4-1-2023

# Irregular Domination in Graphs

Caryn Mays
*Western Michigan University*

# Irregular Domination in Graphs

**Caryn Mays, Ph.D.**

**Western Michigan University, 2023**

Domination in graphs has been a popular area of study due in large degree to its applications to modern society as well as the mathematical beauty of the topic. While this area evidently began with the work of Claude Berge in 1958 and Oystein Ore in 1962, domination did not become an active area of research until 1977 with the appearance of a survey paper by Ernest Cockayne and Stephen Hedetniemi. Since then, a large number of variations of domination have surfaced and provided numerous applications to different areas of science and real-life problems. Among these variations are domination parameters defined in terms of distance which provide a more general setting for domination in graphs.

In this research, we study a variant based around distance of graph domination based around distance referred to as irregular domination. A set $S$ of vertices in a connected graph $G$ is an irregular domination set if the vertices of $S$ can be labeled with distinct positive integers in such a way that for every vertex $u$ of $G$, there is a vertex $v \in S$ such that the distance from $u$ to $v$ is the label assigned to $v$. The minimum cardinality of an irregular dominating set in a graph $G$ is the irregular domination number of $G$. If for every vertex $v \in S$, there is a vertex $u$ of $G$ such that $v$ is the only vertex of $S$ whose distance to $u$ is the label of $v$, then $S$ is a minimal irregular dominating set. A graph $H$ is an irregular domination graph if there exists a graph $G$ with a minimal irregular dominating set $S$ such that $H$ is isomorphic to the subgraph of $G$ induced by $S$. We investigate the irregular domination number of some well-known classes of bipartite graphs and determine all irregular domination paths, cycles, and the familiar graphs of ladders and prisms. Furthermore, we establish characterizations of irregular domination trees, forests, and disconnected graphs. Additionally, we explore connections between irregular dominating sets and irregular domination graphs; other structural results and problems dealing with irregular domination are presented.

# Irregular Domination in Graphs

by

**Caryn Mays**

**A dissertation submitted to the Graduate College
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
Mathematics
Western Michigan University
April 2023**

Doctoral Committee:

Ping Zhang, Ph.D., Chair
Gary Chartrand, Ph.D.
Clifton Ealy, Ph.D.
Dinesh Sarvate, Ph.D.

# ACKNOWLEDGEMENTS

First, I would like to thank my advisor, Dr. Ping Zhang. Without her work and guidance on this project, this dream would have never come to fruition. Her help and teaching has trained me greatly in becoming a true mathematician, full of curiosity and passion. Additionally, I'd like to thank the rest of my committee: Dr. Gary Chartrand, Dr. Clifton Ealy, and Dr. Dinesh Sarvate.

Second, I would like to thank my family, who have never let me believe I was not capable of anything I put my mind to. Their belief in me gave me motivation to keep going when times got hard. They kept my nerdiness in check, both keeping me from becoming too nerdy as well as making sure I was nerdy enough by sharing in my love of learning.

Third, I would like to thank my many supporters at Western Michigan University. To my professors, my teaching supervisors, my classmates and my friends, you made the experience worth-while. Thank you to my teaching supervisors who never let me doubt my abilities and worth for a second. And thank you to my friends who made crying on my kitchen floor in Tennessee before graduate school a laughable thing of the past.

Finally, I thank God for leading me through this experience and bringing me to this point in my life. I look forward to the great plans He has for me.

Caryn Mays

# TABLE OF CONTENTS

Table of Contents – Continued

# LIST OF FIGURES

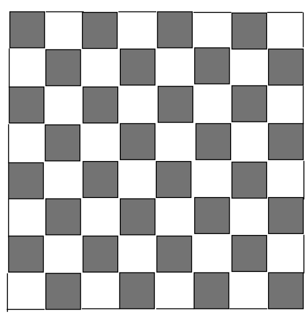
List of Figures – Continued

# Chapter 1

# Introduction

## 1.1 The Game of Chess

The game of chess, being rooted in logic and strategy, has been considered as a mathematical game for centuries. There are numerous puzzles and problems involving chess that have connections to graph theory. Here, we consider several chessboard problems that involve the placement of chess pieces on a standard $8 \times 8$ chessboard.



The best known chessboard problem involves queens. A single move by a queen on a chessboard consists of moving the queen along any number of vacant squares diagonally, horizontally, or vertically. The queen is also said to attack (or dominate) these squares (occupied or not). This is illustrated in Figure 1.1.

The well-known problem involving queens is referred to as

**The Five Queens Problem** *Can five queens be placed on distinct squares of the standard $8 \times 8$ chessboard in such a way that every vacant square can be attacked by at least one of these queens?*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\star$ | | $\star$ | | | | |
| $\star$ | $\star$ | $\star$ | | | | | |
| $\star$ | $Q$ | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ |
| $\star$ | $\star$ | $\star$ | | | | | |
| | $\star$ | | $\star$ | | | | |
| | $\star$ | | | $\star$ | | | |
| | $\star$ | | | | $\star$ | | |
| | $\star$ | | | | | $\star$ | |

Figure 1.1: Squares attacked by a queen $Q$

It is well known that the answer to the question is *yes* and that this cannot be done with four queens. In fact, five queens can be so located on a chessboard such that either (a) each queen can be attacked by another queen or (b) no queen can be attacked by another queen. These two solutions are shown in Figure 1.2. This chessboard problem has been considered the origin of the topic of dominating sets in graphs.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $Q$ | | | | | | | |
| | | | | | | | |
| | | $Q$ | | | | | |
| | | | | | | | |
| | | | $Q$ | | | | |
| | | | | $Q$ | | | |
| | | | | | $Q$ | | |
| | | | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | $Q$ | |
| | | $Q$ | | | | | |
| | | | | | | | |
| | | | | | | | |
| | | | | | | | $Q$ |
| | | | $Q$ | | | | |
| | | | | | | | |
| $Q$ | | | | | | | |

Figure 1.2: Two solutions to the Five Queens Problem

We now consider another chess puzzle introduced in [6]. This problem involves a different kind of chess piece that moves in a different way on a standard $8 \times 8$ chessboard. We refer to such a chess piece as a *princess* (often called *prince*). For an integer $k$ with $1 \leq k \leq 14$, a *$k$-princess* is a chess piece that is permitted to move horizontally and/or vertically a total of exactly $k$ squares (vacant or not) *away from* its current position. The $k$-princess is then said to have *covered* or *attacked* the resulting square to which the $k$-princess has moved. A 14-princess can cover only one square and this can only occur if the 14-princess is located on one of the four corner squares of the chessboard, in which case the 14-princess covers only the opposite corner square. A 3-princess, if properly positioned on a chessboard, can move three squares horizontally or vertically, as a rook can do, or move two squares horizontally or vertically followed by one square in a perpendicular direction, as a knight can do. Figure 1.3 shows the twelve squares (marked $\star$) that a 3-princess $3P$ can attack if $3P$ is placed somewhere on one of the four most central squares of the chessboard.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | $\star$ | | | | |
| | | $\star$ | | $\star$ | | | |
| | $\star$ | | | | $\star$ | | |
| $\star$ | | | $3P$ | | | $\star$ | |
| | $\star$ | | | | $\star$ | | |
| | | $\star$ | | $\star$ | | | |
| | | | $\star$ | | | | |
| | | | | | | | |

Figure 1.3: Squares attacked by a 3-princess on a standard $8 \times 8$ chessboard

This leads us to the following chessboard problem.

**The Fourteen princesses Problem** *Can the fourteen $k$-princesses* ($k = 1, 2, 3, \ldots, 14$) *be placed on distinct squares of a standard* $8 \times 8$ *chessboard in such a way that every square* (*including each occupied square*) *can be covered by at least one princess?*

The answer to the Fourteen princesses Problem is yes and a solution is shown in Figure 1.4 where the location of a $k$-prince on a square is indicated by labeling the square by $k$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | 10 | | | 14 |
| | | | | | | | |
| | | | | | | | 8 |
| | | | | | | | |
| 9 | 5 | 3 | 1 | 4 | 2 | 7 | 6 |
| 11 | | | | | | | |
| | | | | | | | |
| | 13 | | | | | | 12 |

Figure 1.4: A solution to the Fourteen Princesses Problem

## 1.2 Domination in Graphs

The Fourteen Princesses Problem is closely related to the topic of domination in graphs. In recent decades, domination in graphs has become a popular area of study. While this area evidently began with the work of Berge [3] in 1958 and Ore [17] in 1962, domination did not become an active area of research until 1977 with the appearance of a survey paper by Cockayne and Hedetniemi [10]. Since then, a large number of variations and applications of domination have surfaced (see [13, 14]). For a vertex $v$ in a nontrivial connected graph $G$, let $N(v)$ denote the *neighborhood* of $v$ and $N[v] = \{v\} \cup N(v)$ the *closed neighborhood* of $v$. A vertex $v$ in a graph $G$ is said to *dominate* a vertex $u$ if either $u = v$ or $uv \in E(G)$. That is, a vertex $v$ dominates the vertices in its closed neighborhood $N[v]$. A set $S$ of vertices in $G$ is a *dominating set* of $G$ if every vertex of $G$ is dominated by at least one vertex in $S$. The minimum number of vertices in a dominating set of $G$ is the *domination number* $\gamma(G)$ of $G$. A dominating set of cardinality $\gamma(G)$ is called a minimum dominating set.

For example, $S_1 = \{r, u, v, x\}$ and $S_2 = \{t, w, z\}$ are dominating sets for the graph $H$ of order 9 in Figure 1.5. In fact, $S_2$ is a minimum dominating set of $H$ and so $\gamma(H) = 3$. A dominating set $S$ of $G$ is *minimal* if no proper subset of $S$ is a dominating set of $G$. Thus, every minimum dominating set is minimal, while there are minimal dominating sets that are not minimum. For example, $S_1$ is a minimal dominating set of the graph $H$ of Figure 1.5 but $S_1$ is not a minimum dominating set of $H$.

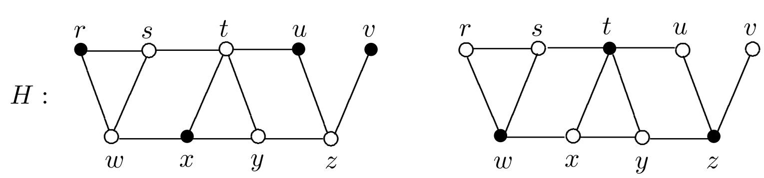



Figure 1.5: Dominating sets in a graph

In their 2023 book, Haynes, Hedetniemi, and Henning [12] presented the major results that have been obtained on what they refer to as the three core concepts of graph domination. One of these concepts is standard domination. A second is independent domination. A set $S$ of vertices in a graph $G$ is an *independent dominating set* of $G$ if it is both an independent set (no two vertices in $S$ are adjacent) and a dominating set of $G$. The third core concept is total domination, introduced by Cockayne, Dawes and Hedetniemi [9] in 1977. In total domination, a vertex $u$ (totally) dominates a vertex $v$ in a graph $G$ if $uv \in E(G)$. That is, $v$ does not dominate itself in total domination. A set $S$ of vertices in a graph $G$ is a *total dominating set* of $G$ if every vertex $v$ of $G$ is totally dominated by some vertex of $S$. The minimum cardinality of a total dominating set of $G$ is the *total domination number* $\gamma_t(G)$ of $G$. A total dominating set of cardinality $\gamma_t(G)$ is called a minimum total dominating set. For example, $T_1 = \{q, r, t, x, v, z\}$ and $T_2 = \{r, s, y, z\}$ are total dominating sets for the graph $F$ of order 10 shown in Figure 1.6. In fact, $T_2$ is a minimum total dominating set of $F$ and so $\gamma_t(F) = 4$.

$F$ :

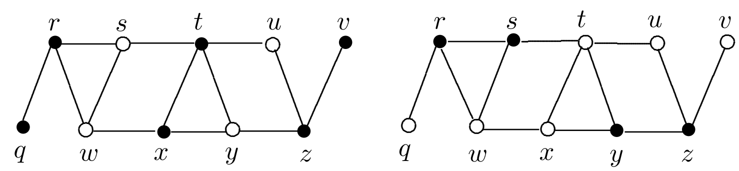



Figure 1.6: Total Dominating sets in a graph

As with the situation of dominating sets, a total dominating set $S$ of a graph $G$ is *minimal* if no proper subset of $S$ is a total dominating set of $G$. Thus, every minimum total dominating set is minimal, while there are minimal total dominating sets that are not minimum. For example, the set $T_1$ is a minimal total dominating set of $F$ in Figure 1.6 but $T_1$ is not a minimum total dominating set of $F$. A graph $G$ has a total dominating set if and only if $G$ has no isolated vertices. The book by Henning and Yeo [15] deals exclusively with total domination in graphs.

Total domination, as well as some other types of domination, can be described with the aid of distance in graphs. We denote the *distance* (the length of a shortest path) between two vertices $u$ and $v$ in a graph $G$ by $d(u, v)$. The greatest distance from a vertex $v$ to another vertex of $G$ is its *eccentricity*, denoted by $e(v)$. The minimum eccentricity among the vertices of $G$ is the *radius* $\text{rad}(G)$ of $G$ and the maximum eccentricity is the *diameter* $\text{diam}(G)$. Therefore, the diameter of $G$ is the maximum distance between any two vertices of $G$. A vertex $v$ with $e(v) = \text{rad}(G)$ is called a *central vertex* of $G$ and a vertex $v$ with $e(v) = \text{diam}(G)$ is called a *peripheral vertex* of $G$. Two vertices $u$ and $v$ of $G$ with $d(u, v) = \text{diam}(G)$ are *antipodal vertices* of $G$. Necessarily, if $u$ and $v$ are antipodal vertices in $G$, then both $u$ and $v$ are peripheral vertices.

In total domination, a vertex $u$ dominates a vertex $v$ if $d(u, v) = 1$. For a total dominating set $S$ in a nontrivial connected graph $G$, one can think of assigning each vertex of $S$ the label 1 and assigning no label to the vertices of $G$ not in $S$. Thus, if $u \in S$, then $u$ is labeled 1, indicating that $u$ dominates all vertices of $G$ whose distance from $u$ is 1. Thus, every vertex of $G$ has distance 1 from at least one vertex of $S$.

In [11], a generalization of (total) domination was introduced called *orbital domination*. For a positive integer $r$ and a vertex $v$ in a connected graph $G$, the *$r$-orbit* $O_r(v)$ of $v$ is $O_r(v) = \{u \in V(G) : d(u, v) = r\}$. A set $S = \{u_1, u_2, \ldots, u_k\}$ of vertices in a nontrivial connected graph $G$ is an *orbital dominating set* of $G$ if each vertex $u_i \in S$ can be labeled with a positive integer $r_i$, where $r_i \leq e(u_i)$, such that $\bigcup_{i=1}^{k} O_{r_i}(u_i) = V(G)$. Thus, if $S$ is an orbital dominating set of $G$, then for every vertex $v$ of $G$, there exists a vertex $u_i$ in $S$ such that $d(u_i, v) = r_i$. Here, $u_i$ is said to *dominate* $v$. The minimum cardinality of an orbital dominating set is called the *orbital domination number* of $G$. This concept has been studied further in [8]. If all labels of an orbital dominating set $S$ are the same positive integer $r$, then $S$ is an *$r$-regular orbital dominating set*. It was shown in [11] that a nontrivial connected graph $G$ has an $r$-regular orbital dominating set if and only if $1 \leq r \leq \text{rad}(G)$. If $r = 1$, then $S$ is a *total dominating set*.

## 1.3 Irregular Domination

In the book [1] various "regularity" concepts are discussed, describing how this can lead to concepts that are in a sense opposite to these, resulting in "irregularity" concepts. In terms of domination, if no two vertices of an orbital dominating set $S$ have the same label, then $S$ is an *irregular orbital dominating set* or, more simply, an *irregular dominating set*. Consequently, a connected graph $G$ has an irregular dominating set if it is possible to assign distinct labels (positive integers) to some vertices of $G$ in such a way that for every vertex $v$ of $G$, there is a labeled vertex $u$ such that $d(u, v)$ is the label assigned to $u$. Such a labeling is called an *irregular* (*dominating*) *labeling* and $u$ is said to *dominate* $v$. While every nontrivial connected graph has an orbital dominating set (indeed, a total dominating set), not every graph has an irregular dominating set. For a nontrivial connected graph $G$ possessing irregular dominating sets, the minimum cardinality of an irregular dominating set of $G$ is referred to as the *irregular domination number* of $G$, denoted by $\tilde{\gamma}(G)$. An irregular dominating set of cardinality $\tilde{\gamma}(G)$ is a *minimum irregular dominating set* and its corresponding irregular dominating labeling is a *minimum irregular dominating labeling* of $G$.

For example, consider the graph $G$ of Figure 1.7. The set $S = \{t, y, z\}$ is an irregular dominating set for $G$, where the corresponding irregular dominating labeling $f : S \to \{1, 2, 3\}$ is defined by $f(t) = 1$, $f(y) = 2$, and $f(z) = 3$. The vertex $t$ dominates $v, w, x$, and $z$, the vertex $y$ dominates $t$ and $w$, and the vertex $z$ dominates $u$ and $y$. Thus, all vertices of $G$ are dominated by the vertices of $S$. The concept of irregular domination was introduced and studied in [4] and studied further in [5, 6].

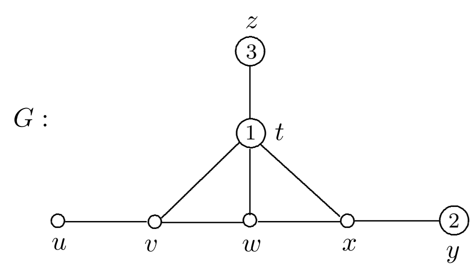



Figure 1.7: An irregular set in a graph

In the area of domination in graphs, a fundamental question for two vertices $u$ and $v$ in a graph $G$ is: Does $u$ dominate $v$? A vertex $u$ in a graph $G$ is said to dominate a vertex $v$ if either $u = v$ or $uv \in E(G)$. A vertex $u$ is said to totally dominate a vertex $v$ in a graph $G$ if $uv \in E(G)$. Thus, $u$ dominates $v$ if and only if $v$ dominates $u$. Also, $u$ totally dominates $v$ if and only if $v$ totally dominates $u$. Hence, both types of domination are symmetric. Further, a vertex $u$ in a graph $G$ always dominates at least one vertex, while a vertex $u$ totally dominates at least one vertex if and only if $u$ is not an isolated vertex in $G$. Here, we consider another type of domination, one that is not symmetric. Indeed, if it should occur that a vertex $u$ dominates a vertex $v$, then $v$ does not dominate $u$. In addition, in the irregular domination we just described, only certain vertices in a graph are designated to be capable of dominating other vertices. Furthermore, it is possible that no vertices in a graph $G$ can be so designated to dominate all vertices of $G$; that is, the graph $G$

does not have an irregular dominating set.

**Observation 1.3.1** *No two vertices in an irregular dominating set of a connected graph dominate each other.*

**Proof.** Let $u$ and $v$ be two vertices in an irregular dominating set of a connected graph. Suppose that $d(u, v) = d$. If one of $u$ and $v$ is labeled $d$, say $u$ is labeled $d$, then $v$ is not labeled $d$. Hence, $u$ dominates $v$ but $v$ cannot dominate $u$. If neither $u$ nor $v$ is labeled $d$, then $u$ cannot dominate $v$ and $v$ cannot dominate $u$. ■

The primary goal in [4, 5, 6] concerns investigating graphs that possess irregular dominating sets and determining the irregular domination numbers of these graphs. The following results appeared in [4, 5, 6] on irregular dominating sets of graphs.

**Theorem 1.3.2** *No connected vertex-transitive graph has an irregular dominating set.*

In particular, the well-known Petersen graph and $n$-cube $Q_n$ do not have an irregular dominating set. Furthermore, if $G$ is a complete graph $K_n$, a complete regular bipartite graph $K_{r,r}$, or a $n$-cycle $C_n$, then $G$ is vertex-transitive and so does not have an irregular dominating set. On the other hand, almost all of nontrivial trees have irregular dominating sets.

**Theorem 1.3.3** *A nontrivial tree $T$ has an irregular orbital labeling if and only if $T$ is none of $P_2, P_6$ or a star.*

**Theorem 1.3.4** *If $T$ is a tree of diameter $d \geq 3$ and $d \neq 5$, then $\tilde{\gamma}(T) \leq \tilde{\gamma}(P_{d+1})$.*

**Proposition 1.3.5** *Let $G$ be a connected graph and let $H$ be a connected subgraph of $G$ such that $d_G(x, y) = d_H(x, y)$ for every two vertices $x$ and $y$ of $H$; that is $H$ is a distance-preserving subgraph of $G$. If $S$ is an irregular dominating set of $G$ with corresponding irregular labeling $f$ where $S \subseteq V(H)$, then $S$ is an irregular dominating set of $H$ with corresponding irregular labeling $f$.*

To illustrate Proposition 1.3.5, we consider the connected graph $G$ of Figure 1.8 and the connected subgraph $H = G - \{x, y, z\}$ of $G$. The subgraph $H$ satisfies the distance-preserving condition described in Proposition 1.3.5. The set $S = \{u_1, u_2, u_4, u_5, v\} \subseteq V(H)$ is an irregular dominating set of $G$ with corresponding irregular labeling $f : S \to [5] = \{1, 2, \ldots, 5\}$ by $f(u_1) = 5$, $f(u_2) = 2$, $f(u_4) = 1$, $f(u_5) = 4$, and $f(v) = 3$. It then follows by Proposition 1.3.5 that $S = \{u_1, u_2, u_4, u_5, v\}$ is also an irregular dominating set of $H$ with corresponding irregular labeling $f$.

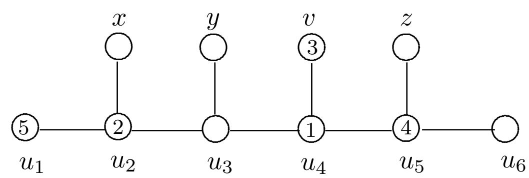



Figure 1.8: Illustrating Proposition 1.3.5

**Theorem 1.3.6** *If $G$ is a connected graph with an irregular dominating set, then $\tilde{\gamma}(G) \geq 3$. An irregular dominating labeling of the graph using exactly three labels must use labels from the set $[3] = \{1, 2, 3\}$.*

By Theorem 1.3.6, the set $S = \{t, y, z\}$ is a minimum irregular dominating set of the graph $G$ of Figure 1.7 and so $\tilde{\gamma}(G) = 3$. Thus, to show that a graph $G$ has irregular domination number 3, it suffices to show that $G$ has an irregular dominating set of cardinality 3. In general, however, it is often challenging to determine the exact value of the irregular domination number of a graph. In particular, if $\tilde{\gamma}(G) = k \geq 4$, then we need to show

(1) the graph $G$ has an irregular dominating set of cardinality $k$ and

(2) for each integer $k'$ with $3 \leq k' \leq k - 1$, there is no irregular dominating set of cardinality $k'$ in $G$.

We now illustrate this concept with two examples.

**Example 1.3.7** *The irregular domination number of the graph $G$ of Figure 1.9 is 4.*

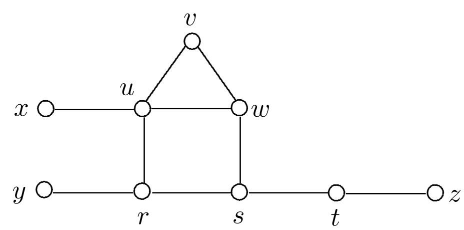



Figure 1.9: The graph $G$

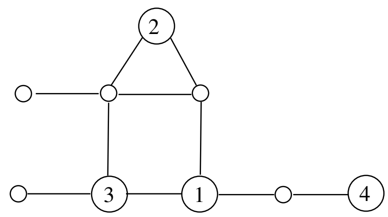



Figure 1.10: An irregular dominating labeling of a graph $G$

**Proof.** Since there is an irregular dominating labeling of $G$ using all labels in the set [4], as shown in Figure 1.10, it follows that $\tilde{\gamma}(G) \leq 4$. We claimed that $\tilde{\gamma}(G) = 4$.

Assume, to the contrary, that $G$ has an irregular dominating labeling $f$ using the labels $1, 2, 3$. First, we show that $v$ must be assigned a label, for suppose that $v$ is not assigned a label by $f$. The graph $H = G - v$ is a distance-preserving bipartite subgraph of $G$ and $\Delta(H) = 3$. Let $U = \{u, y, s, z\}$ and $W = \{w, x, r, t\}$ be the partite sets of $H$. The vertex labeled 1 or 3 dominates at most three vertices in the partite set of $H$ not containing that vertex. The vertex labeled 2 dominates at most three vertices in the same partite set of $H$ containing that vertex. Since each labeled vertex dominates at most three vertices in either $U$ or $W$ and dominates no vertex in the other partite set, it follows that some labeled vertex must dominate all four vertices in either $U$ or $W$. This is impossible. Therefore, the vertex $v$ must be assigned a label. We consider three cases.

*Case* 1. *The vertex $v$ is labeled* 1. Then $v$ dominates $u$ and $w$. Thus, the three vertices in $U - \{u\} = \{y, s, z\}$ must be dominated by a vertex labeled 2 or by a vertex labeled 3. However, there is no vertex in $H$ whose distance from all of $y, s, z$ is a constant and so no vertex can be labeled 2 or 3 to dominate these three vertices, a contradiction.

*Case* 2. *The vertex $v$ is labeled* 2. Then $v$ dominates $x, r$, and $s$. Thus, the three vertices in $U - \{s\} = \{u, y, z\}$ must be dominated by a vertex labeled 1 or by a vertex labeled 3. However, there is no vertex in $H$ whose distance from all of $u, y, z$ is a constant and so no vertex can be labeled 1 or 3 to dominate these three vertices, a contradiction.

*Case* 3. *The vertex $v$ is labeled* 3. Then $v$ dominates $y$ and $t$. Thus, the three vertices in $U - \{y\} = \{u, s, z\}$ must be dominated by a vertex labeled 1 or by a vertex labeled 2. However, there is no vertex in $H$ whose distance from all of $u, s, t$ is a constant and so no vertex can be labeled 1 or 2 to dominate these three vertices, a contradiction.

Therefore, $\tilde{\gamma}(G) = 4$, as claimed. ■

The second example is the graph $H$ of Figure 1.11. This graph is a bipartite graph of diameter 6, order 12, and size 13 with partite sets $U = \{u_1, u_2, \ldots, u_6\}$ and $W = \{w_1, w_2, \ldots, w_6\}$.

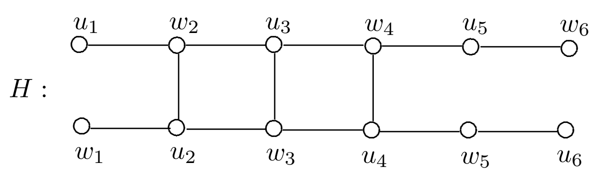



Figure 1.11: A bipartite graph of diameter 6

First, we present some useful information about bipartite graphs in general (see [4, 5]). If $G$ is a connected bipartite graph with partite sets $U$ and $W$, then the distance $d(x, y)$ between two vertices $x$ and $y$ of $G$ is even if and only if $x, y \in U$ or $x, y \in W$. Consequently, $d(x, y)$ is odd if and only if one of $x, y$ belongs to $U$ and the other belongs to $W$. Thus, we have the following observation.

**Observation 1.3.8** *Let $U$ and $W$ be the partite sets of a connected bipartite graph $G$ and let $f$ be an irregular dominating labeling of $G$. If $x$ is a labeled vertex and $f(x) = r$, then*

*(i) $O_r(x) \subseteq U$ or $O_r(x) \subseteq W$ and*

*(ii) $x$ and $O_r(x)$ belong to the same partite set if and only if $r$ is even.*

*Consequently, if a labeled vertex dominates two vertices $u$ and $v$, then $d(u, v)$ is even. Thus, a labeled vertex cannot dominate two vertices $u$ and $w$ if $d(u, w)$ is odd. In particular, a labeled vertex cannot dominate two adjacent vertices of $G$.*

Two vertices $u$ and $w$ in a connected bipartite graph $G$ are said to be of *opposite parity* if $d(u, w)$ is odd, that is, if $u$ and $w$ belong to different partite sets. The two vertices $u$ and $w$ are of the *same parity* if $d(u, w)$ is even, that is, if $u$ and $w$ belong to a same partite set. By Observation 1.3.8, if $f$ is an irregular orbital labeling of a connected bipartite graph $G$ and $x$ is a labeled vertex, then $x$ cannot dominate two vertices of opposite parity in $G$. The following result appeared [6].

