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# On Near-Linear Cellular Automata Over Near Spaces

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Cellular Automata can be considered as examples of massively parallel machines. They are computational mathematical objects consisting of a grid of cells, each of which can exist in a finite number of states. These cells evolve over discrete time steps according to a set of predefined rules based on the states of neighboring cells. The notion of cellular automata was first introduced by Ulam and von Neumann and then popularized by John H. Conway in the 1970s with one of the most famous examples being The Game of Life.

This research builds on and generalizes the work of Tullio Ceccherini-Silberstein and Michel Coornaert, who in 2010 studied cellular automata over vector spaces inspired by the works of Mikhael Gromov regarding the endomorphisms of symbolic algebraic varieties. In this framework, the configuration space $\hom(G, V) = V^G$, where $G$ is a group and $V$ is a vector space over a field $K$, naturally forms a vector space over $K$. Cellular automata are endomorphisms of this vector space, which are continuous and invariant under the natural $G$ action on $V^G$.

In this body of work, we investigate cellular automata whose alphabets are $D^{n\times 1}$, where $D$ is a near-field or an Abraham Adrian Albert (A.A.A) division algebra. Near fields, discovered by Leonard E. Dickson in 1905, generalize both fields and division rings. Hans Zassenhaus noted that finite near fields were "sharply 2-transitive groups". Since then, near fields have been intensively studied in the guise of sharply 2-transitive groups. Specifically, we focus on the structure of near-linear cellular automata, which are cellular automata over groups whose alphabets are near vector spaces, $D^{n\times 1}$ for $D$ a near field or A.A.A division algebra. These cellular automata exhibit near linearity with respect to the induced near space structure on the set of mappings $\hom(G, D^{n\times 1})$.

The main results of this work consist of developing a near-linear analog for the Curtis-Hedlund Theorem, exploring a Garden of Eden-type theorem for near vector spaces, and lastly exploring the correlation between sofic groups and the property of near-linear surjectivity.

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# Chapter 1

# Statement on Main Results

In this dissertation, we generalize and expand upon the works of Tulio Ceccherini-Silberstein and Michel Coornaert. Specifically, we build on the concept of linear cellular automata presented in Chapter 8 of their monograph, *Cellular Automata and Groups* [2]. Within this framework, the configuration space, $\mathrm{Hom}(G, V) = V^G$, where $G$ is a group and $V$ is a vector space over a field $K$, naturally forms a vector space over $K$. Cellular automata are then endomorphisms of this vector space.

The primary focus of this dissertation is to introduce and conceptualize mathematical objects known as $G$-equivariant $D$-endomorphisms of Near Vector Space configuration spaces, where $G$ is any group and $D$ is a near-field or an A.A.A. division algebra. We refer to these objects as *Near-Linear Cellular Automata over Near Spaces.*

The main results of this dissertation can be categorized into three areas. First, we conduct a detailed study of the algebraic properties of near-linear cellular automata over near spaces (Chapter 4). Second, we extend the Garden of Eden Theorem for linear cellular automata to a Garden of Eden Theorem for near-linear cellular automata (Chapter 5). Lastly, we extend a property of all sofic groups within the context of near-linear cellular automata, specifically proving that all sofic groups are near-linearly surjunctive (Chapter 6).

## 1.1 Main Results from Chapter 4

We provide a near-linear analog to the Curtis-Hedlund Theorem [11] , which gives a necessary and sufficient condition for a self-mapping between configuration spaces to be a cellular automaton. In particular, we give a sufficient and necessary condition for a self-mapping between near spaces to be a near-linear cellular automaton.

**Theorem 4.1.4.** Let $G$ be a group and let $V$ be a near space over $D$ where $D$ is a near-field or an A.A.A finite division algebra. Let $\tau : V^G \to V^G$ be a $G$-equivariant and $D$-linear map. Then the following conditions are equivalent:

(a) the map $\tau$ is a near-linear cellular automaton;

(b) the map $\tau$ is uniformly continuous (with respect to the prodiscrete uniform structure on $V^G$);

(c) the map $\tau$ is continuous (with respect to the prodiscrete topology on $V^G$);

(d) the map $\tau$ is continuous (with respect to the prodiscrete topology on $V^G$) at the constant configuration $x = 0$.

The following propositions outline fundamental properties of near-linear cellular automata that are heavily referenced in later chapters.

**Proposition 4.2.4.** Let $G$ be a group and let $V = D^{n\times 1}$ be a near space over $D$ where $D$ is a near-field or an A.A.A finite division algebra. Then, $\mathrm{NLCA}(G, V)$ is a subalgebra of $\mathrm{End}_D(V^G)$.

Note that in the proposition above $\mathrm{NLCA}(G, V)$ refers to the set of all near-linear cellular automata over a group $G$ and near space $V$

**Proposition 4.3.2.** Let $G$ be a group and let $V$ be a near space over $D$ where $D$ is a near-field or an A.A.A finite division algebra. Then $V[G]$, the space of configurations with finite support, is a subspace of $V^G$.

**Proposition 4.3.7.** If $\tau \in \mathrm{NLCA}(G, V)$ then the restriction map $\tau \mid_{V[G]}: V[G] \to V[G]$ is $D$-linear. In other words, $\tau \mid_{V[G]} \in \mathrm{End}_D(V[G])$.

**Proposition 4.3.12.** Let $\tau \in \mathrm{NLCA}(G, V)$. Then the following are equivalent:

(1) $\tau$ is pre-injective.

(2) $\tau|_{V[G]} : V[G] \to V[G]$ is injective.

## 1.2 Main Results from Chapter 5

We provide a near-linear analog to the Garden of Eden Theorem, which gives a sufficient and necessary condition for a cellular automaton over an amenable group to be surjective.

In order to prove such a theorem, we first establish a closed-image property for near-linear cellular automata over near spaces. Specifically,

**Theorem 5.2.1.** Let $G$ be a group and let $V = D^{n\times 1}$ be a finite-dimensional near space over a near-field or A.A.A. division algebra $D$. Then every near-linear cellular automaton $\tau : V^G \to V^G$ has the closed image property with respect to the prodiscrete topology on $V^G$.

Given a set $X$ and a topological space $Y$, one says that a map $f : X \to Y$ has the closed image property if the set $f(X)$ is closed in $Y$ [2]. The closed image property is often used to establish the surjectivity of mappings. In fact, to prove that a map $f : X \to Y$ with the closed image property is surjective, it suffices to show that $f(X)$ is dense in $Y$.

This leads to the following result.

**Theorem 5.3.15.** Let $V = D^{n\times 1}$ be a finite-dimensional near space over a near-field or A.A.A. division algebra $D$ and let $G$ be a countable amenable group. Let $\tau : V^G \to V^G$ be a near-linear cellular automaton. Then the following conditions are equivalent:

(a) $\tau$ is surjective.

(b) $\mathrm{mdim}(\tau(V^G)) = \dim(V)$.

(c) $\tau$ is pre-injective.

## 1.3 Main Results from Chapter 6

Analogous to when cellular automata have finite alphabets, we introduce the following definition.

**Definition 6.2.1.** A group $G$ is said to be *near-linearly surjunctive* (NL-surjunctive) if, for any near-field or A.A.A. division algebra $D$ and any finite-dimensional near space $V = D^{n\times 1}$ over $D$, every injective near-linear cellular automaton $\tau : V^G \to V^G$ is surjective.

We then use results from Chapters 4 and 5 to prove the following.

**Theorem 6.2.6.** Every sofic group is NL-surjunctive.

We note that Theorem 6.2.6 calls upon work established in chapters 4 and 5. In particular the results from theorems 5.2.1 and 5.3.15 are used.

# Chapter 2

# Introduction

In this chapter, we present the notion of a cellular automaton. We fix a group and an arbitrary set, termed the alphabet. A configuration is defined as a mapping from the group to the alphabet, essentially assigning an alphabet element to each group element. The group naturally acts on the configuration set through the shift action. A cellular automaton is a self-mapping of the configuration set, constructed from local rules that commute with the shift. We equip the configuration set with the prodiscrete topology, representing pointwise convergence associated with the discrete topology on the alphabet. It's notable that every cellular automaton is continuous with respect to the prodiscrete topology. Additionally, in the finite alphabet case, any continuous self-mapping of the configuration space that commutes with the shift qualifies as a cellular automaton. Another significant observation in the finite alphabet scenario is that every bijective cellular automaton is invertible; its inverse map also constitutes a cellular automaton.

## 2.1 Cellular Automata over a Group

Cellular automata over groups extend the concept of traditional cellular automata by incorporating group structures into the evolution rules. In this framework, the cells are associated with elements from a group, and the dynamics of the automaton are determined by group operations.

Let $(G, \cdot)$ be a group, also referred to as a *universe*, and let $A$ be a set, also referred to as an *alphabet*.

**Definition 2.1.1.** For a fixed $g \in G$ the mapping

$$L_g : G \times G \to G$$

$$(g, g') \mapsto g \cdot g'$$

for all $g' \in G$, is called the *left multiplication* by $g$ in $G$.

**Remark 2.1.2.** We note that for any $g, g', h \in G$, we have that

$$(L_g \circ L_h)(g') = L_g(L_h(g')) = L_g(h \cdot g') = g \cdot (h \cdot g') = g \cdot h \cdot g' = L_{g \cdot h}(g').$$

Hence, we have that

$$L_g \circ L_h = L_{g \cdot h} \tag{2.1}$$

**Definition 2.1.3.** Let $A$ be a set and let $G$ be a group. Denote by $A^G$, the *space of configurations* which consists of all mappings $x$ from $G$ to $A$. i.e

$$A^G = \{x : G \to A\}.$$

**Remark 2.1.4.** For a given set $A$ and group $G$, we can express $A^G$ as both the Cartesian product of the set $A$ with itself $|G|$ times, denoted as $\prod_{g \in G} A$, and as the set of morphisms from the group $G$ into the set $A$, denoted as $\hom(G, A)$. Hence

$$A^G = \prod_{g \in G} A = \hom(G, A).$$

**Definition 2.1.5.** Let $G$ be a group and $A$ a set. Given $g \in G$ and a mapping (configuration) $x \in A^G$, we define a mapping (configuration) $gx \in A^G$ by the following,

$$gx = x \circ L_{g^{-1}} \tag{2.2}$$

**Remark 2.1.6.** For all $h \in G$, it follows that

$$gx(h) = (x \circ L_{g^{-1}})(h) = x(L_{g^{-1}}(h)) = x(g^{-1} \cdot h).$$

**Theorem 2.1.7.** For $(G, \cdot)$ a group and $A$ a set, the mapping

$$G \times A^G \to A^G$$

$$(g, x) \mapsto gx$$

is a left group action on the space of configurations $A^G$.

