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Dominating Functions in Graphs

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DOMINATING FUNCTIONS IN GRAPHS

by

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DOMINATING FUNCTIONS IN GRAPHS

Maria Talanda-Fisher, Ph.D.

Western Michigan University, 2021

Domination in graphs has become one of the most popular areas of graph theory, no doubt due to its many fascinating problems and applications to modern society, as well as the sheer mathematical beauty of the subject. While this area evidently began with the work by the French mathematician Claude Berge in 1958 and the Norwegian-American mathematician Oystein Ore in 1962, domination did not become an active area of research until 1977 with the appearance of the survey paper by Ernest Cockayne and Stephen Hedetniemi. Since then a large number of variations of domination have surfaced and provided numerous applications to different areas of science and real-life problems. In 1987, Hedetniemi introduced the concept of dominating functions which (a) provided an analytic method of studying this discrete concept, (b) built a connection between domination and graph labelings and colorings, and (c) gave rise to new dominating function parameters. In 2019, Gary Chartrand introduced several variations of dominating functions in connection with some of the best-known concepts in graphs, including irregularity, regularity and chromaticity in graphs. In this research, we study irregular, antiregular, regular and proper dominating functions as well as total dominating functions in graphs. We present structural and extremal results dealing with these domination parameters and establish relationships between the concepts of various dominating functions and some well-known parameters in graphs.

ACKNOWLEDGMENTS

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INTRODUCTION

1.1 A Chessboard Problem

Domination in graphs was not formally defined until the 1960s; however, the concept originated a century earlier with a chessboard problem. A chesspiece is said to *attack* (dominate) any square on a chessboard that it can reach in a single move. In a standard 8×8 chessboard shown in Figure 1.1, the queen is the only chesspiece that can move any number of squares horizontally, vertically or diagonally. So the queen can attack or capture each vacant square that can be reached by moving the queen in one of these directions.

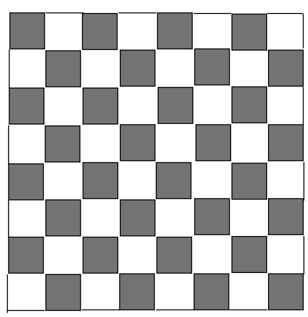

Figure 1.1: A standard 8×8 chessboard

In 1862 Carl Friedrich de Jaenisch [10] looked at determining the minimum number of queens that can be placed on a chessboard so that every square is occupied or attacked (or dominated) by at least one of these queens. It is known that there is no way to place four queens on the squares of a chessboard so that every vacant square can be captured by a queen. The following problem asks whether this can be done with five.

The Five Queens Puzzle *Can five queens be placed on an* 8×8 *chessboard so that every vacant square can be captured by at least one of these queens?*

The answer is *yes*. Figure 1.2 shows one of the solutions to the Five Queens Puzzle. This chessboard problem can be considered the origin of the topic of dominating sets in graphs (see [3, 76], for example).

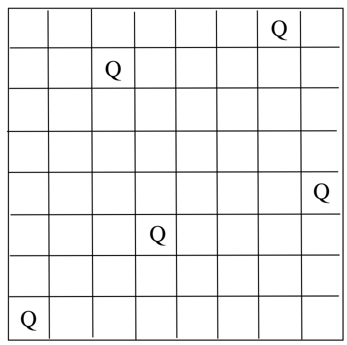

Figure 1.2: A solution to the Five Queens Problem

1.2 Domination in Graphs

In recent decades, domination in graphs has become one of the most popular areas of graph theory, no doubt due to its many fascinating problems and applications to modern society, as well as the sheer mathematical beauty of the subject.

The area of domination in graph theory evidently began with the French mathematician Claude Berge. In 1958 Berge wrote *Théorie des Graphes et Ses Applications* [19], often considered the second book ever written on graph theory. In this book, Berge was first to define the concept of the domination number of a graph (although he didn't use this terminology).

Four years later in 1962, the Norwegian-American mathematician Oystein Ore wrote *Theory of Graphs* [85], the first graph theory book written in English. In this book he introduced the terminology *dominating set* and *domination number*. Ore is one of the best known mathematicians of the mid 20th century and he later became interested in graph theory. Ore once wrote:

> *The theory of graphs is one of the few fields of mathematics with a definite birth date.*

The birth date he was referring to was 1736. This was the date of what is considered the first paper in graph theory, wriiten by Leonhard Euler [11].

Berge and Ore made domination a formal theoretical area of graph theory, but domination was not an active area until the appearance of the survey paper "Towards a theory of domination in graphs" by Ernest Cockayne and Stephen Hedetniemi [9] in 1977. Recently, Haynes, Hedetniemi, and Henning wrote three new books devoted to domination.

The topic of domination is based on a very simple definition: A vertex v in a graph G is said to *dominate* a vertex u if either $u = v$ or $uv \in E(G)$. That is, a vertex v dominates the vertices in its *closed neighborhood* $N[v] = N(v) \cup \{v\}$. A set S of vertices in G is a *dominating set* of G if every vertex of G is dominated by at least one vertex in S. The minimum number of vertices in a dominating set of G is the *domination number* $\gamma(G)$ of G. A dominating set of cardinality $\gamma(G)$ is called a minimum dominating set.

For example, $S_1 = \{r, u, v, x\}$ and $S_2 = \{t, w, z\}$ are dominating sets for the graph H of order 9 in Figure 1.3. In fact, S_2 is a minimum dominating set of H and so $\gamma(H) = 3$.

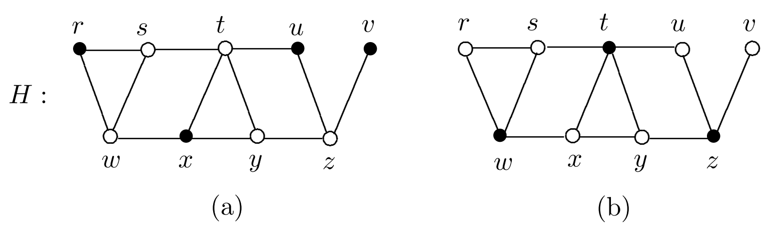

Figure 1.3: Dominating sets in a graph

1.3 Total Domination in Graphs

Over the years many variations of domination have surfaced due to numerous applications to real-life problems. One of the best known variations is total domination, which was introduced by Cockayne, Dawes and Hedetniemi [78]. In this variation, a vertex u dominates a vertex v only if v is a neighbor of u. Hence, a vertex does not totally dominate itself.

We now present this concept formally. A set S of vertices in a graph G is a *total dominating set* for G if every vertex of G is adjacent to some vertex of S. Therefore, a graph G has a total dominating set if and only if G contains no isolated vertices. Furthermore, if S is a total dominating set of G, then the subgraph $G[S]$ induced by S contains no isolated vertices. The minimum cardinality of a total dominating set for G is the *total domination number* $\gamma_t(G)$ of G. A total dominating set of cardinality $\gamma_t(G)$ is called a *minimum total dominating set* or a γ_t*-set* for G. Since every total dominating set of a nontrivial connected graph is also a dominating set, it follows that $\gamma(G) \leq \gamma_t(G)$ for every nontrivial connected graph G. For example, it is known that for each integer $n \geq 3$,

$$\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil.$$

A minimum dominating set in C_n for $6 \leq n \leq 8$ is shown in Figure 1.4.

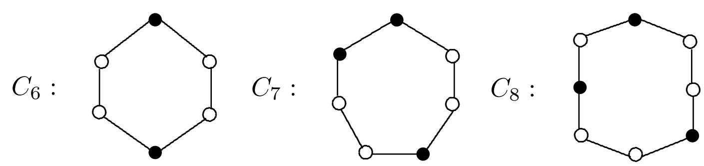

Figure 1.4: A minimum dominating set in C_n for $6 \leq n \leq 8$

On the other hand, it is known that for each integer $n \geq 3$,

$$\gamma_t(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 2 \ (mod\ 4) \\ \frac{n+2}{2} & \text{if } n \equiv 2 \ (mod\ 4). \end{cases}$$

Notice in C_6 in Figure 1.4, the filled in vertices are not adjacent to other filled in vertices so the dominating set shown above is not a total dominating set. A minimum total dominating set in C_n for $6 \leq n \leq 8$ is shown below

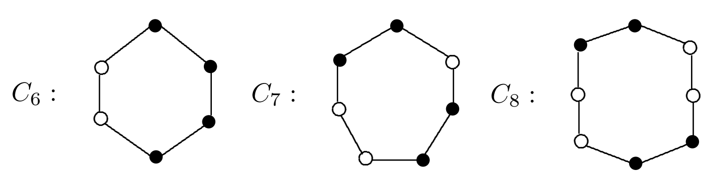

Figure 1.5: A minimum total dominating set in C_n for $6 \leq n \leq 8$

1.4 A Different View of Domination

There is another way that domination and the domination number of a graph G has been looked at. Each function $f : V(G) \to \{0, 1\}$ gives rise to another function

$$c_f : V(G) \to \mathbb{N} \cup \{0\},$$

where $\mathbb{N}$ is the set of positive integers, defined by

$$c_f(v) = \sum_{u \in N[v]} f(u).$$

If $c_f(v) \in \mathbb{N}$ for every vertex v of G, then f is a *dominating function* of G. This concept was formally introduced by Hedetniemi in 1987 (see [76], for example).

If S is a dominating set of G, then the function $f : V(G) \to \{0, 1\}$ defined by $f(v) = 1$ if $v \in S$ and $f(v) = 0$ if $v \in V(G) - S$ is a dominating function of G. For example, the dominating sets S_1 and S_2 of the graph H in Figure 1.3 give rise to the dominating functions f_1 and f_2 defined on the graph H, as shown in Figure 3.2.

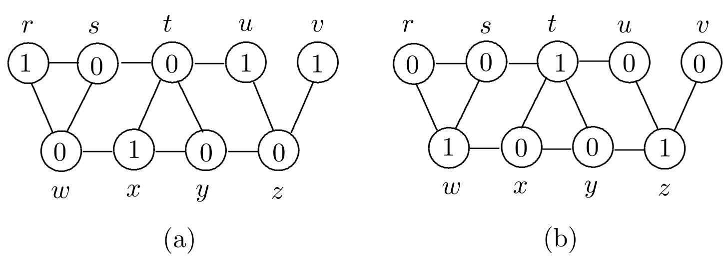

Figure 1.6: Two dominating functions of a graph H

If f is a dominating function of a graph G, then the set

$$\mathcal{I}_f(G) = \{v \in V(G) : f(v) = 1\}$$

is a dominating set of G. If we define the *domination number* $\gamma(f)$ *of a dominating function* f of a graph G as

$$\gamma(f) = \sum_{v \in V(G)} f(v) = |\mathcal{I}_f(G)|,$$

then the domination number $\gamma(G)$ of G can be defined as

$$\gamma(G) = \min\{\gamma(f) : f \text{ is a dominating function of } G\}.$$

Dominating functions have been studied extensively in many research papers. In fact, studying domination by means of dominating functions provides an analytic method to study this discrete concept and gives rise to new dominating function parameters.

In 2019, Gary Chartrand introduced several variations of dominating functions in connection with some of the best-known concepts in graphs, including irregularity, regularity and chromaticity in graphs (see [79, 80, 81]). In this work, we study structural and extremal problems in various dominating functions as well as explore relationships among these domination concepts and traditional domination concepts.

We refer to the books [77, 82] for graph theory notation and terminology not described in this work.

Chapter 2

IRREGULAR & ANTIREGULAR DOMINATING FUNCTIONS

ABSTRACT: We use dominating functions to investigate graphs whose vertices are all dominated by as many different number of vertices as possible. More precisely, a dominating function f of a nontrivial graph G is an irregular dominating function if $c_f(u) \neq c_f(v)$ for every two vertices u and v of G. We show that no graph possesses an irregular dominating function. A dominating function f of a nontrivial graph G of order n is called an antiregular dominating function if there are exactly two vertices u and v of G such that $c_f(u) = c_f(v)$. We show that every antiregular graph has an antiregular dominating function. Furthermore, we prove that for every integer $n \geq 2$, there are exactly $n - 1$ non-isomorphic connected graphs of order n having an antiregular dominating function.

2.1 Introduction

First, we review some basic definitions and notation involving domination and dominating functions in graphs. A vertex v in a graph G is said to *dominate* a vertex u if either $u = v$ or $uv \in E(G)$. That is, a vertex v dominates the vertices in its *closed neighborhood* $N[v] = N(v) \cup \{v\}$. A set S of vertices in G is a *dominating set* of G if every vertex of G is dominated by at least one vertex in S. The minimum number of vertices in a dominating set of G is the *domination number* $\gamma(G)$ of G.

We use another way to look at domination and the domination number of a graph G (see [76]). Let $f : V(G) \to \{0,1\}$ be a function. Then f gives rise to another function

$$c_f : V(G) \to \mathbb{N} \cup \{0\},$$

where $\mathbb{N}$ is the set of positive integers, defined by

$$c_f(v) = \sum_{u \in N[v]} f(u).$$

If $c_f(v) \in \mathbb{N}$ for every vertex v of G, then f is a *dominating function* of G. If f is a dominating function of G, then the set

$$\mathcal{I}_f = \{v \in V(G) : f(v) = 1\}$$

is a dominating set of G. The *domination number* $\gamma(f)$ *of a dominating function* f of a graph G is

$$\gamma(f) = \sum_{v \in V(G)} f(v) = |\mathcal{I}_f|$$

and so the domination number $\gamma(G)$ of G can be defined as

$$\gamma(G) = \min\{\gamma(f) : \ f \text{ is a dominating function of } G\}.$$

2.2 Irregular Graphs and Irregular Dominating Functions

It is graph theory folklore that for every integer $n \geq 2$, there is no graph of order n all of whose vertices have distinct degrees. These non-existent graphs were first looked at formally in [18], where they were called *perfect*. Since the term "perfect" as used by Berge [20] had an entirely different meaning and became common in the literature, perfect graphs in [18] have become known as *irregular graphs*.

Theorem 2.2.1 *For every integer $n \geq 2$, there is no irregular graph of order n.*

This concept of irregularity can be applied to dominating functions. A dominating function $f : V(G) \to \{0,1\}$ of a nontrivial graph G is an *irregular dominating function* if $c_f(u) \neq c_f(v)$ for every two vertices u and v of G. Not only is no graph irregular, but no graph possesses an irregular dominating function. For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. For two integers a and b with $a < b$, let $[a, b] = \{a, a+1, \ldots, b\}$.

Proposition 2.2.2 *No nontrivial connected graph has an irregular dominating function.*

Proof. Assume, to the contrary, that there is a connected graph G of order $n \geq 2$ having an irregular dominating function $f : V(G) \to \{0, 1\}$. Since $1 \leq c_f(v) \leq n$ for every vertex v of G, it follows that $\{c_f(v) : v \in V(G)\} = [n]$. Hence, G has a vertex x of degree $n-1$ such that $f(y) = 1$ for each $y \in N[x]$. However, this implies that $c_f(u) \geq 2$ for every vertex u of G and so $|\{c_f(v) : v \in V(G)\}| \leq n-1$, which is a contradiction. ■

2.3 Antiregular Graphs

Since no graph possesses an irregular dominating function, we will investigate a closely-related type of dominating function. First, we provide some background information in this section. For a graph G, the *degree set* of G is $\mathcal{D}(G) = \{\deg v : v \in V(G)\}$. By Theorem 2.2.1, for each integer $n \geq 2$, there is no graph G of order n whose degree set $\mathcal{D}(G)$ satisfies $|\mathcal{D}(G)| = n$. In 1977, Kapoor, Polimeni, and Wall [84] proved that every set of positive integers is the degree set of some graph. We state this result next.

Theorem 2.3.1 *For every set $S = \{a_1, a_2, \ldots, a_k\}$ of positive integers such that*

$$a_1 < a_2 < \cdots < a_k,$$

there exists a graph G with $\mathcal{D}(G) = S$. Furthermore, the minimum order $\mu(S)$ of such a graph G is $\mu(S) = a_k + 1$.

The following is a consequence of Theorem 7.2.6.

Corollary 2.3.2 *For every two integers k and n with $1 \leq k \leq n-1$, there exists a graph G of order n with $|\mathcal{D}(G)| = k$. In particular, for the set $S = \{n-k, n-k+1, \ldots, n-1\}$, there exists a graph of order n whose degree set is S.*

By Corollary 2.3.2, for each integer $n \geq 2$, there is a graph G of order n such that $|\mathcal{D}(G)| = n-1$. In fact, if a graph G has this property, then its complement $\overline{G}$ has it as well. These graphs have been called many names, but here we refer to them as *antiregular graphs*. Formally, a nontrivial graph G is *antiregular* if exactly two vertices of G have the same degree. The following result is well known (see [77], for example).

Theorem 2.3.3 *For every integer $n \geq 2$, there are exactly two non-isomorphic antiregular graphs of order n, one of which is connected.*

In order to present a few examples of antiregular graphs, we first introduce some additional definitions. Let G be a graph with vertex set $V(G)$. Then the *complement* $\overline{G}$ of G has vertex set $V(G)$ and uv is an edge of $\overline{G}$ if and only if uv is not an edge of G. The *join* $G \vee H$ of two vertex-disjoint graphs G and H has $V(G \vee H) = V(G) \cup V(H)$ and

$$E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

Therefore, for a graph G of order n, the graph $G \vee K_1$ is the graph of order $n+1$ obtained by adding a new vertex v to G and joining v to each vertex of G.

We denote the connected antiregular graph of order n by G_n and so its complement is $\overline{G}_n$. The eight antiregular graphs of order n where $2 \leq n \leq 5$ and the connected antiregular graph G_6 of order 6 are shown in Figure 2.1. Observe that $G_6 = \overline{G}_5 \vee K_1$ is the join of $\overline{G}_5$ and K_1. In fact, $G_n = \overline{G}_{n-1} \vee K_1$ for each integer $n \geq 2$.

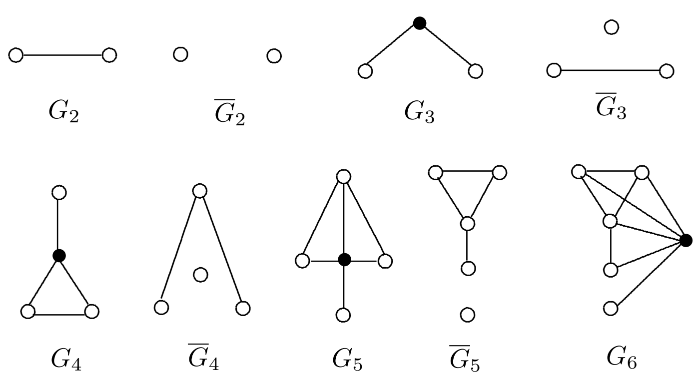

Figure 2.1: The antiregular graphs G_5, $\overline{G}_5$, and G_6

Several structural properties of antiregular graphs are described in [75].

2.4 Antiregular Dominating Functions

A dominating function $f : V(G) \to \{0, 1\}$ of a nontrivial graph G of order n is called an *antiregular dominating function* if $|\{c_f(v) : v \in V(G)\}| = n - 1$, that is, there are exactly two vertices u and v of G such that $c_f(u) = c_f(v)$. First, we make an observation.

Observation 2.4.1 *Let G be a graph of order n and maximum degree Δ. If $\Delta \leq n - 3$, then G does not have an antiregular dominating function.*

Proof. Let $f : V(G) \to \{0, 1\}$ be a dominating function of G. Since $1 \leq c_f(v) \leq \Delta + 1$ for every vertex v of G, it follows that

$$|\{c_f(v) : v \in V(G)\}| \leq \Delta + 1 \leq n - 2$$

and so f is not antiregular. ∎

It is readily seen that the 4-cycle C_4 of order 4 and maximum degree 2 has no antiregular dominating function, Consequently, the converse of Observation 2.4.1 is not true. It may not be surprising that every antiregular graph has a straightforward antiregular dominating function.

Proposition 2.4.2 *Every antiregular graph has an antiregular dominating function.*

Proof. For the unique connected antiregular graph G_n of order n, define the dominating function $f : V(G_n) \to \{0, 1\}$ by $f(v) = 1$ for every vertex v of G_n. Thus, $c_f(v) = 1 + \deg v$ for each $v \in V(G_n)$. Since G_n is antiregular, f is an antiregular dominating function of G_n. Defining the function f in the same manner for its complement $\overline{G}_n$ shows that f is also an antiregular dominating function of $\overline{G}_n$. ∎

There are, however, also non-antiregular graphs having an antiregular dominating function. For example, neither of the two graphs of Figure 2.2 is antiregular, but both have antiregular dominating functions, as shown in Figure 2.2. In fact, for every integer $n \geq 2$, we know the exact number of connected graphs of order n possessing an antiregular dominating function.

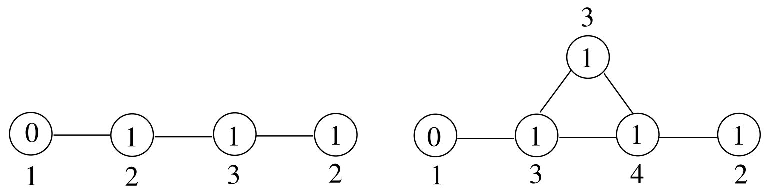

Figure 2.2: Two non-antiregular graphs with an antiregular dominating function

Theorem 2.4.3 *For each integer $n \geq 4$, there are exactly $n-1$ non-isomorphic connected graphs of order n having an antiregular dominating function, one graph of which is antiregular.*

Proof. It is readily seen that the three graphs of order 4 in Figure 2.3 have an antiregular dominating function (also shown in Figure 2.3), where one of these graphs is antiregular and no other connected graphs of order 4 have this property. Hence, we may assume that $n \geq 5$. Let H be a connected graph of order n having an antiregular dominating function f and let $\mathcal{C}_f = \{c_f(v) : v \in V(H)\}$. Since $1 \leq c_f(v) \leq n$ for every vertex v of H and $|\mathcal{C}_f| = n-1$, it follows that $\mathcal{C}_f = [n] - \{r\}$ for some integer r with $1 \leq r \leq n$. We consider two cases.

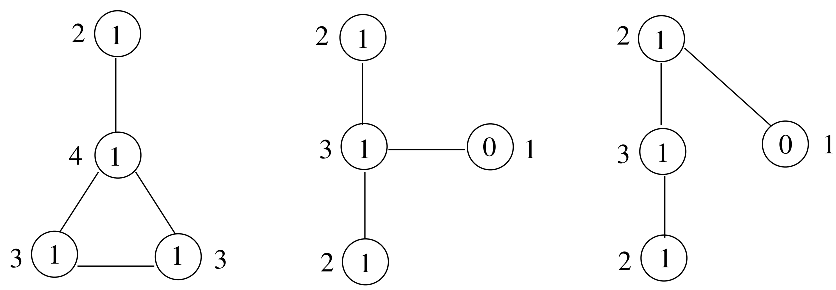

Figure 2.3: Three graphs of order 4 with an antiregular dominating function

Case 1. $1 \leq r \leq n-1$. Thus, there is a vertex u of H such that $c_f(u) = n$. Hence, u is adjacent to every vertex of H and $f(v) = 1$ for all vertices v of H. Therefore, $c_f(v) = 1 + \deg v$ for each vertex v of H. Since $|\mathcal{C}_f| = n-1$, it follows that

$$|\{\deg v : v \in V(H_n)\}| = n - 1.$$

Consequently, $H = G_n$ is the unique connected antiregular graph of order n.

Case 2. $r = n$. Hence, in this case, $\mathcal{C}_f = [n-1]$ and so there is a vertex u of H such that $c_f(u) = n-1$. We consider two possibilities, according to whether $\deg_H u = n-2$ or $\deg_H u = n-1$.

Subcase 2.1. $\deg_H u = n-2$. Since $c_f(u) = n-1$, it follows that $f(v) = 1$ for each vertex $v \in N[u]$. Let $V(H) - N[u] = \{w\}$. First, we claim that $f(w) = 0$, for if $f(w) = 1$, then $c_f(x) = 1 + \deg x$ for each $x \in V(H)$, which implies that H is the unique disconnected antiregular graph $\overline{G}_n$ of order n whose degree set is $[0, n-2]$, which is impossible. Hence, $f(w) = 0$. Therefore, $F = H - w$ is a connected graph of order $n-1$. The restriction g of f to F is a dominating function of F such that $g(x) = 1$ for every vertex $x \in V(F)$. Therefore, $c_f(x) = c_g(x) = 1 + \deg_F x$ for every vertex x of F. Since the degree set of F is $[n-2]$, it follows that $F = G_{n-1}$ is the unique connected antiregular graph of order $n-1$. Since F is connected, $c_g(x) \geq 2$ for every vertex of F. However, since $c_f = [n-1]$ and $f(w) = 0$, it follows that $\deg_H w = 1$ and so w is adjacent to exactly one vertex of F. Regardless of the vertex of F to which w is adjacent, the resulting graph has either two vertices of degree 1 or three vertices of degree 2 (if $n = 5$), so H is a non-antiregular graph of order n with an irregular dominating function. Thus, there are $n-2$ distinct non-antiregular graphs of order n with an antiregular dominating function. This is illustrated in Figure 2.4 for $n = 5$.

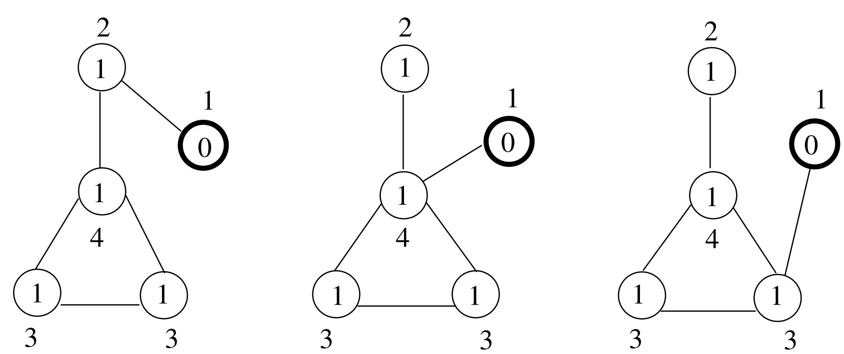

Figure 2.4: Non-antiregular graphs of order 5 with an antiregular dominating function

Subcase 2.2. $\deg_H u = n-1$. First, suppose that $f(u) = 1$. Since $c_f(u) = n-1$, there is exactly one vertex w such that $f(w) = 0$ in H. Then $F = H - w$ is a connected graph of order $n-1$. By the argument used in Subcase 2.1, we see that $F = G_{n-1}$ and w is adjacent to exactly one vertex of F. Hence, there are $n-2$

distinct non-antiregular graphs of order n with an antiregular dominating function. Next, suppose that $f(u) = 0$. Since $c_f(u) = n-1$, it follows that $f(v) = 1$ for each $v \in V(H) - \{u\}$. Then $F = H - u$ is a graph of order $n-1$. The restriction g of f to F is a dominating function of F such that $g(x) = 1$ for each $x \in V(F)$. Since $c_f(x) = c_g(x) = 1 + \deg_F x$ for every $x \in V(F)$ and $1 \leq c_g(x) \leq n-2$, it follows that $\mathcal{C}_g = [n-2]$. This implies that $F = \overline{G}_{n-1}$ is the unique disconnected antiregular graph of order $n-1$ whose degree set is $[0, n-3]$. Adding u to F and joining u to every vertex of F, we see that

$$H = F \vee K_1 = \overline{G}_{n-1} \vee K_1$$

is the unique connected antiregular graph of order n. ■

Chapter 3

REGULAR DOMINATING FUNCTIONS I

ABSTRACT: We now use dominating functions to investigate graphs whose vertices are all dominated by the same number of vertices. More precisely, if f is a dominating function of a graph G such that $c_f(v)$ is the same constant k for every vertex v of G, then f is called a regular (or a k-regular) dominating function of G. We present some preliminary results dealing with properties of regular dominating functions of graphs. In particular, we investigate regular dominating functions of trees. Since the minimum degree of every nontrivial tree is 1, it follows that if a tree has a k-regular dominating function, then $k = 1$ or $k = 2$. We apply algorithmic methods to characterize those trees with a 1-regular dominating function and those with a 2-regular dominating function.

3.1 Introduction

If f is a dominating function of a graph G and $c_f(v)$ is the same constant k for every vertex v of G, then f is called a *regular* (or a *k-regular*) *dominating function* of G. Consequently, if G has a k-regular dominating function, then there is a dominating set S of G such that every vertex of G is dominated by exactly k vertices of S. Since a vertex v whose degree is the minimum degree $\delta(G)$ can be dominated by at most $1 + \delta(G)$ vertices of S, it follows that $1 \leq k \leq 1 + \delta(G)$.

For example, Figures 3.1(a) and 3.1(b) show two dominating functions f and g of a tree T, respectively. For the dominating function f of T in Figure 3.1(a), $c_f(v) = 2$ for every vertex v of T; while for the dominating function g of T in

Figure 3.1(b), $c_g(v) = 1$ for every vertex v of T. Thus, f is 2-regular and g is 1-regular.

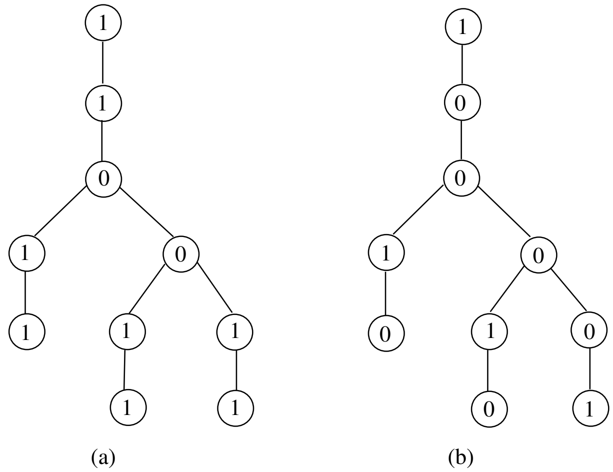

Figure 3.1: Two dominating functions of a tree

While the functions f_1 and f_2 of the graph H shown in Figures 3.2(a) and 3.2(b) are both dominating functions, neither is a regular dominating function. In fact, this graph H has no regular dominating functions. To see this, assume, to the contrary, that H has a regular dominating function $f : V(H) \to \{0, 1\}$ such that f gives rise to a constant function $c_f : V(G) \to \mathbb{N} \cup \{0\}$. Since

$$c_f(r) = f(r) + f(s) + f(w) \text{ and } c_f(w) = f(r) + f(s) + f(w) + f(x),$$

it follows that $f(x) = 0$. Similarly, $f(t) = 0$. Since

$$c_f(v) = f(v) + f(z) \text{ and } c_f(z) = f(v) + f(z) + f(u) + f(y),$$

it follows that $f(u) = f(y) = 0$. Since $c_f(x) = c_f(w)$ forces $f(r) = f(s) = 0$, this implies that $c_f(t) = 0$, which is impossible.

In [76, 82], for example, domination in graphs, and more generally, dominating functions have been shown to have applications to numerous areas including computer science. In this chapter, we investigate regular and irregular domination functions defined on classes of graphs that appear frequently in computer science, including trees.

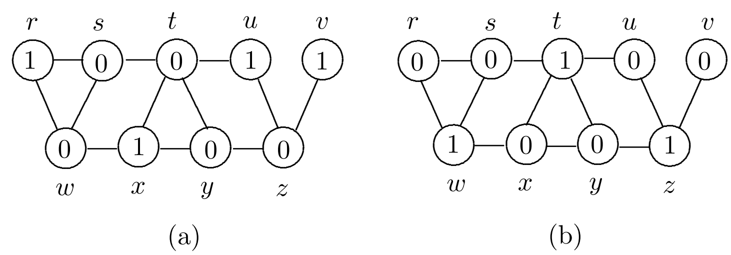

Figure 3.2: Two dominating functions of a graph

3.2 Preliminary Results

We begin by presenting some observations dealing with regular dominating functions of graphs.

Observation 3.2.1 *Let G be a nontrivial connected graph of order n.*

(1) *If G has a vertex v of degree $n-1$, then the function f defined by $f(v) = 1$ and $f(x) = 0$ for each vertex $x \neq v$ is a 1-regular dominating function of G.*

(2) *Let H be a graph with a regular dominating function f and let v be a vertex of H with $f(v) = 0$. If G is the graph obtained by replacing v by an arbitrary graph F where each vertex of F is joined to the neighbors of v, then f can be extended to a regular dominating function of G by defining $f(x) = 0$ for each $x \in V(F)$.*

The next observation provides a useful property of regular dominating functions. A *clique* in a graph G is a complete subgraph of G and a clique is *maximal* if it is not a proper subgraph of any other clique of G.

Lemma 3.2.2 *Let G be a connected graph and let H be a maximal clique in G. If H contains a vertex that is adjacent only to the vertices in H and f is a regular dominating function of G, then f must assign 0 to every vertex in $V(G) - V(H)$ that is adjacent to some vertex of H.*

Proof. Let u be a vertex of H that is adjacent only to the vertices in H and let $x \in V(G) - V(H)$ such that x is adjacent to a vertex v in H. Let X be the set of those neighbors of v not in H. Suppose that f is a regular dominating function of G. Since

$$c_f(u) = \sum_{w \in V(H)} f(w) \text{ and } c_f(v) = \sum_{w \in V(H)} f(w) + \sum_{x \in X} f(x),$$

it follows that $\sum_{x \in X} f(x) = 0$ and so $f(x) = 0$ for every $x \in X$. ■

The next observation limits values of k for which a connected graph may have a k-regular dominating function.

Observation 3.2.3 *Let G be a connected graph with minimum degree $\delta(G) = k$. If G has an r-regular dominating function, then $1 \leq r \leq k+1$. Furthermore, if G is a k-regular graph, then G has a $(k+1)$-regular dominating function.*

Graphs G with $\delta(G) = k$ that are not regular may also have a $(k+1)$-regular dominating function. In fact, more can be said.

Theorem 3.2.4 *For every triple k, δ, Δ of integers with $1 \leq k \leq \delta + 1 \leq \Delta + 1$, there is a connected graph G with $\delta(G) = \delta$ and $\Delta(G) = \Delta$ such that G has a k-regular dominating function.*

Proof. If $\delta = \Delta$, then let $G = K_{\Delta+1}$. For each integer k with $1 \leq k \leq \delta + 1$, the function $f_k : V(G) \to \{0, 1\}$ that assigns 1 to exactly k vertices of G is a k-regular dominating function of G. Thus, we may assume that $1 \leq \delta < \Delta$. We consider two cases, according to whether $k = \delta + 1$ or $1 \leq k \leq \delta$.

Case 1. $k = \delta + 1$. We begin with two vertex-disjoint graphs F and H such that $F \cong K_{\delta+2} - e$ and $H \cong K_\Delta$. Let $V(F) = \{u_1, u_2, \ldots, u_{\delta+2}\}$ where $e = u_1 u_{\delta+2}$ and $V(H) = \{v_1, v_2, \ldots, v_\Delta\}$. The graph G is constructed from F and H by adding the new edge $u_{\delta+2} v_1$ between F and H. Thus, $\delta(G) = \deg u_1 = \delta$ and $\Delta(G) = \deg v_1 = \Delta$. Define the function $f : V(G) \to \{0, 1\}$ by

$$f(x) = \begin{cases} 1 & x = u_i \text{ or } x = v_i \text{ for } 1 \leq i \leq \delta + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $c_f(x) = \delta + 1$ for every $x \in V(G)$, it follows that f is a $(\delta + 1)$-regular dominating function of G.

Case 2. $1 \leq k \leq \delta$. We begin with two vertex-disjoint graphs F and H such that $F \cong K_{\delta+1}$ where $V(F) = \{u_1, u_2, \ldots, u_{\delta+1}\}$ and $H \cong K_{\Delta-k+1}$. The graph G is constructed from F and H by joining each of the k vertices $u_1, u_2, \ldots, u_k$ of F

to every vertex of $V(H)$. Then $\delta(G) = \deg u_{k+1} = \delta$ and $\Delta(G) = \deg v = \Delta$ for each vertex $v \in V(H)$. Define the function $f : V(G) \to \{0, 1\}$ by $f(u_i) = 1$ for $1 \leq i \leq k$ and $f(x) = 0$ for each $x \in V(G) - \{u_1, u_2, \ldots, u_k\}$. Since $c_f(x) = k$ for every $x \in V(G)$, it follows that f is a k-regular dominating function of G. ■

3.3 Trees with Regular Dominating Functions

By Obsevation 3.2.3, if G is a connected graph having a k-regular dominating function, then $1 \leq k \leq \delta(G) + 1$. Thus, if a tree has a k-regular dominating function, then $k = 1$ or $k = 2$. We now investigate trees having a regular dominating function. Thus the primary questions here are the following.

