to characterize the actions or approaches that Paul took to try to get out of the
stuck point, we can summarize them as: try different algebraic manipulation, make sure previous
steps are correct, write down related manipulations and concepts, re-read the problem and clarify
the goal of the proof. In this case, make sure previous steps are correct and re-read the problem
and clarity the goal of the proof seems to be “productive” for Paul.

When Emma and Bob got stuck, they first tried different ways of changing the equation
algebraically but couldn’t go far. Then Emma went back to the statements they wrote in the
beginning and double checked those statements to make sure they were correct so far while Bob
was still trying different algebraic manipulations. After verifying the previous steps were correct,
Emma looked into all of the possible definitions and theorems that they can apply to this situation. But that still didn’t really help them. Thus, they decided to start from the beginning and use a different proof strategy. But the new strategy still didn’t help them to overcome the stuck point of \( p = \frac{a^2}{b^2} \).

If we try to characterize the actions or approaches that Emma and Bob took to try to get out of the stuck point, we can summarize them as: try different algebraic manipulation, look into related definitions or theorems, make sure previous steps are correct, change to a different prove strategy. In this case, make sure previous steps are correct seems to be the only “productive” actions for Emma and Bob.