*Proof.* For any $g, h \in G$, and $x \in A^G$, it follows that

$$\begin{aligned} (g \cdot h)(x) &= x \circ L_{(g \cdot h)^{-1}} \\ &= x \circ L_{h^{-1} \cdot g^{-1}} \\ &= x \circ L_{h^{-1}} \circ L_{g^{-1}} \quad \text{by (1.1)} \\ &= g(x \circ L_{h^{-1}}) \\ &= g(hx). \end{aligned} \tag{2.3}$$

Moreover, if $1 \in G$ is the identity element of $G$ and $I_G : G \to G$ is the identity mapping, it follows that

$$1x = x \circ L_{1^{-1}} = x \circ L_1 = x \circ I_G = x. \tag{2.4}$$

Hence by (1.3) and (1.4) we conclude that the mapping

$$G \times A^G \to A^G$$

$$(g, x) \mapsto gx$$

is a left group action on the space of configurations $A^G$. ☐

**Definition 2.1.8.** The left action of group $G$ on the set $A^G$ is called the *G-Shift* on $A^G$.

We next impose a topology on the space of configuration $A^G$.

**Definition 2.1.9** ([2])**.** For a group $G$ and a set $A$, the *prodiscrete topology* on the space of configurations $A^G$ is the product topology when each factor A is given the discrete topology.

**Remark 2.1.10** ([2])**.** The prodiscrete topology is the smallest topology on the space of configurations, $A^G$, such that the projection mapping,

$$\pi_g : A^G \to A$$

$$\pi_g(x) = x(g)$$

is continuous for every $g \in G$.

Given a subset $F$ of a group $G$ and a mapping $x \in A^G$, let $x\mid_F \in A^F$ denote the restriction of the mapping $x$ to $F$. In other words, $x\mid_F : F \to A$ defined by $x\mid_F (g) = x(g)$ for all $g \in F$.

**Remark 2.1.11.** If $x \in A^G$, a neighborhood base of $x$ is given be the sets

$$V(x, F) = \{y \in A^G : x\mid_F = y\mid_F\} = \bigcap_{g \in F} C(g, x(g)).$$

Here $F$ runs over all finite subsets of the group $G$ and $C(g,a)$ denotes the elementary cylinders

$$C(g,a) = \pi_g^{-1}(\{a\}) = \{x \in A^G : x(g) = a\} \qquad (g \in G, a \in A).$$

**Theorem 2.1.12** ([2])**.** For a group $G$ and a set $A$, the space $A^G$ is Hausdorff and totally disconnected.

**Theorem 2.1.13** ([2])**.** The action of $G$ on $A^G$ is continuous with respect to the prodisrecete topology imposed on $A^G$.

We next give the formal definition of a Cellular Automata over an arbitrary group (universe) and set (alphabet). It is worth noting that in this setting, the elements of the universe are occasionally referred to as cells and the elements of the alphabet are occasionally referred to as states.

**Definition 2.1.14** ([2])**.** A Cellular Automaton over a group $G$ and set $A$ is a map $\tau : A^G \to A^G$ which satisfies the following property:

there exists a finite subset $M \subset G$ and a mapping $\mu : A^M \to A$ such that

$$\tau(x)(g) = \mu((g^{-1}x)|_M) \tag{2.5}$$

for all $x \in A^G, g \in G$, where $(g^{-1}x)|_M$ denotes the restriction of the configuration $g^{-1}x$ to $M$.

**Remark 2.1.15.** Equation (1.5) above can be expressed, by (1.2), in the following way

$$\tau(x)(g) = \mu((x \circ L_g)\mid_M). \tag{2.6}$$

**Proposition 2.1.16.** For any set $A$ and for a group $G$ and a finite subset $M \subseteq G$. The restriction mapping $\varphi : A^G \to A^M, x \mapsto x\mid_M$ is surjective.

*Proof.* For any mapping $x : M \to A$, we want to find a mapping $f : G \to A$ such that $\varphi(f) = x$. Define $f : G \to A$ as follows:

$$f(y) = \begin{cases} x(y) & \text{if } y \in M \\ a & \text{if } y \in G \setminus M \end{cases}$$

where $a$ is some fixed element in $A$. Since $M$ is finite, this is a well-defined function.

Now, let's verify that $\varphi(f) = x$. For $y \in M$, $\varphi(f)(y) = f(y) = x(y)$ by definition. For $y \in G \setminus M$, $\varphi(f)(y) = f(y) = a$, but this does not affect the restriction on $M$ since $y \notin M$. Therefore, $\varphi(f) = x$, and we have found a function $f : G \to A$ such that $\varphi(f) = x$ for any arbitrary $x : M \to A$. This establishes the surjectivity of the restriction mapping $\varphi : A^G \to A^M$. ☐

**Remark 2.1.17.** The proposition above shows that if $M$ is a memory set for the cellular automaton $\tau$, then there is a unique map $\mu : A^M \to A$ which satisfies equation (1.5) or (1.6). Hence, one can say that there exists a map $\mu$ that is the local defining map of the cellular automata $\tau$ that is associated with the memory set $M$.

We now describe some nontrivial examples of cellular automatons.

**Example 2.1.18.** Let $G$ be any group, $A$ a finite set, and $f : A \to A$ a mapping from $A$ into itself. Then the map

$$\tau : A^G \to A^G$$

$$x \mapsto f \circ x$$

is a cellular automaton with finite memory set $M = \{e\}$ where $e$ denotes the identity element on $G$ and local defining map

$$\mu : A^M \to A$$

$$y \mapsto f(y(e)).$$

To see that this is indeed the case, we verify that equation (1.5) is held. We have that

$$\tau(x)(g) = (f \circ x)(g) = f(x(g))$$

for all $x \in A^G$, $g \in G$. On the other hand, we have that

$$\mu((x \circ L_g)|_e) = \mu(x(ge)) = \mu(x)(g) = f(x(g)).$$

Hence $\tau(x)(g) = \mu((g^{-1}x)|_M)$ for all $x \in A^G, g \in G$ as desired.

The next example is considered the classical example of a cellular automaton, Conways's *Game of Life* [18].

**Example 2.1.19.** Let $G = (\mathbb{Z}^2, +) \cong (\mathbb{Z} \times \mathbb{Z}, +)$ and take $M = \{-1, 0, 1\}^2 \subset G$. Moreover, let the set (alphabet) $A = \{0, 1\}$. Consider a mapping

$$\mu : \{0,1\}^{\{-1,0,1\}^2} \to \{0,1\}$$

$$\mu(y) = \begin{cases} 1 & \text{if } \begin{cases} \sum_{m\in M} y(m) = 3 \\ \text{or} \\ \sum_{m\in M} y(m) = 4 \text{ and } y((0,0)) = 1, \end{cases} \\ 0 & \text{if otherwise} \end{cases}$$

for all $y \in \{0,1\}^{\{-1,0,1\}^2}$. The mapping

$$\tau : A^G \to A^G$$

$$\tau : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$$

with a memory set $M$ and local defining map $\mu$ is Cellular Automaton for the Game of Life.

## 2.2 Fundamental Properties of Cellular Automata Over Groups

In this section we give an overview of some classical results of cellular automata over groups.

**Definition 2.2.1.** Let $\mathrm{CA}(G, A)$ be the set of all cellular automata $\tau : A^G \to A^G$.

**Theorem 2.2.2.** The set $\mathrm{CA}(G, A)$ is a monoid for the composition of maps with the identity mapping $\mathrm{Id}_{A^G}$ be the identity element in the monoid.

The following theorem give us a characterization of $\tau \in \mathrm{CA}(G, A)$ provided that $A$ is finite.

**Theorem 2.2.3. (Curtis-Hedlund theorem [2]).** Let $G$ be a group and let $A$ be a finite set. Let $\tau : A^G \to A^G$ be a mapping and equip $A^G$ with its prodiscrete topology. Then, the following are equivalent.

(a) $\tau$ is cellular automata.

(b) $\tau$ is $G$-equivariant and continuous.

We can see the utility of such a theorem in the following example.

**Example 2.2.4.** Let $G$ be a group and $A$ be a finite set. Let $f : A \to A$ be a mapping from $A$ into itself ($A$ is endowed with the discrete topology, hence $f$ is always continuous). Then, the mapping

$$\tau : A^G \to A^G$$

$$\tau(x) = f \circ x$$

is a cellular automata since $\tau$ is continuous (since $\tau = \pi_{g\circ} f$, where $\pi_g$ and $f$ are continuous) and for all

$h \in G$ we have that

$$\tau(gx)(h) = f \circ (gx)(h) = f(x(g^{-1}h)) = g(f \circ x(h)) = g\tau(x)(h).$$

The finiteness in the Curtis-Hedlund theorem is paramount. Infact, if we suppress the hypothesis that $A$ is finite, then it is possible to construct a mapping from $A^G \to A^G$ which is $G$-equivariant and continuous with respect to the prodisrcete topology, but fails to be a cellular automata.

**Example 2.2.5** ([2])**.** Let $G$ be an arbitary infinite group and let $A$, the alphabet set be equal to $G$. Consider the mapping

$$\tau : A^G \to A^G$$

$$\tau(x)(g) = x(g \cdot x(g))$$

for all $x \in A^G$ and $g \in G$. Then, $\tau$ if $G$-equivariant and continuous with respect to the prodiscrete topology on $A^G$. However, $\tau$ is not a cellular automata.

The following theorem allows for the generalization of the Curtis-Hedlund theorem to an infitie alphabet which utilizes the *prodiscrete uniform structure* on a shift space.

**Definition 2.2.6** ([2])**.** Let $G$ be a group and let $A$ be a set. The *prodiscrete uniform structure* on $A^G$ is the product uniform structure obtained by taking the discrete uniform structure on each factor $A^G$.