1. Which trees have both a 1-regular and a 2-regular dominating function?
2. Which trees have a 1-regular but not a 2-regular dominating function?
3. Which trees have a 2-regular but not a 1-regular dominating function?
4. Which trees have neither a 1-regular nor a 2-regular dominating function?

In order to present some preliminary observations concerning regular dominating functions of trees, we first introduce some definitions and notation involving distance in graphs. For two vertices u and v in a connected graph G, the *distance* $d(u, v)$ between u and v is the length of a shortest $u - v$ path in G. For a vertex v in the graph G, the *eccentricity* $e(v)$ of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is its *radius* and the maximum eccentricity is its *diameter*, which are denoted by $\text{rad}(G)$ and $\text{diam}(G)$, respectively. A vertex v in G is a *central vertex* if $e(v) = \text{rad}(G)$; while v is a *peripheral vertex* of G if $e(v) = \text{diam}(G)$. If u and v are two vertices of G such that $d(u, v) = \text{diam}(G)$, then u and v are called *antipodal vertices* of G.

Lemma 3.3.1 *Let T be a tree of order at least 3 with a regular dominating function f.*

- ⋆ *If u is a leaf of T and v is a vertex of T such that $d(u, v) = 2$, then $f(v) = 0$.*
- ⋆ *If w is a vertex of T that is adjacent to two or more leaves of T, then $f(w) = 1$.*

Lemma 3.3.2 *Let T be a tree with a k-regular dominating function f for some integer $k \in \{1, 2\}$. If T contains two adjacent vertices u and v such that $f(u) = f(v) = 0$, then the restrictions of f to the two components T_1 and T_2 of $T - uv$ are k-regular dominating functions of T_1 and T_2.*

We now present a class of trees without any regular dominating function. The *non-leaf minimum degree* $\delta^*(T)$ of a tree T is the minimum degree among the non-leaves of T.

Theorem 3.3.3 *If T is a tree with $\delta^*(T) \geq 3$ and $\mathrm{diam}(T) \geq 3$, then T has neither a 1-regular nor a 2-regular dominating function.*

Proof. Assume, to the contrary, that there exists a tree T with $\delta^*(T) \geq 3$ and $\mathrm{diam}(T) = d \geq 3$ having a regular dominating function f. Let u and v be antipodal vertices of T. Hence, u and v are both end-vertices of T and $d(u, v) = d$. Let

$$P = (u = v_0, v_1, \ldots, v_d = v)$$

be the $u - v$ path in T. First, we show that $f(u) = f(v) = 0$. Since $\deg v_1 \geq 3$, it follows that v_1 has a neighbor u' distinct from $u = v_0$ and v_2. Since $d(u', v_d) = d$, it follows that u' must be a leaf. Because $d(u', u) = 2$, it follows by Lemma 3.3.1 that $f(u) = 0$ (and $f(u') = 0$). Similarly, $f(v) = 0$. Thus, $f(v_1) = 1$. For each integer i with $0 \leq i \leq d$, let $V_i = \{x \in V(G) : \ d(u, x) = i\}$. Thus, $V_0 = \{u\}$ and $v_i \in V_i$ for $0 \leq i \leq d$. Since $f(v) = 0$ and $c_f(v) = f(v) + f(v_{d-1}) \in \mathbb{N}$, it follows that $f(v_{d-1}) = 1$. Therefore, $c_f(v) = 1$ and so $c_f(z) = 1$ for all vertices z of T. Since $c_f(v_{d-1}) = f(v_{d-1}) = 1$, it follows that $f(x) = 0$ for each $x \in N(v_{d-1})$ and so $f(v_{d-2}) = 0$. Since $\deg v_{d-2} \geq 3$, there is a vertex $y \neq v_{d-1}, v_{d-3}$ such that $yv_{d-2} \in E(T)$. Since $c_f(v_{d-2}) = 1$ and $f(v_{d-1}) = 1$, it follows that $f(y) = 0$. Since $f(v_{d-1}) = 1$ and $d(v_{d-1}, y) = 2$, it follows by Lemma 3.3.1 that y is not a leaf. Thus, $\deg y \geq 3$. Since $c_f(y) = 1$ and $f(v_{d-2}) = 0$, there are $y_1, y_2 \in N(y) - \{v_{d-2}\} \subseteq V_d$ such that $f(y_1) = 1$ and $f(y_2) = 0$ where y_1 and y_2 are leaves. However then, $c_f(y_2) = 0$, which is a contradiction. ■

By Theorem 3.3.3, every tree with a regular dominating function must at least one vertex of degree 2. Next, we characterize those trees with a 1-regular dominating function and those with a 2-regular dominating function.

3.4 On 1-Regular Dominating Functions of Trees

First, we present a sufficient condition for a tree to possess a 1-regular dominating function.

Theorem 3.4.1 *If a tree T contains a vertex v such that* $\deg u = 2$ *for every vertex u with* $d(v,u) \equiv 2 \pmod 3$*, then T has a 1-regular dominating function.*

Proof. Assume that T contains a vertex v such that $\deg u = 2$ for every vertex u with $d(v,u) \equiv 2 \pmod 3$. Define the function $f : V(T) \to \{0,1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } d(v,x) \equiv 0 \pmod 3 \\ 0 & \text{if } d(v,x) \not\equiv 0 \pmod 3. \end{cases}$$

We consider T rooted at the vertex v. For each integer i with $0 \le i \le e(v)$ where $e(v)$ is the eccentricity of v, let $V_i = \{x \in V(G) : d(v,x) = i\}$. Thus, $V_0 = \{v\}$ and $V_1 = N(v)$ is the neighborhood of v. Observe that $f(x) = 1$ if $x \in V_i$ where $i \equiv 0 \pmod 3$ and $f(x) = 0$ otherwise.

- ⋆ If $x \in V_i$ where $i \equiv 0 \pmod 3$, then $f(x) = 1$ and x is only adjacent to vertices y with $f(y) = 0$. Hence, $c_f(x) = 1$.
- ⋆ If $x \in V_i$ where $i \equiv 1 \pmod 3$, then $f(x) = 0$ and x is adjacent to exactly one vertex $y \in V_{i-1}$ with $f(y) = 1$ and any other neighbor of x belongs to V_{i+1}. Since $f(z) = 0$ for all $z \in V_{i+1}$, it follows that $c_f(x) = 1$.
- ⋆ If $x \in V_i$ where $i \equiv 2 \pmod 3$, then $f(x) = 0$ and x is adjacent to exactly one vertex $y \in V_{i-1}$ with $f(y) = 0$ and exactly one vertex $z \in V_{i+1}$ with $f(z) = 1$. Hence, $c_f(x) = 1$.

Therefore, $c_f(x) = 1$ for every vertex x of T and so f is a 1-regular dominating function of T. ■

However, the converse of Theorem 3.4.1 is not true. That is, there are trees T with a 1-regular dominating function but containing no vertex v such that $\deg u = 2$ for every vertex u with $d(v,u) \equiv 2 \pmod 3$. The tree T of Figure 3.1 is one such example. As a consequence of Theorem 3.4.1, paths and stars have 1-regular dominating functions.

Corollary 3.4.2 *Every path and star has a* 1*-regular dominating function.*

A *double star* is a tree of diameter 3. Thus, every double star T has exactly two vertices that are not leaves, which are called the *central vertices* of T. By Lemma 3.3.1, there is only one double star with a 1-regular dominating function.

Proposition 3.4.3 *A double star T has a* 1*-regular dominating function if and only if* $T = P_4$.

Next, we characterize those trees having a 1-regular dominating function. To do this, we first introduce some additional terminology. A tree T is called 1*-sequential* if the tree T and a labeling $f : V(T) \to \{0, 1\}$ of T can be constructed by means of the following algorithm.

Algorithm 3.4.4 *Constructing a* 1*-sequential tree T.*

1. We begin with $T_0 = K_1$ where the vertex v_0 of T_0 is labeled 1.
2. The tree T_1 is constructed by adding a vertex v_1 to T_0, joining v_1 to v_0, and defining $f(v_1) = 0$.
3. The tree T_2 is constructed by doing one of the following.
   - (3.1) A vertex v_2 is added to T_1, v_2 is joined to v_0, and we define $f(v_2) = 0$.
   - (3.2) A copy $H_2 = (x_2, y_2)$ of P_2 is added to T_1, x_2 is joined to v_1, and we define $f(x_2) = 0$ and $f(y_2) = 1$.
4. Once a tree T_j, $j \geq 2$, has been constructed, a tree T_{j+1} with an extended 1-regular dominating function f is constructed by doing one of the following.
   - (4.1) A vertex v_{j+1} is added to T_j, v_{j+1} is joined to a vertex labeled 1 in T_j, and we define $f(v_{j+1}) = 0$.
   - (4.2) A copy $H_{j+1} = (x_{j+1}, y_{j+1})$ of P_2 is added to T_j, x_{j+1} is joined to a vertex labeled 0 in T_j, and we define $f(x_{j+1}) = 0$ and $f(y_{j+1}) = 1$.
5. Either repeat Step 4 or stop (resulting in a tree $T_k = T$ for some nonnegative integer k).

Once Algorithm 3.4.4 stops, a sequence of labeled trees $T_0, T_1, T_2, \ldots, T_k$ is constructed, resulting in $T_k \cong T$. Such a tree T is therefore 1-sequential. 1-sequential trees lead us to the next result.

Theorem 3.4.5 *A tree T has a 1-regular dominating function if and only if T is 1-sequential.*

Proof. First, suppose that T is a 1-sequential tree constructed with a labeling $f : V(T) \to \{0, 1\}$. From the construction of T described above, we see that f is a 1-regular dominating function of T. For the converse, assume, to the contrary, that there are trees possessing a 1-regular dominating function that are not 1-sequential. Among these trees, let T be one of minimum order n. Since it is clear that every tree of order 5 or less that possesses a 1-regular dominating function is 1-sequential, it follows that $n \geq 6$. Let f be a 1-regular dominating function of T.

First, suppose that there is a peripheral vertex u of T such that $f(u) = 0$. Then u is a leaf of T. Let v be the neighbor of u in T and so $f(v) = 1$. Then $T' = T - u$ is a tree of order $n - 1$ and the restriction of the 1-regular dominating function f of T to T' is a 1-regular dominating function of T'. Thus, T' is 1-sequential by the induction hypothesis. Since T can be constructed from T' by adding the vertex u to T', joining u to v, and defining $f(u) = 0$, it follows that T is 1-sequential, producing a contradiction.

Next, suppose that all peripheral vertices of T are labeled 1. Let u be a peripheral vertex of T and v be the neighbor of u in T. Thus, $f(u) = 1$ and $f(v) = 0$. Also, if $w \in N_T(v) - \{u\}$, then $f(w) = 0$. We claim that $\deg_T v = 2$. First, u is the only neighbor of v that is a leaf; for otherwise, if there is $w \in N_T(v) - \{u\}$ such that w is a leaf, then $c_f(w) = 0$, which is impossible. Next, since u is a peripheral vertex of T, it follows that v has exactly one neighbor that is not a leaf. Consequently, as claimed, $\deg_T v = 2$. Let x be the neighbor of v distinct from u. We saw that $f(x) = 0$. Then $T'' = T - \{u, v\}$ is a tree of order $n - 2$ and the restriction of the 1-regular dominating function f of T to T'' is a 1-regular dominating function of T''. Hence, T'' is 1-sequential by the induction hypothesis. Since T can be constructed from T'' by adding the copy $H = (u, v)$ of P_2 to T'', joining v to x, and defining $f(u) = 1$ and $f(v) = 0$, it follows that T is 1-sequential, producing a contradiction. ■

It was stated in Corollary 3.4.2 that every path and star has a regular dominating function. In fact, every path and star has a unique 1-regular dominating function. The tree of order 8 of Figure 3.3 has at least two 1-regular dominating functions. It can be shown that it is the only tree of order n with $2 \leq n \leq 9$ having this property. There are other trees having more than one 1-regular dominating function, however.

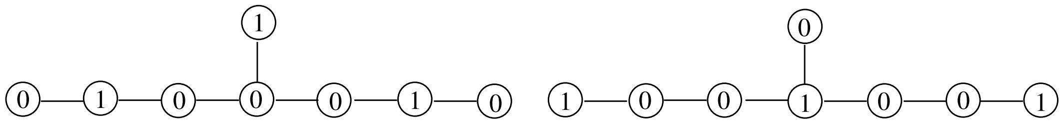

Figure 3.3: The tree T of order 8 with at least two 1-regular dominating functions

Theorem 3.4.6 *For each integer $n \geq 10$, there exists a tree of order n having at least two 1-regular dominating functions.*

Proof. Let $n \geq 10$ be an integer. We consider two cases, according to whether n is even or odd. In each case, let $F = kP_2$ for some positive integer k where (u_i, v_i) is a copy of P_2 in F for $1 \leq i \leq k$.

Case 1. *n is even.* Thus, $n = 10+2k$ for some integer $k \geq 0$. We show that there is a tree T of order $10+2k$ having two distinct 1-regular dominating functions. For $n = 10$, let T be the tree T_0 of order 10 in Figure 3.4 with two distinct 1-regular dominating functions g_1 and g_2, which are also shown in Figure 3.4.

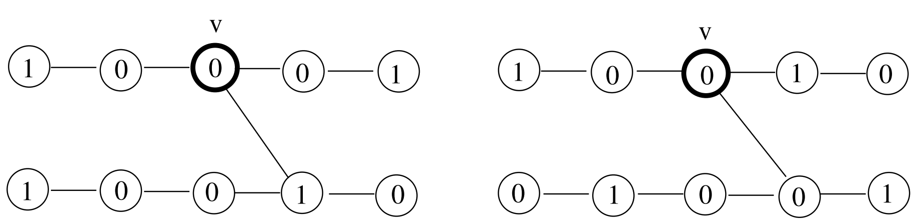

Figure 3.4: The tree T_0 of order 10 in Case 1 with two distinct 1-regular dominating functions

We may therefore assume that $n = 10 + 2k$ for some integer $k \geq 1$. Let T be the tree of order n obtained from the tree T_0 of order 10 and the forest $F = kP_2$ of order $2k$ by joining the vertex v of degree 3 in the tree T_0 that is labeled 0 in both g_1 and g_2 (see Figure 3.4) to each vertex u_i for $1 \leq i \leq k$ in F. For $t = 1, 2$, we define a 1-regular dominating function $f_t : V(T) \to \{0, 1\}$ of T by

$$f_t(x) = \begin{cases} g_t(x) & \text{if } x \in V(T_0) \\ 0 & \text{if } x = u_i \text{ for } 1 \leq i \leq k \\ 1 & \text{if } x = v_i \text{ for } 1 \leq i \leq k. \end{cases}$$

Then f_1 and f_2 are two distinct 1-regular dominating functions of T.

Case 2. *n is odd.* Then $n \geq 11$ and so $n = 11 + 2k$ for some integer $k \geq 0$. We show that there is a tree T of order $11 + 2k$ having two distinct 1-regular dominating functions. For $n = 11$, let T be the tree T_0 of order 11 in Figure 3.5 with two distinct 1-regular dominating functions g_1 and g_2, which are also shown in Figure 3.5.

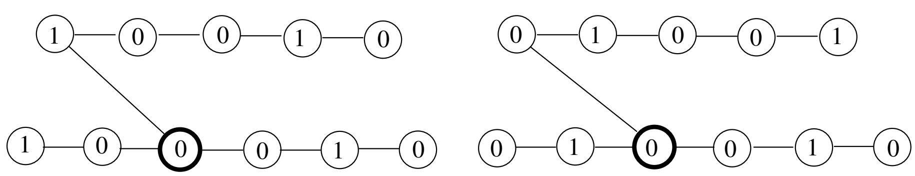

Figure 3.5: The tree T_0 of order 11 in Case 2 with two distinct 1-regular dominating functions

We may therefore assume that that $n = 11 + 2k$ for some integer $k \geq 1$. Let T be the tree of order n obtained from the tree T_0 of order 11 and the forest $F = kP_2$ of order $2k$ by joining the vertex of degree 3 in the tree T_0 (see Figure 3.5) to each vertex u_i for $1 \leq i \leq k$ in F. For $t = 1, 2$, we define a 1-regular dominating function $f_t : V(T) \to \{0, 1\}$ of T by

$$f_t(x) = \begin{cases} g_t(x) & \text{if } x \in V(T_0) \\ 0 & \text{if } x = u_i \text{ for } 1 \leq i \leq k \\ 1 & \text{if } x = v_i \text{ for } 1 \leq i \leq k. \end{cases}$$

Then f_1 and f_2 are two distinct 1-regular dominating functions of T. ■

In fact, more can be concluded. For each integer $n \geq 10$, there are at least two trees of order n having at least two 1-regular dominating functions, as we show next.

Theorem 3.4.7 *For each integer $n \geq 10$, there are at least two trees of order n having at least two 1-regular dominating functions.*

Proof. We consider three cases, according to $n \equiv i \pmod 3$ for $i = 0, 1, 2$. In each case, let $P_{3k} = (v_1, v_2, \ldots, v_{3k})$ be a path of order $3k$. Since we found one tree of order n for each $n \geq 10$ with at least two 1-regular dominating functions in the proof of Theorem 3.4.8, we need only find one more distinct tree of each order with the given property.

Case 1. $i = 0$. Then $n \geq 12$. Thus, $n = 12 + 3k$ for some integer $k \geq 0$. We show that there is a tree T of order $12 + 3k$ having two distinct 1-regular dominating functions. If $n = 12$, let T be the tree T_0 of order 12 in Figure 3.6 with two distinct 1-regular dominating functions g_1 and g_2, also shown in Figure 3.6. If $n = 12 + 3k$ for some integer $k \geq 1$, let T be the tree of order n obtained from the tree T_0 of order 12 and the path P_{3k} of order $3k$ by joining the end-vertex v of the tree T_0 to the end-vertex v_1 of P_{3k}. For $t = 1, 2$, we define the 1-regular dominating function $f_t : V(T) \to \{0, 1\}$ of T by

$$f_t(x) = \begin{cases} g_t(x) & \text{if } x \in V(T_0) \\ 0 & \text{if } x = v_i \text{ and } i \equiv 1, 2 \pmod 3 \\ 1 & \text{if } x = v_i \text{ and } i \equiv 0 \pmod 3). \end{cases}$$

Then f_1 and f_2 are two distinct 1-regular dominating functions of T.

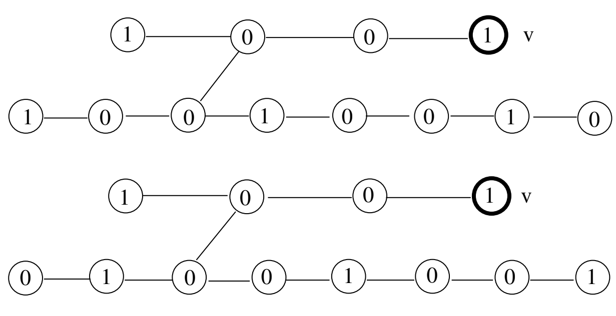

Figure 3.6: The tree T_0 of order 12 in Case 1 with two distinct 1-regular dominating functions

Case 2. $i = 1$. Then $n \geq 10$ and so $n = 10 + 3k$ for some integer $k \geq 0$. We show that there is a tree T of order $10 + 3k$ having two distinct 1-regular dominating functions. If $n = 10$, let T be the tree T_0 of order 10 in Figure 3.7 with two distinct 1-regular dominating functions g_1 and g_2, also shown in Figure 3.7.

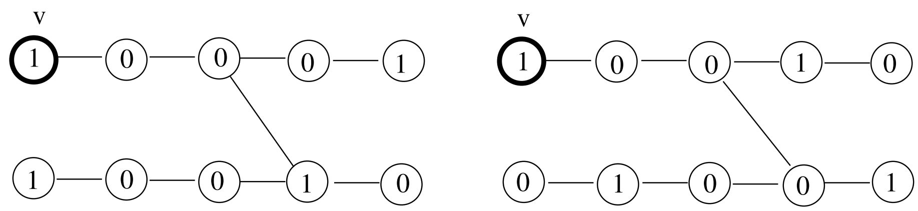

Figure 3.7: The tree T_0 of order 10 in Case 2 with two distinct 1-regular dominating functions

If $n = 10 + 3k$ for some integer $k \geq 1$, let T be the tree of order n obtained from the tree T_0 of order 10 and the path P_{3k} by joining the end-vertex v of the tree T_0 to the end-vertex v_1 of P_{3k}. For $t = 1, 2$, we define the 1-regular dominating function $f_t : V(T) \to \{0, 1\}$ of T by

$$f_t(x) = \begin{cases} g_t(x) & \text{if } x \in V(T_0) \\ 0 & \text{if } x = v_i \text{ and } i \equiv 1, 2 \pmod 3 \\ 1 & \text{if } x = v_i \text{ and } i \equiv 0 \pmod 3. \end{cases}$$

Then f_1 and f_2 are two distinct 1-regular dominating functions of T.

Case 3. $i = 2$. Then $n \geq 11$ and so $n = 11+3k$ for some integer $k \geq 0$. We show that there is a tree T of order $11 + 3k$ having two distinct 1-regular dominating functions. If $n = 11$, let T be the tree T_0 of order 11 in Figure 3.8 with two distinct 1-regular dominating functions g_1 and g_2, also shown in Figure 3.8.

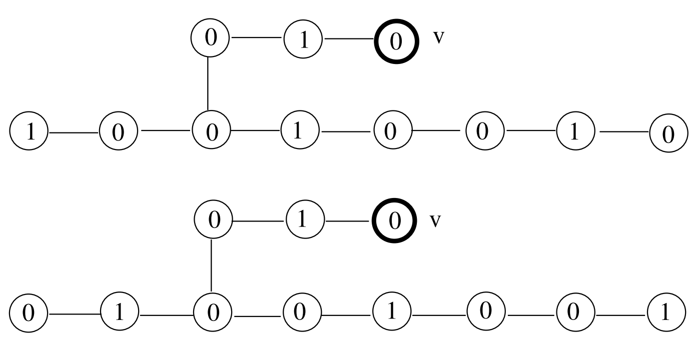

Figure 3.8: The tree T_0 of order 11 in Case 3 with two distinct 1-regular dominating functions

If $n = 11 + 3k$ for some integer $k \geq 1$, let T be the tree of order n obtained from the tree T_0 of order 11 and the path P_{3k} of order $3k$ by joining the end-vertex v of the tree T_0 to the end-vertex v_1 of P_{3k}. For $t = 1, 2$, we define the 1-regular dominating function $f_t : V(T) \to \{0, 1\}$ of T by

$$f_t(x) = \begin{cases} g_t(x) & \text{if } x \in V(T_0) \\ 0 & \text{if } x = v_i \text{ and } i \equiv 1, 0 \pmod{3} \\ 1 & \text{if } x = v_i \text{ and } i \equiv 2 \pmod{3}. \end{cases}$$

Then f_1 and f_2 are two distinct 1-regular dominating functions of T. ■

3.5 On 2-Regular Dominating Functions of Trees

There is a theorem analogous to Theorem 3.4.5 that characterizes those trees having a 2-regular dominating function. To explore this, we first introduce an additional concept. A tree T is called 2-*sequential* if the tree T and a labeling $f : V(T) \to \{0, 1\}$ of T can be constructed by means of the following algorithm.

Algorithm 3.5.1 *Constructing a 2-sequential tree T.*

1. We begin with $T_0 = P_2 = (u_0, v_0)$ where $f(u_0) = f(v_0) = 1$.

2. The tree T_1 is constructed by adding a copy $F_1 = (x_1, y_1, z_1)$ of P_3 to T_0, joining x_1 to a vertex of T_0, and defining $f(x_1) = 0$ and $f(y_1) = f(z_1) = 1$.

3. The tree T_2 is constructed by doing one of the following.

   (3.1) A copy $F_2 = (x_2, y_2, z_2)$ of P_3 is added to T_1, joining x_2 to a vertex labeled 1 in T_1, and defining $f(x_2) = 0$ and $f(y_2) = f(z_2) = 1$.

   (3.2) A copy $H_2 = (u_2, v_2, x_2, y_2, z_2)$ of P_5 is added to T_1, joining x_2 to a vertex labeled 0 in T_1, and defining $f(x_2) = 0$ and $f(w) = 1$ for each $w \in \{u_2, v_2, y_2, z_2\}$.

4. Once a tree T_j, $j \geq 2$, has been constructed, a tree T_{j+1} with an extended 2-regular dominating function f is constructed by doing one of the following.

   (4.1) A copy $F_{j+1} = (x_{j+1}, y_{j+1}, z_{j+1})$ of P_3 is added to T_j, joining x_{j+1} to a vertex labeled 1 in T_j, and defining $f(x_{j+1}) = 0$ and $f(y_{j+1}) = f(z_{j+1}) = 1$.

   (4.2) A copy $H_{j+1} = (u_{j+1}, v_{j+1}, x_{j+1}, y_{j+1}, z_{j+1})$ of P_5 is added to T_j, joining x_{j+1} to a vertex labeled 0 in T_j, and defining $f(x_{j+1}) = 0$ and $f(w)$ for each $w \in \{u_{j+1}, v_{j+1}, y_{j+1}, z_{j+1}\}$.

5. Either repeat Step 4 or stop (resulting in a tree $T_k = T$ for some nonnegative integer k).

Once Algorithm 3.5.1 stops, a sequence $T_0, T_1, T_2, \ldots, T_k$ of labeled trees is constructed, resulting in $T_k \cong T$. Such a tree T is therefore 2-sequential. The proof of the following result is similar to that of Theorem 3.4.5.

Theorem 3.5.2 *A tree T has a 2-regular dominating function if and only if T is 2-sequential.*

Proof. First, suppose that T is a 2-sequential tree constructed with a labeling $f : V(T) \to \{0, 1\}$. From the construction of T described in Algorithm 3.5.1, it follows that the labeling f is a 2-regular dominating function of T. For the converse, assume, to the contrary, that there are trees possessing a 2-regular dominating function that are not 2-sequential. Among these trees, let T be one of minimum order n. Since it is clear that all trees of order at most 10 possessing a 2-regular dominating function are 2-sequential, it follows that $n \geq 11$. Let f be a 2-regular dominating function of T.

Let u be a peripheral vertex of T and v the neighbor of u in T. Thus, $f(u) = f(v) = 1$. Since $c_f(v) = 2$, it follows that $f(w) = 0$ for every $w \in N_T(v) - \{u\}$. Thus, u is the only neighbor of v that is a leaf. Since u is a peripheral vertex of T, it follows that v has exactly one neighbor that is not a leaf. Consequently, $\deg_T v = 2$. Let x be the neighbor of v distinct from u. Thus, $f(x) = 0$. We consider three cases, according to whether $\deg_T x = 2$, $\deg_T x = 3$, or $\deg_T x \geq 4$.

Case 1. $\deg_T x = 2$. Let x' be a neighbor of x distinct from v. Since $c_f(x) = 2$, it follows that $f(x') = 1$. Then $T' = T - \{u, v, x\}$ is a tree of order $n - 3$ and the restriction f' of the 2-regular dominating function f of T to T' is a 2-regular dominating function of T'. Hence, T' is 2-sequential by hypothesis. Since T can be constructed from T' by adding the copy $F = (u, v, x)$ of P_3 to T', joining x to x', and defining $f(x) = 0$ and $f(u) = f(v) = 1$, it follows that T is 2-sequential, producing a contradiction.

Case 2. $\deg_T x = 3$. Let $N_T(x) - \{v\} = \{x_1, x_2\}$. Since $f(v) = 1$, $f(x) = 0$ and $c_f(x) = 2$, it follows that $\{f(x_1), f(x_2)\} = \{0, 1\}$, say $f(x_1) = 1$ and $f(x_2) = 0$. Because $f(x) = 0$, $f(x_1) = 1$ and $c_f(x_1) = 2$, it follows that x_1 is adjacent to a vertex $y_1 \neq x$ such that $f(y_1) = 1$. Furthermore, x_2 is adjacent to two vertices z_1 and z_2 both labeled 1 and each of z_1 and z_2 is adjacent to a vertex labeled 1. Since u is a peripheral vertex of T, it follows that each vertex in $N_T(x_1) - \{x\}$ is a leaf. In fact, $N_T(x_1) - \{x\} = \{y_1\}$. Hence, y_1 is a leaf and $\deg_T x_1 = 2$. Then $T'' = T - \{u, v, x, x_1, y_1\}$ is a tree of order $n - 5$ and the restriction f'' of the 2-regular dominating function f of T to T'' is a 2-regular dominating function of T''. Hence, T'' is 2-sequential by the induction hypothesis. Since T can be constructed from T'' by adding the copy $H = (u, v, x, x_1, y_1)$ of P_5 to T'', joining x to x_2, and defining $f(x) = 0$ and $f(u) = f(v) = f(x_1) = f(y_1) = 1$, it follows that T is 2-sequential, producing a contradiction.

Case 3. $\deg_T x \geq 4$. Let $N_T(x) - \{v\} = \{x_1, x_2, \ldots, x_a\}$ where $a \geq 3$. Since $f(v) = 1$, $f(x) = 0$ and $c_f(x) = 2$, it follows that exactly one of $f(x_i)$, where $1 \leq i \leq a$, is 1, say $f(x_1) = 1$ and $f(x_i) = 0$ for $2 \leq i \leq a$. In particular,

$f(x_2) = f(x_3) = 0$. For $i = 2, 3$, the vertex x_i is adjacent to two distinct vertices y_i and z_i such that $f(y_i) = f(z_i) = 1$. Since $f(x_i) = 0$ and $c_f(y_i) = c_f(z_i) = 2$ where $i = 2, 3$, it follows that each of y_i and z_i is adjacent to a vertex labeled 1 in T. However then, this contradicts the fact that u is a peripheral vertex of T. Hence, Case 3 cannot occur. ■

An argument similar to that used in the proof of Theorem 3.5.2 shows that if a tree has a 2-regular dominating function, then this function is unique.

Theorem 3.5.3 *If a tree T has a 2-regular dominating function, then T has a unique 2-regular dominating function.*

Proof. We proceed by induction on the order n of trees. It is straightforward to see that the statement is true for trees of order n with $2 \leq n \leq 7$. Assume for some integer $n \geq 8$ that this statement is true for all trees of order k where $2 \leq k < n$. Let T be a tree of order n with a 2-regular dominating function f. We show that f is the unique 2-regular dominating function of T.

Let u be a peripheral vertex of T and v the neighbor of u in T. Thus, $f(u) = f(v) = 1$. Let x be the neighbor of v distinct from u. Thus, $f(x) = 0$. We consider three cases, according to whether $\deg_T x = 2$, $\deg_T x = 3$, or $\deg_T x \geq 4$.

Case 1. $\deg_T x = 2$. Let x' be the neighbor of x distinct from v. Then $T' = T - \{u, v, x\}$ is a tree of order $n-3$ and the restriction f' of the 2-regular dominating function f of T to T' is a 2-regular dominating function of T'. By the induction hypothesis, f' is the unique 2-regular dominating function of T'. Since (1) T can be constructed from T' by adding the copy $F = (u, v, x)$ of P_3 to T' and joining x to x' and (2) any 2-regular dominating function of T must assign 0 to x and 1 to u and v, it follows that f is the 2-regular dominating function of T.

Case 2. $\deg_T x = 3$. Let $N_T(x) - \{v\} = \{x_1, x_2\}$. Since $f(v) = 1$, $f(x) = 0$ and $c_f(x) = 2$, it follows that $\{f(x_1), f(x_2)\} = \{0, 1\}$, say $f(x_1) = 1$ and $f(x_2) = 0$. Because $f(x) = 0$, $f(x_1) = 1$ and $c_f(x_1) = 2$, it follows that x_1 is adjacent to a vertex $y_1 \neq x$ such that $f(y_1) = 1$. Furthermore, x_2 is adjacent to two vertices z_1 and z_2 both labeled 1 and each of z_1 and z_2 is adjacent to a vertex labeled 1. Since u is a peripheral vertex of T, it follows that each vertex in $N_T(x_1) - \{x\}$ is a leaf. In fact, $N_T(x_1) - \{x\} = \{y_1\}$. Hence, y_1 is a leaf and $\deg_T x_1 = 2$. Then $T'' = T - \{u, v, x, x_1, y_1\}$ is a tree of order $n - 5$ and the restriction f'' of the 2-regular dominating function f of T to T'' is a 2-regular dominating function of T''.

By the induction hypothesis, f'' is the unique 2-regular dominating function of T''. Since (1) T can be constructed from T'' by adding the copy $H = (u, v, x, x_1, y_1)$ of P_5 to T'', joining x to x_2 and (2) any 2-regular dominating function of T must assign 0 to x and 1 to each of u, v, x_1, y_1, it follows that f is the unique 2-regular dominating function of T.

Case 3. $\deg_T x \geq 4$. Let $N_T(x) - \{v\} = \{x_1, x_2, \ldots, x_a\}$ where $a \geq 3$. Since $f(v) = 1$, $f(x) = 0$ and $c_f(x) = 2$, it follows that exactly one of $f(x_i)$, where $1 \leq i \leq a$, is 1, say $f(x_1) = 1$ and $f(x_i) = 0$ for $2 \leq i \leq a$. In particular, $f(x_2) = f(x_3) = 0$. For $i = 2, 3$, the vertex x_i is adjacent to two distinct vertices y_i and z_i such that $f(y_i) = f(z_i) = 1$. Since $f(x_i) = 0$ and $c_f(y_i) = c_f(z_i) = 2$ where $i = 2, 3$, it follows that each of y_i and z_i is adjacent to a vertex labeled 1 in T. However then, this contradicts the fact that u is a peripheral vertex of T. Hence, Case 3 cannot occur. ■

Next, we show that if a T has 2-regular dominating function, then T has a 1-regular dominating function.

Theorem 3.5.4 *If a tree T has a 2-regular dominating function f_2, then T has a 1-regular dominating function f_1 such that $f_1(x) = 0$ for every $x \in V(T)$ such that $f_2(x) = 0$.*

Proof. We proceed by induction on the order n of a nontrivial tree. For $2 \leq n \leq 5$, the paths P_2 and P_5 are the only trees having a 2-regular dominating function. If $P_2 = (v_1, v_2)$, then f_2 defined by $f_2(v_1) = f_2(v_2) = 1$ is the unique 2-regular dominating function and f_1 defined by $f_1(v_1) = 1$ and $f_1(v_2) = 0$ is a 1-regular dominating function with the desired property. If $P_5 = (v_1, v_2, v_3, v_4, v_5)$, then f_2 defined by $f_2(v_3) = 0$ and $f_2(v_i) = 1$ for $i = 1, 2, 4, 5$ is the unique 2-regular dominating function and f_1 defined by $f_1(v_i) = 0$ for $i = 1, 3, 4$ and $f_1(v_i) = 1$ for $i = 2, 5$ is a 1-regular dominating function with the desired property. Thus, the basis step of the induction holds.

Assume, for an integer $k \geq 6$ that all trees T' of order k' with $2 \leq k' < k$ with a 2-regular dominating function f_2' also have a 1-regular dominating function f_1' such that $f_1'(x) = 0$ for each $x \in V(T')$ with $f_2'(x) = 0$. Let T be a tree of order k. If no tree of order k has a 2-regular dominating function, then the statement is true vacuously. Hence, we may assume that T has a 2-regular dominating function f_2.

Since no star of order 3 or more has a 2-regular dominating function, it follows that T is not a star and so $\mathrm{diam}(T) \geq 3$. We consider two cases.