**Theorem 1.3.9** *If $G$ is a connected bipartite graph with $\tilde{\gamma}(G) = 3$, then $\text{diam}(G) \in \{3, 4\}$.*

Since the diameter of $H$ is 6, it follows that by Theorem 1.3.9 if $f$ is an irregular dominating labeling of $H$, then the set of labels of $f$ is a subset of $[6]$ of cardinality 4 or more. Figure 1.12 shows two irregular dominating labelings of $H$ using five labels. Thus, $\tilde{\gamma}(H) \leq 5$ and so $\tilde{\gamma}(H) = 4$ or $\tilde{\gamma}(H) = 5$. In fact, $\tilde{\gamma}(H) = 5$. In order to verify this fact, we first make the following observations and establish some lemmas.

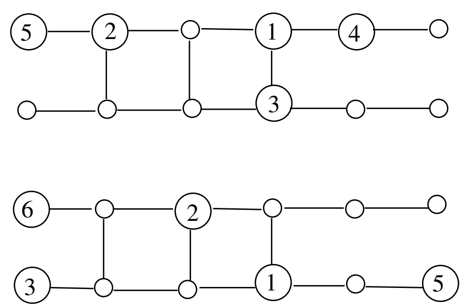



Figure 1.12: Two irregular dominating labelings of the graph $H$

**Observation 1.3.10** *Let there be given an irregular dominating labeling of the graph $H$ of Figure 1.11.*

- ⋆ *Every labeled vertex can dominate only vertices in one partite set of $H$.*

- ⋆ *A vertex labeled* 1 *can dominate at most* 3 *vertices of* $H$.
- ⋆ *A vertex labeled* 2 *can dominate at most* 4 *vertices of* $H$.
- ⋆ *A vertex labeled* 3 *can dominate at most* 3 *vertices of* $H$.
- ⋆ *A vertex labeled* 4 *can dominate at most* 3 *vertices of* $H$.
- ⋆ *A vertex labeled* 5 *can dominate at most* 3 *vertices of* $H$.
- ⋆ *A vertex labeled* 6 *can dominate at most* 1 *vertex of* $H$.

Let $X$ be a set of vertices of a graph $G$. If every vertex in $X$ is dominated by a labeled vertex $v$, then we say that $X$ *is dominated by* $v$. First, we have several useful lemmas, the first of which is a consequence of Observation 1.3.10.

**Lemma 1.3.11** *No partite set of* $H$ *can be dominated by two labeled vertices, one of which is labeled* 6.

**Lemma 1.3.12** *No partite set of* $H$ *can be dominated by two labeled vertices, one of which is labeled* 2.

**Proof.** By Lemma 1.3.11, no partite set of $H$ can be dominated by two vertices labeled 2 and 6. Assume, to the contrary, that there exists a partite set of $H$ that is dominated by two labeled vertices, one of which is labeled 2 and the other is labeled $i$ for some $i \in \{1, 3, 4, 5\}$. By symmetry, we may assume that $W$ is such a partite set of $H$. By Observation 1.3.10, the vertex labeled 2 must dominate at least three vertices and so $f$ assigns the label 2 to one of the vertices $w_2, w_3$, $w_4$.

- ⋆ If $f(w_2) = 2$, then $w_2$ dominates $w_1, w_3, w_4$. A vertex labeled $i$, where $i \in \{1, 3, 4, 5\}$, cannot dominate the three vertices $w_2, w_5, w_6$, a contradiction.
- ⋆ If $f(w_3) = 2$, then $w_3$ dominates $w_1, w_2, w_4, w_5$. A vertex labeled $i$, where $i \in \{1, 3, 4, 5\}$, cannot dominate the two vertices $w_3$ and $w_6$, a contradiction.
- ⋆ If $f(w_4) = 2$, then $w_4$ dominates $w_2, w_3, w_5, w_6$. A vertex labeled $i$, where $i \in \{1, 3, 4, 5\}$, cannot dominate the two vertices $w_1$ and $w_4$, a contradiction. ■

**Lemma 1.3.13** *No partite set of $H$ can be dominated by two vertices labeled* 3 *and* 4 *or by two vertices labeled* 1 *and* 3.

**Proof.** Assume, to the contrary, that there exists a partite set of $H$ that is dominated by two vertices labeled 3 and 4 or by two vertices labeled 1 and 3. By symmetry, we may assume that the partite set $W$ is dominated by one of these two pairs of labeled vertices. Thus, each labeled vertex dominates exactly three vertices of $W$ by Observation 1.3.10. We consider two cases.

*Case* 1. *The partite set $W$ is dominated by two vertices labeled* 3 *and* 4. Since the vertex labeled 4 must dominate three vertices of $W$, it follows that either $f(w_5) = 4$ or $f(w_6) = 4$. If $f(w_5) = 4$, then $w_5$ dominates $w_1, w_2$, and $w_6$. However then, no vertex labeled 3 can dominate the remaining three vertices in $\{w_3, w_4, w_5\}$. If $f(w_6) = 4$, then $w_6$ dominates $w_2, w_3$, and $w_5$. However then, no vertex labeled 3 can dominate the remaining three vertices in $\{w_1, w_4, w_6\}$.

*Case* 2. *The partite set $W$ is dominated by two vertices labeled* 1 *and* 3. Since the vertex labeled 1 must dominate three vertices of $W$, it follows that $f$ assigns the label 1 to one of the vertices $u_2, u_3$, $u_4$. For each $i = 2, 3, 4$, the vertex $u_i$ dominates the three vertices in $N(u_i)$. However then, no vertex labeled 3 can dominate the remaining three vertices in $W - N(u_i)$ for $i = 2, 3, 4$. This is a contradiction. ■

**Lemma 1.3.14** *No partite set of $H$ can be dominated by two vertices labeled* 3 *and* 5 *or by two vertices labeled* 4 *and* 5.

**Proof.** Assume, to the contrary, that there exists a partite set of $H$ that is dominated by two vertices labeled 3 and 5 or by two vertices labeled labeled 4 and 5. By symmetry, we may assume that the partite set $W$ is dominated by one of these two pairs of labeled vertices. Since the vertex labeled 5 must dominate three vertices of $W$, it follows that $f(u_6) = 5$ and $w_6$ dominates $w_1$, $w_2$, and $w_6$. However then, the three vertices in $\{w_3, w_4, w_5\}$ cannot be dominated by a vertex labeled 3 or 4, a contradiction. ■

By Lemmas 1.3.11 – 1.3.14, if $x$ and $y$ are labeled vertices such that $\{f(x), f(y)\}$ contains 6 or is one of the 2-element subsets {1, 2}, {1, 3}, {2, 3}, {2, 4},{2, 5}, {3, 4},{3, 5}, {4, 5} of [5], then $x$ and $y$ cannot dominate a partite set of $H$. The following is a consequence of Lemmas 1.3.11 – 1.3.14.

**Corollary 1.3.15** *Let $f$ be an irregular dominating labeling of $H$. If a partite set of $H$ is dominated by two labeled vertices $x$ and $y$, then $\{f(x), f(y)\} = \{1, 4\}$ or $\{f(x), f(y)\} = \{1, 5\}$.*

We are now prepared to show that $\tilde{\gamma}(H) = 5$ for the graph $H$ of Figure 1.11.

**Example 1.3.16** *The irregular domination number of the graph $H$ of Figure* 1.11 *is* 5.

**Proof.** By Theorem 1.3.9, $\tilde{\gamma}(H) \geq 4$. Since $H$ has irregular dominating labelings using five labels, as shown in Figure 1.12, it follows that $\tilde{\gamma}(H) \leq 5$ and so $\tilde{\gamma}(H) = 4$ or $\tilde{\gamma}(H) = 5$. We show that $\tilde{\gamma}(H) = 5$. Assume, to the contrary, that there exists an irregular dominating labeling $f$ of $H$ using four labels from the set [6]. By Observation 1.3.10, each partite set of $H$ is dominated by two labeled vertices and every labeled vertex can dominate only vertices in one partite set of $H$. By Corollary 1.3.15, the label set of these two vertices is either $\{1, 4\}$ or $\{1, 5\}$, which is impossible. ■

# Chapter 2

# Irregular Domination Numbers of Grids

## 2.1 Revisit the Fourteen Princesses Problem

A standard $8 \times 8$ chessboard can be represented by a graph whose 64 vertices are the squares of the chessboard and where two vertices are adjacent if they are squares having a side in common. The resulting graph is the Cartesian product $P_8 \square P_8$ of the path $P_8$ of order 8 with itself. The Fourteen Princesses Problem can therefore be looked at as a problem in graph theory. The Fourteen Princesses Problem then becomes

> *Is there an irregular dominating labeling of the graph $P_8 \square P_8$ using all 14 labels in the set [14]?*

The solution to the Fourteen Princesses Problem given in Figure 1.4 is the irregular dominating labeling of $P_8 \square P_8$ shown in Figure 2.1 using all 14 labels in the set [14].

The graphs of the type $P_m \square P_n$ are called $m \times n$ *grids*. The following question appeared in [6]:

> For which pairs $m, n$ of positive integers, is there an irregular dominating labeling of $P_m \square P_n$ (in addition to $m = n = 8$)? That is, for which pairs $m, n$ of positive integers, is there a solution to the $m + n - 2$ Princesses Problem on the $m \times n$ chessboard?

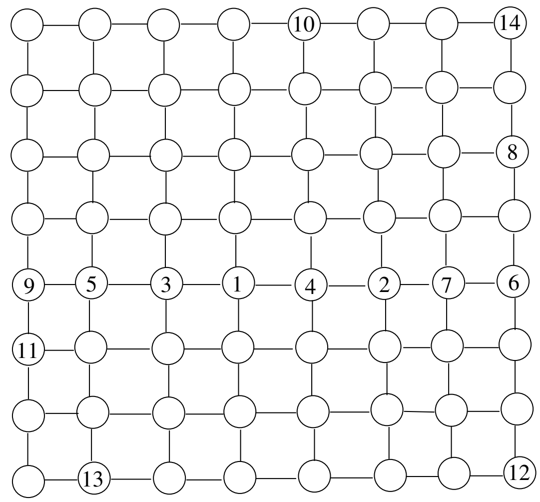



Figure 2.1: An irregular dominating labeling of $P_8 \square P_8$

It was shown in [6] that for infinite classes of grids studied, all have irregular dominating sets with small exceptions, namely $P_2 \square P_2 \cong C_4$ and $P_4 \square P_2$. This suggests the following conjecture that appeared in [6].

**Conjecture 2.1.1** *All* $(m+n-2)$ *Princesses Problems on an* $m \times n$ *chessboard for* $m \geq n \geq 2$ *have a solution except when* $(m, n) \in \{(2, 2), (4, 2)\}$.

Since the solution to the Fourteen Princesses Problem uses all 14 labels in the set [14], it gives rise to the following question that appeared in [6].

**Problem 2.1.2** *What is the minimum number of distinct princesses that can be placed on an* $8 \times 8$ *chessboard so that all squares of a chessboard are covered?*

In terms of graphs, Problem 2.1.2 then becomes

*What is the irregular domination number* $\tilde{\gamma}(P_8 \square P_8)$ *of the graph* $P_8 \square P_8$*?*

## 2.2 The $m \times n$ Grids

For integers $m$ and $n$ with $m \geq n \geq 2$, let $G_{m,n} = P_m \square P_n$ be the $m \times n$ grid of order $mn$. Since $\mathrm{diam}(G_{m,n}) = m+n-2$, it follows by Theorem 1.3.6 that if $\tilde{\gamma}(G_{m,n})$

exists, then

$$3 \leq \tilde{\gamma}(G_{m,n}) \leq m + n - 2. \tag{2.1}$$

Since grids are bipartite graphs, it follows by Theorem 1.3.9 that if $\operatorname{diam}(G_{m,n}) = m+n-2 \geq 5$ or $m+n \geq 7$, then $\tilde{\gamma}(G_{m,n}) \geq 4$. We now determine $\tilde{\gamma}(G_{m,n})$ for some small values of $m$ and $n$, namely $m \geq n \geq 2$ and $m + n \in \{5, 6, 7, 8, 9\}$.

**Proposition 2.2.1** *$\tilde{\gamma}(G_{3,2}) = 3$ and $\tilde{\gamma}(G_{3,3}) = 4$.*

**Proof.** Since there is an irregular dominating labeling of $G_{3,2}$ using all labels from the set [3], it follows by Theorem 1.3.6 that $\tilde{\gamma}(G_{3,2}) = 3$. Thus, it remains to show that $\tilde{\gamma}(G_{3,3}) = 4$. First, there is an irregular dominating labeling of $G_{3,3}$ using all labels from the set [4]. Such a labeling is shown in Figure 2.2. Thus, $\tilde{\gamma}(G_{3,3})$ exists and $\tilde{\gamma}(G_{3,3}) \leq 4$. By Theorem 1.3.6, it follows that $\tilde{\gamma}(G_{3,3}) \geq 3$. Thus, $\tilde{\gamma}(G_{3,3}) = 3$ or $\tilde{\gamma}(G_{3,3}) = 4$. We show that $\tilde{\gamma}(G_{3,3}) = 4$. Label the vertices of $G_{3,3}$ as shown in Figure 2.2. Thus, the partite sets of $G_{3,3}$ are $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w, w_1, w_2, w_3, w_4\}$.

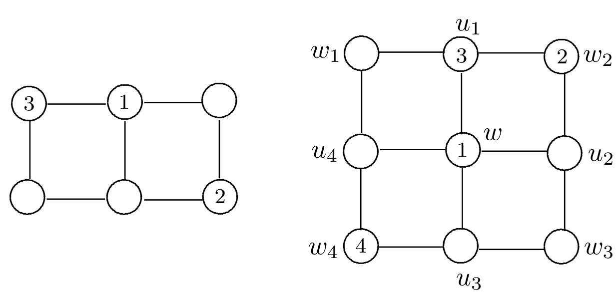



Figure 2.2: Irregular dominating labelings of $G_{3,2}$ and $G_{3,3}$

Assume, to the contrary, that $\tilde{\gamma}(G_{3,3}) = 3$. It then follows by Theorem 1.3.6 that there is an irregular dominating labeling $f$ of $G_{3,3}$ whose label set is [3]. A vertex labeled 1 or 2 can dominate at most four vertices of $G_{3,3}$, while a vertex labeled 3 can dominate at most two vertices of $G$. Since any labeled vertex can only dominate vertices belonging to a single partite set of $G_{3,3}$ and $W$ consists of five vertices, it follows that the partite set $U$ with four vertices must be dominated by a single labeled vertex, namely one labeled 1 or 2. The only such vertex that can satisfy this requirement would be vertex $w$ if labeled 1. This means we then must dominate the partite set $W$ with vertices labeled 2 and 3. In particular, the vertex $w$ must be dominated by a vertex labeled 2. Assume $w_1$ is labeled 2. Then $w_1$ and $w_3$ are un-dominated and cannot be dominated by one vertex labeled 3. Therefore $\tilde{\gamma}(G_{3,3}) \neq 3$ ■

**Proposition 2.2.2** $\tilde{\gamma}(G_{4,3}) = 4$.

**Proof.** Since the diameter of $G_{4,3}$ is 5, it follows by Theorem 1.3.9 that $\tilde{\gamma}(G_{4,3}) \geq 4$. The irregular dominating labeling of $G_{4,3}$ in Figure 2.3 shows that $\tilde{\gamma}(G_{4,3}) \leq 4$. Therefore, $\tilde{\gamma}(G_{4,3}) = 4$. ■

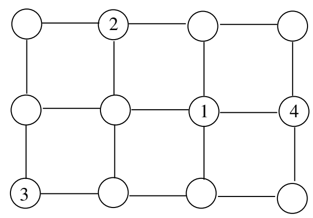



Figure 2.3: An irregular dominating labeling of $G_{4,3}$

**Proposition 2.2.3** $\tilde{\gamma}(G_{4,4}) = 5$.

**Proof.** An irregular dominating labeling of $G_{4,4}$ with label set [5] is shown in Figure 2.4, which shows that $\tilde{\gamma}(G_{4,4}) \leq 5$. We claim that $\tilde{\gamma}(G_{4,4}) = 5$. Label the vertices of $G_{4,4}$ as shown in Figure 2.4. Then $U = \{u_1, u_2, \ldots, u_8\}$ and $W = \{w_1, w_2, \ldots, w_8\}$ are the partite sets of $G_{4,4}$.

Assume, to the contrary, that there exists an irregular dominating labeling $f$ of $G_{4,4}$ using 4 elements of [6]. First, we make the following observations.

- ⋆ A vertex labeled 1 can dominate at most 4 vertices of $G_{4,4}$.
- ⋆ A vertex labeled 2 can dominate 6, 4, or 3 vertices of $G_{4,4}$.
- ⋆ A vertex labeled 3 can dominate at most 4 vertices of $G_{4,4}$.
- ⋆ A vertex labeled 4 can dominate at most 3 vertices of $G_{4,4}$.
- ⋆ A vertex labeled 5 can dominate at most 2 vertices of $G_{4,4}$.

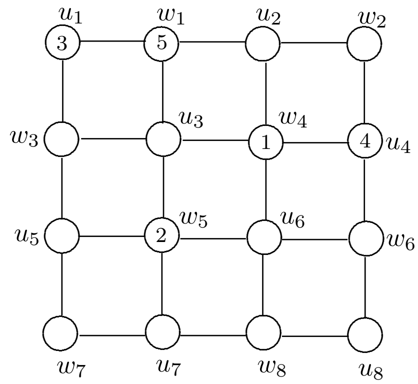



Figure 2.4: An irregular dominating labeling of $G_{4,4}$

$\star$ A vertex labeled 6 can dominate at most 1 vertex of $G_{4,4}$.

By Observation 1.3.8 and the observations above, each partite set of $G_{4,4}$ must be dominated by exactly two labeled vertices. Since $|U| = |W| = 8$, it follows that one partite set of $G_{4,4}$ must be dominated by two vertices labeled 1 and 3 and the other partite set must be dominated by two vertices labeled 2 and $i$ where $i \in \{4, 5\}$. We may assume that $W$ is dominated by two vertices labeled 1 and 3. By symmetry, we can further assume that $f(u_6) = 1$. Thus, $u_6$ dominates all vertices of $W$ except $w_1, w_2, w_3, w_7$. However, $u_6$ is the only vertex whose distance to these four vertices is 3 and $f(u_6) = 1$, which is impossible. ■

**Proposition 2.2.4** $\tilde{\gamma}(G_{5,2}) = 4$.

**Proof.** Since the diameter of $G_{5,2}$ is 5, it follows by Theorem 1.3.9 that $\tilde{\gamma}(G_{5,2}) \geq 4$. The irregular dominating labeling of $G_{5,2}$ in Figure 2.5 shows that $\tilde{\gamma}(G_{5,2}) \leq 4$, Therefore, $\tilde{\gamma}(G_{5,2}) = 4$. ■

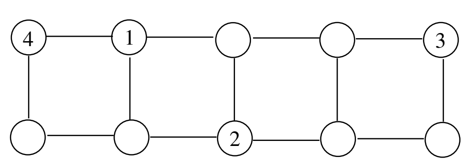



Figure 2.5: An irregular dominating labeling of $G_{5,2}$

**Proposition 2.2.5** $\tilde{\gamma}(G_{5,3}) = 5$.

**Proof.** An irregular dominating labeling of $G_{5,3}$ with label set $[5]$ is shown in Figure 2.6, which shows that $\tilde{\gamma}(G_{5,3}) \leq 5$. We claim that $\tilde{\gamma}(G_{5,3}) = 5$. Label the vertices of $G_{5,3}$ as shown in Figure 2.6. Then $U = \{u_1, u_2, \ldots, u_8\}$ and $W = \{w_1, w_2, \ldots, w_7\}$ are the partite sets of $G_{5,3}$ and $\text{diam}(G_{5,3}) = 6$. Assume, to the contrary, that there exists an irregular dominating labeling $f$ of $G_{5,3}$ using 4 elements of $[6]$.

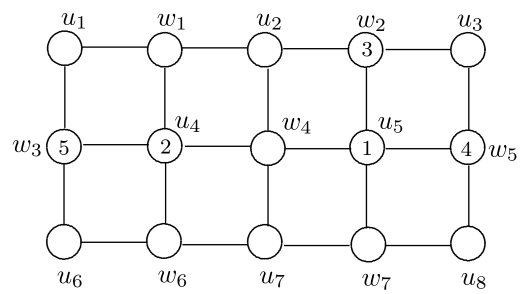



Figure 2.6: An irregular dominating labeling of $G_{5,3}$

First, we make the following observations.

- $\star$ A vertex labeled 1 can dominate at most 4 vertices of $G_{5,3}$.
- $\star$ A vertex labeled 2 can dominate at most 6 vertices of $G_{5,3}$.
- $\star$ A vertex labeled 3 can dominate at most 4 vertices of $G_{5,3}$.
- $\star$ A vertex labeled 4 can dominate at most 3 vertices of $G_{5,3}$.
- $\star$ A vertex labeled 5 can dominate at most 2 vertices of $G_{5,3}$.
- $\star$ A vertex labeled 6 can dominate at most 1 vertex of $G_{5,3}$.

By Observation 1.3.8 and the observations above, it follows that

(1) each partite set of $G_{5,3}$ must be dominated by exactly two labeled vertices and

(2) there must be a vertex labeled 2 that dominates at least 4 vertices of $G_{5,3}$.

First, suppose that a vertex labeled 2 dominates 6 vertices of $G_{5,3}$. Then $f(w_4) = 2$ and so $w_4$ dominates all vertices of $W$ except itself. Since $e(w_4) = 3$ and $f(w_4) = 2$, it follows that $w_4$ can only be dominated by a vertex labeled 1 or 3. This says that the partite set $U$ must be dominated by two labeled vertices whose label set does not contain 2 and contains at most one of the labels 1 and 3. Since $|U| = 8$, this is impossible.

Next, suppose that a vertex labeled 2 dominates 5 vertices of $G_{5,3}$. We may assume that either $f(u_2) = 2$ or $f(u_4) = 2$.

- ⋆ If $f(u_2) = 2$, then $u_2$ dominates all vertices of $U$ except $u_2, u_6$, and $u_8$. Since no labeled vertex (whose label is not 2) can dominate these three vertices, this case cannot occur.

- ⋆ If $f(u_4) = 2$, then $u_4$ dominates all vertices of $U$ except $u_3, u_4$, and $u_8$. Since no labeled vertex (whose label is not 2) can dominate these three vertices, this case cannot occur.

Finally, suppose that a vertex labeled 2 dominates exactly 4 vertices of $G_{5,3}$. Thus, the four labeled vertices must be labeled 1, 2, 3, 4 and the vertex labeled 4 must dominate 3 vertices of $W$. We may assume that $f(w_3) = 4$ and so $w_3$ dominates $w_2, w_5$, and $w_8$. This in turn implies that $f(u_4) = 1$. Thus, $W$ is dominated by the two vertices labeled 1 and 4. Hence, $U$ is dominated by the two vertices labeled 2 and 3 and each labeled vertex dominates 4 vertices of $U$. We can assume that either $f(w_1) = 3$ or $f(w_4) = 3$. In both cases, no unlabeled vertex of $U$ can be labeled 2 so that it dominates the four vertices of $U$ not dominated by the vertex labeled 3. This is a contradiction. ■

**Proposition 2.2.6** $\tilde{\gamma}(G_{5,4}) = 6$.

**Proof.** An irregular dominating labeling of $G_{5,4}$ with label set [6] is shown in Figure 2.7, which shows that $\tilde{\gamma}(G_{5,4}) \leq 6$. We claim that $\tilde{\gamma}(G_{5,4}) = 6$. Label the vertices of $G_{5,4}$ as shown in Figure 2.7. Then $U = \{u_1, u_2, \ldots, u_{10}\}$ and $W = \{w_1, w_2, \ldots, w_{10}\}$ are the partite sets of $G_{5,4}$.

Assume, to the contrary, that there exists an irregular dominating labeling $f$ of $G_{5,4}$ using 5 elements of [7]. First, we make the following observations.

- ⋆ A vertex labeled 1 can dominate at most 4 vertices of $G_{5,4}$.

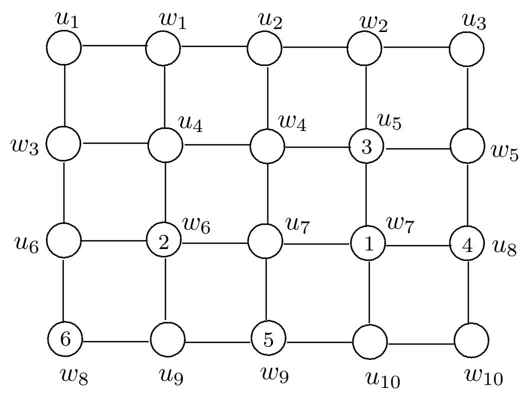



Figure 2.7: An irregular dominating labeling of $G_{5,4}$

- ⋆ A vertex labeled 2 can dominate at most 7 vertices of $G_{5,4}$.
- ⋆ A vertex labeled 3 can dominate at most 6 vertices of $G_{5,4}$.
- ⋆ A vertex labeled 4 can dominate at most 4 vertices of $G_{5,4}$.
- ⋆ A vertex labeled 5 can dominate at most 3 vertices of $G_{5,4}$.
- ⋆ A vertex labeled 6 can dominate at most 2 vertex of $G_{5,4}$.
- ⋆ A vertex labeled 7 can dominate at most 1 vertex of $G_{5,4}$.

By Observation 1.3.8 and the observations above, it follows that (i) some partite set of $G_{5,4}$ must be dominated by exactly two labeled vertices and (ii) a labeled vertex of $G_{5,4}$ must be labeled 2. Let $x$ and $y$ be the two labeled vertices that dominate a partite set of $G_{5,4}$.

First, suppose that $2 \in \{f(x), f(y)\}$. Then the vertex labeled 2 must dominate at least 4 vertices. We consider the following cases.

*Case* 1. *The vertex labeled* 2 *dominates exactly* 7 *vertices of* $G_{5,4}$. We may assume that $f(w_4) = 2$. Thus, $w_4$ dominates $w_1, w_2, w_3, w_5, w_6, w_7$, and $w_9$. Therefore, $w_4$ dominates all vertices of $W$ except $w_4, w_8$, and $w_{10}$. However, no labeled vertex (that is not labeled 2) can dominate these three vertices. Hence, Case 1 cannot occur.

*Case* 2. *The vertex labeled* 2 *dominates exactly* 6 *vertices of* $G_{5,4}$. We may assume that $f(w_6) = 2$. Thus, $w_6$ dominates $w_1, w_3, w_4, w_7, w_8$, and $w_9$. Therefore,

$w_6$ dominates all vertices of $W$ except $w_2, w_5, w_6$, and $w_{10}$. However, no labeled vertex (that is not labeled 2) can dominate these four vertices. Hence, Case 2 cannot occur.

*Case* 3. *The vertex labeled* 2 *dominates exactly* 5 *vertices of* $G_{5,4}$. We may assume that $f(u_2) = 2$. Thus, $u_2$ dominates $u_1, u_3, u_4, u_5$, and $u_7$. Therefore, $u_2$ dominates all vertices of $U$ except $u_2, u_6, u_8, u_9$, and $u_{10}$. However, no labeled vertex can dominate these five vertices. Hence, Case 3 cannot occur.

