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Estimation of Odds Ratio In 2 x 2 Contingency Tables With Small Cell Counts

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ESTIMATION OF ODDS RATIO IN 2×2 CONTINGENCY TABLES WITH SMALL CELL COUNTS

by

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This study is focusing on properties of estimators of odds ratio or its logarithm in case of 2x2 tables with small counts.

The odds ratio represents the odds that an outcome of interest will occur given a particular exposure, compared to the odds of the outcome occurring in the absence of that exposure. Both parameters are often used to quantify the strength of association of two binary variables and are common measurements reported in case-control, cohort, and cross-sectional studies.

Because of their wide applicability, both parameters, odds ratio, and its logarithm, have been intensively studied in the literature. However, most of their desirable properties are based on the asymptotic normality of the estimators which are not necessarily true in case of small sample sizes. In addition, contingency tables with small counts often contain cells with counts that equal zero which makes maximum likelihood estimators of odds ratio and its logarithm undefined. While in many research areas it is possible to collect data of the size needed, there are areas, such as health related multi-center research, where sample size cannot be increased.

We are studying performance of estimators of odds ratio, and its logarithm, for independent 2x2 tables with small counts. Among other applications, our conclusions could also serve as recommendations for comparison of odds ratios in multiple 2x2 tables—a step necessary before performing meta-analysis.

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Chapter 1

INTRODUCTION

The odds ratio (OR) and log(OR) are parameters often used to quantify the strength of association between two binary variables. The OR is defined as the ratio of the odds of an outcome in the presence of a exposure and the odds of the outcome in the absence of that exposure. The odds ratio is the fundamental parameter underlying the analysis of contingency tables by log-linear models. It also has been pointed out that odds ratio is a useful approximation to relative risk in retrospective studies when the incidence of the disease was rare (Walter, 1987). Therefore, it is a common measurement reported in case-control, cohort, and cross-sectional studies. While performance of estimators of there parameters were given a lot of attention in the literature, the main focus was on large date. In this paper, we will evaluate performance of estimators of OR and log(OR) in 2×2 contingency tables with small sample sizes.

Table 1.1: Notation for probabilities

Population	Success	Failure	Total
1	p_1	$1 - p_1$	1.0
2	p_2	$1 - p_2$	1.0

In the cohort design, two populations of individuals are distinguished by a dichotomous exposure variable. The probabilities of disease for individuals in each populations are equal to p_1 and p_2, respectively. Table 1.1 displays notations for probabilities in 2×2 classification tables. With investigation sample sizes n_{1+} and n_{2+}, we organize the observed data as follow:

Table 1.2: Data frame

	Outcome		
Treatment	Success	Failure	Total
1	n_{11}	n_{12}	n_{1+}
2	n_{21}	n_{22}	n_{2+}
Total	n_{+1}	n_{+2}	N

For a probability p_i of success, the *odds* for treatment i are defined as $p_i/(1-p_i)$. Refer to Table 1.1, the odds ratio is denoted by θ and defined as

$$\theta = \frac{p_1(1-p_2)}{p_2(1-p_1)}. \tag{1.1}$$

and its logarithm, log odds ratio, is defined as

$$\psi = \log(\theta) \tag{1.2}$$

Parameter θ has a range from 0 to ∞, while ψ ranges from $-\infty$ to ∞. We are concerned about the statistical properties of estimators of θ and ψ in sparse contingency tables. Not only the asymptotic normality of estimates is not necessarily true in small samples, but there are likely zero counts in the table which leads to undefined estimators, such as maximum likelihood estimator (MLE). These zero entries, due to sampling variation, are called sampling zeros and usually will disappear when the sample size increases. Sometimes tables with zero counts are

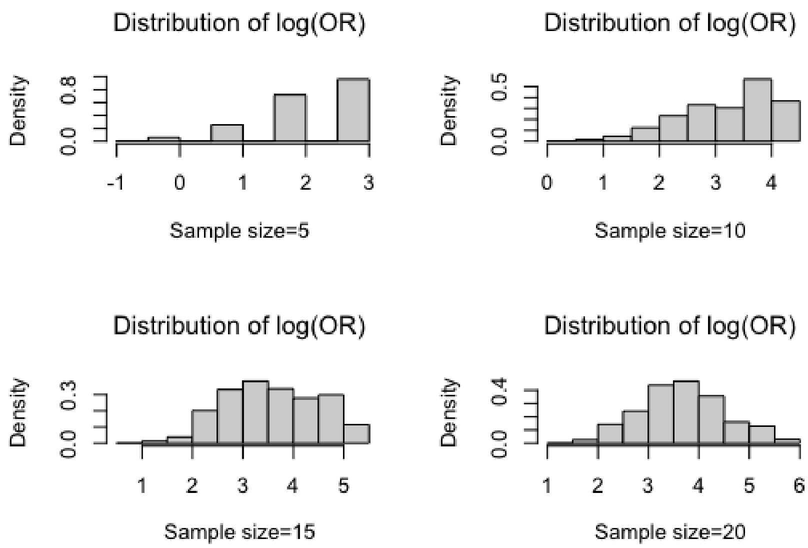

Figure 1.1: Histogram of generated values with p_1=0.8, p_2=0.1

disregarded but it could make the analysis biased. Figures 1.1 and 1.2 illustrate the distribution of estimated log odds ratio for selected values of (p_1, p_2) and small sample sizes. It can easily be seen that the lack of normality disappears when the sample size increases.

We will assume binomial model for each treatment variable so that $n_{11} \sim \text{BIN}(n_{1+}, p_1)$ and $n_{21} \sim BIN(n_{2+}, p_2)$. The probability of at least one zero cell in the table is obtained as:

$$\begin{aligned} P(n_{11}n_{12}n_{21}n_{22} = 0) &= 1 - P(n_{11}n_{12}n_{21}n_{22} \neq 0) \\ &= 1 - [1 - p_1^{n_{1+}} - (1-p_1)^{n_{1+}}] \times [1 - p_2^{n_{2+}} - (1-p_2)^{n_{2+}}] \end{aligned} \tag{1.3}$$

The probability (1.3) depends on sample sizes n_{1+} and n_{2+} and on different values of p_1, p_2. Table 1.3 contains probabilities of one or more zero cells in the table under balanced small samples ($n_{i+} \leq 25$), while Table 1.4 contains such probabilities in selected unbalanced samples. When at least one probability (p_1 or p_2) is close to

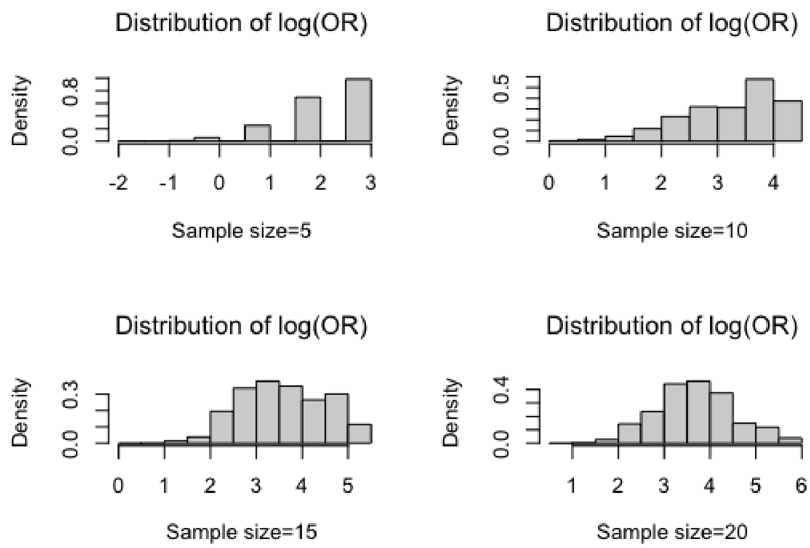

Figure 1.2: Histogram of generated values with p_1=0.9, p_2=0.2

0 or 1, probabilities (1.3) can be large.

Table 1.3: Probabilities of at least one zero in balanced samples

(p_1, p_2)	$n_{i+} = 5$	$n_{i+} = 10$	$n_{i+} = 15$	$n_{i+} = 20$	$n_{i+} = 25$
(0.1,0.5)	0.6161	0.3500	0.2059	0.1216	0.0718
(0.1,0.9)	0.8323	0.5758	0.3694	0.2284	0.1384
(0.2,0.3)	0.4426	0.1326	0.0398	0.0123	0.0039
(0.2,0.8)	0.5484	0.2032	0.0691	0.0229	0.0075
(0.4,0.5)	0.1450	0.0081	0.0005	0.0000	0.0000
(0.5,0.5)	0.1211	0.0039	0.0001	0.0000	0.0000
(0.6,0.7)	0.2435	0.0342	0.0052	0.0008	0.0001
(0.7,0.9)	0.6603	0.3671	0.2097	0.1223	0.0719
(0.8,0.9)	0.7248	0.4186	0.2338	0.1317	0.0753
(0.9,0.3)	0.6603	0.3671	0.2097	0.1223	0.0719

Various types of estimators had been proposed to deal with zero cells when estimating θ or ψ. We will review these estimators and their properties in Chapter 2.

Table 1.4: Probabilities of at least one zero in unbalanced samples

(p_1, p_2)	$n_{1+} = 5$ $n_{2+} = 10$	$n_{1+} = 10$ $n_{2+} = 15$	$n_{1+} = 15$ $n_{2+} = 20$	$n_{1+} = 20$ $n_{2+} = 25$	$n_{1+} = 25$ $n_{2+} = 30$
(0.1,0.5)	0.5913	0.3487	0.2059	0.1216	0.0718
(0.1,0.9)	0.7333	0.4828	0.3024	0.1846	0.1111
(0.2,0.3)	0.3470	0.1116	0.0360	0.0117	0.0038
(0.2,0.8)	0.4001	0.1388	0.0463	0.0153	0.0050
(0.4,0.5)	0.0898	0.0062	0.0005	0.0000	0.0000
(0.5,0.5)	0.0643	0.0020	0.0001	0.0000	0.0000
(0.6,0.7)	0.1138	0.0109	0.0013	0.0002	0.0000
(0.7,0.9)	0.4597	0.2283	0.1257	0.0725	0.0425
(0.8,0.9)	0.5623	0.2912	0.1525	0.0825	0.0460
(0.9,0.3)	0.6021	0.3518	0.2065	0.1217	0.0718

Chapter 2

LITERATURE REVIEW

Review of Estimators

Zero cells in contingency tables are of two types: structural or sampling zeros. Structural zeros occur when it is impossible to observe values for certain combinations of the variable categories. For example, one will never observe a positive count from a male patient suffering from cervical cancer. Sampling zeros are due to sampling variations and can be seen in relatively small sample sizes. Such tables should not be discarded (what is often happening) but should be included in the analysis. They have non-negative expected values and, depending on the model, have zero or positive estimated values. In this paper, we only focus on sampling zeros due to small sample counts in the table.

For 2×2 contingency tables with two independent row variables (treatment) following binomial distribution (see Table 1.1 and 1.2), MLE estimates of p_1 and p_2 are n_{11}/n_{1+}, n_{21}/n_{2+}, and the maximum likelihood estimator of the odds ratio is:

$$\hat{\theta}_{MLE} = \frac{n_{11}n_{22}}{n_{12}n_{21}}. \tag{2.1}$$

The corresponding estimator of log odds ratio is defined as

$$\hat{\psi}_{MLE} = \log\left(\hat{\theta}_{MLE}\right) = \log\frac{n_{11}n_{22}}{n_{12}n_{21}}. \tag{2.2}$$

Estimator (2.1) will be undefined if $n_{12}n_{21} = 0$ and $\hat{\psi}_{MLE}$ is undefined if there is any zero count in particular table. So if $n_{11}n_{12}n_{21}n_{22} = 0$, some modifications need to be applied to the estimators to avoid the problem. The first type of modification is to always add a positive constant, ϵ to each cell. This type of estimator is then defined as:

$$\hat{\theta}_{\epsilon} = \frac{(n_{11}+\epsilon)(n_{22}+\epsilon)}{(n_{12}+\epsilon)(n_{21}+\epsilon)}. \tag{2.3}$$

and the corresponding log estimator of log odds ratio is,

$$\hat{\psi}_{\epsilon} = \log\left(\hat{\theta}_{\epsilon}\right) = \log\frac{(n_{11}+\epsilon)(n_{22}+\epsilon)}{(n_{12}+\epsilon)(n_{21}+\epsilon)}. \tag{2.4}$$

Haldane (1956) suggested that one use 0.5 as the correction applied to each cell. He had shown that the first order bias term can be eliminated by using $\epsilon = 0.5$ based on the Taylor series expansion of $\log(p/(1-p))$. The Haldane's estimator is defined as:

$$\hat{\psi}_{H} = \log\frac{(n_{11}+0.5)(n_{22}+0.5)}{(n_{12}+0.5)(n_{21}+0.5)}. \tag{2.5}$$

Gart et al. (1985) had found the bias of $\log\{(n_{11}+\epsilon)/(n_{21}+\epsilon)\}$ as a estimator of $\log(p/(1-p))$ to be:

$$B_\epsilon(p) = \frac{(p-q)(0.5-\epsilon)}{npq} + \frac{(p-q)}{12(npq)^2}\left\{6\epsilon^2 - 12(1-pq)\epsilon + (5-6pq)\right\} + O(n^{-3})$$

where $q = 1 - p$. (2.6)

In agreement with the result of Haldane (1956), for $\epsilon = 0.5$, the first order bias term vanishes if $n \to \infty$. Pettigrew et al. (1986) provided a method to find the second order term of the bias of $\log\{(n_{11}+0.5)/(n_{21}+0.5)\}$ as a estimator of $\log(p/(1-p))$. Assuming $X \sim BIN(p,n)$, the author defined an estimator of $\log(x/n)$ as

$$l_{0.5}(x) = \log\left[(x+0.5)/(n+0.5)\right].$$

The second order bias item of $l_{0.5}(x)$ is:

$$b_{0.5}(x) = -\{(1-p)(1+p)\}/\left\{24(np)^2\right\} + O(n^{-3}).$$

Thus, the estimator of log odds with $\epsilon = 0.5$ added to both cell counts can be expressed as $L_{0.5}(x) = l_{0.5}(x+0.5) - l_{0.5}(n-x+0.5)$ with the second order bias:

$$b_{0.5}(x) - b_{0.5}(n-x) = (2p-1)/\left\{24(np)^2\right\} + O(n^{-3}). \tag{2.7}$$

Gart et al. (1985) also found the variance of $L_{0.5}(x)$, log odds with $\epsilon = 0.5$ added,

$$Var\left(L_{0.5}(x)\right) = \frac{1}{npq} + \frac{(p-q)^2}{2(npq)^2} + O(n^{-3}). \tag{2.8}$$

Gart (1966) had proposed an unbiased estimator of the $Var\left(L_{0.5}(x)\right)$ except for terms of $O(n^{-3})$:

$$\widehat{Var}\left(L_{0.5}(x)\right) = \frac{1}{x+0.5} + \frac{1}{n-x+0.5}. \tag{2.9}$$

Bedrick (1984) proposed a general estimator of variance of log odds with $\epsilon = 0.5$ added:

$$\widehat{Var}\left(L_{\epsilon}(x)\right) = \frac{x+2\epsilon-0.5}{(x+\epsilon)^2} + \frac{n-x+2\epsilon-0.5}{(n-x+\epsilon)^2}. \tag{2.10}$$

The estimator proposed by Gart et al. (1985) is a special case of the Bedrick's estimator for $\epsilon = 0.5$. Bedrick (1984) also suggested to use $\epsilon = 0.3$ as such estimator had a smaller percentage of bias compared with $\epsilon = 0.5$. Gart and Zweifel (1967) proved that $\widehat{Var}\left(L_{0.5}(x)\right)$ defined in (2.9) tends to overestimate the true variance.

Agresti and Yang (1987) suggested that if adding a constant is necessary to ensure existence of the estimator it is preferable to select a very small one, because of the appearance of adding "fake data". Greenland et al. (2000) had made an adjustment when apply Haldane's correction to avoid the change in overall frequencies. After the correction of 0.5 has been applied to each cell in the table, the counts were multiplied by $\frac{N}{N+0.5\times(num.\,of\,cells)}$. So that the total number of counts is restored to its original total. In the example he used, with six discordant-pair cells and 56 discordant pairs, adding 0.5 to each cell added 3 to the total count, so the counts were multiplied by 56/59 to restore the total discordant-pair count to 56.

Walter (1985) suggested a very similar modification of $\hat{\psi}_{MLE}$ by only adding 0.5 to cells only as necessary, that is when a zero cell occured. He estimated the log odds ratio as:

$$\hat{\psi}_W = \begin{cases} \hat{\psi}_{MLE}; & \text{if } n_{11}n_{12}n_{21}n_{22} \neq 0 \\ \hat{\psi}_H = \log \frac{(n_{11}+0.5)(n_{22}+0.5)}{(n_{12}+0.5)(n_{21}+0.5)}; & \text{otherwise} \end{cases} \tag{2.11}$$

Walter (1985) compared properties of those two modified estimators in a simulation study. The bias of $\hat{\psi}_W$ (2.11) is usually greater than the bias of Haldane's estimator $\hat{\psi}_H$ (2.5), exceptions were samples with a very small sample size ($n = 5$) and small p_2. Also, the bias in $\hat{\psi}_H$ is usually negative, which means that $\hat{\psi}_H$ might underestimate the true value of the parameter. Moreover, $\hat{\psi}_H$ usually has lower MSE than the MSE of estimator $\hat{\psi}_W$, but the differences are quite small.

Jewell (1986) suggested that there were few advantages to work on the original scale of odds ratio rather than log odds ratio. The reason was that odds ratio was easier to interpret. Cornfield (1956) provided a method to estimate the confidence interval of odds ratio without transforming to log scale. Jewell (1986) proposed an adjusted version of $\hat{\theta}_{MLE}$, in which constant 1 was added to counts in denominator only:

$$\hat{\theta}_J = \frac{n_{11}n_{22}}{(n_{12}+1)(n_{21}+1)} \tag{2.12}$$

Using notation from Tables 1.1 and 1.2, the expectation and bias of $\hat{\theta}_J$ are:

$$E\left(\hat{\theta}_J\right) = \theta\left(1 - p_1^{n_{1+}}\right)\left[1 - (1-p_2)^{n_{2+}}\right] \tag{2.13}$$

and

$$B\left(\hat{\theta}_J\right) = E\left(\hat{\theta}_J\right) - \theta = \theta\left[p_1^{n_{1+}} q_2^{n_{2+}} - p_1^{n_{1+}} - q_2^{n_{2+}}\right], \text{ where } q_2 = 1 - p_2, \quad (2.14)$$

respectively. The bias (2.14) in $\hat{\theta}_J$ is small and it decreases fast as n_{1+}, n_{2+} become large. Based on the simulation results, the adjusted estimator, $\hat{\theta}_J$, can effectively reduce the average bias. It is also preferable to $\hat{\theta}_{MLE}$ as point estimator of odds ratio in terms of MSE. The variance of $\hat{\theta}_J$ is asymptotically the same as the estimated variance of $\hat{\theta}_{MLE}$:

$$\widehat{Var}(\hat{\theta}_{MLE}) = \hat{\theta}^2_{MLE}\left(\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}\right). \quad (2.15)$$

However, equation (2.15) tends to overestimate the sample variance of $\hat{\theta}_J$ in small samples.

Becker (1989) made a further comparison of $\hat{\theta}_J$ and $\hat{\theta}_{MLE}$ and adding to Jewell's comparisons of biases in these two estimators. Becker (1989) also compared positions of $\hat{\theta}_J$ (2.12) and $\hat{\theta}_{MLE}$ (2.3) in the confidence intervals obtained as a distance between midpoints of the respective confidence intervals and the estimated values. Then the author compared maximized conditional likelihoods given the Jewell estimates with the unconditional maximum likelihood and concluded under what conditions would he recommend the Jewell estimators. Becker (1989) also provided a formula for the bias in $\hat{\theta}_{MLE}$. :

$$B\left(\hat{\theta}_{MLE}\right) = \sum_{n_{11}=1}^{n_{1+}-1} \sum_{n_{21}=1}^{n_{2+}-1} \frac{n_{11}n_{22}}{n_{12}n_{21}} \times \frac{\binom{n_{1+}}{n_{11}}\binom{n_{2+}}{n_{21}} p_1^{n_{11}}(1-p_1)^{n_{12}} p_2^{n_{21}}(1-p_2)^{n_{22}}}{\left[1-(p_1^{n_{1+}}+(1-p_1)^{n_{1+}})\right]\left[1-(p_2^{n_{2+}}+(1-p_2)^{n_{2+}})\right]} \quad (2.16)$$

To avoid the null value caused by 0 cells, only non-zero tables were used for the estimation. Consequently, the $\hat{\theta}_{MLE}$ could have a substantial bias even for a sample size as large as 80, and the bias will increase as the true odds ratio increases. The bias of $\hat{\theta}_J$ is less than that of $\hat{\theta}_{MLE}$ for the conditions shown in the simulation study, and $\hat{\theta}_J$ has virtually zero bias for $n > 40$ and $p_2 > 0.25$. However, when n_{12} or n_{21} =1,2,3 for odds ratio, the average values of Jewell's estimator are far below the midpoint of the respective confidence intervals. An additional problem with Jewell's estimators is that they are not invariant to switching population 1 and 2 or cases and controls, while the $\hat{\theta}_{MLE}$ are invariant. To use $\hat{\theta}_J$, one should decide which parameter is to be estimated.

Walter and Cook (1991) also have compared statistical properties of $\hat{\theta}_J$ and $\hat{\theta}_{MLE}$. They believed that it is inappropriate to simply discard tables with zero cells and work within a restricted distribution over other tables. To avoid the undefined value caused by the zero cells, the author added $\epsilon = 0.5$ to every cell before calculating $\hat{\theta}_{MLE}$. So this is actually a comparison between Haldane's estimator of log odds ratio, $\hat{\psi}_H$ (2.5), in odds ratio scale, $\hat{\theta}_H$ and $\hat{\theta}_J$. Bias, MSE, and the average absolute error (AAE) were computed and compared between the estimators. Based on the result, $\hat{\theta}_J$ consistently had the smallest bias for all combinations of p_1 and p_2 when $n \geq 25$. When the sample sizes were 5 or 10, $\hat{\theta}_J$ still had smaller average bias in most cases. Absolute bias decreases as sample size increases with p_1 and p_2 fixed. When $n \leq 25$, $\hat{\theta}_J$ was preferred most of time in terms of MSE and AAE. Overall, for the odds ratio, Jewell's estimator usually had smaller bias, MSE, and AAE under most cases. However, agreeing with Becker (1989), if exchange ability with respect to table orientation is important, $\hat{\theta}_{MLE}$ should be considered.

Estimator $\hat{\theta}_J$ (2.12) is always defined, under any values of n_{11}, n_{12}, n_{21}, and n_{22}

but $\log(\hat{\theta}_J)$ is undefined if $n_{11}n_{22} = 0$. Walter and Cook (1991) offered possible modifications to log scale of the Jewell-type estimator, $\hat{\theta}_J$ (2.12). First, $\epsilon = 0.5$ was added to zero cells only when they arose. Second, $\epsilon = 0.5$ was always added to the n_{11} and n_{22} counts when one or both of them were zero. Third, $\epsilon = 0.5$ was always added to the n_{11} or n_{22} cell. The third modification was proved to be the most successful by far in terms of bias, MSE, and AAE. So, the modified Jewell-type estimator of log odd ratio is:

$$\hat{\psi}_J = \log \frac{(n_{11} + 0.5)(n_{22} + 0.5)}{(n_{12} + 1)(n_{21} + 1)} \tag{2.17}$$

which is defined even if $n_{11}n_{12}n_{21}n_{22} = 0$. The author has evaluated the bias, MSE and AAE in $\hat{\psi}_J$ and compared with the same in $\hat{\psi}_H$. For $n \geq 25$, $\hat{\psi}_H$ as smaller bias than $\hat{\psi}_J$ in any combination of probabilities considered. For $n = 5$ and $n = 10$, $\hat{\psi}_H$ appears to have smaller bias under most cases. Two estimators have very similar MSEs and AAE when sample sizes were greater than 25. When sample size was small ($n_{1+} = n_{2+} \leq 10$), the results were very mixed in MSE or AAE especially when n=5. Each of the estimators had the lower MSE or AAE for at least one configuration of p_1 and p_2.

For small samples, another popular method to obtain the estimator of odds ratio was conditional likelihood. It is formed by conditioning on the column totals of the 2×2 table, i.e., row totals, are assumed fixed (Parzen et al., 2002). The probability of observing data (1.2) conditional on all marginal totals $n_{1+}, n_{2+}, n_{+1}, n_{+2}$ remaining fixed is:

$$P\left(n_{11} \mid n_{1+}, n_{2+}, n_{+1}, n_{+2}; \theta\right) = \frac{\binom{n_{1+}}{n_{11}}\binom{n_{2+}}{n_{+1}-n_{11}}\theta^{n_{11}}}{\sum_{i}^{n_{11}}\binom{n_{1+}}{i}\binom{n_{2+}}{n_{+1}-i}\theta^{i}} \tag{2.18}$$

Breslow and Day (1980) found that the conditional maximum likelihood estimator of odds ratio, $\hat{\theta}_C$, the value which maximizes (2.18), is given by the solution to the equation:

$$n_{11} = E\left(n_{11} \mid n_{1+}, n_{2+}, n_{+1}, n_{+2}; \theta\right). \tag{2.19}$$

where E denotes the expectation of the conditional noncentral hypergeometric distribution. Breslow and Day (1980) also have come up with the formula of confidence intervals with lower confidence limit, θ_L satisfying the equation:

$$\alpha/2 = \sum_{i \geq n_{11}} P\left(i \mid n_{1+}, n_{2+}, n_{+1}, n_{+2}; \theta_L\right). \tag{2.20}$$

and upper limit, θ_U, satisfying:

$$\alpha/2 = \sum_{i \leq n_{11}} P\left(i \mid n_{1+}, n_{2+}, n_{+1}, n_{+2}; \theta_U\right). \tag{2.21}$$

Similarly, $\hat{\theta}_C$, is undefined if $n_{12}n_{21} = 0$, $n_{+1} = 0$, or $n_{+2} = 0$. Walter and Cook (1991) had offered following modifications of $\hat{\theta}_C$ according to which $\hat{\theta}_C^*$ was a solution to equation:

- $n_{11} = E[n_{11} \mid n_{1+}, n_{2+}, n_{+1}, n_{+2}; \hat{\theta}_C^*]$, if $n_{11}n_{12}n_{21}n_{22} \neq 0$;

- $n_{11} + 0.5 = E[n_{11} \mid n_{1+}, n_{2+}, n_{+1}, n_{+2}; \hat{\theta}_C^*]$, if $n_{11} = 0$ or $n_{22} = 0$ only;

- $n_{11} - 0.5 = E[n_{11} \mid n_{1+}, n_{2+}, n_{+1}, n_{+2}; \hat{\theta}_C^*]$, if $n_{21} = 0$ or $n_{12} = 0$ only;

- $\hat{\theta}_C^*$ = 1.0, if $n_{+1} = 0$, or $n_{+2} = 0$.

Walter and Cook (1991) evaluated the modified conditional maximum likelihood estimator of odds ratio, $\hat{\theta}_C^*$, and compared with $\hat{\theta}_{MLE}$ as well as their corresponding log odds ratio where 0.5 was added to each cell before estimating $\hat{\theta}_{MLE}$. So it's also a Haldane-type modification in odds ratio scale, $\hat{\psi}_H$. Based on their study, $\hat{\theta}_H$ is preferable in terms of bias. For $\hat{\theta}_H$, the absolute bias usually decreases as sample size increases. While, the trend of bias in $\hat{\theta}_C^*$ sometimes erratic as the sample size changes, in some cases increase trends were observed when $\hat{\theta} > 1$. And the bias in both estimators is larger when θ is large. In terms of MSE, $\hat{\theta}_C^*$ is better than $\hat{\theta}_H$ in term of MSE when $n \leq 25$, especially for the sample sizes as small as 5. $\hat{\theta}_H$ has smaller MSE when sample sizes are large, i.e., $n = 50$, but the difference in MSE is slight under large sample sizes. For log odds ratio, $\hat{\theta}_H$ is overall better in bias and MSE. Estimator $\hat{\psi}_C^*$ only have a slight advantage under some combinations of p_1 and p_1 with the sample sizes equal to 5 or 10.

Some other estimators of odds ratio or log odds ratio were suggested but could be undefined under small sample size. There is a bias-corrected estimator $\hat{\psi}_1$ generated from the special case of logistic regression where there is just one binary independent variable. The estimator is defined as:

$$\hat{\psi}_1 = \hat{\psi}_{MLE} + \frac{1}{2}\left[\frac{n_{12} - n_{11}}{n_{11}n_{12}} + \frac{n_{21} - n_{22}}{n_{21}n_{22}}\right] \tag{2.22}$$

It is the general form of Schaefer's correction applied on log odds ratio (Walter, 1985). The estimator is undefined if there are zero counts in the table. An estimator Haldane-type estimator is based on the correction for the second order bias term:

$$\hat{\psi}_2 = \hat{\psi}_H + \frac{1}{24}\left[\frac{1}{n_{11}^2} - \frac{1}{n_{12}^2} - \frac{1}{n_{21}^2} + \frac{1}{n_{22}^2}\right] \tag{2.23}$$

Another estimator is a special case of Birch's estimator for $k \times 2 \times 2$ tables when k=1.

$$\hat{\psi}_3 = \frac{(n_{11}n_{22} - n_{12}n_{21})(n_{1+} + n_{2+})(n_{1+} + n_{2+} - 1)}{n_1 n_2 (n_{11} + n_{21})(n_{12} + n_{22})} \tag{2.24}$$

Hauck et al. (1982) mentioned that Birch's estimator is an inconsistent estimator of θ.

2.1 Method of Obtaining Confidence Intervals

A precision of estimators is characterized by standard errors (standard deviation). Sampling distributions are usually skewed unless the sample size is very large. For a multinomial sample, these estimators have asymptotic normal distributions around θ. The log transform, having an additive rather than multiplicative structure, converges more rapidly to normality (Agresti, 2012). The estimated standard error of $\hat{\psi}_{MLE}$ is:

$$\hat{\sigma}(\hat{\psi}_{MLE}) = \sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}. \tag{2.25}$$

Since $\hat{\psi}_{MLE}$ is approximately normally distributed, the approximately $(1-\alpha)100\%$ confidence interval is

$$\hat{\psi}_{MLE} \pm z_{\alpha/2}\hat{\sigma}(\hat{\psi}_{MLE}) \tag{2.26}$$

The formula (2.26) was proposed by Woolf (1955). Gart and Thomas (1982) made a correction by adding 0.5 to each cell and turn it into $\hat{\psi}_H$, and the corresponding $(1-\alpha)100\%$ CI is:

$$\hat{\psi}_H \pm \chi_\alpha \hat{\sigma}(\hat{\psi}_H). \tag{2.27}$$

Cornfield (1956) proposed a method to estimate confidence limits of odds ratio without transforming into log odds ratio. Fisher (1962) simplified the Cornfield's method. Considering testing $\theta = \theta_0$, he constructed a statistic having chi-square with one degree of freedom:

$$\chi_0^2 = \{n_{11} - E_{\theta_0}(n_{11})\}^2 \sum_{i=1,j=1}^{2} \{1/E_{\theta_0}(n_{ij})\}, \tag{2.28}$$

where $E_1(\theta_0)$ is the unique solution to the quadratic equation such that $0 \leq E_1(\theta_0) \leq \min(n_{1+}, n_{+1})$, where

$$\theta_0 = \frac{E_{\theta_0}(n_{11})\left[E_{\theta_0}(n_{11}) + n_{+2} - n_{1+}\right]}{\left[n_{1+} - E_{\theta_0}(n_{11})\right]\left[n_{+1} - E_{\theta_0}(n_{11})\right]} \tag{2.29}$$

Confidence limits are found by reversing the process and substituting χ_0^2 in equation (2.28) with $\chi_{\alpha,1}^2$ and solving for the $E_\theta(n_{11})$. Threshold $\chi_{\alpha,1}^2$ is the upper α percentage point of the chi-square distribution.

Miettinen (1976) proposed a formula with a theoretical bias based on chi-square test of association which can be written as

$$\hat{\psi}_{MLE} \pm \chi^2_{\alpha,1}\sqrt{(\hat{\psi}_{MLE})^2/\chi^2} \tag{2.30}$$

where χ^2 represents the 1 d.f. chi-square statistic for testing the hypothesis of no association which is:

$$\chi^2 = \sum_{i,j}^{2} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

Gart and Thomas (1982) compared methods of estimating confidence intervals. They found that Cornfield's method came closest to attaining the nominal confidence coefficient. The logit method was generally displaced to the left for the important case when $\theta > 1$. Thus it tends to underestimate the upper limit (highest possible risk) in the unconditional sample space. The test-based method will yield limits too wide when $\theta = 1$ and too narrow when $\theta > 1$. Although the Cornfield and test-based methods have the same confidence coefficients for $\theta = 1$, the test-based method is more likely to cover distant values of $\theta \neq 1$ when in fact $\theta = 1$. It was concluded that Cornfield's method without the continuity correction was the preferred approximate method in the unconditional sample space.

Chapter 3

METHODOLOGY

3.1 Estimators of ψ_ϵ with Bias Correction Item

Estimators of odds ratio are undefined if either n_{12} or n_{21} counts in the 2×2 contingency tables are zero. Estimator of it logarithm, $\psi = \log \theta$, are undefined if any count in the tables equals zero. Most research proposes adding a constant, ϵ to the counts in the table either if there are zeros, or always. The constants could be equal or not ($\hat{\psi}_J$ (2.17), 0.5 was added to n_{11} and n_{22}, 1 was added to n_{12} and n_{21}). And sometimes they are added to n_{12} and n_{21} only (e.g., estimator (2.1)). Most often $\epsilon = 0.5$, since Haldane (1956) had proved that adding 0.5 to each cell of the table can eliminate the first order bias term of log odds ratio. However, Hitchcock (1962) suggested $\epsilon = 0.25$ for a bioassay logistic model with three dose levels. Hauck et al. (1982) also have used $\epsilon = 0.25$ in $2 \times 2 \times k$ tables to estimate the common odds ratio. We would like to check the properties of $\hat{\psi}_\epsilon$ with different ϵ, compare their bias, variance, AAE and MSE. In addition we will be focused on the small sample sizes when $n_{1+} = n_{2+} = 5$, 10, 15. We will also propose a correction that will be reducing the bias of the estimator.

Let us define ψ_ϵ as:

$$\psi_\epsilon = \log \frac{p_1 + \epsilon/n}{1 - p_1 + \epsilon/n} - \log \frac{p_2 + \epsilon/n}{1 - p_2 + \epsilon/n} \tag{3.1}$$

The difference between ψ_ϵ and ψ is positive, will increase as ϵ increasing but decreases with n. Table 3.1 is offering an example of these trends.

Table 3.1: $p_1 = 0.3, p_2 = 0.5$

ϵ	n	ψ	ψ_ϵ	Difference
$\epsilon = .25$	10	-0.847	-0.802	0.045
	15	-0.847	-0.817	0.031
	20	-0.847	-0.824	0.023
	30	-0.847	-0.832	0.016
$\epsilon = .5$	10	-0.847	-0.762	0.085
	15	-0.847	-0.788	0.059
	20	-0.847	-0.802	0.04
	30	-0.847	0.817	0.030
$\epsilon = .75$	10	-0.847	-0.726	0.121
	15	-0.847	-0.762	0.085
	20	-0.847	-0.782	0.066
	30	-0.847	-0.802	0.045
$\epsilon = 1$	10	-0.847	-0.693	0.154
	15	-0.847	-0.738	0.110
	20	-0.847	-0.762	0.085
	30	-0.847	-0.788	0.059

We have:

$$\begin{aligned} f(\epsilon, n) =& \psi_\epsilon - \psi \\ =& \log \frac{p_1 + \epsilon/n}{1 - p_1 + \epsilon/n} - \log \frac{p_2 + \epsilon/n}{1 - p_2 + \epsilon/n} - \log \frac{p_1}{1 - p_1} + log \frac{p_2}{1 - p_2} \end{aligned}$$

Now, let

$$f_1(\epsilon, n) = \log \frac{p_1 + \epsilon/n}{1 - p_1 + \epsilon/n} - \log \frac{p_1}{1 - p_1}$$

then

$$\lim_{n\to\infty} f_1(\epsilon, n) = 0$$

Similarly, $\lim_{n\to\infty} f_2(\epsilon, n) = 0$, and also $\lim_{n\to\infty} f(\epsilon, n) = 0$, where $f(\epsilon, n) = f_1(\epsilon, n) - f_2(\epsilon, n)$. Since

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \approx x - 1$$

for small values of x, we have:

$$\begin{aligned}
f_1(\epsilon, n) &= \log \frac{(p_1 + \epsilon/n)(1 - p_1)}{(1 - p_1 + \epsilon/n)p_1} \\
&= \log \frac{1 + \frac{\epsilon}{np_1}}{1 + \frac{\epsilon}{n(1-p_1)}} \\
&= \log \left(1 + \frac{\epsilon}{np_1}\right) - \log \left(1 + \frac{\epsilon}{n(1 - p_1)}\right) \\
&\approx \frac{\epsilon}{n}\left(\frac{1}{p_1} - \frac{1}{1 - p_1}\right)
\end{aligned}$$

In the same way, we have

$$f_2(\epsilon, n) \approx \frac{\epsilon}{n}\left(\frac{1}{p_2} - \frac{1}{1 - p_2}\right),$$

and consequently:

$$f(\epsilon, n) \approx \frac{\epsilon}{n}\left[\frac{1}{p_1} - \frac{1}{1 - p_1} - \frac{1}{p_2} + \frac{1}{1 - p_2}\right]. \tag{3.2}$$

Based on the approximation of the difference between ψ and ψ_ϵ, the first proposed estimator with correction is defined as:

$$\hat{\psi}_{\epsilon,A} = \log \frac{(n_{11}+\epsilon)(n_{22}+\epsilon)}{(n_{12}+\epsilon)(n_{22}+\epsilon)} - \frac{2\epsilon}{n} \frac{n_{21}/n - n_{11}/n}{A(1-A)} \tag{3.3}$$

where $A = \max\{\hat{p_1}(1-\hat{p_1}),\ \hat{p_2}(1-\hat{p_2})\}$, $\hat{p_1} = \frac{n_{11}}{n_{1+}}$ and $\hat{p_2} = \frac{n_{21}}{n_{2+}}$. Estimator $\hat{\psi}_A$ can be undefined when more than one zero cell arises in tables with probability that depends on the the value of n and p_1, p_2. Since $0 < p < 1$, then $p(1-p) \leq 0.25$. Thus

$$\begin{aligned}
f(\epsilon, n) &\approx \frac{\epsilon}{n}\left[\frac{1}{p_1} - \frac{1}{1-p_1} - \frac{1}{p_2} + \frac{1}{1-p_2}\right] \\
&= \frac{\epsilon}{n}\left[\frac{1-2p_1}{p_1(1-p_1)} + \frac{2p_2-1}{p_2(1-p_2)}\right] \\
&\geq \frac{\epsilon}{n}\left[4(1-2p_1) + 4(2p_2-1)\right] \\
&= \frac{8\epsilon}{n}(p_2 - p_1)
\end{aligned}$$

This motivates another correction term :

$$\frac{8\epsilon}{n}\left(\frac{n_{21}}{n} - \frac{n_{11}}{n}\right) \tag{3.4}$$

and an estimator that is defined as:

$$\hat{\psi}_{\epsilon,0.5} = \log \frac{(n_{11}+\epsilon)(n_{22}+\epsilon)}{(n_{12}+\epsilon)(n_{22}+\epsilon)} - \frac{8\epsilon}{n}\left(\frac{n_{21}}{n} - \frac{n_{11}}{n}\right) \tag{3.5}$$

The estimator $\hat{\psi}_{\epsilon,0.5}$ is defined under any values of n_{11}, n_{12}, n_{21} and n_{22}.

3.2 Variance of the Estimator $\hat{\psi}_\epsilon$

We are assuming that counts in the table are values of two independent binomial variables (rows).