Case 1. *There are two adjacent vertices* u *and* v *in* T *such that* $f_2(u) = f_2(v) = 0$. Let T_1 and T_2 be the two components of $T - uv$. For $i = 1, 2$, let $f_{i,2}$ be the restriction of f_2 to T_i. By Lemma 3.3.2, $f_{i,2}$ is a 2-regular dominating function of T_i for $i = 1, 2$. Thus, it follows by the induction hypothesis that T_i $(i = 1, 2)$ has a 1-regular dominating function $f_{i,1}$ such that $f_{i,1}(x) = 0$ for each $x \in V(T_i)$ with $f_{i,2}(x) = 0$. Then the function $f_1 : V(T) \to \{0, 1\}$ defined by $f_1(w) = f_{i,1}(w)$ if $w \in V(T_i)$ for $i = 1, 2$ is a 1-regular dominating function of T with the property that $f_1(x) = 0$ for each $x \in V(T)$ with $f_2(x) = 0$.

Case 2. *For every two adjacent vertices* u *and* v *of* T, *either* $f_2(u) = 1$ *or* $f_2(v) = 1$. First, we show that there is a path (v_1, v_2, v_3) in T such that

(i) $\deg v_1 = 1$ and $\deg v_2 = \deg v_3 = 2$ and

(ii) $f_2(v_1) = f_2(v_2) = 1$ and $f_2(v_3) = 0$.

Let v_1 be a peripheral vertex of T. Then v_1 is a leaf of T. Let $v_1v_2, v_2v_3 \in E(T)$ where $v_3 \neq v_1$. Since v_1 is a peripheral vertex of T, it follows that $f_2(v_1) = 1$ and $f_2(v_2) = 1$. Because $d(v_1, v_3) = 2$, it follows by Lemma 3.3.1 that $f_2(v_3) = 0$. We claim that $\deg v_2 = 2$, for suppose, to the contrary, that v_2 is adjacent to a vertex x distinct from v_1 and v_3. Also, since v_1 is a peripheral vertex of T, it follows that x is a leaf of T. Since $d(v_1, x) = 2$, we have $f_2(x) = 0$, contradicting the fact that f_2 is a 2-regular dominating function of T. Hence, $\deg v_2 = 2$, as claimed. Since f_2 is a 2-regular dominating function of T, it follows that v_3 is adjacent to a vertex $v_4 \neq v_2$ such that $f_2(v_4) = 1$. Because $f_2(v_3) = 0$ and v_3 is not adjacent to any vertex whose functional value is 0 (by the hypothesis of Case 2), it follows that $\deg v_3 = 2$. Therefore, the path (v_1, v_2, v_3) has the desired properties (i) and (ii).

Next, let $T' = T - \{v_1, v_2, v_3\}$. Since v_4 is adjacent to a vertex in T' whose functional value is 1, it follows that T' is a nontrivial tree. The restriction f_2' of the 2-regular dominating function f_2 to T' is a 2-regular dominating function of T'. By the induction hypothesis, T' has a 1-regular dominating function f_1' such that $f_1'(x) = 0$ for each $x \in V(T')$ with $f_2'(x) = 0$. There are now two possibilities according to whether $f_1'(v_4) = 1$ or $f_1'(v_4) = 0$. We consider these two situations.

- ⋆ If $f_1'(v_4) = 1$, then we extend f_1' to a 1-regular dominating function f_1 of T by defining $f_1(v_3) = f_1(v_2) = 0$ and $f_1(v_1) = 1$.
- ⋆ If $f_1'(v_4) = 0$, then we extend f_1' to a 1-regular dominating function f_1 of T by defining $f_1(v_3) = f_1(v_1) = 0$ and $f_1(v_2) = 1$.

In either case, $f_1(v_3) = f_2(v_3) = 0$. Therefore, T has a 1-regular dominating function f_1 such that $f_1(x) = 0$ for each $x \in V(T)$ with $f_2(x) = 0$. ■

It may not be surprising that the converse of Theorem 3.5.4 is not true. For example, we saw that every nontrivial path has a 1-regular dominating function. This, however, is not true for 2-regular dominating functions, as shown in the following two results. The following consequence of Lemma 3.2.2 will be useful.

Observation 3.5.5 *If xy is a pendant edge of a nonempty graph G, then $f(x) = f(y) = 1$ for every 2-regular dominating function $f : V(G) \to \{0, 1\}$ of G.*

Proposition 3.5.6 *For an integer $n \geq 2$, the path P_n of order n has a 2-regular dominating function if and only if $n \equiv 2 \pmod 3$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ where $n \geq 2$. First, suppose that $n \equiv 2 \pmod 3$. Then P_n has the 2-regular dominating function f defined by $f(v_i) = 0$ if $i \equiv 0 \pmod 3$ and $f(v_i) = 1$ otherwise. For the converse, assume, to the contrary, that there exists an integer $n \geq 3$ where $n \not\equiv 2 \pmod 3$ such that P_n has a 2-regular dominating function $g : V(P_n) \to \{0, 1\}$. It then follows by Theorem 3.5.3 that g is the unique 2-regular dominating function of P_n. Furthermore, $g(v_1) = g(v_2) = 1$ by Observation 3.5.5. Since $c_g(v_2) = 2$, this forces $g(v_3) = 0$ and so $g(v_4) = g(v_5) = 1$. Extending this argument, we conclude that $g(v_i) = 1$ if $i \equiv 1, 2 \pmod 3$ and $g(v_i) = 0$ if $i \equiv 0 \pmod 3$. However then $c_g(v_n) = 1$ when $n \not\equiv 2 \pmod 3$, which is a contradiction. ■

The following is a consequence of Observation 3.5.5 and Proposition 3.5.6.

Corollary 3.5.7 *Let T be a tree with $2 \leq \operatorname{diam}(T) \leq 4$.*

Then T has a 2-regular dominating function if and only if $T = P_5$.

Proof. If $T = P_5$, then T has a 2-regular dominating function by Proposition 3.5.6. It remains to verify the converse. Assume, to the contrary, that there is a tree $T \neq P_5$ with $2 \leq \text{diam}(T) \leq 4$ such that T has a 2-regular dominating function $f : V(T) \to \{0, 1\}$. By Observation 3.5.5, no vertex of T can be incident with two or more pendant edges. Since P_4 has no 2-regular dominating function by Proposition 3.5.6, it follows that T is neither a star nor a double star. Thus, $\text{diam}(T) = 4$. Let $P = (v_0, v_1, v_2, v_3, v_4)$ be a longest path in T. Then v_0 and v_4 are end-vertices of T. By Observation 3.5.5, $\deg_T v_1 = \deg_T v_3 = 2$. Furthermore, $f(v_i) = 1$ for $i \in \{0, 1, 3, 4\}$ and so $f(v_2) = 0$. Since $T \neq P_5$, it follows that $\deg_T v_2 \geq 3$. Let $X = N(v_3) - \{v_1, v_3\}$, where then $|X| \geq 1$. Let $x \in X$. Because $f(v_1) = f(v_3) = 1$ and $c_f(v_2) = 2$, it follows that $f(x) = 0$ and so x is adjacent to a vertex y distinct from v_2. Since $\text{diam}(T) = 4$, it follows that y is an end-vertex of T. Hence, $f(x) = f(y) = 1$ by Observation 3.5.5. However then, $c_f(v_2) \geq 3$, which is a contradiction. ■

We now determine all those integers $n \geq 2$ for which there is a tree of order n with a 2-regular dominating function. We begin with the following observation.

Observation 3.5.8 *Let n be an integer with $2 \leq n \leq 12$. Then there exists a tree of order n having a 2-regular dominating function if and only if $n \in \{2, 5, 8, 10, 11\}$.*

For $n \in \{2, 5, 8, 10, 11\}$, there exists a tree of order n having a 2-regular dominating function. For $n \in \{2, 5, 8, 11\}$, the path P_n has a 2-regular dominating function and the tree of order 10 in Figure 3.9 has a 2-regular dominating function. For each $n \in \{3, 4, 6, 7, 9, 12\}$, it can be verified that there does not exist a tree of order n having a 2-regular dominating function.

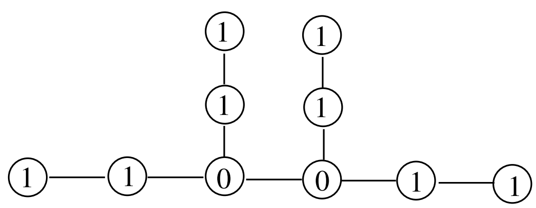

Figure 3.9: A tree of order 10 with a 2-regular dominating function

Theorem 3.5.9 *For every integer $n \geq 13$, there exists a tree T_n of order n such that T_n has a 2-regular dominating function.*

Proof. We proceed by induction on $n \geq 13$. Since each of the trees T_{13}, T_{14}, and T_{15} of Figure 3.10 has a 2-regular dominating function (shown in Figure 3.10), the statement is true for $n \in \{13, 14, 15\}$.

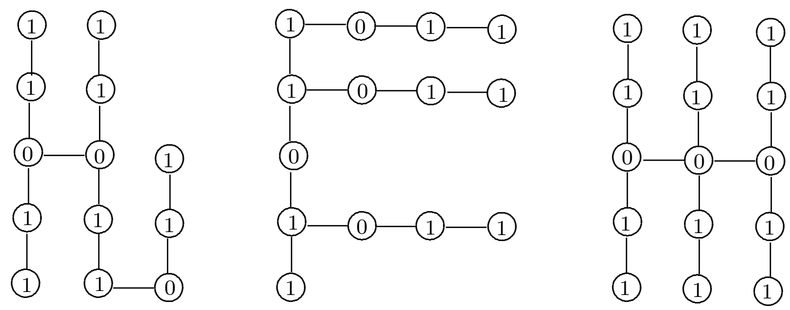

Figure 3.10: Three trees T_{13}, T_{14}, and T_{15} having a 2-regular dominating function

Assume for an integer $n \geq 16$ that there exists a tree T_k of order k for every integer k with $13 \leq k < n$ such that T_k has a 2-regular dominating function. We show that there exists a tree T_n of order n having a 2-regular dominating function. Since $n \geq 16$, it follows that $n-3 \geq 13$. Hence, there exists a tree T' of order $n-3$ having a 2-regular dominating function f'. Let $w \in V(T')$ such that $f'(w) = 1$. Let T_n be the tree of order n obtained by adding the path (x, y, z) to T' and joining x to w. Define the function $f : V(T_n) \to \{0, 1\}$ by

$$f(v) = \begin{cases} f'(v) & \text{if } v \in V(T') \\ 0 & \text{if } v = x \\ 1 & \text{if } v = y \text{ or } v = z. \end{cases}$$

Since f is a 2-regular dominating function, the statement is true for all integers $n \geq 13$. ■

Chapter 4

REGULAR DOMINATING FUNCTIONS II

ABSTRACT: Here, we investigate regular dominating functions of several well-known classes of graphs as well as regular graphs, especially 3-regular (or cubic) graphs. If a cubic graph has a k-regular dominating function, then $1 \le k \le 4$. Moreover every cubic graph has a 4-regular dominating function. The primary question then is whether a cubic graph has a k-regular dominating function for some $k \in \{1, 2, 3\}$. We characterize those connected cubic graphs having a k-regular dominating function for each $k \in \{1, 2, 3\}$. Other results and questions on regular dominating functions of connected graphs in general are also presented.

4.1 Some Properties

We begin by presenting some additional properties of dominating functions in graphs. Recall that if f is a dominating function of G, then we define $\mathcal{I}_f(G)$ as $\{v \in V(G) : f(v) = 1\}$. The following observation is immediate.

Observation 4.1.1 *Let G be a nontrivial connected graph and let $f : V(G) \to \{0, 1\}$ be a dominating function of G. Then*

$$\sum_{v \in V(G)} c_f(v) = \sum_{x \in \mathcal{I}_f(G)} (\deg x + 1) f(x) = \sum_{x \in \mathcal{I}_f(G)} (\deg x + 1).$$

The following is an immediate consequence of Observation 4.1.1.

Proposition 4.1.2 *Let G be a nontrivial connected graph of order n and let $f : V(G) \to \{0,1\}$ be a dominating function of G where $|\mathcal{I}_f(G)| = s$.*

(1) *If G is an r-regular graph for some integer $r \geq 2$, then*

$$\textstyle\sum_{v \in V(G)} c_f(v) = (r+1)s.$$

(2) *If f is a k-regular dominating function for some integer $k \geq 1$, then*

$$\textstyle\sum_{v \in V(G)} c_f(v) = nk.$$

Proposition 4.1.3 *Let G be a nontrivial connected graph. Then G has a k-regular dominating function for some positive integer k if and only if either G is $(k-1)$-regular or G consists of two induced vertex-disjoint subgraphs F and H, where the vertex set of G is partitioned into $V(F)$ and $V(H)$, such that F is $(k-1)$-regular and each vertex of H is adjacent to exactly k vertices in F.*

Proof. First, suppose that G has a k-regular dominating function $f : V(G) \to \{0,1\}$ for some positive integer k. Let $F = G[\mathcal{I}_f]$ and $H = G[\overline{\mathcal{I}}_f]$ (if $\overline{\mathcal{I}}_f \neq \emptyset$). Since $c_f(v) = k$ for each $v \in V(G)$, it follows that every vertex in F is adjacent to exactly $k-1$ vertices in F and so F is $(k-1)$-regular and every vertex in H is adjacent to exactly k vertices in F. Next, we verify the converse. Since the statement is true if G is $(k-1)$-regular, we may assume that G is constructed from two induced vertex-disjoint subgraphs F and H such that F is $(k-1)$-regular and each vertex of H is adjacent to exactly k vertices in F. Then a k-regular dominating function of G can be defined by assigning 0 to each vertex of H and assigning 1 to each vertex of F. ■

The following useful observations are consequences of Proposition 4.1.3.

Corollary 4.1.4 *A nontrivial connected graph G has a 1-regular dominating function if and only if G has a dominating set W such that $d(u,v) \geq 3$ for every two vertices u and v of W.*

Corollary 4.1.5 *Let f be a 2-regular dominating function of a nontrivial connected graph G. Then the edge set of the subgraph $G[\mathcal{I}_f]$ induced by $\mathcal{I}_f$ is a matching of G and so $|\mathcal{I}_f|$ is even.*

4.2 Some Well-known Graphs

We now apply the results obtained earlier to study regular dominating functions in some well-known classes of graphs. The *Cartesian product* $G \square H$ of two graphs G and H has vertex set $V(G \square H) = V(G) \times V(H)$ and two distinct vertices (u, v) and (x, y) of $G \square H$ are adjacent if either (1) $u = x$ and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(G)$. The Cartesian product $G \square K_2$ of a graph G and K_2 is a special case of a more general class of graphs. For an integer $n \geq 2$, the graph $P_n \square K_2$ is often referred to as a *grid graph*. By Corollaries 4.1.4 and 4.1.5, we see that, while the 4-cycle $C_4 = P_2 \square K_2$ has a 3-regular dominating function, it has neither a 1-regular nor a 2-regular dominating function.

Proposition 4.2.1 *For each integer $n \geq 3$, the grid graph $P_n \square K_2$ has a k-regular dominating function for $k \in \{1, 2\}$ if and only if n is odd. Furthermore, $P_n \square K_2$ has no 3-regular dominating function for any integer $n \geq 3$.*

Proof. Let $G = P_n \square K_2$ be constructed from the two copies $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ of the path P_n of order n by adding the edges u_iv_i for $1 \leq i \leq n$. First, we show that if $n \geq 4$ is even, then G does not have a k-regular dominating function for each $k \in \{1, 2\}$. We consider two cases.

Case 1. $k = 1$. Suppose that $f : V(G) \to \{0, 1\}$ is a 1-regular dominating function of G. Observe that exactly one of u_1 and v_1 must be assigned 1 by f, for otherwise, if $f(u_1) = f(v_1) = 1$, then $c_f(u_1) \geq 2$; while if $f(u_1) = f(v_1) = 0$, then $f(u_2) = f(v_2) = 1$ and so $c_f(u_2) \geq 2$, which is impossible in either case. Similarly, exactly one of u_n and v_n must be assigned 1 by f. Let $\mathcal{I}_f = \{v \in V(G) : f(v) = 1\}$. Then $s = |\mathcal{I}_f| \geq 2$. By Observation 4.1.1,

$$2n = \sum_{v \in V(G)} c_f(v) = \sum_{v \in \mathcal{I}_f} (\deg(v) + 1) = 3 \cdot 2 + 4(s - 2).$$

Consequently, $n = 3 + 2(s - 2)$ is odd.

Case 2. $k = 2$. Suppose that $f : V(G) \to \{0, 1\}$ is a 2-regular dominating function of G. Observe that both of u_1 and v_1 must be assigned 1 by f, for otherwise, suppose that $f(u_1) = 0$. Then $f(v_1) = f(u_2) = f(v_2) = 1$ and so $c_f(v_2) \geq 3$, which is impossible. Similarly, both u_n and v_n must be assigned 1 by f. Then $s = |\mathcal{I}_f| \geq 4$. Since the edge set of $G[\mathcal{I}_f]$ is a matching, s is even. By Observation 4.1.1,

$$4n = \sum_{v \in V(G)} c_f(v) = \sum_{v \in \mathcal{I}_f} (\deg(v) + 1) = 3 \cdot 4 + 4(s - 4).$$

Consequently, $n = 3 + (s - 4)$ is odd.

For the converse, assume that $n \geq 3$ is odd. We show that there is both a 1-regular and a 2-regular dominating function of G. Define $f_1 : V(G) \to \{0, 1\}$ by $f_1(u_i) = 1$ for $i \equiv 1 \pmod 4$ and $f_1(v_j) = 0$ for $j \equiv 3 \pmod 4$ and let $f_1(w) = 0$ for any other vertex $w \in V(G)$. Define $f_2 : V(G) \to \{0, 1\}$ by $f_2(u_i) = f_2(v_i) = 1$ if i is odd and $f_2(u_i) = f_2(v_i) = 0$ if i is even. Then f_1 is a 1-regular dominating function of G and f_2 is a 2-regular dominating function of G.

Next, we show that G does not have 3-regular dominating function for each integer $n \geq 3$. Assume, to the contrary, that there is an integer $n \geq 3$ such that $G = P_n \,\square\, K_2$ has a 3-regular dominating function $g : V(G) \to \{0, 1\}$. Since $c_g(u_1) = c_g(v_1) = 3$, it follows that $g(x) = 1$ for each $x \in \{u_1, u_2, v_1, v_2\}$. This forces $g(u_3) = g(v_3) = 0$ so that $c_g(u_3) \leq 2$, which is impossible. ■

The following lemma will be useful in studying regular dominating functions in connected graphs containing many vertices with the same neighborhood.

Lemma 4.2.2 *Let G be a nontrivial connected graph and f a regular dominating function of G. If $u, v \in V(G)$ such that $N(u) = N(v)$, then $f(u) = f(v)$.*

Proof. Let $u, v \in V(G)$ such that $N(u) = N(v) = X$. Since f is a regular dominating function of G, it follows that

$$c_f(u) = f(u) + \textstyle\sum_{x \in X} f(x) = f(v) + \sum_{x \in X} f(x) = c_f(v)$$

and so $f(u) = f(v)$. ■

With the aid of Lemma 4.2.2, we now present a result on regular dominating functions of complete multipartite graphs. Recall that the minimum number of vertices in a dominating set of graph G is the domination number of G, $\gamma(G)$.

Proposition 4.2.3 *A complete multipartite graph G has a regular dominating function if and only if either G is regular or $\gamma(G) = 1$.*

Proof. Let G be a complete multipartite graph of order n. Thus, G has two or more partite sets. Suppose, first, that G is r-regular for some integer $r \geq 1$. Then G has an $(r+1)$-regular dominating function. If $\gamma(G) = 1$, then G contains a vertex that is adjacent to every other vertex in G and so $\Delta(G) = n - 1$. Thus, G has a 1-regular dominating function. For the converse, suppose that G has a k-regular dominating function for some positive integer k. Assume, to the contrary, that G is not regular and $\Delta(G) \neq n-1$, but G has a regular dominating function f. Thus, every partite set of G contains at least two vertices and there are two partite sets U and W such that $|U| \neq |W|$. Let $X = V(G) - (U \cup W)$ and $A = \sum_{x \in X} f(x)$ (where $A = 0$ if $X = \emptyset$). As a consequence of Lemma 4.2.2, we consider two cases.

Case 1. *$f(u) \neq f(w)$ when $u \in U$ and $w \in W$.* We may assume that $f(u) = 1$ for all $u \in U$ and $f(w) = 0$ for all $w \in W$. If $u \in U$ and $w \in W$, then $c_f(u) = 1 + A$ and $c_f(w) = |U| + A$. Since $|U| \geq 2$, it follows that $c_f(u) \neq c_f(w)$, which is impossible.

Case 2. *$f(u) = f(w)$ when $u \in U$ and $w \in W$.* First, suppose that $f(v) = 1$ for all $v \in U \cup W$. If $u \in U$ and $w \in W$, then $c_f(u) = 1 + |W| + A$ and $c_f(w) = 1 + |U| + A$. Since $|U| \neq |W|$, it follows that $c_f(u) \neq c_f(w)$, which is impossible. Next, suppose $f(v) = 0$ for all $v \in U \cup W$. Then $X \neq \emptyset$ and so $A > 0$. Since $c_f(v) = A > 0$ for each $v \in U \cup W$, there is a partite set Y of G such that $f(y) = 1$ for each $y \in Y$. If $y \in Y$, then $c_f(y) = 1 + (A - |Y|)$. Since $|Y| \geq 2$, it follows that $c_f(v) \neq c_f(y)$ for each $v \in U \cup W$ and $y \in Y$, which is a contradiction. ■

4.3 Graphs Having a 1-Regular Dominating Function

The following result describes a property possessed by a 1-regular dominating function.

Theorem 4.3.1 *Let G be a nontrivial connected graph and suppose that f_1 and f_2 are 1-regular dominating functions of G, where $\mathcal{I}_{f_i} = \{v \in V(G) :\ f_i(v) = 1\}$ for $i = 1, 2$. Then $|\mathcal{I}_{f_1}| = |\mathcal{I}_{f_2}|$, that is, the number of vertices assigned 1 by a 1-regular dominating function of G is unique.*

Proof. Observe that $\mathcal{I}_{f_1}$ and $\mathcal{I}_{f_2}$ are independent dominating sets in G. Additionally, every vertex in $V(G) - \mathcal{I}_{f_1}$ is adjacent to exactly one vertex in $\mathcal{I}_{f_1}$ and every vertex in $V(G) - \mathcal{I}_{f_2}$ is adjacent to exactly one vertex in $\mathcal{I}_{f_2}$. Since $\mathcal{I}_{f_1}$ and $\mathcal{I}_{f_2}$ are independent sets, each vertex in $\mathcal{I}_{f_1} - \mathcal{I}_{f_2}$ is adjacent to exactly one vertex in $\mathcal{I}_{f_2} - \mathcal{I}_{f_1}$ and each vertex in $\mathcal{I}_{f_2} - \mathcal{I}_{f_1}$ is adjacent to exactly one vertex in $\mathcal{I}_{f_1} - \mathcal{I}_{f_2}$. This implies that the set of edges between $\mathcal{I}_{f_1} - \mathcal{I}_{f_2}$ and $\mathcal{I}_{f_1} - \mathcal{I}_{f_2}$ form a matching between these sets. It therefore follows that $|\mathcal{I}_{f_1} - \mathcal{I}_{f_2}| = |\mathcal{I}_{f_2} - \mathcal{I}_{f_1}|$. Since

$$|\mathcal{I}_{f_1}| = |\mathcal{I}_{f_1} - \mathcal{I}_{f_2}| + |\mathcal{I}_{f_1} \cap \mathcal{I}_{f_2}| \text{ and } |\mathcal{I}_{f_2}| = |\mathcal{I}_{f_2} - \mathcal{I}_{f_1}| + |\mathcal{I}_{f_1} \cap \mathcal{I}_{f_2}|,$$

it follows that $|\mathcal{I}_{f_1}| = |\mathcal{I}_{f_2}|$. ■

We now characterize those connected graphs having a 1-regular dominating function.

Theorem 4.3.2 *A nontrivial connected graph G has a 1-regular dominating function if and only if there exists a partition $\{V_1, V_2, \ldots, V_t\}$ of $V(G)$ where $|V_i| = n_i$ for $1 \le i \le t$ such that $K_{1,n_i-1} \subseteq G[V_i]$ and the center of K_{1,n_i-1} has degree $n_i - 1$ in G.*

Proof. First, suppose that G has a 1-regular dominating function $f : V(G) \to \{0, 1\}$. Let $\mathcal{I}_f = \{v_1, v_2, \ldots, v_t\}$ for some positive integer t. Since $c_f(v) = 1$ for all $v \in V(G)$, it follows that $f(u) = 0$ for each $u \in N(v_i)$. For $1 \le i \le t$, let $V_i = N[v_i]$ be the closed neighborhood of v_i. Then $|V_i| = 1 + \deg_G v_i = n_i$ and $K_{1,n_i-1} \subseteq G[V_i]$ for $1 \le i \le t$. Furthermore, $\{V_1, V_2, \ldots, V_t\}$ is a partition of $V(G)$.

For the converse, suppose that $\{V_1, V_2, \ldots, V_t\}$ is a partition of $V(G)$ such that $|V_i| = n_i$, $K_{1,n_i-1} \subseteq G[V_i]$, and the center v_i of K_{1,n_i-1} has degree $n_i - 1$ in G for $1 \le i \le t$. Define the function $f : V(G) \to \{0, 1\}$ by $f(v_i) = 1$ for $1 \le i \le t$ and $f(u) = 0$ otherwise. We show that f is a 1-regular dominating function of G. Let $x \in V(G)$. Then $x \in V_i$ for some integer i with $1 \le i \le t$. If $x = v_i$, then $f(x) = 1$ and $f(y) = 0$ for each $y \in N(x) = V_i - \{x\}$, implying that $c_f(x) = 1$. If $x \ne v_i$, then x is adjacent to v_i and x is not adjacent to v_j for each integer j with $1 \le j \le t$ and $j \ne i$, which implies that $c_f(x) = 1$. Consequently, f is a 1-regular dominating function of G. ■

By Theorem 4.3.1, the cardinality of a dominating set W of a graph in Corollary 4.1.4 and the size t of a partition of the vertex set of a graph in Theorem 4.3.2

is unique. On the other hand, the partition of the vertex set in Theorem 4.3.2 may not be unique. For example, the graph G of order 8 in Figure 4.1 has two distinct 1-regular dominating functions, which give rise to two distinct partitions of $V(G)$ as described in Theorem 4.3.2.

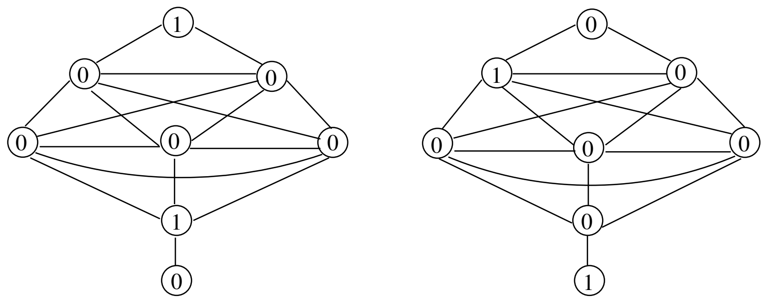

Figure 4.1: The partition of the vertex set in Theorem 4.3.2 is not unique

4.4 Regular Dominating Functions in Regular Graphs

We saw that if G is a connected r-regular graph with a k-regular dominating function, then $1 \leq k \leq r+1$. Furthermore, every connected r-regular graph G has an $(r+1)$-regular dominating function, while there is no guarantee that G has a k-regular dominating function for a given integer k with $1 \leq k \leq r$. In this section, we study those connected r-regular graphs having a k-regular dominating function for a given integer k with $1 \leq k \leq r$.

Let G be a nontrivial connected graph and let $f : V(G) \to \{0,1\}$ be a function of G. The *complementary function* $\overline{f} : V(G) \to \{0,1\}$ is defined by

$$\overline{f}(v) = 1 - f(v) \text{ for every vertex } v \text{ of } G.$$

Observation 4.4.1 *Let G be a nontrivial connected graph. If $f : V(G) \to \{0,1\}$ is a function of G, then $c_f(v) + c_{\overline{f}}(v) = 1 + \deg v$ for each vertex v of G.*

Proof. Let $v \in V(G)$ where $f(v) = i \in \{0,1\}$ and $c_f(v) = k$. Then $\overline{f}(v) = 1 - i$. Since $c_f(v) = k$ and $f(v) = i$, it follows that v is adjacent to $k-i$ vertices labeled 1 by f and so v is adjacent to $\deg v - k + i$ vertices labeled 0 by f. Thus, v is adjacent to $\deg v - k + i$ vertices labeled 1 by $\overline{f}$. Hence,

$$c_{\overline{f}}(v) = (1-i) + (\deg v - k + i) = 1 + \deg v - k.$$

Consequently, $c_f(v) + c_{\overline{f}}(v) = \deg v + 1$. ■

The complementary function of a dominating function of a graph may or may not be a dominating function of the graph. For example, for the dominating function f of the tree T in Figure 4.2(a), its complementary function is not a dominating function of T; while for the dominating function g of the tree T in Figure 4.2(b), its complementary function is a dominating function of T.

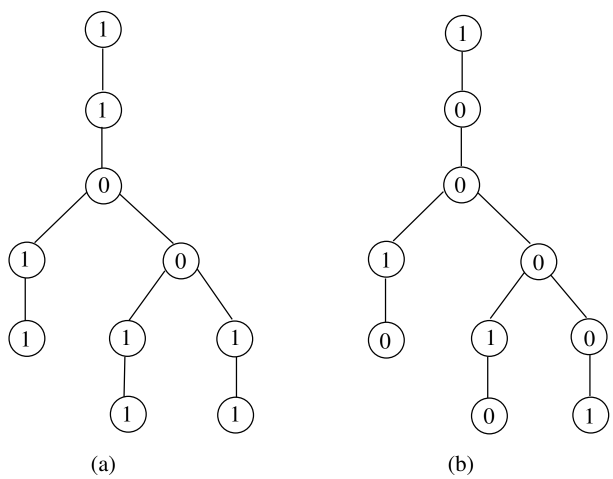

Figure 4.2: Two dominating functions of a tree

The following is an immediate consequence of Observation 4.4.1.

Corollary 4.4.2 *Let G be a nontrivial connected graph. Suppose that f is a dominating function of G such that $\overline{f}$ is also a dominating function of G. Then f and $\overline{f}$ are both regular if and only if G is regular. Furthermore, if G is an r-regular graph and f is a k-regular dominating function of G where $1 \leq k \leq r+1$, then $\overline{f}$ is an $(r+1-k)$-regular dominating function of G.*

For example, Figure 4.3(a) shows a 1-regular dominating function f of C_6. Its complementary function $\overline{f}$, shown in Figure 4.3(b), is a 2-regular dominating function of C_6.

This example illustrates the following result concerning connected 2-regular graphs, namely cycles. Since the cycle C_n of order $n \geq 3$ is 2-regular, it has a 3-regular dominating function. We now determine other values of k for which cycles have a k-regular dominating function.

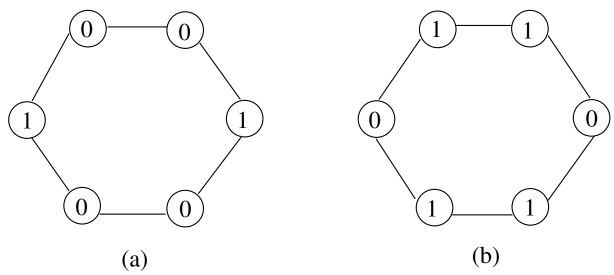

Figure 4.3: Two regular dominating functions of C_6

Proposition 4.4.3 *For $k \in \{1, 2\}$, the cycle C_n of order $n \geq 3$ has a k-regular dominating function if and only if $n \equiv 0 \pmod 3$.*

Proof. First, suppose that C_n has a k-regular dominating function f with $k \in \{1, 2\}$. We show that $n \equiv 0 \pmod 3$. Assume that $|\mathcal{I}_f| = s$. By Proposition 4.1.2, $3s = nk$. If $k = 1$, then $3s = n$; while if $k = 2$, then $3s = 2n$. In either case, $3 \mid n$ and so $n \equiv 0 \pmod 3$. For the converse, suppose that $n \equiv 0 \pmod 3$. We show that C_n has a k-regular dominating function for each $k \in \{1, 2\}$. For $C_n = (v_1, v_2, \ldots, v_n, v_1)$, define a 1-regular dominating function $f : V(G) \to \{0, 1\}$ by

$$f(v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{f} : V(G) \to \{0, 1\}$ is a 2-regular dominating function of C_n by Corollary 4.4.2. ■

Next, we consider connected cubic graphs. As we saw, every such graph has a 4-regular dominating function. In order to characterize those connected cubic graphs having a k-regular dominating function for each $k \in \{1, 2, 3\}$, we first present a necessary condition on the order of such cubic graphs.

Proposition 4.4.4 *Let G be a connected cubic graph of order $n \geq 4$. If G has a k-regular dominating function for some integer $k \in \{1, 2, 3\}$, then $n \equiv 0 \pmod 4$.*

Proof. Let G be a connected cubic graph of order n and let $f : V(G) \to \{0, 1\}$ be a k-regular dominating function where $k \in \{1, 2, 3\}$. Since n is even, $n = 2p$ for some integer $p \geq 2$. Suppose that $|\mathcal{I}_f| = s$. Then $4s = nk = 2pk$ and so $2s = pk$

by Proposition 4.1.2. If $k = 1$, then $2s = p$ and so $n = 4s$; if $k = 2$, then $s = p$ is even and so $n = 2s$; and if $k = 3$, then $2s = 3p$ and so $3 \mid s$. Let $s = 3r$ for some positive integer r. Then $n = 4r$. ■

Consequently, the famous Petersen graph of order 10 has a k-regular dominating function only if $k = 4$. By Proposition 4.4.4, we consider only those connected cubic graphs of order n with $n \equiv 0 \pmod 4$ and determine which such graphs have a k-regular dominating function for some integer $k \in \{1, 2, 3\}$. The following is a consequence of Theorem 4.3.2.

Corollary 4.4.5 *Let G be a connected cubic graph of order $n = 4t$ for some positive integer t. Then G has a 1-regular dominating function if and only if there exists a partition $\{V_1, V_2, \ldots, V_t\}$ of $V(G)$ such that $|V_i| = 4$ and $K_{1,3} \subseteq G[V_i]$ for $1 \leq i \leq t$.*

The following observation is a consequence of Corollary 4.4.5 (and Corollary 4.4.2).

Observation 4.4.6 *For a connected cubic graph G of order $n \geq 4$ where $n \equiv 0 \pmod 4$, a function $f : V(G) \to \{0, 1\}$ is a 1-regular dominating function of G if and only if its complementary function $\overline{f} : V(G) \to \{0, 1\}$ is a 3-regular dominating function of G.*

By Corollary 4.4.5 and Observation 4.4.6, we then have the following.

Corollary 4.4.7 *Let G be a connected cubic graph of order $n = 4t$ for some positive integer t. Then G has a 3-regular dominating function if and only if there exists a partition $\{V_1, V_2, \ldots, V_t\}$ of $V(G)$ such that $|V_i| = 4$ and $K_{1,3} \subseteq G[V_i]$ for $1 \leq i \leq t$.*

Now that we know exactly which cubic graphs have 1-regular dominating functions, we characterize connected cubic graphs having a 2-regular dominating function. For two disjoint sets U and W of vertices in a graph G, let $[U, W]$ be the set of edges joining a vertex of U and a vertex of W in G. The bipartite subgraph of G induced by the set $[U, W]$ of edges in G is denoted by $G[U, W]$.