**Theorem 2.2.7** ([2])**.** Let $A$ be any set and let $G$ be a group. Let $\tau : A^G \to A^G$ be a mapping and equip $A^G$ with its prodiscrete uniform structure. Then, the following are equivalent.

(a) $\tau$ is cellular automata.

(b) $\tau$ is $G$-equivariant and continuous.

The following theorems address the notion of invertibilty or reversibility of a cellular automata.

**Definition 2.2.8** ([2])**.** Let $\tau \in \mathrm{CA}(G, A)$. $\tau$ is invertible (reversible) ) if $\tau$ is bijective and the inverse map $\tau^{-1} : A^G \to A^G$ is also a cellular automaton. This is equivalent to the existence of a cellular automaton $\sigma : A^G \to A^G$ such that $\tau \circ \sigma = \sigma \circ \tau = \mathrm{Id}_{A^G}$.

**Theorem 2.2.9** ([2])**.** Let $A$ be a set and let $G$ be a group. Let $\tau : A^G \to A^G$ be a map and equip $A^G$ with its prodiscrete uniform structure. Then the following conditions are equivalent:

(a) $\tau$ is an invertible cellular automaton;

(b) $\tau$ is a $G$-equivariant uniform automorphism of $A^G$.

**Remark 2.2.10.** Bijective cellular automata over finite alphabets are always invertible.

**Theorem 2.2.11** ([2]). Let $G$ be a group and let $A$ be a finite set. Then every bijective cellular automaton $\tau : A^G \to A^G$ is invertible.

**Theorem 2.2.12** ([2]). Let $G$ be a group and let $A$ be a set. Let $H$ be a subgroup of $G$ and let $\tau \in CA(G, H; A)$. Let $\tau_H \in CA(H; A)$ denote the cellular automaton obtained by restriction of $\tau$ to $H$. Then the following conditions are equivalent:

(a) $\tau$ is invertible;

(b) $\tau_H$ is invertible.

# Chapter 3

# Near-Fields, A.A.A Divsion Algebras and Near Spaces

In 1905, Dickson introduced in his book "Linear Groups with an Exposition of the Galois Field Theory" [4], the revolutionary concept of near-fields. These emerged from his deep dive into field axioms, defining sets with two operations meeting specific criteria, notably distributivity. It's a well-known fact that every field or skew field is a near-field [10].

In 1937, Hans Zassenhaus embarked on an ambitious quest, classifying all finite near-fields, including those first discovered by Dickson. Along the way, he uncovered seven additional "exceptional" cases [10] and established a profound link between finite near-fields and finite sharply double transitive groups. His student, Kalscheuer added to this work by classifying near-fields over the real numbers in 1940.

Meanwhile, in the 1950's, A. Adrian Albert researched the realm of finite division algebras for finite protective planes, which gave rise to the A.A.A Division Algebras.

Drawing inspiration from the construction of vector spaces over fields, we extend this idea to near-fields and A.A.A division algebras. We define a *near vector space*, or more eloquently, a *near space*, formed by taking the Cartesian product of a near-field or an A.A.A division algebra with itself $n$ times.

## 3.1 Near-Fields

Let $D$ be a set that admits an addition, multiplication, and has distinct additive ,$\{0\}$, and multiplicative, $\{1\}$ identities. $D$ is a *near-field* if $D^{\#} = D \setminus \{0\}$ is a group and $D$ has only 1 distributive law [5].

**Definition 3.1.1** ([5])**.** $(\mathcal{L}, \circ)$ is a loop if and only if $\mathcal{L}$ is a non-empty set and $\circ : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that

(1) The equation $a \circ x = b$ and $y \circ a = b$ have unique solutions in $\mathcal{L}$ for each $a, b \in \mathcal{L}$.

(2) There is a 1 in $\mathcal{L}$ such that $a \circ 1 = a = 1 \circ a$ for all $a \in \mathcal{L}$.

**Definition 3.1.2** ([5])**.** $(G, \circ)$ is a group if and only if

(1) $(G, \circ)$ is a loop

(2) $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$.

**Definition 3.1.3** ([5])**.** $(D, +, \circ)$ is a near field if $(D, +)$ is an abelian group, $(D^{\#}, \circ)$ is a group, and for all $x, y, z \in D$, we have that $x \circ (y + z) = x \circ y + x \circ z$.

$K$ is called the *kernel of* $D$ if $K = K(D)$, where

$$K(D) := \{x \in D \mid (y + z) \circ x = y \circ x + z \circ x \, \forall \, y, z \in D\}.$$

$\mathcal{Z}$ is called the *center of* $D$ if $\mathcal{Z} = \mathcal{Z}(D)$, where

$$\mathcal{Z}(D) := \{x \in D \mid x \circ y = y \circ x \, \forall \, y \in D\}.$$

$Z$ is called the *center of* $K$ if

$$Z := \{x \in K \mid x \circ y = y \circ x \, \forall \, y \in K\}.$$

Furthermore, set $V := D^{n \times 1}$, $U := K^{n \times 1}$, and $T := Z^{n \times 1}$ for some $n \in \mathbb{N}$.

**Remark 3.1.4.** We can differentiate between a *left near-field* and a *right near-field*. The definition in 2.1.4 is that of a left near-field. To obtain a right near-field we reqiure $(D, +)$ to be an abelian group, $(D^{\#}, \circ)$ be a group and for all $x, y, z \in D$ we have that $(y + z) \circ x = y \circ x + z \circ x$.

**Theorem 3.1.5** ([5])**.** If $D$ is a near field, then

a) $\mathcal{Z} \subseteq Z \subseteq K$, $Z$ is a field, and $K$ is an associative division algebra;

b) $D$ is a vector space over $Z$ and is a right vector space over $K$;

c) $V$ is a vector space over $Z$ and is a right vector space over $K$.

**Definition 3.1.6** ([5])**.** Let $D$ be a near-field. Set $V = D^{n \times 1}$ for some $n \in \mathbb{N}$. The list

$$\mathcal{A} = A_1, A_2, \cdots, A_h$$

in $V$ is called a *bases* of $V$ if

(1) $A = \sum x_i A_i$ for $x_i \in D$ and for every $A \in V$.

(2) If $A = 0$, then $x_i = 0$ for $i = 1, \cdots, h$.

In the list above, $h$ is called the *length* of $\mathcal{A}$. The *dimension* of $V$ is the length of a bases of $V$. The $x_i$'s are called the *coordinates of $A$ with respect to $\mathcal{A}$*. $\mathcal{A}$ is called a list of *generators* of $V$ if (1) holds. If (2) holds we call $\mathcal{A}$ a *linearly independent* list. We call a basis of $V$ an *$X$-basis of $V$* if the elements of the list $\mathcal{A}$ are in $X$ where $X$ is a subset of $V$.

**Example 3.1.7.** Let $D$ be near-field. Let $E_i$ be the $n \times 1$ matrix whose $i^{th}$ coordinate is 1 and every other coordinate is 0. Take

$$\mathcal{E} = E_1, \cdots, E_n.$$

Then $\mathcal{E}$ is a $T$-basis of $V$ where $T$ is defined as an $n$-tuple of element of the center of $D$.

We now look at a few examples of near-fields.

**Example 3.1.8.** Let $\mathbb{F}$ denote any field or any skew-field, then $\mathbb{F}$ is a near-field.

**Example 3.1.9.** Consider the polynomial $x^2 + 1$ which is irreducible modulo 3. We may therefore use it to generate the Galois Field of order $3^2$, GF($3^2$). In this GF($3^2$), we denote a solution to the equation $x^2 + 1 = 0$ by $j$. Then, the elements of the field may be denoted by $x + yj$ where $x$ and $y$ are integers in $\mathbb{Z}_3$. We take the same symbols $x + yj$ to denote the elements in $N =$ LED($3^2$) (Named after Leonard E. Dickson). (i.e setwise, LED($3^2$) = GF($3^2$)). To force LED($3^2$) to not be GF($3^2$)) algebraically, we keep the same notion of addition but twist the multiplication in the following way: $x \circ_N y = x \circ y$ if $x$ is square in GF($3^2$), else, $x \circ_N y = x \circ y^3$ for all $x, y \in$ LED($3^2$). Then, (LED($3^2$), $+, \circ_N$) is a near-field.

**Example 3.1.10.** The *Kalscheuer Near-Fields* [13], denoted $\mathcal{N}_r$, are defined by twisting the multiplication on the quaternions, $\mathbb{H}$, in the following way: For a given $r \in [0, \infty)$ and for all $a \in \mathbb{H}^{\#}$ define

$$\phi_r(a) := e^{i \cdot r \cdot \ln(|a|)} \in \mathbb{H}^{\#}.$$

Now, define the multiplication, $\circ$, on $\mathcal{N}_r$, by deforming the multiplication on $\mathbb{H}$ by:

$$0 \circ x = 0 \quad \forall x \in \mathbb{H}$$

$$a \circ x := a \cdot \phi_r(a)^{-1} \cdot x \cdot \phi_r(a) \quad \forall a, x \in \mathbb{H}$$

With this twisted multiplication $\mathcal{N}_r$ forms a left near-field.

## 3.2 A.A.A Division Algebras

A division algebra is algebra over a field in which division( either left or right), except by zero, is always possible.

**Definition 3.2.1** ([5])**.** $(D, +, \circ)$ is a division algbera over a field $F$ if $(D, +, \circ)$ is an algebra which is a vector space over $F$ and $(D^{\#}, \circ)$ is a loop.

**Definition 3.2.2.** Let $D$ be a division algebra. The set

$$N_\lambda(D) = \{a \in D \mid a(xy) = (ax)y \text{ for all } x, y \in D\}$$

is called the *left nucleus of* $D$. The set

$$N_\mu(D) = \{a \in D \mid x(ay) = (xa)y \text{ for all } x, y \in D\}$$

is called the *middle nucleus of* $D$. And the set

$$N_\rho(D) = \{a \in D \mid x(ya) = (xy)a \text{ for all } x, y \in D\}$$

is called the *right nucleus of* $D$, and

$$N = N_\lambda \cap N_\mu \cap N_\rho$$

is called the *center* of $D$.