For a Bernoulli Distribution: BIN$(1, p)$, we have

$$f(x) = p^x(1-p)^{1-x}, \;\; x = 0,\, 1,$$

and

$$\log(f(x)) = x \log p + (1-x)\log(1-p).$$

Since

$$\frac{\partial \log f}{\partial p} = \frac{x}{p} - \frac{1-x}{1-p},$$

$$\frac{\partial^2 \log f}{\partial p^2} = \frac{-x}{p^2} - \frac{1-x}{(1-p)^2},$$

and $E(x) = p$, then:

$$\begin{aligned} -E\left[\frac{-x}{p^2} - \frac{1-x}{(1-p)^2}\right] &= \frac{E(x)}{p^2} + \frac{E(1-x)}{(1-p)^2} \\ &= \frac{p}{p^2} + \frac{1-p}{(1-p)^2} \\ &= \frac{1}{p(1-p)} \end{aligned}$$

and the CRLB for unbiased estimators of p is

$$\frac{p(1-p)}{n},$$

and the asymptotic variance is:

$$\frac{[m'(p)]^2}{nI(p)}.$$

For

$$m(p) = \log \frac{p+\epsilon/n}{1-p+\epsilon/n}$$

$$\begin{aligned}
m'(p) &= \frac{1-p+\epsilon/n}{p+\epsilon/n} \cdot \left(\frac{1}{1-p+\epsilon/n} + \frac{p+\epsilon/n}{(1-p+\epsilon/n)^2} \right) \\
&= \frac{1-p+\epsilon/n}{p+\epsilon/n} \cdot \frac{1+2\epsilon/n}{(1-p+\epsilon/n)^2} \\
&= \frac{1+2\epsilon/n}{(p+\epsilon/n)(1-p+\epsilon/n)}
\end{aligned}$$

Then we have:

$$Var(\hat{p}) = \frac{p(1-p)}{n} \cdot \frac{(1+2\epsilon/n)^2}{(p+\epsilon/n)^2(1-p+\epsilon/n)^2}.$$

Now, since

$$\psi_\epsilon = \log \frac{p_1+\epsilon/n}{1-p_1+\epsilon/n} - \log \frac{p_2+\epsilon/n}{1-p_2+\epsilon/n},$$

$Var\left(\psi_\epsilon\right)$ is:

$$
\begin{aligned}
Var\left(\psi_{\epsilon}\right) &= Var\left(\log\frac{p_1+\epsilon/n}{1-p_1+\epsilon/n}\right)+Var\left(\log\frac{p_2+\epsilon/n}{1-p_2+\epsilon/n}\right)\\
&= \frac{p_1(1-p_1)}{n}\cdot\frac{(1+2\epsilon/n)^2}{(p_1+\epsilon/n)^2(1-p_1+\epsilon/n)^2}\\
&+ \frac{p_2(1-p_2)}{n}\cdot\frac{(1+2\epsilon/n)^2}{(p_2+\epsilon/n)^2(1-p_2+\epsilon/n)^2} \qquad (3.6)
\end{aligned}
$$

and:

$$
\begin{aligned}
\widehat{Var}_p\left(\psi_{\epsilon}\right) = \frac{n_{11}n_{12}}{n}\times\frac{(n+2\epsilon)^2}{(n_{11}+\epsilon)^2(n_{12}+\epsilon)^2}\\
+\frac{n_{21}n_{22}}{n}\times\frac{(n+2\epsilon)^2}{(n_{21}+\epsilon)^2(n_{22}+\epsilon)^2} \qquad (3.7)
\end{aligned}
$$

We will be comparing if (3.7) with the commonly used formula based on asymptotic normality:

$$
\widehat{Var}_1\left(\psi_{\epsilon}\right) = \frac{1}{n_{11}+\epsilon}+\frac{1}{n_{12}+\epsilon}+\frac{1}{n_{21}+\epsilon}+\frac{1}{n_{22}+\epsilon} \qquad (3.8)
$$

and with the estimated variance proposed in Bedrick (1984):

$$
\widehat{Var}_2\left(\psi_{\epsilon}\right) = \sum_{i=1,j=1}^{2}\frac{n_{ij}+2\epsilon-0.5}{(n_{ij}+\epsilon)^2}. \qquad (3.9)
$$

Chapter 4

SIMULATION STUDY

4.1 Estimators of ψ for Comparison

We noticed that the performance of the estimators depends on the true value of $|\psi| = |\log\theta|$. And therefore for the trends to become more visible, the tables are arranged in order of magnitude of $|\psi|$ from small to large. Bias, AAE, and MSE of estimators were compared under two scenarios: with ϵ always added to cell counts ($\hat{\psi}_\epsilon$ defined in (2.4)) and when ϵ was added to all counts only when there was at least one zero in the table ($\hat{\psi}^*_\epsilon$). Moreover, we studied the effect of correction term defined in (3.4)).

Each table generation and following computations were repeated 5000 times. Counts in the tables were generated assuming independent binomial distributions for the rows representing treatments (or treatment vs. control) groups $\text{BIN}(n_{1+}, p_1)$, and $\text{BIN}(n_{2+}, p_2)$, respectively. Probabilities p_1 (and p_2) were equal 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95. Total row counts were selected as $n_{1+} = n_{2+} = n = 5, 10, 15$, sometimes also 30 and 50 to verify convergence. All simulations were run in R. Results can be seen in Tables 4.1-4.6. The choice of ϵ will be discussed below.

Table 4.1: Bias of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	-0.0275	-0.0216	-0.0183	-0.0161	-0.0144	-0.0278	-0.0222	-0.0190	-0.0169	-0.0154
$p_1 = 0.1$	n=10	-0.0024	-0.0014	-0.0010	-0.0008	-0.0006	-0.0026	-0.0017	-0.0014	-0.0012	-0.0011
$p_2 = 0.1$	n=15	-0.0117	-0.0089	-0.0074	-0.0063	-0.0056	-0.0106	-0.0070	-0.0048	-0.0032	-0.0019
$\|\psi\| = 0$	n=30	0.0147	0.0099	0.0076	0.0061	0.0052	-0.0044	-0.0051	-0.0056	-0.0059	-0.0061
	n=50	-0.0035	-0.0036	-0.0035	-0.0034	-0.0032	-0.0044	-0.0051	-0.0056	-0.0059	-0.0061
	n=5	-0.0323	-0.0241	-0.0198	-0.0171	-0.0151	-0.0321	-0.0238	-0.0195	-0.0167	-0.0147
$p_1 = 0.2$	n=10	-0.0036	-0.0020	-0.0013	-0.0008	-0.0006	-0.0035	-0.0019	-0.0011	-0.0006	-0.0003
$p_2 = 0.2$	n=15	-0.0067	-0.0066	-0.0063	-0.0060	-0.0058	-0.0075	-0.0081	-0.0084	-0.0087	-0.0089
$\|\psi\| = 0$	n=30	0.0059	0.0057	0.0054	0.0051	0.0048	-0.0010	-0.0010	-0.0010	-0.0010	-0.0010
	n=50	-0.0011	-0.0011	-0.0011	-0.0011	-0.0011	-0.0010	-0.0010	-0.0010	-0.0010	-0.0010
	n=5	-0.0266	-0.0224	-0.0197	-0.0178	-0.0162	-0.0289	-0.0265	-0.0253	-0.0245	-0.0240
$p_1 = 0.3$	n=10	0.0022	0.0019	0.0018	0.0017	0.0016	0.0026	0.0027	0.0029	0.0030	0.0032
$p_2 = 0.3$	n=15	-0.0034	-0.0030	-0.0028	-0.0026	-0.0025	-0.0033	-0.0027	-0.0025	-0.0023	-0.0021
$\|\psi\| = 0$	n=30	0.0049	0.0047	0.0045	0.0043	0.0041	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003
	n=50	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003
	n=5	-0.0303	-0.0256	-0.0226	-0.0203	-0.0186	-0.0318	-0.0282	-0.0259	-0.0243	-0.0230
$p_1 = 0.4$	n=10	-0.0022	-0.0022	-0.0021	-0.0020	-0.0019	-0.0025	-0.0028	-0.0029	-0.0029	-0.0030
$p_2 = 0.4$	n=15	-0.0067	-0.0064	-0.0062	-0.0060	-0.0057	-0.0070	-0.0070	-0.0070	-0.0070	-0.0070
$\|\psi\| = 0$	n=30	0.0053	0.0052	0.0050	0.0049	0.0048	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012
	n=50	-0.0012	-0.0012	-0.0012	-0.0011	-0.0011	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012
	n=5	-0.0472	-0.0389	-0.0339	-0.0303	-0.0276	-0.0494	-0.0429	-0.0394	-0.0369	-0.0352
$p_1 = 0.5$	n=10	-0.0016	-0.0016	-0.0015	-0.0015	-0.0014	-0.0018	-0.0019	-0.0020	-0.0020	-0.0021
$p_2 = 0.5$	n=15	-0.0025	-0.0025	-0.0025	-0.0025	-0.0025	-0.0025	-0.0027	-0.0027	-0.0028	-0.0028
$\|\psi\| = 0$	n=30	0.0052	0.0051	0.0050	0.0049	0.0048	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008
	n=50	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008
	n=5	-0.1117	-0.0338	0.0166	0.0542	0.0840	-0.1459	-0.0946	-0.0656	-0.0458	-0.0309
$p_1 = 0.4$	n=10	-0.0304	-0.0028	0.0202	0.0402	0.0579	-0.0583	-0.0546	-0.0525	-0.0510	-0.0499
$p_2 = 0.5$	n=15	-0.0206	-0.0045	0.0102	0.0238	0.0364	-0.0381	-0.0380	-0.0379	-0.0379	-0.0379
$\|\psi\| = 0.4055$	n=30	-0.0025	0.0048	0.0118	0.0185	0.0250	-0.0100	-0.0100	-0.0100	-0.0100	-0.0100
	n=50	-0.0052	-0.0009	0.0033	0.0075	0.0115	-0.0096	-0.0096	-0.0096	-0.0096	-0.0096
	n=5	-0.0549	0.0144	0.0593	0.0928	0.1193	-0.0843	-0.0378	-0.0113	0.0070	0.0207
$p_1 = 0.5$	n=10	-0.0360	-0.0082	0.0150	0.0351	0.0529	-0.0635	-0.0593	-0.0569	-0.0553	-0.0540
$p_2 = 0.6$	n=15	-0.0113	0.0044	0.0187	0.0320	0.0442	-0.0284	-0.0284	-0.0284	-0.0284	-0.0284
$\|\psi\| = 0.4055$	n=30	-0.0059	0.0015	0.0086	0.0154	0.0219	-0.0135	-0.0135	-0.0135	-0.0135	-0.0135
	n=50	-0.0047	-0.0004	0.0038	0.0080	0.0120	-0.0092	-0.0092	-0.0092	-0.0092	-0.0092
	n=5	-0.0861	0.0016	0.0560	0.0955	0.1262	-0.1141	-0.0479	-0.0110	0.0142	0.0329
$p_1 = 0.3$	n=10	-0.0410	-0.0007	0.0298	0.0549	0.0763	-0.0730	-0.0594	-0.0516	-0.0463	-0.0423
$p_2 = 0.4$	n=15	-0.0296	-0.0062	0.0137	0.0314	0.0473	-0.0533	-0.0508	-0.0494	-0.0484	-0.0476
$\|\psi\| = 0.4418$	n=30	-0.0052	0.0044	0.0135	0.0222	0.0305	-0.0153	-0.0153	-0.0153	-0.0153	-0.0153
	n=50	-0.0070	-0.0013	0.0042	0.0095	0.0147	-0.0128	-0.0128	-0.0128	-0.0128	-0.0128
	n=5	-0.0536	0.0618	0.1286	0.1751	0.2102	-0.0741	0.0255	0.0796	0.1156	0.1420
$p_1 = 0.2$	n=10	-0.0648	0.0137	0.0645	0.1028	0.1334	-0.0991	-0.0481	-0.0200	-0.0011	0.0128
$p_2 = 0.3$	n=15	-0.0571	-0.0068	0.0300	0.0598	0.0849	-0.0924	-0.0715	-0.0598	-0.0519	-0.0459
$\|\psi\| = 0.5390$	n=30	-0.0138	0.0047	0.0213	0.0365	0.0506	-0.0329	-0.0317	-0.0310	-0.0305	-0.0301
	n=50	-0.0111	-0.0009	0.0088	0.0180	0.0269	-0.0217	-0.0217	-0.0217	-0.0217	-0.0217

Table 4.1: Bias of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts (cont.)

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	0.0977	0.2642	0.3539	0.4132	0.4565	0.0883	0.2476	0.3316	0.3863	0.4257
$p_1 = 0.1$	n=10	-0.0527	0.1169	0.2130	0.2789	0.3283	-0.0788	0.0707	0.1506	0.2032	0.2413
$p_2 = 0.2$	n=15	-0.0945	0.0452	0.1307	0.1925	0.2406	-0.1385	-0.0334	0.0239	0.0621	0.0902
$\lvert\psi\rvert = 0.8109$	n=30	-0.0572	0.0088	0.0586	0.0994	0.1344	-0.1083	-0.0849	-0.0717	-0.0628	-0.0560
	n=50	-0.0395	-0.0053	0.0244	0.0509	0.0750	-0.0740	-0.0704	-0.0684	-0.0669	-0.0659
	n=5	-0.1195	0.0195	0.1098	0.1773	0.2309	-0.1795	-0.0873	-0.0346	0.0018	0.0292
$p_1 = 0.4$	n=10	-0.0648	-0.0094	0.0367	0.0768	0.1122	-0.1197	-0.1116	-0.1068	-0.1035	-0.1009
$p_2 = 0.6$	n=15	-0.0294	0.0024	0.0315	0.0583	0.0831	-0.0640	-0.0637	-0.0636	-0.0635	-0.0634
$\lvert\psi\rvert = 0.8109$	n=30	-0.0135	0.0012	0.0153	0.0290	0.0421	-0.0288	-0.0288	-0.0288	-0.0288	-0.0288
	n=50	-0.0092	-0.0005	0.0079	0.0162	0.0243	-0.0180	-0.0180	-0.0180	-0.0180	-0.0180
	n=5	-0.1675	-0.0066	0.0952	0.1700	0.2288	-0.2259	-0.1103	-0.0451	-0.0005	0.0328
$p_1 = 0.3$	n=10	-0.0692	-0.0013	0.0521	0.0971	0.1361	-0.1284	-0.1105	-0.1002	-0.0930	-0.0876
$p_2 = 0.5$	n=15	-0.0435	-0.0043	0.0301	0.0612	0.0895	-0.0844	-0.0817	-0.0801	-0.0790	-0.0782
$\lvert\psi\rvert = 0.8473$	n=30	-0.0130	0.0040	0.0202	0.0358	0.0506	-0.0308	-0.0308	-0.0308	-0.0308	-0.0308
	n=50	-0.0110	-0.0010	0.0087	0.0182	0.0274	-0.0212	-0.0212	-0.0212	-0.0212	-0.0212
	n=5	-0.1145	0.0384	0.1350	0.2060	0.2618	-0.1694	-0.0592	0.0032	0.0458	0.0778
$p_1 = 0.5$	n=10	-0.0792	-0.0108	0.0430	0.0883	0.1276	-0.1383	-0.1200	-0.1095	-0.1022	-0.0966
$p_2 = 0.7$	n=15	-0.0374	0.0011	0.0352	0.0660	0.0941	-0.0784	-0.0764	-0.0752	-0.0744	-0.0737
$\lvert\psi\rvert = 0.8473$	n=30	-0.0160	0.0012	0.0176	0.0333	0.0483	-0.0339	-0.0339	-0.0339	-0.0339	-0.0339
	n=50	-0.0114	-0.0014	0.0083	0.0178	0.0271	-0.0216	-0.0216	-0.0216	-0.0216	-0.0216
	n=5	-0.1132	0.0858	0.2043	0.2883	0.3527	-0.1561	0.0097	0.1015	0.1636	0.2095
$p_1 = 0.2$	n=10	-0.1080	0.0110	0.0925	0.1560	0.2081	-0.1728	-0.1067	-0.0696	-0.0443	-0.0253
$p_2 = 0.4$	n=15	-0.0832	-0.0100	0.0465	0.0938	0.1348	-0.1418	-0.1183	-0.1050	-0.0958	-0.0889
$\lvert\psi\rvert = 0.9808$	n=30	-0.0239	0.0044	0.0304	0.0544	0.0769	-0.0533	-0.0521	-0.0514	-0.0509	-0.0505
	n=50	-0.0177	-0.0019	0.0133	0.0279	0.0420	-0.0342	-0.0342	-0.0342	-0.0342	-0.0342
	n=5	-0.1791	0.0435	0.1855	0.2905	0.3733	-0.2615	-0.1030	-0.0125	0.0499	0.0968
$p_1 = 0.4$	n=10	-0.1080	-0.0121	0.0647	0.1300	0.1869	-0.1941	-0.1713	-0.1580	-0.1487	-0.1414
$p_2 = 0.7$	n=15	-0.0556	-0.0009	0.0480	0.0923	0.1330	-0.1140	-0.1116	-0.1102	-0.1092	-0.1085
$\lvert\psi\rvert = 1.2528$	n=30	-0.0236	0.0009	0.0243	0.0468	0.0684	-0.0492	-0.0492	-0.0492	-0.0492	-0.0492
	n=50	-0.0158	-0.0015	0.0124	0.0261	0.0394	-0.0305	-0.0305	-0.0305	-0.0305	-0.0305
	n=5	-0.1753	0.0467	0.1884	0.2931	0.3757	-0.2567	-0.0979	-0.0072	0.0554	0.1025
$p_1 = 0.3$	n=10	-0.1036	-0.0080	0.0686	0.1336	0.1904	-0.1893	-0.1665	-0.1532	-0.1439	-0.1366
$p_2 = 0.6$	n=15	-0.0523	0.0026	0.0514	0.0957	0.1362	-0.1102	-0.1073	-0.1056	-0.1044	-0.1035
$\lvert\psi\rvert = 1.2528$	n=30	-0.0240	0.0004	0.0238	0.0462	0.0678	-0.0496	-0.0496	-0.0496	-0.0496	-0.0496
	n=50	-0.0149	-0.0006	0.0133	0.0269	0.0402	-0.0296	-0.0296	-0.0296	-0.0296	-0.0296
	n=5	0.0764	0.3500	0.5023	0.6054	0.6819	0.0528	0.3083	0.4460	0.5372	0.6037
$p_1 = 0.1$	n=10	-0.1139	0.1326	0.2788	0.3825	0.4623	-0.1671	0.0376	0.1497	0.2249	0.2804
$p_2 = 0.3$	n=15	-0.1449	0.0449	0.1669	0.2583	0.3313	-0.2186	-0.0881	-0.0154	0.0340	0.0709
$\lvert\psi\rvert = 1.3499$	n=30	-0.0770	0.0078	0.0745	0.1309	0.1801	-0.1474	-0.1227	-0.1087	-0.0989	-0.0915
	n=50	-0.0495	-0.0051	0.0343	0.0701	0.1030	-0.0946	-0.0909	-0.0888	-0.0874	-0.0862
	n=5	-0.1358	0.1243	0.2834	0.3981	0.4871	-0.2040	0.0033	0.1201	0.1999	0.2595
$p_1 = 0.5$	n=10	-0.1404	0.0048	0.1088	0.1919	0.2616	-0.2310	-0.1607	-0.1206	-0.0927	-0.0716
$p_2 = 0.8$	n=15	-0.0878	0.0009	0.0715	0.1318	0.1848	-0.1622	-0.1377	-0.1236	-0.1137	-0.1062
$\lvert\psi\rvert = 1.3863$	n=30	-0.0357	0.0002	0.0335	0.0647	0.0940	-0.0736	-0.0729	-0.0725	-0.0723	-0.0720
	n=50	-0.0214	-0.0012	0.0182	0.0369	0.0551	-0.0423	-0.0423	-0.0423	-0.0423	-0.0423

Table 4.1: Bias of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts (cont.)

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	-0.1946	0.0776	0.2435	0.3628	0.4552	-0.2635	-0.0448	0.0780	0.1619	0.2245
$p_1 = 0.2$	n=10	-0.1362	0.0104	0.1149	0.1982	0.2679	-0.2261	-0.1540	-0.1128	-0.0843	-0.0627
$p_2 = 0.5$	n=15	-0.0971	-0.0081	0.0629	0.1236	0.1769	-0.1723	-0.1482	-0.1342	-0.1245	-0.1171
$\|\psi\| = 1.3863$	n=30	-0.0317	0.0040	0.0371	0.0680	0.0971	-0.0687	-0.0675	-0.0668	-0.0663	-0.0659
	n=50	-0.0218	-0.0016	0.0178	0.0365	0.0546	-0.0426	-0.0426	-0.0426	-0.0426	-0.0426
	n=5	-0.2349	0.0707	0.2641	0.4063	0.5181	-0.3337	-0.1049	0.0267	0.1178	0.1866
$p_1 = 0.3$	n=10	-0.1468	-0.0106	0.0966	0.1868	0.2651	-0.2623	-0.2238	-0.2010	-0.1847	-0.1721
$p_2 = 0.7$	n=15	-0.0785	-0.0007	0.0679	0.1297	0.1860	-0.1600	-0.1549	-0.1518	-0.1496	-0.1479
$\|\psi\| = 1.6946$	n=30	-0.0341	0.0001	0.0328	0.0641	0.0941	-0.0700	-0.0700	-0.0700	-0.0700	-0.0700
	n=50	-0.0216	-0.0017	0.0178	0.0367	0.0552	-0.0421	-0.0421	-0.0421	-0.0421	-0.0421
	n=5	-0.2004	0.1293	0.3339	0.4826	0.5987	-0.2915	-0.0324	0.1155	0.2174	0.2941
$p_1 = 0.4$	n=10	-0.1692	0.0036	0.1305	0.2336	0.3209	-0.2844	-0.2076	-0.1629	-0.1314	-0.1073
$p_2 = 0.8$	n=15	-0.1060	-0.0011	0.0843	0.1581	0.2237	-0.1972	-0.1718	-0.1570	-0.1466	-0.1385
$\|\psi\| = 1.7918$	n=30	-0.0434	-0.0001	0.0403	0.0783	0.1142	-0.0889	-0.0882	-0.0878	-0.0875	-0.0873
	n=50	-0.0258	-0.0014	0.0223	0.0452	0.0674	-0.0511	-0.0511	-0.0511	-0.0511	-0.0511
	n=5	0.0169	0.3740	0.5780	0.7186	0.8243	-0.0221	0.3051	0.4851	0.6059	0.6951
$p_1 = 0.1$	n=10	-0.1571	0.1299	0.3068	0.4357	0.5370	-0.2331	-0.0068	0.1199	0.2064	0.2711
$p_2 = 0.4$	n=15	-0.1710	0.0417	0.1835	0.2923	0.3812	-0.2642	-0.1278	-0.0504	0.0031	0.0434
$\|\psi\| = 1.7918$	n=30	-0.0871	0.0075	0.0835	0.1487	0.2064	-0.1674	-0.1425	-0.1281	-0.1180	-0.1103
	n=50	-0.0562	-0.0061	0.0388	0.0799	0.1181	-0.1070	-0.1033	-0.1012	-0.0997	-0.0986
	n=5	-0.2024	0.1309	0.3367	0.4859	0.6021	-0.2893	-0.0236	0.1279	0.2323	0.3108
$p_1 = 0.2$	n=10	-0.1706	0.0038	0.1314	0.2347	0.3222	-0.2843	-0.2049	-0.1587	-0.1261	-0.1010
$p_2 = 0.6$	n=15	-0.1060	-0.0012	0.0841	0.1580	0.2236	-0.1977	-0.1729	-0.1585	-0.1483	-0.1405
$\|\psi\| = 1.7918$	n=30	-0.0427	0.0004	0.0406	0.0785	0.1142	-0.0875	-0.0863	-0.0855	-0.0850	-0.0846
	n=50	-0.0257	-0.0013	0.0224	0.0452	0.0674	-0.0510	-0.0510	-0.0510	-0.0510	-0.0510
	n=5	-0.0106	0.4101	0.6556	0.8275	0.9581	-0.0635	0.3163	0.5291	0.6740	0.7819
$p_1 = 0.5$	n=10	-0.1907	0.1232	0.3228	0.4716	0.5905	-0.2852	-0.0480	0.0875	0.1815	0.2527
$p_2 = 0.9$	n=15	-0.1707	0.0549	0.2095	0.3306	0.4310	-0.2774	-0.1405	-0.0616	-0.0065	0.0356
$\|\psi\| = 2.1972$	n=30	-0.1076	-0.0009	0.0845	0.1580	0.2232	-0.1938	-0.1624	-0.1442	-0.1314	-0.1216
	n=50	-0.0574	-0.0029	0.0461	0.0912	0.1333	-0.1119	-0.1075	-0.1049	-0.1031	-0.1017
	n=5	-0.0645	0.3658	0.6172	0.7931	0.9269	-0.1200	0.2676	0.4846	0.6323	0.7424
$p_1 = 0.1$	n=10	-0.1853	0.1293	0.3291	0.4780	0.5968	-0.2795	-0.0413	0.0947	0.1889	0.2603
$p_2 = 0.5$	n=15	-0.1850	0.0436	0.1999	0.3221	0.4233	-0.2918	-0.1520	-0.0715	-0.0153	0.0277
$\|\psi\| = 2.1972$	n=30	-0.0948	0.0071	0.0902	0.1623	0.2266	-0.1826	-0.1574	-0.1427	-0.1324	-0.1245
	n=50	-0.0602	-0.0058	0.0433	0.0885	0.1307	-0.1155	-0.1117	-0.1096	-0.1081	-0.1069
	n=5	-0.2619	0.1549	0.4124	0.5991	0.7446	-0.3612	-0.0215	0.1741	0.3096	0.4120
$p_1 = 0.2$	n=10	-0.2138	0.0012	0.1594	0.2879	0.3969	-0.3543	-0.2566	-0.1987	-0.1574	-0.1254
$p_2 = 0.7$	n=15	-0.1321	-0.0044	0.1007	0.1921	0.2735	-0.2466	-0.2188	-0.2023	-0.1905	-0.1813
$\|\psi\| = 2.2336$	n=30	-0.0528	0.0001	0.0497	0.0963	0.1405	-0.1079	-0.1067	-0.1059	-0.1054	-0.1050
	n=50	-0.0324	-0.0023	0.0269	0.0551	0.0824	-0.0635	-0.0635	-0.0635	-0.0635	-0.0635
	n=5	-0.2562	0.1566	0.4125	0.5984	0.7435	-0.3587	-0.0254	0.1667	0.2999	0.4006
$p_1 = 0.3$	n=10	-0.2080	0.0051	0.1624	0.2904	0.3991	-0.3493	-0.2541	-0.1976	-0.1573	-0.1260
$p_2 = 0.8$	n=15	-0.1289	-0.0009	0.1042	0.1955	0.2768	-0.2423	-0.2133	-0.1961	-0.1838	-0.1743
$\|\psi\| = 2.2336$	n=30	-0.0539	-0.0009	0.0487	0.0955	0.1399	-0.1097	-0.1090	-0.1086	-0.1083	-0.1080
	n=50	-0.0316	-0.0015	0.0276	0.0559	0.0832	-0.0627	-0.0627	-0.0627	-0.0627	-0.0627

Table 4.1: Bias of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts (cont.)

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	-0.0751	0.4151	0.7061	0.9120	1.0696	-0.1411	0.2981	0.5481	0.7202	0.8495
$p_1 = 0.4$	n=10	-0.2195	0.1219	0.3445	0.5133	0.6498	-0.3318	-0.0823	0.0628	0.1649	0.2432
$p_2 = 0.9$	n=15	-0.1888	0.0529	0.2223	0.3569	0.4699	-0.3096	-0.1693	-0.0874	-0.0295	0.0152
$\|\psi\| = 2.6027$	n=30	-0.1153	-0.0012	0.0912	0.1716	0.2434	-0.2087	-0.1770	-0.1585	-0.1454	-0.1353
	n=50	-0.0618	-0.0030	0.0502	0.0995	0.1456	-0.1207	-0.1163	-0.1137	-0.1119	-0.1104
	n=5	-0.0723	0.4191	0.7104	0.9162	1.0738	-0.1384	0.3019	0.5522	0.7242	0.8534
$p_1 = 0.1$	n=10	-0.2197	0.1227	0.3456	0.5145	0.6511	-0.3315	-0.0807	0.0651	0.1675	0.2461
$p_2 = 0.6$	n=15	-0.1938	0.0505	0.2211	0.3565	0.4701	-0.3144	-0.1713	-0.0880	-0.0291	0.0163
$\|\psi\| = 2.6027$	n=30	-0.1059	0.0035	0.0938	0.1728	0.2438	-0.2012	-0.1757	-0.1609	-0.1503	-0.1422
	n=50	-0.0641	-0.0054	0.0478	0.0973	0.1435	-0.1238	-0.1200	-0.1178	-0.1163	-0.1151
	n=5	-0.2832	0.2408	0.5608	0.7913	0.9699	-0.3819	0.0655	0.3241	0.5039	0.6399
$p_1 = 0.2$	n=10	-0.2750	0.0168	0.2252	0.3915	0.5309	-0.4361	-0.2777	-0.1828	-0.1147	-0.0616
$p_2 = 0.8$	n=15	-0.1825	-0.0047	0.1370	0.2579	0.3642	-0.3267	-0.2733	-0.2412	-0.2180	-0.1997
$\|\psi\| = 2.7726$	n=30	-0.0726	-0.0009	0.0656	0.1278	0.1863	-0.1475	-0.1455	-0.1443	-0.1434	-0.1427
	n=50	-0.0424	-0.0021	0.0367	0.0742	0.1104	-0.0841	-0.0841	-0.0841	-0.0841	-0.0841
	n=5	-0.1309	0.4424	0.7847	1.0278	1.2144	-0.2035	0.3137	0.6109	0.8168	0.9722
$p_1 = 0.3$	n=10	-0.2583	0.1234	0.3764	0.5701	0.7280	-0.3884	-0.1135	0.0493	0.1653	0.2552
$p_2 = 0.9$	n=15	-0.2117	0.0531	0.2422	0.3943	0.5230	-0.3506	-0.2031	-0.1157	-0.0531	-0.0043
$\|\psi\| = 3.0445$	n=30	-0.1258	-0.0020	0.0997	0.1888	0.2691	-0.2290	-0.1969	-0.1780	-0.1645	-0.1540
	n=50	-0.0676	-0.0031	0.0556	0.1102	0.1615	-0.1323	-0.1278	-0.1252	-0.1233	-0.1219
	n=5	-0.1319	0.4431	0.7861	1.0294	1.2162	-0.2041	0.3150	0.6131	0.8195	0.9752
$p_1 = 0.1$	n=10	-0.2629	0.1201	0.3736	0.5677	0.7258	-0.3938	-0.1184	0.0444	0.1602	0.2499
$p_2 = 0.7$	n=15	-0.2199	0.0473	0.2376	0.3906	0.5199	-0.3593	-0.2098	-0.1213	-0.0581	-0.0089
$\|\psi\| = 3.0445$	n=30	-0.1160	0.0033	0.1028	0.1907	0.2701	-0.2212	-0.1954	-0.1803	-0.1694	-0.1610
	n=50	-0.0708	-0.0064	0.0523	0.1071	0.1586	-0.1362	-0.1325	-0.1302	-0.1286	-0.1274
	n=5	-0.1580	0.5266	0.9330	1.2206	1.4409	-0.2262	0.4054	0.7695	1.0221	1.2130
$p_1 = 0.2$	n=10	-0.3253	0.1352	0.4392	0.6712	0.8598	-0.4688	-0.1257	0.0795	0.2266	0.3410
$p_2 = 0.9$	n=15	-0.2653	0.0493	0.2750	0.4567	0.6104	-0.4281	-0.2506	-0.1435	-0.0659	-0.0050
$\|\psi\| = 3.5835$	n=30	-0.1445	-0.0020	0.1166	0.2211	0.3155	-0.2659	-0.2317	-0.2112	-0.1964	-0.1847
	n=50	-0.0783	-0.0037	0.0646	0.1285	0.1887	-0.1536	-0.1491	-0.1464	-0.1445	-0.1430
	n=5	-0.1532	0.5290	0.9345	1.2216	1.4416	-0.2232	0.4049	0.7670	1.0183	1.2082
$p_1 = 0.1$	n=10	-0.3241	0.1357	0.4394	0.6713	0.8598	-0.4674	-0.1249	0.0801	0.2272	0.3416
$p_2 = 0.8$	n=15	-0.2703	0.0470	0.2739	0.4564	0.6106	-0.4317	-0.2503	-0.1409	-0.0617	0.0005
$\|\psi\| = 3.5835$	n=30	-0.1358	0.0022	0.1187	0.2221	0.3158	-0.2602	-0.2330	-0.2168	-0.2051	-0.1959
	n=50	-0.0808	-0.0063	0.0622	0.1262	0.1866	-0.1568	-0.1530	-0.1507	-0.1491	-0.1478
	n=5	-0.0279	0.8148	1.3067	1.6509	1.9125	-0.0759	0.7298	1.1920	1.5118	1.7529
$p_1 = 0.1$	n=10	-0.3744	0.2540	0.6535	0.9510	1.1887	-0.4970	0.0321	0.3484	0.5749	0.7509
$p_2 = 0.9$	n=15	-0.3532	0.1011	0.4119	0.6552	0.8568	-0.5237	-0.2111	-0.0213	0.1166	0.2250
$\|\psi\| = 4.3944$	n=30	-0.2077	0.0011	0.1697	0.3154	0.4450	-0.3748	-0.3122	-0.2739	-0.2459	-0.2237
	n=50	-0.1168	-0.0079	0.0901	0.1805	0.2648	-0.2258	-0.2171	-0.2118	-0.2079	-0.2048

Table 4.1 compares bias when a constant ϵ is always added to all cell counts ($\hat{\psi}_\epsilon$) with a bias obtained when it is added when at least one of the cell counts is a zero ($\hat{\psi}^*_\epsilon$). In general, a bias increases as values of ϵ increase (in most cases) and it increases monotonically if bias is positive when $\epsilon = 0.25$. As n increases, the bias usually decreases but they are few cases when it increased as sample size changes from 5 to 10. These can be observed for some combinations of (p_1, p_2) with $\epsilon = 0.25$. When $n = 5$, $\hat{\psi}^*_\epsilon$ is less biased compared with $\hat{\psi}_\epsilon$ which could be caused by increased number of possible outcomes in simulations. Considering $n =$10 and 15, it's better to always add ϵ to tables only when $|\psi|$ is small. As $|\psi|$ increases, we start to observe cases in which $\hat{\psi}^*_\epsilon$ is preferred in terms of bias. And when $n \geq 30$, always adding ϵ to cell counts has a smaller bias.

Table 4.2 shows comparison of biases when a constant ϵ is always added to all cell counts ($\hat{\psi}_\epsilon$) and when ϵ is added and correction term is applied ($\hat{\psi}_{\epsilon,0.5}$). For $\hat{\psi}_{\epsilon,0.5}$, the bias increases as values of ϵ increase and $\hat{\psi}_{\epsilon,0.5}$ tends to underestimate the true values of ψ especially when $\epsilon = 0.25, 0.5$. As n increases, the bias usually decreases but there are few cases that it increases when sample size changes from 5 to 10 or from 10 to 15. It could be caused by increased number of possible outcomes in simulations. When $|\psi|$ is large ($\geq$ 1.3499), applying bias correction term can reduce the bias in most cases for $n \leq 15$.

Trends observed in bias (Tables 4.1 and 4.2) are interesting. For $\hat{\psi}_\epsilon$, bias increases with the value of ϵ sometimes it changes from negative to positive. Compared with bias with correction used, it often becomes closer to 0. Also, there is no optimal selections of estimator or ϵ, none of them works best in all cases. Because of these competing trends we decided to focus more on the AAE and MSE in assessment.

Table 4.3 shows comparisons of AAE for two estimators $\hat{\psi}_\epsilon$ and $\hat{\psi}^*_\epsilon$. For small

$|\psi|$, AAE decreases monotonically as the values of ϵ increases which means that using larger values of ϵ leads to a smaller AAE in these combinations of p_1 and p_2. When $|\psi|$ is larger than 2.6027, we start to observe an increasing trend for ϵ. With n increasing, there is usually a decreasing trend in AAE but there are few cases when bias increased as sample size changes from 5 to 10 or from 10 to 15 like $(p_1, p_2) = (0.1, 0.1)$. This is caused by an increase in variance and more likely to happen when both p_1 and p_2 are either close to 0 or 1. When $|\psi|$ is large, adding ϵ only to tables containing zero counts can lower the AAE for $n \leq 15$. In cases with either $|\psi|$ is small or $n = 30, 50$, $\hat{\psi}_\epsilon$ has smaller AAE than $\hat{\psi}^*_\epsilon$.

Table 4.4 shows comparisons of AAE of two estimators $\hat{\psi}_\epsilon$ and $\hat{\psi}_{\epsilon,0.5}$. In general, $\hat{\psi}_{\epsilon,0.5}$ has similar trends in AAE as ϵ increases as estimators $\hat{\psi}_\epsilon$ and $\hat{\psi}^*_\epsilon$. We start to observe an increasing trend when ϵ is greater than 1 with $n = 5$ for $|\psi| > 0.8473$. As n increases, there is usually a decreasing trend on AAE but there are a few cases when AAE increases when sample size changes from 5 to 10 as well. It is caused by an increasing trend in variance. When $|\psi|$ is really large, the correction term reduces the AAE but the thresholds value of $|\psi|$ is higher than for $\hat{\psi}^*_{\epsilon,0.5}$.

Table 4.5 shows comparisons and trends of MSE of $\hat{\psi}_\epsilon$ and $\hat{\psi}^*_\epsilon$. Similar trends as for AAE were observed as ϵ and n changed. MSE of these two estimators in general decreases with increasing ϵ and decreases with sample size n. In few cases, MSE increased when sample size changed from 5 to 10. When $n = 5$, the number of posible values is too limited and variances in some cases increase as n increase. When $|\psi|$ is large, by adding ϵ only to tables containing zero counts produces smaller MSE. Compared with AAE, the thresholds value of $|\psi|$ for which $\hat{\psi}^*_\epsilon$ has smaller MSE is higher.

Table 4.6 shows comparisons and trends of MSE of $\hat{\psi}_\epsilon$ and $\hat{\psi}_{\epsilon,0.5}$. Trends of MSE are similar, MSEs for all three estimators in general decrease with increasing ϵ

and decreases with sample size n. When $|\psi|$ is small, applying the correction term will not help in reducing MSE. When $|\psi|$ is large, applying correction lowers the MSE in some case compared with $\hat{\psi}_\epsilon$ but not as much as with $\hat{\psi}^*_\epsilon$.