Theorem 4.4.8 *Let G be a connected cubic graph of order $n = 4t$ for some positive integer t. Then G has a 2-regular dominating function if and only if there exists a partition $\{U, W\}$ of $V(G)$ where $|U| = |W| = 2t$ such that $G[U] \cong G[W] \cong tK_2$ and $G[U, W]$ is a 2-regular subgraph of G.*

Proof. First, suppose that G has a 2-regular dominating function $f : V(G) \to \{0, 1\}$. Let $U = \mathcal{I}_f$ and $W = V(G) - U$. Then the edge set of $G[U]$ is a matching. Let $w \in W$. Since $c_f(w) = 2$, it follows that w is adjacent to exactly two vertices of U and exactly one vertex in W. This implies that $G[W]$ is a 1-regular subgraph of G and $G[U, W]$ is a 2-regular bipartite subgraph of G. Consequently, $|U| = |W| = 2t = n/2$ and $G[U] \cong G[W] \cong tK_2$.

For the converse, suppose that there exists a partition $\{U, W\}$ of $V(G)$ where $|U| = |W| = 2t$ such that $G[U] \cong G[W] \cong tK_2$ and $G[U, W]$ is a 2-regular subgraph of G. Then the function $f : V(G) \to \{0, 1\}$ defined by $f(u) = 1$ for each $u \in U$ and $f(w) = 0$ for each $w \in W$ is a 2-regular dominating function of G. ■

The following is a consequence of Theorem 4.4.8. A 1-*factor* in a graph G is a 1-regular spanning subgraph of G, also known as a matching.

Corollary 4.4.9 *If G is a connected cubic graph with a 2-regular dominating function, then G has a 1-factor.*

The converse of Corollary 4.4.9 is not true, however. In fact, there is an infinite class of connected cubic graphs having a 1-factor but no 2-regular dominating function, as we show next. For an integer $\ell \geq 3$, let $C_\ell = (z_1, z_2, \ldots, z_\ell, z_1)$ be the cycle of order ℓ. For $1 \leq i \leq \ell$, let L_i be the graph of order 5 obtained from the 5-cycle $(u_i, v_i, w_i, x_i, y_i, u_i)$ by adding two chords $u_i x_i$ and $w_i y_i$. The cubic graph F_ℓ of order 6ℓ is constructed by joining v_i to z_i for $1 \leq i \leq \ell$. The graph F_4 of order 24 is shown in Figure 4.4.

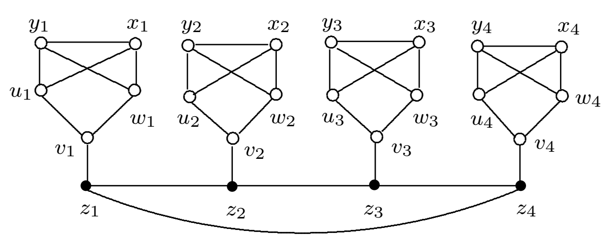

Figure 4.4: The cubic graph F_4

For each integer $\ell \geq 3$, the graph F_ℓ has a 1-factor. If $\ell \geq 3$ is odd, then $6\ell \not\equiv 0 \pmod 4$ and so it follows by Proposition 4.4.4 that F_ℓ does not have any k-regular dominating function for each $k \in \{1, 2, 3\}$. If $\ell \geq 4$ is even, then $6\ell \equiv 0 \pmod 4$. We show, even in this case, that F_ℓ does not have any k-regular dominating function for each $k \in \{1, 2, 3\}$.

Theorem 4.4.10 *For each even integer $\ell \geq 4$, the graph F_ℓ of order $6\ell \equiv 0 \pmod 4$ has no k-regular dominating function for any $k \in \{1,2,3\}$.*

Proof. First, we show that F_ℓ does not have a 1-regular dominating function. Assume, to the contrary, that there is an even integer $\ell \geq 3$ such that F_ℓ has a 1-regular dominating function $f : V(F_\ell) \to \{0,1\}$. Since $N(u_i) = N(w_i)$ and $v_i \in N(u_i) \cap N(w_i)$ for $1 \leq i \leq \ell$, it follows by Lemma 4.2.2 that $f(u_i) = f(w_i) = 0$. In particular, $f(u_1) = f(w_1) = 0$. Hence, exactly one of v_1, x_1, y_1 has f-value 1. If $f(v_1) = 1$, then $f(x_1) = f(y_1) = 0$ and so $c_f(x_1) = c_f(y_1) = 0$, which is impossible. Thus, we may assume that $f(x_1) = 1$ and $f(y_1) = f(v_1) = 0$. This forces $f(z_1) = 1$ and so $f(z_2) = 0$. Since $f(z_2) = f(u_2) = f(w_2) = 0$, it follows that $f(v_2) = 1$. However then, $f(x_2) = f(y_2) = 0$ and so $c_f(x_2) = c_f(y_2) = 0$, which is impossible. Consequently, F_ℓ has no 1-regular dominating function and so F_ℓ has no 3-regular dominating function by Observation 4.4.6.

Next, we show that that F_ℓ does not have a 2-regular dominating function. Assume, to the contrary, that there is an even integer $\ell \geq 3$ such that F_ℓ has a 2-regular dominating function $g : V(H_\ell) \to \{0,1\}$. Again, it follows by Lemma 4.2.2 that $g(u_i) = g(w_i)$ for $1 \leq i \leq \ell$. In particular, $g(u_1) = g(w_1)$.

- ⋆ First, suppose that $g(u_1) = g(w_1) = 1$. Since v_1 is adjacent to u_1 and w_1 and $c_f(v_1) = 2$, it follows that $g(v_1) = g(z_1) = 0$. This forces $g(z_2) = g(z_\ell) = 1$. Since $c_g(v_2) = 2$, it follows that $g(u_2) = g(w_2) = 0$ and $g(v_2) = 1$. Since $c_g(u_2) = c_g(w_2) = 2$, it follows that $\{g(y_2), g(x_2)\} = \{0,1\}$. We may assume that $g(y_2) = 1$ and $g(x_2) = 0$. However then, $c_g(x_2) = 1$, which is impossible.
- ⋆ Next, suppose that $g(u_1) = g(w_1) = 0$. Since $c_g(v_1) = 2$, it follows that $f(v_1) = f(z_1) = 1$. Since $c_g(u_1) = c_g(w_1) = 2$, it follows that $\{g(y_1), g(x_1)\} = \{0,1\}$. We may assume that $g(y_1) = 1$ and $g(x_1) = 0$. However then, $c_g(x_1) = 1$, which is impossible.

Therefore, F_ℓ does not have a 2-regular dominating function. ■

By Theorem 4.4.10, there are connected cubic graphs of order n with $n \equiv 0 \pmod 4$ without any k-regular dominating function for each $k \in \{1,2,3\}$. On the other hand, for each integer $n \geq 4$ with $n \equiv 0 \pmod 4$, there is a connected cubic graph of order n with a k-regular dominating function for each $k \in \{1,2,3\}$.

Theorem 4.4.11 *For each integer $n \geq 4$ with $n \equiv 0 \pmod 4$, there is a connected cubic graph of order n with a k-regular dominating function for each $k \in \{1, 2, 3\}$.*

Proof. Let $n = 4t$ for some positive integer t. Since K_4 is the only cubic graph of order 4 and K_4 has a k-regular dominating function for each $k \in \{1, 2, 3\}$, we may assume that $n \geq 8$. We begin with the n-cycle

$$C_n = (v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}, \ldots, v_{t,1}, v_{t,2}, v_{t,3}, v_{t,4}, v_{1,1}).$$

The cubic graph G is constructed by adding the edges $v_{i,1}v_{i,3}$ and $v_{i,2}v_{i,4}$ for $1 \leq i \leq t$. That is, if $V_i = \{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}\}$ for $1 \leq i \leq t$, then $G[V_i] = K_4 - v_{i,1}v_{i,4}$.

- ⋆ For $k = 1, 3$, let $\{V_1, V_2, \ldots, V_t\}$ be a partition of $V(G)$. Since $|V_i| = 4$ and $K_{1,3} \subseteq G[V_i] = K_4 - e$ for $1 \leq i \leq t$, it follows by Corollary 4.4.5 that G has a k-regular dominating function for $k = 1, 3$. In fact, a 1-regular dominating function $f_1 : V(G) \to \{0, 1\}$ can be defined by $f_1(v_{i,2}) = 1$ for $1 \leq i \leq t$ and $f(x) = 0$ otherwise. Then $f_3 = \overline{f}_1$ is a 3-regular dominating function of G.

- ⋆ For $k = 2$, let $U = \{v_{i,1}, v_{i,2} : 1 \leq i \leq t\}$ and $W = \{v_{i,3}, v_{i,4} : 1 \leq i \leq t\}$. Then $\{U, W\}$ is a partition of $V(G)$. Since $G[U] \cong G[W] \cong tK_2$ and $G[U, W] \cong C_n$, it follows by Theorem 4.4.8 that G has a 2-regular dominating function. In fact, a 2-regular dominating function $f_2 : V(G) \to \{0, 1\}$ can be defined by $f_2(u) = 0$ for each $u \in U$ and $f_2(w) = 1$ for each $w \in W$. The complementary function $\overline{f}_2$ of f_2 is a also 2-regular dominating function of G. ■

By Theorems 4.4.10 and 4.4.11, there are connected cubic graphs of order $n \equiv 0 \pmod 4$ that have no k-regular dominating function for any $k \in \{1, 2, 3\}$ and there are connected cubic graphs of order $n \equiv 0 \pmod 4$ that have k-regular dominating function for each $k \in \{1, 2, 3\}$. We saw that if a connected cubic graph G has a 1-regular dominating function, then G has a 3-regular dominating function and vice versa. However, there are connected cubic graphs (i) having 1-regular or 3-regular dominating functions but no 2-regular dominating functions or (ii) having 2-regular dominating functions but neither a 1-regular nor a 3-regular dominating function. As an example, we consider a well-known class of cubic graphs, namely prisms.

Theorem 4.4.12 *For $n \geq 3$, let $G = C_n \,\Box\, K_2$ be the prism of order $2n$. Then*

(*i*) *G has a 1-regular and a 3-regular dominating function only if $n \equiv 0 \pmod 4$ and*

(*ii*) *G has a 2-regular dominating function only if $n \equiv 0 \pmod 2$.*

Proof. Let $G = C_n \,\Box\, K_2$ be constructed from two copies $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ of the n-cycle by adding the edges u_iv_i for $1 \leq i \leq n$.

To verify (i), first suppose that $n \equiv 0 \pmod 4$. Then a 1-regular dominating function $f : V(G) \to \{0, 1\}$ of G can be defined by

$$f(w) = \begin{cases} 1 & \text{if } w = u_i \text{ where } i \equiv 1 \pmod 4 \\ & \text{or } w = v_j \text{ where } j \equiv 3 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

By Observation 4.4.6, its complementary function $\overline{f} : V(G) \to \{0, 1\}$ is a 3-regular dominating function of G.

Conversely, suppose that G has a 1-regular dominating function. By Proposition 4.4.4, it follows that n is even and so G is a bipartite graph. We claim that $n \equiv 0 \pmod 4$. Assume, to the contrary, that $n \equiv 2 \pmod 4$. Thus, $n = 4p + 2$ for some positive integer p and so the order of G is $n_G = 8p + 4$. Since G has a 1-regular dominating function, it follows by Theorem 4.4.5 that there exists a partition $\{V_1, V_2, \ldots, V_t\}$ of $V(G)$ such that $|V_i| = 4$ and $K_{1,3} \subseteq G[V_i]$ for $1 \leq i \leq t$. Thus, $t = n_G/4 = 2p + 1$. Since G is a bipartite cubic graph and $K_{1,3} \subseteq G[V_i]$ for $1 \leq i \leq t$, it follows that $G[V_i] \cong K_{1,3}$. We may assume that $V_1 = \{u_1, v_1, u_2, u_n\}$. For $1 \leq i \leq t$, let a_i be the subscript of the center of $G[V_i]$. Thus, $a_1 = 1$. We may further assume that $1 = a_1 < a_2 < \cdots < a_t$. Observe that a_2 is the subscript of v_3, a_3 is the subscript of u_5, a_4 is the subscript of v_7, and so on. In general, a_i is the subscript of $u_{1+4(i-1)}$ if i is odd and $1 \leq i \leq t$ and a_i is the subscript of $v_{3+4(i-1)}$ if i is even and $2 \leq i \leq t - 1$. In particular, a_t is the subscript of $u_{8p+3} = u_{n-1}$. However then, $u_n \in V_{t-1} \cap V_t$, which is impossible. Thus, (i) holds.

To verify (ii), first suppose that $n \geq 4$ is even. Then a 2-regular dominating function $f : V(G) \to \{0, 1\}$ of G can be defined by

$$f(w) = \begin{cases} 1 & \text{if } w \in \{u_i, v_i\} \text{ where } i \text{ is odd} \\ 0 & \text{if } w \in \{u_i, v_i\} \text{ where } i \text{ is even.} \end{cases}$$

Conversely, suppose that G has a 2-regular dominating function. Then $2n \equiv 0 \pmod 4$ by Proposition 4.4.4 and so n is even. ■

The following is a consequence of Theorem 4.4.12.

Proposition 4.4.13 *For each integer $n \geq 12$ with $n \equiv 4 \pmod 8$, there is a connected cubic graph of order n that has a 2-regular dominating function but has neither a 1-regular nor a 3-regular dominating function.*

Proof. For $n \geq 12$ with $n \equiv 4 \pmod 8$, let $n = 8p + 4$ for some positive integer p and let $G = C_{4p+2} \,\square\, K_2$ be the prism graph of order n. It then follows by Theorem 4.4.12 that G has a 2-regular dominating function but G has neither a 1-regular nor a 3-regular dominating function. ■

The cubic graph G of order 16 in Figure 4.5 has both a 1-regular and 3-regular dominating function but no 2-regular dominating function. The dominating function f of G shown in Figure 4.5 is 1-regular and $\overline{f}$ is a 3-regular dominating function. It therefore remains to show that G has no 2-regular dominating function. Since G does not have a 1-factor, there is no partition $\{U, W\}$ of $V(G)$ where $|U| = |W| = 8$ such that $G[U] \cong G[W] \cong 4K_2$. Thus, G has no 2-regular dominating function by Theorem 4.4.8.

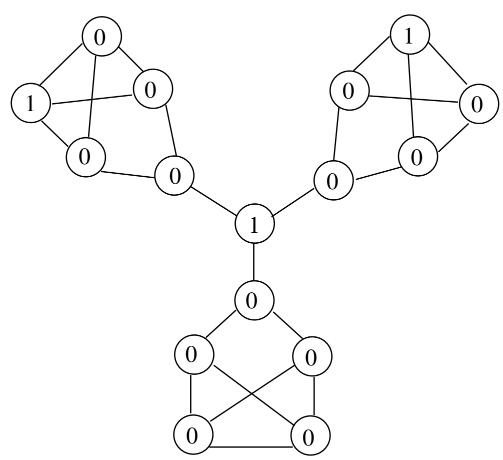

Figure 4.5: A cubic graph with no 2-regular dominating function

While the cubic graph in Figure 4.5 contains a cut-vertex, there are 2-connected cubic graphs of order n with $n \equiv 0 \pmod 4$ having a 1-regular and a 3-regular dominating function but no 2-regular dominating function. For example, let $C_\ell =$

$(a_1, a_2, \ldots, a_\ell, a_1)$ and $C'_\ell = (b_1, b_2, \ldots, b_\ell, b_1)$ be two copies of the cycle of order $\ell \geq 3$. For $1 \leq i \leq \ell$, let L_i be the graph of order 6 obtained from the 6-cycle $(u_i, v_i, w_i, x_i, y_i, z_i, u_i)$ by adding two chords $u_i x_i$ and $w_i z_i$. The cubic graph G_ℓ of order 8ℓ is constructed by joining v_i to a_i and joining y_i to b_i for for $1 \leq i \leq \ell$. The graph G_3 of order 24 is shown in Figure 4.6.

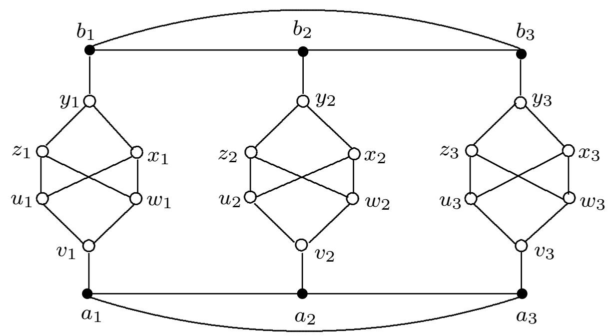

Figure 4.6: A 2-connected cubic graph G_3 of order 24

Proposition 4.4.14 *For each integer $\ell \geq 3$, the 2-connected cubic graph G_ℓ of order 8ℓ has 1-regular and 3-regular dominating functions but no 2-regular dominating function.*

Proof. First, a 1-regular dominating function $g : V(G_\ell) \to \{0, 1\}$ can be defined such that $g(v_i) = g(y_i) = 1$ for $1 \leq i \leq \ell$ and $g(x) = 0$ otherwise. Then $\overline{g}$ is a 3-regular dominating function of G by Observation 4.4.6 . It remains to show that H_ℓ does not have a 2-regular dominating function. Assume, to the contrary, that there is an integer $\ell \geq 3$ such that G_ℓ has a 2-regular dominating function $f : V(G_\ell) \to \{0, 1\}$. Since $N(u_i) = N(w_i)$ and $N(x_i) = N(z_i)$ for $1 \leq i \leq \ell$, it follows by Lemma 4.2.2 that $f(u_i) = f(w_i)$ and $f(x_i) = f(z_i)$. In particular, $f(u_1) = f(w_1)$ and $f(x_1) = f(z_1)$.

- ⋆ First, suppose that $f(u_1) = f(w_1) = 1$. Since v_1 is adjacent to u_1 and w_1 and $c_f(v_1) = 2$, it follows that $f(v_1) = f(a_1) = 0$. Since $c_f(a_1) = 2$, it follows that $f(a_2) = f(a_\ell) = 1$. If $f(v_2) = 1$, then $f(u_2) = f(w_2) = f(x_2) = f(z_2) = 0$. However then, $c_f(u_2) = c_f(w_2) = 1$, which is impossible. Thus, $f(v_2) = 0$ and $f(u_2) = f(w_2) \in \{0, 1\}$. If $f(u_2) = f(w_2) = 0$, then $c_f(v_2) = 1$; while if $f(u_2) = f(w_2) = 1$, then $c_f(v_2) = 3$. A contradiction is produced in either case.

- ⋆ Next, suppose that $f(u_1) = f(w_1) = 0$. Since v_1 is adjacent to u_1 and w_1 and $c_f(v_1) = 2$, it follows that $f(v_1) = f(a_1) = 1$. This forces $f(x_1) = f(z_1) = 0$. However then, $c_f(u_1) = c_f(w_1) = 1$, which is impossible.

Therefore, G_ℓ does not have a 2-regular dominating function. ■

We are therefore left with the following question:

Problem 4.4.15 *Does there exist a 3-connected cubic graph having a 1-regular and a 3-regular dominating function but no 2-regular dominating function?*

Chapter 5

PROPER DOMINATING FUNCTIONS I

ABSTRACT: A dominating function of a graph G is a proper dominating function of G if $c_f(x) \neq c_f(y)$ for every two adjacent vertices x and y of G. Sufficient conditions are obtained under which a graph does or does not have a proper dominating function. For certain classes of graphs, we determined precisely which members possess a proper dominating function.

5.1 Introduction

First, let's review what information we have obtained thus far on dominating functions of graphs. A graph G is *regular* if the vertices of G have the same degree. Regular graphs are among the most-studied graphs in graph theory. If f is a dominating function of a graph G and $c_f(v)$ is the same positive integer k for every vertex v of G, then f is called a *regular* (or k*-regular*) *dominating function* of G. Consequently, if G has a k-regular dominating function, then there is a dominating set S of G such that every vertex of G is dominated by exactly k vertices of S. Not every graph has a regular dominating function. Determining which graphs have a regular dominating function is a primary problem.

If one were to consider a dominating function f of a graph G as giving rise to a vertex coloring c_f of G, then a regular dominating function results in a monochromatic vertex coloring of G (all vertices of G are colored the same). The most studied vertex colorings, and certainly those of greatest interest, are proper vertex colorings, however. A proper (vertex) k-coloring of a graph G is a function

$c : V(G) \to [k]$ for some positive integer k such that $c(u) \neq c(v)$ for every two adjacent vertices u and v. The minimum k for which a proper (vertex) k-coloring of a graph G exists is the *chromatic number* of G and is denoted by $\chi(G)$. This leads us to the following concept in domination.

If f is a dominating function of a graph G such that $c_f(u) \neq c_f(v)$ for each pair u, v of *adjacent* vertices of G, then f is a *proper dominating function* of G. For example, if G is a graph in which every two adjacent vertices have different degrees, then the function that assigns 1 to every vertex of G is a proper dominating function of G. All such graphs G have the property that the set of all vertices of G having a specific degree is independent. For example, if $G = K_{s,t}$ is the complete bipartite graph where $s \neq t$, then the function that assigns 1 to every vertex of G is a proper dominating function of G. The two non-bipartite graphs in Figure 5.1 also have this property, where each vertex is labeled (colored) with its degree. Some people have referred to such graphs as *locally irregular graphs* (see [55], for example).

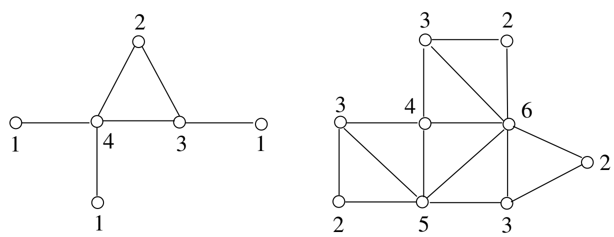

Figure 5.1: Two graphs in which no two adjacent vertices have the same degree

As is the case with regular dominating functions, not every graph has a proper dominating function. There are graphs where one can see quite quickly that they do not possess a proper dominating function. Two vertices u and v in a connected graph G are called *twins* if u and v have the same neighborhood. If u and v are adjacent twins in a graph G, then $c_f(u) = c_f(v)$ for every function f of G. Thus, we have the following observation.

Observation 5.1.1 *If a graph G contains two adjacent twins, then G does not have a proper dominating function.*

As a consequence of Observation 5.1.1, it follows, for example, that no complete graph K_n of order $n \geq 2$ has a proper dominating function. In fact, every dominating function of a complete graph is a regular dominating function. Although complete graphs K_n, $n \geq 2$, are regular graphs that do not have a proper dominating function, regular graphs that are not complete may or may not have a proper dominating function. For example, of the two 3-regular graphs of order 6, the complete bipartite graph $K_{3,3}$ has a proper dominating function, while the prism $C_3 \,\square\, K_2$ does not. The fact that $K_{3,3}$ has a proper dominating function is a consequence of the following result concerning connected bipartite graphs.

Proposition 5.1.2 *Let G be a connected bipartite graph. If each vertex in one of the partite sets of G has degree 2 or more, then G has a proper dominating function. Consequently, if $\delta(G) \geq 2$, then G has a proper dominating function.*

Proof. Let U and W be the partite sets of G such that $\deg u \geq 2$ for every $u \in U$. Define a function $f : V(G) \to \{0, 1\}$ by $f(u) = 0$ for each $u \in U$ and $f(w) = 1$ for each $w \in W$. Since $c_f(u) = \deg u \geq 2$ for each $u \in U$ and $c_f(w) = f(w) = 1$ for each $w \in W$, it follows that f is a proper dominating function of G. ■

This result also shows that every grid has a proper dominating function.

Corollary 5.1.3 *For integers $m, n \geq 2$, the grid $P_m \,\square\, P_n$ has a proper dominating function.*

The fact that the graph $C_3 \,\square\, K_2$ does not have a proper dominating function is a consequence of the following result. The *clique number* $\omega(G)$ of a graph G is the maximum order of a complete subgraph of G.

Proposition 5.1.4 *If G is a connected graph with clique number $\omega(G) \geq k$ for some integer $k \geq 3$ and maximum degree $\Delta(G) \leq 2k - 3$, then G does not have a proper dominating function.*

Proof. Assume, to the contrary, that G contains a complete subgraph H of order $k \geq 3$ with $\Delta(G) \leq 2k-3$ such that G has a proper dominating function $f : V(G) \to \{0, 1\}$. Suppose that f assigns r of the k vertices of H the value 1. If $v \in V(H)$, then

$$c_f(v) \leq r + [\deg v - (k-1)] \leq r + [\Delta(G) - (k-1)] \leq r + k - 2.$$

Consequently, $r \leq c_f(v) \leq r + k - 2$ for every vertex v of H. However then, two vertices of H have the same c_f-value, which is impossible. ■

Proposition 5.1.5 *If a graph G contains a complete subgraph K_r of order $r \geq 3$ such that (i) there are exactly two of vertices of K_r that are adjacent to vertices not in K_r and (ii) each of these two vertices is adjacent to exactly one vertex not in K_r, then G does not have a proper dominating function.*

Proof. Assume, to the contrary, that G has a proper dominating function $f : V(G) \to \{0, 1\}$. Let $p = \sum_{x \in V(K_r)} f(x)$ and let v and w be the only two vertices of K_r satisfying (i) and (ii). Suppose that v is adjacent to the vertex v' not in K_r and w is adjacent to the vertex w' not in K_r. Since v and w cannot be adjacent twins, $v' \neq w'$. Since $c_f(v) = f(v') + p$, $c_f(w) = f(w') + p$, and $vw \in E(G)$, it follows that $\{f(v'), f(w')\} = \{0, 1\}$. We may assume that $f(v') = 0$ and $f(w') = 1$. Since $r \geq 3$, there is $u \in V(K_r) - \{v, w\}$. However then, $c_f(v) = p = c_f(u)$, which is impossible. ■

Proposition 5.1.6 *If a connected graph G contains a set V_1 of vertices of degree 1 such that no two adjacent vertices of its subgraph $G - V_1$ have the same degree, then G has a proper dominating function.*

Proof. We define the function $f : V(G) \to \{0, 1\}$ by assigning 0 to each vertex in V_1 and 1 to each vertex in $V(G) - V_1$. Let $G_1 = G - V_1$. Then $c_f(v) = 1$ if $v \in V_1$ and

$$c_f(v) = 1 + \deg_{G_1} = \deg_G v \geq 2 \text{ if } v \in V(G_1) = V(G) - V_1.$$

Let x and y be two adjacent vertices of G. Then at least one of x and y belongs to $V(G) - V_1$, say $y \in V(G) - V_1$. If $x \in V_1$, then $c_f(x) = 1$ and $c_f(y) \geq 2$ and so $c_f(x) \neq c_f(y)$. If $x \in V(G) - V_1$, then $c_f(x) = 1 + \deg_{G_1} x \neq 1 + \deg_{G_1} y = c_f(y)$. Therefore, f is a proper dominating function of G. ■

As with regular dominating functions, the primary problem is to determine which graphs (especially well-known classes of graphs) have a proper dominating function and which do not.

5.2 Which Prisms Have a Proper Dominating Function?

We saw that the prism $C_3 \,\square\, K_2$ does not have a proper dominating function. We now investigate the other prisms, namely $C_n \,\square\, K_2$, where $n \geq 4$. First, we determine those cycles having a proper dominating function.

Proposition 5.2.1 *For an integer $n \geq 3$, the cycle C_n of order n has a proper dominating function if and only if n is even.*

Proof. By Proposition 5.1.2, it remains to show that if n is odd, then C_n does not have a proper dominating function. Since $C_3 = K_3$ does not have a proper dominating function, we may assume that $n \geq 5$. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Assume, to the contrary, that there is a proper dominating function $g : V(C_n) \to \{0, 1\}$ of C_n. We claim that no two consecutive vertices of C_n can have g-value 0; for otherwise, we may assume that $g(v_1) = g(v_2) = 0$. Since g is a dominating function, it follows that $g(v_n) = g(v_3) = 1$. However then, $c_g(v_1) = c_g(v_2) = 1$, which is a contradiction. Since $n \geq 5$ is odd and no two consecutive vertices of C_n have g-value 0, there are two consecutive vertices on C_n having g-value 1, say $g(v_1) = g(v_2) = 1$. If $g(v_n) = g(v_3)$, then $c_g(v_1) = c_g(v_2)$, which is impossible. Hence, $\{g(v_n), g(v_3)\} = \{0, 1\}$, say $g(v_n) = 1$ and $g(v_3) = 0$, which in turn forces $g(v_4) = 1$. However then, $c_g(v_2) = c_g(v_3) = 2$, which is impossible. ■

We now turn to prisms. We have already seen that $C_3 \,\square\, K_2$ does not have a proper dominating function. This, it turns out, is the exceptional prism.

Proposition 5.2.2 *For each integer $n \geq 4$, the prism $C_n \,\square\, K_2$ has a proper dominating function.*

Proof. For each even integer $n \geq 4$, the prism $C_n \,\square\, K_2$ is a 3-regular bipartite graph and so has a proper dominating function by Proposition 5.1.2. Proper dominating functions of $Q_3 = C_4 \,\square\, K_2$ and $C_6 \,\square\, K_2$ are shown in Figure 5.2, where $f(v)$ is placed inside a vertex v and $c_f(v)$ is placed outside v. Thus, it remains to show for each odd integer $n \geq 5$ that $C_n \,\square\, K_2$ has a proper dominating function.

Suppose that $G = C_n \,\square\, K_2$ is constructed from the two cycles

$$C = (u_1, u_2, \ldots, u_n, u_1) \text{ and } C' = (v_1, v_2, \ldots, v_n, v_1)$$

by adding the edges u_iv_i for $1 \leq i \leq n$.

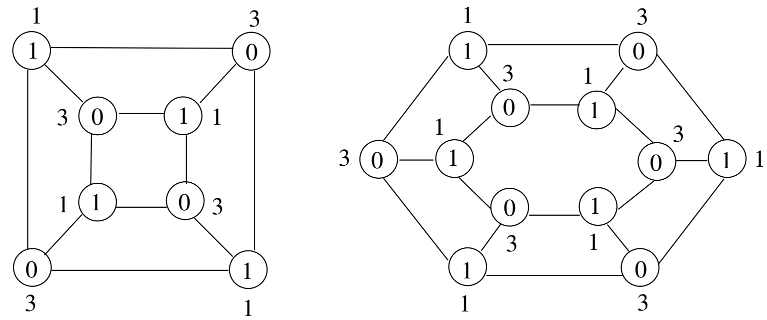

Figure 5.2: Proper dominating functions of $Q_3 = C_4 \,\square\, K_2$ and $C_6 \,\square\, K_2$

We write $n = 2k+3$ for some positive integer k. Define a function $f : V(G) \to \{0,1\}$ of G by

$$\begin{aligned} (f(u_1), f(u_2), \ldots, f(u_n)) &= (0,1,0,1,\ldots,0,1,\ \underline{1,0,1}) \\ (f(v_1), f(v_2), \ldots, f(v_n)) &= (1,0,1,0,\ldots,1,0,\ \underline{1,0,0}), \end{aligned}$$

where there are k pairs $(0,1)$ in the f-values of C and k pairs $(1,0)$ in the f-values of C'. This implies that

$$\begin{aligned} (c_f(u_1), c_f(u_2), \ldots, c_f(u_n)) &= (3,1,3,1,\ldots,3,1,\underline{3,2},\ \underline{3,2,1}) \\ (c_f(v_1), c_f(v_2), \ldots, c_f(v_n)) &= (1,3,1,3,\ldots,1,3,1,3,\ \underline{2,1,2}), \end{aligned}$$

where there are $k-1 \geq 0$ pairs $(3,1)$ in the c_f-values of C and k pairs $(1,3)$ in the c_f-values of C'. This is illustrated in Figure 5.3 for $n=5$ and $n=7$. Therefore, f is a proper dominating function of $C_n \,\square\, K_2$ for all odd integers $n \geq 5$. ■

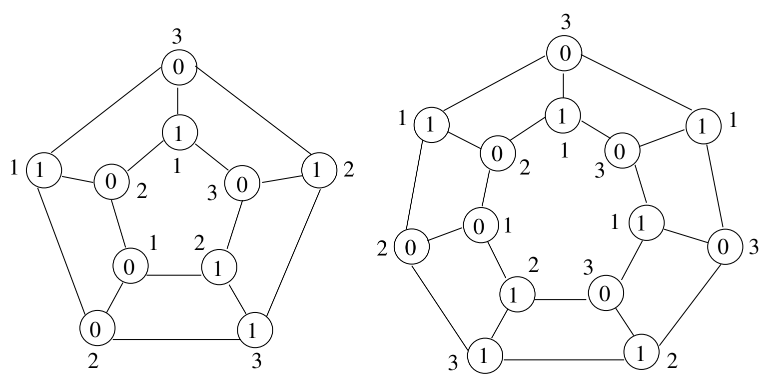

Figure 5.3: Proper dominating functions of $C_5 \,\square\, K_2$ and $C_7 \,\square\, K_2$

We have seen that the prism $C_5 \,\Box\, K_2$ has a proper dominating function. This cubic graph consists of two disjoint 5-cycles C and C', where the set $[V(C), V(C')]$ of edges joining C and C' is a perfect matching in $C_5 \,\Box\, K_2$. Another, even better known, cubic graph constructed in this manner is the famous Petersen graph. Before presenting a result dealing with this graph, it is useful to recall another concept.

Let G be a nontrivial connected graph and let $f : V(G) \to \{0, 1\}$ be a function of G. The *complementary function* $\overline{f} : V(G) \to \{0, 1\}$ is defined by

$$\overline{f}(v) = 1 - f(v) \text{ for every vertex } v \text{ of } G.$$

There are conditions under which the complementary function of a dominating function is a dominating function of the graph.

Proposition 5.2.3 *Let G be a connected graph of order 3 or more. If f is a dominating function of G such that $c_f(v) \le \deg v$ for every vertex v of G, then $\overline{f}$ is also a dominating function of G.*

Proof. We show that $c_{\overline{f}}(v) = \sum_{x \in N[v]} \overline{f}(x) \ge 1$ for every vertex v of G. Let $v \in V(G)$. If $f(v) = 0$, then $\overline{f}(v) = 1$ and so $c_{\overline{f}}(v) \ge \overline{f}(v) = 1$. Thus, we may assume that $f(v) = 1$. Since $c_f(v) \le \deg v$, there exists $u \in N(v)$ such that $f(u) = 0$ and so $\overline{f}(u) = 1$. Hence, $c_{\overline{f}}(v) \ge \overline{f}(u) = 1$. Therefore, $\overline{f}$ is a dominating function of G. ■

The following is a consequence of Proposition 5.2.3.

Proposition 5.2.4 *If $f : V(G) \to \{0, 1\}$ is a proper dominating function of an r-regular graph G where $r \ge 2$ such that $c_f(v) \le r$ for every vertex v of G, then $\overline{f}$ is also a proper dominating function of G.*

Proof. Observe that $c_f(v) + c_{\overline{f}}(v) = 1 + r$ for each vertex v of G. Since $c_f(v) \le r$, it follows by Proposition 5.2.3 that $c_{\overline{f}}(v) \ge 1$ and so $\overline{f}$ is a dominating function of G. Let x and y be adjacent vertices of G. Since $c_f(x) \ne c_f(y)$, it follows that

$$c_{\overline{f}}(x) = 1 + r - c_f(x) \ne 1 + r - c_f(y) = c_{\overline{f}}(y)$$

and so $\overline{f}$ is a proper dominating function of G. ■

With the aid of Proposition 5.2.4, we next show that the famous Petersen graph does not have a proper dominating function.

Theorem 5.2.5 *The Petersen graph does not have a proper dominating function.*

Proof. We label the vertices of the Petersen graph P as shown in Figure 5.4. Assume, to the contrary, that P has a proper dominating function $f : V(P) \to \{0,1\}$. Thus, $c_f(x) \leq 4$ for every vertex v of P. Thus, either $c_f(x) = 4$ for some vertex x of P or $1 \leq c_f(x) \leq 3$ for each $x \in V(P)$. We consider these two cases.