*Case* 4. *The vertex labeled* 2 *dominates exactly* 4 *vertices of* $G_{5,4}$. Thus, $\{f(x), f(y)\} = \{2, 3\}$ and the vertex labeled 3 must dominate 6 vertices. Hence, we may assume that $f(w_4) = 3$. Thus, $w_4$ dominates $u_1, u_3, u_6, u_8, u_9$ and $u_{10}$. Therefore, $w_5$ dominates all vertices of $U$ except $u_2, u_4, u_5$, and $u_7$. However, the vertex labeled 2 cannot dominate these four vertices. Hence, Case 3 cannot occur.

Consequently, $2 \notin \{f(x), f(y)\}$ and so $\{f(x), f(y)\} = \{1, 3\}$ or $\{f(x), f(y)\} = \{3, 4\}$ where the vertex labeled 3 must dominate 6 vertices of $G_{5,4}$ and the vertex labeled 1 or 4 must dominate 4 vertices. As we saw in Case 3, since the vertex labeled 3 dominates 6 vertices, we may assume that $f(w_4) = 3$ and $w_6$ dominates $u_1, u_3, u_6, u_8, u_9$, and $u_{10}$. Therefore, $w_4$ dominates all vertices of $U$ except $u_2, u_4, u_5$, and $u_7$. Since $w_4$ is the only vertex having a constant distance (namely 1) from these four vertices and $f(w_4) = 3$, this is impossible. ■

**Proposition 2.2.7** *For* $m = 6, 7, 8$, $\tilde{\gamma}(G_{m,2}) = 6$.

**Proof.** First, we consider $G_{6,2}$. Since $\text{diam}(G_{6,2}) = 6$, it follows that $\tilde{\gamma}(G_{6,2}) \le 6$. An irregular dominating labeling of $G_{6,2}$ with label set [6] is shown in Figure 2.8. We claim that $\tilde{\gamma}(G_{6,2}) = 6$.

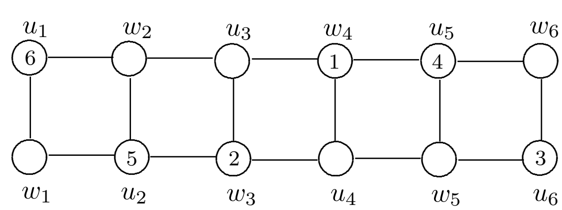



Figure 2.8: An irregular dominating labeling of $G_{6,2}$

Assume, to the contrary, that there exists an irregular dominating labeling $f$ of $G_{6,2}$ using 5 elements of [6]. First, we make the following observations.

- ⋆ A vertex labeled 1 can dominate at most 3 vertices of $G_{6,2}$.
- ⋆ A vertex labeled 2 can dominate at most 4 vertices of $G_{6,2}$.
- ⋆ A vertex labeled 3 can dominate at most 3 vertices of $G_{6,2}$.
- ⋆ A vertex labeled 4 can dominate at most 2 vertices of $G_{6,2}$.
- ⋆ A vertex labeled 5 can dominate at most 2 vertices of $G_{6,2}$.
- ⋆ A vertex labeled 6 can dominate at most 1 vertex of $G_{6,2}$.

Therefore, since $G_{6,2}$ has order 12, some vertex must be labeled 2. First, we verify the following claim.

> **Claim.** No partite set of $G_{6,2}$ can be dominated by two vertices labeled 1 and 3.

Suppose that the claim is false. Then we may assume that the partite set $W$ is dominated by two vertices labeled 1 and 3. Thus, each of these two labeled vertices must dominate three vertices of $W$. We may further assume that $f(u_4) = 3$ and $u_4$ dominates $w_1, w_2$, and $w_6$. However, no unlabeled vertex can be labeled 1 to dominate the remaining vertices $w_3, w_4$, and $w_5$ of $W$, a contradiction.

If the vertex labeled 2 dominates four vertices, then we can assume that $f(u_3) = 2$. Thus, $u_3$ dominates $u_1, u_2, u_4$, and $u_5$. Since no labeled vertex not labeled 2 can dominate the remaining vertices $u_3$ and $u_6$ of $U$, it follows that at least three labeled vertices are required to dominate $U$ and that there are two labeled vertices that dominate $W$. This implies that the vertices labeled 1 and 3 must dominate $W$, which is impossible by the claim. If the vertex labeled 2 dominates three vertices, we can assume that $f(u_2) = 2$ and so $u_2$ dominates $u_1, u_3$, and $u_4$. No labeled vertex not labeled 2 can dominate the remaining vertices $u_2, u_5$, and $u_6$ of $U$. If the vertex labeled 2 only dominates two vertices, then at least three vertices are required to dominate $U$. Again, this implies that the vertices labeled 1 and 3 must dominate $W$, which is impossible by the claim. Therefore, $\tilde{\gamma}(G_{6,2}) = 6$.

The irregular dominating labeling of $G_{6,2}$ in Figure 2.8 can be extended to an irregular dominating labeling of $G_{m,2}$ for $m = 7, 8$ with label set [6]. This is illustrated in Figure 2.9 for $G_{8,2}$ and $G_{7,2} = G_{8,2} - \{u_8, w_8\}$. This implies that $\tilde{\gamma}(G_{n,2}) \leq 6$ for $m = 7, 8$.

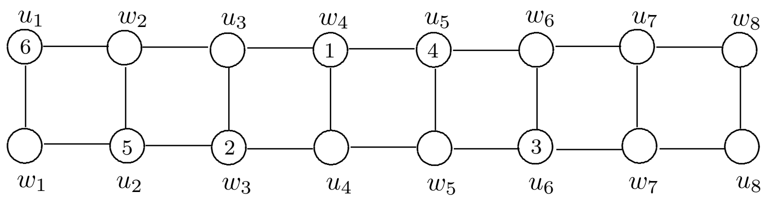
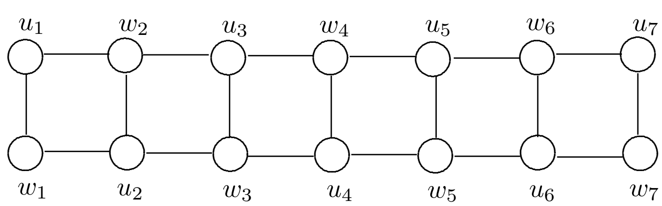



Figure 2.9: An irregular dominating labeling of $G_{8,2}$

It remains to show that $\tilde{\gamma}(G_{m,2}) \geq 6$ for $m = 7, 8$. Assume, to the contrary, that there exists an irregular dominating labeling $f$ of $G_{m,2}$ using 5 elements of [7] where $m = 7, 8$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_m\}$ be the partite sets of $G_{m,2}$. By Observation 1.3.8, there is a partite set of $G_{m,2}$ that is dominated by two labeled vertices. We may assume that $W$ is dominated by two labeled vertices.

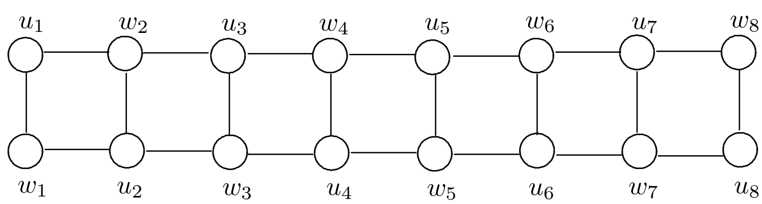
For the graph $G_{7,2}$, we have the following observations.




- $\star$ A vertex labeled 1 can dominate at most 3 vertices of $G_{7,2}$.
- $\star$ A vertex labeled 2 can dominate at most 4 vertices of $G_{7,2}$.
- $\star$ A vertex labeled 3 can dominate at most 4 vertices of $G_{7,2}$.
- $\star$ A vertex labeled 4 can dominate at most 2 vertices of $G_{7,2}$.
- $\star$ A vertex labeled 5 can dominate at most 2 vertices of $G_{7,2}$.
- $\star$ A vertex labeled 6 can dominate at most 2 vertices of $G_{7,2}$.
- $\star$ A vertex labeled 7 can dominate at most 1 vertex of $G_{7,2}$.

Since $|W| = 7$, it follows that $W$ is dominated by two labeled vertices whose label set is $\{1, 2\}$, $\{1, 3\}$, or $\{2, 3\}$. First, assume that a vertex is labeled 1 and the other is labeled 2 or 3. Thus, $f(u_i) = 1$ for $2 \leq i \leq 6$ and $u_i$ dominates each vertex in $N(u_i) = \{w_{i-1}, w_i, w_{i+1}\}$. However then, no unlabeled vertex can be labeled 2 or 3 to dominate the remaining four vertices $W - N(u_i)$, a contradiction. Next, suppose that $W$ is dominated by two vertices labeled 2 and 3. We may assume, without loss of generality, that $f(w_i) = 2$ for $i = 2, 3, 4$. If $f(w_2) = 2$, then $w_2$ dominates $w_1, w_3, w_4$. However, no unlabeled vertex can be labeled 3 to dominate the remaining four vertices $w_2, w_5, w_6$, and $w_7$. If $f(w_3) = 2$, then $w_3$ dominates $w_1, w_2, w_4, w_5$. However, no unlabeled vertex can be labeled 3 to dominate the remaining three vertices $w_3, w_6$, and $w_7$. If $f(w_4) = 2$, then $w_3$ dominates $w_2, w_3, w_5, w_6$. However, no unlabeled vertex can be labeled 3 to dominate the remaining three vertices $w_1, w_4$, and $w_7$, a contradiction. Thus, $\tilde{\gamma}(G_{7,2}) = 6$.

For the graph $G_{8,2}$, we have the following observations.




- ⋆ A vertex labeled 1 can dominate at most 3 vertices of $G_{8,2}$.
- ⋆ A vertex labeled 2 can dominate at most 4 vertices of $G_{8,2}$.
- ⋆ A vertex labeled 3 can dominate at most 4 vertices of $G_{8,2}$.
- ⋆ A vertex labeled 4 can dominate at most 3 vertices of $G_{8,2}$.
- ⋆ A vertex labeled 5 can dominate at most 2 vertices of $G_{8,2}$.
- ⋆ A vertex labeled 6 can dominate at most 2 vertex of $G_{8,2}$.
- ⋆ A vertex labeled 7 can dominate at most 2 vertices of $G_{8,2}$.
- ⋆ A vertex labeled 8 can dominate at most 1 vertex of $G_{8,2}$.

Since $|W| = 8$, it follows that $W$ is dominated by two vertices labeled 2 and 3. An argument similar to the one used in the case of $G_{7,2}$ shows that this is impossible. Thus, $\tilde{\gamma}(G_{8,2}) = 6$. ■

**Proposition 2.2.8** $\tilde{\gamma}(G_{6,3}) = 6$.

**Proof.** An irregular dominating labeling of $G_{6,3}$ with label set [6] is shown in Figure 2.10, which shows that $\tilde{\gamma}(G_{6,3}) \le 6$. We claim that $\tilde{\gamma}(G_{6,3}) = 6$. Label the vertices of $G_{6,3}$ as shown in Figure 2.10. Then $U = \{u_1, u_2, \ldots, u_9\}$ and $W = \{w_1, w_2, \ldots, w_9\}$ are the partite sets of $G_{6,3}$.

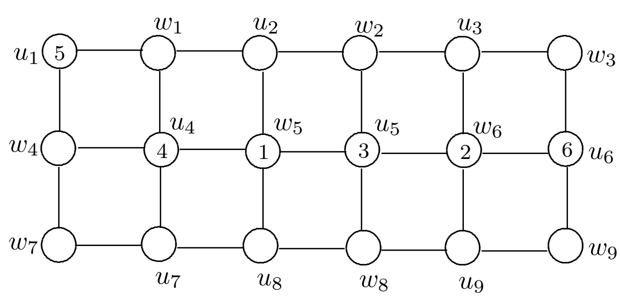



Figure 2.10: An irregular dominating labeling of $G_{6,3}$

Assume, to the contrary, that there exists an irregular dominating labeling $f$ of $G_{6,3}$ using 5 elements of [7]. First, we make the following observations.

- ⋆ A vertex labeled 1 can dominate at most 4 vertices of $G_{6,3}$.
- ⋆ A vertex labeled 2 can dominate at most 6 vertices of $G_{6,3}$.
- ⋆ A vertex labeled 3 can dominate at most 5 vertices of $G_{6,3}$.
- ⋆ A vertex labeled 4 can dominate at most 3 vertices of $G_{6,3}$.
- ⋆ A vertex labeled 5 can dominate at most 3 vertices of $G_{6,3}$.
- ⋆ A vertex labeled 6 can dominate at most 2 vertex of $G_{6,3}$.
- ⋆ A vertex labeled 7 can dominate at most 1 vertex of $G_{6,3}$.

By Observation 1.3.8 and the observations above, it follows that (i) some partite set of $G_{6,3}$ must be dominated by exactly two labeled vertices and (ii) a labeled vertex of $G_{6,3}$ must be labeled 2. Let $x$ and $y$ be the two labeled vertices that dominate a partite set of $G_{6,3}$.

First, suppose that $2 \in \{f(x), f(y)\}$. Then the vertex labeled 2 must dominate at least 4 vertices. There are three possibilities.

*Case* 1. *The vertex labeled* 2 *dominates* 6 *vertices of* $G_{6,3}$. We may assume that $f(w_5) = 2$. Thus, $w_5$ dominates $w_1, w_2, w_4, w_6, w_7$, and $w_8$. Therefore, $w_5$ dominates all vertices of $W$ except $w_3, w_5$, and $w_9$. However, no labeled vertex (that is not labeled 2) can dominate these three vertices. Hence, Case 1 cannot occur.

*Case* 2. *The vertex labeled* 2 *dominates exactly* 5 *vertices of* $G_{6,3}$. We may assume that $f(u_2) = 2$ or $f(u_4) = 2$. If $f(u_2) = 2$, then $u_2$ dominates $u_1, u_3, u_4, u_5$, and $u_8$. Therefore, $u_2$ dominates all vertices of $U$ except $u_2, u_6, u_7$, and $u_9$. However, no labeled vertex can dominate these four vertices. If $f(u_4) = 2$, then $u_4$ dominates $u_1, u_2, u_5, u_7$, and $u_8$. Therefore, $u_4$ dominates all vertices of $U$ except $u_3, u_4, u_6$, and $u_9$. Here as well, no labeled vertex (that is not labeled 2) can dominate these four vertices. Hence, Case 2 cannot occur.

*Case* 3. *The vertex labeled* 2 *dominates exactly* 4 *vertices of* $G_{6,3}$. Since $|U| = |W| = 9$, it follows that $\{f(x), f(y)\} = \{2, 3\}$ and the vertex labeled 3 must dominate 5 vertices. Hence, we may assume that $f(w_5) = 3$. Thus, $w_5$ dominates $u_1, u_3, u_6, u_7$, and $u_9$. Therefore, $w_5$ dominates all vertices of $U$ except $u_2, u_4, u_5$, and $u_8$. However, the vertex labeled 2 cannot dominate these four vertices. Hence, Case 3 cannot occur.

Next, suppose that $2 \notin \{f(x), f(y)\}$. Since $|U| = |W| = 9$, it follows that $\{f(x), f(y)\} = \{1, 3\}$, where the vertex labeled 1 must dominate 4 vertices of $G_{6,3}$ and the vertex labeled 3 must dominate 5 vertices. Since the vertex labeled 3 dominates 5 vertices, we may assume that $f(w_5) = 3$. Thus, $w_5$ dominates $u_1, u_3, u_6, u_7$, and $u_9$. Therefore, $w_5$ dominates all vertices of $U$ except $u_2, u_4, u_5$, and $u_8$. Since $w_5$ is the only vertex whose distance from these four vertices is a constant (namely 1) and $f(w_5) = 3$, this is impossible. ■

By the examples above, we know the exact values of $\tilde{\gamma}(G_{m,n})$ where $5 \leq m + n \leq 9$.

**Proposition 2.2.9** *If $m$ and $n$ are integers with $m \geq n \geq 2$ such that $m+n \in \{5,6,7,8,9\}$ and $(m,n) \neq (4,2)$, then*

$$\tilde{\gamma}(G_{m,n}) = \begin{cases} m+n-2 & \text{if } (m,n) \in \{(3,2),(3,3),(6,2)\} \\ m+n-3 & \text{otherwise.} \end{cases}$$

## 2.3 Comments on Bounds for $\tilde{\gamma}(G_{m,n})$

The following is a consequence of Theorem 1.3.9, (2.1) and Proposition 2.2.9.

**Corollary 2.3.1** *If $m$ and $n$ are integers with $m \geq n \geq 2$ such that $m+n \geq 6$ and $(m,n) \neq (4,2)$, then*

$$4 \leq \tilde{\gamma}(G_{m,n}) \leq m+n-2.$$

In fact, the upper bound $m+n-2 = \operatorname{diam}(G_{m,n})$ in Corollary 2.3.1 can be strict for some pairs $(m,n)$ of integers $m$ and $n$ with $m \geq n \geq 2$ and $m+n$ is large. To illustrate this fact, we present the following result which appeared in [6].

**Proposition 2.3.2** *For integers $m$ and $n$ with $2 \leq n \leq m \leq 8$, the $m \times n$ grid $P_m \,\square\, P_n$ has an irregular dominating labeling except when $(m,n) \in \{(2,2),(4,2)\}$.*

The proof of Proposition 2.3.2 provides an upper bound for $\tilde{\gamma}(G_{m,n})$ that is less than $\operatorname{diam}(G_{m,n}) = m+n-2$ for $(m,n) \in \{(7,6),(7,7),(8,6)\}$.

For $m = 7$, let $G_{7,n} = P_7 \,\square\, P_n$ where $Q_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,7})$ is a copy of $P_7$ in $G_{7,n}$ for $1 \leq i \leq n$ and $u_{i,j}u_{i+1,j} \in E(G_{7,n})$ for $1 \leq i \leq n-1$ and $1 \leq j \leq 7$.

- ⋆ For $n = 6$, an irregular dominating labeling $f_6$ of $G_{7,6}$ can be defined by $f_6(u_{3,1}) = 6$, $f_6(u_{3,3}) = 2$, $f_6(u_{3,4}) = 4$, $f_6(u_{3,5}) = 1$, $f_6(u_{3,6}) = 3$, $f_6(u_{3,7}) = 5$, $f_6(u_{4,7}) = 7$, $f_6(u_{1,4}) = 8$, and $f_6(u_{5,6}) = 9$ with all other vertices of $G_{7,6}$ not labeled.

  Thus, $\tilde{\gamma}(G_{7,6}) \leq 9 = \operatorname{diam}(G_6) - 2$.

* $\star$ For $n = 7$, an irregular dominating labeling $f_7$ of $G_{7,7}$ can be defined by $f_7(u_{4,1}) = 6$, $f_7(u_{4,3}) = 2$, $f_7(u_{4,4}) = 4$, $f_7(u_{4,5}) = 1$, $f_7(u_{4,6}) = 3$, $f_7(u_{4,7}) = 5$, $f_7(u_{5,7}) = 7$, $f_7(u_{2,4}) = 8$, $f_7(u_{7,4}) = 9$, and $f_7(u_{6,7}) = 10$ with all other vertices of $G_{7,7}$ not labeled.

  Thus, $\tilde{\gamma}(G_{7,7}) \leq 9 = \text{diam}(G_{7,7}) - 2$.

For $m = 8$, let $G_{8,6} = P_8 \,\square\, P_6$ where $T_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,8})$ is a copy of $P_8$ in $H_n$ for $1 \leq i \leq 6$ and $u_{i,j}u_{i+1,j} \in E(G_{8,n})$ for $1 \leq i \leq 5$ and $1 \leq j \leq 8$.

* $\star$ For $n = 6$, an irregular dominating labeling $h_6$ of $G_{8,6}$ can be defined by $h_6(u_{1,7}) = 11$, $h_6(u_{2,8}) = 8$, $h_6(u_{4,1}) = 9$, $h_6(u_{4,2}) = 5$, $h_6(u_{4,3}) = 3$, $h_6(u_{4,4}) = 1$, $h_6(u_{4,5}) = 4$, $h_6(u_{4,6}) = 2$, $h_6(u_{4,8}) = 6$, $h_6(u_{6,3}) = 10$, and $h_6(u_{6,4}) = 7$ with all other vertices of $G_{8,6}$ not labeled.

  Thus, $\tilde{\gamma}(G_{8,6}) \leq 11 = \text{diam}(G_{8,6}) - 1$.

Both lower and upper bounds in Corollary 2.3.1 can be improved when $m+n$ is large. To illustrate this fact, we determine $\tilde{\gamma}(G_{m,n})$ for integers $m$ and $n$ with $m \geq n \geq 2$ and $m + n = 10$, namely, $(m,n) \in \{(8,2),(7,3),(6,4),(5,5)\}$. The irregular dominating labelings of $G_{5,5}$, $G_{6,4}$, and $G_{7,4}$ in Figure 2.11 show that

$$\tilde{\gamma}(G_{5,5}) \leq 7,\ \tilde{\gamma}(G_{6,4}) \leq 6, \text{ and } \tilde{\gamma}(G_{7,3}) \leq 6.$$

By Proposition 2.2.7, it follows that $\tilde{\gamma}(G_{8,2}) = 6$. In fact, an extensive case-by-case analysis shows that $\tilde{\gamma}(G_{5,5}) = 7$ and $\tilde{\gamma}(G_{7,3}) = \tilde{\gamma}(G_{8,2}) = \tilde{\gamma}(G_{6,4}) = 6$. Consequently, we have the following result.

**Theorem 2.3.3** *If $m$ and $n$ are integers with $m \geq n \geq 2$ such that $m + n = 10$, then*

$$\tilde{\gamma}(G_{m,n}) = \begin{cases} m+n-3 & \text{if } (m,n) = (5,5) \\ m+n-4 & \text{if } (m,n) \in \{(7,3),(8,2),(6,4)\}. \end{cases}$$

The following is consequence of Theorem 2.3.3.

**Corollary 2.3.4** *If $m$ and $n$ are integers with $m \geq n \geq 2$ such that $m + n = 10$, then*

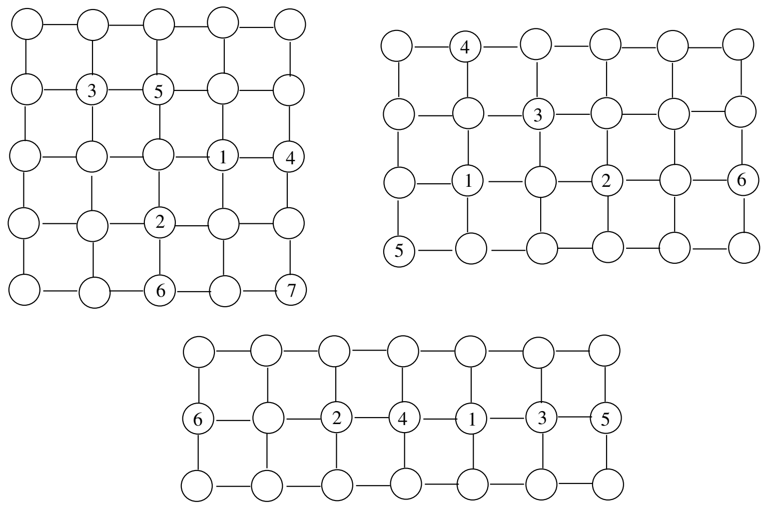


Figure 2.11: Irregular dominating labelings of $G_{5,5}$, $G_{6,4}$, and $G_{7,3}$

$$6 \leq \tilde{\gamma}(G_{m,n}) \leq m + n - 3.$$

Results on the irregular domination numbers of grids that we have obtained thus far suggest the following problem.

**Problem 2.3.5** *For integers $m$ and $n$ with $m \geq n \geq 2$ and $m + n$ is sufficiently large, find better upper and lower bounds in terms of $m$ and $n$.*

# Chapter 3

# Irregular Domination Graphs

## 3.1 Introduction

We have seen that a graph $G$ has a total dominating set if and only if $G$ has no isolated vertices. Recall that a total dominating set $S$ of a graph $G$ is *minimal* if no proper subset of $S$ is a total dominating set of $G$. Thus, every minimum total dominating set is minimal, while there are minimal total dominating sets that are not minimum. In fact, if $S$ is a minimal total dominating set of a graph $G$ without isolated vertices, then $G[S]$ has no isolated vertices. A graph $H$ is *a total domination graph* if there exists a graph $G$ and a minimal total dominating set $S$ of $G$ such that $G[S] \cong H$. If $H$ is a graph without isolated vertices and $G$ is the corona $\mathrm{cor}(H)$ of $H$ (namely, $G$ is obtained by adding a pendant edge at each vertex of $H$), then $V(H)$ is a minimal total dominating set of $G$, which gives the following.

**Observation 3.1.1** *A graph $H$ is a total domination graph if and only if $H$ has no isolated vertices.*

There is a concept analogous to a total domination graph that provides the structure of the subgraph determined by a minimal irregular dominating set in a graph.

## 3.2 Minimal Irregular Dominating Sets

If $S$ is an irregular dominating set of a connected graph $G$ but no proper subset $T$ of $S$ is an irregular dominating set of $G$ (where the label of each vertex of $T$ is that in $S$), then $S$ is a *minimal irregular dominating set*. Equivalently, an irregular dominating set $S$ in a graph is *minimal* if for every vertex $u \in S$, there is a vertex $v$ of $G$ such that $v$ is dominated by $u$ only. Hence, every minimum irregular dominating set is minimal as well. Figure 3.1 shows three different minimal irregular dominating sets of the path $P_9$ of order 9. The irregular dominating sets have cardinalities 6, 7, and 8 from top to bottom. It can be shown that $\tilde{\gamma}(P_9) = 6$. Since the diameter of $P_9$ is 8, these are the only possible cardinalities of minimal irregular dominating sets of $P_9$.

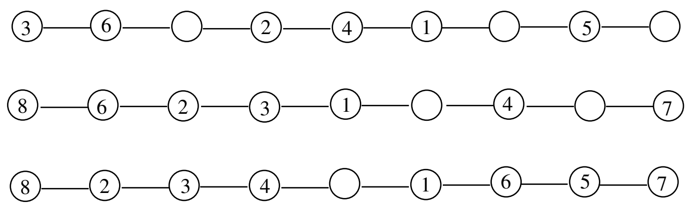



Figure 3.1: Three minimal irregular dominating sets of $P_9$

To illustrate some properties of minimal irregular dominating sets in graphs, we consider the graph $G$ of Figure 3.2. We saw that $\tilde{\gamma}(G) = 4$.

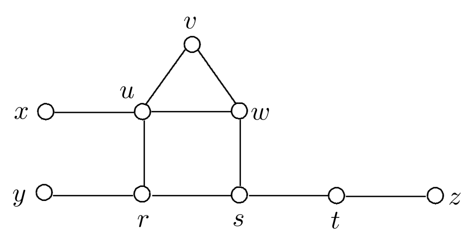



Figure 3.2: The graph $G$

This graph has order 9, radius 3, and diameter 5, where $r, s, w$ are central vertices, while $x$ and $z$ are peripheral vertices. Thus, if $f$ is a minimal irregular dominating labeling of $G$, then the set of labels of $f$ is a subset of [5] of cardinality 4 or more. The labeling of $G$ in Figure 3.3(a) shows that there is a minimal irregular dominating labeling of $G$ using all labels in the set [5]. In fact, there is a minimal irregular dominating labeling of $G$ using all labels in the set [4], as shown in Figure 3.3(b). This brings up the question of which 4-element subsets of [5] can be used for a minimal irregular dominating labeling of $G$.

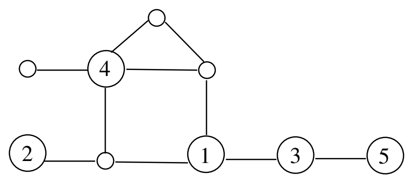



Figure 3.3: Two minimal irregular dominating labelings of a graph $G$

Figure 3.4 shows 4-element subsets of [5] other than [4] that can be used for a minimal irregular dominating labeling of $G$. Not all 4-element subsets of [5] can be used for a minimal irregular dominating labeling of $G$. In fact, the label 2 must be used in every minimal irregular dominating labeling of $G$.