Table 4.2: Bias of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	-0.0275	-0.0216	-0.0183	-0.0161	-0.0144	-0.0293	-0.0252	-0.0237	-0.0232	-0.0233
$p_1 = 0.1$	n=10	-0.0024	-0.0014	-0.0010	-0.0008	-0.0006	-0.0024	-0.0015	-0.0010	-0.0008	-0.0006
$p_2 = 0.1$	n=15	-0.0117	-0.0089	-0.0074	-0.0063	-0.0056	-0.0117	-0.0090	-0.0076	-0.0067	-0.0060
$\|\psi\| = 0$	n=30	0.0147	0.0099	0.0076	0.0061	0.0052	0.0147	0.0100	0.0076	0.0062	0.0052
	n=50	-0.0035	-0.0036	-0.0035	-0.0034	-0.0032	-0.0035	-0.0036	-0.0035	-0.0034	-0.0033
	n=5	-0.0323	-0.0241	-0.0198	-0.0171	-0.0151	-0.0341	-0.0276	-0.0251	-0.0241	-0.0239
$p_1 = 0.2$	n=10	-0.0036	-0.0020	-0.0013	-0.0008	-0.0006	-0.0036	-0.0019	-0.0012	-0.0008	-0.0005
$p_2 = 0.2$	n=15	-0.0067	-0.0066	-0.0063	-0.0060	-0.0058	-0.0068	-0.0069	-0.0068	-0.0067	-0.0066
$\|\psi\| = 0$	n=30	0.0059	0.0057	0.0054	0.0051	0.0048	0.0060	0.0058	0.0055	0.0053	0.0050
	n=50	-0.0011	-0.0011	-0.0011	-0.0011	-0.0011	-0.0011	-0.0011	-0.0011	-0.0011	-0.0011
	n=5	-0.0266	-0.0224	-0.0197	-0.0178	-0.0162	-0.0287	-0.0267	-0.0262	-0.0265	-0.0271
$p_1 = 0.3$	n=10	0.0022	0.0019	0.0018	0.0017	0.0016	0.0022	0.0021	0.0021	0.0020	0.0020
$p_2 = 0.3$	n=15	-0.0034	-0.0030	-0.0028	-0.0026	-0.0025	-0.0035	-0.0032	-0.0030	-0.0030	-0.0029
$\|\psi\| = 0$	n=30	0.0049	0.0047	0.0045	0.0043	0.0041	0.0049	0.0048	0.0046	0.0045	0.0044
	n=50	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003	-0.0003
	n=5	-0.0303	-0.0256	-0.0226	-0.0203	-0.0186	-0.0328	-0.0306	-0.0300	-0.0302	-0.0309
$p_1 = 0.4$	n=10	-0.0022	-0.0022	-0.0021	-0.0020	-0.0019	-0.0023	-0.0024	-0.0024	-0.0024	-0.0024
$p_2 = 0.4$	n=15	-0.0067	-0.0064	-0.0062	-0.0060	-0.0057	-0.0069	-0.0068	-0.0068	-0.0068	-0.0068
$\|\psi\| = 0$	n=30	0.0053	0.0052	0.0050	0.0049	0.0048	0.0054	0.0053	0.0053	0.0052	0.0052
	n=50	-0.0012	-0.0012	-0.0012	-0.0011	-0.0011	-0.0012	-0.0012	-0.0012	-0.0012	-0.0012
	n=5	-0.0472	-0.0389	-0.0339	-0.0303	-0.0276	-0.0508	-0.0462	-0.0448	-0.0449	-0.0457
$p_1 = 0.5$	n=10	-0.0016	-0.0016	-0.0015	-0.0015	-0.0014	-0.0017	-0.0017	-0.0017	-0.0017	-0.0018
$p_2 = 0.5$	n=15	-0.0025	-0.0025	-0.0025	-0.0025	-0.0025	-0.0026	-0.0028	-0.0029	-0.0030	-0.0031
$\|\psi\| = 0$	n=30	0.0052	0.0051	0.0050	0.0049	0.0048	0.0053	0.0053	0.0052	0.0052	0.0052
	n=50	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008
	n=5	-0.1117	-0.0338	0.0166	0.0542	0.0840	-0.1554	-0.1211	-0.1144	-0.1204	-0.1343
$p_1 = 0.4$	n=10	-0.0304	-0.0028	0.0202	0.0402	0.0579	-0.0505	-0.0429	-0.0399	-0.0400	-0.0423
$p_2 = 0.5$	n=15	-0.0206	-0.0045	0.0102	0.0238	0.0364	-0.0341	-0.0315	-0.0302	-0.0301	-0.0310
$\|\psi\| = 0.4055$	n=30	-0.0025	0.0048	0.0118	0.0185	0.0250	-0.0091	-0.0084	-0.0080	-0.0079	-0.0080
	n=50	-0.0052	-0.0009	0.0033	0.0075	0.0115	-0.0092	-0.0089	-0.0087	-0.0085	-0.0085
	n=5	-0.0549	0.0144	0.0593	0.0928	0.1193	-0.0938	-0.0633	-0.0572	-0.0626	-0.0748
$p_1 = 0.5$	n=10	-0.0360	-0.0082	0.0150	0.0351	0.0529	-0.0564	-0.0490	-0.0463	-0.0466	-0.0492
$p_2 = 0.6$	n=15	-0.0113	0.0044	0.0187	0.0320	0.0442	-0.0245	-0.0220	-0.0209	-0.0208	-0.0218
$\|\psi\| = 0.4055$	n=30	-0.0059	0.0015	0.0086	0.0154	0.0219	-0.0125	-0.0118	-0.0114	-0.0112	-0.0113
	n=50	-0.0047	-0.0004	0.0038	0.0080	0.0120	-0.0087	-0.0084	-0.0082	-0.0080	-0.0080
	n=5	-0.0861	0.0016	0.0560	0.0955	0.1262	-0.1281	-0.0823	-0.0699	-0.0724	-0.0836
$p_1 = 0.3$	n=10	-0.0410	-0.0007	0.0298	0.0549	0.0763	-0.0610	-0.0407	-0.0302	-0.0251	-0.0237
$p_2 = 0.4$	n=15	-0.0296	-0.0062	0.0137	0.0314	0.0473	-0.0431	-0.0332	-0.0267	-0.0225	-0.0201
$\|\psi\| = 0.4418$	n=30	-0.0052	0.0044	0.0135	0.0222	0.0305	-0.0118	-0.0089	-0.0064	-0.0043	-0.0026
	n=50	-0.0070	-0.0013	0.0042	0.0095	0.0147	-0.0110	-0.0093	-0.0078	-0.0065	-0.0053
	n=5	-0.0536	0.0618	0.1286	0.1751	0.2102	-0.0956	-0.0221	0.0027	0.0072	0.0004
$p_1 = 0.2$	n=10	-0.0648	0.0137	0.0645	0.1028	0.1334	-0.0847	-0.0261	0.0049	0.0233	0.0341
$p_2 = 0.3$	n=15	-0.0571	-0.0068	0.0300	0.0598	0.0849	-0.0706	-0.0338	-0.0105	0.0057	0.0174
$\|\psi\| = 0.5390$	n=30	-0.0138	0.0047	0.0213	0.0365	0.0506	-0.0204	-0.0085	0.0015	0.0100	0.0175
	n=50	-0.0111	-0.0009	0.0088	0.0180	0.0269	-0.0151	-0.0089	-0.0032	0.0020	0.0069

Table 4.2: Bias of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$) (cont.)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	0.0977	0.2642	0.3539	0.4132	0.4565	0.0555	0.1796	0.2271	0.2442	0.2452
$p_1 = 0.1$	n=10	-0.0527	0.1169	0.2130	0.2789	0.3283	-0.0727	0.0769	0.1529	0.1988	0.2282
$p_2 = 0.2$	n=15	-0.0945	0.0452	0.1307	0.1925	0.2406	-0.1080	0.0182	0.0902	0.1386	0.1733
$\|\psi\| = 0.8109$	n=30	-0.0572	0.0088	0.0586	0.0994	0.1344	-0.0639	-0.0045	0.0386	0.0728	0.1011
	n=50	-0.0395	-0.0053	0.0244	0.0509	0.0750	-0.0435	-0.0133	0.0124	0.0349	0.0550
	n=5	-0.1195	0.0195	0.1098	0.1773	0.2309	-0.1983	-0.1382	-0.1268	-0.1381	-0.1634
$p_1 = 0.4$	n=10	-0.0648	-0.0094	0.0367	0.0768	0.1122	-0.1052	-0.0902	-0.0845	-0.0848	-0.0898
$p_2 = 0.6$	n=15	-0.0294	0.0024	0.0315	0.0583	0.0831	-0.0560	-0.0508	-0.0482	-0.0480	-0.0497
$\|\psi\| = 0.8109$	n=30	-0.0135	0.0012	0.0153	0.0290	0.0421	-0.0268	-0.0255	-0.0246	-0.0243	-0.0245
	n=50	-0.0092	-0.0005	0.0079	0.0162	0.0243	-0.0172	-0.0165	-0.0161	-0.0158	-0.0157
	n=5	-0.1675	-0.0066	0.0952	0.1700	0.2288	-0.2507	-0.1729	-0.1543	-0.1626	-0.1870
$p_1 = 0.3$	n=10	-0.0692	-0.0013	0.0521	0.0971	0.1361	-0.1092	-0.0812	-0.0677	-0.0627	-0.0636
$p_2 = 0.5$	n=15	-0.0435	-0.0043	0.0301	0.0612	0.0895	-0.0703	-0.0579	-0.0502	-0.0459	-0.0444
$\|\psi\| = 0.8473$	n=30	-0.0130	0.0040	0.0202	0.0358	0.0506	-0.0263	-0.0226	-0.0196	-0.0174	-0.0158
	n=50	-0.0110	-0.0010	0.0087	0.0182	0.0274	-0.0190	-0.0170	-0.0153	-0.0138	-0.0126
	n=5	-0.1145	0.0384	0.1350	0.2060	0.2618	-0.1931	-0.1189	-0.1009	-0.1085	-0.1313
$p_1 = 0.5$	n=10	-0.0792	-0.0108	0.0430	0.0883	0.1276	-0.1197	-0.0918	-0.0785	-0.0737	-0.0749
$p_2 = 0.7$	n=15	-0.0374	0.0011	0.0352	0.0660	0.0941	-0.0640	-0.0521	-0.0446	-0.0404	-0.0390
$\|\psi\| = 0.8473$	n=30	-0.0160	0.0012	0.0176	0.0333	0.0483	-0.0293	-0.0254	-0.0223	-0.0200	-0.0182
	n=50	-0.0114	-0.0014	0.0083	0.0178	0.0271	-0.0194	-0.0174	-0.0157	-0.0142	-0.0130
	n=5	-0.1132	0.0858	0.2043	0.2883	0.3527	-0.1949	-0.0777	-0.0410	-0.0388	-0.0561
$p_1 = 0.2$	n=10	-0.1080	0.0110	0.0925	0.1560	0.2081	-0.1479	-0.0689	-0.0273	-0.0039	0.0083
$p_2 = 0.4$	n=15	-0.0832	-0.0100	0.0465	0.0938	0.1348	-0.1101	-0.0638	-0.0342	-0.0139	0.0002
$\|\psi\| = 0.9808$	n=30	-0.0239	0.0044	0.0304	0.0544	0.0769	-0.0372	-0.0221	-0.0095	0.0013	0.0105
	n=50	-0.0177	-0.0019	0.0133	0.0279	0.0420	-0.0257	-0.0179	-0.0108	-0.0041	0.0019
	n=5	-0.1791	0.0435	0.1855	0.2905	0.3733	-0.2977	-0.1938	-0.1704	-0.1841	-0.2199
$p_1 = 0.4$	n=10	-0.1080	-0.0121	0.0647	0.1300	0.1869	-0.1685	-0.1330	-0.1167	-0.1119	-0.1155
$p_2 = 0.7$	n=15	-0.0556	-0.0009	0.0480	0.0923	0.1330	-0.0956	-0.0808	-0.0719	-0.0676	-0.0669
$\|\psi\| = 1.2528$	n=30	-0.0236	0.0009	0.0243	0.0468	0.0684	-0.0436	-0.0391	-0.0356	-0.0331	-0.0315
	n=50	-0.0158	-0.0015	0.0124	0.0261	0.0394	-0.0278	-0.0255	-0.0236	-0.0220	-0.0207
	n=5	-0.1753	0.0467	0.1884	0.2931	0.3757	-0.2937	-0.1900	-0.1667	-0.1803	-0.2161
$p_1 = 0.3$	n=10	-0.1036	-0.0080	0.0686	0.1336	0.1904	-0.1639	-0.1285	-0.1123	-0.1075	-0.1110
$p_2 = 0.6$	n=15	-0.0523	0.0026	0.0514	0.0957	0.1362	-0.0922	-0.0771	-0.0682	-0.0638	-0.0631
$\|\psi\| = 1.2528$	n=30	-0.0240	0.0004	0.0238	0.0462	0.0678	-0.0440	-0.0396	-0.0362	-0.0338	-0.0323
	n=50	-0.0149	-0.0006	0.0133	0.0269	0.0402	-0.0269	-0.0246	-0.0227	-0.0211	-0.0198
	n=5	0.0764	0.3500	0.5023	0.6054	0.6819	-0.0060	0.1851	0.2549	0.2755	0.2695
$p_1 = 0.1$	n=10	-0.1139	0.1326	0.2788	0.3825	0.4623	-0.1538	0.0527	0.1591	0.2229	0.2627
$p_2 = 0.3$	n=15	-0.1449	0.0449	0.1669	0.2583	0.3313	-0.1717	-0.0087	0.0865	0.1510	0.1973
$\|\psi\| = 1.3499$	n=30	-0.0770	0.0078	0.0745	0.1309	0.1801	-0.0903	-0.0188	0.0345	0.0776	0.1136
	n=50	-0.0495	-0.0051	0.0343	0.0701	0.1030	-0.0575	-0.0211	0.0102	0.0380	0.0630
	n=5	-0.1358	0.1243	0.2834	0.3981	0.4871	-0.2546	-0.1134	-0.0731	-0.0772	-0.1070
$p_1 = 0.5$	n=10	-0.1404	0.0048	0.1088	0.1919	0.2616	-0.2008	-0.1160	-0.0724	-0.0497	-0.0404
$p_2 = 0.8$	n=15	-0.0878	0.0009	0.0715	0.1318	0.1848	-0.1278	-0.0790	-0.0483	-0.0280	-0.0149
$\|\psi\| = 1.3863$	n=30	-0.0357	0.0002	0.0335	0.0647	0.0940	-0.0557	-0.0397	-0.0264	-0.0152	-0.0058
	n=50	-0.0214	-0.0012	0.0182	0.0369	0.0551	-0.0334	-0.0252	-0.0178	-0.0111	-0.0050

Table 4.2: Bias of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$) (cont.)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	-0.1946	0.0776	0.2435	0.3628	0.4552	-0.3175	-0.1683	-0.1254	-0.1290	-0.1595
$p_1 = 0.2$	n=10	-0.1362	0.0104	0.1149	0.1982	0.2679	-0.1961	-0.1094	-0.0649	-0.0414	-0.0316
$p_2 = 0.5$	n=15	-0.0971	-0.0081	0.0629	0.1236	0.1769	-0.1373	-0.0885	-0.0577	-0.0372	-0.0240
$\lvert\psi\rvert = 1.3863$	n=30	-0.0317	0.0040	0.0371	0.0680	0.0971	-0.0516	-0.0359	-0.0228	-0.0118	-0.0027
	n=50	-0.0218	-0.0016	0.0178	0.0365	0.0546	-0.0338	-0.0256	-0.0182	-0.0115	-0.0054
	n=5	-0.2349	0.0707	0.2641	0.4063	0.5181	-0.3930	-0.2456	-0.2103	-0.2263	-0.2726
$p_1 = 0.3$	n=10	-0.1468	-0.0106	0.0966	0.1868	0.2651	-0.2272	-0.1713	-0.1445	-0.1347	-0.1368
$p_2 = 0.7$	n=15	-0.0785	-0.0007	0.0679	0.1297	0.1860	-0.1317	-0.1072	-0.0919	-0.0833	-0.0803
$\lvert\psi\rvert = 1.6946$	n=30	-0.0341	0.0001	0.0328	0.0641	0.0941	-0.0608	-0.0532	-0.0472	-0.0426	-0.0392
	n=50	-0.0216	-0.0017	0.0178	0.0367	0.0552	-0.0376	-0.0337	-0.0302	-0.0273	-0.0248
	n=5	-0.2004	0.1293	0.3339	0.4826	0.5987	-0.3592	-0.1884	-0.1426	-0.1528	-0.1956
$p_1 = 0.4$	n=10	-0.1692	0.0036	0.1305	0.2336	0.3209	-0.2496	-0.1572	-0.1106	-0.0879	-0.0810
$p_2 = 0.8$	n=15	-0.1060	-0.0011	0.0843	0.1581	0.2237	-0.1593	-0.1077	-0.0757	-0.0551	-0.0429
$\lvert\psi\rvert = 1.7918$	n=30	-0.0434	-0.0001	0.0403	0.0783	0.1142	-0.0700	-0.0534	-0.0397	-0.0283	-0.0190
	n=50	-0.0258	-0.0014	0.0223	0.0452	0.0674	-0.0418	-0.0333	-0.0257	-0.0188	-0.0126
	n=5	0.0169	0.3740	0.5780	0.7186	0.8243	-0.1054	0.1295	0.2112	0.2295	0.2130
$p_1 = 0.1$	n=10	-0.1571	0.1299	0.3068	0.4357	0.5370	-0.2171	0.0099	0.1268	0.1957	0.2370
$p_2 = 0.4$	n=15	-0.1710	0.0417	0.1835	0.2923	0.3812	-0.2112	-0.0387	0.0628	0.1315	0.1801
$\lvert\psi\rvert = 1.7918$	n=30	-0.0871	0.0075	0.0835	0.1487	0.2064	-0.1070	-0.0324	0.0236	0.0689	0.1066
	n=50	-0.0562	-0.0061	0.0388	0.0799	0.1181	-0.0682	-0.0301	0.0027	0.0319	0.0580
	n=5	-0.2024	0.1309	0.3367	0.4859	0.6021	-0.3605	-0.1854	-0.1377	-0.1467	-0.1886
$p_1 = 0.2$	n=10	-0.1706	0.0038	0.1314	0.2347	0.3222	-0.2508	-0.1567	-0.1094	-0.0863	-0.0790
$p_2 = 0.6$	n=15	-0.1060	-0.0012	0.0841	0.1580	0.2236	-0.1592	-0.1078	-0.0757	-0.0551	-0.0427
$\lvert\psi\rvert = 1.7918$	n=30	-0.0427	0.0004	0.0406	0.0785	0.1142	-0.0694	-0.0529	-0.0394	-0.0282	-0.0191
	n=50	-0.0257	-0.0013	0.0224	0.0452	0.0674	-0.0417	-0.0333	-0.0256	-0.0187	-0.0126
	n=5	-0.0106	0.4101	0.6556	0.8275	0.9581	-0.1699	0.0914	0.1777	0.1902	0.1615
$p_1 = 0.5$	n=10	-0.1907	0.1232	0.3228	0.4716	0.5905	-0.2711	-0.0377	0.0816	0.1499	0.1885
$p_2 = 0.9$	n=15	-0.1707	0.0549	0.2095	0.3306	0.4310	-0.2240	-0.0517	0.0495	0.1173	0.1644
$\lvert\psi\rvert = 2.1972$	n=30	-0.1076	-0.0009	0.0845	0.1580	0.2232	-0.1343	-0.0542	0.0046	0.0515	0.0901
	n=50	-0.0574	-0.0029	0.0461	0.0912	0.1333	-0.0734	-0.0349	-0.0019	0.0272	0.0533
	n=5	-0.0645	0.3658	0.6172	0.7931	0.9269	-0.2280	0.0389	0.1268	0.1393	0.1096
$p_1 = 0.1$	n=10	-0.1853	0.1293	0.3291	0.4780	0.5968	-0.2652	-0.0306	0.0893	0.1581	0.1971
$p_2 = 0.5$	n=15	-0.1850	0.0436	0.1999	0.3221	0.4233	-0.2385	-0.0634	0.0394	0.1081	0.1559
$\lvert\psi\rvert = 2.1972$	n=30	-0.0948	0.0071	0.0902	0.1623	0.2266	-0.1215	-0.0461	0.0103	0.0558	0.0934
	n=50	-0.0602	-0.0058	0.0433	0.0885	0.1307	-0.0762	-0.0378	-0.0048	0.0245	0.0507
	n=5	-0.2619	0.1549	0.4124	0.5991	0.7446	-0.4599	-0.2410	-0.1814	-0.1926	-0.2451
$p_1 = 0.2$	n=10	-0.2138	0.0012	0.1594	0.2879	0.3969	-0.3141	-0.1995	-0.1417	-0.1134	-0.1048
$p_2 = 0.7$	n=15	-0.1321	-0.0044	0.1007	0.1921	0.2735	-0.1988	-0.1378	-0.0994	-0.0747	-0.0599
$\lvert\psi\rvert = 2.2336$	n=30	-0.0528	0.0001	0.0497	0.0963	0.1405	-0.0862	-0.0665	-0.0504	-0.0370	-0.0261
	n=50	-0.0324	-0.0023	0.0269	0.0551	0.0824	-0.0524	-0.0423	-0.0331	-0.0249	-0.0176
	n=5	-0.2562	0.1566	0.4125	0.5984	0.7435	-0.4545	-0.2401	-0.1826	-0.1950	-0.2483
$p_1 = 0.3$	n=10	-0.2080	0.0051	0.1624	0.2904	0.3991	-0.3083	-0.1955	-0.1384	-0.1106	-0.1022
$p_2 = 0.8$	n=15	-0.1289	-0.0009	0.1042	0.1955	0.2768	-0.1955	-0.1341	-0.0956	-0.0709	-0.0562
$\lvert\psi\rvert = 2.2336$	n=30	-0.0539	-0.0009	0.0487	0.0955	0.1399	-0.0872	-0.0676	-0.0513	-0.0378	-0.0268
	n=50	-0.0316	-0.0015	0.0276	0.0559	0.0832	-0.0516	-0.0415	-0.0323	-0.0241	-0.0168

Table 4.2: Bias of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$) (cont.)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	-0.0751	0.4151	0.7061	0.9120	1.0696	-0.2744	0.0164	0.1081	0.1146	0.0729
$p_1 = 0.4$	n=10	-0.2195	0.1219	0.3445	0.5133	0.6498	-0.3199	-0.0788	0.0434	0.1117	0.1479
$p_2 = 0.9$	n=15	-0.1888	0.0529	0.2223	0.3569	0.4699	-0.2555	-0.0805	0.0222	0.0901	0.1364
$\lvert\psi\rvert = 2.6027$	n=30	-0.1153	-0.0012	0.0912	0.1716	0.2434	-0.1486	-0.0678	-0.0087	0.0384	0.0769
	n=50	-0.0618	-0.0030	0.0502	0.0995	0.1456	-0.0818	-0.0430	-0.0098	0.0195	0.0456
	n=5	-0.0723	0.4191	0.7104	0.9162	1.0738	-0.2710	0.0218	0.1144	0.1216	0.0805
$p_1 = 0.1$	n=10	-0.2197	0.1227	0.3456	0.5145	0.6511	-0.3200	-0.0779	0.0448	0.1133	0.1496
$p_2 = 0.6$	n=15	-0.1938	0.0505	0.2211	0.3565	0.4701	-0.2604	-0.0827	0.0214	0.0902	0.1372
$\lvert\psi\rvert = 2.6027$	n=30	-0.1059	0.0035	0.0938	0.1728	0.2438	-0.1393	-0.0632	-0.0063	0.0394	0.0770
	n=50	-0.0641	-0.0054	0.0478	0.0973	0.1435	-0.0841	-0.0454	-0.0121	0.0173	0.0435
	n=5	-0.2832	0.2408	0.5608	0.7913	0.9699	-0.5214	-0.2355	-0.1536	-0.1613	-0.2208
$p_1 = 0.2$	n=10	-0.2750	0.0168	0.2252	0.3915	0.5309	-0.3952	-0.2236	-0.1355	-0.0894	-0.0702
$p_2 = 0.8$	n=15	-0.1825	-0.0047	0.1370	0.2579	0.3642	-0.2625	-0.1647	-0.1031	-0.0622	-0.0359
$\lvert\psi\rvert = 2.7726$	n=30	-0.0726	-0.0009	0.0656	0.1278	0.1863	-0.1126	-0.0809	-0.0544	-0.0322	-0.0137
	n=50	-0.0424	-0.0021	0.0367	0.0742	0.1104	-0.0663	-0.0501	-0.0353	-0.0218	-0.0095
	n=5	-0.1309	0.4424	0.7847	1.0278	1.2144	-0.3698	-0.0353	0.0682	0.0724	0.0202
$p_1 = 0.3$	n=10	-0.2583	0.1234	0.3764	0.5701	0.7280	-0.3786	-0.1172	0.0156	0.0890	0.1266
$p_2 = 0.9$	n=15	-0.2117	0.0531	0.2422	0.3943	0.5230	-0.2917	-0.1068	0.0023	0.0744	0.1231
$\lvert\psi\rvert = 3.0445$	n=30	-0.1258	-0.0020	0.0997	0.1888	0.2691	-0.1658	-0.0820	-0.0203	0.0289	0.0691
	n=50	-0.0676	-0.0031	0.0556	0.1102	0.1615	-0.0916	-0.0511	-0.0164	0.0142	0.0415
	n=5	-0.1319	0.4431	0.7861	1.0294	1.2162	-0.3703	-0.0338	0.0708	0.0757	0.0240
$p_1 = 0.1$	n=10	-0.2629	0.1201	0.3736	0.5677	0.7258	-0.3832	-0.1207	0.0125	0.0862	0.1239
$p_2 = 0.7$	n=15	-0.2199	0.0473	0.2376	0.3906	0.5199	-0.2999	-0.1127	-0.0023	0.0707	0.1200
$\lvert\psi\rvert = 3.0445$	n=30	-0.1160	0.0033	0.1028	0.1907	0.2701	-0.1560	-0.0768	-0.0173	0.0306	0.0700
	n=50	-0.0708	-0.0064	0.0523	0.1071	0.1586	-0.0948	-0.0545	-0.0197	0.0111	0.0385
	n=5	-0.1580	0.5266	0.9330	1.2206	1.4409	-0.4366	-0.0307	0.0971	0.1060	0.0477
$p_1 = 0.2$	n=10	-0.3253	0.1352	0.4392	0.6712	0.8598	-0.4655	-0.1453	0.0184	0.1102	0.1586
$p_2 = 0.9$	n=15	-0.2653	0.0493	0.2750	0.4567	0.6104	-0.3587	-0.1375	-0.0053	0.0831	0.1434
$\lvert\psi\rvert = 3.5835$	n=30	-0.1445	-0.0020	0.1166	0.2211	0.3155	-0.1912	-0.0953	-0.0234	0.0344	0.0822
	n=50	-0.0783	-0.0037	0.0646	0.1285	0.1887	-0.1063	-0.0597	-0.0193	0.0165	0.0487
	n=5	-0.1532	0.5290	0.9345	1.2216	1.4416	-0.4318	-0.0283	0.0985	0.1070	0.0483
$p_1 = 0.1$	n=10	-0.3241	0.1357	0.4394	0.6713	0.8598	-0.4644	-0.1448	0.0186	0.1102	0.1585
$p_2 = 0.8$	n=15	-0.2703	0.0470	0.2739	0.4564	0.6106	-0.3637	-0.1396	-0.0060	0.0831	0.1440
$\lvert\psi\rvert = 3.5835$	n=30	-0.1358	0.0022	0.1187	0.2221	0.3158	-0.1824	-0.0911	-0.0213	0.0354	0.0824
	n=50	-0.0808	-0.0063	0.0622	0.1262	0.1866	-0.1088	-0.0623	-0.0218	0.0142	0.0466
	n=5	-0.0279	0.8148	1.3067	1.6509	1.9125	-0.3471	0.1765	0.3493	0.3743	0.3168
$p_1 = 0.1$	n=10	-0.3744	0.2540	0.6535	0.9510	1.1887	-0.5346	-0.0665	0.1726	0.3098	0.3873
$p_2 = 0.9$	n=15	-0.3532	0.1011	0.4119	0.6552	0.8568	-0.4599	-0.1124	0.0918	0.2284	0.3233
$\lvert\psi\rvert = 4.3944$	n=30	-0.2077	0.0011	0.1697	0.3154	0.4450	-0.2610	-0.1056	0.0096	0.1020	0.1783
	n=50	-0.1168	-0.0079	0.0901	0.1805	0.2648	-0.1488	-0.0719	-0.0059	0.0525	0.1048

Table 4.3: AAE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	1.0180	0.7515	0.6138	0.5256	0.4629	1.0256	0.7649	0.6318	0.5473	0.4876
$p_1 = 0.1$	n=10	1.1414	0.8692	0.7240	0.6288	0.5600	1.1675	0.9151	0.7853	0.7027	0.6443
$p_2 = 0.1$	n=15	1.0386	0.8241	0.7035	0.6214	0.5604	1.0827	0.9016	0.8074	0.7470	0.7041
$\lvert\psi\rvert = 0$	n=30	0.7619	0.6707	0.6091	0.5620	0.5237	0.8156	0.7672	0.7411	0.7238	0.7112
	n=50	0.5780	0.5415	0.5121	0.4870	0.4651	0.6121	0.6046	0.6005	0.5977	0.5956
	n=5	1.2074	0.9376	0.7897	0.6906	0.6177	1.2407	0.9965	0.8689	0.7864	0.7273
$p_1 = 0.2$	n=10	0.9957	0.8388	0.7427	0.6733	0.6192	1.0547	0.9439	0.8851	0.8467	0.8190
$p_2 = 0.2$	n=15	0.8022	0.7154	0.6555	0.6090	0.5707	0.8568	0.8140	0.7908	0.7755	0.7642
$\lvert\psi\rvert = 0$	n=30	0.5296	0.5067	0.4867	0.4689	0.4527	0.5533	0.5515	0.5505	0.5498	0.5493
	n=50	0.4130	0.4034	0.3943	0.3858	0.3777	0.4231	0.4231	0.4231	0.4231	0.4231
	n=5	1.1862	0.9709	0.8424	0.7513	0.6814	1.2502	1.0843	0.9953	0.9368	0.8942
$p_1 = 0.3$	n=10	0.8455	0.7639	0.7048	0.6575	0.6179	0.9058	0.8735	0.8558	0.8440	0.8353
$p_2 = 0.3$	n=15	0.6616	0.6240	0.5925	0.5652	0.5409	0.7000	0.6954	0.6929	0.6911	0.6898
$\lvert\psi\rvert = 0$	n=30	0.4523	0.4411	0.4306	0.4206	0.4112	0.4642	0.4642	0.4642	0.4642	0.4642
	n=50	0.3565	0.3513	0.3464	0.3416	0.3369	0.3618	0.3618	0.3618	0.3618	0.3618
	n=5	1.1127	0.9474	0.8396	0.7590	0.6949	1.1984	1.0997	1.0455	1.0091	0.9824
$p_1 = 0.4$	n=10	0.7693	0.7186	0.6769	0.6410	0.6093	0.8196	0.8119	0.8077	0.8048	0.8026
$p_2 = 0.4$	n=15	0.6023	0.5785	0.5569	0.5371	0.5189	0.6282	0.6279	0.6278	0.6277	0.6276
$\lvert\psi\rvert = 0$	n=30	0.4207	0.4128	0.4053	0.3980	0.3911	0.4289	0.4289	0.4289	0.4289	0.4289
	n=50	0.3330	0.3293	0.3257	0.3222	0.3188	0.3368	0.3368	0.3368	0.3368	0.3368
	n=5	1.0938	0.9443	0.8429	0.7655	0.7031	1.1858	1.1082	1.0648	1.0353	1.0134
$p_1 = 0.5$	n=10	0.7497	0.7062	0.6689	0.6360	0.6065	0.7957	0.7923	0.7904	0.7890	0.7880
$p_2 = 0.5$	n=15	0.5893	0.5684	0.5491	0.5312	0.5144	0.6118	0.6117	0.6116	0.6115	0.6115
$\lvert\psi\rvert = 0$	n=30	0.4113	0.4042	0.3975	0.3909	0.3846	0.4186	0.4186	0.4186	0.4186	0.4186
	n=50	0.3253	0.3220	0.3188	0.3156	0.3125	0.3287	0.3287	0.3287	0.3287	0.3287
	n=5	1.1607	0.9937	0.8824	0.7982	0.7307	1.2540	1.1597	1.1071	1.0714	1.0448
$p_1 = 0.4$	n=10	0.7624	0.7167	0.6796	0.6473	0.6194	0.8105	0.8050	0.8019	0.7998	0.7982
$p_2 = 0.5$	n=15	0.6071	0.5843	0.5635	0.5443	0.5265	0.6318	0.6317	0.6316	0.6316	0.6316
$\lvert\psi\rvert = 0.4055$	n=30	0.4164	0.4092	0.4023	0.3959	0.3897	0.4242	0.4242	0.4242	0.4242	0.4242
	n=50	0.3297	0.3263	0.3231	0.3200	0.3169	0.3333	0.3333	0.3333	0.3333	0.3333
	n=5	1.1817	1.0152	0.9040	0.8198	0.7523	1.2749	1.1809	1.1283	1.0924	1.0657
$p_1 = 0.5$	n=10	0.7516	0.7061	0.6692	0.6372	0.6098	0.7990	0.7931	0.7897	0.7874	0.7856
$p_2 = 0.6$	n=15	0.6109	0.5880	0.5671	0.5478	0.5299	0.6357	0.6354	0.6352	0.6351	0.6351
$\lvert\psi\rvert = 0.4055$	n=30	0.4186	0.4114	0.4046	0.3981	0.3919	0.4263	0.4263	0.4263	0.4263	0.4263
	n=50	0.3278	0.3245	0.3212	0.3181	0.3151	0.3314	0.3314	0.3314	0.3314	0.3314
	n=5	1.2146	1.0143	0.8893	0.7982	0.7270	1.2926	1.1526	1.0762	1.0251	0.9875
$p_1 = 0.3$	n=10	0.8175	0.7509	0.7015	0.6625	0.6293	0.8726	0.8518	0.8402	0.8323	0.8264
$p_2 = 0.4$	n=15	0.6410	0.6098	0.5828	0.5589	0.5372	0.6739	0.6711	0.6696	0.6685	0.6677
$\lvert\psi\rvert = 0.4418$	n=30	0.4376	0.4281	0.4193	0.4110	0.4031	0.4477	0.4477	0.4477	0.4477	0.4477
	n=50	0.3467	0.3423	0.3381	0.3340	0.3301	0.3513	0.3513	0.3513	0.3513	0.3513
	n=5	1.2923	1.0289	0.8785	0.7749	0.6971	1.3420	1.1167	0.9967	0.9180	0.8610
$p_1 = 0.2$	n=10	0.9509	0.8269	0.7470	0.6908	0.6483	1.0089	0.9308	0.8883	0.8601	0.8394
$p_2 = 0.3$	n=15	0.7467	0.6826	0.6360	0.6000	0.5696	0.7947	0.7700	0.7562	0.7469	0.7400
$\lvert\psi\rvert = 0.5390$	n=30	0.4956	0.4783	0.4631	0.4497	0.4376	0.5134	0.5122	0.5115	0.5110	0.5107
	n=50	0.3873	0.3798	0.3728	0.3663	0.3602	0.3952	0.3952	0.3952	0.3952	0.3952

Table 4.3: AAE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts (cont.)

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	1.0938	0.9443	0.8429	0.7655	0.7031	1.1858	1.1082	1.0648	1.0353	1.0134
$p_1 = 0.1$	n=10	0.7497	0.7062	0.6689	0.6360	0.6065	0.7957	0.7923	0.7904	0.7890	0.7880
$p_2 = 0.2$	n=15	0.5893	0.5684	0.5491	0.5312	0.5144	0.6118	0.6117	0.6116	0.6115	0.6115
$\lvert\psi\rvert = 0.8109$	n=30	0.4113	0.4042	0.3975	0.3909	0.3846	0.4186	0.4186	0.4186	0.4186	0.4186
	n=50	0.3253	0.3220	0.3188	0.3156	0.3125	0.3287	0.3287	0.3287	0.3287	0.3287
	n=5	1.1592	0.9936	0.8964	0.8253	0.7706	1.2397	1.1243	1.0589	1.0140	0.9814
$p_1 = 0.4$	n=10	0.7631	0.7167	0.6791	0.6491	0.6228	0.8143	0.8059	0.8010	0.7975	0.7949
$p_2 = 0.6$	n=15	0.6129	0.5904	0.5715	0.5543	0.5389	0.6396	0.6393	0.6391	0.6390	0.6389
$\lvert\psi\rvert = 0.8109$	n=30	0.4232	0.4160	0.4093	0.4031	0.3971	0.4317	0.4317	0.4317	0.4317	0.4317
	n=50	0.3315	0.3281	0.3249	0.3219	0.3190	0.3354	0.3354	0.3354	0.3354	0.3354
	n=5	1.1769	0.9933	0.8835	0.8048	0.7492	1.2538	1.1074	1.0257	0.9702	0.9340
$p_1 = 0.3$	n=10	0.8085	0.7466	0.7031	0.6680	0.6380	0.8596	0.8403	0.8292	0.8215	0.8157
$p_2 = 0.5$	n=15	0.6311	0.6033	0.5793	0.5594	0.5420	0.6626	0.6599	0.6583	0.6572	0.6564
$\lvert\psi\rvert = 0.8473$	n=30	0.4330	0.4239	0.4159	0.4082	0.4014	0.4426	0.4426	0.4426	0.4426	0.4426
	n=50	0.3432	0.3390	0.3350	0.3313	0.3278	0.3476	0.3476	0.3476	0.3476	0.3476
	n=5	1.1855	1.0091	0.9033	0.8272	0.7724	1.2625	1.1227	1.0445	0.9912	0.9556
$p_1 = 0.5$	n=10	0.7980	0.7356	0.6920	0.6569	0.6270	0.8488	0.8288	0.8174	0.8094	0.8035
$p_2 = 0.7$	n=15	0.6396	0.6119	0.5879	0.5679	0.5502	0.6715	0.6692	0.6679	0.6669	0.6662
$\lvert\psi\rvert = 0.8473$	n=30	0.4373	0.4282	0.4200	0.4122	0.4053	0.4470	0.4470	0.4470	0.4470	0.4470
	n=50	0.3410	0.3368	0.3329	0.3292	0.3258	0.3455	0.3455	0.3455	0.3455	0.3455
	n=5	1.2536	1.0157	0.8816	0.8065	0.7559	1.2739	1.0521	0.9310	0.8668	0.8253
$p_1 = 0.2$	n=10	0.9195	0.8092	0.7374	0.6920	0.6565	0.9675	0.8957	0.8560	0.8301	0.8107
$p_2 = 0.4$	n=15	0.7173	0.6598	0.6190	0.5898	0.5664	0.7580	0.7343	0.7209	0.7116	0.7045
$\lvert\psi\rvert = 0.9808$	n=30	0.4793	0.4637	0.4503	0.4393	0.4296	0.4961	0.4948	0.4941	0.4937	0.4933
	n=50	0.3770	0.3704	0.3641	0.3588	0.3539	0.3844	0.3844	0.3844	0.3844	0.3844
	n=5	1.2351	1.0225	0.8964	0.8246	0.7884	1.2925	1.1245	1.0342	0.9725	0.9259
$p_1 = 0.4$	n=10	0.8085	0.7450	0.7020	0.6671	0.6396	0.8629	0.8398	0.8264	0.8169	0.8097
$p_2 = 0.7$	n=15	0.6509	0.6182	0.5900	0.5710	0.5566	0.6854	0.6830	0.6816	0.6806	0.6799
$\lvert\psi\rvert = 1.2528$	n=30	0.4416	0.4326	0.4244	0.4171	0.4102	0.4518	0.4518	0.4518	0.4518	0.4518
	n=50	0.3445	0.3403	0.3364	0.3328	0.3295	0.3493	0.3493	0.3493	0.3493	0.3493
	n=5	1.2420	1.0244	0.8978	0.8266	0.7915	1.2987	1.1252	1.0340	0.9719	0.9249
$p_1 = 0.3$	n=10	0.8067	0.7440	0.7020	0.6680	0.6407	0.8604	0.8376	0.8243	0.8150	0.8078
$p_2 = 0.6$	n=15	0.6483	0.6160	0.5883	0.5698	0.5560	0.6817	0.6788	0.6771	0.6759	0.6750
$\lvert\psi\rvert = 1.2528$	n=30	0.4398	0.4310	0.4231	0.4160	0.4094	0.4501	0.4501	0.4501	0.4501	0.4501
	n=50	0.3452	0.3410	0.3372	0.3335	0.3302	0.3499	0.3499	0.3499	0.3499	0.3499
	n=5	1.2585	0.9495	0.8832	0.8435	0.8172	1.2616	0.9552	0.8871	0.8389	0.8054
$p_1 = 0.1$	n=10	1.0877	0.8782	0.7770	0.7285	0.7149	1.0882	0.8785	0.7776	0.7181	0.6932
$p_2 = 0.3$	n=15	0.9127	0.7642	0.6947	0.6530	0.6278	0.9427	0.8115	0.7401	0.6963	0.6660
$\lvert\psi\rvert = 1.3499$	n=30	0.6230	0.5714	0.5366	0.5154	0.5008	0.6613	0.6367	0.6227	0.6130	0.6057
	n=50	0.4820	0.4606	0.4441	0.4318	0.4222	0.5045	0.5008	0.4987	0.4972	0.4961
	n=5	1.2588	1.0065	0.9014	0.8417	0.8071	1.2986	1.0774	0.9724	0.8997	0.8542
$p_1 = 0.5$	n=10	0.8956	0.7906	0.7245	0.6846	0.6584	0.9377	0.8664	0.8259	0.7986	0.7787
$p_2 = 0.8$	n=15	0.7174	0.6636	0.6293	0.6033	0.5848	0.7605	0.7358	0.7215	0.7116	0.7040
$\lvert\psi\rvert = 1.3863$	n=30	0.4817	0.4658	0.4535	0.4435	0.4353	0.4991	0.4985	0.4981	0.4978	0.4976
	n=50	0.3721	0.3655	0.3600	0.3553	0.3514	0.3797	0.3797	0.3797	0.3797	0.3797

Table 4.3: AAE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts (cont.)