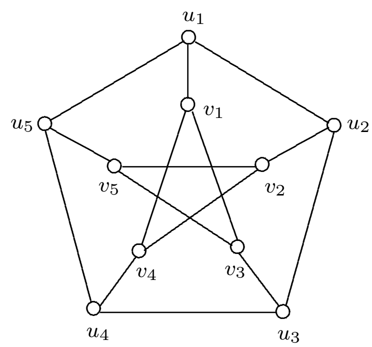

Figure 5.4: The Petersen graph

Case 1. $c_f(x) = 4$ *for some vertex* x *of* P. Since P is vertex-transitive, we may assume that $c_f(u_1) = 4$. Thus, $f(x) = 1$ for each $x \in N[u_1]$ and $2 \leq c_f(x) \leq 3$ for each $x \in N(u_1)$.

Subcase 1.1. $c_f(u_5) = c_f(v_1) = c_f(u_2) = 2$. This implies that the f-value of each remaining vertex of P is 0. However then, $c_f(u_3) = c_f(u_4) = 1$, for example. This is a contradiction.

Subcase 1.2. *Exactly two neighbors of* u_1 *have* c_f*-value* 2, *say* $c_f(u_5) = c_f(u_2) = 2$ *and* $c_f(v_1) = 3$. Then $f(x) = 0$ for each $x \in \{v_2, u_3, v_5, u_4\}$ and $\{f(v_3), f(v_4)\} = \{0,1\}$, say $f(v_3) = 0$ and $f(v_4) = 1$. However then, $c_f(v_3) = f(v_5) = 1$, a contradiction.

Subcase 1.3. *Exactly one neighbor of* u_1 *has* c_f*-value* 2, *say* $c_f(u_5) = c_f(u_2) = 3$ *and* $c_f(v_1) = 2$. Then $f(v_3) = f(v_4) = 0$ and so $\{c_f(v_2), c_f(v_5)\} = \{1,2\}$, say $c_f(v_5) = 1$. This implies that $f(v_5) = f(v_2) = 0$ and so $c_f(v_2) = c_f(v_5) = 1$, a contradiction.

Subcase 1.4. $c_f(u_5) = c_f(v_1) = c_f(u_2) = 3$. Thus, $\{f(v_3), f(v_4)\} = \{0,1\}$, say $f(v_3) = 0$ and $f(v_4) = 1$. Thus, $c_f(u_4) = 2$ or $c_f(u_4) = 4$.

- ⋆ If $c_f(u_4) = 2$, then $f(u_3) = f(u_4) = 0$ and so $c_f(v_4) \in \{2,3\}$, a contradiction.

- ⋆ If $c_f(u_4) = 4$, then $f(u_3) = f(v_4) = 1$ and so $c_f(u_3) = c_f(u_2) = 3$, a contradiction.

Case 2. *$c_f(x) \leq 3$ for each vertex x of P.* This implies that the complementary function $\overline{f}$ of f is also a proper dominating function of P by Proposition 5.2.4. Let $C = (u_1, u_2, u_3, u_4, u_5, u_1)$. We may assume that at least three vertices of C have f-value 1 (for otherwise, we consider $\overline{f}$).

Subcase 2.1. *Three consecutive vertices of C have f-value 1, say $f(u_5) = f(u_1) = f(u_2) = 1$.* Thus, $c_f(u_1) = 3$ and so $c_f(u_5) = c_f(u_2) = 2$. Thus, $f(x) = 0$ for each $x \in \{v_1, v_2, u_3, u_4, v_5\}$. This implies that $\{c_f(v_2), c_f(v_5)\} = \{1, 2\}$, a contradiction.

Subcase 2.2. *No three consecutive vertices of C have f-value 1, say $f(u_4) = f(u_1) = f(u_3) = 1$ and $f(u_2) = f(u_5) = 0$.* We may assume that $c_f(u_4) = 2$ and $c_f(u_3) = 3$. Thus, $f(v_3) = 1$ and $f(v_4) = 0$. Since $c_f(u_5) = 3$, it follows that $f(v_5) = 1$. However then, $c_f(v_3) = c_f(u_3) = 3$, a contradiction. ■

We have seen that $C_5 \square K_2$ has a proper dominating function; while the Petersen graph does not. These two graphs are members of a special class of graphs. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let α be a permutation of the set $S = \{1, 2, \ldots, n\}$. The *permutation graph* $P_\alpha(G)$ of a graph G is the graph of order $2n$ obtained from two copies of G, where the second copy of G is denoted by G' and the vertex v_i in G is denoted by u_i in G' and v_i is joined to the vertex $u_{\alpha(i)}$ in G'. The edges $v_i u_{\alpha(i)}$ are called the *permutation edges* of $P_\alpha(G)$. This concept was introduced by Chartrand and Harary [56]. Therefore, if α is the identity map on S, then $P_\alpha(G) = G \square K_2$. Figure 5.5 shows the four permutation graphs of the 5-cycle C_5 where the graph in Figure 5.5(a) is $C_5 \square K_2$ and the graph in Figure 5.5(d) is the Petersen graph P. Each of these permutation graphs is a 3-regular (or cubic) graph. These four graphs appeared on the cover of the 1969 book *Graph Theory* by Harary [57]. Since the prism $C_5 \square K_2$ has a proper dominating function and the Petersen graph does not have a proper dominating function, the following question remains.

Problem 5.2.6 *Do the two permutation graphs of C_5 of Figure 5.5(b) and Figure 5.5(c) have a proper dominating function?*

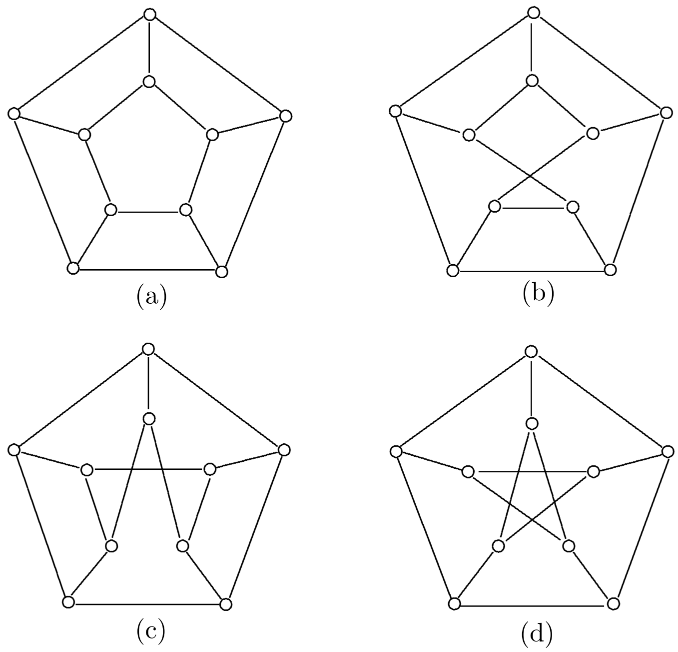

Figure 5.5: The four permutation graphs of C_5

5.3 Cartesian Products of Cycles and Paths

We stated earlier that the bipartite 3-regular graph $K_{3,3}$ of order 6 has a proper dominating function, while the other 3-regular graph of order 6, namely $C_3 \,\Box\, K_2$, does not. As we mentioned, the fact that $K_{3,3}$ has a proper dominating function is an immediate consequence of Proposition 5.1.2. The graph $C_3 \,\Box\, K_2$ is clearly more interesting and gives rise to investigating other graphs suggested by $C_3 \,\Box\, K_2$. Recall the following result (Proposition 5.2.2).

> *For an integer $m \geq 3$, the prism $C_m \,\Box\, K_2$ has a proper dominating function if and only if $m \geq 4$.*

Consequently, the graph $C_3 \,\Box\, K_2$ is the exceptional graph among the prisms $C_m \,\Box\, K_2$. This suggests looking at related classes of graphs defined by a Cartesian product. Next, we consider the 4-regular graphs $C_m \,\Box\, C_3$ for $m \geq 3$, beginning with $C_3 \,\Box\, C_3$.

Proposition 5.3.1 *The graph $C_3 \,\Box\, C_3$ does not have a proper dominating function.*

Proof. The vertices of $G = C_3 \,\Box\, C_3$ are labeled as shown in Figure 5.6.

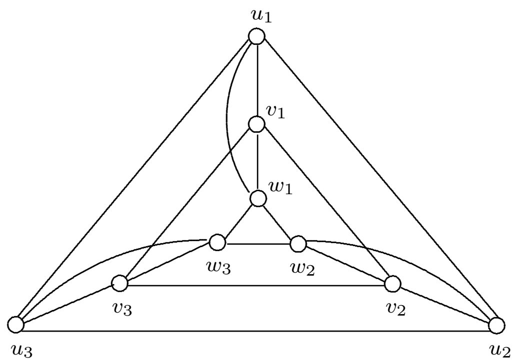

Figure 5.6: The graph $C_3 \,\square\, C_3$

Assume, to the contrary, that G has a proper dominating function $f : V(G) \to \{0,1\}$. Thus, $c_f(x) \leq 5$ for every vertex v of G.

First, we show that no vertex of G has c_f-value 5, for suppose that $c_f(x) = 5$ for some vertex x of G. Since G is vertex-transitive, we may assume that $c_f(u_1) = 5$. Thus, $f(x) = 1$ for each $x \in N[u_1] = \{u_1, u_2, u_3, v_1, w_1\}$ and so $c_f(x) \in \{3, 4\}$ for each $x \in N(u_1) = \{u_2, u_3, v_1, w_1\}$. We may assume that $c_f(v_1) = 4$ and $c_f(w_1) = 3$. This implies that

$$\{f(v_2), f(v_3)\} = \{0, 1\} \text{ and } f(w_2) = f(w_3) = 0.$$

Again, we may assume that $f(v_2) = 0$ and $f(v_3) = 1$. See Figure 5.7. However then, $c_f(v_2) = c_f(u_2) = 3$, which is impossible. This also implies that in every triangle of G, not all three vertices can be assigned 1 by f.

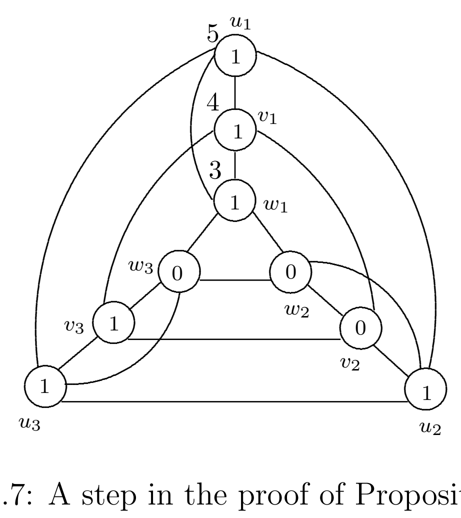

Figure 5.7: A step in the proof of Proposition 5.3.1

Thus, $c_f(x) \leq 4$ for each vertex x of G. This implies that the complementary function $\overline{f}$ of f is also a proper dominating function of G. We may therefore, assume that G has a triangle T where exactly two of the three vertices are labeled 1, say $T = (u_1, v_1, w_1, u_1)$. Thus,

$$\{c_f(u_1), c_f(v_1), c_f(w_1)\} = \{2, 3, 4\},$$

say $c_f(u_1) = 4$, $c_f(v_1) = 3$, and $c_f(w_1) = 2$. If $f(u_1) = 1$, then $\{f(v_1), f(w_1)\} = \{0, 1\}$ and so $f(u_2) = f(u_3) = 1$. However then, $(f(u_1), f(u_2), f(u_3)) = (1, 1, 1)$, which is impossible. Therefore, $f(u_1) = 0$ and so $f(x) = 1$ for each $x \in N(u_1) = \{u_2, u_3, v_1, w_1\}$. Since $c_f(v_1) = 3$ and $c_f(w_1) = 2$, it follows that

$$\{f(v_2), f(v_3)\} = \{0, 1\} \text{ and } f(w_2) = f(w_3) = 0.$$

Again, we may assume that $f(v_2) = 0$ and $f(v_3) = 1$. See Figure 5.8. However then, $c_f(v_3) = c_f(w_3) = 3$, which is impossible. ■

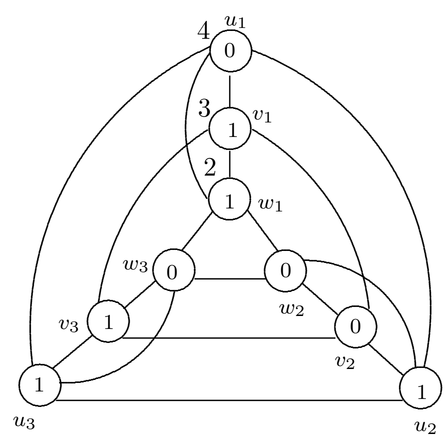

Figure 5.8: Another step in the proof of Proposition 5.3.1

Theorem 5.3.2 *For each integer $n \geq 3$, the graph $C_3 \,\square\, C_n$ has a proper dominating function if and only if $n \equiv 0 \pmod 8$.*

Proof. We have seen that $C_3 \,\square\, C_3$ does not have a proper dominating function, so we may assume that $n \geq 4$. Let $G = C_3 \,\square\, C_n$ where $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ such that u_iu_{i+1}, v_iv_{i+1}, $w_iw_{i+1} \in E(G)$ for $1 \leq i \leq n-1$ and u_1u_n, v_1v_n, $w_1w_n \in E(G)$. In addition, $T_i = (u_i, v_i, w_i, u_i)$ is a triangle in G for $1 \leq i \leq n$. This is shown in Figure 5.9 for the graph $C_3 \,\square\, C_5$.

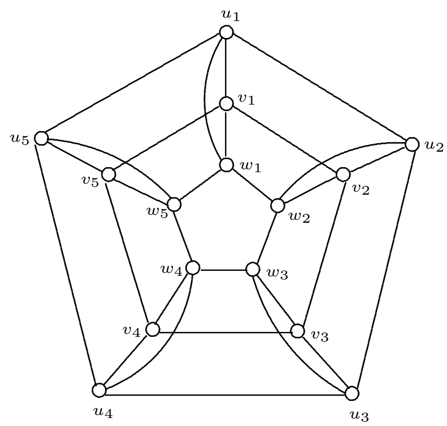

Figure 5.9: The graph $C_3 \,\square\, C_5$

Suppose for an integer $n \geq 4$ that $G = C_3 \,\square\, C_n$ has a proper dominating function f. Thus, $1 \leq c_f(x) \leq 5$ for each vertex x of G. First, we show that $c_f(x) \neq 5$ for every vertex x of G. Assume, to the contrary, that $c_f(x) = 5$ for some vertex x of G. Since G is vertex-transitive, we may assume that $c_f(u_1) = 5$. Hence, $f(x) = 1$ for each $x \in N[u_1]$. Thus, $\{c_f(v_1), c_f(w_1)\} = \{3, 4\}$, say $c_f(v_1) = 4$ and $c_f(w_1) = 3$. This implies that $\{f(v_2), f(v_n)\} = \{0, 1\}$ and $\{f(w_2), f(w_n)\} = \{0\}$. We may further assume that $f(v_2) = 1$ and $f(v_n) = 0$. Since $f(w_n) = 0$, it follows that $3 \leq c_f(u_2), c_f(v_2), c_f(w_2) \leq 4$, which is impossible. Thus, as claimed, there is no vertex x such that $c_f(x) = 5$. In addition, there is no triangle T_i where $f(x) = 1$ for each $x \in V(T_i)$. Similarly, there is no triangle T_i where $f(x) = 0$ for each $x \in V(T_i)$. Consequently, for each triangle T_i, it follows that $\{f(x) : x \in V(T_i)\} = \{0, 1\}$ for $1 \leq i \leq n$. We refer to an i-triangle, $i = 1, 2$, if the triangle has exactly i vertices whose f-value is i.

First, we claim that the triangles in G cannot alternate between 1-triangles and 2-triangles, for if they did alternate, then we can assume that T_1 and T_3 are 2-triangles and T_2 is a 1-triangle. We may assume that

$$(f(u_1), f(v_1), f(w_1)) = (1, 1, 0) \text{ and } (f(u_2), f(v_2), f(w_2)) \in \{(1, 0, 0), (0, 0, 1)\}.$$

In each case, $(f(u_3), f(v_3), f(w_3)) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$.

- ⋆ For $(f(u_2), f(v_2), f(w_2)) = (1, 0, 0)$ and

$$(f(u_3), f(v_3), f(w_3)) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\},$$

there are two vertices in T_2 having the same c_f value in each case, a contradiction.

- ⋆ For $(f(u_2), f(v_2), f(w_2)) = (0, 0, 1)$ and

$$(f(u_3), f(v_3), f(w_3)) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\},$$

there are two vertices in T_2 having the same c_f value in each case, a contradiction.

Therefore, there must be two consecutive 2-triangles, say T_1 and T_2 that are 2-triangles. We may assume that

$$(f(u_1), f(v_1), f(w_1)) = (1, 1, 0) \text{ and } (f(u_2), f(v_2), f(w_2)) \in \{(1, 1, 0), (0, 1, 1)\}.$$

We consider these two cases.

Case 1. $(f(u_2), f(v_2), f(w_2)) = (1, 1, 0)$. For $i = 1, 2$, it follows that $\{c_f(u_i), c_f(v_i)\} = \{3, 4\}$, which implies that $c_f(w_i) \in \{2, 3\}$. However then, $c_f(w_1) = c_f(w_2) = 2$, which is a contradiction.

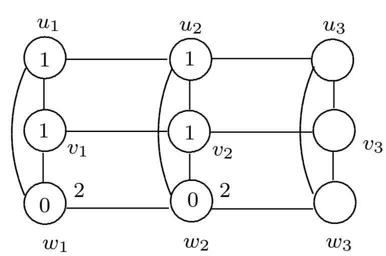

Figure 5.10: A step in Case 1

Case 2. $(f(u_2), f(v_2), f(w_2)) = (0, 1, 1)$. In this case,

$$c_f(u_1), c_f(w_2) \in \{2, 3\} \text{ and } c_f(v_1), c_f(w_1), c_f(u_2), c_f(v_2) \in \{3, 4\}.$$

Since f is a proper dominating function, we must have $c_f(u_1) = c_f(w_2) = 2$. We may assume that $c_f(w_1) = c_f(v_2) = 3$ and $c_f(v_1) = c_f(u_2) = 4$. This implies that $f(u_3) = 1$ and $f(v_3) = f(w_3) = 0$. Hence, we must have $c_f(v_3) = 2$ and so $c_f(u_3) = 1$ and $c_f(w_3) = 3$.

Thus, $f(u_4) = f(v_4) = 0$ and $f(w_4) = 1$. This in turn implies that $c_f(v_4) = 1$ and so $c_f(w_4) = 2$ and $c_f(u_4) = 3$. Continuing in this manner, we arrive at the following.

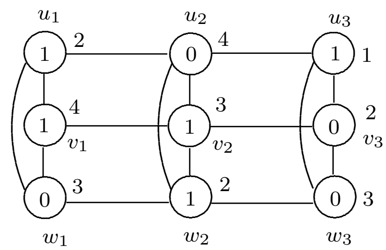

Figure 5.11: A step in Case 2

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	$\cdots$
$f(u_i)$	1	0	1	0	1	0	1	0	1	0	$\cdots$
$f(v_i)$	1	1	0	0	0	1	0	1	1	1	$\cdots$
$f(w_i)$	0	1	0	1	1	1	0	0	0	1	$\cdots$

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	$\cdots$
$c_f(u_i)$	**2**	4	1	3	2	4	1	3	**2**	4	$\cdots$
$c_f(v_i)$	**4**	3	2	1	3	2	3	2	**4**	3	$\cdots$
$c_f(w_i)$	**3**	2	3	2	4	3	2	1	**3**	2	$\cdots$

Consequently, if $n \not\equiv 0 \pmod 8$, then f is not proper dominating function.

For the converse, suppose that $n \equiv 0 \pmod 8$ and so $n = 8k$ for some positive integer k. We define a proper dominating function f of G as follows. First, suppose that $k = 1$. Then for $1 \le i \le 8$, let $f(T_i) = (f(u_i), f(v_i), f(w_i))$ be defined by the following table:

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
$f(u_i)$	1	0	1	0	1	0	1	0
$f(v_i)$	1	1	0	0	0	1	0	1
$f(w_i)$	0	1	0	1	1	1	0	0

Then $c_f(T_i) = (c_f(u_i), c_f(v_i), c_f(w_i))$, $1 \le i \le 8$, is given by the following table.

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
$c_f(u_i)$	2	4	1	3	2	4	1	3
$c_f(v_i)$	4	3	2	1	3	2	3	2
$c_f(w_i)$	3	2	3	2	4	3	2	1

This is shown in Figure 5.12 for $n = 8$.

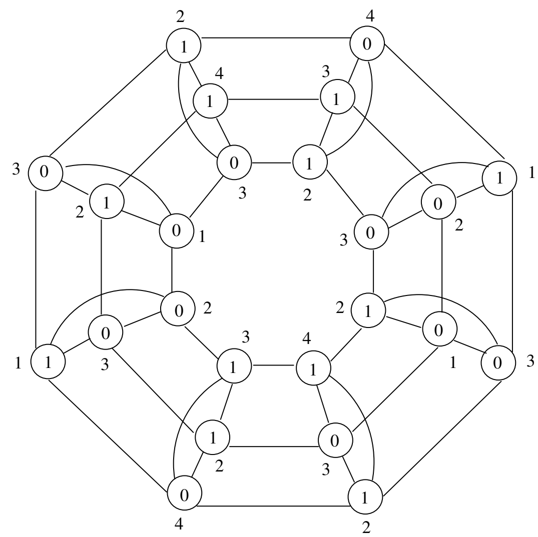

Figure 5.12: A proper dominating function of $C_3 \,\Box\, C_8$

We may now assume that $k \geq 2$. Let i be an integer with $9 \leq i \leq 8k$. If $i \equiv r \pmod 8$, where $1 \leq r \leq 8$, then we define

$$f(T_i) = (f(u_i), f(v_i), f(w_i)) = (f(u_r), f(v_r), f(w_r))$$

and so

$$c_f(T_i) = (c_f(u_i), c_f(v_i), c_f(w_i)) = (c_f(u_r), c_f(v_r), c_f(w_r)).$$

Consequently, f is a proper dominating function of $C_3 \,\Box\, C_n$ for each integer $n \geq 8$ with $n \equiv 0 \pmod 8$. ∎

Proposition 5.3.1 and Theorem 5.6 on the graphs $C_3 \,\Box\, C_n$ bring up the question as to which graphs $C_m \,\Box\, C_n$ have a proper dominating function. The graphs $C_m \,\Box\, C_n$ are often referred to as *toroidal graphs*. We begin with the graphs $C_4 \,\Box\, C_n$ where $n \geq 4$.

Theorem 5.3.3 *For each integer $n \geq 4$, the graph $C_4 \,\Box\, C_n$ has a proper dominating function.*

Proof. By Proposition 5.1.2, we may assume that $n \geq 5$ is odd. Let $G = C_4 \,\Box\, C_n$ where $V(G) = \{u_i, v_i, w_i, x_i : 1 \leq i \leq n\}$ such that u_iu_{i+1}, v_iv_{i+1}, $w_iw_{i+1}, x_ix_{i+1} \in E(G)$ for $1 \leq i \leq n-1$ and u_1u_n, v_1v_n, w_1w_n, $x_1x_n \in E(G)$. In addition,

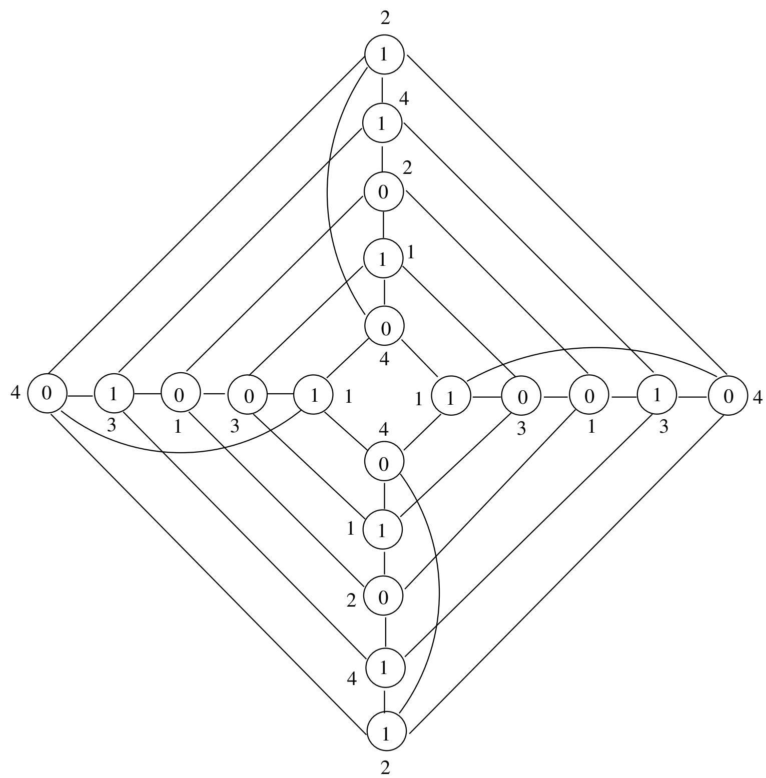

Figure 5.13: A proper dominating function of $C_3 \,\Box\, C_5$

$(u_i, v_i, w_i, x_i, u_i)$ is a 4-cycle in G for $1 \le i \le n$. A proper dominating function of $C_4 \,\Box\, C_5$ is shown in Figure 5.13. Thus, we may assume that $n \ge 7$.

In order to define a proper dominating function of G, we first introduce some additional notation. Let C_u denote the n-cycle $(u_1, u_2, \ldots, u_n, u_1)$ in G. Then the other three corresponding n-cycles in G are denoted by C_v, C_w, and C_x. For a function $f : V(G) \to \{0, 1\}$, let

$$f(C_u) = (f(u_1), f(u_2), \cdots, f(u_n)) \text{ and } c_f(C_u) = (c_f(u_1), c_f(u_2), \cdots, c_f(u_n)).$$

The sequences $f(C_v), f(C_w),\ f(C_x)$ and $c_f(C_v), c_f(C_w),\ c_f(C_x)$ are defined similarly.

Since $n \ge 7$ is odd, it follows that $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$.

⋆ If $n \equiv 1 \pmod 4$ and $n \ge 9$, then $n = 4k + 1$ for some integer $k \ge 2$. Define a proper dominating function $f : V(G) \to \{0, 1\}$ by

$$\begin{aligned} f(C_u) &= (1,1,0,1,0,\ \underline{1,1,0,0},\ \underline{1,1,0,0}, \cdots, \underline{1,1,0,0}) \\ f(C_v) &= (0,1,0,0,1,\ \underline{0,1,0,1},\ \underline{0,1,0,1}, \cdots, \underline{0,1,0,1}) \\ f(C_w) &= (1,1,0,1,0,\ \underline{1,1,0,0},\ \underline{1,1,0,0}, \cdots, \underline{1,1,0,0}) \\ f(C_x) &= (0,1,0,0,1,\ \underline{0,1,0,1},\ \underline{0,1,0,1}, \cdots, \underline{0,1,0,1}), \end{aligned}$$

where $f(C_u) = f(C_w)$ and $f(C_v) = f(C_x)$. There are $k-1$ subsequences $(1, 1, 0, 0)$ in $f(C_u)$ and $f(C_x)$ and $k-1$ subsequences $(0, 1, 0, 1)$ in $f(C_v)$ and $f(C_w)$. Then

$$\begin{aligned} c_f(C_u) &= (2,4,2,1,4,\ \underline{2,4,1,3},\ \underline{2,4,1,3}, \ldots, \underline{2,4,1,3}) \\ c_f(C_v) &= (4,3,1,3,1,\ \underline{4,3,2,1},\ \underline{4,3,2,1}, \ldots, \underline{4,3,2,1}) \\ c_f(C_w) &= (2,4,2,1,4,\ \underline{2,4,1,3},\ \underline{2,4,1,3}, \ldots, \underline{2,4,1,3}) \\ c_f(C_x) &= (4,3,1,3,1,\ \underline{4,3,2,1},\ \underline{4,3,2,1}, \ldots, \underline{4,3,2,1}), \end{aligned}$$

where $c_f(C_u) = c_f(C_w)$ and $c_f(C_v) = c_f(C_x)$. There are $k-1$ subsequences $(2, 4, 1, 3)$ in $c_f(C_u)$ and $c_f(C_x)$ and $k-1$ subsequences $(4, 3, 2, 1)$ in $c_f(C_v)$ and $c_f(C_w)$.

For $n = 9$, the f-values of the vertices of $C_4 \square C_9$ are therefore

$$\begin{aligned} f(C_u) &= (1,1,0,1,0,\ \underline{1,1,0,0}) \\ f(C_v) &= (0,1,0,0,1,\ \underline{0,1,0,1}) \\ f(C_w) &= (1,1,0,1,0,\ \underline{1,1,0,0}) \\ f(C_x) &= (0,1,0,0,1,\ \underline{0,1,0,1}) \end{aligned}$$

and the c_f-values of the vertices of $C_4 \square C_9$ are

$$\begin{aligned} c_f(C_u) &= (2,4,2,1,4,\ \underline{2,4,1,3}) \\ c_f(C_v) &= (4,3,1,3,1,\ \underline{4,3,2,1}) \\ c_f(C_w) &= (2,4,2,1,4,\ \underline{2,4,1,3}) \\ c_f(C_x) &= (4,3,1,3,1,\ \underline{4,3,2,1}). \end{aligned}$$

⋆ If $n \equiv 3 \pmod 4$ and $n \geq 7$, then $n = 4k + 3$ for some positive integer k.

Define a proper dominating function $f : V(G) \to \{0,1\}$ by

$$\begin{aligned}
f(C_u) &= (1,1,0,1,0,\ \underline{1,1,0,0},\ \underline{1,1,0,0},\cdots,\underline{1,1,0,0},\ \mathbf{1},\mathbf{1})\\
f(C_v) &= (0,1,0,0,1,\ \underline{0,1,0,1},\ \underline{0,1,0,1},\cdots,\underline{0,1,0,1},\ \mathbf{0},\mathbf{1})\\
f(C_w) &= (1,1,0,1,0,\ \underline{1,1,0,0},\ \underline{1,1,0,0},\cdots,\underline{1,1,0,0},\ \mathbf{1},\mathbf{1})\\
f(C_x) &= (0,1,0,0,1,\ \underline{0,1,0,1},\ \underline{0,1,0,1},\cdots,\underline{0,1,0,1},\ \mathbf{0},\mathbf{1}),
\end{aligned}$$

where $f(C_u) = f(C_w)$ and $f(C_v) = f(C_x)$. There are $k-1$ subsequences $(1,1,0,0)$ in $f(C_u)$ and $f(C_x)$ and $k-1$ subsequences $(0,1,0,1)$ in $f(C_v)$ and $f(C_w)$. Then

$$\begin{aligned}
c_f(C_u) &= (3,4,2,1,4,\ \underline{2,4,1,3},\ \underline{2,4,1,3},\cdots,\underline{2,4,1,3},\ \mathbf{2},\mathbf{5})\\
c_f(C_v) &= (4,3,1,3,1,\ \underline{4,3,2,1},\ \underline{4,3,2,1},\cdots,\underline{4,3,2,1},\ \mathbf{4},\mathbf{3})\\
c_f(C_w) &= (3,4,2,1,4,\ \underline{2,4,1,3},\ \underline{2,4,1,3},\cdots,\underline{2,4,1,3},\ \mathbf{2},\mathbf{5})\\
c_f(C_x) &= (4,3,1,3,1,\ \underline{4,3,2,1},\ \underline{4,3,2,1},\cdots,\underline{4,3,2,1},\ \mathbf{4},\mathbf{3}),
\end{aligned}$$

where $c_f(C_u) = c_f(C_w)$ and $c_f(C_v) = c_f(C_x)$. There are $k-1$ subsequences $(2,4,1,3)$ in $c_f(C_u)$ and $c_f(C_x)$ and $k-1$ subsequences $(4,3,2,1)$ in $c_f(C_v)$ and $c_f(C_w)$.

For $n = 7$, the f-values of the vertices of $C_4 \square C_7$ are therefore

$$\begin{aligned}
f(C_u) &= (1,1,0,1,0,\ \mathbf{1},\mathbf{1})\\
f(C_v) &= (0,1,0,0,1,\ \mathbf{0},\mathbf{1})\\
f(C_w) &= (1,1,0,1,0,\ \mathbf{1},\mathbf{1})\\
f(C_x) &= (0,1,0,0,1,\ \mathbf{0},\mathbf{1})
\end{aligned}$$

and the c_f-values of the vertices of $C_4 \square C_7$ are

$$\begin{aligned}
c_f(C_u) &= (3,4,2,1,4,\ \mathbf{2},\mathbf{5})\\
c_f(C_v) &= (4,3,1,3,1,\ \mathbf{4},\mathbf{3})\\
c_f(C_w) &= (3,4,2,1,4,\ \mathbf{2},\mathbf{5})\\
c_f(C_x) &= (4,3,1,3,1,\ \mathbf{4},\mathbf{3}).
\end{aligned}$$

This is shown in Figure 5.14.

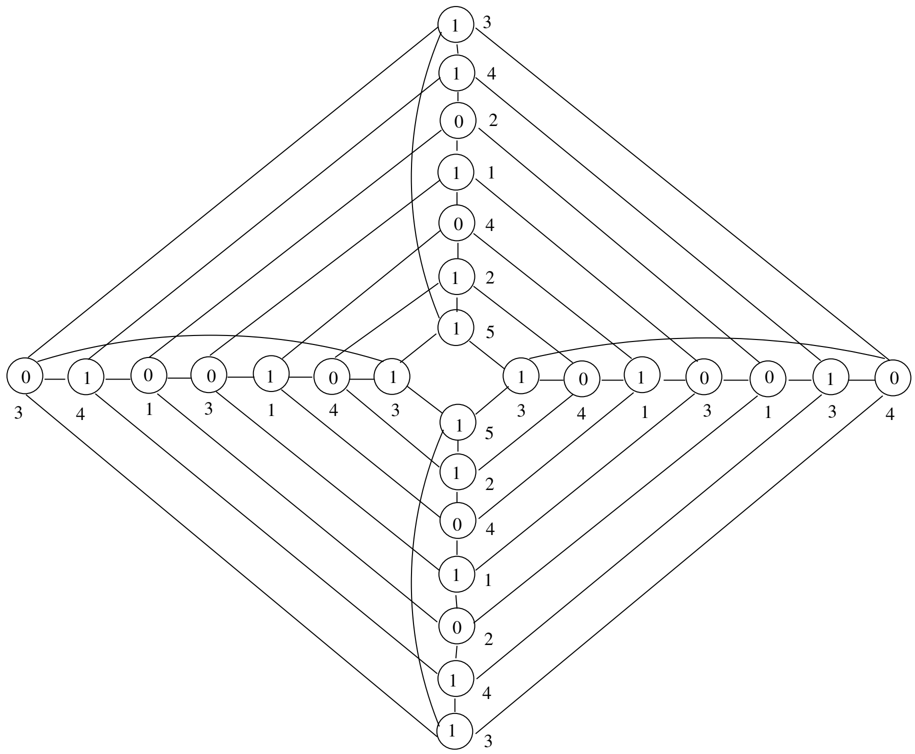

Figure 5.14: A proper dominating function of $C_4 \,\square\, C_4$

Therefore, $C_4 \,\square\, C_n$ has a proper dominating function for each integer $n \geq 5$. ■

With the aid of the proof of Theorem 5.3.3, we are able to prove the following theorem.

Theorem 5.3.4 *If $m, n \geq 4$ are integers, at least one of which is even, then $C_m \,\square\, C_n$ has a proper dominating function.*

Proof. If m and n are both even, then $C_m \,\square\, C_n$ is bipartite and so has a proper dominating function by Proposition 5.1.2. Hence, we may assume that $m \geq 6$ is even and $n \geq 5$ is odd.