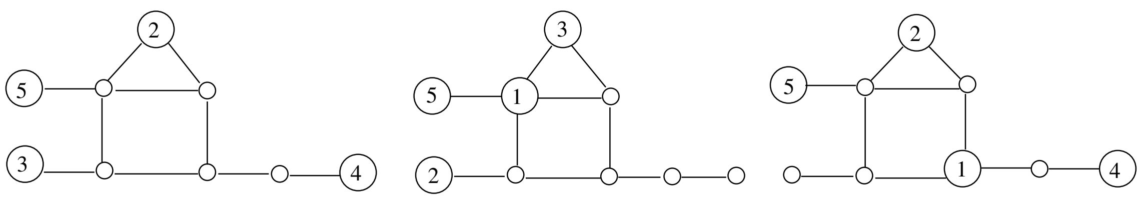



Figure 3.4: Three minimal irregular dominating labelings of a graph $G$

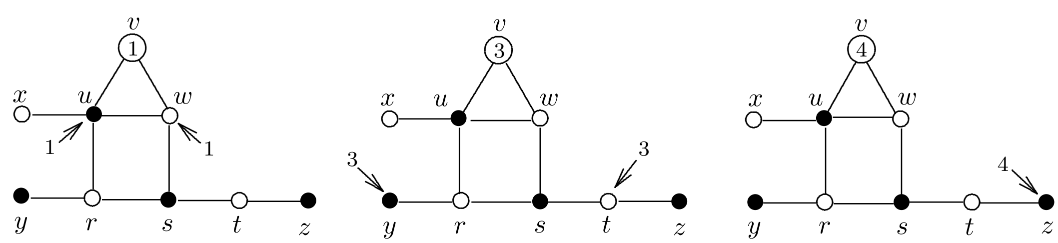
**Proposition 3.2.1** *Every minimal irregular dominating labeling of the graph $G$ of Figure 3.2 using a 4-element subset of [5] must use the label 2. That is, there is no minimal irregular dominating labeling whose set of labels is $\{1, 3, 4, 5\}$.*

**Proof.** Assume, to the contrary, that $G$ has a minimal irregular dominating labeling $f$ whose set of labels is $\{1, 3, 4, 5\}$. We consider two cases, according to whether $f$ assigns a label to the vertex $v$.

*Case 1. No label is assigned to the vertex $v$ by $f$.* The graph $H = G - v$ is a distance-preserving bipartite subgraph of $G$ with partite sets $U = \{u, y, s, z\}$ and $W = \{w, x, r, t\}$. Thus, all four labeled vertices belong to $H$. The vertex labeled 1 or 3 dominates at most three vertices in the partite set of $H$ not containing that vertex, the vertex labeled 5 dominates one vertex in the partite set of $H$ not containing that vertex, and the vertex labeled 4 dominates at most two vertices in the same partite set of $H$ containing that vertex. Consequently, each partite set of $H$ is dominated by exactly two labeled vertices. This says that no partite set can be dominated by the vertices labeled 4 and 5. Therefore, the vertices labeled 4 or 5 must belong to the same partite set.

Since the vertex labeled 5 must dominate one vertex, it follows that $f(x) = 5$ or $f(z) = 5$. If $f(x) = 5$, then $W = \{w, x, r, t\}$ contains the two vertices labeled 4 and 5. Hence, $t$ must be labeled 4 and $t$ only dominates $x$. If $f(z) = 5$, then $U = \{u, y, s, z\}$ contains the two vertices labeled 4 and 5. Hence, either $u$ or $y$ must be labeled 4 and either only dominates $z$. In either case, the vertices labeled 4 and 5 dominate $x$ and $z$. Thus, the vertex labeled 1 must dominate either all vertices of $U - \{z\} = \{u, y, s\}$ or all vertices of $W - \{x\} = \{w, r, t\}$; while the vertex labeled 3 must dominate all vertices of the other set. However, no vertex labeled 3 can dominate all vertices of either $U - \{z\}$ or $W - \{x\}$, a contradiction.

*Case* 2. *A label is assigned to the vertex $v$ by $f$.* Since $f(v) \in \{1, 3, 4\}$, we consider these three possibilities.




*Subcase* 2.1. $f(v) = 1$. Then $v$ dominates $u$ and $w$ and the seven vertices in $V(G) - \{u, w\}$ are dominated by the vertices labeled 3, 4, or 5. Since the vertex labeled 4 can dominate at most two vertices of $U$ or $W$ and the vertex labeled 5 can dominate only one vertex of $U$ or $W$, the vertex labeled 3 must dominate all vertices of $U - \{u\}$ or all vertices of $W - \{w\}$. Since this does not occur, we have a contradiction.

*Subcase* 2.2. $f(v) = 3$. Then $v$ dominates $y$ and $t$ and the seven vertices in $V(G) - \{y, t\}$ are dominated by the vertices labeled 1, 4, or 5. Since $e(r) = e(s) = 3$, it follows that both $r$ and $s$ must be dominated by the vertex labeled 1, which is impossible.

*Subcase* 2.3. $f(v) = 4$. Then $v$ dominates $z$ only. Thus, one labeled vertex must dominate all vertices of $U - \{z\}$. Since these three vertices can only be dominated by the vertex $r$ if $r$ is labeled 1, it follows that $f(r) = 1$. Consequently, all vertices in $W \cup \{v\}$ must be dominated by the vertices labeled 3 and 5. This says that $f(z) = 5$ and $z$ dominates $x$ only. However, no vertex labeled 3 can dominate all vertices of $(W - \{x\}) \cup \{v\}$, a contradiction. ■

Since $\tilde{\gamma}(G) = 4$, it follows that there is no irregular dominating labeling of the graph $G$ of Figure 3.2 using the labels $1, 2, 3$. The following summarize the results above.

**Theorem 3.2.2** *A subset $S$ of [5] is the set of labels of a minimal irregular dominating labeling of the graph $G$ of Figure 3.2 if and only if $|S| \geq 4$ and $S \neq \{1, 3, 4, 5\}$.*

Let's revisit the graph $H$ of Figure 3.5, which is a bipartite graph of diameter 6, order 12, and size 13 with partite sets $U = \{u_1, u_2, \ldots, u_6\}$ and $W = \{w_1, w_2, \ldots, w_6\}$.

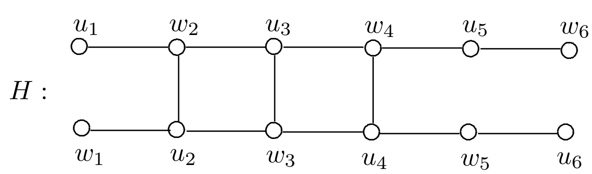



Figure 3.5: A bipartite graph of diameter 6

We now determine all possible label sets of minimal irregular dominating labelings of the graph $H$. First, recall that if $G$ is a connected bipartite graph with $\tilde{\gamma}(G) = 3$, then $\text{diam}(G) \in \{3, 4\}$. Since the diameter of $H$ is 6, it follows that if $f$ is a minimal irregular dominating labeling of $H$, then the set of labels of $f$ is a subset of [6] of cardinality 4 or more. First, we observe in Figure 3.6(a) that there exists a minimal irregular dominating labeling of $H$ using all labels in the set [6]. There is also a minimal irregular dominating labeling of $H$ using all labels in the set [5], shown in Figure 3.6(b). Indeed, there is such a labeling of $H$ with label set $\{1, 2, 3, 4, 6\}$ or $\{1, 2, 3, 5, 6\}$, shown in Figures 3.6(c) and 3.6(d), respectively.

In the minimal irregular dominating labelings of $H$ shown in Figure 3.6(b)–(d), all three label sets are 5-element subsets of [6] containing the subset $\{1, 2, 3\}$.

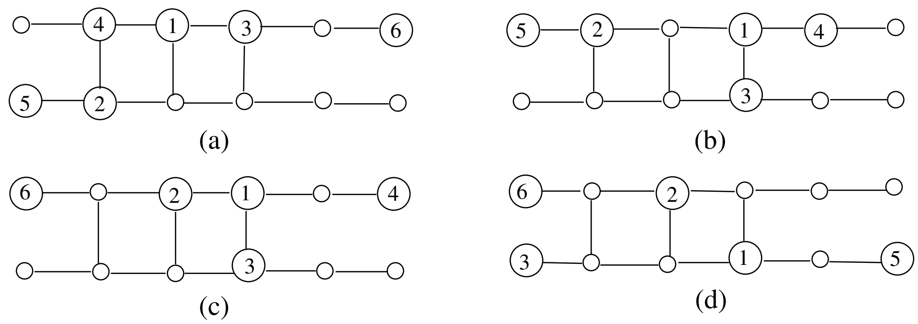



Figure 3.6: Four minimal irregular dominating labelings of the graph $H$

This is not a coincidence as we show next. First, we recall the following observation and facts about this graph $H$.

**Observation 1.** Let there be given a minimal irregular dominating labeling of the graph $H$ of Figure 3.5.

- $\star$ Every labeled vertex can dominate only vertices in one partite set of $H$.
- $\star$ A vertex labeled 1 can dominate at most 3 vertices of $H$.
- $\star$ A vertex labeled 2 can dominate at most 4 vertices of $H$.
- $\star$ A vertex labeled 3 can dominate at most 3 vertices of $H$.
- $\star$ A vertex labeled 4 can dominate at most 3 vertices of $H$.
- $\star$ A vertex labeled 5 can dominate at most 3 vertices of $H$.
- $\star$ A vertex labeled 6 can dominate at most 1 vertex of $H$.

**Fact 1.** The irregular domination number $\tilde{\gamma}(H)$ of the graph $H$ is 5.

**Fact 2.** Let $f$ be an irregular dominating labeling of $H$. If a partite set of $H$ is dominated by two labeled vertices $x$ and $y$, then $\{f(x), f(y)\} = \{1, 4\}$ or $\{f(x), f(y)\} = \{1, 5\}$.

**Proposition 3.2.3** *There exists no minimal irregular dominating labeling of the graph $H$ of Figure 3.5 with label set $\{1, 3, 4, 5, 6\}$.*

**Proof.** Assume, to the contrary, that there exists a minimal irregular dominating labeling $f$ of $H$ with label set $\{1, 3, 4, 5, 6\}$. Consequently, two labeled vertices must dominate one partite set and the remaining three labeled vertices must dominate the other partite set. By symmetry, we may assume that the partite set $W$ is dominated by two labeled vertices and each labeled vertex must dominate three vertices of $W$. By Fact 2, the partite set $W$ can only be dominated by two vertices labeled 1 and 4 or by two vertices labeled 1 and 5. We consider these two cases.

*Case* 1. *The set $W$ is dominated by two vertices labeled* 1 *and* 4. Since the vertex labeled 4 or 1 must dominate three vertices of $W$, it follows that $f$ assigns the label 4 to either $w_5$ or $w_6$ and the label 1 to one of $u_2, u_3$, $u_4$.

- ⋆ If $f(w_5) = 4$, then $w_5$ dominates $w_1, w_2$, and $w_6$. The only vertex that can be labeled 1 which would dominate the remaining three vertices in $\{w_3, w_4, w_5\}$ is $u_4$ and so $f(u_4) = 1$. The partite set $U$ is therefore dominated by the three vertices labeled 3, 5, and 6. By Fact 1, the vertex labeled 6 must dominate one vertex of $U$, namely $f(u_6) = 6$ or $f(u_1) = 6$. If $f(u_6) = 6$, then $u_6$ dominates only $u_1$, Since the vertex labeled 5 must dominate at least two un-dominated vertices, it follows that $f(w_1) = 5$. Then $w_5$ dominates $u_5$ and $u_6$. However then, the three vertices in $\{u_2, u_3, u_4\} = N(w_3)$ cannot be dominated by a vertex labeled 3. If $f(u_1) = 6$, then $u_1$ dominates $u_6$. The vertex labeled 5 must be $w_6$ and so $w_6$ dominates $u_1$ and $u_2$. However, no vertex labeled 3 can dominate $u_3, u_4, u_5$, a contradiction.
- ⋆ If $f(w_6) = 4$, then $w_6$ dominates $w_2, w_3$, and $w_5$. However then, no vertex labeled 1 can dominate the remaining three vertices in $\{w_1, w_4, w_6\}$, a contradiction.

Thus, Case 1 cannot occur.

*Case* 2. *The set $W$ is dominated by two vertices labeled* 1 *and* 5. Since the vertex labeled 5 or 1 must dominate three vertices of $W$, it follows that $f$ assigns the label 5 to $u_6$ and the label 1 to $u_4$. The partite set $U$ must be dominated by the three vertices labeled 3, 4, and 6. By Fact 1, the vertex labeled 6 must dominate one vertex and so $f(u_1) = 6$. The vertex $u_1$ dominates $u_6$. Since no unlabeled vertex of $U$ can be labeled 4 and dominate two vertices of $U$, this is impossible. Thus, Case 2 cannot occur. ∎

**Proposition 3.2.4** *There exists no minimal irregular dominating labeling of the graph $H$ of Figure* 3.5 *with label set* $\{1, 2, 4, 5, 6\}$.

**Proof.** Assume, to the contrary, that there exists a minimal irregular dominating labeling $f$ of $H$ with label set $\{1, 2, 4, 5, 6\}$. By Observation 1, two labeled vertices dominate one partite set and the remaining three labeled vertices dominate the other partite set. By symmetry, we may assume that the partite set $W$ is dominated by two labeled vertices. By Fact 2, the partite set $W$ can only be dominated by two vertices labeled 1 and 4 or by two vertices labeled 1 and 5. We consider these two cases.

*Case* 1. *The partite set $W$ is dominated by the two vertices labeled* 1 *and* 5. Since the vertex labeled 5 or 1 must dominate three vertices of $W$, it follows that $f(u_6) = 5$ and $f(u_4) = 1$. The partite set $U$ must be dominated by the three vertices labeled 2, 4, and 6. By Fact 1, the vertex labeled 6 must dominate one vertex of $U$ and so $f(u_1) = 6$. The vertex $u_1$ dominates $u_6$. Since no un-dominated vertex of $U$ is at distance 4 from $u_4$, the vertex $u_4$ must be dominated by a vertex labeled 2. However, any vertex labeled 2 that dominates $u_4$ fails to dominate a vertex that can be dominated by a vertex that can be labeled 4, a contradiction.

*Case* 2. *The partite set $W$ is dominated by the two vertices labeled* 1 *and* 4. Since the vertex labeled 4 or 1 must dominate three vertices of $W$, it follows that $f$ assigns the label 4 to either $w_5$ or $w_6$. If $f(w_6) = 4$, then no vertex can be labeled 1 that would dominate the un-dominated vertices of $W$. Thus, $f(w_5) = 4$ and $w_5$ dominates $w_1, w_2$, and $w_6$. Thus, $u_4$ is labeled 1 and $u_4$ dominates the three vertices in $\{w_3, w_4, w_5\}$. The partite set $U$ is dominated by the three vertices labeled 3, 5, and 6. By Fact 1, the vertex labeled 6 must dominate one vertex of $U$. Either $f(u_1) = 6$ and $u_1$ dominates $u_6$ or $f(u_6) = 6$ and $u_6$ dominates only $u_1$. Suppose first that $f(u_1) = 6$. Since the vertex labeled 5 must dominate at least two un-dominated vertices, it follows that $f(w_6) = 5$. However, the three un-dominated vertices cannot be dominated by a vertex labeled 3. Thus, $f(u_6) = 6$ and so $f(w_1) = 5$ or $f(w_6) = 5$. Again, the three un-dominated vertices of $U$ cannot be dominated by a vertex labeled 3, a contradiction. ■

**Proposition 3.2.5** *There exists no minimal irregular dominating labeling of the graph $H$ of Figure* 3.5 *with label set* $\{2, 3, 4, 5, 6\}$.

**Proof.** Assume, to the contrary, that there exists a minimal irregular dominating labeling $f$ of $H$ with label set $\{2, 3, 4, 5, 6\}$. By Observation 1, two labeled vertices must dominate one partite set, say $W$, and the remaining three labeled vertices dominate the other partite set. By Fact 2, the partite set $W$ can only be dominated by two vertices labeled 1 and 4 or by two vertices labeled 1 and 5. Since the label 1 is not used by $f$, this is impossible. ■

The following summarize the results above.

**Theorem 3.2.6** *A subset $S$ of $[6]$ is the set of labels of a minimal irregular dominating labeling of the graph $H$ of Figure 3.5 if and only if $|S| \geq 5$ and $[3] \subseteq S$.*

## 3.3 Irregular Domination Graphs

For a graph $G$ with a minimal irregular dominating set $S$, the subgraph $G[S]$ induced by $S$ provides some information on the structural relationship among the vertices of $S$. This subgraph is called the *irregular domination subgraph* of $G$ induced by the minimal irregular dominating set $S$. A graph $H$ is an *irregular domination graph* if there exists a graph $G$ with a minimal irregular dominating set $S$ such that $G[S] \cong H$. For example, in the graph $G = P_4$ of Figure 3.7, the set $S = \{v_1, v_2, v_3\}$ is a minimal irregular dominating set of $G$, where the corresponding labeling assigns the label $i$ to $v_i$ for $1 \leq i \leq 3$. Since $G[S] \cong K_2 + K_1$, it follows that $K_2 + K_1$ is an irregular domination graph.

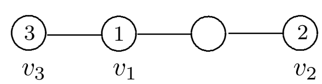



Figure 3.7: A minimal irregular dominating set in $P_4$

In fact, $K_2 + K_1$ is the only irregular domination graph of order 3. To verify this, we first present two observations, the first of which is a consequence of a result obtained by Chartrand, Henning, and Schultz in [8].

**Observation 3.3.1** *If $G$ is a connected graph with an irregular dominating set, then $\tilde{\gamma}(G) \geq 3$. Furthermore, if $S$ is an irregular dominating set of cardinality 3 in $G$, then the three vertices of $S$ are labeled by 1, 2, 3.*

**Proposition 3.3.2** *The graph $K_2 + K_1$ is the only irregular domination graph of order 3.*

**Proof.** That $K_2 + K_1$ is an irregular domination graph of order 3 is shown in Figure 3.7. Assume, to the contrary, that there is an irregular domination graph $H$ of order 3 such that $H \ncong K_2 + K_1$. Then there exists a graph $G$ with a minimal irregular dominating set $S = \{u, v, w\}$ such that $G[S] \cong H$. We may assume, by Observation 3.3.1, that $S$ has an irregular dominating labeling $f$ such that $f(u) = 1$, $f(v) = 2$, and $f(w) = 3$. First, suppose that $H \cong \overline{K}_3$. Then no vertex of $S$ is dominated by $u$. Thus, $v$ and $w$ must dominate each other, which is impossible by Observation 1.3.1. Next, suppose that $H$ is connected. Hence, the distance between every two vertices of $S$ in $G$ is either 1 or 2. Therefore, no vertex of $S$ is dominated by $w$ and so $u$ and $v$ must dominate each other. Once again, this is impossible by Observation 1.3.1. ■

We mentioned earlier that no graph of diameter at most 2 has an irregular dominating set. We have the following corresponding result.

**Proposition 3.3.3** *No connected graph of diameter at most 2 is an irregular domination graph.*

**Proof.** Assume, to the contrary, that there exists a connected graph $H$ with $\operatorname{diam}(H) \leq 2$ that is an irregular domination graph. Then there exists a graph $G$ with a minimal irregular dominating set $S$ such that $G[S] \cong H$. Since $K_2 + K_1$ is the only irregular domination graph of order 3, it follows that the order $n$ of $H$ is at least 4 and so $|S| \geq 4$. Since the distance between every two vertices of $S$ in $G$ is 1 or 2, every vertex of $S$ must be dominated by a vertex of $S$ labeled 1 or 2. Because no vertex of $S$ dominates itself, there are two vertices $u, v \in S$ such that $u$ is labeled 1 and $v$ is labeled 2. However then, $u$ and $v$ must dominate each other, a contradiction by Observation 1.3.1. ■

By Proposition 3.3.3, there is no irregular domination graph of order $n \geq 3$ having a vertex of degree $n - 1$. There is, however, for each pair $\Delta, n$ of integers with $0 \leq \Delta \leq n - 2$ and $n \geq 3$, an irregular domination graph of order $n$ having maximum degree $\Delta$. This is a consequence of the following result.

**Theorem 3.3.4** *If $H$ is a graph of order 4 or more having an isolated vertex, then $H$ is an irregular domination graph.*

**Proof.** Let $H$ be a graph of order 4 or more having an isolated vertex. Then $H = F + K_1$, where the order $n$ of $F$ is at least 3. If $n = 3$, then $F \in \{\overline{K}_3, P_2 + K_1, P_3, K_3\}$. For each graph in Figure 3.8, a minimal irregular dominating labeling is also shown in that figure. Thus, if $F$ is one of $\overline{K}_3$, $P_2 + K_1$, $P_3$, $K_3$, then $F + K_1$ is an irregular domination graph. Hence, we may assume that $F$ is a graph of order $n \geq 4$ and show that there is a graph $G_n$ having a minimal irregular dominating set $S_n$ such that $G_n[S_n] = F + K_1$. We consider two cases.

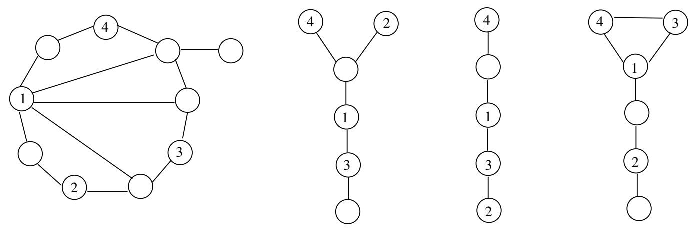


Figure 3.8: A step in the proof of Proposition 3.3.4

*Case* 1. *$F = \overline{K_n}$ is the empty graph of order $n$.* Then $F + K_1 = \overline{K}_{n+1}$ is the empty graph of order $n + 1$. For $n = 4, 5, 6, 7$, the graphs $G_n$ shown in Figure 3.9 have a minimal irregular dominating set $S_n$ (also shown in Figure 3.9 for each graph) where $G[S_n] = \overline{K}_{n+1}$.

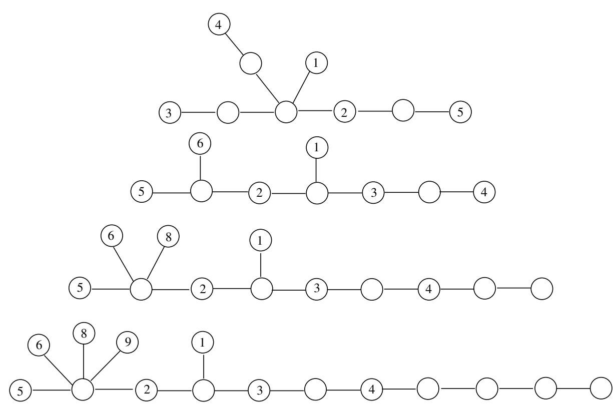


Figure 3.9: The graphs $G_n$ for $n = 4, 5, 6, 7$

For $n \geq 8$, let $G_n$ be the graph obtained from the path $P_{n+3} = (v_1, v_2, \ldots, v_{n+3})$ of order $n+3$ by adding the pendant edge $uv_3$ at $v_3$ and $n-3$ pendant edges $u_iv_1$ at $v_1$

for $1 \leq i \leq n-3$. Then $\mathrm{diam}(G_n) = n+3$. We show that $S_n = \{u, u_1, u_2, \ldots, u_{n-3}, v_2, v_4, v_6\}$ is a minimal irregular dominating set of $G_n$. Define the labeling $f : S_n \to [n+3]$ by assigning the label 1 to $u$, the label 2 to $v_2$, the label 3 to $v_4$, the label 4 to $v_6$, and the $n-3$ distinct labels in $[n+3] - \{1, 2, 3, 4, 7, 10\}$ arbitrarily to the remaining $n-3$ vertices of $S_n - \{u, v_2, v_4, v_6\} = \{u_1, u_2, \ldots, u_{n-3}\}$. Observe that the vertex $v_3$ is only dominated by the vertex $u$ labeled 1, both $v_4$ and $u_i$ where $1 \leq i \leq n-3$ are only dominated by the vertex $v_2$ labeled 2, both $v_7$ and $v_1$ are only dominated by the vertex $v_4$ labeled 3, both $v_{10}$ and $v_2$ are only dominated by the vertex $v_6$ labeled 4, and for $i \in [n+3] - \{1, 2, 3, 4, 7, 10\}$, the vertex $v_i$ is only dominated by the vertex labeled $i$. Since for each labeled vertex $x \in S_n$, there is a vertex $y$ of $G_n$ that is dominated only by $x$, it follows that $S_n$ is a minimal irregular dominating set and $G_n[S_n] = \overline{K}_{n+1}$.

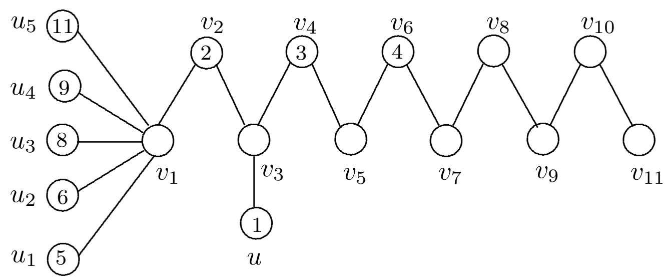



Figure 3.10: A minimal irregular dominating set in the graph $G_8$ in Case 1

*Case* 2. $F \neq \overline{K_n}$. Let $V(F) = \{u_1, u_2, \ldots, u_n\}$. We may assume that $u_1u_2 \in E(F)$. Let $G$ be the graph obtained from $F$, the $n$-path $P_n = (v_1, v_2, \ldots, v_n)$, and the 3-path $P_3 = (w_1, w_2, w_3)$ by adding two new vertices $x$ and $y$ and joining (1) the vertex $x$ to each vertex in $(V(F) - \{u_2\}) \cup \{v_1, w_1\}$ and (2) the vertex $y$ to each vertex in $\{u_1, v_2, w_2\}$. Then $\mathrm{diam}(G) = n+1$. Define the labeling $f : V(F) \cup \{v_1\} \to [n+1]$ by assigning the label 1 to $u_1$, the label 2 to $v_1$, and the label $i$ to $u_{i-1}$ for $3 \leq i \leq n+1$. This labeling is shown in Figure 3.11 for $n = 4$, where the vertices of $F$ are drawn in bold and only the edge $u_1u_2$ of $F$ is drawn, where the dashed line between $u_2$ and $x$ indicates that $u_2$ and $x$ are not adjacent.

Observe that $x$ and $u_2$ are only dominated by the vertex $u_1$ labeled 1, the vertex $u_1$ is only dominated by the vertex $v_1$ labeled 2, the vertex $v_1$ is only dominated by the vertex $u_2$ labeled 3, the vertex $w_3$ is only dominated by the vertex $u_3$ labeled 4, and for $5 \leq i \leq n+1$, the vertex $v_{i-1}$ is only dominated by the vertex $u_{i-1}$ labeled $i$.

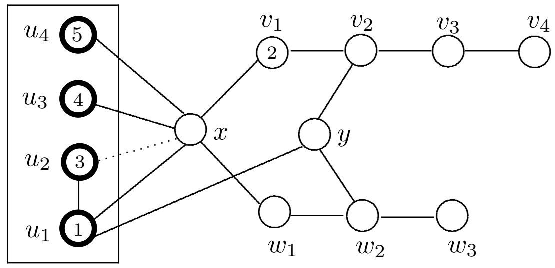



Figure 3.11: A minimal irregular dominating set in the graph $G$ in Case 2

Thus, $S_n = V(F) \cup \{v_1\}$ is a minimal irregular dominating set of $G$. Since $G[S] \cong F + K_1$, it follows that $F + K_1$ is an irregular domination graph. ■

**Corollary 3.3.5** *For each pair $\Delta, n$ of integers with $0 \leq \Delta \leq n-2$ and $n \geq 3$, there exists an irregular domination graph of order $n$ having maximum degree $\Delta$.*

With this information, all irregular domination graphs of order 4 and 5 are determined.

**Proposition 3.3.6** *A graph $H$ of order 4 or 5 is an irregular domination graph if and only if $H$ is disconnected, $H = P_4$, or $H$ is connected and* $\text{diam}(H) \geq 3$.