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	1.2441	0.9792	0.8735	0.8149	0.7811	1.2861	1.0540	0.9468	0.8725	0.8253
$p_1 = 0.2$	n=10	0.9100	0.8035	0.7368	0.6958	0.6683	0.9523	0.8798	0.8391	0.8115	0.7914
$p_2 = 0.5$	n=15	0.7105	0.6558	0.6203	0.5930	0.5737	0.7548	0.7306	0.7167	0.7070	0.6996
$\|\psi\| = 1.3863$	n=30	0.4750	0.4591	0.4471	0.4374	0.4292	0.4917	0.4905	0.4897	0.4892	0.4889
	n=50	0.3737	0.3672	0.3616	0.3567	0.3526	0.3811	0.3811	0.3811	0.3811	0.3811
	n=5	1.2582	1.0490	0.9334	0.8702	0.8249	1.2834	1.0593	0.9314	0.8578	0.8036
$p_1 = 0.3$	n=10	0.8460	0.7731	0.7222	0.6865	0.6584	0.9102	0.8716	0.8488	0.8325	0.8201
$p_2 = 0.7$	n=15	0.6774	0.6434	0.6175	0.5962	0.5784	0.7179	0.7128	0.7097	0.7075	0.7058
$\|\psi\| = 1.6946$	n=30	0.4564	0.4464	0.4376	0.4300	0.4229	0.4695	0.4695	0.4695	0.4695	0.4695
	n=50	0.3571	0.3524	0.3483	0.3446	0.3410	0.3630	0.3630	0.3630	0.3630	0.3630
	n=5	1.2664	1.0357	0.9171	0.8637	0.8453	1.2478	1.0024	0.8717	0.8083	0.7596
$p_1 = 0.4$	n=10	0.9067	0.7972	0.7349	0.6950	0.6790	0.9483	0.8715	0.8272	0.7969	0.7745
$p_2 = 0.8$	n=15	0.7298	0.6721	0.6349	0.6102	0.5972	0.7697	0.7444	0.7296	0.7191	0.7111
$\|\psi\| = 1.7918$	n=30	0.4865	0.4697	0.4580	0.4486	0.4426	0.5047	0.5040	0.5036	0.5033	0.5031
	n=50	0.3753	0.3687	0.3636	0.3591	0.3560	0.3835	0.3835	0.3835	0.3835	0.3835
	n=5	1.1745	0.9745	0.8946	0.9157	0.9354	1.1527	0.9359	0.8426	0.8527	0.8579
$p_1 = 0.1$	n=10	1.0584	0.8541	0.7750	0.7453	0.7431	1.0452	0.8308	0.7327	0.6804	0.6576
$p_2 = 0.4$	n=15	0.8867	0.7480	0.6793	0.6523	0.6389	0.9099	0.7744	0.7000	0.6542	0.6239
$\|\psi\| = 1.7918$	n=30	0.6100	0.5577	0.5285	0.5097	0.4996	0.6471	0.6221	0.6077	0.5977	0.5901
	n=50	0.4736	0.4523	0.4375	0.4264	0.4191	0.4960	0.4923	0.4901	0.4886	0.4875
	n=5	1.2684	1.0395	0.9227	0.8707	0.8510	1.2458	0.9990	0.8675	0.8032	0.7539
$p_1 = 0.2$	n=10	0.9087	0.7990	0.7364	0.6957	0.6794	0.9477	0.8687	0.8233	0.7921	0.7690
$p_2 = 0.6$	n=15	0.7231	0.6651	0.6280	0.6039	0.5904	0.7645	0.7397	0.7253	0.7153	0.7076
$\|\psi\| = 1.7918$	n=30	0.4831	0.4662	0.4549	0.4460	0.4399	0.5009	0.4996	0.4989	0.4984	0.4980
	n=50	0.3752	0.3686	0.3634	0.3590	0.3559	0.3832	0.3832	0.3832	0.3832	0.3832
	n=5	1.2100	0.9745	0.9583	0.9891	1.0523	1.1901	0.9391	0.8902	0.8940	0.9345
$p_1 = 0.5$	n=10	1.0423	0.8375	0.7705	0.7553	0.7621	1.0223	0.8000	0.6982	0.6511	0.6268
$p_2 = 0.9$	n=15	0.8896	0.7565	0.6973	0.6735	0.6710	0.9062	0.7697	0.6936	0.6439	0.6105
$\|\psi\| = 2.1972$	n=30	0.6220	0.5662	0.5366	0.5184	0.5107	0.6573	0.6260	0.6078	0.5950	0.5852
	n=50	0.4671	0.4459	0.4315	0.4221	0.4164	0.4893	0.4849	0.4823	0.4805	0.4791
	n=5	1.1871	0.9454	0.9208	0.9520	1.0211	1.1688	0.9128	0.8539	0.8568	0.9022
$p_1 = 0.1$	n=10	1.0531	0.8490	0.7786	0.7619	0.7685	1.0329	0.8115	0.7071	0.6595	0.6359
$p_2 = 0.5$	n=15	0.8851	0.7471	0.6846	0.6598	0.6599	0.9046	0.7655	0.6881	0.6383	0.6078
$\|\psi\| = 2.1972$	n=30	0.6067	0.5547	0.5268	0.5098	0.5036	0.6448	0.6196	0.6049	0.5946	0.5868
	n=50	0.4707	0.4498	0.4355	0.4259	0.4199	0.4933	0.4896	0.4874	0.4859	0.4847
	n=5	1.3235	1.0291	0.9343	0.9179	0.9328	1.3161	1.0051	0.8484	0.7809	0.7527
$p_1 = 0.2$	n=10	0.9492	0.8226	0.7594	0.7207	0.7138	0.9909	0.8935	0.8363	0.7970	0.7677
$p_2 = 0.7$	n=15	0.7532	0.6872	0.6497	0.6324	0.6200	0.8083	0.7805	0.7640	0.7523	0.7433
$\|\psi\| = 2.2336$	n=30	0.4987	0.4821	0.4685	0.4603	0.4567	0.5187	0.5174	0.5167	0.5162	0.5157
	n=50	0.3872	0.3799	0.3738	0.3703	0.3681	0.3956	0.3956	0.3956	0.3956	0.3956
	n=5	1.3235	1.0296	0.9372	0.9203	0.9337	1.3188	1.0097	0.8535	0.7839	0.7530
$p_1 = 0.3$	n=10	0.9451	0.8205	0.7585	0.7211	0.7150	0.9874	0.8924	0.8364	0.7978	0.7693
$p_2 = 0.8$	n=15	0.7559	0.6913	0.6553	0.6392	0.6277	0.8081	0.7791	0.7619	0.7496	0.7401
$\|\psi\| = 2.2336$	n=30	0.5011	0.4844	0.4703	0.4621	0.4586	0.5213	0.5206	0.5202	0.5199	0.5196
	n=50	0.3881	0.3805	0.3744	0.3707	0.3685	0.3967	0.3967	0.3967	0.3967	0.3967

Table 4.3: AAE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts (cont.)

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	1.1883	0.9867	0.9941	1.0391	1.1261	1.1511	0.8985	0.8649	0.8762	0.9348
$p_1 = 0.4$	n=10	1.0538	0.8425	0.7804	0.7774	0.8039	1.0328	0.7983	0.6895	0.6455	0.6313
$p_2 = 0.9$	n=15	0.8965	0.7609	0.7074	0.6857	0.6923	0.9144	0.7747	0.6956	0.6439	0.6108
$\lvert\psi\rvert = 2.6027$	n=30	0.6260	0.5701	0.5410	0.5256	0.5211	0.6620	0.6303	0.6118	0.5987	0.5886
	n=50	0.4700	0.4487	0.4346	0.4263	0.4226	0.4924	0.4880	0.4854	0.4836	0.4822
	n=5	1.1884	0.9871	0.9983	1.0476	1.1344	1.1511	0.8987	0.8689	0.8844	0.9428
$p_1 = 0.1$	n=10	1.0494	0.8407	0.7796	0.7762	0.8027	1.0285	0.7947	0.6854	0.6395	0.6244
$p_2 = 0.6$	n=15	0.8965	0.7557	0.7000	0.6797	0.6894	0.9188	0.7767	0.6970	0.6479	0.6196
$\lvert\psi\rvert = 2.6027$	n=30	0.6140	0.5625	0.5342	0.5199	0.5163	0.6521	0.6266	0.6118	0.6012	0.5931
	n=50	0.4722	0.4511	0.4372	0.4290	0.4254	0.4953	0.4916	0.4894	0.4878	0.4866
	n=5	1.3580	1.0424	0.9684	0.9580	1.0617	1.2593	0.8672	0.7317	0.6706	0.7316
$p_1 = 0.2$	n=10	1.0295	0.8781	0.7988	0.7692	0.7861	1.0693	0.9114	0.8189	0.7558	0.7136
$p_2 = 0.8$	n=15	0.8260	0.7451	0.6932	0.6731	0.6732	0.8845	0.8312	0.7990	0.7759	0.7581
$\lvert\psi\rvert = 2.7726$	n=30	0.5408	0.5189	0.5024	0.4931	0.4931	0.5683	0.5662	0.5650	0.5642	0.5635
	n=50	0.4155	0.4070	0.3999	0.3956	0.3952	0.4272	0.4272	0.4272	0.4272	0.4272
	n=5	1.2580	1.0272	0.9897	1.1352	1.2492	1.1855	0.8985	0.8160	0.9242	1.0070
$p_1 = 0.3$	n=10	1.0946	0.8599	0.8034	0.8137	0.8614	1.0641	0.8033	0.6859	0.6385	0.6343
$p_2 = 0.9$	n=15	0.9210	0.7856	0.7265	0.7165	0.7276	0.9425	0.7954	0.7113	0.6573	0.6244
$\lvert\psi\rvert = 3.0445$	n=30	0.6392	0.5823	0.5538	0.5393	0.5400	0.6767	0.6446	0.6257	0.6122	0.6017
	n=50	0.4803	0.4584	0.4452	0.4369	0.4356	0.5039	0.4994	0.4968	0.4949	0.4935
	n=5	1.2466	1.0215	0.9895	1.1359	1.2507	1.1743	0.8934	0.8165	0.9260	1.0097
$p_1 = 0.1$	n=10	1.0919	0.8589	0.8011	0.8096	0.8581	1.0631	0.8052	0.6859	0.6359	0.6319
$p_2 = 0.7$	n=15	0.9259	0.7836	0.7197	0.7100	0.7231	0.9518	0.8027	0.7186	0.6678	0.6395
$\lvert\psi\rvert = 3.0445$	n=30	0.6276	0.5752	0.5479	0.5340	0.5355	0.6673	0.6415	0.6263	0.6154	0.6071
	n=50	0.4820	0.4604	0.4475	0.4394	0.4382	0.5063	0.5026	0.5003	0.4987	0.4975
	n=5	1.2640	0.9853	1.1185	1.2206	1.4409	1.1957	0.8641	0.9550	1.0221	1.2130
$p_1 = 0.2$	n=10	1.1789	0.9134	0.8517	0.8830	0.9599	1.1178	0.8006	0.6629	0.6093	0.6121
$p_2 = 0.9$	n=15	0.9871	0.8255	0.7709	0.7549	0.7934	1.0131	0.8362	0.7341	0.6690	0.6328
$\lvert\psi\rvert = 3.5835$	n=30	0.6765	0.6116	0.5850	0.5732	0.5771	0.7205	0.6862	0.6657	0.6509	0.6393
	n=50	0.5045	0.4811	0.4682	0.4610	0.4622	0.5313	0.5267	0.5241	0.5221	0.5206
	n=5	1.2602	0.9780	1.1161	1.2216	1.4416	1.1903	0.8539	0.9486	1.0183	1.2082
$p_1 = 0.1$	n=10	1.1760	0.9138	0.8495	0.8790	0.9573	1.1133	0.7980	0.6573	0.6019	0.6061
$p_2 = 0.8$	n=15	0.9964	0.8292	0.7706	0.7562	0.7963	1.0235	0.8428	0.7396	0.6788	0.6467
$\lvert\psi\rvert = 3.5835$	n=30	0.6660	0.6056	0.5807	0.5697	0.5742	0.7134	0.6863	0.6700	0.6583	0.6492
	n=50	0.5073	0.4838	0.4710	0.4636	0.4651	0.5351	0.5312	0.5289	0.5273	0.5260
	n=5	1.1325	1.0896	1.3067	1.6509	1.9125	1.0846	1.0046	1.1920	1.5118	1.7529
$p_1 = 0.1$	n=10	1.2773	0.9468	0.8884	1.0523	1.1887	1.1546	0.7248	0.5833	0.6762	0.7509
$p_2 = 0.9$	n=15	1.1545	0.9092	0.8379	0.8683	0.9453	1.1246	0.8164	0.6582	0.5831	0.5670
$\lvert\psi\rvert = 4.3944$	n=30	0.7884	0.6981	0.6542	0.6492	0.6657	0.8484	0.7858	0.7476	0.7198	0.6986
	n=50	0.5885	0.5517	0.5317	0.5280	0.5312	0.6289	0.6202	0.6149	0.6110	0.6079

Table 4.4: AAE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	1.0180	0.7515	0.6138	0.5256	0.4629	1.0699	0.8552	0.7694	0.7330	0.7222
$p_1 = 0.1$	n=10	1.1414	0.8692	0.7240	0.6288	0.5600	1.1618	0.9100	0.7852	0.7104	0.6620
$p_2 = 0.1$	n=15	1.0386	0.8241	0.7035	0.6214	0.5604	1.0497	0.8461	0.7365	0.6655	0.6155
$\|\psi\| = 0$	n=30	0.7619	0.6707	0.6091	0.5620	0.5237	0.7659	0.6787	0.6211	0.5780	0.5437
	n=50	0.5780	0.5415	0.5121	0.4870	0.4651	0.5800	0.5454	0.5179	0.4947	0.4746
	n=5	1.2074	0.9376	0.7897	0.6906	0.6177	1.2824	1.0877	1.0148	0.9908	0.9929
$p_1 = 0.2$	n=10	0.9957	0.8388	0.7427	0.6733	0.6192	1.0237	0.8948	0.8267	0.7853	0.7592
$p_2 = 0.2$	n=15	0.8022	0.7154	0.6555	0.6090	0.5707	0.8174	0.7458	0.7012	0.6699	0.6469
$\|\psi\| = 0$	n=30	0.5296	0.5067	0.4867	0.4689	0.4527	0.5350	0.5174	0.5029	0.4904	0.4796
	n=50	0.4130	0.4034	0.3943	0.3858	0.3777	0.4155	0.4085	0.4021	0.3961	0.3905
	n=5	1.1862	0.9709	0.8424	0.7513	0.6814	1.2749	1.1482	1.1084	1.1059	1.1246
$p_1 = 0.3$	n=10	0.8455	0.7639	0.7048	0.6575	0.6179	0.8779	0.8287	0.8021	0.7873	0.7801
$p_2 = 0.3$	n=15	0.6616	0.6240	0.5925	0.5652	0.5409	0.6791	0.6591	0.6452	0.6354	0.6287
$\|\psi\| = 0$	n=30	0.4523	0.4411	0.4306	0.4206	0.4112	0.4585	0.4535	0.4492	0.4454	0.4422
	n=50	0.3565	0.3513	0.3464	0.3416	0.3369	0.3594	0.3573	0.3552	0.3534	0.3517
	n=5	1.1127	0.9474	0.8396	0.7590	0.6949	1.2072	1.1364	1.1231	1.1369	1.1673
$p_1 = 0.4$	n=10	0.7693	0.7186	0.6769	0.6410	0.6093	0.8042	0.7884	0.7816	0.7805	0.7837
$p_2 = 0.4$	n=15	0.6023	0.5785	0.5569	0.5371	0.5189	0.6209	0.6159	0.6130	0.6119	0.6123
$\|\psi\| = 0$	n=30	0.4207	0.4128	0.4053	0.3980	0.3911	0.4273	0.4261	0.4252	0.4246	0.4243
	n=50	0.3330	0.3293	0.3257	0.3222	0.3188	0.3362	0.3356	0.3352	0.3349	0.3346
	n=5	1.0938	0.9443	0.8429	0.7655	0.7031	1.1908	1.1383	1.1340	1.1536	1.1883
$p_1 = 0.5$	n=10	0.7497	0.7062	0.6689	0.6360	0.6065	0.7853	0.7775	0.7758	0.7785	0.7847
$p_2 = 0.5$	n=15	0.5893	0.5684	0.5491	0.5312	0.5144	0.6084	0.6067	0.6065	0.6077	0.6102
$\|\psi\| = 0$	n=30	0.4113	0.4042	0.3975	0.3909	0.3846	0.4180	0.4178	0.4178	0.4180	0.4185
	n=50	0.3253	0.3220	0.3188	0.3156	0.3125	0.3286	0.3285	0.3285	0.3286	0.3287
	n=5	1.1607	0.9937	0.8824	0.7982	0.7307	1.2635	1.1993	1.1908	1.2093	1.2446
$p_1 = 0.4$	n=10	0.7624	0.7167	0.6796	0.6473	0.6194	0.7970	0.7852	0.7813	0.7827	0.7883
$p_2 = 0.5$	n=15	0.6071	0.5843	0.5635	0.5443	0.5265	0.6264	0.6230	0.6215	0.6216	0.6231
$\|\psi\| = 0.4055$	n=30	0.4164	0.4092	0.4023	0.3959	0.3897	0.4231	0.4224	0.4219	0.4217	0.4218
	n=50	0.3297	0.3263	0.3231	0.3200	0.3169	0.3329	0.3326	0.3323	0.3322	0.3321
	n=5	1.1817	1.0152	0.9040	0.8198	0.7523	1.2846	1.2209	1.2126	1.2312	1.2666
$p_1 = 0.5$	n=10	0.7516	0.7061	0.6692	0.6372	0.6098	0.7859	0.7739	0.7699	0.7712	0.7767
$p_2 = 0.6$	n=15	0.6109	0.5880	0.5671	0.5478	0.5299	0.6302	0.6267	0.6251	0.6251	0.6265
$\|\psi\| = 0.4055$	n=30	0.4186	0.4114	0.4046	0.3981	0.3919	0.4252	0.4244	0.4239	0.4237	0.4237
	n=50	0.3278	0.3245	0.3212	0.3181	0.3151	0.3310	0.3307	0.3304	0.3303	0.3302
	n=5	1.2146	1.0143	0.8893	0.7982	0.7270	1.3127	1.2103	1.1834	1.1904	1.2172
$p_1 = 0.3$	n=10	0.8175	0.7509	0.7015	0.6625	0.6293	0.8502	0.8163	0.7985	0.7895	0.7867
$p_2 = 0.4$	n=15	0.6410	0.6098	0.5828	0.5589	0.5372	0.6597	0.6469	0.6386	0.6332	0.6302
$\|\psi\| = 0.4418$	n=30	0.4376	0.4281	0.4193	0.4110	0.4031	0.4440	0.4409	0.4382	0.4360	0.4341
	n=50	0.3467	0.3423	0.3381	0.3340	0.3301	0.3497	0.3483	0.3471	0.3459	0.3449
	n=5	1.2923	1.0289	0.8785	0.7749	0.6971	1.3820	1.2083	1.1476	1.1336	1.1453
$p_1 = 0.2$	n=10	0.9509	0.8269	0.7470	0.6908	0.6483	0.9802	0.8856	0.8350	0.8046	0.7863
$p_2 = 0.3$	n=15	0.7467	0.6826	0.6360	0.6000	0.5696	0.7637	0.7163	0.6860	0.6653	0.6506
$\|\psi\| = 0.5390$	n=30	0.4956	0.4783	0.4631	0.4497	0.4376	0.5013	0.4897	0.4802	0.4720	0.4649
	n=50	0.3873	0.3798	0.3728	0.3663	0.3602	0.3900	0.3853	0.3809	0.3769	0.3732

Table 4.4: AAE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$) (cont.)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	1.2776	0.9761	0.8231	0.7246	0.6789	1.3495	1.1128	1.0209	0.9857	0.9802
$p_1 = 0.1$	n=10	1.1485	0.9123	0.7836	0.7033	0.6624	1.1714	0.9580	0.8520	0.7888	0.7544
$p_2 = 0.2$	n=15	0.9727	0.8084	0.7129	0.6495	0.6118	0.9854	0.8323	0.7489	0.6953	0.6626
$\|\psi\| = 0.8109$	n=30	0.6559	0.5993	0.5592	0.5289	0.5067	0.6605	0.6080	0.5717	0.5447	0.5245
	n=50	0.5060	0.4825	0.4630	0.4472	0.4341	0.5082	0.4865	0.4690	0.4546	0.4427
	n=5	1.1592	0.9936	0.8964	0.8253	0.7706	1.2570	1.1808	1.1698	1.1900	1.2297
$p_1 = 0.4$	n=10	0.7631	0.7167	0.6791	0.6491	0.6228	0.7987	0.7828	0.7765	0.7767	0.7816
$p_2 = 0.6$	n=15	0.6129	0.5904	0.5715	0.5543	0.5389	0.6318	0.6265	0.6238	0.6230	0.6241
$\|\psi\| = 0.8109$	n=30	0.4232	0.4160	0.4093	0.4031	0.3971	0.4300	0.4288	0.4279	0.4273	0.4271
	n=50	0.3315	0.3281	0.3249	0.3219	0.3190	0.3348	0.3342	0.3338	0.3334	0.3331
	n=5	1.1769	0.9933	0.8835	0.8048	0.7492	1.2776	1.1729	1.1475	1.1614	1.1970
$p_1 = 0.3$	n=10	0.8085	0.7466	0.7031	0.6680	0.6380	0.8406	0.8106	0.7952	0.7887	0.7886
$p_2 = 0.5$	n=15	0.6311	0.6033	0.5793	0.5594	0.5420	0.6498	0.6382	0.6306	0.6259	0.6234
$\|\psi\| = 0.8473$	n=30	0.4330	0.4239	0.4159	0.4082	0.4014	0.4394	0.4368	0.4346	0.4327	0.4313
	n=50	0.3432	0.3390	0.3350	0.3313	0.3278	0.3463	0.3451	0.3440	0.3430	0.3422
	n=5	1.1855	1.0091	0.9033	0.8272	0.7724	1.2848	1.1854	1.1620	1.1765	1.2121
$p_1 = 0.5$	n=10	0.7980	0.7356	0.6920	0.6569	0.6270	0.8302	0.7999	0.7843	0.7778	0.7777
$p_2 = 0.7$	n=15	0.6396	0.6119	0.5879	0.5679	0.5502	0.6581	0.6464	0.6387	0.6338	0.6311
$\|\psi\| = 0.8473$	n=30	0.4373	0.4282	0.4200	0.4122	0.4053	0.4437	0.4408	0.4384	0.4365	0.4349
	n=50	0.3410	0.3368	0.3329	0.3292	0.3258	0.3441	0.3429	0.3418	0.3409	0.3400
	n=5	1.2536	1.0157	0.8816	0.8065	0.7559	1.3314	1.1713	1.1150	1.1010	1.1102
$p_1 = 0.2$	n=10	0.9195	0.8092	0.7374	0.6920	0.6565	0.9495	0.8691	0.8271	0.8036	0.7905
$p_2 = 0.4$	n=15	0.7173	0.6598	0.6190	0.5898	0.5664	0.7339	0.6930	0.6677	0.6510	0.6393
$\|\psi\| = 0.9808$	n=30	0.4793	0.4637	0.4503	0.4393	0.4296	0.4855	0.4754	0.4675	0.4608	0.4551
	n=50	0.3770	0.3704	0.3641	0.3588	0.3539	0.3799	0.3758	0.3722	0.3688	0.3658
	n=5	1.2351	1.0225	0.8964	0.8246	0.7884	1.3366	1.2255	1.1955	1.2030	1.2310
$p_1 = 0.4$	n=10	0.8085	0.7450	0.7020	0.6671	0.6396	0.8409	0.8065	0.7892	0.7826	0.7831
$p_2 = 0.7$	n=15	0.6509	0.6182	0.5900	0.5710	0.5566	0.6706	0.6576	0.6489	0.6433	0.6401
$\|\psi\| = 1.2528$	n=30	0.4416	0.4326	0.4244	0.4171	0.4102	0.4479	0.4446	0.4418	0.4394	0.4375
	n=50	0.3445	0.3403	0.3364	0.3328	0.3295	0.3477	0.3462	0.3449	0.3437	0.3427
	n=5	1.2420	1.0244	0.8978	0.8266	0.7915	1.3441	1.2286	1.1967	1.2039	1.2317
$p_1 = 0.3$	n=10	0.8067	0.7440	0.7020	0.6680	0.6407	0.8389	0.8048	0.7877	0.7812	0.7818
$p_2 = 0.6$	n=15	0.6483	0.6160	0.5883	0.5698	0.5560	0.6675	0.6543	0.6456	0.6399	0.6367
$\|\psi\| = 1.2528$	n=30	0.4398	0.4310	0.4231	0.4160	0.4094	0.4463	0.4431	0.4404	0.4382	0.4363
	n=50	0.3452	0.3410	0.3372	0.3335	0.3302	0.3483	0.3468	0.3455	0.3444	0.3433
	n=5	1.2585	0.9495	0.8832	0.8435	0.8172	1.3404	1.0927	1.0271	1.0263	1.0371
$p_1 = 0.1$	n=10	1.0877	0.8782	0.7770	0.7285	0.7149	1.1118	0.9226	0.8436	0.7970	0.7677
$p_2 = 0.3$	n=15	0.9127	0.7642	0.6947	0.6530	0.6278	0.9283	0.7888	0.7203	0.6837	0.6626
$\|\psi\| = 1.3499$	n=30	0.6230	0.5714	0.5366	0.5154	0.5008	0.6280	0.5801	0.5487	0.5258	0.5092
	n=50	0.4820	0.4606	0.4441	0.4318	0.4222	0.4843	0.4646	0.4493	0.4366	0.4270
	n=5	1.2588	1.0065	0.9014	0.8417	0.8071	1.3620	1.2052	1.1648	1.1675	1.1912
$p_1 = 0.5$	n=10	0.8956	0.7906	0.7245	0.6846	0.6584	0.9250	0.8494	0.8103	0.7878	0.7756
$p_2 = 0.8$	n=15	0.7174	0.6636	0.6293	0.6033	0.5848	0.7361	0.6948	0.6712	0.6554	0.6447
$\|\psi\| = 1.3863$	n=30	0.4817	0.4658	0.4535	0.4435	0.4353	0.4881	0.4780	0.4697	0.4629	0.4573
	n=50	0.3721	0.3655	0.3600	0.3553	0.3514	0.3751	0.3710	0.3674	0.3643	0.3614

Table 4.4: AAE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$) (cont.)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	1.2441	0.9792	0.8735	0.8149	0.7811	1.3521	1.1861	1.1444	1.1478	1.1728
$p_1 = 0.2$	n=10	0.9100	0.8035	0.7368	0.6958	0.6683	0.9398	0.8629	0.8237	0.8011	0.7889
$p_2 = 0.5$	n=15	0.7105	0.6558	0.6203	0.5930	0.5737	0.7300	0.6889	0.6656	0.6499	0.6394
$\|\psi\| = 1.3863$	n=30	0.4750	0.4591	0.4471	0.4374	0.4292	0.4815	0.4717	0.4636	0.4571	0.4518
	n=50	0.3737	0.3672	0.3616	0.3567	0.3526	0.3767	0.3727	0.3691	0.3660	0.3631
	n=5	1.2582	1.0490	0.9334	0.8702	0.8249	1.3490	1.2030	1.1659	1.1755	1.2127
$p_1 = 0.3$	n=10	0.8460	0.7731	0.7222	0.6865	0.6584	0.8826	0.8303	0.8021	0.7883	0.7843
$p_2 = 0.7$	n=15	0.6774	0.6434	0.6175	0.5962	0.5784	0.6943	0.6730	0.6583	0.6478	0.6415
$\|\psi\| = 1.6946$	n=30	0.4564	0.4464	0.4376	0.4300	0.4229	0.4631	0.4576	0.4530	0.4491	0.4458
	n=50	0.3571	0.3524	0.3483	0.3446	0.3410	0.3603	0.3579	0.3558	0.3538	0.3520
	n=5	1.2664	1.0357	0.9171	0.8637	0.8453	1.3408	1.1780	1.1206	1.1250	1.1593
$p_1 = 0.4$	n=10	0.9067	0.7972	0.7349	0.6950	0.6790	0.9358	0.8553	0.8131	0.7883	0.7741
$p_2 = 0.8$	n=15	0.7298	0.6721	0.6349	0.6102	0.5972	0.7461	0.7046	0.6785	0.6605	0.6477
$\|\psi\| = 1.7918$	n=30	0.4865	0.4697	0.4580	0.4486	0.4426	0.4931	0.4825	0.4737	0.4664	0.4603
	n=50	0.3753	0.3687	0.3636	0.3591	0.3560	0.3787	0.3743	0.3704	0.3670	0.3639
	n=5	1.1745	0.9745	0.8946	0.9157	0.9354	1.2569	1.0939	1.0462	1.0417	1.0591
$p_1 = 0.1$	n=10	1.0584	0.8541	0.7750	0.7453	0.7431	1.0834	0.9042	0.8296	0.7916	0.7685
$p_2 = 0.4$	n=15	0.8867	0.7480	0.6793	0.6523	0.6389	0.8999	0.7721	0.7074	0.6688	0.6448
$\|\psi\| = 1.7918$	n=30	0.6100	0.5577	0.5285	0.5097	0.4996	0.6155	0.5665	0.5360	0.5147	0.5001
	n=50	0.4736	0.4523	0.4375	0.4264	0.4191	0.4761	0.4558	0.4412	0.4297	0.4204
	n=5	1.2684	1.0395	0.9227	0.8707	0.8510	1.3419	1.1771	1.1191	1.1231	1.1571
$p_1 = 0.2$	n=10	0.9087	0.7990	0.7364	0.6957	0.6794	0.9372	0.8559	0.8136	0.7889	0.7746
$p_2 = 0.6$	n=15	0.7231	0.6651	0.6280	0.6039	0.5904	0.7398	0.6983	0.6722	0.6541	0.6414
$\|\psi\| = 1.7918$	n=30	0.4831	0.4662	0.4549	0.4460	0.4399	0.4901	0.4796	0.4711	0.4640	0.4582
	n=50	0.3752	0.3686	0.3634	0.3590	0.3559	0.3784	0.3741	0.3703	0.3670	0.3639
	n=5	1.2100	0.9745	0.9583	0.9891	1.0523	1.3025	1.1087	1.0398	1.0417	1.0528
$p_1 = 0.5$	n=10	1.0423	0.8375	0.7705	0.7553	0.7621	1.0687	0.8871	0.8096	0.7749	0.7546
$p_2 = 0.9$	n=15	0.8896	0.7565	0.6973	0.6735	0.6710	0.9044	0.7738	0.7104	0.6778	0.6562
$\|\psi\| = 2.1972$	n=30	0.6220	0.5662	0.5366	0.5184	0.5107	0.6273	0.5741	0.5407	0.5183	0.5025
	n=50	0.4671	0.4459	0.4315	0.4221	0.4164	0.4698	0.4494	0.4342	0.4228	0.4133
	n=5	1.1871	0.9454	0.9208	0.9520	1.0211	1.2821	1.0884	1.0200	1.0214	1.0335
$p_1 = 0.1$	n=10	1.0531	0.8490	0.7786	0.7619	0.7685	1.0796	0.8992	0.8211	0.7849	0.7635
$p_2 = 0.5$	n=15	0.8851	0.7471	0.6846	0.6598	0.6599	0.9012	0.7667	0.7011	0.6673	0.6448
$\|\psi\| = 2.1972$	n=30	0.6067	0.5547	0.5268	0.5098	0.5036	0.6125	0.5635	0.5322	0.5110	0.4960
	n=50	0.4707	0.4498	0.4355	0.4259	0.4199	0.4733	0.4531	0.4381	0.4268	0.4173
	n=5	1.3235	1.0291	0.9343	0.9179	0.9328	1.4346	1.2242	1.1537	1.1437	1.1593
$p_1 = 0.2$	n=10	0.9492	0.8226	0.7594	0.7207	0.7138	0.9786	0.8788	0.8254	0.7933	0.7737
$p_2 = 0.7$	n=15	0.7532	0.6872	0.6497	0.6324	0.6200	0.7742	0.7243	0.6929	0.6709	0.6553
$\|\psi\| = 2.2336$	n=30	0.4987	0.4821	0.4685	0.4603	0.4567	0.5056	0.4931	0.4831	0.4746	0.4673
	n=50	0.3872	0.3799	0.3738	0.3703	0.3681	0.3902	0.3853	0.3807	0.3765	0.3727
	n=5	1.3235	1.0296	0.9372	0.9203	0.9337	1.4356	1.2265	1.1566	1.1471	1.1633
$p_1 = 0.3$	n=10	0.9451	0.8205	0.7585	0.7211	0.7150	0.9744	0.8763	0.8236	0.7918	0.7724
$p_2 = 0.8$	n=15	0.7559	0.6913	0.6553	0.6392	0.6277	0.7754	0.7258	0.6945	0.6726	0.6569
$\|\psi\| = 2.2336$	n=30	0.5011	0.4844	0.4703	0.4621	0.4586	0.5075	0.4951	0.4850	0.4762	0.4687
	n=50	0.3881	0.3805	0.3744	0.3707	0.3685	0.3913	0.3862	0.3816	0.3773	0.3734

Table 4.4: AAE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$) (cont.)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	1.1883	0.9867	0.9941	1.0391	1.1261	1.2965	1.0985	1.0567	1.0537	1.0727
$p_1 = 0.4$	n=10	1.0538	0.8425	0.7804	0.7774	0.8039	1.0850	0.8951	0.8118	0.7741	0.7513
$p_2 = 0.9$	n=15	0.8965	0.7609	0.7074	0.6857	0.6923	0.9108	0.7830	0.7176	0.6775	0.6519
$\|\psi\| = 2.6027$	n=30	0.6260	0.5701	0.5410	0.5256	0.5211	0.6315	0.5776	0.5436	0.5198	0.5031
	n=50	0.4700	0.4487	0.4346	0.4263	0.4226	0.4728	0.4520	0.4363	0.4244	0.4147
	n=5	1.1884	0.9871	0.9983	1.0476	1.1344	1.2944	1.0987	1.0565	1.0533	1.0721
$p_1 = 0.1$	n=10	1.0494	0.8407	0.7796	0.7762	0.8027	1.0811	0.8921	0.8093	0.7715	0.7489
$p_2 = 0.6$	n=15	0.8965	0.7557	0.7000	0.6797	0.6894	0.9120	0.7801	0.7126	0.6717	0.6469
$\|\psi\| = 2.6027$	n=30	0.6140	0.5625	0.5342	0.5199	0.5163	0.6198	0.5706	0.5390	0.5165	0.5002
	n=50	0.4722	0.4511	0.4372	0.4290	0.4254	0.4750	0.4544	0.4386	0.4267	0.4170
	n=5	1.3580	1.0424	0.9684	0.9580	1.0617	1.4380	1.1838	1.0869	1.0567	1.0863
$p_1 = 0.2$	n=10	1.0295	0.8781	0.7988	0.7692	0.7861	1.0676	0.9181	0.8395	0.7921	0.7636
$p_2 = 0.8$	n=15	0.8260	0.7451	0.6932	0.6731	0.6732	0.8450	0.7642	0.7174	0.6859	0.6625
$\|\psi\| = 2.7726$	n=30	0.5408	0.5189	0.5024	0.4931	0.4931	0.5466	0.5279	0.5126	0.4998	0.4889
	n=50	0.4155	0.4070	0.3999	0.3956	0.3952	0.4185	0.4110	0.4042	0.3984	0.3931
	n=5	1.2580	1.0272	0.9897	1.1352	1.2492	1.3597	1.1093	1.0174	0.9910	0.9947
$p_1 = 0.3$	n=10	1.0946	0.8599	0.8034	0.8137	0.8614	1.1256	0.9174	0.8248	0.7786	0.7494
$p_2 = 0.9$	n=15	0.9210	0.7856	0.7265	0.7165	0.7276	0.9416	0.7990	0.7274	0.6865	0.6587
$\|\psi\| = 3.0445$	n=30	0.6392	0.5823	0.5538	0.5393	0.5400	0.6450	0.5893	0.5545	0.5288	0.5107
	n=50	0.4803	0.4584	0.4452	0.4369	0.4356	0.4833	0.4619	0.4459	0.4325	0.4226
	n=5	1.2466	1.0215	0.9895	1.1359	1.2507	1.3481	1.0999	1.0091	0.9833	0.9872
$p_1 = 0.1$	n=10	1.0919	0.8589	0.8011	0.8096	0.8581	1.1234	0.9168	0.8247	0.7776	0.7480
$p_2 = 0.7$	n=15	0.9259	0.7836	0.7197	0.7100	0.7231	0.9479	0.8010	0.7268	0.6845	0.6561
$\|\psi\| = 3.0445$	n=30	0.6276	0.5752	0.5479	0.5340	0.5355	0.6337	0.5828	0.5503	0.5258	0.5084
	n=50	0.4820	0.4604	0.4475	0.4394	0.4382	0.4850	0.4637	0.4476	0.4343	0.4245
	n=5	1.2640	0.9853	1.1185	1.2206	1.4409	1.3784	1.1134	1.0217	0.9914	0.9940
$p_1 = 0.2$	n=10	1.1789	0.9134	0.8517	0.8830	0.9599	1.2081	0.9560	0.8468	0.7838	0.7466
$p_2 = 0.9$	n=15	0.9871	0.8255	0.7709	0.7549	0.7934	1.0057	0.8384	0.7516	0.6967	0.6678
$\|\psi\| = 3.5835$	n=30	0.6765	0.6116	0.5850	0.5732	0.5771	0.6827	0.6213	0.5798	0.5510	0.5313
	n=50	0.5045	0.4811	0.4682	0.4610	0.4622	0.5079	0.4840	0.4660	0.4521	0.4421
	n=5	1.2602	0.9780	1.1161	1.2216	1.4416	1.3747	1.1090	1.0166	0.9863	0.9895
$p_1 = 0.1$	n=10	1.1760	0.9138	0.8495	0.8790	0.9573	1.2044	0.9560	0.8468	0.7834	0.7458
$p_2 = 0.8$	n=15	0.9964	0.8292	0.7706	0.7562	0.7963	1.0157	0.8440	0.7547	0.6991	0.6694
$\|\psi\| = 3.5835$	n=30	0.6660	0.6056	0.5807	0.5697	0.5742	0.6725	0.6160	0.5764	0.5489	0.5300
	n=50	0.5073	0.4838	0.4710	0.4636	0.4651	0.5108	0.4868	0.4686	0.4545	0.4444
	n=5	1.1325	1.0896	1.3067	1.6509	1.9125	1.2385	0.9992	0.9514	0.9147	0.8814
$p_1 = 0.1$	n=10	1.2773	0.9468	0.8884	1.0523	1.1887	1.3199	0.9727	0.8399	0.7910	0.7568
$p_2 = 0.9$	n=15	1.1545	0.9092	0.8379	0.8683	0.9453	1.1619	0.9217	0.8032	0.7333	0.6978
$\|\psi\| = 4.3944$	n=30	0.7884	0.6981	0.6542	0.6492	0.6657	0.7923	0.7006	0.6443	0.6032	0.5810
	n=50	0.5885	0.5517	0.5317	0.5280	0.5312	0.5907	0.5540	0.5265	0.5059	0.4929

Table 4.5: MSE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	2.0541	1.1269	0.7571	0.5587	0.4357	2.0705	1.1545	0.7926	0.6001	0.4816
$p_1 = 0.1$	n=10	2.2445	1.2848	0.8893	0.6716	0.5341	2.2947	1.3690	0.9976	0.7981	0.6745
$p_2 = 0.1$	n=15	1.9230	1.1631	0.8340	0.6460	0.5237	2.0094	1.3084	1.0216	0.8655	0.7679
$\lvert\psi\rvert = 0$	n=30	1.1158	0.8038	0.6413	0.5355	0.4594	1.2301	1.0004	0.8999	0.8425	0.8054
	n=50	0.5996	0.5032	0.4399	0.3921	0.3538	0.6714	0.6307	0.6122	0.6013	0.5940
	n=5	2.5638	1.5311	1.0854	0.8314	0.6666	2.6438	1.6656	1.2591	1.0346	0.8924
$p_1 = 0.2$	n=10	1.7918	1.2078	0.9275	0.7544	0.6347	1.9283	1.4407	1.2321	1.1146	1.0389
$p_2 = 0.2$	n=15	1.1900	0.8875	0.7243	0.6151	0.5349	1.3139	1.1021	1.0083	0.9542	0.9188
$\lvert\psi\rvert = 0$	n=30	0.4862	0.4359	0.3975	0.3660	0.3391	0.5341	0.5233	0.5182	0.5152	0.5132
	n=50	0.2729	0.2595	0.2473	0.2362	0.2258	0.2876	0.2876	0.2876	0.2876	0.2876
	n=5	2.5193	1.6369	1.2183	0.9638	0.7906	2.6919	1.9282	1.5956	1.4058	1.2827
$p_1 = 0.3$	n=10	1.2847	1.0010	0.8356	0.7194	0.6312	1.4391	1.2706	1.1947	1.1503	1.1208
$p_2 = 0.3$	n=15	0.7509	0.6523	0.5812	0.5248	0.4781	0.8420	0.8154	0.8031	0.7958	0.7908
$\lvert\psi\rvert = 0$	n=30	0.3360	0.3187	0.3030	0.2886	0.2753	0.3550	0.3550	0.3550	0.3550	0.3550
	n=50	0.2005	0.1946	0.1890	0.1837	0.1786	0.2068	0.2068	0.2068	0.2068	0.2068
	n=5	2.2745	1.5821	1.2212	0.9886	0.8239	2.5257	2.0072	1.7728	1.6357	1.5452
$p_1 = 0.4$	n=10	1.0166	0.8671	0.7610	0.6778	0.6098	1.1502	1.1060	1.0854	1.0731	1.0648
$p_2 = 0.4$	n=15	0.6010	0.5514	0.5091	0.4722	0.4396	0.6570	0.6554	0.6547	0.6542	0.6539
$\lvert\psi\rvert = 0$	n=30	0.2860	0.2752	0.2650	0.2554	0.2464	0.2975	0.2975	0.2975	0.2975	0.2975
	n=50	0.1736	0.1697	0.1660	0.1623	0.1589	0.1776	0.1776	0.1776	0.1776	0.1776
	n=5	2.2319	1.5897	1.2418	1.0126	0.8481	2.5061	2.0542	1.8451	1.7208	1.6379
$p_1 = 0.5$	n=10	0.9597	0.8376	0.7448	0.6693	0.6062	1.0817	1.0580	1.0464	1.0393	1.0343
$p_2 = 0.5$	n=15	0.5661	0.5249	0.4886	0.4563	0.4273	0.6115	0.6104	0.6098	0.6094	0.6092
$\lvert\psi\rvert = 0$	n=30	0.2732	0.2638	0.2549	0.2465	0.2385	0.2831	0.2831	0.2831	0.2831	0.2831
	n=50	0.1653	0.1620	0.1587	0.1555	0.1525	0.1688	0.1688	0.1688	0.1688	0.1688
	n=5	2.2733	1.5885	1.2314	1.0025	0.8418	2.5254	2.0131	1.7801	1.6438	1.5541
$p_1 = 0.4$	n=10	0.9840	0.8486	0.7502	0.6726	0.6090	1.1132	1.0796	1.0637	1.0540	1.0473
$p_2 = 0.5$	n=15	0.5850	0.5395	0.5003	0.4661	0.4360	0.6373	0.6367	0.6365	0.6363	0.6362
$\lvert\psi\rvert = 0.4055$	n=30	0.2787	0.2687	0.2593	0.2506	0.2424	0.2894	0.2894	0.2894	0.2894	0.2894
	n=50	0.1692	0.1656	0.1621	0.1588	0.1556	0.1730	0.1730	0.1730	0.1730	0.1730
	n=5	2.3622	1.6593	1.2910	1.0542	0.8875	2.6222	2.0975	1.8575	1.7166	1.6236
$p_1 = 0.5$	n=10	0.9691	0.8326	0.7345	0.6575	0.5948	1.0950	1.0575	1.0395	1.0285	1.0210
$p_2 = 0.6$	n=15	0.5939	0.5473	0.5076	0.4730	0.4426	0.6461	0.6444	0.6436	0.6431	0.6427
$\lvert\psi\rvert = 0.4055$	n=30	0.2785	0.2684	0.2590	0.2502	0.2419	0.2894	0.2894	0.2894	0.2894	0.2894
	n=50	0.1724	0.1687	0.1651	0.1617	0.1585	0.1763	0.1763	0.1763	0.1763	0.1763
	n=5	2.3992	1.6079	1.2224	0.9852	0.8233	2.5996	1.9445	1.6565	1.4918	1.3852
$p_1 = 0.3$	n=10	1.1534	0.9346	0.7996	0.7019	0.6265	1.2970	1.1865	1.1362	1.1066	1.0869
$p_2 = 0.4$	n=15	0.6785	0.6024	0.5454	0.4993	0.4609	0.7526	0.7361	0.7283	0.7237	0.7205
$\lvert\psi\rvert = 0.4418$	n=30	0.3093	0.2954	0.2828	0.2712	0.2606	0.3246	0.3246	0.3246	0.3246	0.3246
	n=50	0.1875	0.1826	0.1779	0.1735	0.1693	0.1928	0.1928	0.1928	0.1928	0.1928
	n=5	2.5627	1.5992	1.1765	0.9345	0.7778	2.6686	1.7758	1.4027	1.1970	1.0675
$p_1 = 0.2$	n=10	1.5504	1.1108	0.8908	0.7519	0.6545	1.6908	1.3507	1.2045	1.1222	1.0696
$p_2 = 0.3$	n=15	0.9726	0.7704	0.6544	0.5743	0.5145	1.0817	0.9597	0.9051	0.8735	0.8528
$\lvert\psi\rvert = 0.5390$	n=30	0.4127	0.3779	0.3511	0.3289	0.3101	0.4457	0.4383	0.4348	0.4327	0.4313
	n=50	0.2358	0.2261	0.2174	0.2094	0.2022	0.2464	0.2464	0.2464	0.2464	0.2464

Table 4.5: MSE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts (cont.)