Let $G = C_m \,\square\, C_n$, where $V(G) = \{u_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$, and G consists of n mutually disjoint m-cycles

$$H_j = (u_{1,j}, u_{2,j}, \ldots, u_{m,j}, u_{1,j}) \text{ for } 1 \leq j \leq n$$

and m mutually disjoint n-cycles

$$F_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,n}, u_{i,1}) \text{ for } 1 \leq i \leq m.$$

The vertices in the n-cycles F_i, $1 \leq i \leq m$, are labeled by the function f as in the proof of Theorem 5.3.3 according to (1) whether $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ and (2) i is odd or i is even. (That is, if i is odd, then $f(F_i) = f(C_u)$ in the proof of Theorem 5.3.3; while if i is even, then $f(F_i) = f(C_v)$ in the proof of Theorem 5.3.3.) The c_f-values of each cycle F_i, $1 \leq i \leq m$, are precisely those as in the proof of Theorem 5.3.3. Consequently, f is a proper dominating function of G. ■

We now illustrate Theorem 5.3.4 for $C_6 \,\square\, C_7$ and $C_6 \,\square\, C_9$.

⋆ For $n = 7$, where $n \equiv 3 \pmod 4$, the f-values of the vertices of $C_6 \,\square\, C_7$ are therefore

$$
\begin{aligned}
f(F_1) &= (1,1,0,1,0,\ 1,1)\\
f(F_2) &= (0,1,0,0,1,\ 0,1)\\
f(F_3) &= (1,1,0,1,0,\ 1,1)\\
f(F_4) &= (0,1,0,0,1,\ 0,1)\\
f(F_5) &= (1,1,0,1,0,\ 1,1)\\
f(F_6) &= (0,1,0,0,1,\ 0,1)
\end{aligned}
$$

and the c_f-values of the vertices of $C_4 \,\square\, C_7$ are

$$
\begin{aligned}
c_f(F_1) &= (3,4,2,1,4,\ 2,5)\\
c_f(F_2) &= (4,3,1,3,1,\ 4,3)\\
c_f(F_3) &= (3,4,2,1,4,\ 2,5)\\
c_f(F_4) &= (4,3,1,3,1,\ 4,3)\\
c_f(F_5) &= (3,4,2,1,4,\ 2,5)\\
c_f(F_6) &= (4,3,1,3,1,\ 4,3).
\end{aligned}
$$

⋆ For $n = 9$ where $n \equiv 1 \pmod 4$, the f-values of the vertices of $C_6 \,\square\, C_9$ are therefore

$$\begin{aligned}
f(F_1) &= (1,1,0,1,0,\ \underline{1,1,0,0})\\
f(F_2) &= (0,1,0,0,1,\ \underline{0,1,0,1})\\
f(F_3) &= (1,1,0,1,0,\ \underline{1,1,0,0})\\
f(F_4) &= (0,1,0,0,1,\ \underline{0,1,0,1})\\
f(F_5) &= (1,1,0,1,0,\ \underline{1,1,0,0})\\
f(F_6) &= (0,1,0,0,1,\ \underline{0,1,0,1})
\end{aligned}$$

and the c_f-values of the vertices of $C_4 \square C_9$ are

$$\begin{aligned}
c_f(F_1) &= (2,4,2,1,4,\ \underline{2,4,1,3})\\
c_f(F_2) &= (4,3,1,3,1,\ \underline{4,3,2,1})\\
c_f(F_3) &= (2,4,2,1,4,\ \underline{2,4,1,3})\\
c_f(F_4) &= (4,3,1,3,1,\ \underline{4,3,2,1})\\
c_f(F_5) &= (2,4,2,1,4,\ \underline{2,4,1,3})\\
c_f(F_6) &= (4,3,1,3,1,\ \underline{4,3,2,1}).
\end{aligned}$$

As a consequence of the results obtained for the graphs $C_m \square C_n$, the only such graphs for which it is unknown whether there is a proper dominating function are those where $m, n \geq 5$ and m and n are both odd.

Problem 5.3.5 *For odd integers $m, n \geq 5$, does $C_m \square C_n$ have a proper dominating function? In particular, does $C_5 \square C_5$ have a proper dominating function?*

The graph $C_3 \square K_2$ can also be expressed as $C_3 \square P_2$. Looking at the graph $C_3 \square K_2$ in this manner suggests considering another class of graphs to investigate, namely $C_3 \square P_n$ for $n \geq 3$.

Proposition 5.3.6 *For each integer $n \geq 3$, the graph $C_3 \square P_n$ does not have a proper dominating function.*

Proof. Let $G = C_3 \square P_n$ where $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ such that u_iu_{i+1}, v_iv_{i+1}, $w_iw_{i+1} \in E(G)$ for $1 \leq i \leq n-1$. In addition, $T_i = (u_i, v_i, w_i, u_i)$ is a triangle in G for $1 \leq i \leq n$. The graphs $C_3 \square P_3$ and $C_3 \square P_4$ are shown in

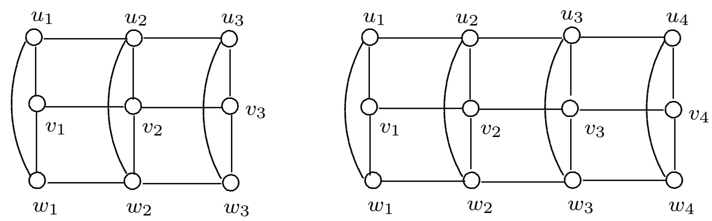

Figure 5.15: The graphs $C_3 \square P_3$ and $C_3 \square P_4$

Figure 5.15. Assume, to the contrary, that $G = C_3 \square P_n$ has a proper dominating function $f : V(G) \to \{0,1\}$ for some integer $n \geq 3$.

First, we show that if $\deg x = 3$, then $c_f(x) \leq 3$; for otherwise, we may assume that $c_f(u_1) = 4$. Then $f(x) = 1$ for each $x \in N[u_1] = \{u_1, u_2, v_1, w_1\}$. However then, $c_f(u_1), c_f(v_1), c_f(w_1) \in \{3,4\}$, which is impossible. This also shows that at most two vertices of $T_1 = (u_1, v_1, w_1, u_1)$ have f-value 1. If exactly two vertices of T_1 have f-value 1, then $c_f(u_1), c_f(v_1), c_f(w_1) \in \{2,3\}$; if exactly one vertex of T_1 has f-value 1, then $c_f(u_1), c_f(v_1), c_f(w_1) \in \{1,2\}$; and if no vertex of T_1 has f-value 1, then $c_f(u_1) = c_f(v_1) = c_f(w_1) = 1$, which is impossible in each case. ■

Since $P_m \square P_n$ is bipartite for all $m, n \geq 2$ and $C_m \square P_n$ is bipartite for all even integers $m \geq 4$ and integers $n \geq 2$, all of these graphs have a proper dominating function by Proposition 5.1.2. However, this suggests investigating the graphs $C_m \square P_n$ for odd integers $m \geq 5$ and all integers $n \geq 3$. The graphs $C_m \square P_n$ are often called *cylindrical graphs*. We begin with $C_5 \square P_n$ for odd integers $n \geq 3$.

Proposition 5.3.7 *For each odd integer $n \geq 3$, the graph $C_5 \square P_n$ has a proper dominating function.*

Proof. Let $G = C_5 \square P_n$, where $V(G) = \{u_{i,j} : 1 \leq i \leq 5, 1 \leq j \leq n\}$, and G consists of n mutually disjoint 5-cycles $F_j = (u_{1,j}, u_{2,j}, \ldots, u_{5,j}, u_{1,j})$ for $1 \leq j \leq n$ and five mutually disjoint n-paths $(u_{i,1}, u_{i,2}, \ldots, u_{i,n})$ for $1 \leq i \leq 5$. Define a proper dominating function $f : V(G) \to \{0,1\}$ by

$$f(F_j) = \begin{cases} (0,1,0,0,1) & \text{if } j \text{ is odd and } 1 \leq j \leq n \\ (1,1,0,1,0) & \text{if } j \text{ is even and } 2 \leq j \leq n-1. \end{cases}$$

Then the c_f-values of the vertices of $C_5 \,\Box\, P_n$ are

$$c_f(f_j) = \begin{cases} (3,2,1,2,1) & \text{if } j = 1, n \\ (2,4,2,1,3) & \text{if } j \text{ is even and } 2 \le j \le n-1 \\ (4,3,1,3,1) & \text{if } j \text{ is odd and } 3 \le j \le n-2. \end{cases}$$

For example, the proper dominating function f is illustrated in Figure 5.16 for the graph $C_5 \,\Box\, P_5$.

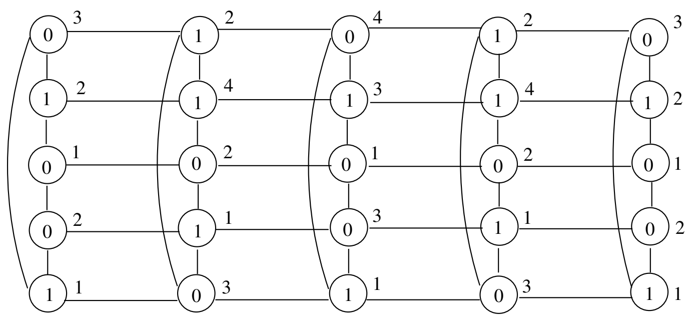

Figure 5.16: A proper dominating function of $C_5 \,\Box\, P_5$

Thus, $C_5 \,\Box\, P_n$ has a proper dominating function for each odd integer $n \ge 3$. ■

A number of questions remain however.

Problem 5.3.8 *For an even integer $n \ge 4$, does $C_5 \,\Box\, P_n$ have a proper dominating function? In particular, does $C_5 \,\Box\, P_4$ have a proper dominating function?*

Problem 5.3.9 *For an odd integer $m \ge 7$ and an integer $n \ge 3$, does $C_m \,\Box\, P_n$ have a proper dominating function? In particular, does $C_7 \,\Box\, P_3$ have a proper dominating function?*

Chapter 6

PROPER DOMINATING FUNCTIONS II

ABSTRACT: We study proper dominating functions of trees. Sufficient conditions are obtained under which a tree has or does not have a proper dominating function. For several classes of trees, we determine precisely which members possess a proper dominating function.

6.1 Introduction

We saw (in Proposition 5.1.2) that if each vertex in one of the partite sets of connected bipartite graph G has degree 2 or more, then G has a proper dominating function. Consequently, if G is a bipartite graph with $\delta(G) \geq 2$, then G has a proper dominating function. This brings up the question of what can be said if G is a bipartite graph with $\delta(G) = 1$. The primary class of bipartite graphs with this property are trees. The following result is a consequence of Proposition 5.1.2 on bipartite graphs.

Proposition 6.1.1 *If T is a tree such that all end-vertices of T belong to the same partite set, then T has a proper dominating function.*

An immediate consequence of Proposition 6.1.1 is that every star of order 3 or more has a proper dominating function. One of the simplest classes of trees is the class of paths.

Proposition 6.1.2 *For an integer $n \geq 3$, the path P_n has a proper dominating function if and only if $n = 4$ or n is odd.*

Proof. If $n \geq 3$ is odd, then the two end-vertices of P_n belong to a same partite set. It then follows by Proposition 6.1.1 that P_n has a proper dominating function for odd integers n.

For the converse, suppose that $P_n = (v_1, v_2, \ldots, v_n)$ is a path of even order $n \geq 4$. For $n = 4$, the function f with $f(v_1) = f(v_2) = f(v_3) = 1$ and $f(v_4) = 0$ is a proper dominating function. Thus, we may assume that $n \geq 6$. Assume, to the contrary, that there is a proper dominating function $g : V(P_n) \to \{0, 1\}$ of P_n for some even integer $n \geq 6$. First, we claim that no two consecutive vertices of P_n can have g-value 0. Since g is a dominating function, it is impossible that $g(v_1) = g(v_2) = 0$ or $g(v_{n-1}) = g(v_n) = 0$. Thus, we may assume that $g(v_i) = g(v_{i+1}) = 0$ where $2 \leq i \leq n-2$. This forces $g(v_{i-1}) = g(v_{i+2}) = 1$ and so $c_g(v_i) = c_g(v_{i+1}) = 1$, a contradiction. Necessarily, there is a vertex v_i, $2 \leq i \leq n-1$, with $g(v_i) = 0$. Thus, $g(v_{i-1}) = g(v_{i+1}) = 1$, which implies that $c_g(v_i) = 2$ and $c_g(v_{i-1}) = c_g(v_{i+1}) = 1$. This in turns implies that $g(v_i) = g(v_j)$ if and only if i and j are of the same parity. We may therefore assume that $g(v_i) = 0$ if i is odd and $1 \leq i \leq n-1$ and $g(v_i) = 1$ if i is even and $2 \leq i \leq n$. However then, $c_g(v_1) = c_g(v_2) = 1$, which is impossible. ■

A path $P_3 = (x, y, z)$ in a connected graph G of order 4 or more is called a *pendant* 3*-path at* z in G if x is an end-vertex in G and y has degree 2 in G. In this case, z is referred to as the *terminal vertex* of P_3. Clearly, z is not an end-vertex in G. For example, the tree of Figure 6.1 has exactly three pendant 3-paths, two at the vertex z_1 and one at the vertex z_2.

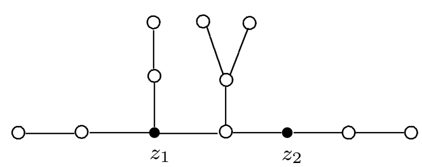

Figure 6.1: Illustrating pendant 3-paths and their terminal vertices

If $P_3 = (x, y, z)$ is a pendant 3-path at z in a connected graph G of at least order 4, then every proper dominating function f of G must assign 1 to the terminal vertex z; for otherwise, $c_f(x) = c_f(y)$, which is impossible. We will see that this following observation is useful.

Observation 6.1.3 *If f is a proper dominating function of a connected graph G of order 4 or more, then f assigns 1 to the terminal vertex of each pendant 3-path in G.*

Proposition 6.1.4 *Let G be a connected graph of order 4 or more and let f be a proper dominating function of G. If u and v are two adjacent vertices of G such that $N[u] - N[v] = \{x\}$ and $N[v] - N[u] = \{y\}$, then $f(x) \neq f(y)$. In particular, if (x, u, v, y) is a 4-path in G such that $\deg_G u = \deg_G v = 2$, then $f(x) \neq f(y)$.*

Proof. Assume, to the contrary, that $f(x) = f(y)$. Let $W = N[u] - \{x\} = N[v] - \{y\}$ and let $p = \sum_{w \in W} f(w)$. Then $c_f(u) = f(x) + p = f(y) + p = c_f(v)$, which is impossible. ■

The following is a consequence of Proposition 6.1.4.

Corollary 6.1.5 *If a graph G contains two adjacent vertices u and v such that $N[u] - N[v] = \{x\}$ and $N[v] - N[u] = \{y\}$ and each of x and y is the terminal vertex of a pendant 3-path, then G does not have a proper dominating function.*

The graph G of Figure 6.2 does not have a proper dominating function by Corollary 6.1.5.

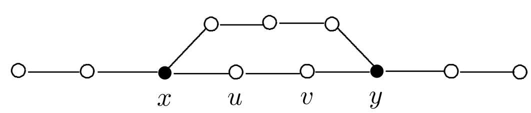

Figure 6.2: A graph possessing no proper dominating function

We saw in Proposition 6.1.1 that if T is a tree whose end-vertices all belong to the same partite set, then T has a proper dominating function. The converse of Proposition 6.1.1 is not true, however, since the path P_4 has a proper dominating function, as stated in Proposition 6.1.2. In fact, there is an infinite class of trees which includes P_4 whose end-vertices belong to different partite sets and having a proper dominating function. A *double star* is a tree of diameter 3. Thus, every double star T has exactly two vertices that are not leaves, which are referred to as the *central vertices* of T.

Proposition 6.1.6 *Every double star has a proper dominating function.*

Proof. Let T denote a double star whose central vertices are u and v, and let w be an end-vertex that is adjacent to u. Define a function $f : V(T) \to \{0,1\}$ by $f(u) = f(v) = f(w) = 1$ and $f(x) = 0$ for all remaining vertices of T. Since $c_f(u) = 3$, $c_f(v) = c_f(w) = 2$, and $c_f(x) = 1$ for each $x \in V(T) - \{u, v, w\}$, it follows that f is a proper dominating function. ∎

In what follows, we consider some familiar classes of trees and determine which trees in these classes have a proper dominating function.

6.2 A Class of Trees Having Proper Dominating Functions

In this section, we consider a class of trees where every member possesses a proper dominating function. The *non-leaf minimum degree* $\delta^*(T)$ of a tree T is the minimum degree among the non-leaves of T. A tree T is often referred to as r-*regular* for some integer $r \geq 2$ if every non-leaf of T has degree r. In particular, a 3-regular tree is a *cubic* tree. A *caterpillar* T is a tree of order 3 or more, the removal of whose leaves produces a path called the *spine* of T. A star is therefore a caterpillar with a trivial spine and a double star is a caterpillar with spine P_2.

Proposition 6.2.1 *Every caterpillar T with $\delta^*(T) \geq 3$ has a proper dominating function.*

Proof. Let T be a caterpillar of diameter d. Since all stars and double stars have a proper dominating function, we may assume that $d \geq 4$. Let $(u_0, u_1, \ldots, u_{d-1}, u_d)$ be a path of length d in T. We consider two cases.

Case 1. *T is a cubic caterpillar.* For $1 \leq i \leq d-1$, let v_i be the end-vertex adjacent to u_i. Define a function $f : V(T) \to \{0,1\}$ of T by

$$f(x) = \begin{cases} 1 & \text{if } x = u_i \text{ for } 1 \leq i \leq d \\ & \text{or } x = v_j \text{ for odd integer } j \text{ with } 3 \leq j \leq d-1 \\ 0 & \text{otherwise.} \end{cases}$$

This is illustrated in Figure 6.3 for $d = 6, 7$. It remains to show that c_f is proper.

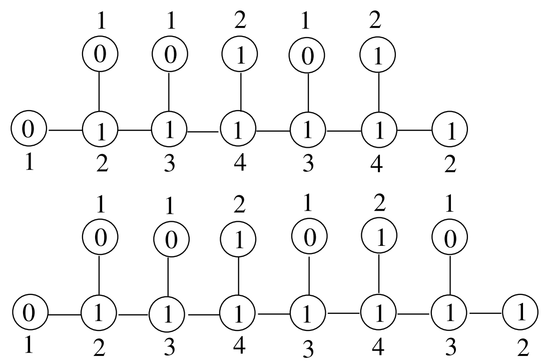

Figure 6.3: Proper dominating functions for $d = 6, 7$ in Case 1

- ⋆ If d is even, say $d = 2k$ for some integer $k \geq 2$, then

$$\begin{aligned}(c_f(u_0), c_f(u_1), \ldots, c_f(u_d)) &= (\underline{1, 2},\ 3, 4, 3, 4, \ldots, 3, 4,\ \underline{2})\\ (c_f(v_1), c_f(v_2), \ldots, c_f(v_{d-1})) &= (\underline{1}, 1, 2, 1, 2, \ldots, 1, 2, 1, 2),\end{aligned}$$

  where there are $k - 1 \geq 0$ pairs $(3, 4)$ in the $c_f(u_i)$-values for $0 \leq i \leq d$ and $k - 1$ pairs $(1, 2)$ in the $c_f(v_i)$-values for $1 \leq i \leq d - 1$.

- ⋆ If d is odd, say $d = 2k + 1$ for some integer $k \geq 2$, then

$$\begin{aligned}(c_f(u_0), c_f(u_1), \ldots, c_f(u_d)) &= (\underline{1, 2},\ 3, 4, 3, 4, \ldots, 3, 4,\ \underline{3, 2})\\ (c_f(v_1), c_f(v_2), \ldots, c_f(v_{d-1})) &= (\underline{1}, 1, 2, 1, 2, \ldots, 1, 2, 1, 2,\ \underline{1}),\end{aligned}$$

  where there are $k - 1 \geq 0$ pairs $(3, 4)$ in the $c_f(u_i)$-values for $0 \leq i \leq d$ and $k - 1$ pairs $(1, 2)$ in the $c_f(v_i)$-values for $1 \leq i \leq d - 1$.

Thus, f is a proper dominating function of T.

Case 2. *T is not a cubic caterpillar.* Since $\delta^*(T) \geq 3$, it follows that T contains a cubic subcaterpillar T_0 as described in Case 1 and let f be the proper dominating function of T_0 defined in Case 1. The function f then can be extended to a proper dominating function g of T by defining $g(v) = f(v)$ if $v \in V(T_0)$ and $g(v) = 0$ for each $v \in V(T) - V(T_0)$. ■

6.3 Starlike Trees

By Proposition 6.2.1, every caterpillar T with $\delta^*(T) \geq 3$ has a proper dominating function. The condition that $\delta^*(T) \geq 3$ in Proposition 6.2.1 for a caterpillar T to possess a proper dominating function is needed since $\delta^*(T) = 2$ for the caterpillar T of Figure 6.4 but this caterpillar has no proper dominating function.

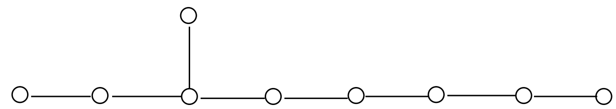

Figure 6.4: A caterpillar possessing no proper dominating function

We will soon see why this tree fails to have a proper dominating function. The caterpillar T of Figure 6.4 has the added characteristic of possessing only one vertex of degree greater than 2. It is the class of trees containing a unique vertex of degree greater than 2 that we now consider. The simplest members of this class are the stars $K_{1,n-1}$ of order $n \geq 4$, all of which have a proper dominating function by Proposition 6.1.1. All other members of this class are obtained by subdividing the edges of a star of order 4 or more. As we will see, many members of this class possess a proper dominating function and many do not. We present a complete solution for this class of trees.

A tree T is *starlike* if T is obtained by subdividing the edges of a star of order 4 or more. Thus, the caterpillar shown in Figure 6.4 is a starlike tree. Equivalently, a tree T is starlike if and only if T has exactly one vertex whose degree is greater than 2. This vertex is referred to as the *center* of T. The branches of T at the center are called the *arms* of T. Thus, if v is the center of a starlike tree T and $\deg v = r \geq 3$, then T contains r arms where v is an end-vertex in each arm. The sum of the lengths of the arms of T is the size of T. An arm is *even* if its length is even; while an arm is *odd* if its length is odd. For example, the tree T shown in Figure 6.5 is a starlike tree obtained by subdividing the edges of the star $K_{1,3}$. Its center is v_4; T has two even arms and one odd arm, one arm of each of the lengths 2, 3, and 4.

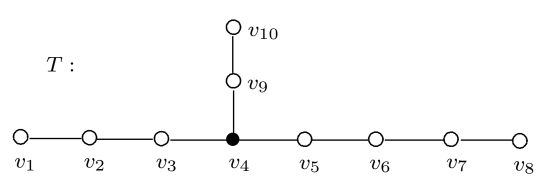

Figure 6.5: A starlike tree T

The primary problem here is determining which starlike trees possess a proper dominating function. While all stars have a proper dominating function, this is not the case for starlike trees. We give one such example.

Example 6.3.1 *The starlike tree T in Figure* 6.5 *does not have a proper dominating function.*

Proof. Assume, to the contrary, that T has a proper dominating function $f : V(T) \to \{0, 1\}$. The tree T has three pendant 3-paths, namely (v_1, v_2, v_3), (v_{10}, v_9, v_4) and (v_8, v_7, v_6). Thus, each of the vertices v_3, v_4, v_6 is the terminal vertex of a pendant 3-path in T. It then follows by Observation 6.1.3 that $f(v_3) = f(v_4) = f(v_6) = 1$. We consider two cases, depending on the value of $f(v_5)$.

Case 1. $f(v_5) = 1$. Then $c_f(v_5) = 3$, which forces $f(v_7) = 0$ and $f(v_8) = 1$. However then, $c_f(v_6) = c_f(v_7) = 2$, a contradiction.

Case 2. $f(v_5) = 0$. Then $c_f(v_5) = 2$. If $f(v_7) = 1$, then $c_f(v_5) = c_f(v_6) = 2$, which is impossible – necessarily then $f(v_7) = 0$ and so $f(v_8) = 1$. Since $c_f(v_5) = 2$, it requires that $c_f(v_4) = 3$ and so $c_f(v_3) = 2$. Consequently, $f(v_2) = 0$ and $f(v_1) = 1$. However then, $c_f(v_2) = c_f(v_3) = 2$, which is impossible. ■

The starlike tree T in Example 6.3.1 has two even arms and one odd arm and so T has arms of different parity. On the other hand, if all arms in a starlike tree have the same parity, then all end-vertices of this tree belong to the same partite set. Thus the following is a consequence of Proposition 6.1.1.

Corollary 6.3.2 *If T is a starlike tree of maximum degree* 3 *or more and having arms of the same parity, then T has a proper dominating function.*

Proposition 6.3.3 *If T is a starlike tree with diameter at most* 3*, then T has a proper dominating function.*

If T is a tree with diameter at most 3, then T is either a star or a double star. If T is a star, then T is bipartite with all end vertices in the same partite set so by Proposition 6.1.1, T has a proper dominating function. If T is a double star, then Proposition 6.1.6 shows that T has a proper dominating function.

By Corollary 6.3.2 and Propositions 6.3.3, we only consider starlike trees of diameter at least 4 having arms of different parity. First, we present some useful observations. The following is a consequence of the proof of Proposition 6.1.2.

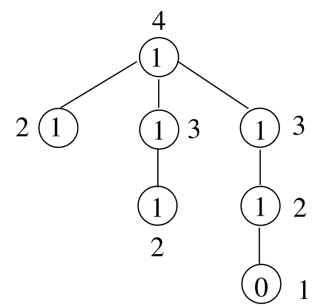

Figure 6.6: A proper dominating function of a starlike tree

Observation 6.3.4 *Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order n, where $n = 4$ or n is odd. For a proper dominating function f of P_n, let*

$$S_f = (f(v_1), f(v_2), \ldots, f(v_n)).$$

(*a*) *If $n = 4$, then $S_f = (0,1,1,1)$ or $S_f = (1,1,1,0)$.*

(*b*) *If $n = 3$, then $S_f = (1,1,1)$ or $S_f = (1,0,1)$.*

(*c*) *If $n = 5$, then $S_f = (0,1,1,1,0)$ or $S_f = (1,0,1,0,1)$.*

(*d*) *If n is odd and $n \neq 5$, then $S_f = (1,0,1,0,\ldots,1,0,1)$.*

Lemma 6.3.5 *Let T be a starlike tree whose center is v with $\deg v \geq 3$. If T has a proper dominating function f such that $f(v) = 0$, then each arm of T has odd length or length 4. For $u \in N(v)$, let ℓ_u be the length of the arm A_u containing the edge vu in T.*

(*i*) *If ℓ_u is odd, then $f(u)$ can be 0 or 1 when $\ell_u = 5$ and $f(u) = 1$ when $\ell_u \neq 5$.*

(*ii*) *If ℓ_u is even, then $\ell_u = 4$ and $f(u)$ can be 0 or 1.*

Proof. For $u \in N(v)$, let $A_u = (v, u = v_1, v_2, \ldots, v_{\ell_u})$ be the arm of length ℓ_u at v and let $P_{\ell_u} = A_u - v = (v_1, v_2, \ldots, v_{\ell_u})$. Since $f(v) = 0$, the restriction of f to P_{ℓ_u} is also a proper dominating function of P_{ℓ_u}. It then follows by Proposition 6.1.2 that either ℓ_u is odd or $\ell_u = 4$. Therefore, (i) and (ii) are consequences of Observation 6.3.4. ■

Proposition 6.3.6 *Let T be a starlike tree whose center is v with $\deg v \geq 3$. If each arm of T has odd length or length 4, then T has a proper dominating function.*

Proof. For $u \in N(v)$, let A_u be the arm of length ℓ_u at v and let

$$P_{\ell_u} = A_u - v = (u = v_1, v_2, \ldots, v_{\ell_u}).$$

We define a function $f : V(T) \to \{0, 1\}$ as follows.

- $\star$ Let $f(v) = 0$.
- $\star$ If $\ell_u = 4$, let $(f(v_1), f(v_2), f(v_3), f(v_4)) = (1, 1, 1, 0)$.
- $\star$ If ℓ_u is odd, then $(f(v_1), f(v_2), \ldots, f(v_{\ell_u})) = (1, 0, 1, 0, \ldots, 1, 0, 1)$.

Since $f(v) = 0$ and $f(u) = 1$ for each $u \in N(v)$, it follows that $c_f(v) = \deg v \geq 3$ and $c_f(u) \in \{1, 2\}$ for each $u \in N(v)$. Furthermore, the restriction of f to each arm of T is a proper dominating function of the arm. Therefore, f is a proper dominating function of T. ■

Lemma 6.3.7 *Let T be a starlike tree whose center is v with* $\deg v \geq 3$. *If T has a proper dominating function f such that $f(v) = 1$, then each arm of T has even length or length* 1 *or* 3. *For $u \in N(v)$, let ℓ_u be the length of the arm A_u containing the edge vu in T.*

(i) If ℓ_u is even, then $f(u)$ can be 0 *or* 1 *when $\ell_u = 2$ and $f(u) = 0$ when $\ell_u \neq 2$.*

(ii) If $\ell_u = 1$, then $f(u)$ can be 0 *or* 1.

(iii) If $\ell_u = 3$, then $f(u) = 1$.

Proof. For $u \in N(v)$, let $A_u = (v, u = v_1, v_2, \ldots, v_{\ell_u})$ be the arm of length ℓ_u at v and let $P_{\ell_u} = A_u - v = (u = v_1, v_2, \ldots, v_{\ell_u})$. Assume, to the contrary, that ℓ_u is odd and $\ell_u \geq 5$. Then

$$P_{\ell_u - 1} = A_u - \{v, u\} = (v_2, v_3, \ldots, v_{\ell_u})$$

is a subpath of P_{ℓ_u} of even order $\ell_u - 1 \geq 4$. We consider two cases, according to whether $f(u) = 0$ or $f(u) = 1$.

Case 1. $f(u) = 0$. Since $f(u) = 0$, the restriction of f to $P_{\ell_u - 1}$ is a proper dominating function of $P_{\ell_u - 1}$. Since $\ell_u - 1 \geq 4$ is even, it follows by Proposition 6.1.2 that $\ell_u - 1 = 4$. Hence, $P_{\ell_u - 1} = P_4 = (v_2, v_3, v_4, v_5)$. By Observation 6.3.4, either

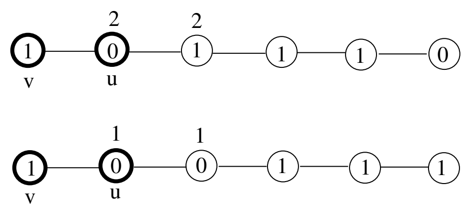

Figure 6.7: Illustrating Case 1

$(f(v_2), f(v_3), f(v_4), f(v_5)) = (0, 1, 1, 1)$ or $((f(v_2), f(v_3), f(v_4), f(v_5)) = (1, 1, 1, 0)$, as indicated below. In either case, $c_f(v_1) = c_f(v_2)$, which is impossible.

Case 2. $f(u) = 1$. First, suppose that $f(v_2) = 0$ and so $c_f(u) = c_f(v_1) = 2$. This implies that $f(v_3) = 0$ and so A_u has two adjacent vertices whose f-value is 0, which is impossible. Next, suppose that $f(v_2) = 1$ and so $c_f(u) = c_f(v_1) = 3$. This forces $f(v_3) = 0$ and results in $c_f(v_2) = 2$. Since $c_f(v_3) \neq 2$, this forces $f(v_4) = 0$, once again, resulting in two adjacent vertices whose f-value is 0, an impossibility.

The statements (i), (ii), and (iii) are then consequences of Observation 6.3.4. ■

The starlike tree T in Figure 6.5 has two arms of length at most 3. As we saw, T has no proper dominating function.

Proposition 6.3.8 *Let T be a starlike tree whose center is v with $\deg v \geq 3$. If each arm of T has even length or length 1 or 3 and at least three arms of length at most 3, then T has a proper dominating function.*

Proof. For $u \in N(v)$, let A_u be the arm of length ℓ_u at v and let

$$P_{\ell_u} = A_u - v = (u = v_1, v_2, \ldots, v_{\ell_u}).$$

We define a function $f : V(T) \to \{0, 1\}$ as follows.

- ⋆ Let $f(v) = 1$.
- ⋆ If $\ell_u = 1$, let $f(u) = f(v_1) = 1$.
- ⋆ If $\ell_u = 2$, let $(f(v_1), f(v_2)) = (1, 1)$.
- ⋆ If $\ell_u = 3$, let $(f(u) = f(v_1), f(v_2), f(v_3)) = (1, 1, 0)$.
- ⋆ If ℓ_u is even, then $(f(v_1), f(v_2), \ldots, f(v_{\ell_u})) = (0, 1, 0, 1, \ldots, 0, 1)$.

Since $f(v) = f(u) = 1$ for each $u \in N(v)$ such that $1 \leq \ell_u \leq 3$ and T has at least three arms of length at most 3, it follows that $c_f(v_i) \in \{1, 2, 3\}$ for $1 \leq i \leq \ell_u$.

If $\ell_u = 1$, then $c_f(u) = c_f(v_1) = 2$.

If $\ell_u = 2$, then $(c_f(u) = c_f(v_1), c_f(v_2)) = (3, 2)$.

If $\ell_u = 3$, then $(c_f(u) = c_f(v_1), c_f(v_2), c_f(v_3)) = (3, 2, 1)$,

If $\ell_u \geq 4$ is even, then $(c_f(u) = c_f(v_1), c_f(v_2), \ldots, c_f(v_{\ell_u})) = (2, 1, 2, 1, \ldots, 2, 1)$.

The functions f and c_f are indicated for the starlike tree in Figure 6.8. Therefore, f is a proper dominating function of T. ■

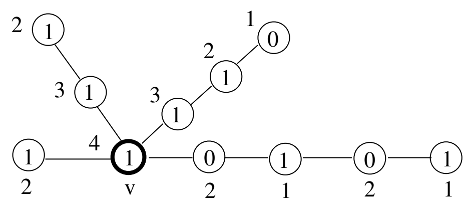

Figure 6.8: A starlike tree

Proposition 6.3.9 *Let T be a starlike tree whose center is v with $\deg v \geq 3$. If each arm of T has even length or length 1, at least two of which have length 1 or 2, then T has a proper dominating function.*

Proof. For $u \in N(v)$, let A_u be the arm of length ℓ_u at v and let

$$P_{\ell_u} = A_u - v = (u = v_1, v_2, \ldots, v_{\ell_u}).$$

We define a function $f : V(T) \to \{0, 1\}$ as follows.

- ⋆ Let $f(v) = 1$.
- ⋆ If $\ell_u = 1$, let $f(u) = f(v_1) = 1$.
- ⋆ If $\ell_u = 2$, let $(f(v_1), f(v_2)) = (1, 0)$.
- ⋆ If $\ell_u \geq 4$, then $(f(v_1), f(v_2), \ldots, f(v_{\ell_u})) = (0, 1, 0, 1, \ldots, 0, 1)$.

Since $f(v) = f(u) = 1$ for each $u \in N(v)$ such that $1 \leq \ell_u \leq 2$ and T has at least two arms of length 1 or 2, it follows that $c_f(v_i) \in \{1, 2\}$ for $1 \leq i \leq \ell_u$.

If $\ell_u = 1$, then $c_f(u) = c_f(v_1) = 2$.

If $\ell_u = 2$, then $(c_f(u) = c_f(v_1), c_f(v_2)) = (2, 1)$.

If $\ell_u \geq 4$, then $(c_f(u) = c_f(v_1), c_f(v_2), \ldots, c_f(v_{\ell_u})) = (2, 1, 2, 1, \ldots, 2, 1)$.

The functions f and c_f are indicated for the starlike tree in Figure 6.9. Therefore, f is a proper dominating function of T. ■

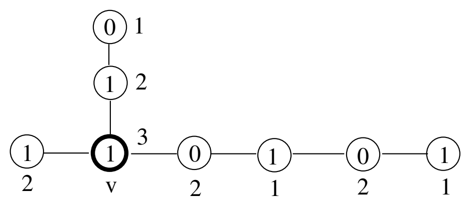

Figure 6.9: A starlike tree

We begin with those starlike trees having at least three arms of length at most 3.