**Definition 3.2.3.** Let $(D, +, \circ)$ be a division algebra. $K$ is called the *kernel of* $D$ if $K = K(D)$ where

$$K(D) := \{x \in D \mid (y+z) \circ x = y \circ x + z \circ x,\, x \circ (y+z) = x \circ y + x \circ z \,\forall\, y, z \in D\}.$$

$Z$ is called the *center of* $K$ if

$$Z := \{x \in K \mid x \in N \text{ and } x \circ y = y \circ x \,\forall\, y \in D\} \cup \{0\}.$$

Furthermore, set $V := D^{n\times 1}$, $U := K^{n\times 1}$, and $T := Z^{n\times 1}$ for some $n \in \mathbb{N}$.

For the purposes of this paper, we will focus on a specific class of division algebras known as A. A. Albert Algebras [9], A.A.A for short, which were first coined by mathematician Abraham Adrian Albert.

**Definition 3.2.4** ([9])**.** Let $p$ be an odd prime. Let $r > 1$ be an odd integer. Define $(D, +, \circ)$ by $(D, +) = (GF(p^r), +)$. First define $\Box$ on $D$ as $x \Box y = 2^{-1}(xy^P + x^p y)$. Then $(D^{\#}, \Box)$ is a quasi group. Now Define $\alpha$, a permutation of $D^{\#}$, by $\alpha(x) := u$ if $x = u \Box 1$. We finally define $x \circ y = \alpha(x) \Box \alpha(y)$. We note that $\alpha(1) = 1$, hence $(D^{\#}, \circ)$ is a loop with unity 1. As $(GF(p)^{\#}, \Box)$ is a subloop of $(D^{\#}, \Box)$, we have that $(D, +, \circ) = \text{A.A.A}(p^r)$ is a division algebra over $GF(p)$ which is never is never isomorphic to $GF(p^r)$.

**Remark 3.2.5.** When $D$ is an A.A.A division algebra, the center of $D$ is $\mathbb{Z}_p$, the fixed field of the Frobenius automorphism.

**Definition 3.2.6** ([5])**.** Let $D$ be an A.A.A divsion algebra. Set $V = D^{n\times 1}$ for some $n \in \mathbb{N}$. The list

$$\mathcal{A} = A_1, A_2, \cdots, A_h$$

in $V$ is called a *bases* of $V$ if

(1) $A = \sum x_i A_i$ for $x_i \in D$ and for every $A \in V$.

(2) If $A = 0$, then $x_i = 0$ for $i = 1, \cdots, h$.

In the list above, $h$ is called the *length* of $\mathcal{A}$. The *dimension* of $V$ is the length of a bases of $V$. The $x_i$'s are called the *coordinates of $A$ with respect to $\mathcal{A}$*. $\mathcal{A}$ is called a list of *generators* of $V$ if (1) holds. If (2) holds we call $\mathcal{A}$ a *linearly independent* list. We call a basis of $V$ an *$X$-basis of $V$* if the elements of the list $\mathcal{A}$ are in $X$ where $X$ is a subset of $V$.

**Example 3.2.7.** Let $D$ be an A.A.A division algebra. Let $E_i$ be the $n \times 1$ matrix whose $i^{th}$ coordinate is 1 and every other coordinate is 0. Take

$$\mathcal{E} = E_1, \cdots, E_n.$$

Then $\mathcal{E}$ is a $T$-basis of $V$ where $T$ is defined as an $n$-tuple of element of the center of $D$.

## 3.3 Near Spaces

**Definition 3.3.1.** Let $D$ be a near-field or an A.A.A. division algebra. Then $(V, +, \cdot)$ is called a *Near Space over $D$* if and only if the following conditions hold:

1. $V = \prod_{i\in I} D = D^I$, where $I$ is a finite or infinite indexing set.

2. The operation $+ : V \times V \to V$ is defined point-wise, that is, for $A, B \in V$,

$$(A + B)(i) = A(i) + B(i) \quad \text{for all } i \in I.$$

3. The operations $\cdot : D \times V \to V$ and $\cdot : V \times D \to V$ are defined point-wise, that is, for $x \in D$ and $A \in V$,

$$(x \cdot A)(i) = x \cdot A(i) \quad \text{and} \quad (A \cdot x)(i) = A(i) \cdot x \quad \text{for all } i \in I.$$

**Definition 3.3.2.** Let $M$ be a nonempty subset of $V$.

1. $M$ is called an *L-subspace* of $V$ if $(M, +)$ is an abelian subgroup of $(V, +)$ and $x \cdot A \in M$ for any $x \in D$ and $A \in M$.

2. $M$ is called a *subspace* of $V$ if $(M, +)$ is an abelian subgroup of $(V, +)$, $x \cdot A \in M$, and $A \cdot x \in M$ for any $x \in D$ and $A \in M$.

We next focus on the notion of a mapping between near spaces that preserve the structure of the space. We will refer to these mappings as *near-linear* mappings or *near-linear* transformations.

**Definition 3.3.3.** Let $V$ and $W$ be near spaces over some near-field or A.A.A. division algebra $D$. A *near-linear map* from $V$ to $W$ is a function $T : V \to W$ with the following properties:

1. **Additivity:**
$$T(u + v) = T(u) + T(v) \quad \text{for all } u, v \in V;$$

2. **Left homogeneity:**
$$T(av) = aT(v) \quad \text{for all } a \in D, v \in V;$$

3. **Right homogeneity:**
$$T(va) = T(v)a \quad \text{for all } a \in D, v \in V.$$

**Remark 3.3.4.** If $T$ is a near-linear map from a near space $V$ over $D$ into itself, we call $T$ a $D$-endomorphism of $V$ denoted $\mathrm{End}_D(V)$.

**Example 3.3.5.** Let $T$ be a linear transformation form a vector space $V$ to a vector space $W$, then $T$ is a near-linear transformation.

We next look at some examples of near spaces which are not vector spaces

**Example 3.3.6.** Let $D = \mathrm{LED}(3^2)$ as in example 2.1.9. Set $V = D^n$ for some integer $n$. Then $V$ is a finite dimensional near space over $D$ which is in fact not a vector space.

**Example 3.3.7.** Let $D = \mathrm{A.A.A}(p^r)$ which is a division algebra over $GF(p)$ (see definition 2.2.3). Set $V = D^n$ for some integer $n$. Then $V$ is a finite dimensional near space over $D$ which is in fact not a vector space.

# Chapter 4

# Near Space Near-Linear Cellular Automata

Within this chapter, we investigate the realm of *near-linear cellular automata*, where the alphabet is structured as a near space and exhibits near-linearity within the induced near vector space structure on configuration sets. When the alphabet near vector space and the underlying group are established, the set of near-linear cellular automata forms a subalgebra within the endomorphism algebra of configuration spaces. A notable trait of near-linear cellular automata is their capacity to produce results with finite support when applied to configurations with finite support. Additionally, a near-linear cellular automaton's behavior is fully determined by its actions on finitely-supported configurations, and it is pre-injective if and only if its operations on such configurations are injective. A characterization of near-linear cellular automata is also given. For this Chapter, let $D$ denote either a near-field or an $A.A.A$ finite division algebra.

## 4.1 Near-Linear Cellular Automata

**Theorem 4.1.1.** Let $G$ be a group and let $V = D^{n\times 1}$ be a near space over $D$. The set

$$V^G := \{x : G \to V = D^{n\times 1}\} = \prod_{g\in G} V$$

consisting of all configurations (maps) $x : G \to V$ over $G$ and alphabet $V$ has a natural structure of a near space over $D$ in which addition and scalar multiplication are defined below:

$$(x + y)(g) = x(g) + y(g); \quad (dx)(g) = d \cdot x(g); \quad (xd)(g) = x(g) \cdot d$$

for all $x, y \in V^G, d \in D, g \in G$.

*Proof.* Let $I = \{1, \cdots, n\}$ for some natural number $n$. Then we have that

$$V^G = (D^I)^G = \prod_{g \in G} (\prod_{i \in I} D) = \prod_{g \in G} \prod_{i \in I} D = \prod_{(g,i) \in G \times I} D = D^{|I| \times |G|}.$$

Hence, $V^G$, alongside the operations

$$(x + y)(g) = x(g) + y(g), \quad (dx)(g) = d \cdot x(g), \quad (xd)(g) = x(g) \cdot d$$

for all $x, y \in V^G$, $d \in D$, and $g \in G$, forms a near space. □

**Definition 4.1.2.** A Near Space Cellular Automata, *near-linear cellular automata*, over a group $G$ and alphabet $V = D^{n \times 1}$ is cellular automata $\tau : V^G \to V^G$ which is $D$-linear i.e

$$\tau(x + y) = \tau(x) + \tau(y); \quad \tau(dx) = d \cdot \tau(x); \quad \tau(xd) = \tau(x) \cdot d$$

for all $x, y \in V^G, d \in D$.

**Proposition 4.1.3.** Let $G$ be a group and let $V = D^{n \times 1}$ be a near space over $D$. Let $\tau : V^G \to V^G$ be a cellular automata with a finite memory set $S \subset G$ and local defining map $\mu : V^S \to V$. Let $K$ denote the kernel of $D$ and $Z$ denote the center of $K$. Then

(a) $\tau$ is a near-linear cellular automata if and only if $\mu$ is $D$-linear.

(b) If $\tau$ is a near-linear cellular automata, then $\tau$ is right linear over $K$ with the same $\mu$.

(c) If $\tau$ is a near-linear cellular automata, then $\tau$ is linear over $Z$ with the same $\mu$.