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	2.2897	1.3836	1.0367	0.8589	0.7543	2.3161	1.4266	1.0907	0.9204	0.8211
$p_1 = 0.1$	n=10	2.0115	1.2551	0.9505	0.7884	0.6903	2.0745	1.3583	1.0809	0.9379	0.8534
$p_2 = 0.2$	n=15	1.5846	1.0359	0.8011	0.6707	0.5893	1.6641	1.1664	0.9658	0.8593	0.7948
$\|\psi\| = 0.8109$	n=30	0.7792	0.6079	0.5160	0.4564	0.4147	0.8628	0.7499	0.7002	0.6718	0.6536
	n=50	0.4365	0.3816	0.3449	0.3176	0.2964	0.4839	0.4651	0.4565	0.4514	0.4481
	n=5	2.4081	1.6581	1.2885	1.0632	0.9123	2.6309	2.0271	1.7582	1.6045	1.5060
$p_1 = 0.4$	n=10	1.0016	0.8473	0.7433	0.6658	0.6056	1.1355	1.0822	1.0563	1.0405	1.0296
$p_2 = 0.6$	n=15	0.6112	0.5599	0.5179	0.4828	0.4533	0.6710	0.6689	0.6678	0.6672	0.6667
$\|\psi\| = 0.8109$	n=30	0.2841	0.2732	0.2634	0.2545	0.2465	0.2962	0.2962	0.2962	0.2962	0.2962
	n=50	0.1766	0.1725	0.1688	0.1653	0.1621	0.1809	0.1809	0.1809	0.1809	0.1809
	n=5	2.4008	1.6094	1.2355	1.0139	0.8689	2.5939	1.9277	1.6389	1.4772	1.3752
$p_1 = 0.3$	n=10	1.1233	0.9164	0.7904	0.7017	0.6353	1.2634	1.1593	1.1111	1.0825	1.0635
$p_2 = 0.5$	n=15	0.6625	0.5898	0.5366	0.4949	0.4612	0.7347	0.7188	0.7113	0.7068	0.7037
$\|\psi\| = 0.8473$	n=30	0.3021	0.2888	0.2772	0.2668	0.2576	0.3170	0.3170	0.3170	0.3170	0.3170
	n=50	0.1832	0.1785	0.1741	0.1701	0.1664	0.1884	0.1884	0.1884	0.1884	0.1884
	n=5	2.4275	1.6494	1.2778	1.0560	0.9097	2.6274	1.9796	1.6973	1.5387	1.4384
$p_1 = 0.5$	n=10	1.1054	0.8975	0.7719	0.6838	0.6181	1.2419	1.1339	1.0839	1.0542	1.0344
$p_2 = 0.7$	n=15	0.6716	0.5996	0.5464	0.5043	0.4703	0.7457	0.7320	0.7256	0.7216	0.7190
$\|\psi\| = 0.8473$	n=30	0.3032	0.2896	0.2776	0.2670	0.2576	0.3186	0.3186	0.3186	0.3186	0.3186
	n=50	0.1871	0.1822	0.1777	0.1736	0.1698	0.1924	0.1924	0.1924	0.1924	0.1924
	n=5	2.4582	1.5804	1.2068	1.0022	0.8769	2.5651	1.7539	1.4235	1.2477	1.1418
$p_1 = 0.2$	n=10	1.4240	1.0438	0.8580	0.7449	0.6694	1.5484	1.2527	1.1266	1.0566	1.0127
$p_2 = 0.4$	n=15	0.9013	0.7193	0.6187	0.5524	0.5054	0.9951	0.8797	0.8278	0.7978	0.7782
$\|\psi\| = 0.9808$	n=30	0.3860	0.3543	0.3309	0.3125	0.2977	0.4162	0.4087	0.4052	0.4030	0.4015
	n=50	0.2231	0.2143	0.2065	0.1998	0.1939	0.2332	0.2332	0.2332	0.2332	0.2332
	n=10	1.1399	0.9119	0.7816	0.6962	0.6376	1.2872	1.1605	1.1015	1.0664	1.0431
$p_1 = 0.4$	n=15	0.6889	0.6114	0.5563	0.5152	0.4841	0.7726	0.7581	0.7512	0.7469	0.7440
$p_2 = 0.7$	n=30	0.3089	0.2944	0.2822	0.2719	0.2633	0.3260	0.3260	0.3260	0.3260	0.3260
$\|\psi\| = 1.2528$	n=50	0.1913	0.1860	0.1814	0.1773	0.1738	0.1973	0.1973	0.1973	0.1973	0.1973
	n=5	2.5356	1.6753	1.2997	1.0958	0.9752	2.6821	1.9058	1.5791	1.4030	1.2969
$p_1 = 0.3$	n=10	1.1317	0.9083	0.7801	0.6959	0.6381	1.2801	1.1590	1.1026	1.0691	1.0471
$p_2 = 0.6$	n=15	0.6873	0.6088	0.5539	0.5131	0.4825	0.7685	0.7513	0.7431	0.7381	0.7347
$\|\psi\| = 1.2528$	n=30	0.3072	0.2929	0.2809	0.2707	0.2623	0.3239	0.3239	0.3239	0.3239	0.3239
	n=50	0.1912	0.1860	0.1813	0.1773	0.1738	0.1971	0.1971	0.1971	0.1971	0.1971
	n=5	2.2783	1.4980	1.2365	1.1255	1.0767	2.2977	1.5252	1.2655	1.1532	1.1011
$p_1 = 0.1$	n=10	1.7721	1.1593	0.9401	0.8422	0.7968	1.8093	1.2127	0.9986	0.8992	0.8486
$p_2 = 0.3$	n=15	1.3770	0.9195	0.7415	0.6563	0.6137	1.4277	0.9940	0.8247	0.7391	0.6906
$\|\psi\| = 1.3499$	n=30	0.7097	0.5507	0.4718	0.4257	0.3979	0.7818	0.6686	0.6187	0.5903	0.5722
	n=50	0.4004	0.3484	0.3156	0.2930	0.2771	0.4456	0.4263	0.4175	0.4122	0.4088
	n=5	2.4564	1.6097	1.2721	1.1043	1.0149	2.5502	1.7522	1.4386	1.2806	1.1924
$p_1 = 0.5$	n=10	1.3741	1.0056	0.8332	0.7357	0.6772	1.4918	1.1962	1.0691	0.9988	0.9552
$p_2 = 0.8$	n=15	0.9055	0.7229	0.6260	0.5658	0.5266	0.9998	0.8808	0.8273	0.7964	0.7765
$\|\psi\| = 1.3863$	n=30	0.3824	0.3517	0.3292	0.3121	0.2992	0.4162	0.4117	0.4096	0.4082	0.4073
	n=50	0.2253	0.2162	0.2085	0.2021	0.1968	0.2362	0.2362	0.2362	0.2362	0.2362

Table 4.5: MSE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts (cont.)

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	2.4488	1.5719	1.2258	1.0554	0.9657	2.5359	1.7029	1.3772	1.2137	1.1229
$p_1 = 0.2$	n=10	1.3951	1.0238	0.8504	0.7522	0.6932	1.5135	1.2161	1.0889	1.0189	0.9756
$p_2 = 0.5$	n=15	0.8868	0.7067	0.6112	0.5518	0.5132	0.9797	0.8623	0.8092	0.7785	0.7585
$\|\psi\| = 1.3863$	n=30	0.3790	0.3476	0.3255	0.3089	0.2965	0.4096	0.4021	0.3985	0.3963	0.3948
	n=50	0.2189	0.2101	0.2028	0.1967	0.1917	0.2292	0.2292	0.2292	0.2292	0.2292
	n=5	2.6213	1.6718	1.3101	1.1463	1.0733	2.6944	1.7668	1.4009	1.2188	1.1202
$p_1 = 0.3$	n=10	1.2718	0.9721	0.8198	0.7321	0.6817	1.4327	1.2323	1.1399	1.0859	1.0509
$p_2 = 0.7$	n=15	0.7674	0.6613	0.5938	0.5488	0.5193	0.8748	0.8438	0.8289	0.8198	0.8136
$\|\psi\| = 1.6946$	n=30	0.3320	0.3139	0.2996	0.2886	0.2804	0.3544	0.3544	0.3544	0.3544	0.3544
	n=50	0.2061	0.1996	0.1941	0.1896	0.1861	0.2139	0.2139	0.2139	0.2139	0.2139
	n=5	2.5176	1.6110	1.2927	1.1649	1.1216	2.5585	1.6525	1.3150	1.1584	1.0816
$p_1 = 0.4$	n=10	1.4102	1.0185	0.8447	0.7558	0.7118	1.5335	1.2096	1.0693	0.9918	0.9444
$p_2 = 0.8$	n=15	0.9245	0.7347	0.6370	0.5803	0.5477	1.0292	0.9067	0.8512	0.8191	0.7983
$\|\psi\| = 1.7918$	n=30	0.3885	0.3567	0.3341	0.3179	0.3068	0.4248	0.4203	0.4181	0.4167	0.4158
	n=50	0.2297	0.2201	0.2123	0.2062	0.2015	0.2414	0.2414	0.2414	0.2414	0.2414
	n=5	2.1608	1.4949	1.3248	1.2918	1.3111	2.1589	1.4799	1.2921	1.2398	1.2395
$p_1 = 0.1$	n=10	1.6509	1.0924	0.9199	0.8655	0.8613	1.6578	1.0865	0.8912	0.8091	0.7747
$p_2 = 0.4$	n=15	1.3105	0.8681	0.7104	0.6477	0.6290	1.3423	0.9032	0.7329	0.6485	0.6026
$\|\psi\| = 1.7918$	n=30	0.6833	0.5265	0.4522	0.4124	0.3921	0.7543	0.6398	0.5891	0.5603	0.5419
	n=50	0.3889	0.3370	0.3053	0.2846	0.2713	0.4347	0.4151	0.4060	0.4006	0.3970
	n=5	2.5553	1.6307	1.3069	1.1768	1.1323	2.5917	1.6656	1.3219	1.1628	1.0852
$p_1 = 0.2$	n=10	1.4289	1.0259	0.8482	0.7578	0.7130	1.5453	1.2052	1.0575	0.9757	0.9255
$p_2 = 0.6$	n=15	0.9043	0.7200	0.6253	0.5706	0.5395	1.0076	0.8894	0.8361	0.8053	0.7854
$\|\psi\| = 1.7918$	n=30	0.3849	0.3522	0.3298	0.3141	0.3034	0.4180	0.4104	0.4067	0.4044	0.4029
	n=50	0.2286	0.2192	0.2116	0.2055	0.2010	0.2401	0.2401	0.2401	0.2401	0.2401
	n=5	2.2032	1.5669	1.4598	1.4946	1.5798	2.1680	1.4879	1.3353	1.3257	1.3687
$p_1 = 0.5$	n=10	1.6049	1.0510	0.9004	0.8751	0.9034	1.5940	1.0052	0.8080	0.7302	0.7035
$p_2 = 0.9$	n=15	1.3101	0.8715	0.7238	0.6755	0.6739	1.3359	0.8899	0.7161	0.6306	0.5851
$\|\psi\| = 2.1972$	n=30	0.7203	0.5398	0.4581	0.4172	0.3991	0.7906	0.6498	0.5877	0.5527	0.5306
	n=50	0.3914	0.3366	0.3047	0.2850	0.2735	0.4361	0.4121	0.4009	0.3942	0.3898
	n=5	2.1340	1.4834	1.3740	1.4098	1.4971	2.0983	1.4026	1.2462	1.2359	1.2794
$p_1 = 0.1$	n=10	1.6271	1.0733	0.9225	0.8969	0.9249	1.6169	1.0290	0.8321	0.7547	0.7282
$p_2 = 0.5$	n=15	1.2989	0.8563	0.7078	0.6594	0.6580	1.3261	0.8772	0.7033	0.6184	0.5737
$\|\psi\| = 2.1972$	n=30	0.6769	0.5197	0.4473	0.4113	0.3960	0.7495	0.6338	0.5824	0.5532	0.5346
	n=50	0.3848	0.3327	0.3016	0.2823	0.2709	0.4316	0.4118	0.4026	0.3972	0.3936
	n=5	2.6566	1.6374	1.3435	1.2735	1.2968	2.6230	1.5427	1.1777	1.0342	0.9853
$p_1 = 0.2$	n=10	1.5740	1.0891	0.8910	0.8045	0.7761	1.6976	1.2660	1.0790	0.9769	0.9158
$p_2 = 0.7$	n=15	0.9855	0.7712	0.6654	0.6098	0.5844	1.1149	0.9785	0.9160	0.8797	0.8563
$\|\psi\| = 2.2336$	n=30	0.4097	0.3729	0.3487	0.3329	0.3237	0.4495	0.4417	0.4380	0.4357	0.4341
	n=50	0.2439	0.2330	0.2246	0.2184	0.2142	0.2576	0.2576	0.2576	0.2576	0.2576
	n=5	2.6551	1.6374	1.3424	1.2714	1.2939	2.6204	1.5396	1.1714	1.0246	0.9728
$p_1 = 0.3$	n=10	1.5462	1.0787	0.8869	0.8034	0.7766	1.6762	1.2663	1.0888	0.9921	0.9344
$p_2 = 0.8$	n=15	1.0043	0.7839	0.6754	0.6184	0.5921	1.1323	0.9887	0.9228	0.8844	0.8597
$\|\psi\| = 2.2336$	n=30	0.4122	0.3763	0.3520	0.3359	0.3263	0.4549	0.4503	0.4480	0.4466	0.4456
	n=50	0.2447	0.2336	0.2251	0.2189	0.2147	0.2585	0.2585	0.2585	0.2585	0.2585

Table 4.5: MSE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ added to tables containing zero counts (cont.)

		ϵ added to all tables					ϵ added to tables with zero counts				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	2.2310	1.5642	1.5145	1.6258	1.7904	2.1476	1.3944	1.2614	1.2950	1.3877
$p_1 = 0.4$	n=10	1.6417	1.0625	0.9205	0.9182	0.9767	1.6229	0.9922	0.7820	0.7032	0.6814
$p_2 = 0.9$	n=15	1.3334	0.8836	0.7386	0.7007	0.7143	1.3637	0.9029	0.7227	0.6348	0.5892
$\|\psi\| = 2.6027$	n=30	0.7270	0.5444	0.4633	0.4252	0.4117	0.8006	0.6583	0.5953	0.5597	0.5373
	n=50	0.3961	0.3405	0.3087	0.2900	0.2801	0.4425	0.4183	0.4070	0.4002	0.3957
	n=5	2.2390	1.5703	1.5217	1.6341	1.7994	2.1555	1.4002	1.2683	1.3029	1.3963
$p_1 = 0.1$	n=10	1.6532	1.0670	0.9233	0.9205	0.9790	1.6304	0.9901	0.7764	0.6958	0.6730
$p_2 = 0.6$	n=15	1.3220	0.8754	0.7328	0.6966	0.7115	1.3576	0.9040	0.7295	0.6460	0.6040
$\|\psi\| = 2.6027$	n=30	0.6774	0.5200	0.4490	0.4160	0.4054	0.7559	0.6425	0.5925	0.5643	0.5466
	n=50	0.3973	0.3429	0.3112	0.2921	0.2820	0.4458	0.4244	0.4143	0.4082	0.4040
	n=5	2.6816	1.6121	1.4163	1.4701	1.6174	2.5604	1.3605	1.0369	0.9701	1.0052
$p_1 = 0.2$	n=10	1.8601	1.1974	0.9671	0.8971	0.9065	1.9422	1.2824	1.0098	0.8707	0.7951
$p_2 = 0.8$	n=15	1.2281	0.8944	0.7499	0.6880	0.6735	1.3757	1.1163	0.9988	0.9319	0.8901
$\|\psi\| = 2.7726$	n=30	0.4908	0.4354	0.4016	0.3822	0.3739	0.5527	0.5398	0.5335	0.5296	0.5269
	n=50	0.2827	0.2670	0.2558	0.2483	0.2444	0.3032	0.3032	0.3032	0.3032	0.3032
	n=5	2.3582	1.6078	1.6238	1.8325	2.0996	2.2219	1.3401	1.2342	1.3316	1.4974
$p_1 = 0.3$	n=10	1.7839	1.1249	0.9779	0.9988	1.0940	1.7530	1.0193	0.7753	0.6880	0.6702
$p_2 = 0.9$	n=15	1.4170	0.9328	0.7825	0.7536	0.7848	1.4596	0.9629	0.7672	0.6724	0.6248
$\|\psi\| = 3.0445$	n=30	0.7514	0.5637	0.4819	0.4463	0.4378	0.8326	0.6883	0.6243	0.5881	0.5653
	n=50	0.4112	0.3538	0.3217	0.3037	0.2956	0.4607	0.4364	0.4250	0.4182	0.4136
	n=5	2.3274	1.5927	1.6163	1.8294	2.0992	2.1937	1.3297	1.2331	1.3363	1.5061
$p_1 = 0.1$	n=10	1.8034	1.1303	0.9787	0.9976	1.0917	1.7739	1.0268	0.7785	0.6893	0.6704
$p_2 = 0.7$	n=15	1.4081	0.9264	0.7773	0.7492	0.7809	1.4612	0.9748	0.7863	0.6970	0.6536
$\|\psi\| = 3.0445$	n=30	0.7025	0.5401	0.4684	0.4379	0.4323	0.7891	0.6742	0.6233	0.5947	0.5767
	n=50	0.4139	0.3572	0.3248	0.3063	0.2977	0.4657	0.4439	0.4335	0.4272	0.4229
	n=5	2.3880	1.6377	1.8134	2.2011	2.6400	2.2056	1.2879	1.3129	1.5653	1.8829
$p_1 = 0.2$	n=10	2.1046	1.2456	1.0842	1.1485	1.3103	2.0307	1.0482	0.7399	0.6469	0.6483
$p_2 = 0.9$	n=15	1.6530	1.0448	0.8674	0.8491	0.9101	1.6981	1.0605	0.8096	0.6909	0.6349
$\|\psi\| = 3.5835$	n=30	0.8327	0.6229	0.5335	0.4990	0.4977	0.9338	0.7765	0.7058	0.6656	0.6404
	n=50	0.4502	0.3874	0.3530	0.3353	0.3297	0.5083	0.4837	0.4722	0.4653	0.4607
	n=5	2.3420	1.6137	1.7977	2.1896	2.6310	2.1566	1.2578	1.2881	1.5421	1.8597
$p_1 = 0.1$	n=10	2.0916	1.2398	1.0810	1.1465	1.3088	2.0175	1.0424	0.7367	0.6449	0.6471
$p_2 = 0.8$	n=15	1.6604	1.0503	0.8722	0.8538	0.9148	1.7102	1.0744	0.8262	0.7101	0.6566
$\|\psi\| = 3.5835$	n=30	0.7875	0.6034	0.5235	0.4936	0.4947	0.8985	0.7741	0.7185	0.6870	0.6674
	n=50	0.4537	0.3915	0.3566	0.3383	0.3322	0.5141	0.4917	0.4810	0.4744	0.4700
	n=5	2.0871	1.8077	2.4756	3.2920	4.0994	1.8940	1.4459	1.9664	2.6533	3.3463
$p_1 = 0.1$	n=10	2.3343	1.3145	1.2897	1.5547	1.9294	2.1539	0.9284	0.6912	0.7459	0.9167
$p_2 = 0.9$	n=15	2.1067	1.2080	1.0279	1.0936	1.2725	2.0541	1.0290	0.6832	0.5619	0.5430
$\|\psi\| = 4.3944$	n=30	1.1354	0.7894	0.6599	0.6273	0.6514	1.2821	0.9963	0.8697	0.8000	0.7587
	n=50	0.6236	0.5114	0.4548	0.4305	0.4290	0.7256	0.6756	0.6515	0.6369	0.6271

Table 4.6: MSE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	2.0541	1.1269	0.7571	0.5587	0.4357	2.2708	1.4655	1.1986	1.0979	1.0735
$p_1 = 0.1$	n=10	2.2445	1.2848	0.8893	0.6716	0.5341	2.3226	1.4075	1.0470	0.8597	0.7499
$p_2 = 0.1$	n=15	1.9230	1.1631	0.8340	0.6460	0.5237	1.9602	1.2233	0.9124	0.7400	0.6316
$\|\psi\| = 0$	n=30	1.1158	0.8038	0.6413	0.5355	0.4594	1.1257	0.8213	0.6652	0.5650	0.4939
	n=50	0.5996	0.5032	0.4399	0.3921	0.3538	0.6031	0.5097	0.4492	0.4038	0.3679
	n=5	2.5638	1.5311	1.0854	0.8314	0.6666	2.8843	2.0592	1.7966	1.7196	1.7340
$p_1 = 0.2$	n=10	1.7918	1.2078	0.9275	0.7544	0.6347	1.8835	1.3659	1.1429	1.0222	0.9518
$p_2 = 0.2$	n=15	1.1900	0.8875	0.7243	0.6151	0.5349	1.2296	0.9588	0.8235	0.7397	0.6833
$\|\psi\| = 0$	n=30	0.4862	0.4359	0.3975	0.3660	0.3391	0.4954	0.4536	0.4233	0.3992	0.3794
	n=50	0.2729	0.2595	0.2473	0.2362	0.2258	0.2762	0.2660	0.2569	0.2486	0.2412
	n=5	2.5193	1.6369	1.2183	0.9638	0.7906	2.8872	2.2718	2.0961	2.0790	2.1471
$p_1 = 0.3$	n=10	1.2847	1.0010	0.8356	0.7194	0.6312	1.3762	1.1695	1.0745	1.0246	1.0002
$p_2 = 0.3$	n=15	0.7509	0.6523	0.5812	0.5248	0.4781	0.7888	0.7247	0.6858	0.6600	0.6428
$\|\psi\| = 0$	n=30	0.3360	0.3187	0.3030	0.2886	0.2753	0.3450	0.3365	0.3292	0.3230	0.3178
	n=50	0.2005	0.1946	0.1890	0.1837	0.1786	0.2038	0.2012	0.1987	0.1965	0.1944
	n=5	2.2745	1.5821	1.2212	0.9886	0.8239	2.6486	2.2495	2.1608	2.1960	2.3041
$p_1 = 0.4$	n=10	1.0166	0.8671	0.7610	0.6778	0.6098	1.1061	1.0379	1.0084	0.9989	1.0026
$p_2 = 0.4$	n=15	0.6010	0.5514	0.5091	0.4722	0.4396	0.6380	0.6236	0.6150	0.6108	0.6101
$\|\psi\| = 0$	n=30	0.2860	0.2752	0.2650	0.2554	0.2464	0.2950	0.2930	0.2915	0.2904	0.2897
	n=50	0.1736	0.1697	0.1660	0.1623	0.1589	0.1769	0.1762	0.1757	0.1753	0.1750
	n=5	2.2319	1.5897	1.2418	1.0126	0.8481	2.6143	2.2788	2.2169	2.2699	2.3930
$p_1 = 0.5$	n=10	0.9597	0.8376	0.7448	0.6693	0.6062	1.0493	1.0100	0.9959	0.9964	1.0077
$p_2 = 0.5$	n=15	0.5661	0.5249	0.4886	0.4563	0.4273	0.6029	0.5969	0.5948	0.5958	0.5994
$\|\psi\| = 0$	n=30	0.2732	0.2638	0.2549	0.2465	0.2385	0.2822	0.2817	0.2815	0.2817	0.2821
	n=50	0.1653	0.1620	0.1587	0.1555	0.1525	0.1686	0.1685	0.1684	0.1685	0.1686
	n=5	2.2733	1.5885	1.2314	1.0025	0.8418	2.6630	2.2791	2.1995	2.2434	2.3613
$p_1 = 0.4$	n=10	0.9840	0.8486	0.7502	0.6726	0.6090	1.0748	1.0214	0.9998	0.9955	1.0033
$p_2 = 0.5$	n=15	0.5850	0.5395	0.5003	0.4661	0.4360	0.6227	0.6128	0.6075	0.6059	0.6074
$\|\psi\| = 0.4055$	n=30	0.2787	0.2687	0.2593	0.2506	0.2424	0.2878	0.2866	0.2858	0.2853	0.2852
	n=50	0.1692	0.1656	0.1621	0.1588	0.1556	0.1726	0.1722	0.1719	0.1717	0.1716
	n=5	2.3622	1.6593	1.2910	1.0542	0.8875	2.7635	2.3716	2.2902	2.3355	2.4569
$p_1 = 0.5$	n=10	0.9691	0.8326	0.7345	0.6575	0.5948	1.0582	1.0019	0.9789	0.9735	0.9804
$p_2 = 0.6$	n=15	0.5939	0.5473	0.5076	0.4730	0.4426	0.6318	0.6211	0.6155	0.6137	0.6151
$\|\psi\| = 0.4055$	n=30	0.2785	0.2684	0.2590	0.2502	0.2419	0.2876	0.2863	0.2855	0.2850	0.2848
	n=50	0.1724	0.1687	0.1651	0.1617	0.1585	0.1758	0.1754	0.1751	0.1749	0.1748
	n=5	2.3992	1.6079	1.2224	0.9852	0.8233	2.7791	2.2660	2.1332	2.1434	2.2337
$p_1 = 0.3$	n=10	1.1534	0.9346	0.7996	0.7019	0.6265	1.2459	1.1060	1.0429	1.0128	1.0023
$p_2 = 0.4$	n=15	0.6785	0.6024	0.5454	0.4993	0.4609	0.7170	0.6757	0.6511	0.6357	0.6265
$\|\psi\| = 0.4418$	n=30	0.3093	0.2954	0.2828	0.2712	0.2606	0.3184	0.3132	0.3089	0.3054	0.3025
	n=50	0.1875	0.1826	0.1779	0.1735	0.1693	0.1909	0.1892	0.1877	0.1864	0.1852
	n=5	2.5627	1.5992	1.1765	0.9345	0.7778	2.9156	2.1813	1.9597	1.9120	1.9531
$p_1 = 0.2$	n=10	1.5504	1.1108	0.8908	0.7519	0.6545	1.6454	1.2755	1.1151	1.0300	0.9830
$p_2 = 0.3$	n=15	0.9726	0.7704	0.6544	0.5743	0.5145	1.0132	0.8435	0.7557	0.7010	0.6644
$\|\psi\| = 0.5390$	n=30	0.4127	0.3779	0.3511	0.3289	0.3101	0.4221	0.3958	0.3767	0.3616	0.3494
	n=50	0.2358	0.2261	0.2174	0.2094	0.2022	0.2392	0.2327	0.2269	0.2217	0.2171

Table 4.6: MSE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$) (cont.)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	2.2897	1.3836	1.0367	0.8589	0.7543	2.5491	1.7754	1.5345	1.4557	1.4521
$p_1 = 0.1$	n=10	2.0115	1.2551	0.9505	0.7884	0.6903	2.0986	1.3873	1.1145	0.9774	0.9002
$p_2 = 0.2$	n=15	1.5846	1.0359	0.8011	0.6707	0.5893	1.6260	1.1003	0.8812	0.7625	0.6898
$\|\psi\| = 0.8109$	n=30	0.7792	0.6079	0.5160	0.4564	0.4147	0.7895	0.6254	0.5389	0.4832	0.4443
	n=50	0.4365	0.3816	0.3449	0.3176	0.2964	0.4402	0.3883	0.3539	0.3283	0.3085
	n=5	2.4081	1.6581	1.2885	1.0632	0.9123	2.8243	2.3727	2.2706	2.3067	2.4243
$p_1 = 0.4$	n=10	1.0016	0.8473	0.7433	0.6658	0.6056	1.0955	1.0213	0.9891	0.9783	0.9816
$p_2 = 0.6$	n=15	0.6112	0.5599	0.5179	0.4828	0.4533	0.6511	0.6358	0.6268	0.6225	0.6219
$\|\psi\| = 0.8109$	n=30	0.2841	0.2732	0.2634	0.2545	0.2465	0.2936	0.2916	0.2901	0.2891	0.2884
	n=50	0.1766	0.1725	0.1688	0.1653	0.1621	0.1801	0.1795	0.1789	0.1784	0.1781
	n=5	2.4008	1.6094	1.2355	1.0139	0.8689	2.8094	2.2988	2.1733	2.1934	2.2969
$p_1 = 0.3$	n=10	1.1233	0.9164	0.7904	0.7017	0.6353	1.2207	1.0930	1.0366	1.0114	1.0050
$p_2 = 0.5$	n=15	0.6625	0.5898	0.5366	0.4949	0.4612	0.7029	0.6654	0.6437	0.6307	0.6236
$\|\psi\| = 0.8473$	n=30	0.3021	0.2888	0.2772	0.2668	0.2576	0.3116	0.3071	0.3035	0.3005	0.2982
	n=50	0.1832	0.1785	0.1741	0.1701	0.1664	0.1867	0.1853	0.1840	0.1828	0.1818
	n=5	2.4275	1.6494	1.2778	1.0560	0.9097	2.8363	2.3421	2.2226	2.2462	2.3522
$p_1 = 0.5$	n=10	1.1054	0.8975	0.7719	0.6838	0.6181	1.2016	1.0716	1.0142	0.9884	0.9815
$p_2 = 0.7$	n=15	0.6716	0.5996	0.5464	0.5043	0.4703	0.7123	0.6760	0.6545	0.6415	0.6344
$\|\psi\| = 0.8473$	n=30	0.3032	0.2896	0.2776	0.2670	0.2576	0.3128	0.3080	0.3040	0.3009	0.2983
	n=50	0.1871	0.1822	0.1777	0.1736	0.1698	0.1907	0.1891	0.1878	0.1866	0.1855
	n=5	2.4582	1.5804	1.2068	1.0022	0.8769	2.8343	2.1827	1.9999	1.9780	2.0409
$p_1 = 0.2$	n=10	1.4240	1.0438	0.8580	0.7449	0.6694	1.5252	1.2138	1.0829	1.0167	0.9833
$p_2 = 0.4$	n=15	0.9013	0.7193	0.6187	0.5524	0.5054	0.9448	0.7950	0.7204	0.6754	0.6467
$\|\psi\| = 0.9808$	n=30	0.3860	0.3543	0.3309	0.3125	0.2977	0.3960	0.3725	0.3562	0.3437	0.3337
	n=50	0.2231	0.2143	0.2065	0.1998	0.1939	0.2267	0.2211	0.2161	0.2118	0.2079
	n=5	2.4867	1.6511	1.2846	1.0851	0.9670	2.9239	2.3547	2.2113	2.2259	2.3310
$p_1 = 0.4$	n=10	1.1399	0.9119	0.7816	0.6962	0.6376	1.2446	1.0940	1.0267	0.9951	0.9848
$p_2 = 0.7$	n=15	0.6889	0.6114	0.5563	0.5152	0.4841	0.7331	0.6914	0.6662	0.6504	0.6412
$\|\psi\| = 1.2528$	n=30	0.3089	0.2944	0.2822	0.2719	0.2633	0.3192	0.3136	0.3090	0.3052	0.3022
	n=50	0.1913	0.1860	0.1814	0.1773	0.1738	0.1952	0.1934	0.1917	0.1902	0.1889
	n=5	2.5356	1.6753	1.2997	1.0958	0.9752	2.9756	2.3819	2.2292	2.2389	2.3410
$p_1 = 0.3$	n=10	1.1317	0.9083	0.7801	0.6959	0.6381	1.2356	1.0892	1.0234	0.9926	0.9827
$p_2 = 0.6$	n=15	0.6873	0.6088	0.5539	0.5131	0.4825	0.7309	0.6879	0.6624	0.6465	0.6373
$\|\psi\| = 1.2528$	n=30	0.3072	0.2929	0.2809	0.2707	0.2623	0.3175	0.3122	0.3078	0.3042	0.3013
	n=50	0.1912	0.1860	0.1813	0.1773	0.1738	0.1950	0.1932	0.1916	0.1902	0.1889
	n=5	2.2783	1.4980	1.2365	1.1255	1.0767	2.5639	1.8955	1.7078	1.6614	1.6808
$p_1 = 0.1$	n=10	1.7721	1.1593	0.9401	0.8422	0.7968	1.8676	1.2902	1.0862	0.9925	0.9450
$p_2 = 0.3$	n=15	1.3770	0.9195	0.7415	0.6563	0.6137	1.4234	0.9843	0.8132	0.7275	0.6796
$\|\psi\| = 1.3499$	n=30	0.7097	0.5507	0.4718	0.4257	0.3979	0.7214	0.5687	0.4926	0.4467	0.4170
	n=50	0.4004	0.3484	0.3156	0.2930	0.2771	0.4046	0.3553	0.3240	0.3018	0.2854
	n=5	2.4564	1.6097	1.2721	1.1043	1.0149	2.8631	2.2323	2.0642	2.0557	2.1338
$p_1 = 0.5$	n=10	1.3741	1.0056	0.8332	0.7357	0.6772	1.4836	1.1810	1.0551	0.9926	0.9622
$p_2 = 0.8$	n=15	0.9055	0.7229	0.6260	0.5658	0.5266	0.9533	0.8025	0.7283	0.6842	0.6564
$\|\psi\| = 1.3863$	n=30	0.3824	0.3517	0.3292	0.3121	0.2992	0.3933	0.3712	0.3550	0.3424	0.3324
	n=50	0.2253	0.2162	0.2085	0.2021	0.1968	0.2294	0.2235	0.2183	0.2138	0.2099

Table 4.6: MSE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$) (cont.)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	2.4488	1.5719	1.2258	1.0554	0.9657	2.8638	2.2034	2.0261	2.0138	2.0907
$p_1 = 0.2$	n=10	1.3951	1.0238	0.8504	0.7522	0.6932	1.5056	1.2011	1.0750	1.0126	0.9824
$p_2 = 0.5$	n=15	0.8868	0.7067	0.6112	0.5518	0.5132	0.9345	0.7864	0.7135	0.6703	0.6431
$\lvert\psi\rvert = 1.3863$	n=30	0.3790	0.3476	0.3255	0.3089	0.2965	0.3898	0.3667	0.3507	0.3386	0.3291
	n=50	0.2189	0.2101	0.2028	0.1967	0.1917	0.2228	0.2173	0.2124	0.2082	0.2045
	n=5	2.6213	1.6718	1.3101	1.1463	1.0733	3.0954	2.3736	2.1784	2.1679	2.2602
$p_1 = 0.3$	n=10	1.2718	0.9721	0.8198	0.7321	0.6817	1.3905	1.1644	1.0623	1.0104	0.9868
$p_2 = 0.7$	n=15	0.7674	0.6613	0.5938	0.5488	0.5193	0.8171	0.7462	0.7037	0.6760	0.6580
$\lvert\psi\rvert = 1.6946$	n=30	0.3320	0.3139	0.2996	0.2886	0.2804	0.3435	0.3344	0.3268	0.3204	0.3151
	n=50	0.2061	0.1996	0.1941	0.1896	0.1861	0.2105	0.2074	0.2046	0.2021	0.1998
	n=5	2.5176	1.6110	1.2927	1.1649	1.1216	2.9624	2.2421	2.0468	2.0282	2.1057
$p_1 = 0.4$	n=10	1.4102	1.0185	0.8447	0.7558	0.7118	1.5324	1.2038	1.0657	0.9961	0.9614
$p_2 = 0.8$	n=15	0.9245	0.7347	0.6370	0.5803	0.5477	0.9778	0.8196	0.7404	0.6927	0.6622
$\lvert\psi\rvert = 1.7918$	n=30	0.3885	0.3567	0.3341	0.3179	0.3068	0.4007	0.3773	0.3602	0.3468	0.3360
	n=50	0.2297	0.2201	0.2123	0.2062	0.2015	0.2342	0.2278	0.2223	0.2175	0.2132
	n=5	2.1608	1.4949	1.3248	1.2918	1.3111	2.4677	1.8764	1.7284	1.7050	1.7405
$p_1 = 0.1$	n=10	1.6509	1.0924	0.9199	0.8655	0.8613	1.7573	1.2227	1.0449	0.9692	0.9342
$p_2 = 0.4$	n=15	1.3105	0.8681	0.7104	0.6477	0.6290	1.3633	0.9343	0.7731	0.6961	0.6554
$\lvert\psi\rvert = 1.7918$	n=30	0.6833	0.5265	0.4522	0.4124	0.3921	0.6966	0.5452	0.4710	0.4273	0.3998
	n=50	0.3889	0.3370	0.3053	0.2846	0.2713	0.3938	0.3444	0.3134	0.2916	0.2759
	n=5	2.5553	1.6307	1.3069	1.1768	1.1323	3.0036	2.2651	2.0637	2.0421	2.1175
$p_1 = 0.2$	n=10	1.4289	1.0259	0.8482	0.7578	0.7130	1.5516	1.2111	1.0685	0.9965	0.9604
$p_2 = 0.6$	n=15	0.9043	0.7200	0.6253	0.5706	0.5395	0.9569	0.8037	0.7271	0.6811	0.6518
$\lvert\psi\rvert = 1.7918$	n=30	0.3849	0.3522	0.3298	0.3141	0.3034	0.3970	0.3727	0.3558	0.3428	0.3324
	n=50	0.2286	0.2192	0.2116	0.2055	0.2010	0.2331	0.2269	0.2216	0.2168	0.2126
	n=5	2.2032	1.5669	1.4598	1.4946	1.5798	2.5408	1.9370	1.7921	1.7726	1.8137
$p_1 = 0.5$	n=10	1.6049	1.0510	0.9004	0.8751	0.9034	1.7235	1.1806	1.0014	0.9255	0.8901
$p_2 = 0.9$	n=15	1.3101	0.8715	0.7238	0.6755	0.6739	1.3694	0.9392	0.7772	0.6996	0.6585
$\lvert\psi\rvert = 2.1972$	n=30	0.7203	0.5398	0.4581	0.4172	0.3991	0.7362	0.5603	0.4759	0.4268	0.3960
	n=50	0.3914	0.3366	0.3047	0.2850	0.2735	0.3969	0.3444	0.3122	0.2899	0.2739
	n=5	2.1340	1.4834	1.3740	1.4098	1.4971	2.4793	1.8617	1.7135	1.6936	1.7358
$p_1 = 0.1$	n=10	1.6271	1.0733	0.9225	0.8969	0.9249	1.7467	1.2046	1.0261	0.9506	0.9156
$p_2 = 0.5$	n=15	1.2989	0.8563	0.7078	0.6594	0.6580	1.3588	0.9250	0.7622	0.6846	0.6437
$\lvert\psi\rvert = 2.1972$	n=30	0.6769	0.5197	0.4473	0.4113	0.3960	0.6920	0.5394	0.4645	0.4204	0.3925
	n=50	0.3848	0.3327	0.3016	0.2823	0.2709	0.3903	0.3406	0.3093	0.2874	0.2715
	n=5	2.6566	1.6374	1.3435	1.2735	1.2968	3.1502	2.2641	2.0161	1.9727	2.0379
$p_1 = 0.2$	n=10	1.5740	1.0891	0.8910	0.8045	0.7761	1.7166	1.2882	1.1068	1.0129	0.9626
$p_2 = 0.7$	n=15	0.9855	0.7712	0.6654	0.6098	0.5844	1.0465	0.8626	0.7678	0.7095	0.6710
$\lvert\psi\rvert = 2.2336$	n=30	0.4097	0.3729	0.3487	0.3329	0.3237	0.4235	0.3950	0.3746	0.3587	0.3458
	n=50	0.2439	0.2330	0.2246	0.2184	0.2142	0.2490	0.2414	0.2348	0.2289	0.2237
	n=5	2.6551	1.6374	1.3424	1.2714	1.2939	3.1465	2.2620	2.0135	1.9697	2.0347
$p_1 = 0.3$	n=10	1.5462	1.0787	0.8869	0.8034	0.7766	1.6872	1.2759	1.1009	1.0102	0.9617
$p_2 = 0.8$	n=15	1.0043	0.7839	0.6754	0.6184	0.5921	1.0654	0.8750	0.7774	0.7173	0.6778
$\lvert\psi\rvert = 2.2336$	n=30	0.4122	0.3763	0.3520	0.3359	0.3263	0.4260	0.3987	0.3782	0.3620	0.3489
	n=50	0.2447	0.2336	0.2251	0.2189	0.2147	0.2497	0.2420	0.2353	0.2293	0.2241

Table 4.6: MSE of $\hat{\psi}_\epsilon$ with ϵ added to all tables vs. with ϵ and correction added to all tables ($\hat{\psi}_{\epsilon,0.5}$) (cont.)