Theorem 6.3.10 *Let T be a starlike tree whose center is v with $\deg v \geq 3$ and diameter at least 4 such that T has arms of different parity and at least three arms of length 1, 2 or 3. Then T has a proper dominating function if and only if*

(i) *each arm of T has odd length or length 4 or*

(ii) *each arm of T has even length or length 1 or 3.*

Proof. By Propositions 6.3.6 and 6.3.8, it remains for such starlike trees only to verify the necessity of this statement. Suppose that T satisfies neither of (i) nor (ii). This implies that T has an even arm whose length is not 4 and an arm of odd length 5 or more. We show that T does not have a proper dominating function. Assume, to the contrary, that T has a proper dominating function $f : V(T) \to \{0, 1\}$. Since T has an even arm whose length is not 4, it follows by Lemma 6.3.5 that $f(v) = 1$. However then, all odd arms of T have length 1 or 3 by Lemma 6.3.7, which is a contradiction. ■

We now consider those starlike trees without any arm of length at most 3.

Theorem 6.3.11 *Let T be a starlike tree whose center is v with $\deg v \geq 3$ such that T has arms of different parity and no arm has length 1, 2, or 3. Then T has a proper dominating function if and only if each arm of T has odd length or length 4.*

Proof. By Proposition 6.3.6, if each arm of T has odd length or length 4, then T has a proper dominating function. For the converse, suppose that T does not satisfy the hypothesis and assume, to the contrary, that T has a proper dominating function $f : V(T) \to \{0, 1\}$. Since T does not satisfy the hypothesis, it follows that T has an even arm whose length is not 4. By Lemma 6.3.5, $f(v) = 1$. However then, all odd arms of T have length 1 or 3 by Lemma 6.3.7, which is a contradiction. ■

Next, we consider those starlike trees having exactly one arm of length at most 3. We begin with starlike trees having exactly one arm of length 1 and all other arms have length 4 or more.

Proposition 6.3.12 *Let T be a starlike tree whose center is v with $\deg v \geq 3$ and having arms of different parity. Suppose that T has exactly one arm of length 1 with all other arms of (even or odd) length 4 or more. Then T has a proper dominating function if and only if all even arms of T have length 4.*

Proof. Let $x \in N(v)$ such that $A_x = (v, x)$ has length 1. Thus, for each $u \in N(v)$, the arm A_u at v has length $\ell_u \geq 4$. By Proposition 6.3.6, if all even arms of T have length 4, then T has a proper dominating function. For the converse, suppose that T has an even arm of length 6 or more and a proper dominating function $f : V(T) \to \{0, 1\}$. By Lemma 6.3.5, $f(v) = 1$. It then follows by Lemma 6.3.7 that A_x is the only odd arm in T and $f(u) = 0$ for $u \in N(v) - \{x\}$. However then, $c_f(v) = c_f(x)$, which is impossible. ■

The situation in Proposition 6.3.12 is the same if a starlike tree has exactly one arm of length 3 and all other arms have length at least 4.

Proposition 6.3.13 *Let T be a starlike tree whose center is v with $\deg v \geq 3$ and having arms of different parity. Suppose that T has exactly one arm of length 3 with all other arms of (even or odd) length 4 or more. Then T has a proper dominating function if and only if all even arms of T have length 4.*

Proof. Let $x \in N(v)$ such that $A_x = (v, x = x_1, x_2, x_3)$ has length 3. Thus, for each $u \in N(v)$, the arm A_u at v has length $\ell_u \geq 4$. By Proposition 6.3.6, if all even arms of T have length 4, then T has a proper dominating function. For the converse, suppose that T has an even arm of length 6 or more and a proper dominating function $f : V(T) \to \{0, 1\}$. By Lemma 6.3.5, $f(v) = 1$. It then follows by Lemma 6.3.7 that A_x is the only odd arm in T and $f(x) = 1$ and $f(u) = 0$ for each $u \in N(v) - \{x\}$. Let $u \in N(v) - \{x\}$. Since no two adjacent vertices (different from v) on an arm can both have f-value 0, there exists a neighbor v' of u distinct from v such that $f(v') = 1$. However then, $c_f(v) = c_f(u) = 2$, which is impossible. ■

The situation is completely different for a starlike tree having arms of different parity, some of which has length 2 and others having length 4 or more.

Proposition 6.3.14 *Let T be a starlike tree whose center is v with $\deg v \geq 3$ and having arms of different parity. If T has an arm of length 2 with all other arms of (even or odd) length 4 or more, then T has no proper dominating function.*

Proof. Let $x \in N(v)$ such that the arm A_x has length 2. Thus A_x is a pendant P_3 at v. Assume, to the contrary, that T has a proper dominating function $f : V(T) \to \{0, 1\}$. Then $f(v) = 1$. It follows by Lemma 6.3.7 that each odd arm of T must have length 1 or 3, which is impossible. ■

The following is a consequence of the proof of Proposition 6.3.14 (or Lemma 6.3.7).

Corollary 6.3.15 *Let T be a starlike tree whose center is v with $\deg v \geq 3$ and having arms of different parity. If T has an arm of length 2 and T has a proper dominating function, then every odd arm of T has length 1 or 3.*

First, we determine which starlike trees having arms of different parity and an arm of length 2 possess a proper dominating function.

Theorem 6.3.16 *Let T be a starlike tree whose center is v with $\deg v \geq 3$ and having arms of different parity and an arm of length 2. Then T has a proper dominating function if and only if T satisfies the following conditions (a) and (b):*

(a) each odd arm of T has length 1 or 3 and

(b) if T has an arm of length 3, then T has at least three arms of length 1, 2, or 3.

Proof. First, suppose that T satisfies (a) and (b).

⋆ If each odd arm of T has length 1, then T has at least two arms of length at most 2 and so T has a proper dominating function by Proposition 6.3.9.

⋆ If T has an arm of length 3, then T has at least three arms of length 1, 2, or 3 and so T has a proper dominating function by Proposition 6.3.8.

For the converse, suppose that T does not satisfy (a) or does not satisfy (b) and has a proper dominating function $f : V(T) \to \{0, 1\}$. By Corollary 6.3.15, each arm of T has even length or length 1 or 3. So, (a) is satisfied and (b) is not satisfied. If T does not have an arm of length 3, then (b) is true vacuously. Hence, T has an arm of length 3 but T has at most two arms of length 1, 2 or 3. Since T has an arm of length 2, it follows that T has exactly one odd arm and this odd arm has length 3. Therefore, T has exactly one arm of length 2, exactly one arm of length 3, and all other arms have even length 4 or more. Let $x, y \in N(v)$ such that the length of A_x is 2 and the length of A_y is 3, say $A_x = (v, x, x')$ and $A_y = (v, y, y', y'')$. For each $u \in N(v) - \{x, y\} \neq \emptyset$, the arm A_u has even length 4 or more, say $A_u = (v, u = v_1, v_2, \ldots, v_{\ell_u})$ where ℓ_u is even and $\ell_u \geq 4$. By Lemma 6.3.7, it follows that

$$(f(v), f(x), f(x')) \in \{(1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$
$$(f(v), f(y), f(y'), f(y'')) = (1, 1, 1, 0)$$
$$(f(v), f(v_1), f(v_2), \ldots, f(v_{\ell_u})) = (1, 0, 1, 0, 1, \ldots, 0, 1).$$

The functions f and c_f are indicated for the starlike tree in Figure 6.10. Thus, $c_f(x) \in \{2, 3\}$, $c_f(y) = 3$ and $c_f(u) = 2$. If $f(x) = 0$, then $c_f(v) = c_f(u) = 2$; while if $f(x) = 1$, then $c_f(v) = c_f(y) = 3$. In either case, a contradiction is produced. ■

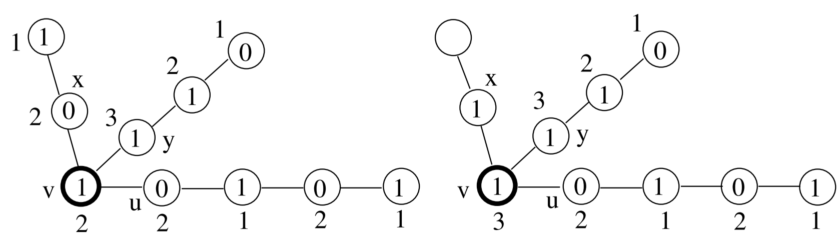

Figure 6.10: A starlike tree

We now consider starlike trees having arms of different parity but no arm of length 2. In order to do this, we first present a result that is similar to Propositions 6.3.12 and 6.3.13.

Proposition 6.3.17 *Let T be a starlike tree whose center is v with $\deg v \geq 3$ and having arms of different parity. Suppose that T has two arms of length 1 or 3, at least one of which has length 3, with all other arms of (even or odd) length 4 or more. Then T has a proper dominating function if and only if all even arms of T have length 4.*

Proof. Let $x, y \in N(v)$ such that A_x has length 1 or 3 and $A_y = (v, y = y_1, y_2, y_3)$ has length 3. If all even arms of T have length 4, then T has a proper dominating function by Proposition 6.3.6. For the converse, suppose that T has an even arm of length 6 or more and a proper dominating function $f : V(T) \to \{0, 1\}$. By Lemma 6.3.5, $f(v) = 1$. It then follows by Lemma 6.3.7 that (i) A_x and A_y are the only odd arms in T and (ii)$f(y) = 1$ and $f(u) = 0$ for each $u \in N(v) - \{x, y\} \neq \emptyset$. For $u \in N(v) - \{x, y\}$, let $A_u = (v, u = v_1, v_2, \ldots, v_{\ell_u})$ where then ℓ_u is even and $\ell_u \geq 4$. Since $f(u) = f(v_1) = 0$ and no two adjacent vertices (different from v) on an arm can both have f-value 0, it follows that $f(v_2) = 1$ and so $c_f(u) = 2$. This forces $f(x) = 1$ and so $c_f(v) = 3$. If $f(y_2) = 1$, then $c_f(v) = c_f(y) = 3$, while if $f(y_2) = 0$, then $f(y_3) = 1$ and so $c_f(y) = c_f(y_2) = 2$, which is impossible in either case. ■

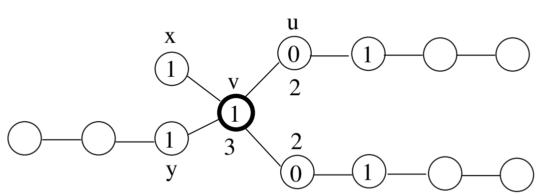

Figure 6.11: A starlike tree

Theorem 6.3.18 *Let T be a starlike tree whose center is v with $\deg v \geq 3$ and having arms of different parity but no arms of length 2. Then T has a proper dominating function if and only if T satisfies one of the following conditions:*

(i) Each even arm of T has length 4.

(ii) Some even arm of T has length 6 or more and each odd arm has length 1 or 3 such that either T has at least two arms of length 1 or T has at least three odd arms, each of which has length 1 or 3.

Proof. If T satisfies (i), then T has a proper dominating function by Proposition 6.3.6. If T satisfies (ii), then T has a proper dominating function by Proposition 6.3.9 and Theorem 6.3.10.

For the converse, suppose that T satisfies neither (i) nor (ii), but T has a proper dominating function $f : V(T) \to \{0, 1\}$. Since T does not satisfy (i), it follows that T has an even arm of length 6 or more. Let $x \in N(v)$ such that A_x is an even arm of length $\ell_x \geq 6$. It then follows by Lemma 6.3.5 that $f(v) = 1$ and so $f(x) = 0$ by Lemma 6.3.7(i). Since $f(v) = 1$, it follows by Lemma 6.3.7 that each odd arm of T has length 1 or 3. Since T does not satisfy (ii), it follows that T has one or two odd arms, each of which has length 1 or 3 and at least one of which has length 3. However then, all even arms must have length 4 by Propositions 6.3.12, 6.3.13, and 6.3.17. This contradicts the assumption that T has an even arm of length 6 or more. ■

By Corollary 6.3.2, Propositions 6.3.3 and 6.3.9, and Theorems 6.3.16 and 6.3.18, we have the following information. Let T be a starlike tree with maximum degree 3 or more.

(1) If $\text{diam}(T) \leq 3$ or all arms of T have the same parity, then T has a proper dominating function.

(2) Suppose that $\text{diam}(T) \geq 4$ and T has arms of different parity.

(2.1) Assume that T has an arm of length 2.

Then T has a proper dominating function if and only if each arm of T has even length or length 1 or 3 and if T has an arm of length 3, then T has at least three arms of length 1, 2, or 3.

(2.2) Assume that T has no an arm of length 2.

Then T has a proper dominating function if and only if

(i) each arm of T has odd length or length 4 or

(ii) some even arm of T has length 6 or more and each odd arm has length 1 or 3 such that either T has two arms of length 1 or T has at least three odd arms of length 1 or 3.

We are now prepared to present a characterization of all starlike trees T with maximum degree 3 or more such that T has a proper dominating function.

Theorem 6.3.19 *A starlike tree T with maximum degree 3 or more has a proper dominating function if and only if T satisfies any of the following conditions.*

(*a*) *The diameter of T is at most 3.*

(*b*) *All arms of T have the same parity.*

(*d*) *Each arm of T has odd length or length 4.*

(*c*) *Some even arm of T has length 2 or length 6 or more and each odd arm has length 1 or 3 such that either T has at least two arms of length at most 2 or T has at least three arms of length at most 3,*

Chapter 7

TOTAL DOMINATING FUNCTIONS

ABSTRACT: Let G be a nontrivial connected graph. A set S of vertices in G is a total dominating set of G if every vertex of G is adjacent to some vertex of S. The minimum cardinality of a total dominating set of G is the total domination number $\gamma_t(G)$ of G. Total domination has also been looked at in another way. A function $h : V(G) \to \{0, 1\}$ is a total dominating function of a graph G if $\sigma_h(v) = \sum_{u \in N(v)} h(u) \geq 1$ for every vertex v of G. A total dominating function h of a nontrivial graph G is an irregular total dominating function if $\sigma_h(u) \neq \sigma_h(v)$ for every two vertices u and v of G. No graph possesses an irregular total dominating function. A total dominating function h of a nontrivial graph G of order n is called an antiregular total dominating function if there are exactly two vertices u and v of G such that $\sigma_h(u) = \sigma_h(v)$. We show that for every integer $n \geq 3$, there are exactly two non-isomorphic graphs of order n having an antiregular total dominating function. If h is a total dominating function of a graph G such that $\sigma_h(v)$ is the same constant k for every vertex v of G, then h is called a regular (or a k-regular) total dominating function of G. We present some preliminary results dealing with properties of regular total dominating functions of graphs. In particular, we investigate regular total dominating functions of trees. Since the minimum degree of every nontrivial tree is 1, it follows that if a tree has a k-regular total dominating function, then $k = 1$. We apply algorithmic methods to characterize those trees with a 1-regular total dominating function. We also investigate regular total dominating functions of several well-known classes of regular graphs. We also present other results and questions on regular dominating

functions of connected graphs in general.

7.1 Introduction

We have seen that a vertex u dominates a vertex v in a graph if either $u = v$ or v is a neighbor of u. However, there are a number of variations of domination. One of the best known variations is total domination, which was introduced by Cockayne, Dawes and Hedetniemi [78]. In this variation, a vertex u dominates a vertex v only if v is a neighbor of u. Hence, a vertex does not dominate itself.

Let G be a graph. A set S of vertices in G is a *total dominating set* for G if every vertex of G is adjacent to some vertex of S. In particular, a vertex of S must be adjacent to another vertex of S. Therefore, a graph G has a total dominating set if and only if G contains no isolated vertices. Furthermore, if S is a total dominating set of G, then the subgraph $G[S]$ induced by S contains no isolated vertices. Therefore, in this chapter, we will assume that all graphs under consideration have no isolated vertices. The minimum cardinality of a total dominating set for G is the *total domination number* $\gamma_t(G)$ of G. A total dominating set of cardinality $\gamma_t(G)$ is called a *minimum total dominating set* or a γ_t*-set* for G. Since every total dominating set of a nontrivial connected graph is also a dominating set, $\gamma(G) \leq \gamma_t(G)$ for every nontrivial connected graph G.

For example, for the graph G of Figure 7.1, the set $\{u_1, v, w, v_4\}$ is a minimum total dominating set of G and so $\gamma_t(G) = 4$. On the other hand, the set $\{u, v, w\}$ is a minimum dominating set of G and so $\gamma(G) = 3$.

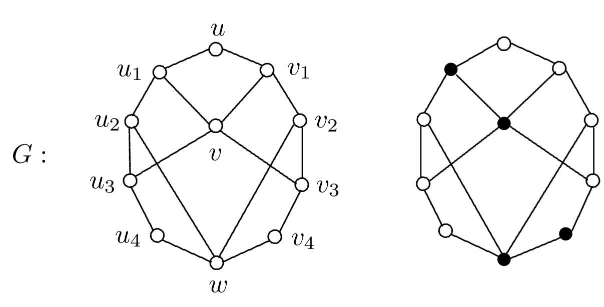

Figure 7.1: A minimum total dominating set in a graph

There is also another way to look at total domination and the total domination number of a graph G. Let $h : V(G) \to \{0, 1\}$ be a function. Then h gives rise to another function $\sigma_h : V(G) \to \mathbb{N} \cup \{0\}$ defined by

$$\sigma_h(v) = \sum_{u \in N(v)} h(u).$$

(The function σ_h can also be denoted by σ if the function h under consideration is clear.) Thus, $0 \leq \sigma_h(v) \leq \deg v \leq \Delta(G)$ for every vertex v of G. If $\sigma_h(v) \geq 1$ for every vertex v of G, then h is called a *total dominating function* of G. If h is a total dominating function of G, then the set $\mathcal{I}_h(G) = \{v \in V(G) : h(v) = 1\}$ is a total dominating set of G. On the other hand, if S is a total dominating set in G, then the function h that assigns 1 to each vertex of S and 0 to each vertex in $\overline{S} = V(G) - S$ is a total dominating function of G with $\mathcal{I}_h = S$. This is illustrated in Figure 7.2 where a total dominating function of the graph G of Figure 7.1 is obtained from a total dominating set in G.

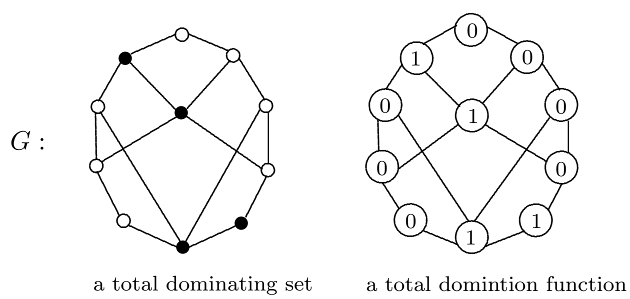

Figure 7.2: A total dominating function in a graph

Consequently, if we define the *total domination number* $\gamma_t(h)$ *of a total dominating function* h of a graph G as

$$\gamma_t(h) = \sum_{v \in V(G)} h(v) = |\mathcal{I}_h(G)|,$$

then the total domination number $\gamma_t(G)$ of G can be defined as

$$\gamma_t(G) = \min\{\gamma(h) : \ h \text{ is a total dominating function of } G\}.$$

First, we present some preliminary results on total dominating functions of graphs.

Proposition 7.1.1 *Let G be a nontrivial connected graph and let $h : V(G) \to \{0,1\}$ be a total dominating function of G. Then*

$$\sum_{v \in V(G)} \sigma_h(v) = \sum_{v \in V(G)} \deg v \cdot h(v) = \sum_{x \in \mathcal{I}_h(G)} \deg x.$$

Proof. Let $\mathcal{I}_h(G) = \{v \in V(G) : h(v) = 1\}$ be the set of all vertices of G whose h-value is 1. Denote the sum of the σ_h-values of all vertices of G by $\Sigma_h(G) = \sum_{v \in V(G)} \sigma_h(v)$. For each $v \in V(G)$, its h-value $h(v)$ is counted exactly $\deg v$ times in $\Sigma_h(G)$, once in $\sigma_h(u)$ for each neighbor u of v and so the result follows. ■

Let G be a nontrivial connected graph and let $h : V(G) \to \{0,1\}$ be a total dominating function of G. The *complementary function* $\overline{h} : V(G) \to \{0,1\}$ is defined by

$$\overline{h}(v) = 1 - h(v) \text{ for every vertex } v \text{ of } G.$$

Proposition 7.1.2 *Let G be a nontrivial connected graph. If $h : V(G) \to \{0,1\}$ is a total dominating function of G, then $\sigma_h(v) + \sigma_{\overline{h}}(v) = \deg v$ for each vertex v of G.*

Proof. Let $v \in V(G)$ where $h(v) = i \in \{0,1\}$ and $\sigma_h(v) = k$. Then $\overline{h}(v) = 1 - i$. Since $\sigma_h(v) = k$, it follows that v is adjacent to k vertices labeled 1 by h and so v is adjacent to $\deg v - k$ vertices labeled 0 by h. Thus, v is adjacent to $\deg v - k$ vertices labeled 1 by $\overline{h}$. Hence, $\sigma_{\overline{h}}(v) = \deg v - k$. Consequently, $\sigma_h(v) + \sigma_{\overline{h}}(v) = \deg v$. ■

The following corollaries are consequences of Proposition 7.1.2.

Corollary 7.1.3 *Let G be a nontrivial connected graph. If $h : V(G) \to \{0,1\}$ is a total dominating function of G such that $\sigma_h(v) \leq \deg v - 1$ for each vertex v of G, then its complementary function $\overline{h}$ is also a total dominating function of G.*

Proof. Since $\sigma_h(v) \leq \deg v - 1$ for each vertex v of G, it follows by Proposition 7.1.2 that $\sigma_{\overline{h}}(v) = \deg v - \sigma_h(v) \geq 1$ and so $\overline{h}$ is also a total dominating function of G. ■

Corollary 7.1.4 *If h is a total dominating function of a graph G with $\delta(G) = 1$, then its complementary function $\overline{h}$ is not a total dominating function of G.*

Proof. Let v be an end-vertex of G. Since $\sigma_h(v) = 1$, it follows by Proposition 7.1.2 that $\sigma_{\overline{h}}(v) = 0$. Thus, $\overline{h}$ is not a total dominating function of G. ■

7.2 Irregular and Antiregular Total Dominating Functions

Let G be a connected graph with no isolated vertices. If h is a total dominating function of G such that $\sigma_h(u) \neq \sigma_h(v)$ for every two vertices u and v of G, then h is called an *irregular total dominating function* of G. As is the case with irregular dominating functions, no graph possesses an irregular total dominating function.

Proposition 7.2.1 *No nontrivial connected graph possesses an irregular total dominating function.*

Proof. Assume, to the contrary, that there is a connected graph G of order $n \geq 2$ having an irregular total dominating function $h : V(G) \to \{0, 1\}$. Since $1 \leq \sigma_h(v) \leq n-1$ for every vertex v of G, it follows that $\{\sigma_h(v) : v \in V(G)\} \subseteq [n-1]$ and so there exist two distinct vertices x and y of G such that $\sigma_h(x) = \sigma_h(y)$, which is a contradiction. ■

Recall that a nontrivial graph G is *antiregular* if exactly two vertices of G have the same degree. Furthermore, for every integer $n \geq 2$, there are exactly two non-isomorphic antiregular graphs of order n, one of which is connected. We denote the connected antiregular graph of order n by G_n and so its complement is $\overline{G}_n$. Since the disconnected antiregular graph $\overline{G}_n$ of order $n \geq 2$ contains an isolated vertex, it follows that $\overline{G}_n$ does not have a total dominating function.

A total dominating function $h : V(G) \to \{0, 1\}$ of a nontrivial connected graph G of order n is called an *antiregular total dominating function* if

$$|\{\sigma_h(v) : v \in V(G)\}| = n - 1,$$

that is, there are exactly two vertices x and y of G such that $\sigma_h(x) = \sigma_h(y)$. Consequently, if h is an antiregular total dominating function of G, then

$$\{\sigma_h(v) : v \in V(G)\} = [n-1].$$

It may not be surprising that every connected antiregular graph has a simple antiregular total dominating function.

Proposition 7.2.2 *Every connected antiregular graph has an antiregular total dominating function.*

Proof. For the unique connected antiregular graph G_n of order $n \geq 2$, define the total dominating function $f : V(G_n) \to \{0, 1\}$ by $f(v) = 1$ for every vertex v of G_n. Thus, $\sigma_f(v) = \deg v \geq 1$ for each $v \in V(G_n)$. Since G_n is antiregular, f is an antiregular total dominating function of G_n. ■

There are also non-antiregular graphs having an antiregular total dominating function. For example, the three graphs of Figure 7.3 are not antiregular, but each has an antiregular total dominating function, as shown in Figure 7.3.

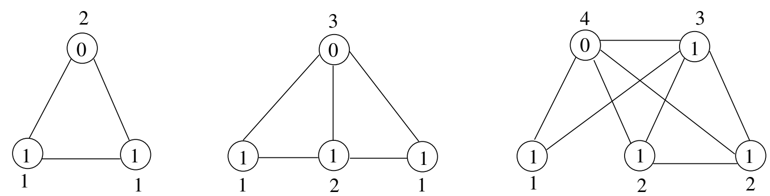

Figure 7.3: Non-antiregular graphs with an antiregular total dominating function

These examples give rise to the following question.

Which non-antiregular graphs of order 3 *or more have an antiregular total dominating function?*

Each of the three graphs of order $n \in \{3, 4, 5\}$ in Figure 7.3 has maximum degree $n - 1$ and minimum degree 2. As we show next, this is true for all non-antiregular graphs of order 3 or more that have an antiregular total dominating function.

Proposition 7.2.3 *If a non-antiregular graph G of order $n \geq 3$ has an antiregular total dominating function, then $\Delta(G) = n - 1$ and $\delta(G) = 2$.*

Proof. Let G be a non-antiregular graph of order $n \geq 3$ and let $h : V(G) \to \{0, 1\}$ be an antiregular total dominating function of G. Then $\Sigma_h = \{\sigma_h(x) : x \in V(G)\} = [n - 1]$. Assume, to the contrary, that either $\Delta(G) \leq n - 2$ or $\delta(G) \neq 2$. First, suppose that $\Delta(G) \leq n - 2$. Since $1 \leq \sigma_h(v) \leq \Delta(G) \leq n - 2$ for every vertex v of G, it follows that $\Sigma_h \subseteq [n - 2]$, which is a contradiction. Consequently, $\Delta(G) = n - 1$.

Next, suppose that $\delta(G) \neq 2$. We consider two cases, according to whether $\delta(G) = 1$ or $\delta(G) \geq 3$.

Case 1. $\delta(G) = 1$. Let u be an end-vertex of G and let $v \in V(G)$ such that $\sigma_h(v) = n-1$. Then $\deg v = n-1$ and $h(x) = 1$ for each $x \in N(v) = V(G) - \{v\}$. On the other hand, the end-vertex u is adjacent only to v and so $h(v) = 1$. Therefore, $h(x) = 1$ for every vertex x of G and so $\sigma_h(x) = \deg x$ for every vertex x of G. However then, $\mathcal{D}(G) = \{\deg x : x \in V(G)\} = \Sigma_h = [n-1]$. This implies that G is an antiregular graph, which is a contradiction.

Case 2. $\delta(G) \geq 3$. Again, let $v \in V(G)$ such that $\sigma_h(v) = n-1$. Then $\deg v = n-1$ and $h(x) = 1$ for each $x \in N(v) = V(G) - \{v\}$. Since $\deg x \geq 3$ for each vertex x of G and $h(x) \in \{0,1\}$, it follows that $\sigma_h(x) \geq 2$ for every vertex x of G. Consequently, $\Sigma_h \subseteq \{2, 3, \ldots, n-1\}$, which is a contradiction.

Therefore, $\Delta(G) = n-1$ and $\delta(G) = 2$, as claimed. ■

By Proposition 7.2.3, if G is a non-antiregular graph of order $n \geq 3$ such that $\Delta(G) \leq n-2$ or $\delta(G) \neq 2$, then G does not have an antiregular total dominating function. There are other degree conditions that prevent a graph from having an antiregular total dominating function.

Proposition 7.2.4 *Let G be a connected graph. If for some $k \in \mathcal{D}(G)$, G has at least $k+2$ vertices of degree k or if there exist two distinct integers $k_1, k_2 \in \mathcal{D}(G)$ such that G has at least $k_1 + 1$ vertices of degree k_1 and at least $k_2 + 1$ vertices of degree k_2, then G has no antiregular total dominating function.*

Proof. First, suppose that G is a connected graph that has at least $k+2$ vertices of degree k for some $k \in \mathcal{D}(G)$. Let $v \in V(G)$ be a vertex of degree k. Then $\sigma_h(v) \in [k]$. Since G has at least $k+2$ vertices of degree k, it follows that either there exist $u, v, w \in V(G)$ with $\deg u = \deg v = \deg w = k$ and $\sigma_h(u) = \sigma_h(v) = \sigma_h(w)$ or there exist $u, v, w, x \in V(G)$ with $\deg u = \deg v = \deg w = \deg x = k$ and $\sigma_h(u) = \sigma_h(v)$ and $\sigma_h(w) = \sigma_h(x)$. So, G does not have an antiregular total dominating function.

Next, assume that there exist two distinct integers $k_1, k_2 \in \mathcal{D}(G)$ such that G has at least $k_1 + 1$ vertices of degree k_1 and at least $k_2 + 1$ vertices of degree k_2. Let $u \in V(G)$ be a vertex of degree k_1 and $v \in V(G)$ be a vertex of degree k_2 so $\sigma_h(u) \in [k_1]$ and $\sigma_h(v) \in [k_2]$. Since G has at least $k_1 + 1$ vertices of degree k_1 and at least $k_2 + 1$ vertices of degree k_2, there exist $w, x, y, z \in V(G)$ with $\deg w = \deg x = k_1$, $\deg y = \deg z = k_2$ and $\sigma_h(w) = \sigma_h(x)$ and $\sigma_h(y) = \sigma_h(z)$. So, G does not have an antiregular total dominating function. ■

In fact, for every integer $n \geq 3$, there are exactly two graphs of order n possessing an antiregular total dominating function, one of which is the connected antiregular graph G_n of order n. In order to state the next result, we first recall the *join* $G = F \vee H$ of two vertex-disjoint graphs F and H has vertex set $V(G) = V(F) \cup V(H)$ and edge set $E(G) = E(F) \cup E(H) \cup \{uv : u \in V(F), v \in V(H)\}$. In particular, the graph $F \vee K_1$ is constructed by adding a new vertex and joining this vertex to every vertex of F.

Theorem 7.2.5 *For each integer $n \geq 3$, there are exactly two graphs of order n with an antiregular total dominating function, one of which is G_n and the other is $G_{n-1} \vee K_1$.*

Proof. It is readily seen that $G_3 = P_3$ and $K_3 = G_2 \vee K_1 = P_2 \vee K_1$ (shown in Figure 7.3) are the only graphs of order 3 that have an antiregular total dominating function. Let H be a graph of order $n \geq 4$ having an antiregular total dominating function h. Then $\Sigma_h = \{\sigma_h(v) : v \in V(H)\} = [n-1]$. Let $u \in V(H)$ such that $\sigma_h(u) = n-1$. Then $\deg u = n-1$ and $h(x) = 1$ for each $x \in N(u) = V(H) - \{u\}$. Since $h(u) \in \{0, 1\}$, we consider these two cases.

Case 1. $h(u) = 1$. Therefore, $h(x) = 1$ for every vertex x of H and so $\sigma_h(x) = \deg x$ for every vertex x of H. Since $\mathcal{D}(H) = \{\deg x : x \in V(H)\} = \Sigma_h = [n-1]$, it follows that $H = G_n$ is the connected antiregular graph of order n.

Case 2. $h(u) = 0$. Therefore, u is the only vertex of H whose h-value is 0. This implies that $1 \leq \sigma_h(x) = \deg_H x - 1 \leq n-2$ for every vertex x of $V(H) - \{u\}$. Thus, u is the only vertex of H whose σ_h-value is $n-1$ and so

$$\{\sigma_h(x) : x \in V(H) - \{u\}\} = \{\deg_H x - 1 : x \in V(H) - \{u\}\} = [n-2].$$

Then $H' = H - u$ is a connected subgraph of H and $\deg_{H'} x = \deg_H x - 1$ for each vertex x of H'. Since the order of H' is $n-1$ and $\mathcal{D}(H') = \{\deg_{H'} x : x \in V(H')\} = [n-2]$, it follows that H' is the connected antiregular graph G_{n-1} of order $n-1$. Therefore, $H = H' \vee K_1 = G_{n-1} \vee K_1$.

Therefore, the two graphs of order $n \geq 3$ with an antiregular total dominating function are G_n and $G_{n-1} \vee K_1$. ■

If $n = 4$, then G_4 and $G_3 \vee K_1 = P_3 \vee K_1$ (which is also shown in Figure 7.3) are the only connected graphs of order 4 that have an antiregular total dominating function. If $n = 5$, then G_5 and $G_4 \vee K_1$ (which is also shown in Figure 7.3) are the only connected graphs of order 5 that have an antiregular total dominating function. These four graphs are shown in Figure 7.4 together with an antiregular total dominating function for each of them.

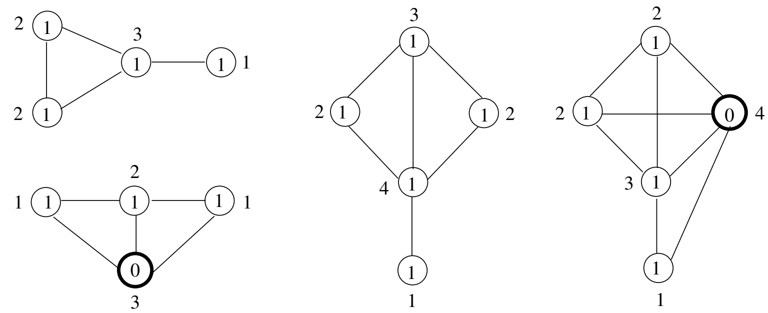

Figure 7.4: The graphs of order 4 or 5 with an antiregular total dominating function

By Theorem 7.2.5, for each integer $n \geq 3$, there is a graph G of order n having a total dominating function h such that $\{\sigma_h(v) : v \in V(G)\} = [n-1]$. This gives rise to the following question:

> *For a given nonempty set S of positive integers, does there exist a graph G with a total dominating function h such that $\{\sigma_h(v) : v \in V(G)\} = S$?*

It turns out that this question has an affirmative answer. In 1977, Kapoor, Polimeni, and Wall [84] proved that every set of positive integers is the degree set of some graph.

Theorem 7.2.6 *For every set $S = \{a_1, a_2, \ldots, a_k\}$ of positive integers with*

$$a_1 < a_2 < \cdots < a_k,$$

there exists a graph G with $\mathcal{D}(G) = S$. Furthermore, the minimum order $\mu(S)$ of such a graph G is $\mu(S) = a_k + 1$.

As an immediate consequence of Theorem 7.2.6, every nonempty set of positive integers is realizable as the set of σ_h-values of a total dominating function h of some graph.

Corollary 7.2.7 *For every set $S = \{a_1, a_2, \ldots, a_k\}$ of positive integers with*

$$a_1 < a_2 < \cdots < a_k,$$

there exists a graph G of order $a_k + 1$ having a total dominating function h such that

$$\{\sigma_h(v) : v \in V(G)\} = S.$$

Proof. Let $S = \{a_1, a_2, \ldots, a_k\}$ be any set of positive integers with $a_1 < a_2 < \cdots < a_k$. By Theorem 7.2.6, there exists a graph G of order $a_k + 1$ such that $\mathcal{D}(G) = S$. Let $h : V(G) \to \{0, 1\}$ be defined by $h(v) = 1$ for every $v \in V(G)$. Since $\sigma_h(v) = \deg(v) \geq 1$ for every $v \in V(G)$, it follows that h is a total dominating function of G and $\{\sigma_h(v) : v \in V(G)\} = \mathcal{D}(G) = S$. ■

For each pair k, n of integers with $1 \leq k \leq n-1$, let $I_{k,n} = \{n-k, n-k+1, \ldots, n-1\}$ be the set of k consecutive positive integers. By Theorem 7.2.6, there exists a graph G of order n whose degree set is $I_{k,n}$. Consequently, there exists a graph G of order n having a total dominating function h such that $\{\sigma_h(v) : v \in V(G)\} = I_{k,n}$. This observation gives rise to the fact that every interval of positive integers is realizable as the set of σ_h-values of a total dominating function h of some graph.