*Proof.* (a) Suppose that $\tau$ is a near-linear cellular automata. Let $y, y' \in V^S$ and let $d \in D$. Let $x, x'$ be two mappings in $V^G$ that extend $y$ and $y'$ respectively. In other words $x\,|_S = y$ and $x'\,|_S = y'$. Recall that

$$\tau(x)(g) = \mu((g^{-1}x)\,|_S)$$

can be written in the following way

$$\tau(x)(g) = \mu((x \circ L_g)\,|_S)$$

for all $g \in G$. Hence the equation above yields that

$$\tau(x)(1_G) = \mu(x\,|_S)$$

Thus it follows that

$$\begin{aligned}
\mu(y+y') &= \mu(x\mid_S + x'\mid_S) \\
&= \tau(x+x')(1_G) \\
&= (\tau(x)+\tau(x'))(1_G) \quad (\text{as } \tau \text{ is } D- \text{ linear}) \\
&= \tau(x)(1_G) + \tau(x')(1_G) \\
&= \mu(x\mid_S) + \mu(x'\mid_S) \\
&= \mu(y) + \mu(y')
\end{aligned} \tag{4.1}$$

Moreover, as

$$(dx)\mid_S = dy; \quad \text{and} \quad (xd)\mid_S = yd$$

for any $d \in D$. Hence we have that

$$\mu(dy) = \mu((dx)\mid_S) = \tau(dx)(1_G) = d\cdot\tau(x)(1_G) = d\cdot\mu(x\mid_S) = d\cdot\mu(y).$$

And

$$\mu(yd) = \mu((xd)\mid_S) = \tau(xd)(1_G) = \tau(x)(1_G)\cdot d = \mu(x\mid_S)\cdot d = \mu(y)\cdot d.$$

This shows that $\mu$ is $D$-linear, as desired.

On the other hand, suppose that $\mu$ $D$-linear. Then, for all $x, x' \in V^G, d \in D, g \in G$,

$$\begin{aligned}
\tau(x+x')(g) &= \mu(g^{-1}(x+x')\mid_S) \\
&= \mu(g^{-1}(x)\mid_S + g^{-1}(x')\mid_S) \\
&= \mu(g^{-1}(x)\mid_S) + \mu(g^{-1}(x')\mid_S) \\
&= \tau(x)(g) + \tau(x')(g) \\
&= (\tau(x)+\tau(x'))(g)
\end{aligned} \tag{4.2}$$

Moreover, we have that

$$\begin{aligned}
\tau(dx)(g) &= \mu(g^{-1}(dx)\mid_S) \\
&= \mu(d(g^{-1}x)\mid_S) \\
&= d\cdot\mu((g^{-1}x)\mid_S) \\
&= d\cdot\tau(x)(g)
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\tau(xd)(g) &= \mu(g^{-1}(xd) \mid_S) \\ &= \mu((g^{-1}x)d \mid_S) \\ &= \mu((g^{-1}x) \mid_S) \cdot d \\ &= \tau(x)(g) \cdot d\end{aligned} \tag{4.4}$$

Therefore we can conclude that $\tau$ is $D$- linear as desired. □

(b) Let $\tau$ be a near space linear cellular automata. Since $K \subset D$, it follows that $\tau$ restricted to $K$ is right linear over $K$. □

(c) Let $\tau$ be a near space linear cellular automata. Since $Z \subset K \subset D$, it follows that $\tau$ restricted to $Z$ is linear over $Z$. □

Our next result will show the near linear analogue of the Curtis-Hedlund theorem (2.2.3).

**Theorem 4.1.4.** Let $G$ be a group and let $V$ be a near space over $D$. Let $\tau : V^G \to V^G$ be a $G$-equivariant and $D$-linear map. Then the following conditions are equivalent:

(a) the map $\tau$ is a near-linear cellular automaton;

(b) the map $\tau$ is uniformly continuous (with respect to the prodiscrete uniform structure on $V^G$);

(c) the map $\tau$ is continuous (with respect to the prodiscrete topology on $V^G$);

(d) the map $\tau$ is continuous (with respect to the prodiscrete topology on $V^G$) at the constant configuration $x = 0$.

*Proof.*

$(a \implies b)$

Assume that $\tau : V^G \to V^G$ is a near linear cellular automaton with memory set $S$ and local defining map $\mu : V^S \to V$ such that

$$\tau(x)(g) = \mu((g^{-1}x) \mid_S) = \mu((x \circ L_g) \mid_S)$$

for all $x \in V^G$ and $g \in G$. By assumption $\tau$ is $D$- linear and $\tau$ is also $G$-equivariant ($\tau(gx) = g\tau(x)$ for all $g \in G$ and $x \in V^G$). We note that $\tau(x)(g)$ depends only on the restriction of the mapping $x$ to the set $gS$ since $(g^{-1}x)(s) = x(gs)$ for all $s \in S$. Let $x, \in V^G$ such that

$$x \mid_{gS} = y \mid_{gS}$$

for some $g \in G$. Hence

$$\tau(x)(g) = \mu((g^{-1}x)|_S) = \mu((g^{-1}y)|_S) = \tau(y)(g).$$

Now, suppose that $x$ and $y$ coincide on $\Omega S$ for some subset $\Omega \subset G$. That is, for all $g \in \Omega$ and $s \in S$, we have $(g^{-1}x)(s) = (g^{-1}y)(s)$. It follows then that $\tau(x)$ and $\tau(y)$ coincide on $\Omega$. To see this, we need to show that for all $g \in \Omega$, we have $\tau(x)(g) = \tau(y)(g)$. Let $g \in \Omega$. Since $gS \subseteq \Omega S$, we know that $x$ and $y$ coincide on $gS$, and therefore $\tau(x)(g) = \tau(y)(g)$ by the argument above. Thus, we have shown that $\tau(x)$ and $\tau(y)$ coincide on $\Omega$.

We note that $\Omega S$ is a finite set whenever $\Omega$ and $S$ are finite, since it is the Cartesian product of two finite sets. Recall that a base of entourages for the prodiscrete uniform structure on $V^G$ is given by the sets $W_\Omega \subset V^G \times V^G$, where

$$W_\Omega = \{(x, y) \in V^G \times V^G \mid x \mid_\Omega = y \mid_\Omega\}$$

and $\Omega$ runs over all finite subsets of $G$. Hence we have that

$$(\tau \times \tau)(W_{\Omega S} \subset W_\Omega)$$

for every finite subset $\Omega$ of $G$. Since the sets $W_\Omega$, such that $\Omega$ runs over all finite subsets of $G$, form a base of entourages for the prodiscrete uniform structure, we can conlcude that $\tau$ is uniformly continuous.

$(b \implies c)$

We have that the topology associated with the prodiscrete uniform structure is the prodiscrete topology and moreover, every uniformly continuous map is continuous.

$(c \implies d)$

Clear.

$(d \implies a)$

Suppose that the mapping $\tau$ is continuous at 0. Consider the map $V^G \to V$ defined by $x \mapsto \tau(x)(1_G)$. We claim that this mapping is continuous at 0. To see this we note that for any $g$ in $G$, the map that assigns to each configuration in $V^G$ its value at $g$, i.e., $x \to x(g)$, is continuous with respect to the prodiscrete topology on $V^G$. This topology is generated by the sets of the form $U = \{x \in V^G \mid x(g) \in V\}$ for some $g$ in $G$ and some open subset of V. To see why this is true, note that if $U$ is such a set, then its preimage under the projection map $x \to x(g)$ is the set $\{x \in V^G \mid x(g) \in V\}$, which is open in the product topology on $V^G$. Therefore, the projection map is continuous. Now, using the $G$-equivariance of $\tau$, we can write

$$\tau(x)(g) = \tau(g^{-1}x)(1_G)$$

for all $x$ in $V^G$ and $g$ in $G$. This shows that the map $x \to \tau(x)(1_G)$ can be expressed as a composition of continuous maps, namely the projection maps $x \to x(g)$ followed by the map $g^{-1}x \to \tau(g^{-1}x)(1_G)$. Since the composition of continuous maps is continuous, we can conclude that the map is also continuous.

Now, by the continuity of this mapping, we have that there exits a finite subset $S \subset G$ such that if $x \in V^G$ satisfies $x(s) = 0$ for all $s \in S$, then $\tau(x)(1_G) = 0$.

Now, as $\tau$ is $D$-linear, we have that if two configurations $x$ and $y$ coincide on $S$ then

$$\tau(x)(1_G) = \tau(y)(1_G).$$

Thus there exists a $D$-linear map $\mu$ from $V^S$ to $V$ such that

$$\tau(x)(1_G) = \mu(x \mid_S)$$

for all $x$ in $V^G$. Since $\tau$ is $G$-equivariant, we have that

$$\tau(x)(g) = \tau(g^{-1}x)(1_G) = \mu((g^{-1}x) \mid_S)$$

for all $x \in V^G$ and $g \in G$. Hence the map $\tau$ is a near linear cellular automaton cellular with memory set $S$ and local defining map $\mu$. $\square$

**Example 4.1.5.** Let $G$ be a group, and let $V = D^{n\times 1}$ be a near space over $D$. Let $f \in \mathrm{End}_D(N)$. Then the mapping

$$\tau : V^G \to V^G$$

$$f \circ x$$

for all $x \in V^G$ is a near-linear cellular automata.

**Example 4.1.6.** Let $G$ be a group and let $V = D^{n\times 1}$ be a near space over $D$. Moreover, let $f \in \mathrm{End}_D(V)$. Consider the mapping $\tau : V^G \to V^G$ defined in the following way; $\tau(x) = f \circ x$ for all configuration $x \in V^G$. Then it follows that $\tau$ is a near-linear cellular automata.

## 4.2 The Algebra of Near-Linear Cellular Automata

Recall that if $F$ is a field, and $A$ is a vector space over $F$ equipped with an additional binary operation from $A \times A$ to $A$, denoted here by $\cdot$, then $A$ is an algebra over $F$, also known as an $F$-algebra, if the following identities hold for all elements $x, y, z \in A$ and all elements $a, b \in F$:

1. **Right distributivity:** $(x + y) \cdot z = x \cdot z + y \cdot z$
2. **Left distributivity:** $z \cdot (x + y) = z \cdot x + z \cdot y$
3. **Compatibility with scalars:** $(ax) \cdot (by) = (ab)(x \cdot y)$

Recall that if $B$ is a subset of $A$ and $F$-Algebra such that $B$ is a sub vector space and a sub ring of $A$, then $B$ is a subalgerba of $A$. We now wish to naturally extend these definition to our notion of near spaces and near fields.