		ϵ added to all tables					ϵ and correction added all tables				
	n	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$	$\epsilon = 0.25$	$\epsilon = 0.5$	$\epsilon = 0.75$	$\epsilon = 1$	$\epsilon = 1.25$
	n=5	2.2310	1.5642	1.5145	1.6258	1.7904	2.6044	1.9085	1.7357	1.7025	1.7368
$p_1 = 0.4$	n=10	1.6417	1.0625	0.9205	0.9182	0.9767	1.7770	1.1958	0.9996	0.9133	0.8704
$p_2 = 0.9$	n=15	1.3334	0.8836	0.7386	0.7007	0.7143	1.4014	0.9553	0.7843	0.7005	0.6545
$\|\psi\| = 2.6027$	n=30	0.7270	0.5444	0.4633	0.4252	0.4117	0.7452	0.5664	0.4800	0.4291	0.3968
	n=50	0.3961	0.3405	0.3087	0.2900	0.2801	0.4024	0.3490	0.3160	0.2930	0.2763
	n=5	2.2390	1.5703	1.5217	1.6341	1.7994	2.6110	1.9114	1.7381	1.7044	1.7377
$p_1 = 0.1$	n=10	1.6532	1.0670	0.9233	0.9205	0.9790	1.7887	1.2002	1.0018	0.9148	0.8716
$p_2 = 0.6$	n=15	1.3220	0.8754	0.7328	0.6966	0.7115	1.3899	0.9469	0.7781	0.6959	0.6509
$\|\psi\| = 2.6027$	n=30	0.6774	0.5200	0.4490	0.4160	0.4054	0.6948	0.5414	0.4652	0.4197	0.3906
	n=50	0.3973	0.3429	0.3112	0.2921	0.2820	0.4038	0.3516	0.3187	0.2956	0.2786
	n=5	2.6816	1.6121	1.4163	1.4701	1.6174	3.1984	2.1455	1.8471	1.7713	1.8085
$p_1 = 0.2$	n=10	1.8601	1.1974	0.9671	0.8971	0.9065	2.0309	1.4024	1.1459	1.0138	0.9396
$p_2 = 0.8$	n=15	1.2281	0.8944	0.7499	0.6880	0.6735	1.3038	0.9937	0.8422	0.7516	0.6923
$\|\psi\| = 2.7726$	n=30	0.4908	0.4354	0.4016	0.3822	0.3739	0.5075	0.4597	0.4260	0.4003	0.3799
	n=50	0.2827	0.2670	0.2558	0.2483	0.2444	0.2887	0.2762	0.2654	0.2561	0.2480
	n=5	2.3582	1.6078	1.6238	1.8325	2.0996	2.7762	1.9093	1.6841	1.6228	1.6389
$p_1 = 0.3$	n=10	1.7839	1.1249	0.9779	0.9988	1.0940	1.9425	1.2641	1.0280	0.9199	0.8625
$p_2 = 0.9$	n=15	1.4170	0.9328	0.7825	0.7536	0.7848	1.4958	1.0092	0.8173	0.7205	0.6652
$\|\psi\| = 3.0445$	n=30	0.7514	0.5637	0.4819	0.4463	0.4378	0.7725	0.5878	0.4971	0.4431	0.4082
	n=50	0.4112	0.3538	0.3217	0.3037	0.2956	0.4185	0.3631	0.3285	0.3042	0.2865
	n=5	2.3274	1.5927	1.6163	1.8294	2.0992	2.7427	1.8899	1.6708	1.6126	1.6300
$p_1 = 0.1$	n=10	1.8034	1.1303	0.9787	0.9976	1.0917	1.9634	1.2714	1.0310	0.9211	0.8628
$p_2 = 0.7$	n=15	1.4081	0.9264	0.7773	0.7492	0.7809	1.4878	1.0043	0.8141	0.7183	0.6638
$\|\psi\| = 3.0445$	n=30	0.7025	0.5401	0.4684	0.4379	0.4323	0.7227	0.5633	0.4829	0.4342	0.4024
	n=50	0.4139	0.3572	0.3248	0.3063	0.2977	0.4213	0.3668	0.3321	0.3075	0.2893
	n=5	2.3880	1.6377	1.8134	2.2011	2.6400	2.8286	1.8046	1.5416	1.4519	1.4373
$p_1 = 0.2$	n=10	2.1046	1.2456	1.0842	1.1485	1.3103	2.2983	1.3851	1.0731	0.9315	0.8547
$p_2 = 0.9$	n=15	1.6530	1.0448	0.8674	0.8491	0.9101	1.7504	1.1282	0.8821	0.7585	0.6881
$\|\psi\| = 3.5835$	n=30	0.8327	0.6229	0.5335	0.4990	0.4977	0.8579	0.6495	0.5451	0.4824	0.4421
	n=50	0.4502	0.3874	0.3530	0.3353	0.3297	0.4588	0.3976	0.3587	0.3313	0.3114
	n=5	2.3420	1.6137	1.7977	2.1896	2.6310	2.7744	1.7692	1.5121	1.4245	1.4103
$p_1 = 0.1$	n=10	2.0916	1.2398	1.0810	1.1465	1.3088	2.2849	1.3790	1.0696	0.9293	0.8534
$p_2 = 0.8$	n=15	1.6604	1.0503	0.8722	0.8538	0.9148	1.7588	1.1347	0.8881	0.7642	0.6937
$\|\psi\| = 3.5835$	n=30	0.7875	0.6034	0.5235	0.4936	0.4947	0.8118	0.6292	0.5346	0.4768	0.4392
	n=50	0.4537	0.3915	0.3566	0.3383	0.3322	0.4625	0.4020	0.3629	0.3350	0.3146
	n=5	2.0871	1.8077	2.4756	3.2920	4.0994	2.4266	1.5181	1.3371	1.2525	1.1875
$p_1 = 0.1$	n=10	2.3343	1.3145	1.2897	1.5547	1.9294	2.5555	1.3728	1.0445	0.9274	0.8741
$p_2 = 0.9$	n=15	2.1067	1.2080	1.0279	1.0936	1.2725	2.2317	1.2722	0.9471	0.8130	0.7535
$\|\psi\| = 4.3944$	n=30	1.1354	0.7894	0.6599	0.6273	0.6514	1.1701	0.8177	0.6548	0.5674	0.5194
	n=50	0.6236	0.5114	0.4548	0.4305	0.4290	0.6356	0.5232	0.4561	0.4126	0.3841

4.2 Variances of Estimators of ψ_ϵ

With 2×2 tables generated under different p_1,p_2 and n, estimates of $Var(\hat{\psi})$ were calculated with different estimators and compared with the actual value of the variance. Table 4.7 shows an example of the percentage of bias of three estimators of variance when $n \leq 15$. In this part of the simulation study, we only considered three values of ϵ: 0.5, 0.75 and 1, based on our conclusions of the comparison of estimators of ψ.

According to the result, the proposed estimator of variance, $\widehat{Var}_p(\psi)$ (3.7) tends to underestimate the true variance regardless of $|\psi|$. The percentage of bias decreases with increasing n and also with ϵ. The performance of $\widehat{Var}_1(\psi)$ (3.8) is highly related to $|\psi|$. When $|\psi|$ is 0 or large, it tends to overestimate the true variance especially for $n = 5$. $\widehat{Var}_2(\psi)$ (3.9) almost always overestimates the sample variance. When $\epsilon = 0.5$, $\widehat{Var}_1(\psi)$ and $\widehat{Var}_2(\psi)$ coincide. When $\epsilon > 0.5$, $\widehat{Var}_2(\psi)$ has large percentage of bias than $\widehat{Var}_1(\psi)$.

In other words, obtaining confidence intervals using variance/standard deviation estimated by $\widehat{Var}_p(\psi)$ could produce a narrower width and likely losing information. However, the confidence interval obtained based on $\widehat{Var}_1(\psi)$ or $\widehat{Var}_2(\psi)$ could be too wide especially when $n = 5$ for ψ either equal to 0 or very large.

Table 4.7: Bias of Variance Estimators in Percentage with different ϵ

		$\widehat{Var}_p(\hat{\psi}_\epsilon)$			$\widehat{Var}_1(\hat{\psi}_\epsilon)$			$\widehat{Var}_2(\hat{\psi}_\epsilon)$		
	n	0.5	0.75	1	0.5	0.75	1	0.5	0.75	1
$p_1 = 0.1$	n=5	190.98	216.86	246.11	-54.58	-46.87	-41.56	190.98	296.48	374.22
$p_2 = 0.1$	n=10	79.52	97.15	115.78	-49.32	-41.71	-36.55	79.52	140.81	186.65
$\|\psi\| = 0$	n=15	45.49	59.30	73.61	-44.34	-37.46	-32.92	45.49	90.25	124.18
$p_1 = 0.2$	n=5	60.47	75.95	92.63	-48.41	-41.82	-37.53	60.47	111.85	151.46
$p_2 = 0.2$	n=10	17.22	27.96	38.59	-36.17	-30.66	-27.20	17.22	46.52	69.54
$\|\psi\| = 0$	n=15	4.98	13.47	21.51	-26.29	-22.66	-20.45	4.98	25.10	41.62
$p_1 = 0.3$	n=5	17.20	29.57	42.06	-36.22	-31.62	-28.87	17.20	48.61	75.00
$p_2 = 0.4$	n=10	0.39	7.60	14.48	-21.40	-19.08	-17.70	0.39	16.21	29.97
$\|\psi\| = 0.4418$	n=15	-2.76	2.14	6.78	-14.34	-13.41	-12.83	-2.76	7.22	16.36
$p_1 = 0.5$	n=5	12.55	24.48	36.47	-36.89	-32.59	-30.02	12.55	42.27	67.42
$p_2 = 0.7$	n=10	-1.15	5.92	12.63	-21.24	-19.03	-17.70	-1.15	13.98	27.27
$\|\psi\| = 0.8573$	n=15	2.91	7.81	12.49	-8.49	-7.75	-7.28	2.91	12.91	22.16
$p_1 = 0.2$	n=5	33.95	46.57	59.92	-41.12	-35.63	-32.23	33.95	71.86	102.13
$p_2 = 0.4$	n=10	9.69	17.94	25.99	-28.33	-24.04	-21.47	9.69	31.31	48.52
$\|\psi\| = 0.9808$	n=15	2.34	8.73	14.68	-19.92	-17.32	-15.80	2.34	17.16	29.39
$p_1 = 0.3$	n=5	42.04	55.95	70.68	-43.75	-37.68	-33.85	42.04	84.66	118.29
$p_2 = 0.8$	n=10	9.78	19.25	28.40	-31.16	-26.42	-23.50	9.78	33.77	52.96
$\|\psi\| = 2.2336$	n=15	6.09	13.74	20.89	-18.96	-15.80	-13.87	6.09	23.29	37.63
$p_1 = 0.1$	n=5	86.32	100.05	116.01	-46.77	-40.34	-36.24	86.32	142.75	184.59
$p_2 = 0.7$	n=10	46.51	55.75	65.73	-38.17	-31.69	-27.66	46.51	83.33	109.43
$\|\psi\| = 3.0445$	n=15	30.07	38.95	47.80	-32.68	-26.34	-22.41	30.07	59.88	81.25
$p_1 = 0.2$	n=5	109.44	126.93	146.78	-52.56	-45.63	-41.02	109.44	179.22	230.91
$p_2 = 0.9$	n=10	50.80	63.49	76.75	-42.14	-35.17	-30.67	50.80	94.57	126.91
$\|\psi\| = 3.5835$	n=15	32.01	43.08	54.20	-34.97	-28.40	-24.19	32.01	65.98	91.52

[1] $\widehat{Var}_p(\hat{\psi}_\epsilon)$: Proposed estimator of variance defined as equation (3.7).
[2] $\widehat{Var}_1(\hat{\psi}_\epsilon)$: Estimator based assumption of asymptotic normality defined as equation (3.8).
[3] $\widehat{Var}_2(\hat{\psi}_\epsilon)$: Estimator proposed by Berdick defined as equation (3.9).

Chapter 5

Results and Future Research

5.1 Results

Bias Bias is increasing with the value of ϵ increasing from 0.25 to 1.25. In most cases, bias is negative for $\epsilon = 0.25$ and getting closer to 0 and then becoming positive. In these cases, the absolute value of average bias is not increasing monotonically as ϵ increases. Considering n=5, 10, 15, 30, and 50, absolute values of bias decrease with the increasing n in most cases. To lower the bias if n is small ($n = 5$), it is better to add ϵ only if there are zeros in the table or always add epsilon to cell counts and apply correction (3.5). For larger n ($n = 10, 15$) it is better to always add epsilon to cell counts when $|\psi|$ is small ($|\psi| < 1.2528$). When $|\psi|$ is large ($|\psi| \geq 1.2528$), it is better to add epsilon to cell counts and follow with correction or add epsilon only when there is a zero count for n =10 and 15.

AAE In general, one can observe a decreasing trend in AAE with increasing ϵ but eventually the trend will start to reverse. This is caused by the bias that increases with epsilon and is more obvious when $|\psi|$ is large, values of ϵ that have the smallest AAE are usually among 0.5, 0.75 and 1. AAE decreases with n increasing (from 5 to 50). When $|\psi|$ is low ($|\psi| < 1.7918$), AAE is lower if ϵ is always added to the cell counts and correction is not used. When the value of $|\psi|$ is large ($|\psi| \geq 1.7918$) then adding ϵ only when there are zeros in the table would lower AAE. When the value of $|\psi|$ is even larger ($|\psi| \geq 3.0445$) then always adding ϵ and applying correction term usually lowers AAE.

MSE Similarly to AAE, one can observe a decreasing trend in MSE with increasing ϵ but eventually the trend will start to reverse. The selections of ϵ that minimizes MSE are usually among 0.5, 0.75 and 1. MSE decreases with increasing n in most cases. When the $|\psi|$ value is low ($|\psi| < 2.1972$), MSE is lower if ϵ is always added to the cell counts and correction is not used. When the value of $|\psi|$ is large ($|\psi| \geq 2.1972$) then adding ϵ only when there are zeros in the table would lower MSE. Adding ϵ and applying correction term can lower MSE when $|\psi| \geq 4.3944$.

The results mentioned above are based on the value of ψ. Therefore, for these results to be meaningful we need to make recommendations for the practical use when ψ is unknown (estimated). In practice, a researcher has one, or few, 2×2 tables. Then, to estimate ψ with the lowest MSE and AAE he/she should:

- Find out if $|\psi|$ is large (≥ 1.7918) or low (< 1.7918).
- Add ϵ to all cell counts when ψ is low while for ψ large use the MLE estimator (2.2) if there are no zero counts while add ϵ when there is at least one cell count equal zero.
- If $|\psi|$ is very large (≥ 3), one could also add ϵ to all cell counts regardless if any of them is a zero and use correction (3.5).

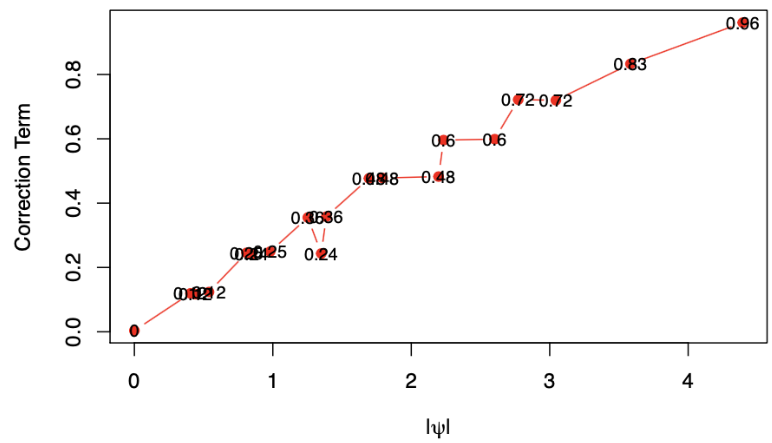

Figure 5.1: n=5, Correction term (3.4), $\epsilon = 0.75$, an average in 5000 repetitions.

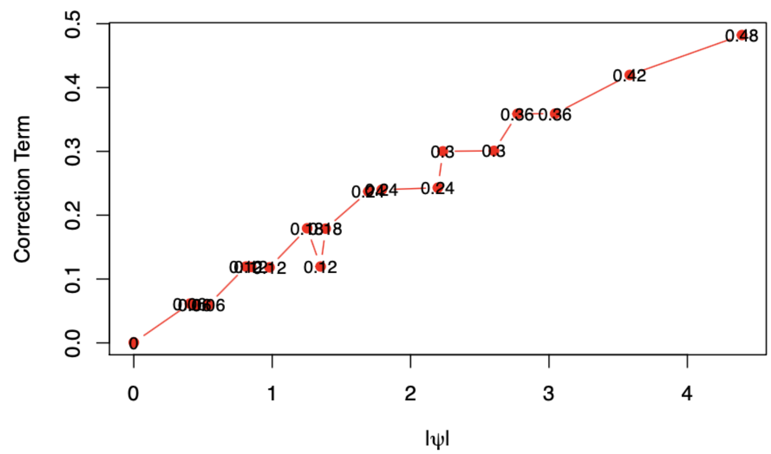

Figure 5.2: n=10 Correction term (3.4), $\epsilon = 0.75$, average in 5000 repetitions.

Two questions still need to be answered:

- How to find out if $|\psi|$ is large or low?
- What value of ϵ should be used?

The value of ψ is strongly correlated with the difference between the estimator $\hat{\psi}_\epsilon$ and $\hat{\psi}_{\epsilon,0.5}$ - so with the correction term (3.4). The trend, independent of the value of ϵ can be seen in Figures 5.1 - 5.4. Of course it does not need to be true for any particular 2×2 table and related estimators, and further research is needed here. Another recommendation is related to the value of ϵ to be added. Since

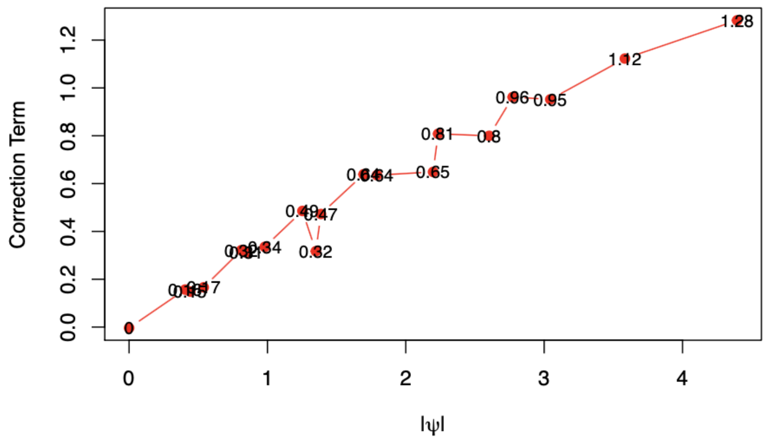

Figure 5.3: n=5, Correction term (3.4), $\epsilon = 1$, an average in 5000 repetitions.

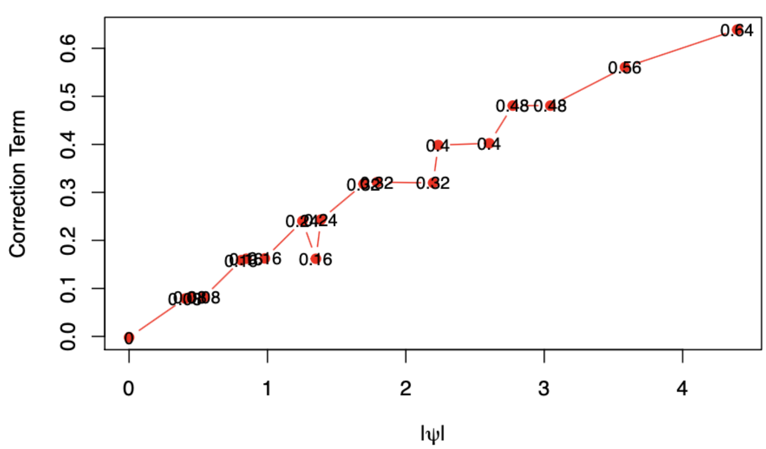

Figure 5.4: n=10, Correction term (3.4), $\epsilon = 1$, an average in 5000 repetitions.

AAE and MSE first decrease with increasing ϵ and then start to increase, we were looking for a value of ϵ that minimizes AAE and MSE. Such value is not the same for all pairs p_1, p_2.

Table 5.1: Frequencies of values of ϵ minimizing AAE and MSE

ϵ	AAE	MSE
0.5	5/31	5/30
0.75	12/31	12/30
1	14/31	13/30

Table 5.1 shows proportions of values of ϵ that minimize AAE and MSE for 31 pairs of (p_1, p_2). Based on these results we would suggest that either $\epsilon = 0.75$ or $\epsilon = 1$ should be used. While $\epsilon = 0.5$ is the most common selection of ϵ in statistical practice, its use is not recommended considering AAE and MSE of the estimators. While $\epsilon = 0.5$ could reduce the bias in some pairs of p_1 and p_2 (large n), in small sample sizes, the selection of ϵ is erratic. To make a suggestion on the choice of ϵ value for small tables, one could base the choice on the performance of the other two characteristics (AAE and MSE). For larger values of n ($n > 15$) the value of ϵ makes no difference in bias, AAE, and MSE what can be seen in Tables 4.1-4.6.

Based on the results of simulations, we noticed that the performance of the estimators depends on the true value of $|\psi|$. For different values of $|\psi|$, different estimators or strategies should be used in order to have a less biased estimation. Therefore it motivated our idea of combined strategies of using estimator in estimating $|\psi|$. In the combined strategies, the estimator selection was determined by the comparison of the estimated $|\psi|$ and the values of thresholds we found in the results of previous simulation. By using the suggested values of ϵ we suggested to users, for ϵ=0.75 and 1, we tried several combined strategies:

- $\hat{\psi}_{Com1}$: The true value of $|\psi|$ is estimated by the correction term calculated. When the correction term is less than then threshold (0.49 for n=5, 0.3 for n=10, 0.2 for n=15), ϵ is added to all tables ($\hat{\psi}_{\epsilon}$), otherwise ϵ is only added when there are zero counts in the table.

- $\hat{\psi}_{Com2}$: The true value of $|\psi|$ is estimated by the correction term calculated. When the correction term is less than then threshold (0.84 for n=5, 0.42 for n=10, 0.32 for n=15), ϵ is added to all tables ($\hat{\psi}_{\epsilon}$), otherwise ϵ is always added and correction term (3.5) is applied.

- $\hat{\psi}_{Com3}$: The true values of $|\psi|$ is estimated by $\hat{\psi}_{\epsilon}$. When the correction term is less than then threshold (3.57 for n=5, 3.5735 for n=10, 3.49 for n=15), ϵ was added to all tables ($\hat{\psi}_{\epsilon}$), otherwise ϵ was always added and applying correction term at the same time ($\hat{\psi}_{\epsilon,0.5}$).

- $\hat{\psi}_{Com4}$: The true value of $|\psi|$ is estimated by the correction term calculated. When the correction term is less than then threshold 1 (0.72 for n=5, 0.36 for n=10, 0.24 for n=15), ϵ is added to all tables ($\hat{\psi}_{\epsilon}$). When the value is larger than thresholds 2 (0.84 for n=5, 0.42 for n=10, 0.28 for n=15), ϵ is always added and correction terms is applied at the same time ($\hat{\psi}_{\epsilon,0.5}$). When the estimated value is in between, ϵ is only added when there are zero counts in the table..

- $\hat{\psi}_{Com5}$: The true value of $|\psi|$ is estimated by $\hat{\psi}_{\epsilon}$ calculated from the table. When the correction term is less than then threshold (3.4835 for n=5, 3.5735 for n=10, 3.9 for n=15), ϵ was added to all tables ($\hat{\psi}_{\epsilon}$), otherwise ϵ is added to tables contains zero counts and correction term is applied.

Table 5.2 to 5.5 show examples of how the estimators with combined strategies work for pairs of (p_1, p_2). Since $|\psi|$ is estimated either by estimators or based on the value of correction term, it is very likely to have incorrect selection of strategies for a specific table. This kind of variation increases the MSE or AAE in many cases especially when p_1 and p_2 are close to 0.5 or $n = 5$. So in these cases, MSE or AAE are not reduced by the using the combined estimators. The tables that contains all combinations of (p_1, p_2) can be found in Appendix.

Table 5.2: AAE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.6260	0.6432	0.6262	0.6260	0.6260	0.6262	0.6260
$p_2 = 0.1$	n=10	0.7101	0.7700	0.7101	0.7101	0.7101	0.7101	0.7101
$\lvert\psi\rvert = 0$	n=15	0.7283	0.8318	0.7283	0.7283	0.7283	0.7283	0.7283
$p_1 = 0.2$	n=5	0.8142	0.8934	0.8167	0.8155	0.8144	0.8180	0.8144
$p_2 = 0.2$	n=10	0.7325	0.8730	0.7338	0.7325	0.7325	0.7326	0.7325
$\lvert\psi\rvert = 0$	n=15	0.6707	0.8097	0.6708	0.6707	0.6707	0.6707	0.6707
$p_1 = 0.3$	n=5	0.9035	1.0900	0.9230	0.9218	0.9049	0.9414	0.9049
$p_2 = 0.4$	n=10	0.6846	0.8193	0.7004	0.6862	0.6847	0.6898	0.6847
$\lvert\psi\rvert = 0.4418$	n=15	0.6003	0.6885	0.6045	0.6003	0.6004	0.6014	0.6004
$p_1 = 0.5$	n=5	0.8905	1.0327	0.9363	0.9313	0.8951	0.9771	0.8951
$p_2 = 0.7$	n=10	0.6936	0.8206	0.7454	0.7008	0.6942	0.7171	0.6942
$\lvert\psi\rvert = 0.8573$	n=15	0.5845	0.6638	0.5991	0.5845	0.5849	0.5911	0.5849
$p_1 = 0.2$	n=5	0.8895	0.9427	0.9122	0.9212	0.8922	0.9438	0.8922
$p_2 = 0.4$	n=10	0.7193	0.8370	0.7509	0.7234	0.7198	0.7306	0.7198
$\lvert\psi\rvert = 0.9808$	n=15	0.6359	0.7419	0.6481	0.6360	0.6363	0.6405	0.6363
$p_1 = 0.3$	n=5	0.9231	0.8373	0.9657	1.1625	0.9889	1.2050	0.9889
$p_2 = 0.8$	n=10	0.7576	0.8383	0.9208	0.8608	0.7869	0.9662	0.7869
$\lvert\psi\rvert = 2.2336$	n=15	0.6493	0.7595	0.8360	0.6614	0.6658	0.7828	0.6594
$p_1 = 0.1$	n=5	0.9835	0.8120	0.8909	1.3439	1.0967	1.2513	1.0967
$p_2 = 0.7$	n=10	0.8024	0.6851	0.7350	0.9540	0.8813	0.8567	0.8813
$\lvert\psi\rvert = 3.0445$	n=15	0.7148	0.7085	0.7702	0.7527	0.7791	0.8067	0.7623
$p_1 = 0.2$	n=5	1.1125	0.9506	1.0066	1.2909	1.3386	1.1849	1.3386
$p_2 = 0.9$	n=10	0.8490	0.6642	0.6816	0.9604	1.0080	0.8248	1.0080
$\lvert\psi\rvert = 3.5835$	n=15	0.7604	0.7360	0.7531	0.8684	0.8856	0.7171	0.8322

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table 5.3: MSE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_{\epsilon}$	$\hat{\psi}^*_{\epsilon}$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.7749	0.8094	0.7756	0.7749	0.7749	0.7756	0.7749
$p_2 = 0.1$	n=10	0.8583	0.9594	0.8583	0.8583	0.8583	0.8583	0.8583
$\|\psi\| = 0$	n=15	0.8794	1.0682	0.8794	0.8794	0.8794	0.8794	0.8794
$p_1 = 0.2$	n=5	1.1234	1.2964	1.1352	1.1328	1.1256	1.1447	1.1256
$p_2 = 0.2$	n=10	0.9031	1.1970	0.9092	0.9031	0.9031	0.9038	0.9031
$\|\psi\| = 0$	n=15	0.7580	1.0589	0.7586	0.7580	0.7580	0.7580	0.7580
$p_1 = 0.3$	n=5	1.2518	1.6871	1.3277	1.3679	1.2640	1.4438	1.2640
$p_2 = 0.4$	n=10	0.7777	1.0956	0.8399	0.7873	0.7786	0.8024	0.7786
$\|\psi\| = 0.4418$	n=15	0.5830	0.7737	0.6022	0.5830	0.5838	0.5888	0.5838
$p_1 = 0.5$	n=5	1.2441	1.6578	1.3849	1.4719	1.2798	1.6126	1.2798
$p_2 = 0.7$	n=10	0.7737	1.0989	0.9469	0.8095	0.7774	0.8706	0.7774
$\|\psi\| = 0.8573$	n=15	0.5485	0.7244	0.6039	0.5485	0.5508	0.5774	0.5508
$p_1 = 0.2$	n=5	1.2214	1.4486	1.2850	1.3879	1.2409	1.4515	1.2409
$p_2 = 0.4$	n=10	0.8298	1.0876	0.9258	0.8501	0.8330	0.8742	0.8330
$\|\psi\| = 0.9808$	n=15	0.6533	0.8787	0.6971	0.6537	0.6557	0.6724	0.6557
$p_1 = 0.3$	n=5	1.3133	1.1469	1.3463	2.0794	1.6344	2.1124	1.6344
$p_2 = 0.8$	n=10	0.8911	1.0934	1.2157	1.1784	1.0181	1.2884	1.0181
$\|\psi\| = 2.2336$	n=15	0.6753	0.9106	0.9996	0.7261	0.7416	0.9243	0.7188
$p_1 = 0.1$	n=5	1.6061	1.2277	1.4839	2.2127	1.9753	2.0905	1.9753
$p_2 = 0.7$	n=10	0.9790	0.7797	0.9189	1.2704	1.1924	1.1880	1.1924
$\|\psi\| = 3.0445$	n=15	0.7712	0.7672	0.8924	0.8760	0.9242	0.9314	0.8914
$p_1 = 0.2$	n=5	1.7968	1.3025	1.5428	2.2442	2.2900	1.9902	2.290
$p_2 = 0.9$	n=10	1.0821	0.7434	0.8119	1.3360	1.3950	1.1662	1.395
$\|\psi\| = 3.5835$	n=15	0.8541	0.8145	0.8690	1.0499	1.0637	0.9042	1.012

[1] $\hat{\psi}_{\epsilon}$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_{\epsilon}$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

5.2 Future Research

As a part of future work, we will try to find a reliable way to estimate the values of ψ based on a specific table. The method of using the difference between $\hat{\psi}_{\epsilon}$ and $\hat{\psi}_{\epsilon,0.5}$ (the bias correction term) to estimate the true values of ψ need further investigation. We will also study the properties of estimators under unbalanced samples, when $n_{1+} \neq n_{2+}$. Resampling techniques like bootstrap could be a way of better estimating the threshold values and is certainly worthy investigating. Bayesian ap-

Table 5.4: AAE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_{\epsilon}$	$\hat{\psi}^*_{\epsilon}$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.5359	0.5567	0.5381	0.5359	0.5359	0.5359	0.5359
$p_2 = 0.1$	n=10	0.6160	0.6882	0.6160	0.6160	0.6160	0.6160	0.6160
$\|\psi\| = 0$	n=15	0.6432	0.7683	0.6432	0.6432	0.6432	0.6432	0.6432
$p_1 = 0.2$	n=5	0.7119	0.8077	0.7273	0.7137	0.7122	0.7137	0.7122
$p_2 = 0.2$	n=10	0.6635	0.8347	0.6651	0.6635	0.6635	0.6637	0.6635
$\|\psi\| = 0$	n=15	0.6231	0.7940	0.6233	0.6231	0.6231	0.6231	0.6231
$p_1 = 0.3$	n=5	0.8105	1.0369	0.8979	0.8349	0.8124	0.8349	0.8124
$p_2 = 0.4$	n=10	0.6458	0.8105	0.6654	0.6480	0.6459	0.6522	0.6459
$\|\psi\| = 0.4418$	n=15	0.5753	0.6868	0.5806	0.5753	0.5754	0.5767	0.5754
$p_1 = 0.5$	n=5	0.8153	0.9802	0.9622	0.8697	0.8214	0.8697	0.8214
$p_2 = 0.7$	n=10	0.6583	0.8130	0.7227	0.6679	0.6583	0.6876	0.6583
$\|\psi\| = 0.8573$	n=15	0.5637	0.6622	0.5820	0.5637	0.5639	0.5720	0.5639
$p_1 = 0.2$	n=5	0.8134	0.8781	0.9052	0.8555	0.8169	0.8555	0.8169
$p_2 = 0.4$	n=10	0.6741	0.8107	0.7131	0.6796	0.6742	0.6882	0.6742
$\|\psi\| = 0.9808$	n=15	0.6058	0.7326	0.6210	0.6058	0.6062	0.6115	0.6062
$p_1 = 0.3$	n=5	0.9072	0.7704	0.8035	1.2263	0.9949	1.2263	0.9949
$p_2 = 0.8$	n=10	0.7184	0.7989	0.9025	0.8560	0.7361	0.9759	0.7361
$\|\psi\| = 2.2336$	n=15	0.6313	0.7459	0.8481	0.6369	0.6440	0.8016	0.6425
$p_1 = 0.1$	n=5	1.1338	0.9257	0.9454	1.4442	1.2849	1.4442	1.2849
$p_2 = 0.7$	n=10	0.8088	0.6351	0.6971	0.9872	0.8564	0.8582	0.8564
$\|\psi\| = 3.0445$	n=15	0.7032	0.6551	0.7326	0.7212	0.7545	0.7856	0.7495
$p_1 = 0.2$	n=5	1.2161	1.0197	1.0306	1.3525	1.5176	1.3525	1.5176
$p_2 = 0.9$	n=10	0.8766	0.6062	0.6280	0.9328	1.0073	0.8049	1.0073
$\|\psi\| = 3.5835$	n=15	0.7463	0.6759	0.6975	0.8172	0.8556	0.6713	0.8334

[1] $\hat{\psi}_{\epsilon}$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_{\epsilon}$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

proach is another statistical method that could be applied in our study in estimating ψ. Instead of using the frequencies, one could use a proper prior distribution. Also, we would like to explore how to obtain confidence intervals based on one 2×2 contingency table and use them to detect heterogeneity of odds ratio in several 2×2 tables with small counts.

Table 5.5: MSE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.5717	0.6119	0.5785	0.5717	0.5717	0.5717	0.5717
$p_2 = 0.1$	n=10	0.6462	0.7641	0.6462	0.6462	0.6462	0.6462	0.6462
$\lvert\psi\rvert = 0$	n=15	0.6812	0.9021	0.6812	0.6812	0.6812	0.6812	0.6812
$p_1 = 0.2$	n=5	0.8600	1.0621	0.9108	0.8720	0.8628	0.8720	0.8628
$p_2 = 0.2$	n=10	0.7326	1.0797	0.7400	0.7326	0.7326	0.7334	0.7326
$\lvert\psi\rvert = 0$	n=15	0.6445	1.0038	0.6453	0.6445	0.6445	0.6445	0.6445
$p_1 = 0.3$	n=5	1.0085	1.5166	1.2312	1.1546	1.0236	1.1546	1.0236
$p_2 = 0.4$	n=10	0.6799	1.0618	0.7548	0.6920	0.6810	0.7097	0.6810
$\lvert\psi\rvert = 0.4418$	n=15	0.5322	0.7663	0.5556	0.5322	0.5328	0.5394	0.5328
$p_1 = 0.5$	n=5	1.0293	1.5048	1.3098	1.3134	1.0734	1.3134	1.0734
$p_2 = 0.7$	n=10	0.6857	1.0721	0.8927	0.7307	0.6857	0.8019	0.6857
$\lvert\psi\rvert = 0.8573$	n=15	0.5043	0.7176	0.5718	0.5043	0.5052	0.5396	0.5052
$p_1 = 0.2$	n=5	1.0157	1.2738	1.1549	1.2227	1.0396	1.2227	1.0396
$p_2 = 0.4$	n=10	0.7171	1.0153	0.8308	0.7424	0.7181	0.7701	0.7181
$\lvert\psi\rvert = 0.9808$	n=15	0.5834	0.8485	0.6362	0.5834	0.5852	0.6065	0.5852
$p_1 = 0.3$	n=5	1.2435	1.0036	1.0806	2.1366	1.6206	2.1366	1.6206
$p_2 = 0.8$	n=10	0.8020	0.9914	1.1528	1.1378	0.8823	1.2532	0.8823
$\lvert\psi\rvert = 2.2336$	n=15	0.6152	0.8670	0.9889	0.6414	0.6664	0.9073	0.6604
$p_1 = 0.1$	n=5	1.8236	1.3363	1.4214	2.4322	2.2282	2.4322	2.2282
$p_2 = 0.7$	n=10	0.9954	0.6896	0.8680	1.2817	1.1338	1.1814	1.1338
$\lvert\psi\rvert = 3.0445$	n=15	0.7391	0.6721	0.8367	0.7971	0.8602	0.8902	0.8480
$p_1 = 0.2$	n=5	2.1853	1.5575	1.6180	2.4566	2.6678	2.4566	2.6678
$p_2 = 0.9$	n=10	1.1426	0.6469	0.7345	1.3029	1.3993	1.0966	1.3993
$\lvert\psi\rvert = 3.5835$	n=15	0.8399	0.7031	0.7739	0.9908	1.0262	0.8111	0.9962