**Proof.** First, suppose that $H$ is a graph of order 4. We show that $H$ is an irregular domination graph if and only if $H$ is disconnected or $H = P_4$. By Proposition 3.3.3, if $H$ is a connected graph of order 4 with $\text{diam}(H) \leq 2$, then $H$ is not an irregular domination graph. Thus, it remains to verify the converse. By Theorem 3.3.4, if $H$ is one of $\overline{K}_4$, $P_2 + 2K_1$, $P_3 + K_1$, $K_3 + K_1$, then $H$ is an irregular domination graph. For each graph in Figure 3.12, a minimal irregular dominating labeling is also shown in that figure. Thus, $2P_2$ and $P_4$ are irregular domination graphs.

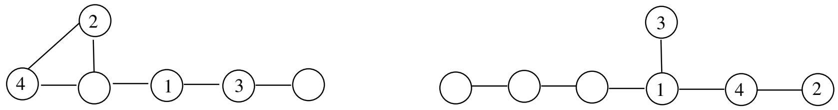



Figure 3.12: A step in the proof of Proposition 3.3.6

Next, suppose that $H$ is a graph of order 5. We show that $H$ is an irregular domination graph if and only if $H$ is disconnected or $H$ is connected and $\text{diam}(H) \geq$

3. By Proposition 3.3.3, if $H$ is a (connected) graph of order 5 with $\text{diam}(H) \leq 2$, then $H$ is not an irregular domination graph. Thus, it remains to verify the converse. Suppose that $H$ is a graph of order 5 that is either disconnected or is a connected graph of diameter at least 3. We show that there is a graph $G$ with a minimal irregular dominating set $S$ such that $G[S] = H$. By Theorem 3.3.4, we may assume that $H$ does not contain isolated vertices. Thus, $H$ is one of the eight graphs $H_1, H_2, \ldots, H_8$ in Figure 3.13.

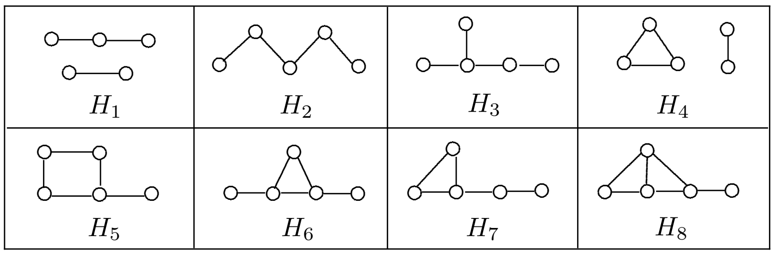



Figure 3.13: Eight graphs of order 5

If $H = H_i$ where $1 \leq i \leq 8$, then the graph $G_i$ in Figure 3.14 has a minimal irregular dominating set (also shown in Figure 3.14) that induces a subgraph of $G_i$ isomorphic to $H_i$. ■

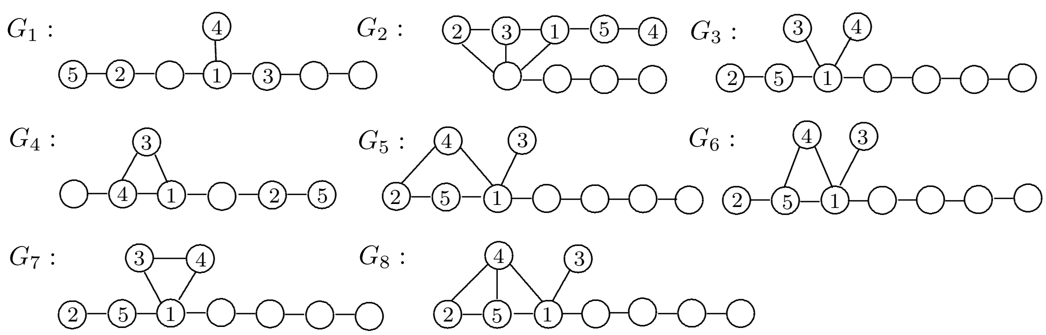



Figure 3.14: The eight graphs $G_i$ for $1 \leq i \leq 8$

## 3.4 Irregular Domination Graphs with $\tilde{\gamma}(G) = 3$

**Observation 3.4.1** *If $G$ is a connected graph with an irregular dominating set, then $\tilde{\gamma}(G) \geq 3$. Furthermore, if $S$ is an irregular dominating set of cardinality 3 in $G$, then the three vertices of $S$ are labeled by $1, 2, 3$.*

By Observation 3.4.1, if $G$ is a connected graph with an irregular dominating set, then $\tilde{\gamma}(G) \geq 3$. Next, we show that every connected graph $G$ with $\tilde{\gamma}(G) = 3$ is an irregular domination graph. In order to establish this fact, we first present a lemma which provides a special structure in a connected graph having irregular domination number 3.

**Lemma 3.4.2** *If $G$ is a connected graph with $\tilde{\gamma}(G) = 3$, then $G$ possesses an irregular dominating labeling $f$ and a geodesic of length* 3 *whose vertices are labeled by $f$ as shown in Figure* 3.15*. Consequently, each labeled vertex is dominated by exactly one labeled vertex. Furthermore, the vertex labeled* 3 *is dominated by a vertex labeled* 1*, the vertex labeled* 2 *is dominated by a vertex labeled* 3*, and the vertex labeled* 1 *is dominated by a vertex labeled* 2*.*

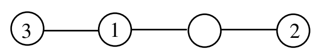



Figure 3.15: A labeled geodesic of length 3 in Lemma 3.4.2

**Proof.** By Observation 3.4.1, the labels used by a minimum irregular dominating labeling $f$ of $G$ are $1, 2$, and 3. Let $u \in V(G)$ such that $f(u) = 3$. Since $u$ cannot dominate itself, $u$ is dominated by a vertex $z$ labeled 1 or 2.

- ⋆ First, suppose that $f(z) = 2$. Then $d(u, z) = 2$ and $G$ contains a path $(u, y, z)$ where $uz \notin E(G)$. Since $f(u) = 3$ and $f(z) = 2$, neither $y$ nor $z$ is dominated by $u$ or $z$. This implies that $y$ and $z$ are dominated by a vertex $x$ labeled 1 and so $d(x, y) = d(x, z) = 1$. See Figure 3.16(a). Hence, $(u, y, x)$ is a path of length 2 in $G$ and so $d(x, u) \leq 2$. This implies that $x$ cannot be dominated by any of the three labeled vertices $u, z$, or $x$, which is a contradiction.

- ⋆ Thus, $z$ cannot be labeled 2 and so $f(z) = 1$. Then $uz \in E(G)$. Since neither $u$ nor $z$ dominates $z$, it follows that $z$ must be dominated by a vertex $w$ labeled 2 and so $d(w, z) = 2$. Thus, $G$ contains a path $(w, x, z)$ where $wz \notin E(G)$. Since $w$ cannot be dominated by itself, $w$ must be dominated by $z$ or $u$. However, since $f(z) = 1$ and $d(w, z) = 2$, it follows that $w$ cannot be dominated by $z$. Hence, $w$ is dominated by $u$ and $d(u, w) = 3$. See Figure 3.16(b). Thus, the path $(u, z, x, w)$ is a $u - w$ geodesic of length 3 in $G$ whose vertices are labeled by $f$ as shown in Figure 3.15. ■

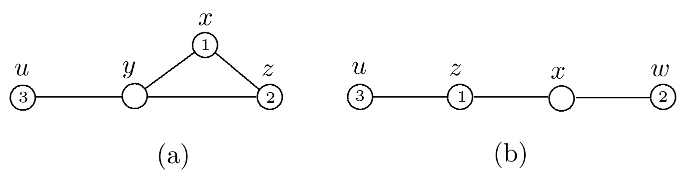



Figure 3.16: Two situations in the proof of Proposition 3.4.2

With the aid of Observation 3.4.1 and Lemma 3.4.2, we are able to verify the following.

**Theorem 3.4.3** *Every connected graph $G$ with $\tilde{\gamma}(G) = 3$ is an irregular domination graph.*

**Proof.** Let $G$ be a connected graph with $\tilde{\gamma}(G) = 3$. By Observation 3.4.1, there is a minimum (and minimal) irregular dominating labeling $f_0$ of $G$ using the labels $1, 2,$ and 3. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and we may assume that $f_0(u_i) = i$ for $i = 1, 2, 3$. By Lemma 3.4.2, there is a $u_3 - u_2$ geodesic $P = (u_3, u_1, u, u_2)$ of length 3 in $G$. Furthermore, the vertex labeled 3 is only dominated by a vertex labeled 1, the vertex labeled 2 is only dominated by a vertex labeled 3, and the vertex labeled 1 is only dominated by a vertex labeled 2. Let $H$ be the graph obtained from $G$ and the path $P_{n-1} = (v_1, v_2, \ldots, v_{n-1})$ of order $n - 1$ by adding the edge $u_1v_1$. We show that $V(G)$ is a minimal irregular dominating set of $H$, which implies that $G$ is an irregular domination graph. For $i = 1, 2, 3, 4$, let

$$X_i = \{x \in V(G) : d(u_1, x) = i\}.$$

Since every vertex of $G$ must be dominated by a vertex in $\{u_1, u_2, u_3\}$, it follows that $V(G) - \{u_1\} = X_1 \cup X_2 \cup X_3 \cup X_4$. While $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$, it is possible that $X_3 = \emptyset$ or $X_4 = \emptyset$. Furthermore, if $x \in X_i$ where $i = 1, 2, 3, 4$ and $1 \leq j \leq n - 1$, then

$$d(x, v_j) = d(x, u_1) + d(u_1, v_j) = i + j.$$

The graph $H$ is shown in Figure 3.17, where the $u_3 - u_2$ geodesic $P = (u_3, u_1, u, u_2)$ of $G$ is drawn in bold.

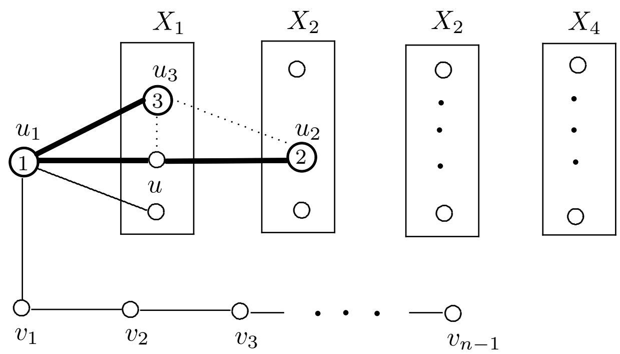



Figure 3.17: Constructing the graph $H$

For $n = 1, 2, 3, 4$, let $n_1 = |X_1 - \{u_3\}| \geq 1$, $n_2 = |X_2 - \{u_2\}| \geq 0$, and let $n_i = |X_i| \geq 0$ for $i = 3, 4$. Then $n_1+n_2+n_3+n_4 = n-3$ and so $n_1+n_2+n_3+n_4+3 = n$. We consider two cases, according to whether $n_2 = 0$ or $n_2 \geq 1$.

*Case* 1. $n_2 = 0$. Then $n_1 + n_3 + n_4 + 3 = n$. We define a labeling $f : V(G) \to [n + 3]$ by $f(u_i) = i$ for $i = 1, 2, 3$ and assigning

- $\star$ the $n_1$ labels in $[4, n_1 + 3]$ to the $n_1$ vertices in $X_1 - \{u_3\}$,
- $\star$ the $n_3$ labels in $[n_1 + 6, n_1 + n_3 + 5]$ to the $n_3$ vertices in $X_3$ if $n_3 \neq 0$, and
- $\star$ the $n_4$ labels in $[n_1 + n_3 + 7, n_1 + n_3 + n_4 + 6]$ to the $n_4$ vertices in $X_4$ if $n_4 \neq 0$.

Consequently, the set of the labels of $f$ is $[n + 6] - \{n_1 + 4, n_1 + 5, n_1 + n_3 + 6\}$. It remains to show that $f$ is a minimal irregular dominating labeling of $H$.

- Since $f_0$ is a minimum irregular dominating labeling of $G$ and $f(u_i) = f_0(u_i) = i$ for $i = 1, 2, 3$, there is a vertex in $G$ that is only dominated by the vertex $u_i$ for each integer $i$ with $i = 1, 2, 3$. Furthermore, the vertex $v_1$ is only dominated by the vertex $u_1$ labeled 1 and the vertex $v_2$ is only dominated by the vertex $u_3$ labeled 3.
- For each label $j$ with $4 \leq j \leq n_1 + 3$, the vertex $v_{j-1}$ is only dominated by the vertex labeled $j$.
- For each label $j$ with $n_1+6 \leq j \leq n_1+n_3+5$, the vertex $v_{j-3}$ is only dominated by the vertex labeled $j$.

- For each label $j$ with $n_1+n_3+7 \leq j \leq n_1+n_3+n_4+6 = n+3$, the vertex $v_{j-4}$ is only dominated by the vertex labeled $j$.

This is illustrated for the graph $G$ of order $n = 7$ shown in Figure 3.18, where $n_1 = 2$, $n_3 = n_4 = 1$. Notice that since $P = (u_3, u_1, u, u_2)$ is a geodesic in $G$, if $u_2$ is adjacent to a vertex $x \in X_3$, then $x$ is only dominated by the vertex labeled 3 in $G$. Since $u_3u, u_3u_2 \notin E(G)$ and $n_2 = 0$, it follows that $u_3$ is adjacent to a vertex in $X_1 - \{u_3, u\}$ and so $n_1 = |X_1 - \{u_3\}| \geq 2$.

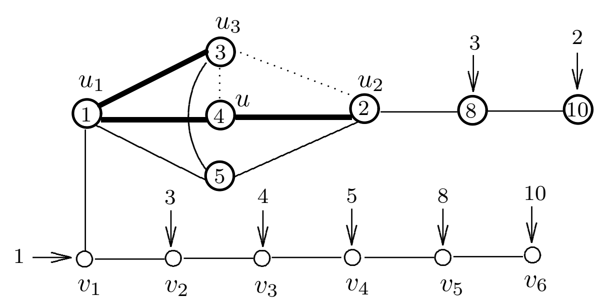



Figure 3.18: Illustrating the proof of Case 1

*Case* 2. $n_2 \geq 1$. We define a labeling $f : V(G) \to [n+3]$ by $f(u_i) = i$ for $i = 1, 2, 3$ and assigning

$\star$ the $n_1$ labels in $[4, n_1+3]$ to the $n_1$ vertices in $X_1 - \{u_3\}$,

$\star$ the $n_2$ labels in $[n_1+5, n_1+n_2+4]$ to the $n_2$ vertices in $X_2 - \{u_2\}$,

$\star$ the $n_3$ labels in $[n_1+n_2+6, n_1+n_2+n_3+5]$ to the $n_3$ vertices in $X_3$ if $n_3 \neq 0$, and

$\star$ the $n_4$ labels in $[n_1+n_2+n_3+7, n_1+n_2+n_3+n_4+6]$ to the $n_4$ vertices in $X_4$ if $n_4 \neq 0$.

Consequently, the set of the labels of $f$ is $[n+3]-\{n_1+4, n_1+n_2+5, n_1+n_2+n_3+6\}$. It remains to show that $f$ is a minimal irregular dominating labeling of $H$.

- Since $f_0$ is a minimum irregular dominating labeling of $G$ and $f(u_i) = f_0(u_i) = i$ for $i = 1, 2, 3$, there is a vertex in $G$ that is only dominated by the vertex $u_i$ for each integer $i$ with $i = 1, 2, 3$. Furthermore, the vertex $v_1$ is only dominated by the vertex $u_1$ labeled 1 and the vertex $v_2$ is only dominated by the vertex $u_3$ labeled 3.

- For each label $j$ with $4 \le j \le n_1 + 3$, the vertex $v_{j-1}$ is only dominated by the vertex labeled $j$.
- For each label $j$ with $n_1+5 \le j \le n_1+n_2+4$, the vertex $v_{j-2}$ is only dominated by the vertex labeled $j$.
- For each label $j$ with $n_1 + n_2 + 6 \le j \le n_1 + n_2 + n_3 + 5$, the vertex $v_{j-3}$ is only dominated by the vertex labeled $j$.
- For each label $j$ with $n_1+n_2+n_3+7 \le j \le n_1+n_2+n_3+n_4+6$, the vertex $v_{j-4}$ is only dominated by the vertex labeled $j$.

This is illustrated for the graph $G$ of order $n = 9$ shown in Figure 3.19, where $n_1 = 2$, $n_2 = 1$, $n_3 = 2$, and $n_4 = 1$.

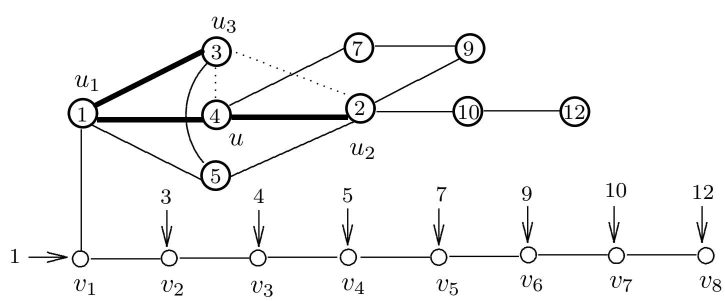



Figure 3.19: Illustrating the proof of Case 2

In each case, $f$ is a minimal irregular dominating labeling of $H$ and so $V(G)$ is the minimal irregular dominating set for $H$. Therefore, $G$ is an irregular domination graph. ■

The following is a consequence of Theorem 3.4.3.

**Corollary 3.4.4** *If $G$ is a connected graph $G$ with* $\mathrm{diam}(G) = 3$ *such that $G$ contains an irregular dominating set, then $G$ is an irregular domination graph.*

**Proof.** Let $G$ be a connected graph with $\mathrm{diam}(G) = 3$ such that $G$ contains an irregular dominating set. Since $\tilde{\gamma}(G) \leq \mathrm{diam}(G) = 3$, it follows by Observation 3.4.1 that $\tilde{\gamma}(G) = 3$. Therefore, $G$ is an irregular domination graph by Theorem 3.4.3. ■

Notice that the condition that the graph $G$ contains an irregular dominating set in Corollary 3.4.4 is necessary. As we will see later, the 6-cycle $C_6$ and the cube $Q_3$ are both connected graphs of diameter 3 neither of which is an irregular domination graph.

## 3.5 Irregular Domination Paths and Cycles

We now determine all paths and cycles that are irregular domination graphs, beginning with paths. By Proposition 3.3.3, the paths $P_2$ and $P_3$ are not irregular domination graphs. These are, however, the exceptions for paths.

**Theorem 3.5.1** *For each integer $n \geq 4$, the path $P_n$ is an irregular domination graph.*

**Proof.** By Proposition 3.3.6, both $P_4$ and $P_5$ are irregular domination graphs. For the path $P_6$, consider the graph $G$ in Figure 3.20. For the set $S = \{u_1, u_2, \ldots, u_6\}$ of vertices of $G$, the vertex $u_1$ is the only vertex of $S$ that dominates $u_4$, the vertex $u_2$ is the only vertex of $S$ that dominates $u_3$, the vertex $u_3$ is the only vertex of $S$ that dominates $x$, the vertex $u_4$ is the only vertex of $S$ that dominates $u_2$, the vertex $u_5$ is the only vertex of $S$ that dominates $y$, and the vertex $u_6$ is the only vertex of $S$ that dominates $z$, Therefore, $S$ is a minimal irregular dominating set and $G[S] \cong P_6$. Therefore, $P_6$ is an irregular domination graph.

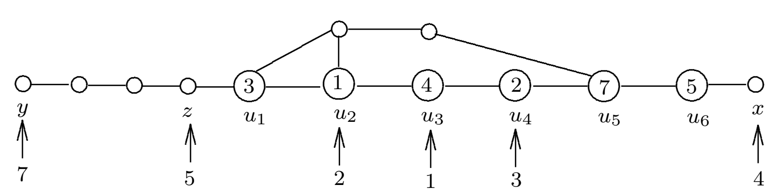



Figure 3.20: The graph $G$ in the proof of Theorem 3.5.1

Next, we show that for each integer $n \geq 7$, there is a graph $G_n$ having a minimal irregular dominating set $S_n$ such that $G_n[S_n] \cong P_n$. First, suppose that $n =$

11 and let $G_{11}$ be the graph shown in Figure 3.21, where $\operatorname{diam}(G_{11}) = d(w_1, v_5) = 13$. Let $S_{11} = \{u_1, u_2, \ldots, u_{11}\}$ with the corresponding irregular dominating labeling $f_{11}$ shown in Figure 3.21. Observe that the vertex $x$ is only dominated by the vertex $u_5$ labeled 1, the vertex $u_5$ is only dominated by the vertex $u_3$ labeled 2, the vertex $u_3$ is only dominated by the vertex $u_6$ labeled 3, the vertex $v_1$ is only dominated by the vertex $u_4$ labeled 4, the vertex $u_2$ is only dominated by the vertex $u_7$ labeled 5, the vertex $u_7$ is only dominated by the vertex $u_1$ labeled 6, the vertex $w_1$ is only dominated by the vertex $u_2$ labeled 7, and for $8 \le i \le 11$, the vertex $v_{i-6}$ is only dominated by the vertex $u_i$ labeled $i$. Furthermore, every vertex of $G$ is dominated by at least one vertex in $S_{11}$. Since $S_{11}$ is a minimal irregular dominating set of $G_{11}$ and $G_{11}[S_{11}] = P_{11}$, it follows that $P_{11}$ is an irregular domination graph.

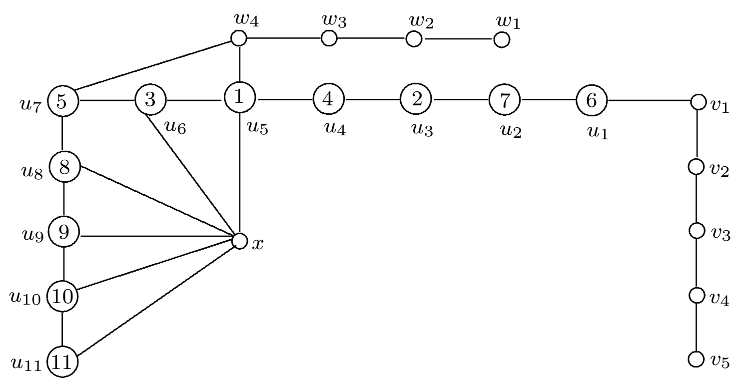



Figure 3.21: The graph $G_{11}$ in the proof of Theorem 3.5.1

It remains to show that if $n \ge 7$ and $n \ne 11$, then $P_n$ is an irregular domination graph. We consider two cases, according to whether $7 \le n \le 10$ or $n \ge 12$.

*Case* 1. $7 \le n \le 10$. Beginning with $G_{11}$, we construct the graph $G_n$ from the graph $G_{n+1}$ recursively as follows. For $n = 10$, let $G_{10}$ be the graph obtained from $G_{11}$ by deleting the vertices $u_{11}$ and $v_5$, for $n = 9$, let $G_9$ be the graph obtained from $G_{10}$ by deleting the vertices $u_{10}$ and $v_4$, for $n = 8$, let $G_8$ be the graph obtained from $G_9$ by deleting the vertices $u_9$, and $v_3$, and for $n = 7$, let $G_7$ be the graph obtained from $G_8$ by deleting the vertices $u_8$, $x$, and $v_2$. For $n = 7, 8, 9, 10$, let $S_n = \{u_1, u_2, \ldots, u_n\}$ and let $f_n(u_i) = f_{11}(u_i)$ for $1 \le i \le n$. Then every vertex of $G_n$ is dominated by at least one vertex in $S_n$. Furthermore, $G_n$ $(7 \le n \le 10)$ is a distance-preserving subgraph of $G_{11}$. Thus, for each $u \in S_n$, there is a vertex of $G_n$ that is only dominated by $u$. Therefore, $S_n$ is a minimal irregular dominating set of $G_n$ and $G_n[S_n] = P_n$. The graphs $G_7$ and $G_8$ are shown in Figure 3.22.

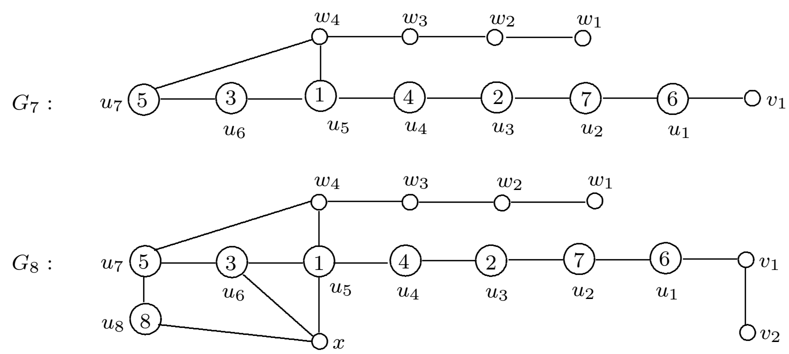



Figure 3.22: The graphs $G_7$ and $G_8$ in Case 1

*Case* 2. $n \geq 12$. Let $G_n$ be the graph obtained from $G_{11}$, the path $(u_{12}, u_{13}, \ldots, u_n)$ of order $n-11$, and the path $(v_6, v_7, \ldots, v_{n-5})$ of order $n-10$ by (1) joining each vertex $u_i$ to $x$ for $12 \leq i \leq n$ and (2) joining $v_5$ to $v_6$. Then $\text{diam}(G_n) = d(w_1, v_{n-5}) = 8 + (n-5) = n+3$. Let $S_n = \{u_1, u_2, \ldots, u_n\}$. Define a labeling $f_n : S_n \to [n+1]$ by $f_n(u_i) = f_{11}(u_i)$ for $1 \leq i \leq 11$ and $f_n(u_i) = i+1$ for $12 \leq i \leq n$. This is illustrated in Figure 3.23.

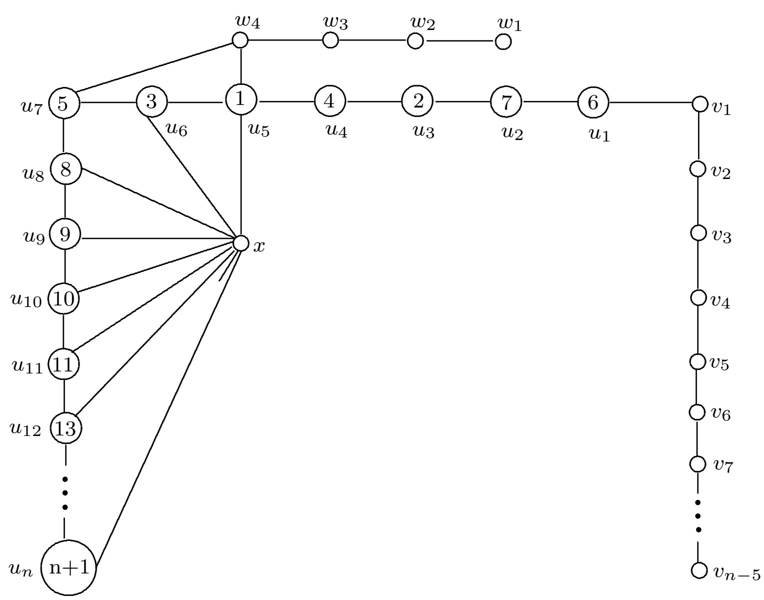



Figure 3.23: The graph $G_n$ in Case 2 for $n \geq 12$

First, every vertex of $G_n$ is dominated by at least one vertex in $S_n$ and so $S_n$ is an irregular dominating set of $G_n$. Since $G_{11}$ is a distance-preserving subgraph of $G_n$, it follows that if $u_i \in S_n$, where $1 \leq i \leq 11$, then there is a vertex of $G_{11} \subseteq G_n$ that is only dominated by $u_i$. Furthermore, for $7 \leq i \leq n-5$, the vertex $v_i$ is only dominated by $u_{i+5}$ labeled $i+6$. Therefore, $S_n$ is a minimal irregular dominating set of $G_n$ and $G_n[S_n] \cong P_n$. ■

By Theorem 3.5.1, we now know exactly which paths are irregular domination graphs.