**Definition 4.2.1.** Let $D$ be a near-field or an A.A.A. division algebra and let $V = D^{n \times 1}$ be the near space over $D$ equipped with an additional binary operation from $V \times V$ to $V$, denoted here by $\cdot$. Then $V$ is a *near algebra* over $D$, also known as a $D$-algebra, if the following identities hold for all elements $x, y, z \in V$ and all elements $a, b \in D$:

1. **Left distributivity:** $z \cdot (x + y) = z \cdot x + z \cdot y$
2. **Compatibility with scalars on the left:** $(ax) \cdot (by) = (ab)(x \cdot y)$
3. **Compatibility with scalars on the right:** $(xa) \cdot (yb) = (x \cdot y)(ab)$

**Definition 4.2.2.** Let $B$ be a subset of a $D$-algebra $A$. Then $B$ is called a subalgebra of $A$ if $B$ is both a subspace and a subring (with only left distribution) of $A$.

Let $V = D^{n \times 1}$ be a near space over the near field $D$. Let $\text{End}_D(V)$ denote the set of endomorphisms (structure preserving maps from $V$ to $V$). We note that $\text{End}_D(V)$ admits a natural near space structure of a $D$-algebra where

$$(\varphi + \varphi')(v) = \varphi(v) + \varphi'(v),$$

$$(\varphi\varphi')(v) = (\varphi \circ \varphi')(v) = \varphi(\varphi'(v))$$

and

$$(d\varphi)(v) = d\varphi(v); \quad (\varphi d)(v) = \varphi(v)d$$

for all $\varphi, \varphi' \in \text{End}_D(V), d \in D$, and $v \in V$. We note that the identity mapping acts as the unit element ind $\text{End}_D(V)$.

Let $G$ be a group and let $V = D^{n\times 1}$ be a near space over $D$. We denote by $\mathrm{NLCA}(G,V)$ the set of all near linear cellular automata with respect to the group $G$ and near space $V$. It follows from definition 7 that

$$\mathrm{NLCA}(G,V) \subset \mathrm{End}_D(V^G).$$

Before we state our next Proposition, consider the following lemma.

**Lemma 4.2.3.** Let $G$ be a group and let $A$ be a set. Let $\tau_1$ and $\tau_2$ be cellular automta over $G$ and $A$ with memeory sets $S_1$ and $S_2$ respectively. Then $S = S_1 \cup S_2$ is a memory set of the cellular automata $\tau_1 + \tau_2$.

*Proof.* Let $\tau_1$ and $\tau_2$ be two cellular automata with memory sets $S_1$ and $S_2$ respectively. We want to show that $S = S_1 \cup S_2$ is a memory set for the addition of these automata.

Let $\mu_1 : A^{S_1} \to A$ and $\mu_2 : A^{S_2} \to A$ be the maps defined on the restricted configurations for $\tau_1$ and $\tau_2$ respectively, as per the given definition of a cellular automaton.

We define a new map $\mu : A^S \to A$ as follows:

$$\mu(y) = \begin{cases} \mu_1(y|_{S_1}) & \text{if } y|_{S_1} \text{ is defined} \\ \mu_2(y|_{S_2}) & \text{if } y|_{S_1} \text{ is not defined} \end{cases}$$

where $y|_{S_1}$ and $y|_{S_2}$ denote the restrictions of $y$ to $S_1$ and $S_2$ respectively.

For any $x \in A^G$ and $g \in G$, we have:

$$\begin{aligned} (\tau_1 + \tau_2)(x)(g) &= \tau_1(x)(g) + \tau_2(x)(g) \quad \text{(Addition of two automata)} \\ &= \mu_1((g^{-1}x)|_{S_1}) + \mu_2((g^{-1}x)|_{S_2}) \end{aligned}$$

Now, consider the restriction $(g^{-1}x)|_S$:

- If $(g^{-1}x)|_{S_1}$ is defined, then $(g^{-1}x)|_{S_1} = (g^{-1}x)|_S$, because $S_1 \subseteq S$. In this case, we have:

$$\begin{aligned} \mu((g^{-1}x)|_S) &= \mu_1((g^{-1}x)|_{S_1}) \quad \text{(By definition of } \mu) \\ &= \tau_1(x)(g) \quad \text{(By definition of } \mu_1) \end{aligned}$$

- If $(g^{-1}x)|_{S_1}$ is not defined, then $(g^{-1}x)|_{S_1} \neq (g^{-1}x)|_S$, because $S_1 \nsubseteq S$. In this case, we have:

$$\begin{aligned} \mu((g^{-1}x)|_S) &= \mu_2((g^{-1}x)|_{S_2}) \quad \text{(By definition of } \mu) \\ &= \tau_2(x)(g) \quad \text{(By definition of } \mu_2) \end{aligned}$$

Therefore, in both cases, we have $\tau_1(x)(g) + \tau_2(x)(g) = \mu((g^{-1}x)|_S)$, which satisfies the property required for a memory set.

Hence, we have shown that $S = S_1 \cup S_2$ is a memory set for the addition of the two cellular automata $\tau_1$ and $\tau_2$, as the map $\mu$ defined on the restricted configurations satisfies the required property. □

**Proposition 4.2.4.** Let $G$ be a group and let $V$ be a near space over $D$. Then, $\mathrm{NLCA}(G; V)$ is a subalgebra of $\mathrm{End}_D(V^G)$.

*Proof.* Let $\tau_1, \tau_2 \in \mathrm{NLCA}(G, V)$ with finite memory sets $S_1$ and $S_2$ respectively. Then $S = S_1 \cup S_2$ is a memory set of the cellular automata $\tau_1 + \tau_2$ by lemma 12.Now, let $\mu_1 : V^S \to V$ and $\mu_2 : V^S \to V$ be the corresponding local defining maps of $\tau_1$ and $\tau_2$ respectively and set $\mu = \mu_1 + \mu_2$. Note that by proposition 8(a), $\mu_1$ and $\mu_2$ is $D$-linear and hence $\mu$ is $D$-linear. For all $x \in V^G$ and $g \in G$ we have

$$\begin{aligned} (\tau_1 + \tau_2)(x)(g) &= \tau_1(x)(g) + \tau_2(x)(g) \\ &= \mu_1(g^{-1}x|_S) + \mu_2(g^{-1}x|_S) \\ &= \mu(g^{-1}x|_S). \end{aligned}$$

Hence $\tau_1 + \tau_2$ is a neaer cellular automaton with memory set $S$ and local defining map $\mu$. Since the map $1 + 2$ is $D$-linear, we have that $\tau_1 + \tau_2 \in \mathrm{NLCA}(G; V)$.

Moreover, let $d \in D$ and let $\tau \in \mathrm{NLCA}(G, V)$ with a finite memory set $S$ and $D$-linear local defining map $\mu$. Then it follows that

$$(d\tau)(x)(g) = d\tau(x)(g) = d\mu(g^{-1}x|_S) = (d\mu)(g^{-1}x|_S)$$

and

$$(\tau d)(x)(g) = \tau(x)(g)d = \mu(g^{-1}x|_S)d = (\mu d)(g^{-1}x|_S).$$

Thus the $D$-linear mappings $d\tau$ and $\tau d$ are cellular automaton with memory set $S$ and local defining map $d\mu$ and $\mu d$ respectively. It follows that $d\tau, \tau d \in \mathrm{NLCA}(G, V)$.

Lastly, we note that the identity configuration on $V^G$ is in $\mathrm{NLCA}(G, V)$. Moreover, since the composition of two CA's is a CA, we have that $\tau_1 \circ \tau_2 = \tau_1\tau_2 \in \mathrm{NLCA}(G, V)$. So we conclude that $\mathrm{NLCA}(G; V)$ is a subalgebra of $\mathrm{End}_D(V^G)$. □

## 4.3 Configurations of Finite Support For NLCA

Let $G$ be a group and let $V$ be a near space over $D$.

**Definition 4.3.1.** The support of a mapping $x \in V^G$ is the set

$$\{g \in G \mid x(g) \neq 0_V\}.$$

We denote by $V[G]$ the subset of $V^G$ consisting of all mappings with finite support.

**Proposition 4.3.2.** Let $G$ be a group and let $V$ be a near space over $D$. Then $V[G]$ is a subspace of $V^G$.

*Proof.* We first note that the set $V[G]$ is non empty as the zero mapping, $0(g) = 0_V$ for all $g \in G$, is clearly in $V[G]$.

Now, let $x, y \in V[G]$ such that $S$ is the support of $x$ and $T$ is the support if $y$ where $|S| < \infty$ and $|T| < \infty$. Let's denote the union of their supports as $S \cup T$, which is also a finite subset of $G$ because it's a union of two finite sets. Now, consider the sum $x + y$, which is also a configuration in $V^G$, defined as $(x + y)(g) = x(g) + y(g)$ for all $g$ in $G$, where $x(g)$ and $y(g)$ denote the values of $x$ and $y$ at the element $g$ in $G$, respectively.

To show that $x + y$ has finite support, we need to show that the set of elements of $G$ where $(x + y)$ is nonzero is finite. Now, for any element $g$ in $G$, we have two cases to consider:

Case 1: $g \in S \cup T$. This means that $g$ is in the union of the supports of $x$ and $y$, i.e., $g \in S$ or $g \in T$. Without loss of generality, let's say $g \in S$. This implies that $x(g) \neq 0_V$, because $g$ is in the support of $x$, which is a finite set. Therefore, $(x + y)(g) = x(g) + y(g) \neq 0_V$, because $x(g)$ is nonzero.

Case 2: $g \notin S \cup T$. This means that $g$ is not in the union of the supports of $x$ and $y$, i.e., $g \notin S$ and $g \notin T$. This implies that $x(g) = 0_V$ and $y(g) = 0_V$, because $g$ is not in the support of $x$ or $y$, which are finite sets. Therefore, $(x + y)(g) = x(g) + y(g) = 0_V + 0_V = 0_V$, which is the zero vector in $V$.

In both cases, we can see that $(x + y)(g)$ is either nonzero for finitely many elements $g$ in $G$ (Case

1) or zero for all other elements $g$ in $G$ (Case 2). Therefore, the set of elements of $G$ where $(x + y)$ is nonzero is finite. Hence $(x + y) \in V[G]$.

Now, Let $x$ be a configuration in $V[G]$ and let $d$ be a scalar in $D$. We need to show that $ax$ is also a configuration in $V[G]$.