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Appendix A

R code

```
### Table generate
tableobt5 <- function(p1, p2) {
        x <- matrix(0, 5000, 11) for (i in 1:5000) {
                x[i, 1] = rbinom(1, 5, p1)
                x[i, 2] = rbinom(1, 5, p2)
                x[i, 3] = rbinom(1, 10, p1)
                x[i, 4] = rbinom(1, 10, p2)
                x[i, 5] = rbinom(1, 15, p1)
                x[i, 6] = rbinom(1, 15, p2)
                x[, 7] = log(p1 * (1 - p2)/p2/(1 - p1)) x[i, 8] = rbinom(1, 30, p1)
                x[i, 9] = rbinom(1, 30, p2) x[i, 10] = rbinom(1, 50, p1) x[i, 11] =
                     rbinom(1, 50, p2)
        }
        return(x) }

###Estimators

## \epsilon=0.25
nci_0.25<-function(a){
        tv<-mean(a[,7])
        esth<-matrix(0,5000,5)
        estp<-matrix(0,5000,5)
        ci1<-matrix(0,5,4)
        ci2<-matrix(0,5,4)
        for (i in 1:5000) {
```

```
            esth[i,1]<-log((a[i,1]+0.25)*(5-a[i,2]+0.25)/(a[i,2]+0.25)/(5-a[i
                ,1]+0.25))
            esth[i,2]<-log((a[i,3]+0.25)*(10-a[i,4]+0.25)/(a[i,4]+0.25)/(10-a[i
                ,3]+0.25))
            esth[i,3]<-log((a[i,5]+0.25)*(15-a[i,6]+0.25)/(a[i,6]+0.25)/(15-a[i
                ,5]+0.25))
            esth[i,4]<-log((a[i,8]+0.25)*(30-a[i,9]+0.25)/(a[i,9]+0.25)/(30-a[i
                ,8]+0.25))
            esth[i,5]<-log((a[i,10]+0.25)*(50-a[i,11]+0.25)/(a[i,11]+0.25)/(50-
                a[i,10]+0.25))
            estp[i,1]<-log((a[i,1]+0.25)*(5-a[i,2]+0.25)/(a[i,2]+0.25)/(5-a[i
                ,1]+0.25))-8*0.25/5^2*(a[i,2]-a[i,1])
            estp[i,2]<-log((a[i,3]+0.25)*(10-a[i,4]+0.25)/(a[i,4]+0.25)/(10-a[i
                ,3]+0.25))-8*0.25/10^2*(a[i,4]-a[i,3])
            estp[i,3]<-log((a[i,5]+0.25)*(15-a[i,6]+0.25)/(a[i,6]+0.25)/(15-a[i
                ,5]+0.25))-8*0.25/15^2*(a[i,6]-a[i,5])
            estp[i,4]<-log((a[i,8]+0.25)*(30-a[i,9]+0.25)/(a[i,9]+0.25)/(30-a[i
                ,8]+0.25))-8*0.25/30^2*(a[i,9]-a[i,8])
            estp[i,5]<-log((a[i,10]+0.25)*(50-a[i,11]+0.25)/(a[i,11]+0.25)/(50-
                a[i,10]+0.25))-8*0.25/50^2*(a[i,11]-a[i,10])
        }
        std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
        avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
        AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
        for (j in 1:5) {
            ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
            ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
        }
        tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
            [3,],ci1[4,],ci1[5,])
        compare_table<-round(tmp1, digits = 4)
        colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
        return(compare_table)
}

### \epsilon=0.5
nci_0.5<-function(a){
        tv<-mean(a[,7])
        esth<-matrix(0,5000,5)
```

```
    estp<-matrix(0,5000,5)
    ci1<-matrix(0,5,4)
    ci2<-matrix(0,5,4)
    for (i in 1:5000) {
        esth[i,1]<-log((a[i,1]+0.5)*(5-a[i,2]+0.5)/(a[i,2]+0.5)/(5-a[i
            ,1]+0.5))
        esth[i,2]<-log((a[i,3]+0.5)*(10-a[i,4]+0.5)/(a[i,4]+0.5)/(10-a[i
            ,3]+0.5))
        esth[i,3]<-log((a[i,5]+0.5)*(15-a[i,6]+0.5)/(a[i,6]+0.5)/(15-a[i
            ,5]+0.5))
        esth[i,4]<-log((a[i,8]+0.5)*(30-a[i,9]+0.5)/(a[i,9]+0.5)/(30-a[i
            ,8]+0.5))
        esth[i,5]<-log((a[i,10]+0.5)*(50-a[i,11]+0.5)/(a[i,11]+0.5)/(50-a[i
            ,10]+0.5))
        estp[i,1]<-log((a[i,1]+0.5)*(5-a[i,2]+0.5)/(a[i,2]+0.5)/(5-a[i
            ,1]+0.5))-8*0.5/5^2*(a[i,2]-a[i,1])
        estp[i,2]<-log((a[i,3]+0.5)*(10-a[i,4]+0.5)/(a[i,4]+0.5)/(10-a[i
            ,3]+0.5))-8*0.5/10^2*(a[i,4]-a[i,3])
        estp[i,3]<-log((a[i,5]+0.5)*(15-a[i,6]+0.5)/(a[i,6]+0.5)/(15-a[i
            ,5]+0.5))-8*0.5/15^2*(a[i,6]-a[i,5])
        estp[i,4]<-log((a[i,8]+0.5)*(30-a[i,9]+0.5)/(a[i,9]+0.5)/(30-a[i
            ,8]+0.5))-8*0.5/30^2*(a[i,9]-a[i,8])
        estp[i,5]<-log((a[i,10]+0.5)*(50-a[i,11]+0.5)/(a[i,11]+0.5)/(50-a[i
            ,10]+0.5))-8*0.5/50^2*(a[i,11]-a[i,10])
    }
    std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
    avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
    AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
    for (j in 1:5) {
        ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
        ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
    }
    tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
        [3,],ci1[4,],ci1[5,])
    compare_table<-round(tmp1, digits = 4)
    colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
    return(compare_table)
}
```

```
## \epsilon=0.75
nci_0.75<-function(a){
	tv<-mean(a[,7])
	esth<-matrix(0,5000,5)
	estp<-matrix(0,5000,5)
	ci1<-matrix(0,5,4)
	ci2<-matrix(0,5,4)
	for (i in 1:5000) {
		esth[i,1]<-log((a[i,1]+0.75)*(5-a[i,2]+0.75)/(a[i,2]+0.75)/(5-a[i
		    ,1]+0.75))
		esth[i,2]<-log((a[i,3]+0.75)*(10-a[i,4]+0.75)/(a[i,4]+0.75)/(10-a[i
		    ,3]+0.75))
		esth[i,3]<-log((a[i,5]+0.75)*(15-a[i,6]+0.75)/(a[i,6]+0.75)/(15-a[i
		    ,5]+0.75))
		esth[i,4]<-log((a[i,8]+0.75)*(30-a[i,9]+0.75)/(a[i,9]+0.75)/(30-a[i
		    ,8]+0.75))
		esth[i,5]<-log((a[i,10]+0.75)*(50-a[i,11]+0.75)/(a[i,11]+0.75)/(50-
		    a[i,10]+0.75))
		estp[i,1]<-log((a[i,1]+0.75)*(5-a[i,2]+0.75)/(a[i,2]+0.75)/(5-a[i
		    ,1]+0.75))-8*0.75/5^2*(a[i,2]-a[i,1])
		estp[i,2]<-log((a[i,3]+0.75)*(10-a[i,4]+0.75)/(a[i,4]+0.75)/(10-a[i
		    ,3]+0.75))-8*0.75/10^2*(a[i,4]-a[i,3])
		estp[i,3]<-log((a[i,5]+0.75)*(15-a[i,6]+0.75)/(a[i,6]+0.75)/(15-a[i
		    ,5]+0.75))-8*0.75/15^2*(a[i,6]-a[i,5])
		estp[i,4]<-log((a[i,8]+0.75)*(30-a[i,9]+0.75)/(a[i,9]+0.75)/(30-a[i
		    ,8]+0.75))-8*0.75/30^2*(a[i,9]-a[i,8])
		estp[i,5]<-log((a[i,10]+0.75)*(50-a[i,11]+0.75)/(a[i,11]+0.75)/(50-
		    a[i,10]+0.75))-8*0.75/50^2*(a[i,11]-a[i,10])
	}
	std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
	avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
	AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
	for (j in 1:5) {
		ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
		ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
	}
	tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
	    [3,],ci1[4,],ci1[5,])
```

```
        compare_table<-round(tmp1, digits = 4)
        colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
        return(compare_table)
}


## \epsilon=1
nci_1<-function(a){
        tv<-mean(a[,7])
        esth<-matrix(0,5000,5)
        estp<-matrix(0,5000,5)
        ci1<-matrix(0,5,4)
        ci2<-matrix(0,5,4)
        for (i in 1:5000) {
                esth[i,1]<-log((a[i,1]+1)*(5-a[i,2]+1)/(a[i,2]+1)/(5-a[i,1]+1))
                esth[i,2]<-log((a[i,3]+1)*(10-a[i,4]+1)/(a[i,4]+1)/(10-a[i,3]+1))
                esth[i,3]<-log((a[i,5]+1)*(15-a[i,6]+1)/(a[i,6]+1)/(15-a[i,5]+1))
                esth[i,4]<-log((a[i,8]+1)*(30-a[i,9]+1)/(a[i,9]+1)/(30-a[i,8]+1))
                esth[i,5]<-log((a[i,10]+1)*(50-a[i,11]+1)/(a[i,11]+1)/(50-a[i
                    ,10]+1))
                estp[i,1]<-log((a[i,1]+1)*(5-a[i,2]+1)/(a[i,2]+1)/(5-a[i,1]+1))-8*1
                    /5^2*(a[i,2]-a[i,1])
                estp[i,2]<-log((a[i,3]+1)*(10-a[i,4]+1)/(a[i,4]+1)/(10-a[i,3]+1))-8
                    *1/10^2*(a[i,4]-a[i,3])
                estp[i,3]<-log((a[i,5]+1)*(15-a[i,6]+1)/(a[i,6]+1)/(15-a[i,5]+1))-8
                    *1/15^2*(a[i,6]-a[i,5])
                estp[i,4]<-log((a[i,8]+1)*(30-a[i,9]+1)/(a[i,9]+1)/(30-a[i,8]+1))-8
                    *1/30^2*(a[i,9]-a[i,8])
                estp[i,5]<-log((a[i,10]+1)*(50-a[i,11]+1)/(a[i,11]+1)/(50-a[i
                    ,10]+1))-8*1/50^2*(a[i,11]-a[i,10])
        }
        std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
        avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
        AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
        for (j in 1:5) {
                ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
                ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
        }
```

```
        tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
            [3,],ci1[4,],ci1[5,])
        compare_table<-round(tmp1, digits = 4)
        colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
        return(compare_table)
}



## \epsilon=1.25
nci_1.25<-function(a){
        tv<-mean(a[,7])
        esth<-matrix(0,5000,5)
        estp<-matrix(0,5000,5)
        ci1<-matrix(0,5,4)
        ci2<-matrix(0,5,4)
        for (i in 1:5000) {
                esth[i,1]<-log((a[i,1]+1.25)*(5-a[i,2]+1.25)/(a[i,2]+1.25)/(5-a[i
                    ,1]+1.25))
                esth[i,2]<-log((a[i,3]+1.25)*(10-a[i,4]+1.25)/(a[i,4]+1.25)/(10-a[i
                    ,3]+1.25))
                esth[i,3]<-log((a[i,5]+1.25)*(15-a[i,6]+1.25)/(a[i,6]+1.25)/(15-a[i
                    ,5]+1.25))
                esth[i,4]<-log((a[i,8]+1.25)*(30-a[i,9]+1.25)/(a[i,9]+1.25)/(30-a[i
                    ,8]+1.25))
                esth[i,5]<-log((a[i,10]+1.25)*(50-a[i,11]+1.25)/(a[i,11]+1.25)/(50-
                    a[i,10]+1.25))
                estp[i,1]<-log((a[i,1]+1.25)*(5-a[i,2]+1.25)/(a[i,2]+1.25)/(5-a[i
                    ,1]+1.25))-8*1.25/5^2*(a[i,2]-a[i,1])
                estp[i,2]<-log((a[i,3]+1.25)*(10-a[i,4]+1.25)/(a[i,4]+1.25)/(10-a[i
                    ,3]+1.25))-8*1.25/10^2*(a[i,4]-a[i,3])
                estp[i,3]<-log((a[i,5]+1.25)*(15-a[i,6]+1.25)/(a[i,6]+1.25)/(15-a[i
                    ,5]+1.25))-8*1.25/15^2*(a[i,6]-a[i,5])
                estp[i,4]<-log((a[i,8]+1.25)*(30-a[i,9]+1.25)/(a[i,9]+1.25)/(30-a[i
                    ,8]+1.25))-8*1.25/30^2*(a[i,9]-a[i,8])
                estp[i,5]<-log((a[i,10]+1.25)*(50-a[i,11]+1.25)/(a[i,11]+1.25)/(50-
                    a[i,10]+1.25))-8*1.25/50^2*(a[i,11]-a[i,10])
        }
        std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
        avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
```

```
        AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
        for (j in 1:5) {
                ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
                ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
        }
        tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
            [3,],ci1[4,],ci1[5,])
        compare_table<-round(tmp1, digits = 4)
        colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
        return(compare_table)
}

###Estimators
## \epsilon=0.25
nci1_0.25<-function(a){
        tv<-mean(a[,7])
        esth<-matrix(0,5000,5)
        estp<-matrix(0,5000,5)
        ci1<-matrix(0,5,4)
        ci2<-matrix(0,5,4)
        for (i in 1:5000) {
                if(a[i,1]*(5-a[i,1])*a[i,2]*(5-a[i,2])==0){
                        esth[i,1]<-log((a[i,1]+0.25)*(5-a[i,2]+0.25)/(a[i,2]+0.25)/
                            (5-a[i,1]+0.25))
                        estp[i,1]<-log((a[i,1]+0.25)*(5-a[i,2]+0.25)/(a[i,2]+0.25)/
                            (5-a[i,1]+0.25))-8*0.25/5^2*(a[i,2]-a[i,1])
                }else{
                        esth[i,1]=estp[i,1]=log((a[i,1])*(5-a[i,2])/(a[i,2])/(5-a[i
                            ,1]))
                }
                if(a[i,3]*(10-a[i,3])*a[i,4]*(10-a[i,4])==0){
                        esth[i,2]<-log((a[i,3]+0.25)*(10-a[i,4]+0.25)/(a[i,4]+0.25)
                            /(10-a[i,3]+0.25))
                        estp[i,2]<-log((a[i,3]+0.25)*(10-a[i,4]+0.25)/(a[i,4]+0.25)
                            /(10-a[i,3]+0.25))-8*0.25/10^2*(a[i,4]-a[i,3])
                }else{
                        esth[i,2]=estp[i,2]= log((a[i,3])*(10-a[i,4])/(a[i,4])/(10-
                            a[i,3]))
                }
```

```
        if(a[i,5]*(15-a[i,5])*a[i,6]*(15-a[i,6])==0){
                esth[i,3]<-log((a[i,5]+0.25)*(15-a[i,6]+0.25)/(a[i,6]+0.25)
                    /(15-a[i,5]+0.25))
                estp[i,3]<-log((a[i,5]+0.25)*(15-a[i,6]+0.25)/(a[i,6]+0.25)
                    /(15-a[i,5]+0.25))-8*0.25/15^2*(a[i,6]-a[i,5])
        }else{
                esth[i,3]=estp[i,3]=log((a[i,5])*(15-a[i,6])/(a[i,6])/(15-a
                    [i,5]))
        }
        if(a[i,8]*(30-a[i,8])*a[i,9]*(30-a[i,9])==0){
                esth[i,4]<-log((a[i,8]+0.25)*(30-a[i,9]+0.25)/(a[i,9]+0.25)
                    /(30-a[i,8]+0.25))
                estp[i,4]<-log((a[i,8]+0.25)*(30-a[i,9]+0.25)/(a[i,9]+0.25)
                    /(30-a[i,8]+0.25))-8*0.25/30^2*(a[i,9]-a[i,8])
        }else{
                esth[i,4]=estp[i,4]= log((a[i,8])*(30-a[i,9])/(a[i,9])/(30-
                    a[i,8]))
        }
        if(a[i,10]*(50-a[i,10])*a[i,11]*(50-a[i,11])==0){
                esth[i,5]<-log((a[i,10]+0.25)*(50-a[i,11]+0.25)/(a[i
                    ,11]+0.25)/(50-a[i,10]+0.25))
                estp[i,5]<-log((a[i,10]+0.25)*(50-a[i,11]+0.25)/(a[i
                    ,11]+0.25)/(50-a[i,10]+0.25))-8*0.25/50^2*(a[i,11]-a[i
                    ,10])
        }else{
                esth[i,5]=estp[i,5]= log((a[i,10])*(50-a[i,11])/(a[i,11])/
                    (50-a[i,10]))
        }
}
std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
for (j in 1:5) {
        ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
        ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
}
tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
    [3,],ci1[4,],ci1[5,])
compare_table<-round(tmp1, digits = 4)
```

```
        colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
        return(compare_table)
}

nci1_0.5<-function(a){
        tv<-mean(a[,7])
        esth<-matrix(0,5000,5)
        estp<-matrix(0,5000,5)
        ci1<-matrix(0,5,4)
        ci2<-matrix(0,5,4)
        for (i in 1:5000) {
                if(a[i,1]*(5-a[i,1])*a[i,2]*(5-a[i,2])==0){
                        esth[i,1]<-log((a[i,1]+0.5)*(5-a[i,2]+0.5)/(a[i,2]+0.5)/(5-
                            a[i,1]+0.5))
                        estp[i,1]<-log((a[i,1]+0.5)*(5-a[i,2]+0.5)/(a[i,2]+0.5)/(5-
                            a[i,1]+0.5))-8*0.5/5^2*(a[i,2]-a[i,1])
                }else{
                        esth[i,1]=estp[i,1]=log((a[i,1])*(5-a[i,2])/(a[i,2])/(5-a[i
                            ,1]))
                }
                if(a[i,3]*(10-a[i,3])*a[i,4]*(10-a[i,4])==0){
                        esth[i,2]<-log((a[i,3]+0.5)*(10-a[i,4]+0.5)/(a[i,4]+0.5)/
                            (10-a[i,3]+0.5))
                        estp[i,2]<-log((a[i,3]+0.5)*(10-a[i,4]+0.5)/(a[i,4]+0.5)/
                            (10-a[i,3]+0.5))-8*0.5/10^2*(a[i,4]-a[i,3])
                }else{
                        esth[i,2]=estp[i,2]= log((a[i,3])*(10-a[i,4])/(a[i,4])/(10-
                            a[i,3]))
                }
                if(a[i,5]*(15-a[i,5])*a[i,6]*(15-a[i,6])==0){
                        esth[i,3]<-log((a[i,5]+0.5)*(15-a[i,6]+0.5)/(a[i,6]+0.5)/
                            (15-a[i,5]+0.5))
                        estp[i,3]<-log((a[i,5]+0.5)*(15-a[i,6]+0.5)/(a[i,6]+0.5)/
                            (15-a[i,5]+0.5))-8*0.5/15^2*(a[i,6]-a[i,5])
                }else{
                        esth[i,3]=estp[i,3]=log((a[i,5])*(15-a[i,6])/(a[i,6])/(15-a
                            [i,5]))
                }
                if(a[i,8]*(30-a[i,8])*a[i,9]*(30-a[i,9])==0){
```

```
                    esth[i,4]<-log((a[i,8]+0.5)*(30-a[i,9]+0.5)/(a[i,9]+0.5)/
                        (30-a[i,8]+0.5))
                    estp[i,4]<-log((a[i,8]+0.5)*(30-a[i,9]+0.5)/(a[i,9]+0.5)/
                        (30-a[i,8]+0.5))-8*0.5/30^2*(a[i,9]-a[i,8])
            }else{
                    esth[i,4]=estp[i,4]= log((a[i,8])*(30-a[i,9])/(a[i,9])/(30-
                        a[i,8]))
            }
            if(a[i,10]*(50-a[i,10])*a[i,11]*(50-a[i,11])==0){
                    esth[i,5]<-log((a[i,10]+0.5)*(50-a[i,11]+0.5)/(a[i,11]+0.5)
                        /(50-a[i,10]+0.5))
                    estp[i,5]<-log((a[i,10]+0.5)*(50-a[i,11]+0.5)/(a[i,11]+0.5)
                        /(50-a[i,10]+0.5))-8*0.5/50^2*(a[i,11]-a[i,10])
            }else{
                    esth[i,5]=estp[i,5]= log((a[i,10])*(50-a[i,11])/(a[i,11])/
                        (50-a[i,10]))
            }
        }
        std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
        avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
        AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
        for (j in 1:5) {
                ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
                ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
        }
        tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
            [3,],ci1[4,],ci1[5,])
        compare_table<-round(tmp1, digits = 4)
        colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
        return(compare_table)
}

nci1_0.75<-function(a){
        tv<-mean(a[,7])
        esth<-matrix(0,5000,5)
        estp<-matrix(0,5000,5)
        ci1<-matrix(0,5,4)
        ci2<-matrix(0,5,4)
        for (i in 1:5000) {
```

```
if(a[i,1]*(5-a[i,1])*a[i,2]*(5-a[i,2])==0){
        esth[i,1]<-log((a[i,1]+0.75)*(5-a[i,2]+0.75)/(a[i,2]+0.75)/
            (5-a[i,1]+0.75))
        estp[i,1]<-log((a[i,1]+0.75)*(5-a[i,2]+0.75)/(a[i,2]+0.75)/
            (5-a[i,1]+0.75))-8*0.75/5^2*(a[i,2]-a[i,1])
}else{
        esth[i,1]=estp[i,1]=log((a[i,1])*(5-a[i,2])/(a[i,2])/(5-a[i
            ,1]))
}
if(a[i,3]*(10-a[i,3])*a[i,4]*(10-a[i,4])==0){
        esth[i,2]<-log((a[i,3]+0.75)*(10-a[i,4]+0.75)/(a[i,4]+0.75)
            /(10-a[i,3]+0.75))
        estp[i,2]<-log((a[i,3]+0.75)*(10-a[i,4]+0.75)/(a[i,4]+0.75)
            /(10-a[i,3]+0.75))-8*0.75/10^2*(a[i,4]-a[i,3])
}else{
        esth[i,2]=estp[i,2]= log((a[i,3])*(10-a[i,4])/(a[i,4])/(10-
            a[i,3]))
}
if(a[i,5]*(15-a[i,5])*a[i,6]*(15-a[i,6])==0){
        esth[i,3]<-log((a[i,5]+0.75)*(15-a[i,6]+0.75)/(a[i,6]+0.75)
            /(15-a[i,5]+0.75))
        estp[i,3]<-log((a[i,5]+0.75)*(15-a[i,6]+0.75)/(a[i,6]+0.75)
            /(15-a[i,5]+0.75))-8*0.75/15^2*(a[i,6]-a[i,5])
}else{
        esth[i,3]=estp[i,3]=log((a[i,5])*(15-a[i,6])/(a[i,6])/(15-a
            [i,5]))
}
if(a[i,8]*(30-a[i,8])*a[i,9]*(30-a[i,9])==0){
        esth[i,4]<-log((a[i,8]+0.75)*(30-a[i,9]+0.75)/(a[i,9]+0.75)
            /(30-a[i,8]+0.75))
        estp[i,4]<-log((a[i,8]+0.75)*(30-a[i,9]+0.75)/(a[i,9]+0.75)
            /(30-a[i,8]+0.75))-8*0.75/30^2*(a[i,9]-a[i,8])
}else{
        esth[i,4]=estp[i,4]= log((a[i,8])*(30-a[i,9])/(a[i,9])/(30-
            a[i,8]))
}
if(a[i,10]*(50-a[i,10])*a[i,11]*(50-a[i,11])==0){
        esth[i,5]<-log((a[i,10]+0.75)*(50-a[i,11]+0.75)/(a[i
            ,11]+0.75)/(50-a[i,10]+0.75))
```

```
                        estp[i,5]<-log((a[i,10]+0.75)*(50-a[i,11]+0.75)/(a[i
                            ,11]+0.75)/(50-a[i,10]+0.75))-8*0.75/50^2*(a[i,11]-a[i
                            ,10])
                }else{
                        esth[i,5]=estp[i,5]= log((a[i,10])*(50-a[i,11])/(a[i,11])/
                            (50-a[i,10]))
                }
        }
        std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
        avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
        AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
        for (j in 1:5) {
                ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
                ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
        }
        tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
            [3,],ci1[4,],ci1[5,])
        compare_table<-round(tmp1, digits = 4)
        colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
        return(compare_table)
}

nci1_1<-function(a){
        tv<-mean(a[,7])
        esth<-matrix(0,5000,5)
        estp<-matrix(0,5000,5)
        ci1<-matrix(0,5,4)
        ci2<-matrix(0,5,4)
        for (i in 1:5000) {
                if(a[i,1]*(5-a[i,1])*a[i,2]*(5-a[i,2])==0){
                        esth[i,1]<-log((a[i,1]+1)*(5-a[i,2]+1)/(a[i,2]+1)/(5-a[i
                            ,1]+1))
                        estp[i,1]<-log((a[i,1]+1)*(5-a[i,2]+1)/(a[i,2]+1)/(5-a[i
                            ,1]+1))-8*1/5^2*(a[i,2]-a[i,1])
                }else{
                        esth[i,1]=estp[i,1]=log((a[i,1])*(5-a[i,2])/(a[i,2])/(5-a[i
                            ,1]))
                }
                if(a[i,3]*(10-a[i,3])*a[i,4]*(10-a[i,4])==0){
```

```
            esth[i,2]<-log((a[i,3]+1)*(10-a[i,4]+1)/(a[i,4]+1)/(10-a[i
                ,3]+1))
            estp[i,2]<-log((a[i,3]+1)*(10-a[i,4]+1)/(a[i,4]+1)/(10-a[i
                ,3]+1))-8*1/10^2*(a[i,4]-a[i,3])
        }else{
            esth[i,2]=estp[i,2]= log((a[i,3])*(10-a[i,4])/(a[i,4])/(10-
                a[i,3]))
        }
        if(a[i,5]*(15-a[i,5])*a[i,6]*(15-a[i,6])==0){
            esth[i,3]<-log((a[i,5]+1)*(15-a[i,6]+1)/(a[i,6]+1)/(15-a[i
                ,5]+1))
            estp[i,3]<-log((a[i,5]+1)*(15-a[i,6]+1)/(a[i,6]+1)/(15-a[i
                ,5]+1))-8*1/15^2*(a[i,6]-a[i,5])
        }else{
            esth[i,3]=estp[i,3]=log((a[i,5])*(15-a[i,6])/(a[i,6])/(15-a
                [i,5]))
        }
        if(a[i,8]*(30-a[i,8])*a[i,9]*(30-a[i,9])==0){
            esth[i,4]<-log((a[i,8]+1)*(30-a[i,9]+1)/(a[i,9]+1)/(30-a[i
                ,8]+1))
            estp[i,4]<-log((a[i,8]+1)*(30-a[i,9]+1)/(a[i,9]+1)/(30-a[i
                ,8]+1))-8*1/30^2*(a[i,9]-a[i,8])
        }else{
            esth[i,4]=estp[i,4]= log((a[i,8])*(30-a[i,9])/(a[i,9])/(30-
                a[i,8]))
        }
        if(a[i,10]*(50-a[i,10])*a[i,11]*(50-a[i,11])==0){
            esth[i,5]<-log((a[i,10]+1)*(50-a[i,11]+1)/(a[i,11]+1)/(50-a
                [i,10]+1))
            estp[i,5]<-log((a[i,10]+1)*(50-a[i,11]+1)/(a[i,11]+1)/(50-a
                [i,10]+1))-8*1/50^2*(a[i,11]-a[i,10])
        }else{
            esth[i,5]=estp[i,5]= log((a[i,10])*(50-a[i,11])/(a[i,11])/
                (50-a[i,10]))
        }
}
std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
```

```
        for (j in 1:5) {
                ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
                ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
        }
        tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
            [3,],ci1[4,],ci1[5,])
        compare_table<-round(tmp1, digits = 4)
        colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
        return(compare_table)
}

nci1_1.25<-function(a){
        tv<-mean(a[,7])
        esth<-matrix(0,5000,5)
        estp<-matrix(0,5000,5)
        ci1<-matrix(0,5,4)
        ci2<-matrix(0,5,4)
        for (i in 1:5000) {
                if(a[i,1]*(5-a[i,1])*a[i,2]*(5-a[i,2])==0){
                        esth[i,1]<-log((a[i,1]+1.25)*(5-a[i,2]+1.25)/(a[i,2]+1.25)/
                            (5-a[i,1]+1.25))
                        estp[i,1]<-log((a[i,1]+1.25)*(5-a[i,2]+1.25)/(a[i,2]+1.25)/
                            (5-a[i,1]+1.25))-8*1.25/5^2*(a[i,2]-a[i,1])
                }else{
                        esth[i,1]=estp[i,1]=log((a[i,1])*(5-a[i,2])/(a[i,2])/(5-a[i
                            ,1]))
                }
                if(a[i,3]*(10-a[i,3])*a[i,4]*(10-a[i,4])==0){
                        esth[i,2]<-log((a[i,3]+1.25)*(10-a[i,4]+1.25)/(a[i,4]+1.25)
                            /(10-a[i,3]+1.25))
                        estp[i,2]<-log((a[i,3]+1.25)*(10-a[i,4]+1.25)/(a[i,4]+1.25)
                            /(10-a[i,3]+1.25))-8*1.25/10^2*(a[i,4]-a[i,3])
                }else{
                        esth[i,2]=estp[i,2]= log((a[i,3])*(10-a[i,4])/(a[i,4])/(10-
                            a[i,3]))
                }
                if(a[i,5]*(15-a[i,5])*a[i,6]*(15-a[i,6])==0){
                        esth[i,3]<-log((a[i,5]+1.25)*(15-a[i,6]+1.25)/(a[i,6]+1.25)
                            /(15-a[i,5]+1.25))
```

```
                estp[i,3]<-log((a[i,5]+1.25)*(15-a[i,6]+1.25)/(a[i,6]+1.25)
                    /(15-a[i,5]+1.25))-8*1.25/15^2*(a[i,6]-a[i,5])
            }else{
                esth[i,3]=estp[i,3]=log((a[i,5])*(15-a[i,6])/(a[i,6])/(15-a
                    [i,5]))
            }
            if(a[i,8]*(30-a[i,8])*a[i,9]*(30-a[i,9])==0){
                esth[i,4]<-log((a[i,8]+1.25)*(30-a[i,9]+1.25)/(a[i,9]+1.25)
                    /(30-a[i,8]+1.25))
                estp[i,4]<-log((a[i,8]+1.25)*(30-a[i,9]+1.25)/(a[i,9]+1.25)
                    /(30-a[i,8]+1.25))-8*1.25/30^2*(a[i,9]-a[i,8])
            }else{
                esth[i,4]=estp[i,4]= log((a[i,8])*(30-a[i,9])/(a[i,9])/(30-
                    a[i,8]))
            }
            if(a[i,10]*(50-a[i,10])*a[i,11]*(50-a[i,11])==0){
                esth[i,5]<-log((a[i,10]+1.25)*(50-a[i,11]+1.25)/(a[i
                    ,11]+1.25)/(50-a[i,10]+1.25))
                estp[i,5]<-log((a[i,10]+1.25)*(50-a[i,11]+1.25)/(a[i
                    ,11]+1.25)/(50-a[i,10]+1.25))-8*1.25/50^2*(a[i,11]-a[i
                    ,10])
            }else{
                esth[i,5]=estp[i,4]= log((a[i,10])*(50-a[i,11])/(a[i,11])/
                    (50-a[i,10]))
            }
        }
        std1<-apply(esth, 2, sd);std2<-apply(estp, 2, sd)
        avg1<-apply(esth, 2, mean);avg2<-apply(estp, 2, mean)
        AAE1<-apply(abs(tv-esth),2,mean);AAE2<-apply(abs(tv-estp),2,mean)
        for (j in 1:5) {
            ci1[j,]<-c(tv-avg1[j],std1[j]^2,(tv-avg1[j])^2+std1[j]^2,AAE1[j])
            ci2[j,]<-c(tv-avg2[j],std2[j]^2,(tv-avg2[j])^2+std2[j]^2,AAE2[j])
        }
        tmp1<-rbind(ci2[1,],ci2[2,],ci2[3,],ci2[4,],ci2[5,],ci1[1,],ci1[2,],ci1
            [3,],ci1[4,],ci1[5,])
        compare_table<-round(tmp1, digits = 4)
        colnames(compare_table) <- c("Bias","Variance","MSE","AAE")
        return(compare_table)
}
```

Appendix B

Tables of AAE and MSE of Combined Estimators

Table B.1: AAE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.6260	0.6432	0.6262	0.6260	0.6260	0.6262	0.6260
$p_2 = 0.1$	n=10	0.7101	0.7700	0.7101	0.7101	0.7101	0.7101	0.7101
$\|\psi\| = 0$	n=15	0.7283	0.8318	0.7283	0.7283	0.7283	0.7283	0.7283
$p_1 = 0.2$	n=5	0.8142	0.8934	0.8167	0.8155	0.8144	0.8180	0.8144
$p_2 = 0.2$	n=10	0.7325	0.8730	0.7338	0.7325	0.7325	0.7326	0.7325
$\|\psi\| = 0$	n=15	0.6707	0.8097	0.6708	0.6707	0.6707	0.6707	0.6707
	n=5	0.8620	1.0140	0.8698	0.8678	0.8625	0.8755	0.8625
$p_1 = 0.3$	n=10	0.6939	0.8387	0.6977	0.6940	0.6940	0.6949	0.6940
$p_2 = 0.3$ $\|\psi\| = 0$	n=15	0.6083	0.7088	0.6089	0.6083	0.6083	0.6084	0.6083
$p_1 = 0.4$	n=5	0.8578	1.0685	0.8742	0.8650	0.8590	0.8815	0.8590
$p_2 = 0.4$	n=10	0.6587	0.7870	0.6668	0.6589	0.6587	0.6609	0.6587
$\|\psi\| = 0$	n=15	0.5763	0.6501	0.5774	0.5763	0.5763	0.5767	0.5763
$p_1 = 0.5$	n=5	0.8499	1.0814	0.8717	0.8598	0.8519	0.8815	0.8519
$p_2 = 0.5$	n=10	0.6481	0.7670	0.6574	0.6485	0.6481	0.6510	0.6481
$\|\psi\| = 0$	n=15	0.5667	0.6319	0.5680	0.5667	0.5667	0.5669	0.5667
$p_1 = 0.4$	n=5	0.8997	1.1273	0.9326	0.9235	0.9021	0.9564	0.9021
$p_2 = 0.5$	n=10	0.6608	0.7826	0.6834	0.6625	0.6608	0.6707	0.6608
$\|\psi\| = 0.4055$	n=15	0.5825	0.6534	0.5873	0.5825	0.5825	0.5844	0.5825
$p_1 = 0.5$	n=5	0.8847	1.1104	0.9163	0.9066	0.8873	0.9382	0.8873
$p_2 = 0.6$	n=10	0.6682	0.7902	0.6932	0.6706	0.6682	0.6783	0.6682
$\|\psi\| = 0.4055$	n=15	0.5666	0.6358	0.5714	0.5666	0.5666	0.5688	0.5666
$p_1 = 0.3$	n=5	0.9035	1.0900	0.9230	0.9218	0.9049	0.9414	0.9049
$p_2 = 0.4$	n=10	0.6846	0.8193	0.7004	0.6862	0.6847	0.6898	0.6847
$\|\psi\| = 0.4418$	n=15	0.6003	0.6885	0.6045	0.6003	0.6004	0.6014	0.6004
$p_1 = 0.2$	n=5	0.8946	1.0111	0.9030	0.9062	0.8949	0.9146	0.8949
$p_2 = 0.3$	n=10	0.7342	0.8729	0.7429	0.7348	0.7343	0.7369	0.7343
$\|\psi\| = 0.5390$	n=15	0.6517	0.7739	0.6529	0.6517	0.6517	0.6524	0.6517

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.1: AAE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.8485	0.8702	0.8505	0.8523	0.8485	0.8543	0.8485
$p_2 = 0.2$	n=10	0.7721	0.8136	0.7736	0.7721	0.7721	0.7721	0.7721
$\|\psi\| = 0.8109$	n=15	0.7351	0.8201	0.7353	0.7351	0.7351	0.7351	0.7351
$p_1 = 0.4$	n=5	0.8855	1.0534	0.9348	0.9293	0.8917	0.9786	0.8917
$p_2 = 0.6$	n=10	0.6780	0.8032	0.7354	0.6852	0.6789	0.7049	0.6789
$\|\psi\| = 0.8109$	n=15	0.5710	0.6404	0.5866	0.5710	0.5712	0.5782	0.5712
$p_1 = 0.3$	n=5	0.8983	1.0389	0.9430	0.9431	0.9046	0.9878	0.9046
$p_2 = 0.5$	n=10	0.6856	0.8097	0.7325	0.6922	0.6866	0.7053	0.6866
$\|\psi\| = 0.8473$	n=15	0.5975	0.6784	0.6134	0.5977	0.5979	0.6046	0.5978
$p_1 = 0.5$	n=5	0.8905	1.0327	0.9363	0.9313	0.8951	0.9771	0.8951
$p_2 = 0.7$	n=10	0.6936	0.8206	0.7454	0.7008	0.6942	0.7171	0.6942
$\|\psi\| = 0.8473$	n=15	0.5845	0.6638	0.5991	0.5845	0.5849	0.5911	0.5849
$p_1 = 0.2$	n=5	0.8895	0.9427	0.9122	0.9212	0.8922	0.9438	0.8922
$p_2 = 0.4$	n=10	0.7193	0.8370	0.7509	0.7234	0.7198	0.7306	0.7198
$\|\psi\| = 0.9808$	n=15	0.6359	0.7419	0.6481	0.6360	0.6363	0.6405	0.6363
$p_1 = 0.4$	n=5	0.8920	1.0302	0.9627	0.9747	0.9037	1.0455	0.9037
$p_2 = 0.7$	n=10	0.7038	0.8300	0.8127	0.7236	0.7071	0.7633	0.7071
$\|\psi\| = 1.2528$	n=15	0.5870	0.6775	0.6319	0.5877	0.5884	0.6094	0.5879
$p_1 = 0.3$	n=5	0.8920	1.0302	0.9497	0.9657	0.8987	1.0316	0.8987
$p_2 = 0.6$	n=10	0.7038	0.8300	0.8094	0.7207	0.7052	0.7607	0.7052
$\|\psi\| = 1.2528$	n=15	0.5870	0.6775	0.6322	0.5879	0.5884	0.6080	0.5876
$p_1 = 0.1$	n=5	0.8992	0.9011	0.9063	0.9178	0.9011	0.9250	0.9011
$p_2 = 0.3$	n=10	0.7635	0.7617	0.7715	0.7655	0.7638	0.7664	0.7638
$\|\psi\| = 1.3499$	n=15	0.7163	0.7626	0.7214	0.7163	0.7166	0.7171	0.7166
$p_1 = 0.5$	n=5	0.8843	0.9547	0.9369	0.9569	0.8959	1.0095	0.8959
$p_2 = 0.8$	n=10	0.7222	0.8276	0.8110	0.7397	0.7258	0.7660	0.7258
$\|\psi\| = 1.3863$	n=15	0.6236	0.7191	0.6633	0.6239	0.6259	0.6413	0.6257

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.1: AAE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.2$	n=5	0.8920	0.9621	0.9432	0.9687	0.9047	1.0199	0.9047
$p_2 = 0.5$	n=10	0.7185	0.8218	0.8060	0.7342	0.7210	0.7574	0.7210
$\|\psi\| = 1.3863$	n=15	0.6371	0.7369	0.6814	0.6379	0.6394	0.6583	0.6390
$p_1 = 0.3$	n=5	0.9246	0.9184	1.0175	1.0738	0.9555	1.1667	0.9555
$p_2 = 0.7$	n=10	0.7250	0.8504	0.9143	0.7741	0.7356	0.8482	0.7356
$\|\psi\| = 1.6946$	n=15	0.6135	0.7056	0.7202	0.6170	0.6187	0.6756	0.6162
$p_1 = 0.4$	n=5	0.9090	0.8645	0.9929	1.0459	0.9366	1.1298	0.9366
$p_2 = 0.8$	n=10	0.7334	0.8296	0.9061	0.7798	0.7447	0.8420	0.7447
$\|\psi\| = 1.7918$	n=15	0.6298	0.7260	0.7343	0.6327	0.6361	0.6897	0.6347
$p_1 = 0.1$	n=5	0.9051	0.8548	0.9230	0.9585	0.9120	0.9765	0.9120
$p_2 = 0.4$	n=10	0.7576	0.7124	0.7971	0.7674	0.7602	0.7715	0.7602
$\|\psi\| = 1.7918$	n=15	0.7011	0.7241	0.7253	0.7015	0.7035	0.7121	0.7034
$p_1 = 0.2$	n=5	0.9049	0.8553	0.9866	1.0389	0.9344	1.1206	0.9344
$p_2 = 0.6$	n=10	0.7360	0.8282	0.9046	0.7808	0.7473	0.8410	0.7473
$\|\psi\| = 1.7918$	n=15	0.6280	0.7274	0.7330	0.6311	0.6337	0.6878	0.6319
$p_1 = 0.5$	n=5	0.9485	0.8823	0.9697	1.0691	0.9704	1.0902	0.9704
$p_2 = 0.9$	n=10	0.7668	0.6988	0.8352	0.8050	0.7779	0.8279	0.7779
$\|\psi\| = 2.1972$	n=15	0.6861	0.6926	0.7630	0.6880	0.6951	0.7270	0.6943
$p_1 = 0.1$	n=5	0.9411	0.8708	0.9590	1.0674	0.9631	1.0853	0.9631
$p_2 = 0.5$	n=10	0.7606	0.6879	0.8231	0.7952	0.7694	0.8156	0.7694
$\|\psi\| = 2.1972$	n=15	0.7058	0.7123	0.7825	0.7076	0.7165	0.7460	0.7160
$p_1 = 0.2$	n=5	0.9270	0.8438	0.9704	1.1660	0.9894	1.2094	0.9894
$p_2 = 0.7$	n=10	0.7619	0.8416	0.9243	0.8651	0.7918	0.9695	0.7918
$\|\psi\| = 2.2336$	n=15	0.6479	0.7620	0.8392	0.6592	0.6644	0.7843	0.6572
$p_1 = 0.3$	n=5	0.9231	0.8373	0.9657	1.1625	0.9889	1.2050	0.9889
$p_2 = 0.8$	n=10	0.7576	0.8383	0.9208	0.8608	0.7869	0.9662	0.7869
$\|\psi\| = 2.2336$	n=15	0.6493	0.7595	0.8360	0.6614	0.6658	0.7828	0.6594

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.1: AAE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.4$	n=5	0.9923	0.8633	0.9531	1.2162	1.0442	1.1770	1.0442
$p_2 = 0.9$	n=10	0.7785	0.6919	0.7877	0.8752	0.8102	0.8822	0.8102
$\|\psi\| = 2.6027$	n=15	0.6980	0.6958	0.8086	0.7076	0.7252	0.8021	0.7215
$p_1 = 0.1$	n=5	0.9877	0.8592	0.9487	1.2024	1.0388	1.1634	1.0388
$p_2 = 0.6$	n=10	0.7780	0.6846	0.7816	0.8724	0.8091	0.8748	0.8091
$\|\psi\| = 2.6027$	n=15	0.6976	0.6920	0.8045	0.7068	0.7244	0.7996	0.7213
$p_1 = 0.2$	n=5	0.9560	0.7274	0.8288	1.3324	1.0784	1.2052	1.0784
$p_2 = 0.8$	n=10	0.7954	0.8255	0.8691	0.9887	0.8656	0.9724	0.8656
$\|\psi\| = 2.7726$	n=15	0.6878	0.7966	0.8448	0.7292	0.7350	0.8306	0.7097
$p_1 = 0.3$	n=5	0.9855	0.8116	0.8910	1.3499	1.1076	1.2554	1.1076
$p_2 = 0.9$	n=10	0.8032	0.6888	0.7368	0.9540	0.8825	0.8594	0.8825
$\|\psi\| = 3.0445$	n=15	0.7155	0.7117	0.7725	0.7534	0.7774	0.8100	0.7607
$p_1 = 0.1$	n=5	0.9835	0.8120	0.8909	1.3439	1.0967	1.2513	1.0967
$p_2 = 0.7$	n=10	0.8024	0.6851	0.7350	0.9540	0.8813	0.8567	0.8813
$\|\psi\| = 3.0445$	n=15	0.7148	0.7085	0.7702	0.7527	0.7791	0.8067	0.7623
$p_1 = 0.2$	n=5	1.1125	0.9506	1.0066	1.2909	1.3386	1.1849	1.3386
$p_2 = 0.9$	n=10	0.8490	0.6642	0.6816	0.9604	1.0080	0.8248	1.0080
$\|\psi\| = 3.5835$	n=15	0.7604	0.7360	0.7531	0.8684	0.8856	0.7171	0.8322
$p_1 = 0.1$	n=5	1.1119	0.9522	1.0065	1.2856	1.3330	1.1801	1.3330
$p_2 = 0.8$	n=10	0.8487	0.6557	0.6708	0.9591	1.0093	0.8225	1.0093
$\|\psi\| = 3.5835$	n=15	0.7622	0.7330	0.7504	0.8701	0.8881	0.7205	0.8365
$p_1 = 0.1$	n=5	1.3077	1.1977	1.2233	1.1117	1.4959	1.0273	1.4959
$p_2 = 0.9$	n=10	0.8859	0.5808	0.5838	0.8829	1.0364	0.8157	1.0364
$\|\psi\| = 4.3944$	n=15	0.8248	0.6507	0.6525	0.8889	0.8823	0.7716	0.9297