Proposition 7.2.8 *For each interval $\mathcal{I}$ of positive integers, there exists a graph G having a total dominating function h such that $\{\sigma_h(v) : v \in V(G)\} = \mathcal{I}$.*

7.3 Regular Total Dominating Functions

Let G be a nontrivial connected graph. If h is a total dominating function of G such that $\sigma_h(v) = k \in \mathbb{N}$ for every vertex v of G, then h is a *k-regular total dominating function* of G. As is the case with regular dominating functions, not every graph with no isolated vertices has a regular total dominating function.

For example, the graph G of Figure 7.5 has no k-regular total dominating function for any positive integer k. To see this, assume, to the contrary, that G has a k-regular total dominating function $h : V(G) \to \{0, 1\}$ for some positive integer k. Since u is only dominated by v and x is only dominated by w, it follows that $h(v) = h(w) = 1$. Since $\sigma_h(u) = \sigma_h(x) = 1$, it follows that $k = 1$. On the other hand, $\sigma_h(y) = h(v) + h(w) = 2$, which is impossible.

The main question here is determining which graphs (especially those belonging to well-known classes of graphs) have a k-regular total dominating function for

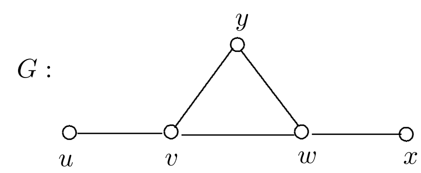

Figure 7.5: A graph G with no regular total dominating function

some $k \in \mathbb{N}$ and which do not. We begin by presenting some preliminary results dealing with regular total dominating functions of graphs.

Observation 7.3.1 *Let G be a nontrivial connected graph.*

(1) *If G is an r-regular graph for some positive integer r, then the function h defined by $h(x) = 1$ for each vertex x of G is an r-regular total dominating function of G.*

(2) *If G has a k-regular total dominating function, then $1 \leq k \leq \delta(G)$.*

(3) *Let H be a graph with a regular total dominating function h and let v be a vertex of H with $h(v) = 0$. If G is the graph obtained by replacing v by an arbitrary graph F where each vertex of F is joined to the neighbors of v, then h can be extended to a regular total dominating function of G by defining $h(x) = 0$ for each $x \in V(F)$.*

Proposition 7.3.2 *For a positive integer k, a graph G has a k-regular total dominating function if and only if either G is k-regular or G consists of two induced vertex-disjoint subgraphs F and H such that F is k-regular and each vertex of H is adjacent to exactly k vertices in F.*

Proof. First, suppose that G has a k-regular total dominating function $h : V(G) \to \{0, 1\}$ for some positive integer k. To simplify the notation, we let

$$\mathcal{I}_h = \{v \in V(G) : h(v) = 1\} \text{ and } \overline{\mathcal{I}_h} = V(G) - \mathcal{I}_h = \{v \in V(G) : h(v) = 0\}.$$

Next, let $F = G[\mathcal{I}_h]$ and $H = G[\overline{\mathcal{I}_h}]$ (if $\overline{\mathcal{I}_h} \neq \emptyset$). Since $\sigma_h(v) = k$ for each $v \in V(G)$, it follows that every vertex in F is adjacent to exactly k vertices in F and so F is k-regular and every vertex in H is adjacent to exactly k vertices in F.

Next, we verify the converse. Since the statement is true if G is k-regular, we may assume that G is constructed from two induced vertex-disjoint subgraphs F

and H such that F is k-regular and each vertex of H is adjacent to exactly k vertices in F. Then a k-regular total dominating function of G can be defined by assigning 0 to each vertex of H and assigning 1 to each vertex of F. ■

- ⋆ If $k = 1$ in Proposition 7.3.2, then a nonempty graph G has a 1-regular total dominating function if and only if F is 1-regular (or $E(F)$ is a matching in G) and each vertex in H is adjacent to exactly one vertex of F.
- ⋆ If $k = 2$ in Proposition 7.3.2, then G has a 2-regular dominating function if and only if F is 2-regular (or the subgraph induced by $E(F)$ is a union of cycles) and each vertex in H is adjacent to exactly two vertices of F.

The following is a consequence of Proposition 7.1.1.

Proposition 7.3.3 *Let G be a nontrivial connected graph of order n, let $h : V(G) \to \{0, 1\}$ be a total dominating function of G, and suppose that $|\mathcal{I}_h(G)| = s \geq 2$.*

(1) *If G is an r-regular graph for some integer $r \geq 2$, then*

$$\textstyle\sum_{v \in V(G)} \sigma_h(v) = rs.$$

(2) *If h is a k-regular total dominating function for some integer $k \geq 1$, then*

$$\textstyle\sum_{v \in V(G)} \sigma_h(v) = nk.$$

Proof. By Proposition 7.1.1, it follows that if G is an r-regular graph for some integer $r \geq 2$, then

$$\sum_{v \in V(G)} \sigma_h(v) = \sum_{x \in \mathcal{I}_h(G)} \deg x = \sum_{x \in \mathcal{I}_h(G)} r = rs$$

and so (1) holds. Also, by Proposition 7.1.1, if h is a k-regular total dominating function of a graph G for some integer $k \geq 1$, then

$$\sum_{v \in V(G)} \sigma_h(v) = \sum_{v \in V(G)} k = nk$$

and so (2) holds. ■

Observation 7.3.4 *Let G be a connected graph having a* 1*-regular total dominating function h and let u be an end-vertex of G.*

- ⋆ *If v is a vertex of G that is adjacent to u, then $h(v) = 1$.*
- ⋆ *If w is a vertex of G such that $d(u, w) = 3$, then $h(w) = 0$.*

Proposition 7.3.5 *If a connected graph G contains two end-vertices x and y such that $d(x, y) = 4$, then G does not have any* 1*-regular total dominating function.*

Proof. Let (x, u, v, w, y) be an $x - y$ path in G. By Observation 7.3.4, any 1-regular total dominating function h must assign 1 to u and w. However then, $\sigma_h(v) \geq 2$, which is impossible. ∎

7.4 Regular Total Dominating Functions in Trees

By Observation 7.3.1, if a connected graph G has a k-regular total dominating function, then $1 \leq k \leq \delta(G)$. Hence, if T is a nontrivial tree, then the only possible regular total dominating function for T is a 1-regular total dominating function. All trees of diameter 3 or less have a 1-regular total dominating function.

Proposition 7.4.1 *Every star and double star has a* 1*-regular total dominating function.*

Proof. Let T be a star of order 2 or more. Suppose that v is the central vertex of T and w is an end-vertex of T. Then the function $h : V(T) \to \{0, 1\}$ defined by $h(v) = h(w) = 1$ and $h(x) = 0$ for each $x \in V(T) - \{v, w\}$ is a 1-regular total dominating function. If T is a double star with central vertices u and v, then the function $h : V(T) \to \{0, 1\}$ defined by $h(u) = h(v) = 1$ and $h(x) = 0$ for each $x \in V(T) - \{u, v\}$ is a 1-regular total dominating function. ∎

It is known that every nontrivial path has a 1-regular dominating function. This is not true for 1-regular total dominating function.

Proposition 7.4.2 *A path P_n of order $n \geq 2$ has a* 1*-regular total dominating function if and only if $n \not\equiv 1 \pmod 4$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. First, suppose that $n \not\equiv 1 \pmod 4$. For $n \equiv 0 \pmod 4$, define $h : V(P_n) \to \{0, 1\}$ by

$$h(v_i) = \begin{cases} 0 & \text{if } i \equiv 0, 1 \pmod 4 \\ 1 & \text{if } i \equiv 2, 3 \pmod 4. \end{cases}$$

For $n \equiv 2, 3 \pmod 4$, define $h : V(P_n) \to \{0, 1\}$ by

$$h(v_i) = \begin{cases} 0 & \text{if } i \equiv 0, 3 \pmod 4 \\ 1 & \text{if } i \equiv 1, 2 \pmod 4. \end{cases}$$

In each case, h is a 1-regular total dominating function of P_n.

For the converse, assume, to the contrary, that there is an integer $n \geq 5$ with $n \equiv 1 \pmod 4$ such that P_n has a 1-regular total dominating function $h : V(P_n) \to \{0, 1\}$. Let $\mathcal{I}_h = \{v \in V(G) :\ h(v) = 1\}$. Since n is odd and

$$\sum_{v \in V(G)} \sigma_h(v) = \sum_{x \in \mathcal{I}_h} \deg x = n$$

by Proposition 7.1.1 and Proposition 7.3.3, it follows that exactly one of v_1 and v_n belongs to $\mathcal{I}_h$, say $v_1 \in \mathcal{I}_h$ and $v_n \notin \mathcal{I}_h$. Since $v_1 \in \mathcal{I}_h$, this forces

$$h(v_i) = \begin{cases} 1 & \text{if } i \equiv 1, 2 \pmod 4 \\ 0 & \text{if } i \equiv 3, 0 \pmod 4 \end{cases}$$

and so $v_n \in \mathcal{I}_h$, which is a contradiction. ■

The following is a consequence of Proposition 7.3.5.

Proposition 7.4.3 *If a tree T contains two end-vertices x and y such that $d(x, y) = 4$, then T does not have any 1-regular total dominating function. In particular, if $\operatorname{diam}(T) = 4$, then T does not have any 1-regular total dominating function.*

All nontrivial trees of order 6 or less are shown in Figure 7.6. By Propositions 7.4.1, 7.4.2 and 7.4.3, the three trees of diameter 4 in Figure 7.6 are the only nontrivial trees of order 6 or less that have no 1-regular total dominating functions.

The following is a consequence of Proposition 7.4.2.

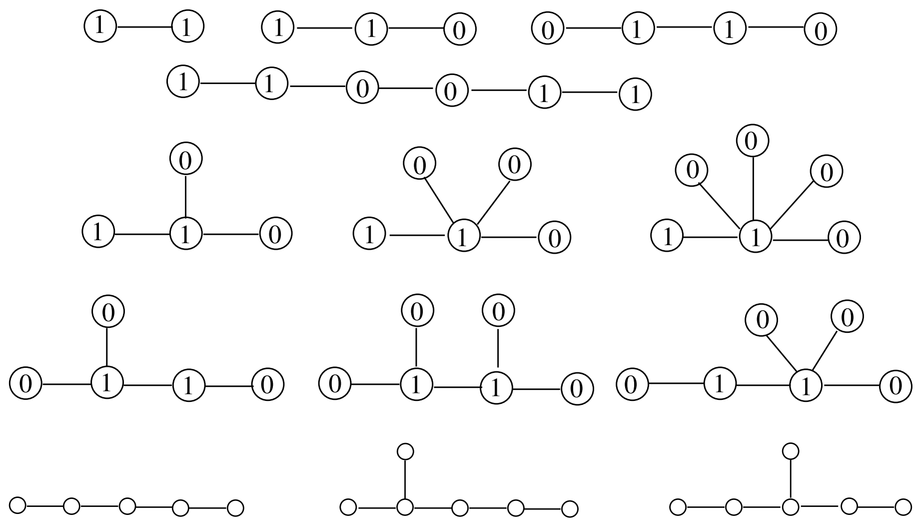

Figure 7.6: Nontrivial trees of order 6 or less

Corollary 7.4.4 *If G is a path with* $\mathrm{diam}(G) \equiv 0 \pmod 4$*, then G does not have a* 1*-regular total dominating function.*

Corollary 7.4.4 gives rise to the following question:

> *If T is a tree with diameter $d \geq 1$ where $d \not\equiv 0 \pmod 4$, does T have a* 1*-regular total dominating function?*

The answer depends on the structure of the tree in question, as we see later in this section. However, every positive integer d distinct from 4 can be realized as the diameter of a tree T having a 1-regular total dominating function.

Proposition 7.4.5 *For every positive integer d distinct from* 4*, there exists a tree of diameter d having a* 1*-regular total dominating function.*

Proof. If d is a positive integer such that $d \not\equiv 0 \pmod 4$, then the path P_{d+1} of diameter d has a 1-regular total dominating function by Proposition 7.4.2. Thus, we may assume that $d \equiv 0 \pmod 4$ and $d \geq 8$. Then $d = 4q$ for some integer $q \geq 2$. We show that there is a tree T_q of diameter $4q$ that has a 1-regular total dominating function h_q. Figure 7.7 shows the tree T_2 of diameter 8 having a 1-regular total dominating function h_2. Thus, the statement is true for $q = 2$.

For an integer $q \geq 3$, let T_q be the tree obtained from the tree T_2 of Figure 7.7 and the path $P_{4q-8} = (v_1, v_2, \ldots, v_{4q-8})$ of order $4q - 8$ by adding the edge vv_1

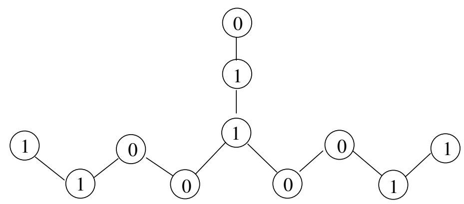

Figure 7.7: A tree of diameter 8 having a 1-regular total dominating function

where v is an end-vertex with $h_2(v) = 1$. Then $\operatorname{diam}(T_q) = 8 + (4q - 8) = 4q$. The function $h_q : V(T_q) \to \{0, 1\}$ of T_q defined by $h_q(x) = h_2(x)$ if $x \in V(T_2)$ and

$$h_q(v_i) = \begin{cases} 0 & \text{if } i \equiv 1, 2 \pmod 4 \\ 1 & \text{if } i \equiv 0, 3 \pmod 4 \end{cases}$$

is a 1-regular total dominating function of T_q. This is illustrated in Figure 7.8 for the tree T_3 of diameter 12 together with the 1-regular total dominating function h_3. ■

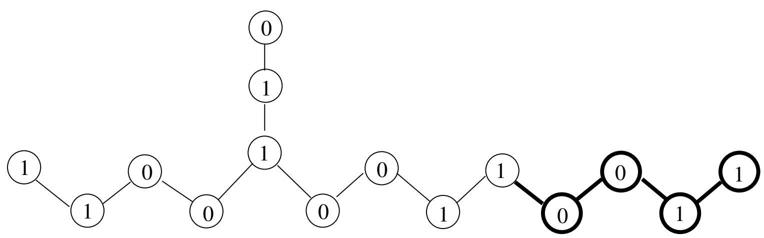

Figure 7.8: A tree of diameter 12 having a 1-regular total dominating function

In fact, if T is a tree with a 1-regular total dominating function h and v is a vertex of T with $h(v) = 1$, then the tree T' obtained from T by adding a new vertex u and joining u to the vertex v also has a 1-regular total dominating function. For example, a 1-regular total dominating function h' of T' can be defined by $h'(x) = h(x)$ if $x \in V(T)$ and $h'(u) = 0$. Consequently, for every positive integer $d \geq 2$ distinct from 4, there exist infinitely many trees of diameter d having a 1-regular total dominating function.

Next, we characterize those trees having a 1-regular total dominating function. To do this, we first introduce additional terminology. A tree T is called *totally 1-sequential* if the tree T with a 1-regular total dominating function $h : V(T) \to \{0, 1\}$ can be constructed by means of the following algorithm.

Algorithm 7.4.6 *Constructing a totally* 1*-sequential tree* T.

1. We begin with $T_0 = K_2$ whose two vertices are labeled 1.

2. Once a tree T_j, $j \geq 0$, has been constructed with a 1-regular total dominating function h, a tree T_{j+1} with an extended 1-regular total dominating function h is constructed by performing one of the following steps:

   (2.1) A vertex v is added to T_j where v is joined to a vertex labeled 1 in T_j and $h(v)$ is defined to be 0.

   (2.2) A path (x, y, z) of order 3 is added to T_j where x is joined to a vertex labeled 0 in T_j, $h(x)$ is defined to be 0, and $h(y)$ and $h(z)$ are defined to be 1.

3. Either repeat Step 2 or stop, resulting in a tree $T = T_k$ for some nonnegative integer k.

Once Algorithm 7.4.6 stops, a sequence of labeled trees $T_0, T_1, T_2, \ldots, T_k$ is constructed, resulting in $T_k \cong T$. Such a tree T is therefore *totally* 1*-sequential*. Our interest in totally 1-sequential trees is due to the following result.

Theorem 7.4.7 *A tree* T *has a* 1*-regular total dominating function if and only if* T *is totally* 1*-sequential.*

Proof. First, if T is a totally 1-sequential tree, then from the manner in which T is constructed, we see that the defined function h is a 1-regular total dominating function of T. For the converse, assume, to the contrary, that there are trees possessing a 1-regular total dominating function that are not totally 1-sequential. Among all such trees, let T be one of minimum order n. Since it is clear that all trees of order 6 or less that possess a 1-regular total dominating function are totally 1-sequential, it follows that $n \geq 7$. Let h be a 1-regular total dominating function of T.

First, suppose that T contains an end-vertex u such that $h(u) = 0$. Suppose that u is adjacent to the vertex v in T. Necessarily, $h(v) = 1$. The restriction of the 1-regular dominating function h to the tree $T - u$ of order $n - 1$ is a 1-regular total dominating function of $T - u$. Thus, $T - u$ is totally 1-sequential. Adding the vertex u to $T - u$, joining u to v, and defining $h(u) = 0$ shows that T is totally

1-sequential, producing a contradiction. Hence, we may assume that $h(w) = 1$ for every end-vertex w of T.

Let z be a peripheral vertex of T. Since z is an end-vertex of T, it follows that $h(z) = 1$. Let y be the neighbor of z in T. Thus, $h(y) = 1$. If y is adjacent to an end-vertex v distinct from z, then $h(v) = 0$, contrary to our assumption. Thus, z is the only end-vertex adjacent to y. Since z is a peripheral vertex of T, it follows that $\deg_T y = 2$. Let (x, y, z) be a path in T. Then $h(x) = 0$. Since $\sigma_h(x) = 1$ and $h(y) = 1$, it follows that every neighbor of x distinct from y must be labeled 0 by h. We claim that $\deg_T x = 2$, for suppose that $\deg_T x \geq 3$. Let u and w be two neighbors of x distinct from y. Thus, $h(u) = h(w) = 0$. Since $\sigma_h(u) = \sigma_h(w) = 1$, it follows that each of u and w has a neighbor, u' and w', respectively, such that $h(u') = h(w') = 1$. Since $\sigma_h(u') = \sigma_h(w') = 1$, it follows that each of u' and w' has a neighbor u'' and w'', respectively, such that $h(u'') = h(w'') = 1$. This, however, contradicts the fact that z is a peripheral vertex of T. Therefore, as claimed, $\deg_T x = 2$. Let u be the neighbor of x distinct from y. We saw that $h(u) = 0$. Then $T' = T - \{x, y, z\}$ is a tree of order $n - 3$ and the restriction of the 1-regular total dominating function h of T to T' is a 1-regular total dominating function of T'. Hence, T' is totally 1-sequential. Adding the path (x, y, z) to T', joining x to u, and defining $h(x) = 0$ and $h(y) = h(z) = 1$ shows that T is totally 1-sequential, producing a contradiction. ■

Every star has a similar 1-regular total dominating function and every double star has a unique 1-regular total dominating function. If a path has a 1-regular total dominating function, then this 1-regular total dominating function is unique. However, there are trees having distinct 1-regular total dominating functions. Figure 7.9 shows the (only) tree of order 8 that has two distinct 1-regular total dominating functions.

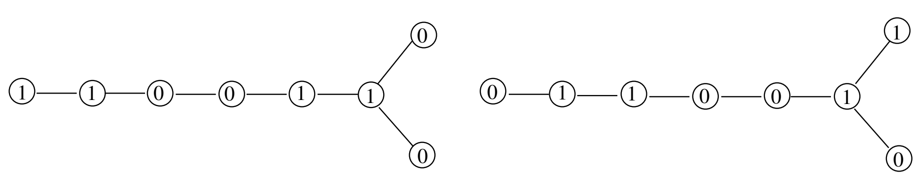

Figure 7.9: The tree of order 8 with distinct 1-regular total dominating functions

By Algorithm 7.4.6, we can add vertices v to the tree T of Figure 7.9 with a 1-regular total dominating function h (where v is joined to a vertex labeled 1 and

$h(v)$ is defined to be 0) to produce trees of order $n \geq 9$ with distinct 1-regular total dominating functions. Also, by Algorithm 7.4.6, we add paths (x, y, z) the tree T of Figure 7.9 with a 1-regular total dominating function h (where x is joined to a vertex labeled 0 in T, $h(x)$ is defined to be 0, and $h(y)$ and $h(z)$ are defined to be 1) to produce trees of order $n \geq 9$ with distinct 1-regular total dominating functions. For example, two trees of order 11 are shown in Figure 7.10 each of which has distinct 1-regular total dominating functions.

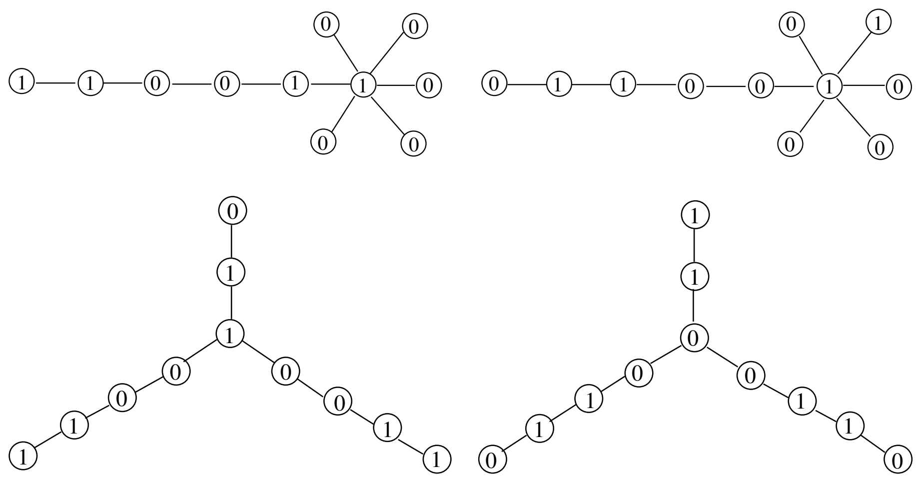

Figure 7.10: Two trees of order 11 two distinct 1-regular total dominating functions

Proposition 7.4.8 *For each integer $n \geq 9$, there are two trees of order n having distinct 1-regular total dominating functions.*

7.5 Regular Total Dominating Functions in Regular Graphs

The following is an immediate consequence of Proposition 7.1.2.

Corollary 7.5.1 *Let G be a nontrivial connected graph. Suppose that h is a total dominating function of G such that $\overline{h}$ is also a total dominating function of G. Then h and $\overline{h}$ are both regular if and only if G is regular. Furthermore, if G is an r-regular graph and h is a k-regular total dominating function of G where $1 \leq k \leq r$, then $\overline{h}$ is an $(r-k)$-regular total dominating function of G.*

By Corollary 7.5.1, we have the following corollary.

Corollary 7.5.2 *For a nontrivial connected r-regular graph G for some integer $r \geq 2$, a function $h : V(G) \to \{0,1\}$ is a 1-regular total dominating function of G if and only if $\overline{h} : V(G) \to \{0,1\}$ is an $(r-1)$-regular total dominating function of G.*

For each integer $n \geq 3$, every cycle C_n of order n is 2-regular and thus has a 2-regular total dominating function. Next, we determine those n-cycles having a 1-regular total dominating function.

Proposition 7.5.3 *For an integer $n \geq 3$, the n-cycle C_n has a 1-regular total dominating function if and only if $n \equiv 0 \pmod 4$.*

Proof. First, suppose that C_n has a 1-regular total dominating function h. Let $\mathcal{I}_h = \{v \in V(G) : \ h(v) = 1\}$. Since the edge set of $G[\mathcal{I}_h]$ is a matching, it follows that $|\mathcal{I}_h| = s$ is even. Since C_n is a 2-regular graph of order n and s is even, it follows by Observation 7.3.3 that

$$n = \sum_{v \in V(G)} \sigma_h(v) = 2s \equiv 0 \pmod 4.$$

For the converse, suppose that $n \equiv 0 \pmod 4$ and let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. A 1-regular total dominating function $f : V(C_n) \to \{0,1\}$ of G can be defined by $f(v_i) = 1$ if $i \equiv 0, 1 \pmod 4$ and $f(v_i) = 0$ if $i \equiv 2, 3 \pmod 4$. ■

It is known that for each integer $k \in \{1,2\}$, the cycle C_n of order $n \geq 3$ has a k-regular dominating function if and only if $n \equiv 0 \pmod 3$. In particular, C_n has a 2-regular dominating function if and only if $n \equiv 0 \pmod 3$. Thus, C_6 has a 2-regular dominating function. On the other hand, C_6 does not have a 1-regular total dominating function by Proposition 7.5.3. Hence, a 2-regular dominating function of C_6 is not a 1-regular total dominating function of C_6. For example, the 2-regular dominating function f of C_6 of Figure 7.11 is not a 1-regular total dominating function of C_6. In fact, more can be said.

First, we make an observation.

Observation 7.5.4 *Let G be a nontrivial connected graph and let $f : V(G) \to \{0,1\}$ be a dominating function of G. Since $c_f(v) = \sum_{u \in N[v]} f(u)$ and $\sigma_f(v) = \sum_{u \in N(v)} f(u)$, it follows that $c_f(v) = f(v) + \sigma_f(v)$. Consequently, if $f(v) = 0$, then $c_f(v) = \sigma_f(v)$, while if $f(v) = 0$, then $c_f(v) = \sigma_f(v) + 1$.*

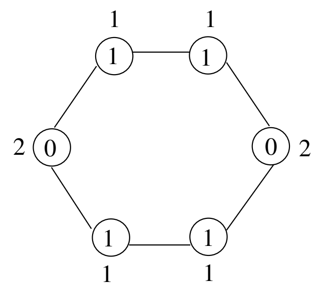

Figure 7.11: A 2-regular dominating function of C_6 that is not a 1-regular total dominating function of C_6

The following is an immediate consequence of Observation 7.5.4.

Proposition 7.5.5 *Let G be a nontrivial connected graph and let $k \geq 2$ be an integer. If $f : V(G) \to \{0, 1\}$ is a k-regular dominating function of G such that $f(v) = 0$ for at least one vertex v of G, then f is not a $(k-1)$-regular total dominating function of G.*

Proof. Let v be a vertex of G such that $f(v) = 0$. Since $c_f(v) = k$, it follows that v is adjacent to k vertices whose f-value are 1. However then, $\sigma_f(v) = k$ and so f is not a $(k-1)$-regular total dominating function of G. ■

While every connected cubic graph has a 3-regular total dominating function, there is no guarantee that such a cubic graph has a k-regular total dominating function for $k \in \{1, 2\}$. We first present a necessary condition on the order of a cubic graph to possess a k-regular total dominating function for $k \in \{1, 2\}$.

Lemma 7.5.6 *If G is a connected cubic graph of order n having a k-regular total dominating function where $k \in \{1, 2\}$, then $n \equiv 0 \pmod 6$.*

Proof. Suppose that G has a k-regular total dominating function h, where $k \in \{1, 2\}$. Let $|\{v \in V(G) : h(v) = 1\}| = s$. Since G is a 3-regular graph of order n, it follows by Observation 7.3.3 that

$$nk = \sum_{v \in V(G)} \sigma_h(v) = 3s.$$

If $k = 1$, then $n = 3s$, while if $k = 2$, then $2n = 3s$. In either case, $n \equiv 0 \pmod 3$. Since n is even, it follows that $n \equiv 0 \pmod 6$. ■

As a consequence of Lemma 7.5.6, no cubic graph of order 10 has a 1-regular total dominating function or 2-regular total dominating function. In particular, the Petersen graph has neither a 1-regular total dominating function nor a 2-regular total dominating function.

Corollary 7.5.7 *The Petersen graph has a k-regular total dominating function if and only if $k = 3$.*

One of the best known classes of cubic graphs is that of the prisms $C_n \,\Box\, K_2$.

Proposition 7.5.8 *Let $n \geq 3$ and $k \in \{1, 2\}$. Then $C_n \,\Box\, K_2$ has a k-regular total dominating function if and only if $n \equiv 0 \pmod 3$.*

Proof. Let $G = C_n \,\Box\, K_2$, where $n \geq 3$. First, suppose that G has a k-regular total dominating function h, where $k \in \{1, 2\}$. Since G is a 3-regular graph of order $2n$, it follows by Lemma 7.5.6 that $2n \equiv 0 \pmod 3$ and so $n \equiv 0 \pmod 3$.

For the converse, suppose that $n \equiv 0 \pmod 3$. Let G be constructed from two copies $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ of the n-cycle by adding the edges $u_i v_{i+1}$ for $1 \leq i \leq n$. The function $h_1 : V(G) \to \{0, 1\}$ of G defined by

$$h_1(w) = \begin{cases} 1 & \text{if } w = u_i \text{ or } w = v_i \text{ where } i \equiv 1 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

is a 1-regular total dominating function of G and $\overline{h}_1$ is a 2-regular total dominating function of G by Corollary 7.5.1. This is illustrated in Figure 7.12 for $C_6 \,\Box\, K_2$. ■

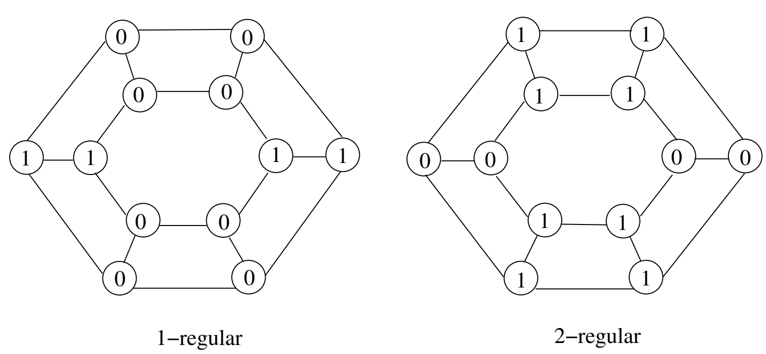

Figure 7.12: Regular total dominating functions of $C_6 \,\Box\, K_2$

A class of graphs related to the prisms $C_n \,\Box\, K_2$ are the graphs $P_n \,\Box\, K_2$, sometimes referred to as ladders.

Proposition 7.5.9 *For each positive integer n, the ladder graph $P_n \,\Box\, K_2$ has a 1-regular total dominating function.*

Proof. Let $G = P_n \,\Box\, K_2$ be constructed from the two copies $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ of the path P_n of order n by adding the edges u_iv_i for $1 \leq i \leq n$.

- ⋆ If $n \equiv 1 \pmod 3$, then the function $h : V(G) \to \{0, 1\}$ defined by

$$h(w) = \begin{cases} 1 & \text{if } w = u_i \text{ or } w = v_i \text{ where } i \equiv 1 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

  is a 1-regular total dominating function of G.

- ⋆ If $n \not\equiv 1 \pmod 3$, then the function $h : V(G) \to \{0, 1\}$ defined by

$$h(w) = \begin{cases} 1 & \text{if } w = u_i \text{ or } w = v_i \text{ where } i \equiv 2 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

  is a 1-regular total dominating function of G.

The 1-regular total dominating function h, as defined above, is illustrated in Figure 7.13 for $P_n \,\Box\, K_2$ where $n \in \{5, 6, 7\}$. ■

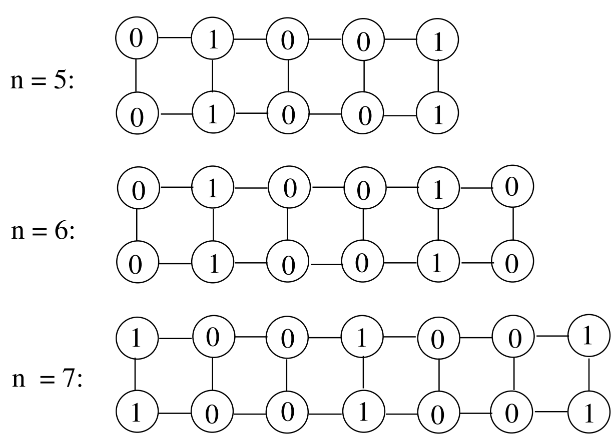

Figure 7.13: The 1-regular total dominating function of $P_n \,\Box\, K_2$ defined in the proof of Proposition 7.5.9 for $n = 5, 6, 7$

Proposition 7.5.10 *For a positive integer n, the ladder graph $P_n \square K_2$ has a 2-regular total dominating function if and only if $n \equiv 2 \pmod 3$.*

Proof. Let $G = P_n \square K_2$ be constructed from the two copies $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ of the path P_n of order n by adding the edges u_iv_i for $1 \leq i \leq n$. If $n \equiv 2 \pmod 3$, then the function $h : V(G) \to \{0, 1\}$ defined by

$$h(w) = \begin{cases} 0 & \text{if } w = u_i \text{ or } w = v_i \text{ where } i \equiv 0 \pmod 3 \\ 1 & \text{otherwise} \end{cases}$$

is a 2-regular total dominating function of G. This 2-regular total dominating function h is illustrated in Figure 7.14 for $P_n \square K_2$ where $n = 5$ and $n = 8$.

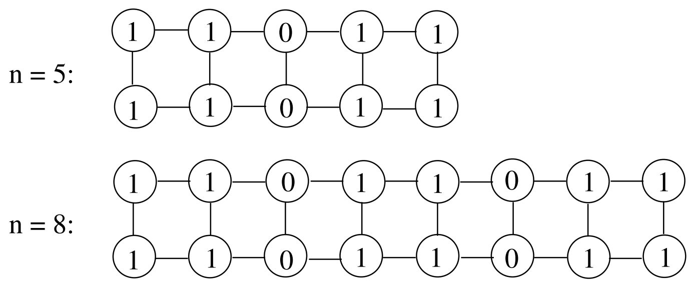

Figure 7.14: 2-regular total dominating functions of $P_n \square K_2$ for $n = 5, 8$

For the converse, assume, to the contrary, that there is a positive integer n such that $n \not\equiv 2 \pmod 3$ and $G = P_n \square K_2$ has a 2-regular total dominating function h. Because $P_1 \square K_2 = K_2$ does not have a 2-regular total dominating function, it follows that $n \geq 3$. Since $\deg u_1 = \deg v_1 = 2$ and $\sigma_h(u_1) = \sigma_h(v_1) = 2$, it follows that $h(x) = 1$ for each $x \in N(u_1) \cup N(v_1) = \{u_1, u_2, v_1, v_2\}$. Because $\sigma_h(u_2) = \sigma_h(v_2) = 2$, we have $h(u_3) = h(v_3) = 0$. If $n = 3$, then $\sigma_h(u_3) = \sigma_h(v_3) = 1$, a contradiction. Thus, $n \geq 4$ and $h(u_4) = h(v_4) = 1$. If $n = 4$, then $\sigma_h(u_4) = \sigma_h(v_4) = 1$, a contradiction. Thus, $n \geq 6$ and $h(u_5) = h(v_5) = h(u_6) = h(v_6) = 1$. Continuing in this fashion, we see that $h(u_i) = h(v_i) = 0$ if $i \equiv 0 \pmod 3$ and $h(u_i) = h(v_i) = 1$ if $i \not\equiv 0 \pmod 3$. If $n \equiv 0 \pmod 3$, then $N(v_n) = \{u_n, v_{n-1}\}$ and $h(u_n) = 0$, which implies that $\sigma_h(v_n) = 1$; while if $n \equiv 1 \pmod 3$, then $N(v_n) = \{u_n, v_{n-1}\}$ and $h(v_{n-1}) = 0$, which implies that $\sigma_h(v_n) = 1$. A contradiction is produced in either case. Therefore, if $n \not\equiv 2 \pmod 3$, then $P_n \square K_2$ does not have a 2-regular total dominating function. ■

We conclude with the following question.

Problem 7.5.11 *Under what conditions do both the graphs G and $G \,\square\, K_2$ have a k-regular total dominating function for some positive integer k?*

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