Since $x$ is a configuration in $V[G]$, it has finite support, denoted as $S$, which is a finite subset of $G$. This means that there are only finitely many elements in $G$ where $x$ is nonzero, i.e., $x(g) \neq 0_V$, for finitely many $g$ in $G$.

Now, consider the scalar multiplication $dx$, which is also a configuration in $V^G$, defined as $(dx)(g) = d \cdot x(g)$ for all $g$ in $G$, where $x(g)$ denotes the value of $x$ at the element $g$ in $G$.

To show that $dx$ has finite support, we need to show that the set of elements of $G$ where $(dx)$ is nonzero is finite.

Now, for any element $g$ in $G$, we have two cases to consider:

Case 1: $g \in S$ This means that $g$ is in the support of $x$, i.e., $g \in S$. Therefore, $x(g) \neq 0_V$, because $g$ is in the support of $x$, which is a finite set. This implies that $d \cdot x(g)$ is nonzero, because $d$ is a nonzero scalar in $D$. Therefore, $(dx)(g) = d \cdot x(g) \neq 0_V$, because $d \cdot x(g)$ is nonzero.

Case 2: $g \notin S$ This means that $g$ is not in the support of $x$, i.e., $g \notin S$. This implies that $x(g) = 0_V$, because $g$ is not in the support of $x$, which is a finite set. Therefore, $(dx)(g) = d \cdot x(g) = d \cdot 0_V = 0_V$, which is the zero vector in $V$.

In both cases, we can see that $(dx)(g)$ is either nonzero for finitely many elements $g$ in $G$ (Case 1) or zero for all other elements $g$ in $G$ (Case 2). Therefore, the set of elements of $G$ where $(dx)$ is nonzero is finite. Since $dx$ has finite support, we can conclude that $dx$ belongs to $V[G]$.

A similar argument shows that $xd \in V[G]$. Hence $V[G]$ is a subspace of $V^G$ as desired. $\square$

**Proposition 4.3.3.** $V[G]$ is dense in $V^G$ for the prodiscrete topology.

*Proof.* In the prodiscrete topology, if $x \in V^G$, a neighborhood base of $x$ is given by the set

$$W = \{y \in V^G \mid y\mid_\Omega = x\mid_\Omega\}$$

for $\Omega$ a finite subset of $G$. Let $y \in W$ and define

$$\bar{y}(g) = \begin{cases} 0 & \text{if } g \notin \Omega \\ y(g) & \text{if } g \in \Omega \end{cases}$$

Hence $\bar{y}(g) \in V[G]$ so $V[G]$ is dense in $V^G$. □

**Definition 4.3.4.** Two mappings $x, x' \in V^G$ are almost equal if and only if the set

$$\{g \in G \mid x(g) \neq x'(g)\}$$

is finite.

**Proposition 4.3.5.** Let $G$ be a group and let $V$ be a near space over the near field $D$. Let $x, x' \in V^G$. Then $x, x'$ are almost equal if and only $x - x' \in V[G]$.

*Proof.* Let $x, x' \in V^G$. Then by definition Let $x, x'$ are almost equal if and if the set

$$\{g \in G \mid x(g) \neq x'(g)\}$$

is finite. However, it follows that

$$\{g \in G \mid x(g) \neq x'(g)\} = \{g \in G \mid (x - x')(g) \neq 0_V\}$$

Hence, $x, x'$ are almost equal if and only $x - x' \in V[G]$. □

**Proposition 4.3.6.** Let $G$ be a group and let $V$ be a near space over $D$. Let $\tau$ be a nearl linear cellular automata over $G$ and $V$. Then one has that $\tau(V[G]) \subset V[G]$.

*Proof.* Let $x \in V[G]$ with support $T$. Note that for any $x \in V[G]$ with support $T$, the definition of $\tau$ tells us that $\tau(x)(g) = \mu((g^{-1}x)|_S)$, where $S$ is a finite subset of $G$ and $\mu : V^S \to V$ is the corresponding local defining map for $\tau$.

Now, since $x$ has finite support, we can let $T$ be the support of $x$, which is also a finite subset of $G$. Then, for any $g \in G$, if $g^{-1}T \cap S = \varnothing$, we have $(g^{-1}x)|_S = 0$. This is because $g^{-1}x$ has support contained in $g^{-1}T$, which does not intersect $S$. Therefore, we can write $\tau(x)(g) = \mu((g^{-1}x)|_S) = 0$ if $g^{-1}T \cap S = \varnothing$.

Since $S$ is finite, there are only finitely many $g \in G$ for which $g^{-1}T \cap S \neq \varnothing$. Let $U$ be the set of such $g$, i.e., $U = g \in G : g^{-1}T \cap S \neq \varnothing$. Then, we have shown that $\tau(x)(g) = 0$ for all $g \in G \setminus U$. Therefore, the support of $\tau(x)$ is contained in $U$.

Finally, we note that $U$ is a finite set because both $T$ and $S$ are finite, so $g^{-1}T$ intersects $S$ for only finitely many $g$. Therefore, we can write $U = TS^{-1}$, the product of two finite sets. This shows that the support of $\tau(x)$ is contained in a finite set, so $\tau(x) \in V[G]$.

Hence, we have shown that $\tau(V[G]) \subseteq V[G]$ as required. □

**Remark 4.3.7.** If $\tau \in NLCA(G, V)$ then the restriction map

$$\tau \mid_{V[G]} = V[G] \to V[G]$$

is $D$-linear. In other words $\tau \mid_{V[G]} \in \mathrm{End}_D(V[G])$.

*Proof.* To show that $\tau|_{V[G]}$ is $D$-linear, we need to verify three properties:

1. For any $u, v \in V[G]$, we have $\tau|_{V[G]}(u + v) = \tau|_{V[G]}(u) + \tau|_{V[G]}(v)$.
2. For any $\alpha \in D$ and $u \in V[G]$, we have $\tau|_{V[G]}(\alpha u) = \alpha\tau|_{V[G]}(u)$.
3. For any $\alpha \in D$ and $u \in V[G]$, we have $\tau|_{V[G]}(u\alpha) = \tau|_{V[G]}(u)\alpha$.

Let $u, v \in V[G]$ be arbitrary. Since $u, v \in V[G]$, they are configurations in $V^G$ with finite support. Therefore, $\tau(u), \tau(v) \in V^G$ as well, since $\tau$ is defined on $V^G$.

Now, consider $\tau|_{V[G]}(u + v)$. By definition, this is $\tau(u + v)$ restricted to $V[G]$. Using the definition of $\tau$ as a near linear cellular automaton, we have:

$$\begin{aligned}
\tau|_{V[G]}(u + v) &= \tau(u + v)_{V[G]} \\
&= (\tau(u) + \tau(v))|_{V[G]} && \text{by additivity of } \tau \\
&= \tau(u)|_{V[G]} + \tau(v)|_{V[G]} && \text{by restriction of } V[G] \\
&= \tau|_{V[G]}(u) + \tau|_{V[G]}(v).
\end{aligned}$$

So, we have shown that $\tau|_{V[G]}$ is additive.

Let $\alpha \in D$ and $u \in V[G]$ be arbitrary. Again, since $u \in V[G]$, it is a configuration in $V^G$ with finite support. Therefore, $\tau(u) \in V^G$ as well.

Now, consider $\tau|_{V[G]}(\alpha u)$. By definition, this is $\tau(\alpha u)$ restricted to $V[G]$. Using the definition of $\tau$ as a near linear cellular automaton, we have:

$$\begin{aligned}
\tau|_{V[G]}(\alpha u) &= \tau(\alpha u)|_{V[G]} \\
&= (\alpha\tau(u))|_{V[G]} && \text{by homogeneity of } \tau \\
&= \alpha(\tau(u)|_{V[G]}) && \text{by restriction of } V[G] \\
&= \alpha\tau|_{V[G]}(u).
\end{aligned}$$

Similarly, let $\alpha \in D$ and $u \in V[G]$ be arbitrary. Again, since $u \in V[G]$, it is a configuration in $V^G$ with finite support. Therefore, $\tau(u) \in V^G$ as well.

Now, consider $\tau|_{V[G]}(u\alpha)$. By definition, this is $\tau(u\alpha)$ restricted to $V[G]$. Using the definition of $\tau$ as a near linear cellular automaton, we have:

$$\begin{aligned}
\tau|_{V[G]}(u\alpha) &= \tau(u\alpha)|_{V[G]} \\
&= (\tau(u)\alpha)|_{V[G]} && \text{by homogeneity of } \tau \\
&= (\tau(u)|_{V[G]})\alpha && \text{by restriction of } V[G] \\
&= \tau|_{V[G]}(\alpha).
\end{aligned}$$

So, we have shown that $\tau|_{V[G]}$ is homogeneous. Since $\tau|_{V[G]}$ satisfies both the additivity and homogeneity properties, it is a $D$-linear map. Therefore, $\tau|_{V[G]} \in \mathrm{End}_D(V[G])$. □

**Definition 4.3.8.** Let $N$ and $M$ be two near algebras over $D$. A map $F : N \to M$ is called a $D$-algebra homomorphism if $F$ is both a near space homomorphism (i.e., a $D$-linear map) and a ring homomorphism.

**Proposition 4.3.9.** Let $G$ be a group and let $V = D^{n\times 1}$ be a near space over a near field $D$. The map $\Lambda : \mathrm{NLCA}(G, V) \to \mathrm{End}_D(V[G])$ defined by $\Lambda(\tau) = \tau|_{V[G]}$ is an injective $D$-algebra homomorphism.

*Proof.* It suffices to show that $\Lambda$ is injective as $F$ is a $D$-algebra homomorphism by virtue of the structures on $\mathrm{NLCA}(V, G)$ and $\mathrm{End}_D(V[G])$.

To show that $\Lambda$ is injective, we need to prove that if $\Lambda(\tau_1) = \Lambda(\tau_2)$ for some $\tau_1, \tau_2 \in \mathrm{NLCA}(G, V)$, then $\tau_1 = \tau_2$. In other words, if the restrictions of $\tau_1$ and $\tau_2$ to $V[G]$ are equal, then $\tau_1$ and $\tau_2$ themselves are equal.