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.2: AAE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.5359	0.5567	0.5381	0.5359	0.5359	0.5359	0.5359
$p_2 = 0.1$	n=10	0.6160	0.6882	0.6160	0.6160	0.6160	0.6160	0.6160
$\lvert\psi\rvert = 0$	n=15	0.6432	0.7683	0.6432	0.6432	0.6432	0.6432	0.6432
$p_1 = 0.2$	n=5	0.7119	0.8077	0.7273	0.7137	0.7122	0.7137	0.7122
$p_2 = 0.2$	n=10	0.6635	0.8347	0.6651	0.6635	0.6635	0.6637	0.6635
$\lvert\psi\rvert = 0$	n=15	0.6231	0.7940	0.6233	0.6231	0.6231	0.6231	0.6231
$p_1 = 0.3$	n=5	0.7684	0.9527	0.8117	0.7760	0.7690	0.7760	0.7690
$p_2 = 0.3$	n=10	0.6466	0.8254	0.6513	0.6467	0.6466	0.6479	0.6466
$\lvert\psi\rvert = 0$	n=15	0.5797	0.7059	0.5804	0.5797	0.5797	0.5798	0.5797
$p_1 = 0.4$	n=5	0.7756	1.0316	0.8464	0.7852	0.7772	0.7852	0.7772
$p_2 = 0.4$	n=10	0.6237	0.7843	0.6337	0.6239	0.6237	0.6264	0.6237
$\lvert\psi\rvert = 0$	n=15	0.5556	0.6498	0.5569	0.5556	0.5556	0.5560	0.5556
$p_1 = 0.5$	n=5	0.7725	1.0539	0.8569	0.7857	0.7750	0.7857	0.7750
$p_2 = 0.5$	n=10	0.6166	0.7663	0.6281	0.6170	0.6166	0.6202	0.6166
$\lvert\psi\rvert = 0$	n=15	0.5481	0.6319	0.5497	0.5481	0.5481	0.5484	0.5481
$p_1 = 0.4$	n=5	0.8156	1.0923	0.9362	0.8474	0.8188	0.8474	0.8188
$p_2 = 0.5$	n=10	0.6291	0.7808	0.6573	0.6314	0.6291	0.6414	0.6291
$\lvert\psi\rvert = 0.4055$	n=15	0.5625	0.6533	0.5685	0.5625	0.5625	0.5649	0.5625
$p_1 = 0.5$	n=5	0.8024	1.0767	0.9175	0.8316	0.8059	0.8316	0.8059
$p_2 = 0.6$	n=10	0.6365	0.7886	0.6676	0.6397	0.6365	0.6491	0.6365
$\lvert\psi\rvert = 0.4055$	n=15	0.5470	0.6356	0.5531	0.5470	0.5470	0.5497	0.5470
$p_1 = 0.3$	n=5	0.8105	1.0369	0.8979	0.8349	0.8124	0.8349	0.8124
$p_2 = 0.4$	n=10	0.6458	0.8105	0.6654	0.6480	0.6459	0.6522	0.6459
$\lvert\psi\rvert = 0.4418$	n=15	0.5753	0.6868	0.5806	0.5753	0.5754	0.5767	0.5754
$p_1 = 0.2$	n=5	0.7899	0.9310	0.8366	0.8053	0.7902	0.8053	0.7902
$p_2 = 0.3$	n=10	0.6777	0.8435	0.6884	0.6784	0.6777	0.6810	0.6777
$\lvert\psi\rvert = 0.5390$	n=15	0.6145	0.7642	0.6160	0.6145	0.6145	0.6154	0.6145

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.2: AAE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.7471	0.7687	0.7582	0.7523	0.7471	0.7523	0.7471
$p_2 = 0.2$	n=10	0.6911	0.7405	0.6929	0.6912	0.6911	0.6911	0.6911
$\|\psi\| = 0.8109$	n=15	0.6696	0.7721	0.6698	0.6696	0.6696	0.6696	0.6696
$p_1 = 0.4$	n=5	0.8152	1.0099	0.9813	0.8735	0.8235	0.8735	0.8235
$p_2 = 0.6$	n=10	0.6475	0.8002	0.7189	0.6570	0.6475	0.6811	0.6475
$\|\psi\| = 0.8109$	n=15	0.5533	0.6401	0.5730	0.5533	0.5534	0.5624	0.5534
$p_1 = 0.3$	n=5	0.8196	0.9831	0.9660	0.8794	0.8279	0.8794	0.8279
$p_2 = 0.5$	n=10	0.6504	0.8011	0.7087	0.6592	0.6505	0.6750	0.6505
$\|\psi\| = 0.8473$	n=15	0.5764	0.6768	0.5964	0.5764	0.5766	0.5853	0.5766
$p_1 = 0.5$	n=5	0.8153	0.9802	0.9622	0.8697	0.8214	0.8697	0.8214
$p_2 = 0.7$	n=10	0.6583	0.8130	0.7227	0.6679	0.6583	0.6876	0.6583
$\|\psi\| = 0.8473$	n=15	0.5637	0.6622	0.5820	0.5637	0.5639	0.5720	0.5639
$p_1 = 0.2$	n=5	0.8134	0.8781	0.9052	0.8555	0.8169	0.8555	0.8169
$p_2 = 0.4$	n=10	0.6741	0.8107	0.7131	0.6796	0.6742	0.6882	0.6742
$\|\psi\| = 0.9808$	n=15	0.6058	0.7326	0.6210	0.6058	0.6062	0.6115	0.6062
$p_1 = 0.4$	n=5	0.8205	0.9688	1.0047	0.9308	0.8362	0.9308	0.8362
$p_2 = 0.7$	n=10	0.6682	0.8202	0.8036	0.6946	0.6692	0.7427	0.6692
$\|\psi\| = 1.2528$	n=15	0.5675	0.6757	0.6242	0.5675	0.5683	0.5959	0.5683
$p_1 = 0.3$	n=5	0.8205	0.9688	0.9938	0.9216	0.8323	0.9216	0.8323
$p_2 = 0.6$	n=10	0.6682	0.8202	0.8003	0.6917	0.6677	0.7401	0.6677
$\|\psi\| = 1.2528$	n=15	0.5675	0.6757	0.6245	0.5676	0.5681	0.5941	0.5679
$p_1 = 0.1$	n=5	0.8589	0.8530	0.8805	0.8837	0.8614	0.8837	0.8614
$p_2 = 0.3$	n=10	0.7150	0.7012	0.7249	0.7177	0.7150	0.7186	0.7150
$\|\psi\| = 1.3499$	n=15	0.6714	0.7165	0.6778	0.6714	0.6714	0.6725	0.6714
$p_1 = 0.5$	n=5	0.8270	0.8844	0.9263	0.9237	0.8423	0.9237	0.8423
$p_2 = 0.8$	n=10	0.6813	0.8009	0.7915	0.7047	0.6822	0.7360	0.6822
$\|\psi\| = 1.3863$	n=15	0.5959	0.7087	0.6456	0.5959	0.5972	0.6183	0.5972

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.2: AAE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.2$	n=5	0.8298	0.8872	0.9271	0.9320	0.8468	0.9320	0.8468
$p_2 = 0.5$	n=10	0.6772	0.7937	0.7856	0.6981	0.6780	0.7257	0.6780
$\|\psi\| = 1.3863$	n=15	0.6095	0.7273	0.6650	0.6096	0.6106	0.6362	0.6106
$p_1 = 0.3$	n=5	0.8624	0.8452	0.8876	1.0614	0.9037	1.0614	0.9037
$p_2 = 0.7$	n=10	0.6875	0.8325	0.9229	0.7530	0.6940	0.8420	0.6940
$\|\psi\| = 1.6946$	n=15	0.5910	0.7021	0.7257	0.5925	0.5944	0.6700	0.5938
$p_1 = 0.4$	n=5	0.8568	0.8023	0.8450	1.0392	0.8936	1.0392	0.8936
$p_2 = 0.8$	n=10	0.6923	0.7995	0.9065	0.7541	0.6976	0.8284	0.6976
$\|\psi\| = 1.7918$	n=15	0.6046	0.7148	0.7359	0.6053	0.6092	0.6806	0.6090
$p_1 = 0.1$	n=5	0.9230	0.8620	0.8958	0.9942	0.9323	0.9942	0.9323
$p_2 = 0.4$	n=10	0.7284	0.6595	0.7772	0.7414	0.7288	0.7458	0.7288
$\|\psi\| = 1.7918$	n=15	0.6710	0.6760	0.7010	0.6710	0.6718	0.6847	0.6718
$p_1 = 0.2$	n=5	0.8549	0.7942	0.8357	1.0335	0.8942	1.0335	0.8942
$p_2 = 0.6$	n=10	0.6939	0.7961	0.9031	0.7536	0.6991	0.8256	0.6991
$\|\psi\| = 1.7918$	n=15	0.6027	0.7172	0.7345	0.6041	0.6061	0.6784	0.6056
$p_1 = 0.5$	n=5	0.9829	0.8888	0.9259	1.1436	1.0120	1.1436	1.0120
$p_2 = 0.9$	n=10	0.7504	0.6509	0.8189	0.8013	0.7529	0.8273	0.7529
$\|\psi\| = 2.1972$	n=15	0.6627	0.6447	0.7547	0.6632	0.6674	0.7141	0.6672
$p_1 = 0.1$	n=5	0.9706	0.8736	0.9096	1.1390	1.0000	1.1390	1.0000
$p_2 = 0.5$	n=10	0.7439	0.6396	0.8061	0.7900	0.7472	0.8129	0.7472
$\|\psi\| = 2.1972$	n=15	0.6785	0.6616	0.7707	0.6793	0.6844	0.7291	0.6842
$p_1 = 0.2$	n=5	0.9113	0.7771	0.8134	1.2300	0.9945	1.2300	0.9945
$p_2 = 0.7$	n=10	0.7211	0.8004	0.9043	0.8588	0.7395	0.9777	0.7395
$\|\psi\| = 2.2336$	n=15	0.6298	0.7495	0.8524	0.6353	0.6415	0.8036	0.6393
$p_1 = 0.3$	n=5	0.9072	0.7704	0.8035	1.2263	0.9949	1.2263	0.9949
$p_2 = 0.8$	n=10	0.7184	0.7989	0.9025	0.8560	0.7361	0.9759	0.7361
$\|\psi\| = 2.2336$	n=15	0.6313	0.7459	0.8481	0.6369	0.6440	0.8016	0.6425

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.2: AAE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_{\epsilon}$	$\hat{\psi}^*_{\epsilon}$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.4$	n=5	1.0380	0.8749	0.9045	1.3365	1.1071	1.3365	1.1071
$p_2 = 0.9$	n=10	0.7742	0.6465	0.7651	0.9031	0.7886	0.8860	0.7886
$\|\psi\| = 2.6027$	n=15	0.6769	0.6461	0.7923	0.6804	0.6956	0.8024	0.6945
$p_1 = 0.1$	n=5	1.0395	0.8771	0.9079	1.3258	1.1076	1.3258	1.1076
$p_2 = 0.6$	n=10	0.7734	0.6384	0.7585	0.8993	0.7869	0.8777	0.7869
$\|\psi\| = 2.6027$	n=15	0.6753	0.6410	0.7864	0.6800	0.6940	0.7988	0.6931
$p_1 = 0.2$	n=5	0.9490	0.6714	0.6933	1.4168	1.1122	1.4168	1.1122
$p_2 = 0.8$	n=10	0.7634	0.7615	0.8164	0.9996	0.8149	0.9562	0.8149
$\|\psi\| = 2.7726$	n=15	0.6671	0.7729	0.8345	0.6873	0.7005	0.8268	0.6932
$p_1 = 0.3$	n=5	1.1300	0.9190	0.9400	1.4490	1.2929	1.4490	1.2929
$p_2 = 0.9$	n=10	0.8103	0.6382	0.6979	0.9860	0.8576	0.8602	0.8576
$\|\psi\| = 3.0445$	n=15	0.7062	0.6599	0.7364	0.7251	0.7580	0.7906	0.7531
$p_1 = 0.1$	n=5	1.1338	0.9257	0.9454	1.4442	1.2849	1.4442	1.2849
$p_2 = 0.7$	n=10	0.8088	0.6351	0.6971	0.9872	0.8564	0.8582	0.8564
$\|\psi\| = 3.0445$	n=15	0.7032	0.6551	0.7326	0.7212	0.7545	0.7856	0.7495
$p_1 = 0.2$	n=5	1.2161	1.0197	1.0306	1.3525	1.5176	1.3525	1.5176
$p_2 = 0.9$	n=10	0.8766	0.6062	0.6280	0.9328	1.0073	0.8049	1.0073
$\|\psi\| = 3.5835$	n=15	0.7463	0.6759	0.6975	0.8172	0.8556	0.6713	0.8334
$p_1 = 0.1$	n=5	1.2193	1.0255	1.0356	1.3503	1.5140	1.3503	1.5140
$p_2 = 0.8$	n=10	0.8754	0.5987	0.6176	0.9318	1.0074	0.8044	1.0074
$\|\psi\| = 3.5835$	n=15	0.7460	0.6703	0.6923	0.8166	0.8580	0.6737	0.8366
$p_1 = 0.1$	n=5	1.6519	1.5185	1.5219	1.1323	1.6445	1.1323	1.6445
$p_2 = 0.9$	n=10	1.0494	0.6732	0.6770	0.8451	1.1359	0.7699	1.1359
$\|\psi\| = 4.3944$	n=15	0.8570	0.5790	0.5813	0.9508	0.9190	0.6982	0.9508

[1] $\hat{\psi}_{\epsilon}$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_{\epsilon}$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.3: MSE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.7749	0.8094	0.7756	0.7749	0.7749	0.7756	0.7749
$p_2 = 0.1$	n=10	0.8583	0.9594	0.8583	0.8583	0.8583	0.8583	0.8583
$\lvert\psi\rvert = 0$	n=15	0.8794	1.0682	0.8794	0.8794	0.8794	0.8794	0.8794
$p_1 = 0.2$	n=5	1.1234	1.2964	1.1352	1.1328	1.1256	1.1447	1.1256
$p_2 = 0.2$	n=10	0.9031	1.1970	0.9092	0.9031	0.9031	0.9038	0.9031
$\lvert\psi\rvert = 0$	n=15	0.7580	1.0589	0.7586	0.7580	0.7580	0.7580	0.7580
$p_1 = 0.3$	n=5	1.2529	1.6269	1.2899	1.2934	1.2574	1.3304	1.2574
$p_2 = 0.3$	n=10	0.8241	1.1574	0.8416	0.8249	0.8249	0.8298	0.8249
$\lvert\psi\rvert = 0$	n=15	0.6206	0.8447	0.6237	0.6206	0.6206	0.6213	0.6206
$p_1 = 0.4$	n=5	1.2536	1.8198	1.3320	1.3062	1.2648	1.3847	1.2648
$p_2 = 0.4$	n=10	0.7341	1.0491	0.7722	0.7354	0.7341	0.7460	0.7341
$\lvert\psi\rvert = 0$	n=15	0.5448	0.7013	0.5502	0.5448	0.5448	0.5468	0.5448
$p_1 = 0.5$	n=5	1.2411	1.8827	1.3447	1.3140	1.2590	1.4175	1.2590
$p_2 = 0.5$	n=10	0.7074	1.0038	0.7520	0.7097	0.7074	0.7237	0.7074
$\lvert\psi\rvert = 0$	n=15	0.5229	0.6558	0.5296	0.5229	0.5229	0.5244	0.5229
$p_1 = 0.4$	n=5	1.2605	1.8380	1.3907	1.4139	1.2813	1.5441	1.2813
$p_2 = 0.5$	n=10	0.7191	1.0286	0.8125	0.7293	0.7198	0.7675	0.7198
$\lvert\psi\rvert = 0.4055$	n=15	0.5368	0.6841	0.5580	0.5368	0.5368	0.5461	0.5368
$p_1 = 0.5$	n=5	1.2396	1.8061	1.3649	1.3802	1.2633	1.5055	1.2633
$p_2 = 0.6$	n=10	0.7295	1.0437	0.8317	0.7430	0.7295	0.7791	0.7295
$\lvert\psi\rvert = 0.4055$	n=15	0.5122	0.6521	0.5337	0.5122	0.5125	0.5229	0.5125
$p_1 = 0.3$	n=5	1.2518	1.6871	1.3277	1.3679	1.2640	1.4438	1.2640
$p_2 = 0.4$	n=10	0.7777	1.0956	0.8399	0.7873	0.7786	0.8024	0.7786
$\lvert\psi\rvert = 0.4418$	n=15	0.5830	0.7737	0.6022	0.5830	0.5838	0.5888	0.5838
$p_1 = 0.2$	n=5	1.2013	1.4332	1.2322	1.2707	1.2032	1.3016	1.2032
$p_2 = 0.3$	n=10	0.8712	1.1660	0.9041	0.8743	0.8719	0.8835	0.8719
$\lvert\psi\rvert = 0.5390$	n=15	0.6899	0.9508	0.6956	0.6899	0.6903	0.6934	0.6903

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.3: MSE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	1.0933	1.1474	1.0997	1.1141	1.0933	1.1204	1.0933
$p_2 = 0.2$	n=10	0.9185	1.0417	0.9231	0.9190	0.9185	0.9185	0.9185
$\|\psi\| = 0.8109$	n=15	0.8379	1.0192	0.8388	0.8379	0.8379	0.8379	0.8379
$p_1 = 0.4$	n=5	1.2586	1.7371	1.4139	1.5088	1.3084	1.6641	1.3084
$p_2 = 0.6$	n=10	0.7437	1.0750	0.9414	0.7802	0.7493	0.8569	0.7493
$\|\psi\| = 0.8109$	n=15	0.5243	0.6795	0.5835	0.5243	0.5254	0.5553	0.5254
$p_1 = 0.3$	n=5	1.2581	1.6695	1.3955	1.5118	1.3058	1.6492	1.3058
$p_2 = 0.5$	n=10	0.7639	1.0726	0.9179	0.7984	0.7709	0.8457	0.7709
$\|\psi\| = 0.8473$	n=15	0.5749	0.7571	0.6337	0.5762	0.5772	0.6056	0.5768
$p_1 = 0.5$	n=5	1.2441	1.6578	1.3849	1.4719	1.2798	1.6126	1.2798
$p_2 = 0.7$	n=10	0.7737	1.0989	0.9469	0.8095	0.7774	0.8706	0.7774
$\|\psi\| = 0.8473$	n=15	0.5485	0.7244	0.6039	0.5485	0.5508	0.5774	0.5508
$p_1 = 0.2$	n=5	1.2214	1.4486	1.2850	1.3879	1.2409	1.4515	1.2409
$p_2 = 0.4$	n=10	0.8298	1.0876	0.9258	0.8501	0.8330	0.8742	0.8330
$\|\psi\| = 0.9808$	n=15	0.6533	0.8787	0.6971	0.6537	0.6557	0.6724	0.6557
$p_1 = 0.4$	n=5	1.2675	1.5588	1.4277	1.6688	1.3492	1.8290	1.3492
$p_2 = 0.7$	n=10	0.7893	1.1260	1.0841	0.8766	0.8084	0.9932	0.8084
$\|\psi\| = 1.2528$	n=15	0.5600	0.7527	0.6960	0.5640	0.5679	0.6393	0.5652
$p_1 = 0.3$	n=5	1.2675	1.5588	1.4145	1.6703	1.3670	1.8196	1.3670
$p_2 = 0.6$	n=10	0.7893	1.1260	1.0734	0.8687	0.8058	0.9881	0.8058
$\|\psi\| = 1.2528$	n=15	0.5600	0.7527	0.7002	0.5677	0.5707	0.6373	0.5664
$p_1 = 0.1$	n=5	1.2802	1.3153	1.2949	1.3653	1.2929	1.3801	1.2929
$p_2 = 0.3$	n=10	0.9167	0.9592	0.9345	0.9261	0.9183	0.9257	0.9183
$\|\psi\| = 1.3499$	n=15	0.7800	0.8734	0.7948	0.7800	0.7815	0.7829	0.7815
$p_1 = 0.5$	n=5	1.2372	1.3994	1.3422	1.5730	1.3129	1.6780	1.3129
$p_2 = 0.8$	n=10	0.8267	1.0654	1.0348	0.9009	0.8469	0.9678	0.8469
$\|\psi\| = 1.3863$	n=15	0.6218	0.8171	0.7340	0.6235	0.6334	0.6813	0.6324

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.3: MSE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.2$	n=5	1.2441	1.4044	1.3463	1.6000	1.3277	1.7023	1.3277
$p_2 = 0.5$	n=10	0.8178	1.0532	1.0224	0.8844	0.8316	0.9415	0.8316
$\|\psi\| = 1.3863$	n=15	0.6458	0.8602	0.7735	0.6501	0.6578	0.7179	0.6555
$p_1 = 0.3$	n=5	1.2883	1.3684	1.4166	1.9036	1.4728	2.0319	1.4728
$p_2 = 0.7$	n=10	0.8319	1.1526	1.2111	1.0111	0.8885	1.1678	0.8885
$\|\psi\| = 1.6946$	n=15	0.6000	0.8289	0.8501	0.6176	0.6255	0.7689	0.6139
$p_1 = 0.4$	n=5	1.2731	1.2989	1.3727	1.8089	1.4322	1.9085	1.4322
$p_2 = 0.8$	n=10	0.8447	1.0764	1.1497	1.0083	0.9013	1.1223	0.9013
$\|\psi\| = 1.7918$	n=15	0.6348	0.8437	0.8632	0.6497	0.6641	0.7927	0.6581
$p_1 = 0.1$	n=5	1.3602	1.3380	1.3816	1.5606	1.4004	1.5819	1.4004
$p_2 = 0.4$	n=10	0.8858	0.8484	0.9395	0.9219	0.8976	0.9221	0.8976
$\|\psi\| = 1.7918$	n=15	0.7470	0.7829	0.8002	0.7487	0.7569	0.7769	0.7567
$p_1 = 0.2$	n=5	1.2818	1.3001	1.3788	1.8122	1.4520	1.9091	1.4520
$p_2 = 0.6$	n=10	0.8465	1.0659	1.1408	1.0057	0.9026	1.1135	0.9026
$\|\psi\| = 1.7918$	n=15	0.6292	0.8464	0.8618	0.6447	0.6559	0.7842	0.6476
$p_1 = 0.5$	n=5	1.4372	1.3131	1.4536	1.8057	1.5454	1.8221	1.5454
$p_2 = 0.9$	n=10	0.8956	0.8102	0.9834	1.0066	0.9392	1.0099	0.9392
$\|\psi\| = 2.1972$	n=15	0.7120	0.7185	0.8261	0.7202	0.7441	0.7952	0.7413
$p_1 = 0.1$	n=5	1.4127	1.2879	1.4266	1.7970	1.5221	1.8110	1.5221
$p_2 = 0.5$	n=10	0.8848	0.7909	0.9637	0.9875	0.9209	0.9829	0.9209
$\|\psi\| = 2.1972$	n=15	0.7454	0.7536	0.8626	0.7530	0.7835	0.8301	0.7815
$p_1 = 0.2$	n=5	1.3263	1.1575	1.3599	2.0838	1.6309	2.1175	1.6309
$p_2 = 0.7$	n=10	0.8989	1.0954	1.2183	1.1866	1.0283	1.2966	1.0283
$\|\psi\| = 2.2336$	n=15	0.6677	0.9182	1.0075	0.7148	0.7328	0.9208	0.7071
$p_1 = 0.3$	n=5	1.3133	1.1469	1.3463	2.0794	1.6344	2.1124	1.6344
$p_2 = 0.8$	n=10	0.8911	1.0934	1.2157	1.1784	1.0181	1.2884	1.0181
$\|\psi\| = 2.2336$	n=15	0.6753	0.9106	0.9996	0.7261	0.7416	0.9243	0.7188

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.3: MSE of Combined Strategies with $\epsilon = 0.75$

	n	$\hat{\psi}_{\epsilon}$	$\hat{\psi}^{*}_{\epsilon}$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.4$	n=5	1.5132	1.2668	1.4828	2.0422	1.7280	2.0117	1.7280
$p_2 = 0.9$	n=10	0.9213	0.7921	0.9810	1.1340	1.0287	1.1116	1.0287
$\|\psi\| = 2.6027$	n=15	0.7278	0.7277	0.8852	0.7618	0.8083	0.8804	0.7979
$p_1 = 0.1$	n=5	1.5018	1.2541	1.4715	2.0123	1.7136	1.9820	1.7136
$p_2 = 0.6$	n=10	0.9161	0.7759	0.9677	1.1250	1.0205	1.0959	1.0205
$\|\psi\| = 2.6027$	n=15	0.7290	0.7212	0.8799	0.7637	0.8101	0.8814	0.8013
$p_1 = 0.2$	n=5	1.3909	1.0214	1.2923	2.2339	1.8565	2.1353	1.8565
$p_2 = 0.8$	n=10	0.9653	1.0220	1.1315	1.3439	1.2067	1.3573	1.2067
$\|\psi\| = 2.7726$	n=15	0.7460	0.9894	1.0818	0.8756	0.8895	0.9910	0.8265
$p_1 = 0.3$	n=5	1.6034	1.2193	1.4788	2.2342	2.0016	2.1096	2.0016
$p_2 = 0.9$	n=10	0.9807	0.7850	0.9216	1.2718	1.1955	1.1924	1.1955
$\|\psi\| = 3.0445$	n=15	0.7732	0.7699	0.8946	0.8770	0.9216	0.9344	0.8891
$p_1 = 0.1$	n=5	1.6061	1.2277	1.4839	2.2127	1.9753	2.0905	1.9753
$p_2 = 0.7$	n=10	0.9790	0.7797	0.9189	1.2704	1.1924	1.1880	1.1924
$\|\psi\| = 3.0445$	n=15	0.7712	0.7672	0.8924	0.8760	0.9242	0.9314	0.8914
$p_1 = 0.2$	n=5	1.7968	1.3025	1.5428	2.2442	2.2900	1.9902	2.290
$p_2 = 0.9$	n=10	1.0821	0.7434	0.8119	1.3360	1.3950	1.1662	1.395
$\|\psi\| = 3.5835$	n=15	0.8541	0.8145	0.8690	1.0499	1.0637	0.9042	1.012
$p_1 = 0.1$	n=5	1.7798	1.2942	1.5269	2.2165	2.2619	1.9636	2.2619
$p_2 = 0.8$	n=10	1.0754	0.7313	0.7919	1.3284	1.3891	1.1581	1.3891
$\|\psi\| = 3.5835$	n=15	0.8596	0.8115	0.8654	1.0596	1.0743	0.9134	1.0244
$p_1 = 0.1$	n=5	2.4680	1.9792	2.1288	2.0185	2.6940	1.6794	2.6940
$p_2 = 0.9$	n=10	1.2896	0.6904	0.7072	1.2060	1.5086	1.0128	1.5086
$\|\psi\| = 4.3944$	n=15	1.0047	0.6662	0.6748	1.1499	1.1454	0.8826	1.1986

[1] $\hat{\psi}_{\epsilon}$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^{*}_{\epsilon}$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.4: MSE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.5717	0.6119	0.5785	0.5717	0.5717	0.5717	0.5717
$p_2 = 0.1$	n=10	0.6462	0.7641	0.6462	0.6462	0.6462	0.6462	0.6462
$\|\psi\| = 0$	n=15	0.6812	0.9021	0.6812	0.6812	0.6812	0.6812	0.6812
$p_1 = 0.2$	n=5	0.8600	1.0621	0.9108	0.8720	0.8628	0.8720	0.8628
$p_2 = 0.2$	n=10	0.7326	1.0797	0.7400	0.7326	0.7326	0.7334	0.7326
$\|\psi\| = 0$	n=15	0.6445	1.0038	0.6453	0.6445	0.6445	0.6445	0.6445
$p_1 = 0.3$	n=5	0.9904	1.4285	1.1355	1.0418	0.9960	1.0418	0.9960
$p_2 = 0.3$	n=10	0.7065	1.1064	0.7277	0.7075	0.7065	0.7134	0.7065
$\|\psi\| = 0$	n=15	0.5584	0.8323	0.5622	0.5584	0.5584	0.5592	0.5584
$p_1 = 0.4$	n=5	1.0154	1.6795	1.2597	1.0821	1.0294	1.0821	1.0294
$p_2 = 0.4$	n=10	0.6533	1.0367	0.6994	0.6549	0.6533	0.6677	0.6533
$\|\psi\| = 0$	n=15	0.5043	0.6998	0.5110	0.5043	0.5043	0.5068	0.5043
$p_1 = 0.5$	n=5	1.0146	1.7675	1.3098	1.1068	1.0370	1.1068	1.0370
$p_2 = 0.5$	n=10	0.6371	1.0004	0.6911	0.6400	0.6371	0.6568	0.6371
$\|\psi\| = 0$	n=15	0.4881	0.6558	0.4964	0.4881	0.4881	0.4901	0.4881
$p_1 = 0.4$	n=5	1.0293	1.7044	1.3570	1.2225	1.0553	1.2225	1.0553
$p_2 = 0.5$	n=10	0.6443	1.0204	0.7569	0.6572	0.6443	0.7025	0.6443
$\|\psi\| = 0.4055$	n=15	0.4992	0.6832	0.5255	0.4992	0.4992	0.5108	0.4992
$p_1 = 0.5$	n=5	1.0143	1.6764	1.3276	1.1914	1.0439	1.1914	1.0439
$p_2 = 0.6$	n=10	0.6545	1.0368	0.7779	0.6717	0.6545	0.7142	0.6545
$\|\psi\| = 0.4055$	n=15	0.4765	0.6512	0.5030	0.4765	0.4765	0.4897	0.4765
$p_1 = 0.3$	n=5	1.0085	1.5166	1.2312	1.1546	1.0236	1.1546	1.0236
$p_2 = 0.4$	n=10	0.6799	1.0618	0.7548	0.6920	0.6810	0.7097	0.6810
$\|\psi\| = 0.4418$	n=15	0.5322	0.7663	0.5556	0.5322	0.5328	0.5394	0.5328
$p_1 = 0.2$	n=5	0.9548	1.2243	1.0607	1.0421	0.9572	1.0421	0.9572
$p_2 = 0.3$	n=10	0.7316	1.0791	0.7708	0.7355	0.7316	0.7463	0.7316
$\|\psi\| = 0.5390$	n=15	0.6052	0.9167	0.6121	0.6052	0.6052	0.6094	0.6052

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.4: MSE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_{\epsilon}$	$\hat{\psi}^{*}_{\epsilon}$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.1$	n=5	0.9045	0.9663	0.9236	0.9305	0.9045	0.9305	0.9045
$p_2 = 0.2$	n=10	0.7595	0.9003	0.7649	0.7601	0.7595	0.7595	0.7595
$\lvert\psi\rvert = 0.8109$	n=15	0.7007	0.9090	0.7018	0.7007	0.7007	0.7007	0.7007
$p_1 = 0.4$	n=5	1.0415	1.5928	1.3661	1.3536	1.1030	1.3536	1.1030
$p_2 = 0.6$	n=10	0.6662	1.0626	0.9029	0.7123	0.6662	0.8025	0.6662
$\lvert\psi\rvert = 0.8109$	n=15	0.4872	0.6777	0.5597	0.4872	0.4877	0.5253	0.4877
$p_1 = 0.3$	n=5	1.0339	1.5068	1.3116	1.3502	1.0928	1.3502	1.0928
$p_2 = 0.5$	n=10	0.6754	1.0417	0.8593	0.7185	0.6764	0.7736	0.6764
$\lvert\psi\rvert = 0.8473$	n=15	0.5285	0.7500	0.6004	0.5285	0.5301	0.5661	0.5301
$p_1 = 0.5$	n=5	1.0293	1.5048	1.3098	1.3134	1.0734	1.3134	1.0734
$p_2 = 0.7$	n=10	0.6857	1.0721	0.8927	0.7307	0.6857	0.8019	0.6857
$\lvert\psi\rvert = 0.8473$	n=15	0.5043	0.7176	0.5718	0.5043	0.5052	0.5396	0.5052
$p_1 = 0.2$	n=5	1.0157	1.2738	1.1549	1.2227	1.0396	1.2227	1.0396
$p_2 = 0.4$	n=10	0.7171	1.0153	0.8308	0.7424	0.7181	0.7701	0.7181
$\lvert\psi\rvert = 0.9808$	n=15	0.5834	0.8485	0.6362	0.5834	0.5852	0.6065	0.5852
$p_1 = 0.4$	n=5	1.0734	1.3944	1.3114	1.5677	1.1732	1.5677	1.1732
$p_2 = 0.7$	n=10	0.7020	1.0922	1.0502	0.8104	0.7083	0.9454	0.7083
$\lvert\psi\rvert = 1.2528$	n=15	0.5159	0.7449	0.6809	0.5159	0.5202	0.6126	0.5202
$p_1 = 0.3$	n=5	1.0734	1.3944	1.2952	1.5665	1.1920	1.5665	1.1920
$p_2 = 0.6$	n=10	0.7020	1.0922	1.0385	0.8020	0.7064	0.9402	0.7064
$\lvert\psi\rvert = 1.2528$	n=15	0.5159	0.7449	0.6856	0.5202	0.5227	0.6100	0.5216
$p_1 = 0.1$	n=5	1.1632	1.1986	1.1873	1.2677	1.1787	1.2677	1.1787
$p_2 = 0.3$	n=10	0.8179	0.8558	0.8386	0.8292	0.8179	0.8285	0.8179
$\lvert\psi\rvert = 1.3499$	n=15	0.6877	0.7830	0.7053	0.6877	0.6877	0.6912	0.6877
$p_1 = 0.5$	n=5	1.0755	1.2465	1.2132	1.4870	1.1676	1.4870	1.1676
$p_2 = 0.8$	n=10	0.7291	0.9956	0.9724	0.8200	0.7343	0.8962	0.7343
$\lvert\psi\rvert = 1.3863$	n=15	0.5607	0.7842	0.6951	0.5607	0.5670	0.6326	0.5670

[1] $\hat{\psi}_{\epsilon}$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^{*}_{\epsilon}$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.4: MSE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.2$	n=5	1.0741	1.2434	1.2086	1.5103	1.1758	1.5103	1.1758
$p_2 = 0.5$	n=10	0.7203	0.9826	0.9592	0.8019	0.7246	0.8670	0.7246
$\|\psi\| = 1.3863$	n=15	0.5831	0.8296	0.7362	0.5837	0.5887	0.6702	0.5887
$p_1 = 0.3$	n=5	1.1267	1.1872	1.2123	1.8704	1.3488	1.8704	1.3488
$p_2 = 0.7$	n=10	0.7393	1.0940	1.1769	0.9575	0.7746	1.1343	0.7746
$\|\psi\| = 1.6946$	n=15	0.5507	0.8143	0.8499	0.5591	0.5675	0.7549	0.5647
$p_1 = 0.4$	n=5	1.1505	1.1486	1.1878	1.7948	1.3413	1.7948	1.3413
$p_2 = 0.8$	n=10	0.7536	0.9979	1.1017	0.9507	0.7817	1.0781	0.7817
$\|\psi\| = 1.7918$	n=15	0.5764	0.8088	0.8466	0.5803	0.5974	0.7653	0.5960
$p_1 = 0.1$	n=5	1.3245	1.2857	1.3180	1.5655	1.3726	1.5655	1.3726
$p_2 = 0.4$	n=10	0.8325	0.7656	0.8925	0.8752	0.8347	0.8748	0.8347
$\|\psi\| = 1.7918$	n=15	0.6784	0.6956	0.7405	0.6784	0.6820	0.7137	0.6820
$p_1 = 0.2$	n=5	1.1568	1.1463	1.1886	1.7945	1.3609	1.7945	1.3609
$p_2 = 0.6$	n=10	0.7545	0.9844	1.0902	0.9462	0.7815	1.0666	0.7815
$\|\psi\| = 1.7918$	n=15	0.5728	0.8148	0.8481	0.5805	0.5892	0.7584	0.5869
$p_1 = 0.5$	n=5	1.4772	1.3079	1.3920	1.9075	1.6046	1.9075	1.6046
$p_2 = 0.9$	n=10	0.8696	0.7324	0.9605	0.9972	0.8805	0.9988	0.8805
$\|\psi\| = 2.1972$	n=15	0.6659	0.6379	0.7951	0.6684	0.6829	0.7622	0.6822
$p_1 = 0.1$	n=5	1.4490	1.2800	1.3592	1.8978	1.5778	1.8978	1.5778
$p_2 = 0.5$	n=10	0.8602	0.7140	0.9413	0.9780	0.8749	0.9710	0.8749
$\|\psi\| = 2.1972$	n=15	0.6910	0.6655	0.8240	0.6944	0.7120	0.7893	0.7113
$p_1 = 0.2$	n=5	1.2586	1.0161	1.1041	2.1415	1.6164	2.1415	1.6164
$p_2 = 0.7$	n=10	0.8084	0.9912	1.1533	1.1446	0.8912	1.2601	0.8912
$\|\psi\| = 2.2336$	n=15	0.6094	0.8789	1.0012	0.6349	0.6564	0.9062	0.6474
$p_1 = 0.3$	n=5	1.2435	1.0036	1.0806	2.1366	1.6206	2.1366	1.6206
$p_2 = 0.8$	n=10	0.8020	0.9914	1.1528	1.1378	0.8823	1.2532	0.8823
$\|\psi\| = 2.2336$	n=15	0.6152	0.8670	0.9889	0.6414	0.6664	0.9073	0.6604

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

Table B.4: MSE of Combined Strategies with $\epsilon = 1$

	n	$\hat{\psi}_\epsilon$	$\hat{\psi}^*_\epsilon$	$\hat{\psi}_{Com1}$	$\hat{\psi}_{Com2}$	$\hat{\psi}_{Com3}$	$\hat{\psi}_{Com4}$	$\hat{\psi}_{Com5}$
$p_1 = 0.4$	n=5	1.6281	1.3047	1.4017	2.2164	1.8743	2.2164	1.8743
$p_2 = 0.9$	n=10	0.9161	0.7113	0.9556	1.1486	0.9694	1.1192	0.9694
$\lvert\psi\rvert = 2.6027$	n=15	0.6915	0.6446	0.8578	0.7053	0.7464	0.8602	0.7428
$p_1 = 0.1$	n=5	1.6214	1.2966	1.3989	2.1895	1.8642	2.1895	1.8642
$p_2 = 0.6$	n=10	0.9124	0.6960	0.9438	1.1404	0.9619	1.1043	0.9619
$\lvert\psi\rvert = 2.6027$	n=15	0.6900	0.6344	0.8486	0.7091	0.7465	0.8587	0.7438
$p_1 = 0.2$	n=5	1.4464	0.9597	1.0425	2.3623	1.9723	2.3623	1.9723
$p_2 = 0.8$	n=10	0.8912	0.8811	1.0229	1.3018	1.0726	1.2975	1.0726
$\lvert\psi\rvert = 2.7726$	n=15	0.6830	0.9204	1.0426	0.7573	0.7910	0.9490	0.7692
$p_1 = 0.3$	n=5	1.8119	1.3175	1.4086	2.4483	2.2483	2.4483	2.2483
$p_2 = 0.9$	n=10	0.9960	0.6927	0.8678	1.2819	1.1345	1.1834	1.1345
$\lvert\psi\rvert = 3.0445$	n=15	0.7448	0.6783	0.8423	0.8040	0.8656	0.8966	0.8537
$p_1 = 0.1$	n=5	1.8236	1.3363	1.4214	2.4322	2.2282	2.4322	2.2282
$p_2 = 0.7$	n=10	0.9954	0.6896	0.8680	1.2817	1.1338	1.1814	1.1338
$\lvert\psi\rvert = 3.0445$	n=15	0.7391	0.6721	0.8367	0.7971	0.8602	0.8902	0.8480
$p_1 = 0.2$	n=5	2.1853	1.5575	1.6180	2.4566	2.6678	2.4566	2.6678
$p_2 = 0.9$	n=10	1.1426	0.6469	0.7345	1.3029	1.3993	1.0966	1.3993
$\lvert\psi\rvert = 3.5835$	n=15	0.8399	0.7031	0.7739	0.9908	1.0262	0.8111	0.9962
$p_1 = 0.1$	n=5	2.1752	1.5582	1.6148	2.4374	2.6470	2.4374	2.6470
$p_2 = 0.8$	n=10	1.1375	0.6375	0.7149	1.2965	1.3941	1.0905	1.3941
$\lvert\psi\rvert = 3.5835$	n=15	0.8391	0.6934	0.7635	0.9928	1.0311	0.8141	1.0021
$p_1 = 0.1$	n=5	3.2883	2.6751	2.6996	2.2047	3.2768	2.2047	3.2768
$p_2 = 0.9$	n=10	1.5510	0.7411	0.7626	1.1469	1.7028	0.9247	1.7028
$\lvert\psi\rvert = 4.3944$	n=15	1.0750	0.5525	0.5636	1.2275	1.2081	0.7505	1.2229

[1] $\hat{\psi}_\epsilon$: Estimator with ϵ added to all tables.
[2] $\hat{\psi}^*_\epsilon$: Estimator with ϵ added only to tables contains zero counts.
[3] $\hat{\psi}_{Com1}$-$\hat{\psi}_{Com5}$: Estimator with combined strategies.

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