**Corollary 3.5.2** *A path $P_n$ of order $n \geq 2$ is an irregular domination graph if and only if $n \notin \{2, 3\}$.*

Next, we turn our attention to cycles. Since $\text{diam}(C_n) \leq 2$ for $n = 3, 4, 5$, it follows by Proposition 3.3.3 that $C_n$ is not an irregular domination graph if $n = 3, 4, 5$. Before proceeding further with our discussion of cycles, we present the following result.

**Proposition 3.5.3** *Let $H$ be an $r$-regular graph, $r \geq 2$, of diameter 3 with the property that for each vertex $x$ of $H$, there is exactly one vertex $y$ such that $d(x, y) = 3$. Then $H$ is not an irregular domination graph.*

**Proof.** Assume, to the contrary, that such a graph $H$ is an irregular domination graph. Then there exists a graph $G$ with a minimal irregular dominating set $S$ such that $G[S] \cong H$. Let $f$ be an irregular dominating labeling of $S$. Since the distance between every two vertices of $S$ in $G$ is 1, 2, or 3, every vertex of $S$ must be dominated by a vertex of $S$ labeled 1, 2 or 3. We consider two cases.

*Case* 1. *No vertex of $S$ is dominated by a vertex labeled* 3. Thus, every vertex of $S$ is dominated by a vertex of $S$ labeled 1 or 2. Since no vertex of $S$ can dominate itself, there are vertices $u, v \in S$ such that $f(u) = 1$ and $f(v) = 2$. However then, $u$ and $v$ dominate each other, which is impossible by Observation 1.3.1.

*Case* 2. *There is a vertex of $S$ that is dominated by a vertex labeled* 3. Let $w, z \in S$ such that $f(w) = 3$ and $z$ is dominated by $w$. Thus, $d_G(w, z) = 3$ and $z$ is the only vertex of $S$ that is dominated by $w$. Therefore, every vertex of $S - \{z\}$ is dominated by a vertex labeled 1 or 2. Since no vertex of $S$ can dominate itself, there are vertices $u, v \in S$ such that $f(u) = 1$ and $f(v) = 2$. If neither $u$ nor $v$

is $z$, then $u$ and $v$ dominate each other, which is impossible by Observation 1.3.1. Thus, either $u = z$ or $v = z$. First, suppose that $u = z$, that is, $f(z) = 1$ and $f(v) = 2$. Since only $v$ can dominate $w$, it follows that $d(v, w) = 2$ and there is a $v - w$ geodesic $(v, x, w)$ in $G$. However then, $x$ cannot be dominated by any of the three labeled vertices $v, w, z$ of $S$, which is impossible. Next, suppose that $v = z$, that is, $f(u) = 1$ and $f(z) = 2$. Since $z$ is the only vertex of $S$ that is dominated by $w$, it follows that $w$ cannot dominate any neighbor of $z$ in $S$. Furthermore, $z$ cannot dominate any of its neighbor. This implies that $u$ must dominate each of the $r$ neighbors of $z$ in $S$ as well as the vertex $w$. Thus, $u$ must dominate at least $r+1$ vertices of $S$, which says that $u$ is adjacent to at least $r+1$ vertices of $S$. Since $G[S]$ is $r$-regular, this is impossible. ■

By Proposition 3.5.3, the 6-cycle $C_6$ is not an irregular domination graph. The 7-cycle $C_7$ has diameter 3 but for each vertex $x$ of $C_7$, there are two vertices $y$ such that $d(x, y) = 3$. Not only is $C_7$ an irregular domination graph, $C_n$ is an irregular domination graph for every integer $n \geq 7$.

**Theorem 3.5.4** *For each integer $n \geq 7$, the cycle $C_n$ is an irregular domination graph.*

**Proof.** We show for each integer $n \geq 7$ that there is a graph $G_n$ having a minimal irregular dominating set $S_n$ such that $G_n[S_n] \cong C_n$. First, suppose that $n = 7$. Let $G_7$ be the graph shown in Figure 3.24. Then $\text{diam}(G_7) = d(u_3, v_6) = d(u_5, v_6) = 8$. Let $S_7 = \{u_1, u_2, \ldots, u_7\}$ with the corresponding irregular dominating labeling $f_7$ as shown in Figure 3.24. Observe that the vertex $x$ is only dominated by the vertex $u_2$ labeled 1, the vertex $v_2$ is only dominated by the vertex $u_7$ labeled 2, the vertex $v_1$ is only dominated by the vertex $u_3$ labeled 3, the vertex $v_3$ is only dominated by the vertex $u_4$ labeled 4, the vertex $v_4$ is only dominated by the vertex $u_6$ labeled 5, the vertex $v_6$ is only dominated by the vertex $u_1$ labeled 6, and the vertex $v_5$ is only dominated by the vertex $u_5$ labeled 7. Furthermore, every vertex of $G_7$ is dominated by at least one vertex in $S_7$. Therefore, $S_7$ is a minimal irregular dominating set of $G_7$ and $G_7[S_7] \cong C_7$.

Next, suppose that $n = 8$. Let $G_8$ be the graph shown in Figure 3.25, where $\text{diam}(G_8) = d(u_7, v_6) = 10$. Let $S_8 = \{u_1, u_2, \ldots, u_8\}$ with the corresponding irregular dominating labeling $f_8$ shown in Figure 3.25. Observe that the vertex $x$ is only

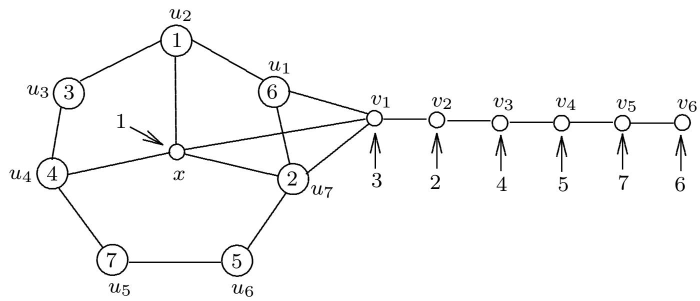



Figure 3.24: Showing that $C_7$ is an irregular domination graph

dominated by the vertex $u_5$ labeled 1, the vertex $v_1$ is only dominated by the vertex $u_4$ labeled 2, the vertex $v_2$ is only dominated by the vertex $u_2$ labeled 3, the vertex $u_3$ is only dominated by the vertex $u_7$ labeled 4, the vertex $v_5$ is only dominated by the vertex $u_3$ labeled 5, the vertex $v_3$ is only dominated by the vertex $u_6$ labeled 6, the vertex $v_4$ is only dominated by the vertex $u_8$ labeled 7, and the vertex $v_6$ is only dominated by the vertex $u_1$ labeled 8. Furthermore, every vertex of $G_8$ is dominated by at least one vertex in $S_8$. Therefore, $S_8$ is a minimal irregular dominating set of $G_8$ and $G_8[S_8] \cong C_8$.

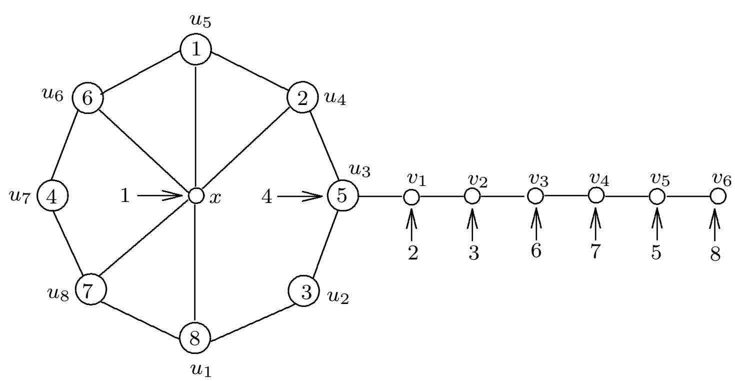



Figure 3.25: Showing that $C_8$ is an irregular domination graph

Finally, suppose that $n \geq 9$. Let $G_n$ be the graph obtained from $G_8$ by (1) replacing the edge $u_8u_1$ by the path $Q = (u_8, x_1, x_2, \ldots, x_{n-8}, u_1)$ and joining $x$ to each vertex of $Q_1$, (2) adding two new vertices $w_1$ and $w_2$ and five new edges $u_5w_1$, $u_7w_1$, $w_1w_2$, $w_2x$, $w_2u_2$, and (3) adding the path $(y_1, y_2, \ldots, y_{n-8})$ at $v_6$ by joining $v_6$ to $y_1$. Then $\text{diam}(G_n) = d(u_7, y_{n-8}) = n + 2$. Let $S_n = \{u_1, u_2, \ldots, u_8, x_1, x_2, \ldots, x_{n-8}\}$. Define a labeling $f_n : S_n \to [n+1]$ by $f_n(u_i) = f_8(u_i)$ for $1 \leq i \leq 8$ and $f(x_i) = 9+i$

for $1 \le i \le n-8$. This is illustrated in Figure 3.26 for $n = 10$.

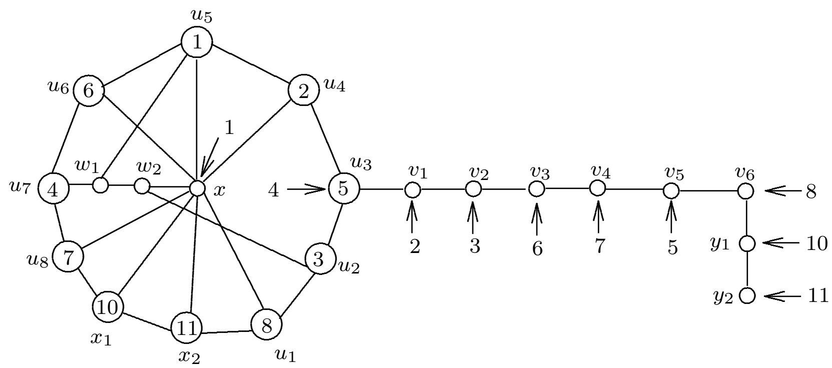



Figure 3.26: Showing that $C_{10}$ is an irregular domination graph

First, every vertex of $G_n$ is dominated by at least one vertex in $S_n$ and so $S_n$ is an irregular dominating set of $G_n$. Since $G_8$ is a distance-preserving subgraph of $G_n$, it follows that if $u_i \in S_n$, where $1 \le i \le 8$, then there is a vertex of $G_8 \subseteq G_n$ that is only dominated by $u_i$. Furthermore, for $1 \le i \le n-8$, the vertex $y_i$ is only dominated by $x_i$ labeled $9+i$. Therefore, $S_n$ is a minimal irregular dominating set of $G_n$ and $G_n[S_n] \cong C_n$. ■

As another illustration of the proof of Theorem 3.5.4 for the case when $n \ge 9$, the graph $G_{12}$ is shown in Figure 3.27 together with a minimal irregular dominating set $S_{12}$ and the corresponding irregular dominating labeling $f_{12}$ of $G_{12}$.

**Corollary 3.5.5** *A cycle $C_n$ of order $n \ge 3$ is an irregular domination graph if and only if $n \ge 7$.*

It was mentioned that no connected vertex-transitive graph has an irregular dominating set; consequently, no cycle has an irregular dominating set. By Corollary 3.5.5, however, almost all cycles are irregular domination graphs. That is, a graph may be an irregular domination graph even if it fails to have an irregular dominating set.

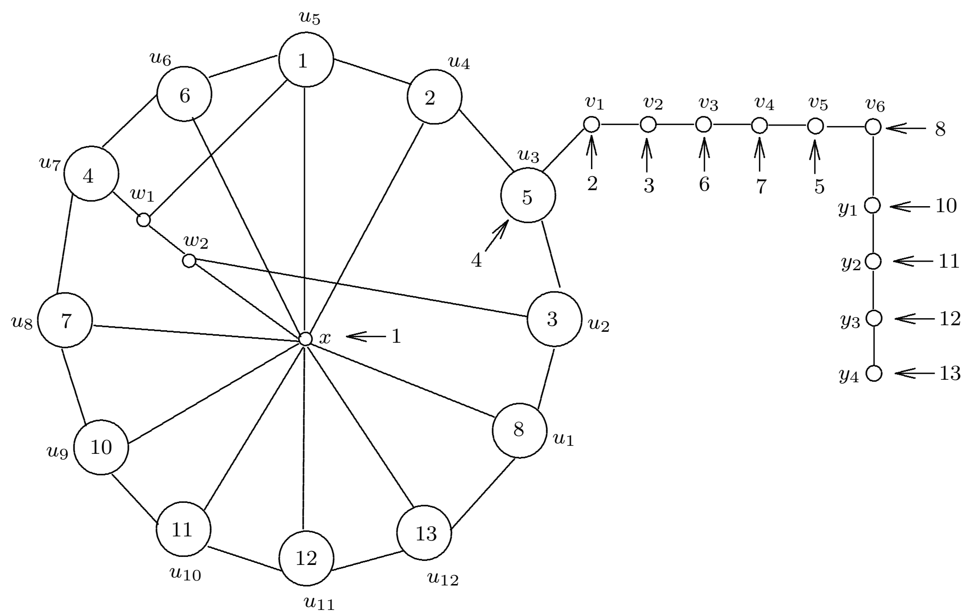



Figure 3.27: The graph $C_{12}$ in the proof of Theorem 3.5.4

## 3.6 Irregular Domination Grids and Prisms

We now turn our attention to two well-known classes of graphs constructed from paths and cycles. For two vertex-disjoint graphs $G$ and $H$, the *Cartesian product* $G \square H$ of $G$ and $H$ has vertex set $V(G \square H) = V(G) \times V(H)$ and two distinct vertices $(u, v)$ and $(x, y)$ of $G \square H$ are adjacent if either (1) $u = x$ and $vy \in E(H)$ or (2) $v = y$ and $ux \in E(G)$. For an integer $n \geq 2$, the graph $P_n \square K_2$ is often referred to as a *ladder* and for each integer $n \geq 3$, the graph $C_n \square K_2$ is referred to as a *prism*.

We saw that a path $P_n$ of order $n \geq 2$ is an irregular domination graph if and only if $n \notin \{2, 3\}$. While $P_2 \square K_2 = C_4$ is not an irregular dominating graph, $P_n \square K_2$ is an irregular domination graph for all $n \geq 3$.

**Theorem 3.6.1** *For each integer $n \geq 3$, the ladder $P_n \square K_2$ is an irregular domination graph.*

**Proof.** For an integer $n \geq 3$, let $H_n = P_n \square K_2$ where $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are two vertex-disjoint copies of an $n$-path in $H_n$ and $u_iv_i \in E(H_n)$

for $1 \leq i \leq n$. Then $\text{diam}(H_n) = d(u_1, v_n) = d(v_1, u_n) = n$. We show that there is a graph $G_n$ having a minimal irregular dominating set $S_n$ with corresponding labeling $f_n$ such that $G_n[S_n] \cong H_n$.

For $n = 3, 4$, the graph $G_n$ is shown in Figure 3.28. Let $S_n = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ with the corresponding labeling $f_n$ also shown in Figure 3.28.

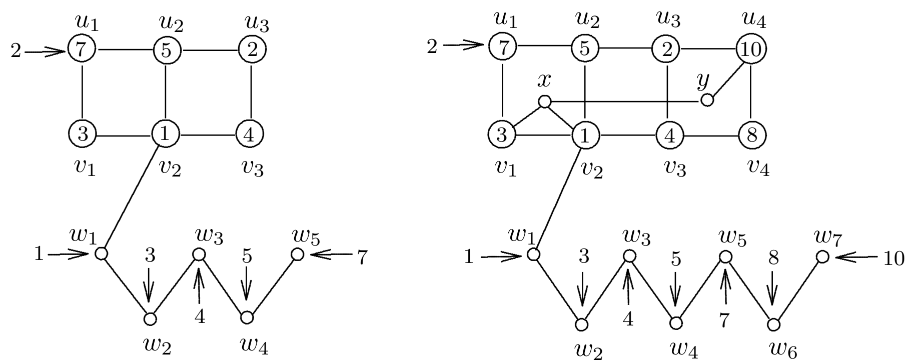



Figure 3.28: The graphs $G_3$ and $G_4$ in the proof of Theorem 3.6.1

- ⋆ In $G_3$, the vertex $w_1$ is only dominated by the vertex labeled 1, the vertex $u_1$ is only dominated by the vertex labeled 2, the vertex $w_2$ is only dominated by the vertex labeled 3, the vertex $w_3$ is only dominated by the vertex labeled 4, the vertex $w_4$ is only dominated by the vertex labeled 5, and the vertex $w_5$ is only dominated by the vertex labeled 7.

- ⋆ In $G_4$, the vertex $w_1$ is only dominated by the vertex labeled 1, the vertex $u_1$ is only dominated by the vertex labeled 2, the vertex $w_2$ is only dominated by the vertex labeled 3, the vertex $w_3$ is only dominated by the vertex labeled 4, the vertex $w_4$ is only dominated by the vertex labeled 5, the vertex $w_5$ is only dominated by the vertex labeled 7, the vertex $w_6$ is only dominated by the vertex labeled 8, and the vertex $w_7$ is only dominated by the vertex labeled 10.

Furthermore, every vertex of $G_n$, $n = 3, 4$, is dominated by at least one vertex in $S_n$. Thus, $S_n$ is a minimal irregular dominating set of $G_n$ and $G_n[S_n] \cong H_n$ for $n = 3, 4$.

For $n \geq 5$, let $G_n$ be the graph constructed from $H_n$ and the path $P = (w_1, w_2, \ldots, w_{2n-1})$ of order $2n - 1$ by (1) adding two new vertices $x$ and $y$ and the four new edges $xv_1, xv_2, xy, yu_4$, (2) joining $y$ to both $u_i$ and $v_i$ for $5 \leq i \leq n$, and (3) joining $w_1$ to $v_2$. Thus, $V(G_n) = V(H_n) \cup \{x, y\} \cup V(P)$ and

$$E(G_n) = E(H_n) \cup E(P) \cup \{v_2w_1, xv_1, xv_2, xy, yu_4\} \cup \{yu_i, yv_i : 5 \leq i \leq n\}.$$

Then $\mathrm{diam}(G_n) = d(u_1, w_{2n-1}) = 2n + 2$. Notice that $G_4$ is a distance-preserving subgraph of $G_n$. Let $S_n = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$. We define a labeling $f_n : S_n \to [2n + 2]$ by extending the labeling $f_4$ of $G_4$ shown in Figure 3.28; that is, we define $f_n(u_i) = f_4(u_i)$ and $f_n(v_i) = f_4(v_i)$ for $1 \leq i \leq 4$ (see Figure 3.28) and $f_n(u_i) = 2i + 2$ for $5 \leq i \leq n$ and $f_n(v_i) = 2i + 1$ for $5 \leq i \leq n$. Thus, the set of labels used by $f_n$ is $[2n+2] - \{6, 9\}$. The graph $G_7$ is shown in Figure 3.29 together with the corresponding labeling $f_7$ of $G_7$.

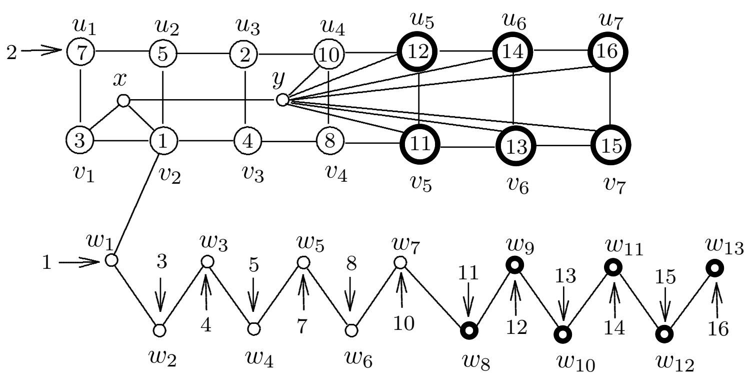



Figure 3.29: The graph $G_7$ in the proof of Theorem 3.6.1

Since $G_4$ is a distance-preserving subgraph of $G_n$, it follows that for each $\ell \in \{1, 2, 3, 4, 5, 7, 8, 10\}$, there is a vertex of $G_4 \subseteq G_n$ that is only dominated by the vertex labeled $\ell$. Furthermore, if $\ell \in \{11, 12, \ldots, 2n + 2\}$, then the vertex $w_{\ell-3}$ is only dominated by the vertex labeled $\ell$. Also, every vertex of $G_n$ is dominated by at least one vertex in $S_n$. In particular, if $5 \leq i \leq n$, then $u_i$ and $v_i$ are dominated by $v_1$ labeled 3. Therefore, $S_n$ is a minimal irregular dominating set of $G_n$ and $G_n[S_n] \cong P_n \,\square\, K_2$. ■

The graphs $P_m \,\square\, P_n$ for $m, n \geq 2$ are commonly referred to as *grids*. Thus, ladders form a subset of the grids. While it is an open problem to determine which of these graphs are irregular domination graphs, it is known that $P_m \,\square\, P_n$ is an irregular domination graph for each pair $m, n$ of integers with $2 \leq m \leq n \leq 4$.

That $P_m \square P_n$ where $(m, n) \in \{(3,3), (3,4), (4,4)\}$ are irregular domination graphs is shown in Figure 3.30.

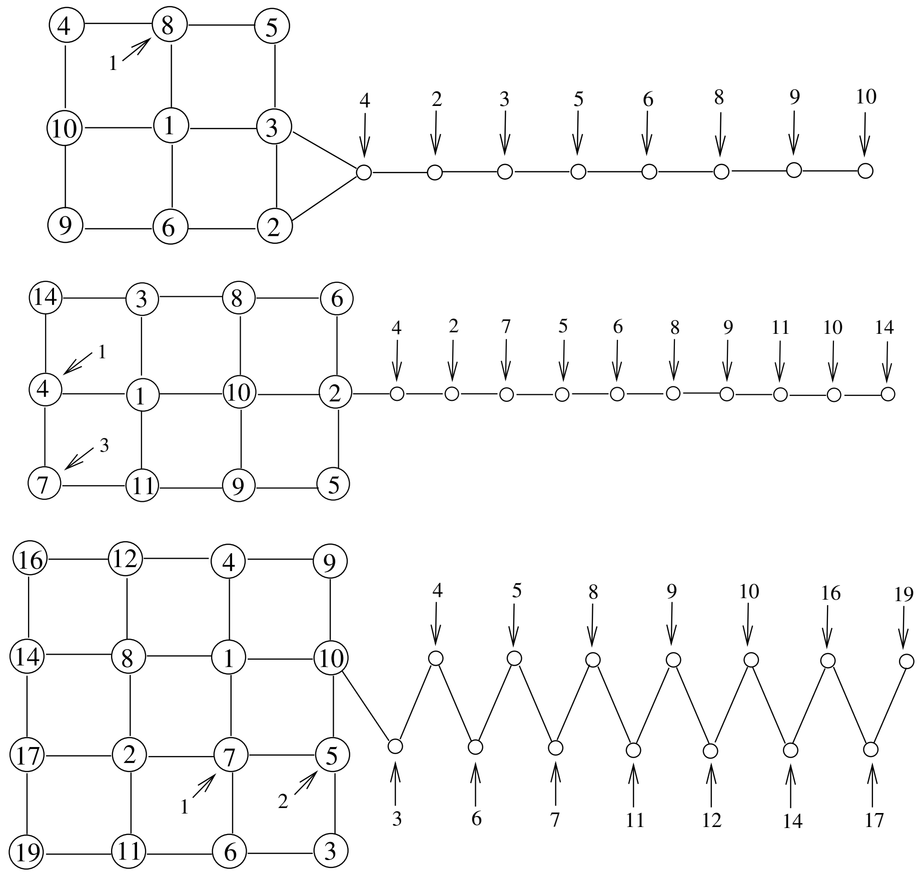


Figure 3.30: Showing that $P_3 \square P_3$, $P_3 \square P_4$, and $P_4 \square P_4$ are irregular domination graphs

We now turn our attention to prisms. First, we determine those prisms that are not irregular domination graphs.

**Proposition 3.6.2** *For $n = 3, 4, 5$, the prism $C_n \square K_2$ is not an irregular domination graph.*

**Proof.** Since $\text{diam}(C_3 \square K_2) = 2$, it follows that $C_3 \square K_2$ is not an irregular domination graph by Proposition 3.3.3. Since $C_4 \square K_2$ is a 3-regular graph of diameter 3 with the property that for each vertex $x$ of $H$, there is exactly one vertex $y$ such that $d(x, y) = 3$, it follows by Proposition 3.5.3 that $C_4 \square K_2$ is not an irregular domination graph. It remains to consider $C_5 \,\square\, K_2$. Let $H = C_5 \,\square\, K_2$ where $(u_1, u_2, \dots, u_5, u_1)$ and $(v_1, v_2, \dots, v_5, v_1)$ are two vertex-disjoint copies of a 5-cycle in $H$ and $u_iv_i \in E(H)$ for $1 \le i \le 5$. Assume, to the contrary, that $H$ is an irregular domination graph. Then there is a graph $G$ with a minimal irregular dominating set $S$ with corresponding irregular dominating labeling $f$ such that $G[S] \cong H$. Since $\mathrm{diam}(H) = 3$, each vertex of $S$ is dominated by a vertex of $S$ labeled 1, 2, or 3. A vertex labeled 3 in $S$ dominates at most two vertices of $S$, a vertex labeled 2 in $S$ dominates at least four and at most six vertices of $S$, and a vertex labeled 1 in $S$ dominates exactly three vertices of $S$. Since $S$ has ten vertices, all three labels 1, 2, and 3 must be used. Furthermore, the vertex labeled 2 dominates at least five vertices of $S$. Suppose, without loss of generality, that $f(u_1) = 2$.

*Case* 1. *$u_1$ dominates exactly five vertices of $S$.* We may assume that $u_1$ dominates $v_2$, $v_4$, $v_5$, $u_3$, $u_4$ and $u_1$ does not dominate $v_3$. Therefore, there is a vertex $x \notin S$ such that $x$ is a neighbor of both $u_1$ and $v_4$. Furthermore, there is no vertex $y \notin S$ such that $y$ is a neighbor of both $u_1$ and $v_3$. In this case, every vertex of $S$ is dominated by exactly one vertex labeled 1, 2, or 3. Since $v_2$ is the only unlabeled vertex whose three neighbors are not dominated, it follows that $f(v_2) = 1$. Since $u_1$ and $u_5$ are the only vertices not dominated by a vertex labeled 1 or 2, it follows that $u_1$ and $u_5$ must be dominated by a vertex labeled 3, which implies that $f(v_3) = 3$. Since $d_H(x, w) \le 3$ for each $w \in V(H)$, it follows that $x$ is not dominated by any labeled vertex, which is a contradiction.

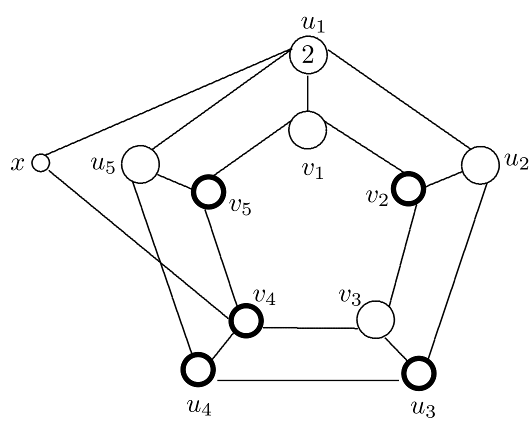



Figure 3.31: A step in the proof of Proposition 3.6.2 in Case 1

*Case* 2. *$u_1$ dominates exactly six vertices of $S$.* Then $u_1$ dominates $v_2, v_3, v_4, v_5, u_3, u_4$.

Thus, there is a vertex $x \notin S$ such that $x$ is a neighbor of both $u_1$ and $v_4$ and there is a vertex $y \notin S$ such that $y$ is a neighbor of both $u_1$ and $v_3$, where possibly $x = y$. Since the vertex labeled 1 must dominate at least two unlabeled vertices, we may assume that $f(v_2) = 1$ and so $v_2$ dominates $u_2$ and $v_1$. Therefore, the vertex $z$ labeled 3 must dominate $u_1$ and $u_5$. However, no such vertex $z$ has this property. This is a contradiction. Therefore, $C_5 \square K_2$ is not an irregular domination graph. ■

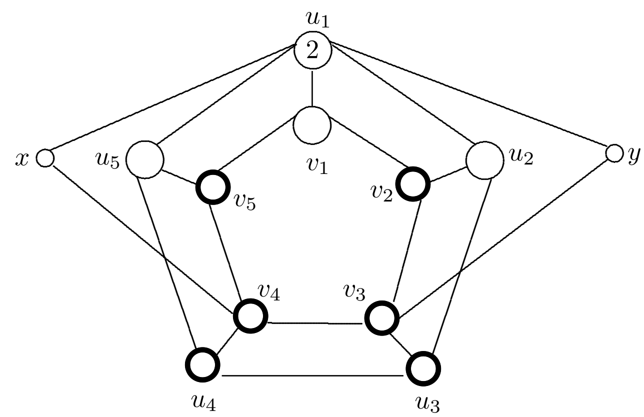



Figure 3.32: A step in the proof of Proposition 3.6.2 in Case 2

**Theorem 3.6.3** *For each integer $n \geq 6$, the prism $C_n \square K_2$ is an irregular domination graph.*

**Proof.** For $n \geq 6$, the diameter $C_n \square K_2$ is $\text{diam}(C_n \square K_2) = \text{diam}(C_n) + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Let $H_n = C_n \square K_2$ where $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ are two vertex-disjoint copies of an $n$-cycle in $H_n$ and $u_i v_i \in E(H_n)$ for $1 \leq i \leq n$. We show for each integer $n \geq 6$ that there is a graph $G_n$ having a minimal irregular dominating set $S_n$ such that $G_n[S_n] \cong H_n$.

First, suppose that $n = 6$ and $\text{diam}(H_6) = 4$. Let $G_6$ be the graph shown in Figure 3.33. Let $S_6 = \{u_1, u_2, \ldots, u_6, v_1, v_2, \ldots, v_6\}$ with the corresponding irregular dominating labeling $f_6$ as shown in Figure 3.33. Observe that

- ⋆ the vertex $y_1$ is only dominated by the vertex $u_1$ labeled 1,
- ⋆ the vertex $u_1$ is only dominated by the vertex $v_6$ labeled 2,
- ⋆ for $i = 3, 4, 5$, the vertex $y_{i-1}$ is only dominated by the vertex labeled $i$,

* $\star$ for $i = 7, 8, \ldots, 12$, the vertex $y_{i-2}$ is only dominated by the vertex labeled $i$, and the vertex $y_{11}$ is only dominated by the vertex labeled 14.

Furthermore, every vertex of $G_6$ is dominated by at least one vertex in $S_6$. Therefore, $S_6$ is a minimal irregular dominating set of $G_6$ and $G_6[S_6] \cong H_6 = C_6 \,\square\, K_2$.

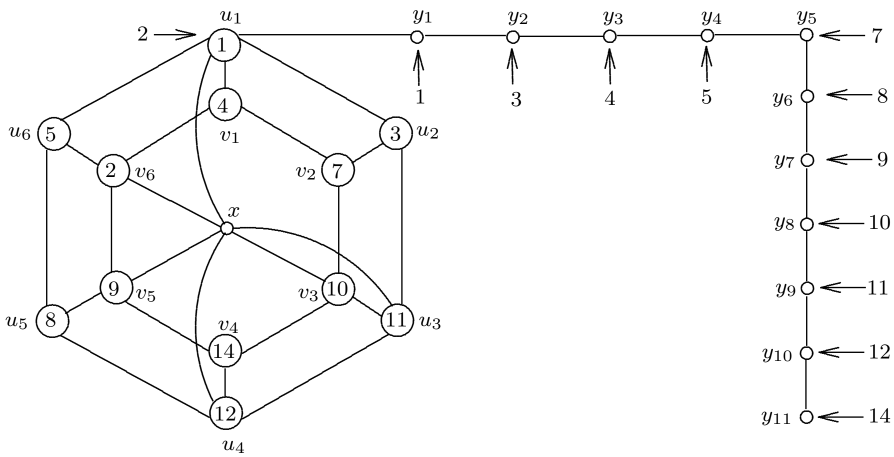



Figure 3.33: The graph $G_6$ the proof of Theorem 3.6.3

First, suppose that $n = 7$. Let $F_7$ be the graph constructed in the proof of Theorem 3.5.4 with

$$V(F_7) = \{u_1, u_2, \ldots, u_7\} \cup \{x, y_1, y_2, \ldots, y_6\},$$

where $(u_1, u_2, \ldots, u_7, u_1) = C_7$ and $(y_1, y_2, \ldots, y_6) = P_6$, and

$$E(F_7) = E(C_7) \cup E(P_6) \cup \{xu_2, xu_4, xu_7, xy_1, u_1y_1, u_7y_1\}.$$

Then $T_7 = \{u_1, u_2, \ldots, u_7\}$ is a minimal irregular dominating set of $F_7$ with corresponding irregular dominating labeling $g_7$ defined by $g_7(u_1) = 6$, $g_7(u_2) = 1$, $g_7(u_3) = 3$, $g_7(u_4) = 4$, $g_7(u_5) = 7$, $g_7(u_6) = 5$, and $g_7(u_7) = 2$ such that $F_7[T_7] = C_7 = (u_1, u_2, \ldots, u_7, u_1)$. Let $G_7$ be the graph obtained from $F_7$ by (1) adding the 7-cycle $(v_1, v_2, \ldots, v_7, v_1)$ and joining $v_i$ to $u_i$ for $1 \le i \le 7$, (2) joining $x$ to $v_i$ for $1 \le i \le 7$, and (3) adding the 7-path $(z_1, z_2, \ldots, z_7)$ and joining $z_1$ to $y_6$. Then $\text{diam}(G_7) = d(u_3, z_7) = 15$. Let $S_7 = \{u_1, u_2, \ldots, u_7\} \cup \{v_1, v_2, \ldots, v_7\}$. Define a labeling $f_7 : S_7 \to [14]$ by $f_7(u_i) = g_7(u_i)$ for $1 \le i \le 7$ and $f(v_i) = 7 + i$ for $1 \le i \le 7$. The graph $G_7$ is shown in Figure 3.34 where the labeling $f_7$ of $G_7$ is also shown. Observe that

- $\star$ the vertex $x$ is only dominated by the vertex $u_2$ labeled 1,
- $\star$ the vertex $y_2$ is only dominated by the vertex $u_7$ labeled 2,
- $\star$ the vertex $y_1$ is only dominated by the vertex $u_3$ labeled 3,
- $\star$ the vertex $y_3$ is only dominated by the vertex $u_4$ labeled 4,
- $\star$ the vertex $y_4$ is only dominated by the vertex $u_6$ labeled 5,
- $\star$ the vertex $y_6$ is only dominated by the vertex $u_1$ labeled 6,
- $\star$ the vertex $y_5$ is only dominated by the vertex $u_5$ labeled 7, and
- $\star$ the vertex $z_i$ is only dominated by the vertex $v_i$ labeled $7+i$ for $1 \leq i \leq 7$.

Furthermore, every vertex of $G_7$ is dominated by at least one vertex in $S_7$. In particular, the vertex $v_7$ is dominated by $u_3$ labeled 3 and if $i \neq 7$, then $v_i$ is dominated by $u_7$ labeled 2. Therefore, $S_7$ is a minimal irregular dominating set of $G_7$ and $G_7[S_7] \cong H_7 = C_7 \square K_2$.

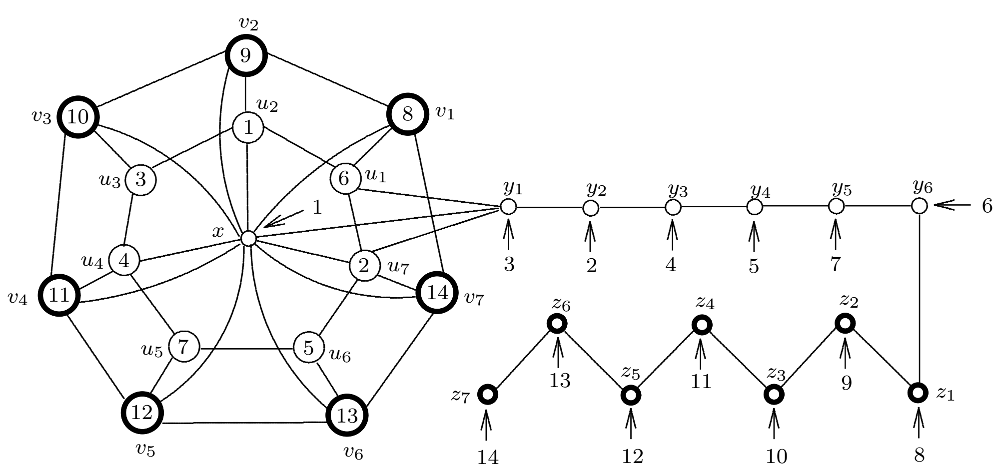


Figure 3.34: The graph $G_7$ the proof of Theorem 3.6.3

Next, suppose that $n = 8$. Let $F_8$ be the graph constructed in the proof of Theorem 3.5.4 with

$$V(F_8) = \{u_1, u_2, \ldots, u_8\} \cup \{x, y_1, y_2, \ldots, y_6\},$$

where $(u_1, u_2, \ldots, u_8, u_1) = C_8$ and $(y_1, y_2, \ldots, y_6) = P_6$, and

$$E(F_8) = E(C_7) \cup E(P_6) \cup \{xu_i : \ i \in [8] - \{2, 3, 7\}\} \cup \{u_3 y_1\}.$$

Then $T_8 = \{u_1, u_2, \ldots, u_8\}$ is a minimal irregular dominating set of $F_8$ with corresponding irregular dominating labeling $g_8$ defined by $g_8(u_1) = 6$, $g_8(u_2) = 1$, $g_8(u_3) = 3$, $g_8(u_4) = 4$, $g_8(u_5) = 7$, $g_8(u_6) = 5$, $g_8(u_7) = 2$, and $g_8(u_8) = 2$ such that $F_8[T_8] = C_8 = (u_1, u_2, \ldots, u_8, u_1)$. Let $G_8$ be the graph obtained from $F_8$ by

(1) adding the 8-cycle $(v_1, v_2, \ldots, v_8, v_1)$ and joining $v_i$ to $u_i$ for $1 \le i \le 8$,

(2) joining $x$ to $v_i$ for $i \in [8] - \{4\}$ (and so $xv_4 \notin E(G_4)$), and

(3) adding the 8-path $(z_1, z_2, \ldots, z_8)$ and joining $z_1$ to $y_6$.

Then $\text{diam}(G_8) = d(u_7, z_8) = 18$. Let $S_8 = \{u_1, u_2, \ldots, u_8\} \cup \{v_1, v_2, \ldots, v_8\}$. Define a labeling $f_8 : S_8 \to [17]$ by $f_8(u_i) = g_8(u_i)$ for $1 \le i \le 8$, $f_8(v_1) = 14$, $f_8(v_2) = 10$, $f_8(v_3) = 11$, $f_8(v_4) = 9$, $f_8(v_5) = 12$, $f_8(v_6) = 17$, $f_8(v_7) = 16$, $f_8(v_8) = 15$. The graph $G_8$ is shown in Figure 3.35 where the labeling $f_8$ of $G_8$ is also shown. Observe that

- $\star$ the vertex $x$ is only dominated by the vertex $u_2$ labeled 1,
- $\star$ the vertex $y_1$ is only dominated by the vertex $u_4$ labeled 2,
- $\star$ the vertex $y_2$ is only dominated by the vertex $u_2$ labeled 3,
- $\star$ the vertex $u_3$ is only dominated by the vertex $u_7$ labeled 4,
- $\star$ the vertex $y_5$ is only dominated by the vertex $u_3$ labeled 5,
- $\star$ the vertex $y_3$ is only dominated by the vertex $u_6$ labeled 6,
- $\star$ the vertex $y_4$ is only dominated by the vertex $u_8$ labeled 7,
- $\star$ the vertex $y_6$ is only dominated by the vertex $u_1$ labeled 8,
- $\star$ the vertex $z_1$ is only dominated by the vertex $v_4$ labeled 9,
- $\star$ the vertex $z_2$ is only dominated by the vertex $v_2$ labeled 10,
- $\star$ the vertex $z_4$ is only dominated by the vertex $v_3$ labeled 11,

⋆ the vertex $z_3$ is only dominated by the vertex $v_5$ labeled 12, and

⋆ the vertex $z_i$ is only dominated by the vertex labeled $i+9$ for $5 \leq i \leq 8$.

Furthermore, every vertex of $G_8$ is dominated by at least one vertex in $S_8$. In particular, the vertex $v_4$ is dominated by $u_2$ labeled 3 and if $i \neq 4$, then $v_i$ is dominated by $u_4$ labeled 2. Therefore, $S_8$ is a minimal irregular dominating set of $G_8$ and $G_8[S_8] \cong H_8 = C_8 \,\square\, K_2$.

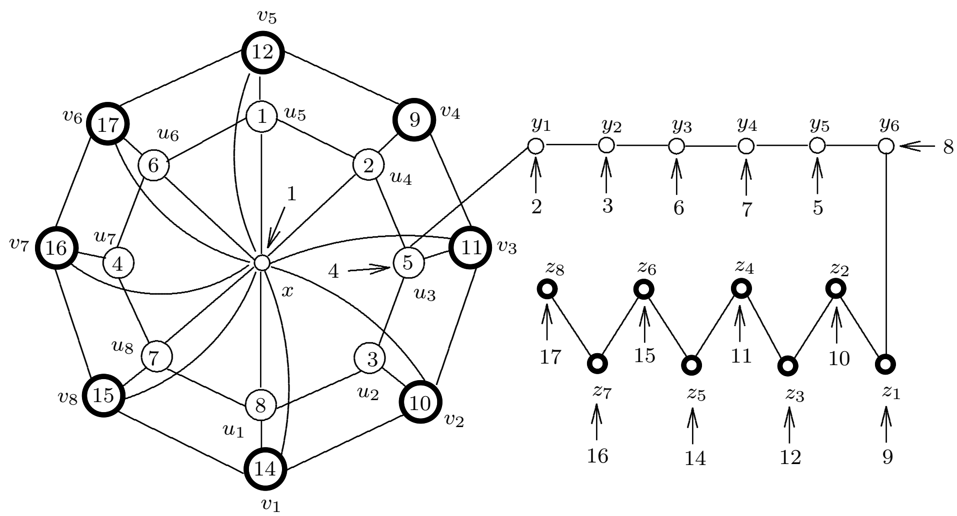



Figure 3.35: The graph $G_8$ the proof of Theorem 3.6.3

Finally, suppose that $n \geq 9$. Let $F_n$ be the graph (constructed in the proof of Theorem 3.5.4) with

$$V(F_n) = \{u_1, u_2, \ldots, u_n\} \cup \{w_1, w_2, x, y_1, y_2, \ldots, y_{n-2}\},$$

where $C_n = (u_1, u_2, \ldots, u_n, u_1)$, $P_{n-2} = (y_1, y_2, \ldots, y_{n-2})$, and

$$E(F_n) = E(C_n) \cup E(P_{n-2}) \cup \{u_5w_1, u_7w_1, w_1w_2, w_2x, w_2u_2, u_3y_1\} \cup \{xu_i : i \in [n] - \{2, 3, 7\}\}.$$

Let $G_n$ be the graph obtained from $F_n$ by (1) adding the $n$-cycle $(v_1, v_2, \ldots, v_n, v_1)$ and joining $v_i$ to $u_i$ for $1 \leq i \leq n$, (2) joining $x$ to $v_i$ for each $i \in [n]-\{4\}$ (and so $xv_4 \notin E(G_n)$), and (3) adding the $n$-path $(y_{n-1}, y_n, y_{n+1}, \ldots, y_{2n-2})$ and joining $y_{n-1}$ to $y_{n-2}$. Then $\text{diam}(G_n) = d(u_7, z_n) = 2n+1$. Let $S_n = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$.

Define a labeling $f_n : S_n \to [2n{+}1]$ by $f_n(u_1) = 8$, $f_n(u_2) = 3$, $f_n(u_3) = 5$, $f_n(u_4) = 2$, $f_n(u_5) = 1$, $f_n(u_6) = 6$, $f_n(u_7) = 4$, $f_n(u_8) = 7$, $f_n(u_9) = 10$, and $f_n(u_i) = 4 + i$ for $10 \leq i \leq n$, and $f_n(v_1) = n + 5$, $f_n(v_2) = 11$, $f_n(v_3) = 9$, $f_n(v_4) = 12$, and $f_n(v_i) = n+1+i$ for $5 \leq i \leq n$. Thus, the set of labels used by $f_n$ is $[2n+1] - \{13\}$. The graph $G_{10}$ is shown in Figure 3.36 where the labeling $f_{10}$ is also shown and the set of labels used by $f_{10}$ is $[21] - \{13\}$.

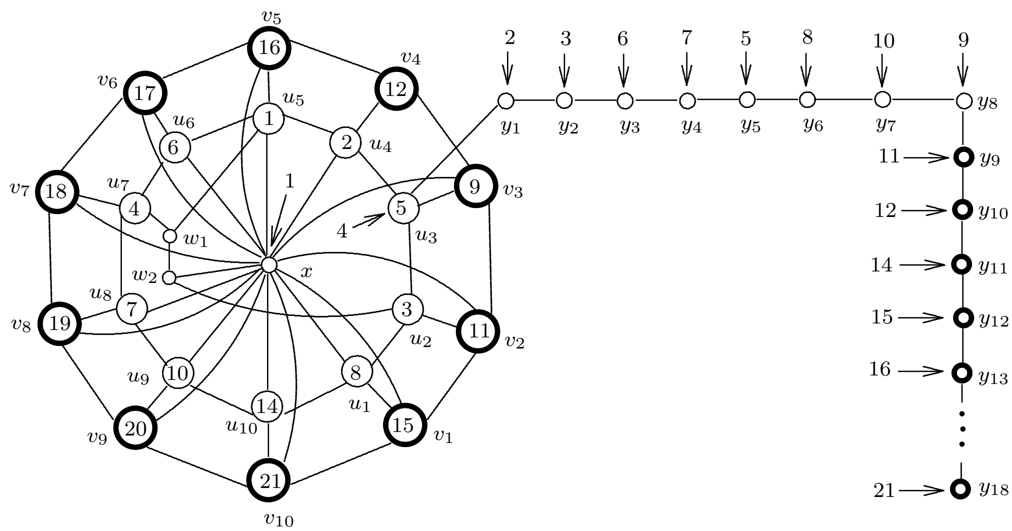


Figure 3.36: The graph $G_{10}$ the proof of Theorem 3.6.3

Observe that

- ⋆ the vertex $x$ is only dominated by the vertex $u_2$ labeled 1,
- ⋆ the vertex $y_1$ is only dominated by the vertex $u_4$ labeled 2,
- ⋆ the vertex $y_2$ is only dominated by the vertex $u_2$ labeled 3,
- ⋆ the vertex $u_3$ is only dominated by the vertex $u_7$ labeled 4,
- ⋆ the vertex $y_5$ is only dominated by the vertex $u_3$ labeled 5,
- ⋆ the vertex $y_3$ is only dominated by the vertex $u_6$ labeled 6,
- ⋆ the vertex $y_4$ is only dominated by the vertex $u_8$ labeled 7,
- ⋆ the vertex $y_6$ is only dominated by the vertex $u_1$ labeled 8,

- ⋆ the vertex $y_8$ is only dominated by the vertex $v_4$ labeled 9,
- ⋆ the vertex $y_7$ is only dominated by the vertex $v_2$ labeled 10,
- ⋆ the vertex $y_9$ is only dominated by the vertex $v_3$ labeled 11,
- ⋆ the vertex $y_{10}$ is only dominated by the vertex $v_5$ labeled 12, and
- ⋆ the vertex $y_i$ is only dominated by the vertex labeled $3+i$ for $11 \leq i \leq 2n-2$.

As another example, the graph $G_{12}$ is shown in Figure 3.37.

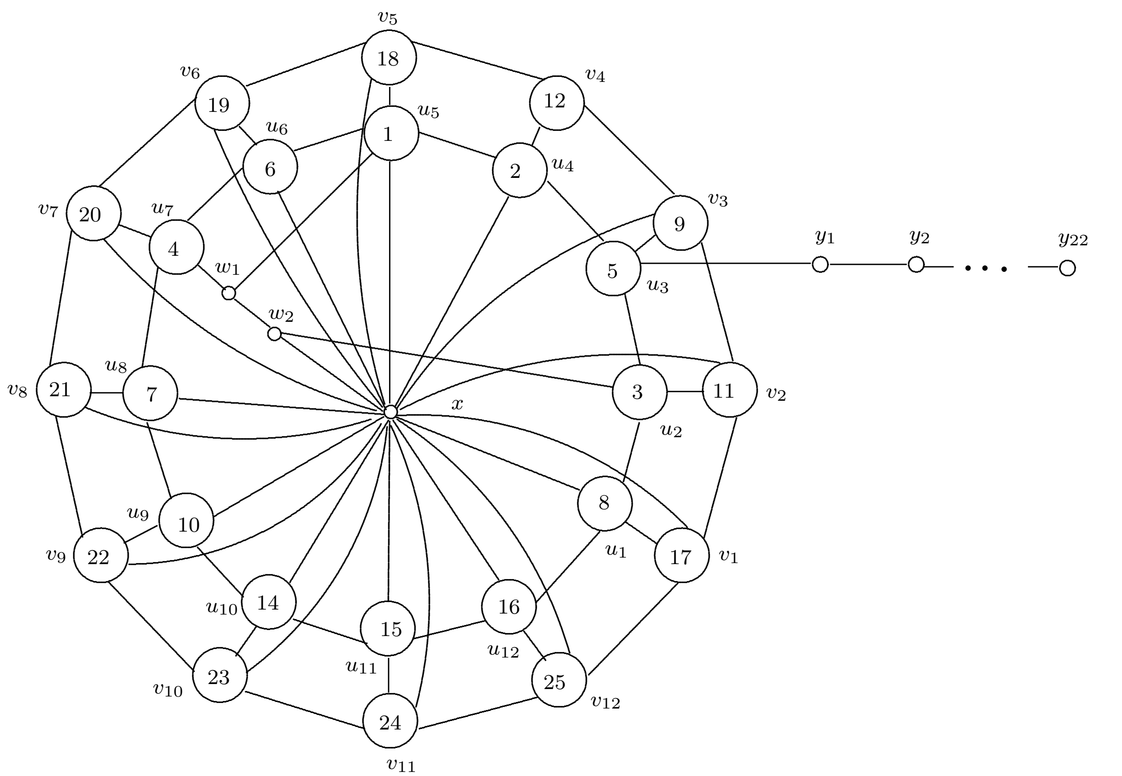



Figure 3.37: The graph $G_{12}$ in the proof of Theorem 3.6.3

Furthermore, every vertex of $G_n$ is dominated by at least one vertex in $S_n$. In particular, the vertex $v_4$ is dominated by $u_2$ labeled 3 and if $i \neq 4$, then $v_i$ is dominated by $u_4$ labeled 2. Therefore, $S_n$ is a minimal irregular dominating set of $G_n$ and $G_n[S_n] \cong H_n = C_n \,\square\, K_2$. ■

The following is a consequence of Proposition 3.6.2 and Theorem 3.6.3.

**Corollary 3.6.4** *The prism $C_n \square K_2$ is an irregular domination graph if and only of $n \geq 6$.*

# Chapter 4

# Irregular Domination Forests and Disconnected Graphs

## 4.1 Introduction

A graph $T$ is a *tree* if $T$ is connected and contains no cycles. A graph $F$ is a *forest* if each component of $F$ is a tree. We saw in Corollary 3.5.2 that every path of diameter at least 3 is an irregular domination graph. It is our goal here to first determine all irregular domination trees and forests and then all irregular domination disconnected graphs.

## 4.2 Forests

By Theorem 3.3.4, If $H$ is a graph of order 3 or more, then $H + K_1$ is an irregular domination graph. Consequently, the following is a consequence of Theorem 3.3.4.

**Corollary 4.2.1** *If $F$ is a forest of order 4 or more having an isolated vertex, then $F$ is an irregular domination graph.*

The following result gives another sufficient condition for a forest to be an irregular domination graph.

**Theorem 4.2.2** *If $F$ is a forest containing a component of diameter 3 or more, then $F$ is an irregular domination graph.*

**Proof.** Let $F$ be a forest of order $n \geq 4$ containing a component of diameter 3 or more. Then $F$ contains a path $P = (u_1, u_2, u_3, u_4)$, where $u_1$ is an end-vertex of $F$. We consider two cases depending on whether $\deg_F u_2 = 2$ or $\deg_F u_2 \geq 3$.

*Case* 1. $\deg_F u_2 = 2$ . Let $X = V(F) - V(P)$. Let $G$ be the graph obtained from $F$ by adding (a) a path $(y_1, y_2, \ldots, y_{n-1})$ of order $n-1$ and joining $y_1$ to $u_2$ and (b) a vertex $z$ and joining $z$ to every vertex in $\{u_2\} \cup X$. The graph $G$ is shown in Figure 4.1. Then $d_G(x, u_2) = 2$ and the diameter of $G$ is $\text{diam}(G) = d(x, y_{n-1}) = 2 + (n-1) = n+1$ for each $x \in X$.

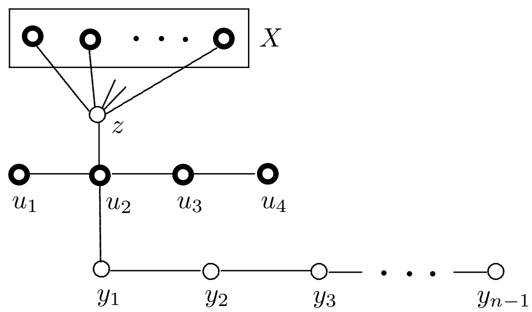



Figure 4.1: The graph $G$ in the proof of Case 1 of Theorem 4.2.2

Let $S = V(F)$. We define a labeling $f : S \to [n+1]$ of $G$ as follows:

- ⋆ $f(u_1) = 3$, $f(u_2) = 1$, $f(u_3) = 4$, $f(u_4) = 2$, and
- ⋆ the $n-4$ labels in the set $[6, n+1]$ are assigned arbitrarily to the $n-4$ vertices in $X$.

Hence, the set of labels assigned to the vertices of $S$ by $f$ is $[n+1] - \{5\}$. This is shown in Figure 4.2 for $n = 8$, where the vertices $u_1$ and $u_4$ may be adjacent to some vertices of $X$. It remains to show that $S$ is a minimal irregular dominating set $S$. Since

(1) each vertex in $V(F)$ is dominated by a vertex labeled $i$ for some $i \in [3]$ and $u_2$ is only dominated by the vertex labeled 2,

(2) the vertex $z$ is only dominated by the vertex $u_2$ labeled 1,

(3) the vertex $y_1$ is only dominated by the vertex labeled 1, the vertex $y_2$ is only dominated by the vertex labeled 3, the vertex $y_3$ is only dominated by the vertex labeled 4, and

(4) for $4 \le i \le n-1$, the vertex $y_i$ is only dominated by the vertex labeled $i+2$,

it follows that $S$ is a minimal irregular dominating set of $G$ and $G[S] \cong F$.

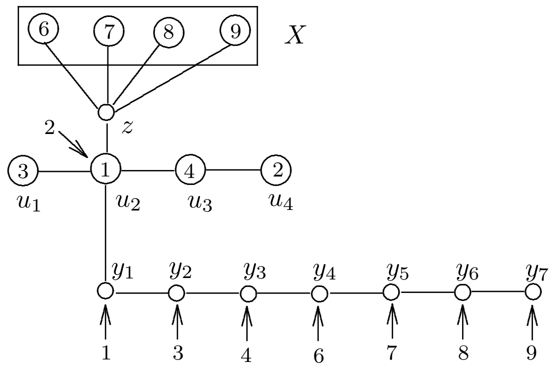



Figure 4.2: The graph $G$ in the proof of Case 1 for $n = 8$

*Case* 2. $\deg_F u_2 = t \ge 3$. Let $W = N_F(u_2) - \{u_1, u_3\}$ be the set of those $t-2$ neighbors of $u_2$ that do not belong to $P$ and let $X = V(F) - (N_T(u_2) \cup \{u_2, u_4\})$. Then $|X| = n - (t+2) = n - t - 2$. The graph $G$ is obtained from $F$ by adding (a) a path $(y_1, y_2, \ldots, y_{n-1})$ of order $n-1$ and joining $y_1$ to $u_2$ and (b) a vertex $z$ and joining $z$ to every vertex in $\{u_2\} \cup X$. The graph $G$ is shown in Figure 4.3. Thus, $d_G(x, u_2) = 2$ and $\operatorname{diam}(G) = d(x, y_{n-1}) = 2 + (n-1) = n+1$ each $x \in X$.

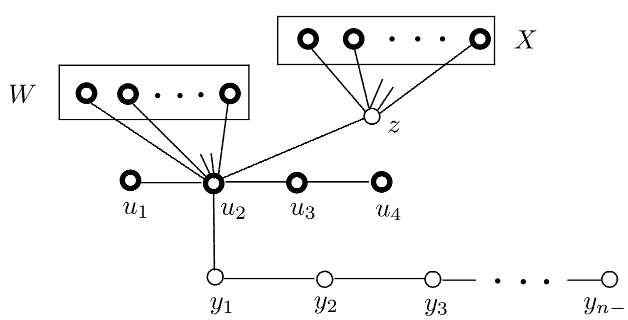



Figure 4.3: The graph $G$ in the proof of Case 2 of Theorem 4.2.2

Let $S = V(F)$. We define a labeling $f : S \to [n+1]$ of $G$ as follows:

- ⋆ $f(u_1) = 3,\ f(u_2) = 1,\ f(u_3) = 4,\ f(u_4) = 2,$
- ⋆ the $t-2$ labels in the set $[5, t+2]$ are assigned arbitrarily to the $t-2$ vertices in $W = N_T(u_2) - \{u_1, u_3\}$, and
- ⋆ the $n-t-2$ labels in the set $[t+4, n+1]$ are assigned arbitrarily to the $n-t-2$ vertices in $X$.

Hence, the set of labels assigned to the vertices of $S$ by $f$ is $[n+1] - \{t+3\}$. This is shown in Figure 4.4 for $n = 11$ and $t = 5$.

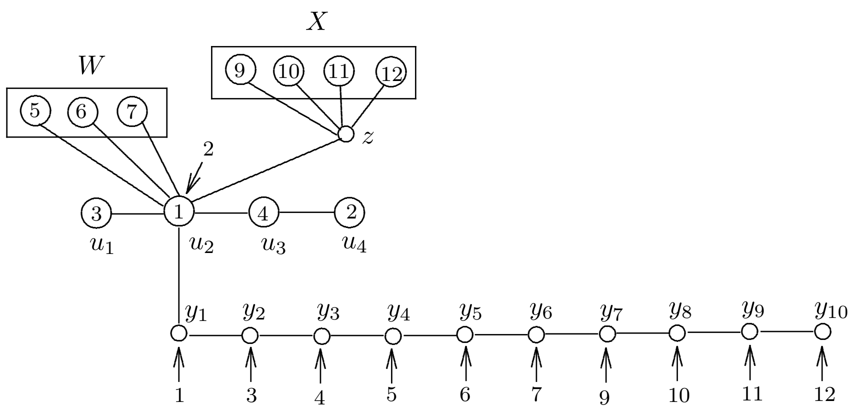



Figure 4.4: The graph $G$ in Case 2 for $n = 11$ and $t = 5$

It remains to show that $S$ is a minimal irregular dominating set $S$. Since

(1) each vertex in $V(F)$ is dominated by a vertex labeled $i$ for some $i \in [3]$ and $u_2$ is only dominated by the vertex labeled 2,

(2) the vertex $z$ is only dominated by the vertex $u_2$ labeled 1,

(3) the vertex $y_1$ is only dominated by the vertex labeled 1, the vertex $y_2$ is only dominated by the vertex labeled 3, the vertex $y_3$ is only dominated by the vertex labeled 4,

(4) for $4 \le i \le t+1$, the vertex $y_i$ is only dominated by the vertex labeled $i+1$, and

(5) for $t+2 \le i \le n-1$, the vertex $y_i$ is only dominated by the vertex labeled $i+2$,

it follows that $S$ is a minimal irregular dominating set of $G$ and $G[S] \cong F$. ■

Since no tree of diameter 1 or 2 is an irregular domination graph, the following result is a consequence of Theorem 4.2.2.

**Corollary 4.2.3** *A tree $T$ is an irregular domination graph if and only if* $\mathrm{diam}(T) \geq 3$.

By Theorem 4.2.2, only those forests having at least two components, each of which is either $K_2$ or a star, remain to be considered. If every component of a disconnected forest $F$ is $K_2$, then $F$ is an irregular domination graph, as we show next.

**Theorem 4.2.4** *If $F = kK_2$ for some integer $k \geq 2$, then $F$ is an irregular domination graph.*

**Proof.** For an integer $k \geq 2$, let $E(kK_2) = \{u_iv_i : 1 \leq i \leq k\}$ be the set of the $k$ edges of $2K_2$. We show that there exists a graph $G_k$ with a minimal dominating set $S_k$ such that $G_k[S_k] \cong kK_2$ (although $2K_2$ is an irregular domination graph by Proposition 3.3.6). For $k = 2$, let $G_2$ be the graph of diameter 5 shown in Figure 4.5. Let $S_2 = \{u_1, u_2, v_1, v_2\}$ where the corresponding labeling $f_2 : S_2 \to \{1, 2, 3, 4\}$ is also shown in Figure 4.5.

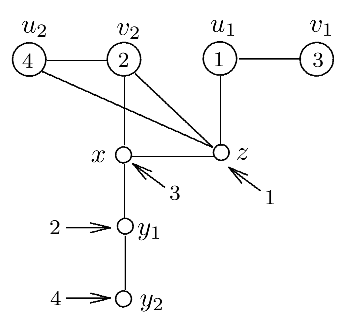



Figure 4.5: The graph $G_2$ in the proof of Theorem 4.2.4

Since (1) each vertex in $V(2K_2)$ is dominated by a vertex labeled $i$ for some $i \in [3]$, (2) the vertex $z$ is only dominated by the vertex labeled 1, the vertex $x$ is only dominated by the vertex labeled 3 and the vertex $y_1$ is only dominated by the vertex labeled 2, and the vertex $y_2$ is only dominated by the vertex labeled 4, it follows that $S_2$ is a minimal dominating set of $G_2$ and $G_2[S_2] \cong 2K_2$. Thus, $2K_2$ is an irregular domination graph.

For $k \geq 3$, let $H_i$ be a triangle with vertex set $\{u_i, v_i, w_i\}$ for $3 \leq i \leq k$. We construct a graph $G_k$ from the graph $G_2$ and the triangles $H_i$ $(3 \leq i \leq k)$ by (a) joining the vertex $w_i$ of $H_i$ to the vertex $x$ of $G_2$ for $3 \leq i \leq k$ and (b) adding a path $(y_3, y_4, \cdots, y_{2k-2})$ of order $2k-4$ and joining $y_3$ to $y_2$. Thus, $\mathrm{diam}(G_k) = d_G(v_1, y_{2k-2}) = 2k+1$. Let $S_k = V(kK_2)$ and $W = \{u_i, v_i : 3 \leq i \leq k\}$. We define a labeling $f_k : S_k \to [2k]$ as follows:

- $\star$ $f_k(v) = f_2(v)$ if $v \in V(G_2)$ and
- $\star$ the $2k-4$ labels in the set $[5, 2k]$ are assigned arbitrarily to the $2k-4$ vertices in $W$.

This is shown in Figure 4.6 for $k = 4$.

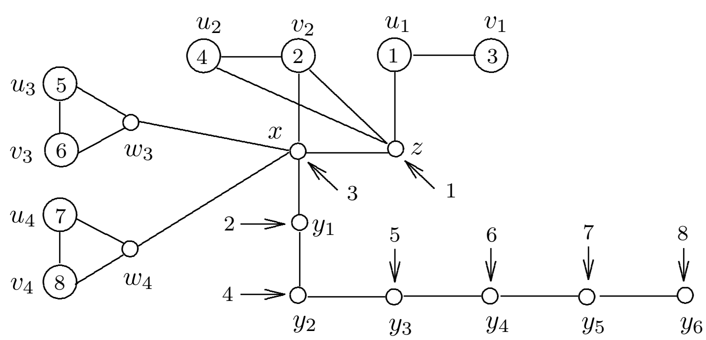



Figure 4.6: The graph $G_4$ in the proof of Theorem 4.2.4

Hence, the set of labels assigned to the vertices of $S_k$ by $f_k$ is $[2k]$. It remains to show that $S_k$ is a minimal dominating set of $G_k$. Since

(1) each vertex in $V(kK_2)$ is dominated by a vertex labeled $i$ for some $i \in [4]$,

(2) the vertex $z$ is only dominated by the vertex labeled 1, the vertex $x$ is only dominated by the vertex labeled 3 and the vertex $y_1$ is only dominated by the vertex labeled 2, and

(3) for $2 \leq i \leq 2k-2$, the vertex $y_i$ is only dominated by the vertex labeled $i+2$,

it follows that $S_k$ is a minimal irregular dominating set of $G_k$ and $G_k[S_k] \cong 2K_2$. ■

By Theorems 4.2.2 and 4.2.4, only one situation remains, namely when all components of a disconnected forest are stars or $K_2$ and at least one component is a star.

**Theorem 4.2.5** *Let $F = T_1 + T_2 + \cdots + T_k$ be a forest with $k \geq 2$ components $T_i$ where $1 \leq i \leq k$, each of which is either $K_2$ or a star. If $F$ contains at least one star, then $F$ is an irregular domination graph.*

**Proof.** Let $F = T_1 + T_2 + \cdots + T_k$ be a forest of order $n$, where $T_1$ is a star and each tree $T_i$, $2 \leq i \leq k$, is either $K_2$ or a star and $u_i \in V(T_i)$ such that $\deg_{T_i} u_i = \Delta(T_i)$ for $1 \leq i \leq k$. Thus, if $T_i$ is a star, then $u_i$ is the center of $T_i$ for $1 \leq i \leq k$. We consider two cases, depending on $k = 2$ or $k \geq 3$.

*Case* 1. $k = 2$. Let $G$ be obtained from $F$ by adding a path $(y_1, y_2, \ldots, y_{n-1})$ of order $n - 1$ and joining $y_1$ to $u_1$ and $u_2$. Let $v_{1,1}$ and $v_{1,2}$ be two neighbors of $u_1$ in $T_1$ and let $v_{2,1}$ be a neighbor of $u_2$ in $T_2$. Then the diameter of $G$ is $\text{diam}(G) = d_G(v_{1,1}, y_{n-1}) = n$. Let $X = V(F) - \{u_1, u_2, v_{1,1}, v_{1,2}, v_{2,1}\}$. Then $|X| = n - 5$. Let $S = V(F)$. We define a labeling $f : S \to [n]$ as follows:

- ⋆ $f(u_1) = 1$, $f(u_2) = 2$, $f(v_{1,1}) = 3$, $f(v_{1,2}) = 4$, $f(v_{2,1}) = 5$ and
- ⋆ the $n - 5$ labels in $[6, n]$ are assigned arbitrarily to the $n - 5$ vertices of $X$.

Hence, the set of labels assigned to the vertices of $S$ by $f$ is $[n]$. This is shown in Figure 4.7 where $T_1 = K_{1,3}$ and $T_2 = K_2$.

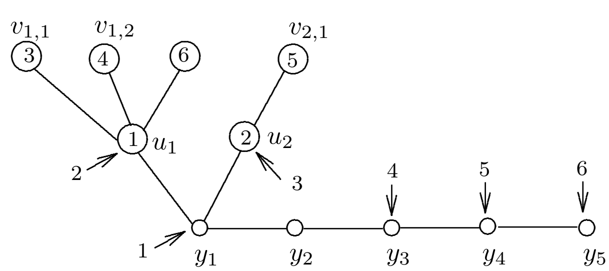



Figure 4.7: The graph $G$ in the proof of Case 1 of Theorem 4.2.5

It remains to show that $S$ is a minimal irregular dominating set of $G$. Since

(1) each vertex in $V(F)$ is dominated by a vertex labeled $i$ for some $i \in [4]$, the vertex $u_1$ is only dominated by the vertex labeled 2, and the vertex $u_2$ is only dominated by the vertex labeled 3,

(2) the vertex $y_1$ is only dominated by the vertex labeled 1 and the vertex $y_2$ is dominated by the vertex labeled 2 or 3, and

(3) for $3 \leq i \leq n-1$, the vertex $y_i$ is only dominated by the vertex labeled $i+1$.

it follows that $S$ is a minimal irregular dominating set of $G$ and $G[S] \cong F$.

*Case* 2. $k \geq 3$. Let $G$ be obtained from $F$ by adding (a) a vertex $z$ and joining $z$ to every vertex in $\{u_3, u_4, \ldots, u_k\}$ and (b) a path $(y_1, y_2, \ldots, y_n)$ of order $n$ and joining $y_1$ to each vertex in $\{z, u_1, u_2\}$. Let $v_{1,1}$ and $v_{1,2}$ be two neighbors of $u_1$ in $T_1$ and let $v_{2,1}$ be a neighbor of $u_2$ in $T_2$. Then the diameter of $G$ is $\operatorname{diam}(G) = d_G(v_{3,1}, y_n) = n+1$, where $v_{3,1}$ is a neighbor of $u_3$ in $T_3$. Let

$$\begin{aligned} W &= V(F) - (V(T_1) \cup V(T_2) \cup \{u_3, u_4, \ldots, u_k\}) \text{ and} \\ X &= V(T_1) \cup V(T_2) - \{u_1, u_2, v_{1,1}, v_{1,2}, v_{2,1}\}. \end{aligned}$$

Let $p = |V(T_1) \cup V(T_2)|$. Then $|W| = p-5$ and $|X| = n-p-(k-2) = n-p-k+2$. Let $S = V(F)$. We define a labeling $f : S \to [n+1]$ as follows:

- ⋆ $f(u_1) = 1$, $f(u_2) = 2$, $f(v_{1,1}) = 3$, $f(v_{1,2}) = 4$, $f(v_{2,1}) = 5$,
- ⋆ the $p-5$ labels in $[6, p]$ are assigned arbitrarily to the $p-5$ vertices of $X$,
- ⋆ the $k-2$ labels in $[p+1, p+(k-2)]$ are assigned arbitrarily to the $k-2$ vertices of $\{u_3, u_4, \ldots, u_k\}$, and
- ⋆ the $n-p-k+2$ labels in $[p+k, n+1]$ are assigned arbitrarily to the $n-p-k+2$ vertices of $W$.

Hence, the set of labels assigned to the vertices of $S$ by $f$ is $[n+1] - \{p+k-1\}$. This is shown in Figure 4.8 where $T_1 = K_{1,3}$, $T_2 = K_2$, $T_3 = K_{1,2}$, and $T_4 = K_{1,3}$. Thus, $k = 4$, $p = 6$, and $n = 13$.

It remains to show that $S$ is a minimal irregular dominating set of $G$. Since

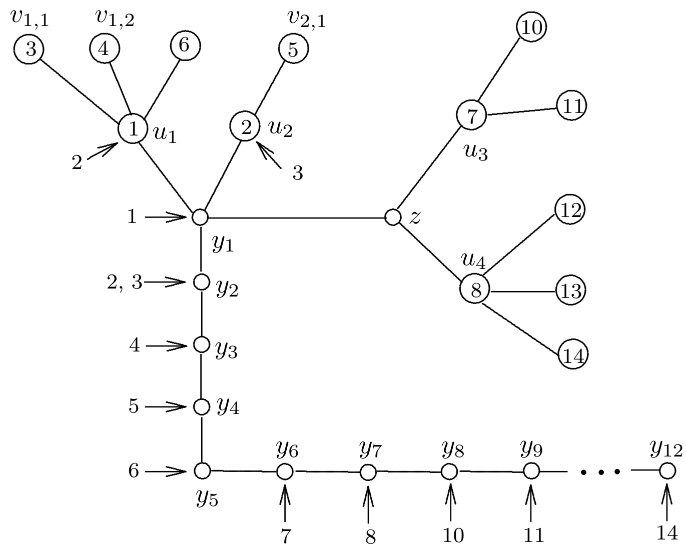



Figure 4.8: The graph $G$ in the proof of Case 2 of Theorem 4.2.5

(1) each vertex in $V(F)$ is dominated by a vertex labeled $i$ for some $i \in [4]$ and $u_2$ is only dominated by the vertex labeled 2,

(2) the vertex $z$ is only dominated by the vertex $u_2$ labeled 1,

(3) the vertex $y_1$ is only dominated by the vertex labeled 1, the vertex $y_2$ is only dominated by the vertex labeled 3, the vertex $y_3$ is only dominated by the vertex labeled 4,

(4) for $4 \leq i \leq t+1$, the vertex $y_i$ is only dominated by the vertex labeled $i+1$, and

(5) for $t+2 \leq i \leq n-1$, the vertex $y_i$ is only dominated by the vertex labeled $i+2$,

it follows that $S$ is a minimal irregular dominating set of $G$ and $G[S] \cong F$. ■

Observe that the graph constructed in each case of the proof of Theorems 4.2.5 is a tree. Thus, if $F$ is a disconnected forest in which each component is either $K_2$ or a star and at least one component is a star, then there is a tree $T$ with a minimal irregular dominating set $S$ such that $T[S] \cong F$. We are now able to characterize all forests that are irregular domination graphs.

**Corollary 4.2.6** *A forest $F$ is an irregular domination graph if and only if*

(1) *$F$ is a tree of diameter 3 or more,*

(2) $F \cong K_1 + K_2$ *or*

(3) *$F$ is disconnected of order 4 or more.*

## 4.3 Disconnected Graphs

As a consequence of Theorem 3.3.4, Corollary 4.2.3, and arguments used in the proofs of Theorems 4.2.2, 4.2.4, and 4.2.5, every disconnected graph of order 4 or more, in which at least one component is a tree, is an irregular domination graph. In fact, more can be said about disconnected irregular domination graphs in general.

**Theorem 4.3.1** *A disconnected graph in which at least one component has order 3 or more is an irregular domination graph.*

**Proof.** Let $H = H_1 + H_2 + \cdots + H_k$ be a disconnected graph of order $n$ consisting of $k \geq 2$ components $H_1, H_2, \ldots, H_k$, where $H_1$ has order 3 or more. Let $u \in V(H_1)$ such that $\deg_{H_1} u \geq 2$ and let $w \in V(H_2)$. Next, let $W = V(H) - V(H_1)$ and let $X = V(H_1) - N[u]$, where possibly $X = \emptyset$. A graph $G$ is constructed from $H$ by adding

(a) a vertex $z$ and joining $z$ to each vertex in $W$ and

(b) a path $(z_1, z_2, y_1, y_2, \ldots, y_{n-2})$ of order $n$ and joining $z_1$ to each vertex in $\{u, w\} \cup X$ and joining $z_2$ to $u$ and $z$.

The graph $G$ is shown in Figure 4.9, where any edge joining a vertex of $N(u)$ and a vertex of $X$ is not drawn as well as any edge joining vertices in $X$, $N(u)$, or $W$. The diameter of $G$ is $\text{diam}(G) = d_G(w, y_{n-2}) = n$ for each $w \in W$.

Let $u_1$ and $u_2$ be two neighbors of $u$. We define a labeling $f : V(H) \to [n]$ of $G$ by

$$f(u) = 1,\ f(w) = 2,\ f(u_1) = 3,\ f(u_2) = 4.$$

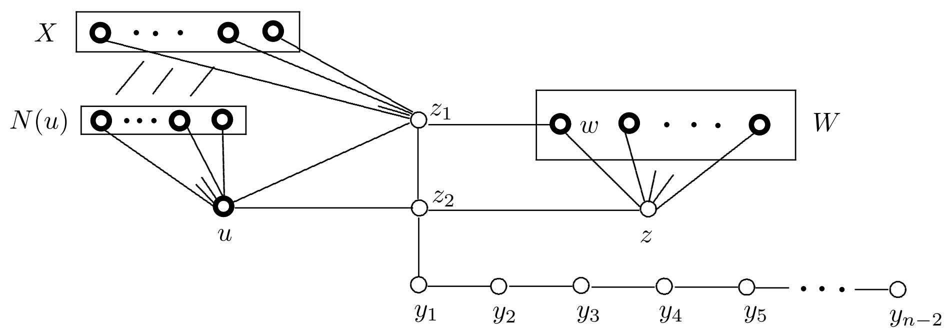



Figure 4.9: The graph $G$ in the proof of Theorem 4.3.1

The $n-4$ labels in the set $[5,n]$ are assigned arbitrarily to the $n-4$ vertices in the set $V(H)-\{u,u_1,u_2,w\}$. Thus, the set of labels assigned to the vertices of $V(H)$ by $f$ is $[n]$. The graph $G$ is shown in Figure 4.10 for a graph $H$ of order $n=10$, where any edge joining a vertex of $N(u)$ and a vertex of $X$ is not drawn as well as any edge joining vertices in $X$, $N(u)$, or $W$.

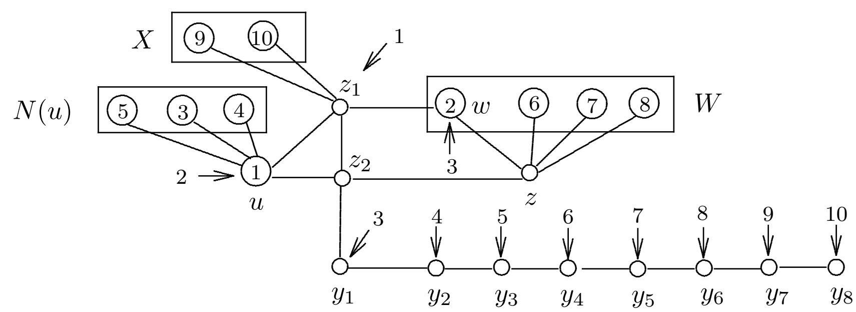



Figure 4.10: The graph $G$ in the proof of Theorem 4.3.1

It remains to show that $S$ is a minimal irregular dominating set of $G$. Since

(1) each vertex in $V(H)$ is dominated by a vertex labeled $i$ for some $i \in [4]$, each vertex in $X$ is dominated by the vertex labeled 2, the vertex $u$ is only dominated by the vertex labeled 2, and the vertex $w$ is only dominated by the vertex labeled 3,

(2) the vertex $z_1$ is only dominated by the vertex labeled 1 and the vertex $z_2$ is dominated by the vertex labeled 1, and

(3) for $1 \leq i \leq n-2$, the vertex $y_i$ is only dominated by the vertex labeled $i+2$,

it follows that $S$ is a minimal irregular dominating set of $G$ and $G[S] \cong H$. ■

Observe that Theorem 4.2.5 is, in fact, a corollary of Theorem 4.3.1. The following is a consequence of Theorems 3.3.4, 4.2.4, and 4.3.1.

**Corollary 4.3.2** *Every disconnected graph of order* 4 *or more is an irregular domination graph.*

**Proof.** Let $G$ be a disconnected graph of order 4 or more. If $G$ contains an isolated vertex, then $G$ is an irregular domination graph by Theorem 3.3.4. Thus, we may assume that every component of $G$ has order 2 or more. If every component of $G$ is $K_2$, then $G$ is an irregular domination graph by Theorem 4.2.4. If at least component of $G$ has order 3 or more, then $G$ is an irregular domination graph by Theorem 4.3.1. ■

Since (a) there is no irregular domination graph of order 2, (b) the graph $K_2 + K_1$ is the only irregular domination graph of order 3 by Proposition 3.3.2, and (c) every disconnected graph of order 4 or more is an irregular domination graph by Corollary 4.3.2, we are now able to characterize all disconnected graphs that are irregular domination graphs.

**Theorem 4.3.3** *A disconnected graph $G$ is an irregular domination graph if and only if $G$ is neither $2K_1$ nor $3K_1$.*

## 4.4 Closing Comments

By Proposition 3.3.3, if $G$ is a connected graph with $\text{diam}(G) \leq 2$, then $G$ is not an irregular domination graph. By Theorem 3.5.3, there is an infinite class of graphs of diameter 3 that are not irregular domination graphs. By Corollary 4.2.3, every tree of diameter 3 (a double star) is an irregular domination graph and so there is an infinite class of connected graphs of diameter 3 that are irregular domination graphs. In fact, more can be said. The *eccentricity* $e(v)$ of a vertex $v$ of a connected graph $G$ is the distance between $v$ and a vertex farthest from $v$ in $G$. If $e(v) = \text{diam}(G)$, then $v$ is a *peripheral vertex* of $G$. The following is a consequence of the proof of Theorem 4.2.2.

**Corollary 4.4.1** *Let $G$ be a connected graph of diameter 3 or more. If $G$ contains an end-vertex that is a peripheral vertex of $G$, then $G$ is an irregular domination graph.*

By Corollary 4.4.1, if $G$ is a connected graph with $\mathrm{diam}(G) = 3$ having a peripheral vertex of degree 1, then $G$ is an irregular domination graph. This is also true for all connected graphs of diameter 4. We know of no connected graph of diameter 4 or more, however, that is not an irregular domination graph. By Corollary 4.4.1, if $G$ is a connected graph of diameter 4 or more that is not an irregular domination graph, then no end-vertex of $G$ is a peripheral vertex of $G$. We close by stating the following conjecture.

**Conjecture 4.4.2** *Every connected graph of diameter 4 or more is an irregular domination graph.*

# Bibliography


[1] A. Ali, G. Chartrand, and P. Zhang, *Irregularity in Graphs.* Springer, New York (2021).

[2] A. Ali, G. Chartrand, and P. Zhang, On irregular and antiregular domination in graphs. *Electronic J. Math.* **2** (2021), 26–36.

[3] C. Berge, Sur le couplage maximum d'un graphe. *C. R. Acad. Sci. Paris* **247** (1958) 258-259.

[4] P. Broe, G. Chartrand, and P. Zhang, Irregular orbital domination in graphs. *International Journal of Computer Mathematics: Computer Systems Theory.* **7** (2022) 68-79.

[5] P. Broe, G. Chartrand, and P. Zhang, Irregular domination in trees. *Electronic J. Math.* **1** (2021) 89-100.

[6] G. Chartrand and P. Zhang, A chessboard problem and irregular domination. *Bull. Inst. Combin Appl.*. To appear.

[7] G. Chartrand, T. W. Haynes, M. A. Henning, and P. Zhang, *From Domination to Coloring: Stephen Hedetniemi's Graph Theory and Beyond.* Springer, New York (2019).

[8] G. Chartrand, M. A. Henning, and K. Schultz, On orbital domination numbers of graphs. *J. Combin. Math. Combin. Comput.* **37** (2001), 3–26.

[9] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, Total domination in graphs. *Networks* **10**(3) (1977), 211–219.

[10] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs. *Networks* **7** (1977), 247–261.

[11] L. Hayes, K. Schultz, and J. Yates, Universal domination sequences of graphs. *Util. Math.* **54** (1998), 193–209.

[12] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning, *Domination in Graphs: Core Concepts.* Springer, New York (2022).

[13] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, New York (1998).

[14] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York (1998).

[15] M. A. Henning and A. Yeo, *Total Domination in Graphs*, Springer, New York (2013).

[16] C. Mays and P. Zhang, Irregular domination graphs. *Contrib. Math.* **6** (2022), 5–14.

[17] O. Ore, *Theory of Graphs.* Amer. Math. Soc. Colloq. Pub., Providence, RI (1962).

[18] W. W. Rouse Ball, *Mathematical Recreations & Essays*, revision by H.S.M. Coxeter of the original 1892 Edition, Chapter 6, Minimum Pieces Problem. Macmillan, London, 3rd Edition, 1939.