Let $\tau_1, \tau_2 \in \mathrm{NLCA}(G, V)$ such that $\Lambda(\tau_1) = \Lambda(\tau_2)$. This means that $\tau_1|_{V[G]} = \tau_2|_{V[G]}$ as maps from $V[G]$ to $V[G]$.

To show that $\tau_1 = \tau_2$, we need to show that they are equal on all configurations $x \in V^G$, i.e., $\tau_1(x) = \tau_2(x)$ for all $x \in V^G$.

Let $x \in V^G$ be an arbitrary configuration. Since $x \in V^G$, we can write $x$ as a function from $G$ to $V$ with finite support, i.e., $x : G \to V$ such that $x(g) = 0$ for all but finitely many $g \in G$.

Now, let's consider the action of $\tau_1$ and $\tau_2$ on $x$. By definition of $\Lambda$, we know that $\tau_1|_{V[G]} = \tau_2|_{V[G]}$, which means that both $\tau_1$ and $\tau_2$ act the same way on the configurations in $V[G]$. Since $x \in V^G$, we know that $x$ is also in $V[G]$ because $x$ has finite support.

Thus, we have $\tau_1(x) = \tau_1|_{V[G]}(x) = \tau_2|_{V[G]}(x) = \tau_2(x)$. Since $x$ was an arbitrary configuration in $V^G$, we have shown that $\tau_1(x) = \tau_2(x)$ for all $x \in V^G$, which implies that $\tau_1 = \tau_2$.

Therefore, we have proved that $\Lambda$ is an injective $D$-algebra homomorphism, as desired. □

**Definition 4.3.10.** Let $X$ be a Cartesian product. A function $f : X \to X$ is pre-injective if it sends distinct, almost equal elements into distinct elements.

Let $G$ be a group and let $A$ be a set. Given a subset $X \subset A^G$ a set $Z$ and a map $f : X \to Z$ is called pre-injective if it satisfies the following conditions: if $x_1, x_2 \in X$ are almost equal and $f(x_1) = f(x_2)$, then $x_1 = x_2$.

**Remark 4.3.11.** If $f$ is in injective then $f$ is pre-ijective. Moreover, if the cardinality of the group $G$ is finite, then pre-injectivity of $f$ implies injectivity.

**Proposition 4.3.12.** Let $\tau \in \mathrm{NLCA}(G, V)$. Then the following are equivalent.

(1) $\tau$ is pre-injective.

(2) $\tau|_{V[G]} : V[G] \to V[G]$ is injective.

*Proof.*

$(1 \implies 2)$

Let $\tau$ be pre-injective. Let $x \in \ker(\tau|_{V[G]})$. Then it follows that $x \in V[G]$ and $\tau(x) = \tau|_{V[G]}(x) = 0$. We note that the zero configuration, $0(g) = 0$ for all $g \in G$, and $x$ are almost equal. More over, by the $D$-linearity of $\tau$ we have that $\tau(0) = 0$. Hence, by our assumption that $\tau$ is pre-injective, this the implies that $x = 0$. Therefore the $\tau|_{V[G]}$ has a trivial kernel and thus is injective.

$(2 \implies 1)$

Suppose that $\tau|_{V[G]}$ injective. Let $x, x' \in V^G$ be two mappings which are almost equal and such that their images under $\tau$ agree. i.e $\tau(x) = \tau(x')$. Then by proposition 18, we have that $x - x' \in V[G]$ and

$$\tau|_{V[G]}(x - x') = \tau(x - x') = \tau(x) - \tau(x') = 0.$$

Since $\tau|_{V[G]}$ is injective, this implies then that $x - x' = 0$. So $x = x'$. Hence $\tau$ is pre-injective as desired. □

## 4.4 Restriction and Induction of Near-Linear Ceulluar Automata

In this section, we show that the processes of restriction and induction for cellular automata preserve near-linearity properties.

**Definition 4.4.1.** Let $G$ be a group and let $V = D^{n\times 1}$ be a near space over $D$. We denote by $\mathrm{NLCA}(G, H; V) = \mathrm{NLCA}(G; V) \cap \mathrm{CA}(G, H; V)$ the set of all near-linear cellular automata $\tau : V^G \to V^G$ admitting a memory set $S$ such that $S \subset H$.

**Proposition 4.4.2.** Let $\tau \in \mathrm{CA}(G, H; V)$. Then, $\tau \in \mathrm{NLCA}(G, H; V)$ if and only if $\tau_H \in \mathrm{NLCA}(H; V)$.

*Proof.* It follows from the definitions of restriction and induction that a cellular automaton $\tau \in \mathrm{CA}(G, H; V)$ is near-linear if and only if $\tau_H \in CA(H; V)$ is near-linear. □

**Remark 4.4.3.** Let $\mathcal{A}$ and $\mathcal{B}$ be two near algebras over a near-feild or division algebra $D$. A map $F : \mathcal{A} \to \mathcal{B}$ is called a $D$-algebra isomorphism if $F$ is a bijective $D$-algebra homomorphism. It is clear that if $F : \mathcal{A} \to \mathcal{B}$ is a $D$-algebra isomorphism, then its inverse map $F^{-1} : B \to \mathcal{A}$ is also a $D$-algebra isomorphism.

**Proposition 4.4.4.** The set $\mathrm{NLCA}(G, H; V)$ is a subalgebra of $\mathrm{NLCA}(G; V)$. Moreover, the map $\tau \mapsto \tau_H$ is a $D$-algebra isomorphism from $\mathrm{NLCA}(G, H; V)$ onto $\mathrm{NLCA}(H; V)$ whose inverse is the map $\sigma \mapsto \sigma^G$.

*Proof.* Let $\tau_1, \tau_2 \in \mathrm{NLCA}(G, H; V)$ with memory sets $S_1, S_2 \subset H$ respectively. Then the near-linear cellular automaton $\tau_1 + \tau_2$ admits $S_1 \cup S_2$ as a memory set. As $S_1 \cup S_2 \subset H$, we have $\tau_1 + \tau_2 \in$

$\mathrm{NLCA}(G,H;V)$. If $d \in D$ and $\tau \in \mathrm{NLCA}(G,H;V)$, with memory set $S \subset H$, then $S$ is also a memory set for $d\tau$ and therefore $d\tau \in \mathrm{NLCA}(G,H;V)$. Similarly, if $d \in D$ and $\tau \in \mathrm{NLCA}(G,H;V)$, with memory set $S \subset H$, then $S$ is also a memory set for $\tau d$ and therefore $\tau d \in \mathrm{NLCA}(G,H;V)$. This shows that $\mathrm{NLCA}(G,H;V)$ is a subspace of $\mathrm{NLCA}(G;V)$. Since $\mathrm{NLCA}(G,H;V)$ is a submonoid of $\mathrm{NLCA}(G;V)$, we deduce that $\mathrm{NLCA}(G,H;V)$ is a subalgebra of $\mathrm{NLCA}(G;V)$.

For simplicity's sake, denote by $\Phi : \mathrm{NLCA}(G,H;V) \to \mathrm{NLCA}(H;V)$ and $\Psi : \mathrm{NLCA}(H;V) \to \mathrm{NLCA}(G,H;V)$ the maps defined by $\Phi(\tau) = \tau_H$ and $\Psi(\sigma) = \sigma^G$ respectively. It is clear from the definitions that $\Psi \circ \Phi : \mathrm{NLCA}(G,H;V) \to \mathrm{NLCA}(G,H;V)$ and $\Phi \circ \Psi : LCA(H;V) \to \mathrm{NLCA}(H;V)$ are the identity maps. Therefore, $\Phi$ is bijective with inverse $\Psi$. It remains to show that $\Phi$ is a $D$-algebra homomorphism.

Let $\tau_1, \tau_2 \in \mathrm{NLCA}(G,H;V)$ and $d_1, d_2 \in D$. Let $x \in V^H$ and let $x' \in V^G$ extending $x$. We have

$$\Phi(d_1\tau_1 + d_2\tau_2)(x)(h) = (d_1\tau_1 + d_2\tau_2)(x')(h) = d_1\tau_1(x')(h) + d_2\tau_2(x')(h)$$

for all $h \in H$. Moreover, we have that

$$\Phi(\tau_1 d_1 + \tau_2 d_2)(x)(h) = (\tau_1 d_1 + \tau_2 d_2)(x')(h) = \tau_1 d_1(x')(h) + \tau_2 d_2(x')(h)$$

for all $h \in H$.

We deduce that $\Phi(d_1\tau_1 + d_2\tau_2)(x) = (d_1\Phi(\tau_1) + d_2\Phi(\tau_2))(x)$ and $\Phi(\tau_1 d_1 + \tau_2 d_2)(x) = (\Phi(\tau_1)d_1 + \Phi(\tau_2)d_2)(x)$ for all $x \in V^H$, that is, $\Phi(d_1\tau_1 + d_2\tau_2) = d_1\Phi(\tau_1) + d_2\Phi(\tau_2)$ and $\Phi(\tau_1 d_1 + \tau_2 d_2) = \Phi(\tau_1)d_1 + \Phi(\tau_2)d_2$. This shows that $\Phi$ is $D$-linear.

Lastly, it follows that $\Phi(\mathrm{Id}_{V^G}) = \mathrm{Id}_{V^H}$ and $\Phi(\tau_1\tau_2) = \Phi(\tau_1)\Phi(\tau_2)$ for all $\tau_1, \tau_2 \in \mathrm{NLCA}(G,H;V)$. We have shown that $\Phi$ is a $D$-algebra isomorphism. $\square$

# Chapter 5

# The Garden of Eden Theorem for Near-Linear Cellular Automata

The Garden of Eden theorem, also recognized as the Moore-Myhill theorem, represents a fundamental principle in cellular automata theory. It asserts that a cellular automaton achieves surjectivity if and only if it adheres to a weakened form of injectivity termed pre-injectivity. Moore and Myhill initially formulated this theorem in the early 1960s, focusing on cellular automata with finite alphabets operating over the $Z^d$ groups. Moore's work, presented in [15], demonstrated that surjectivity implies pre-injectivity for such automata, while shortly thereafter, Myhill's contribution [16] established the converse. These seminal works, each published in 1963 as separate papers, laid the foundation for this significant theorem. The biblical allusion associated with the Moore-Myhill theorem arises from the terminology adopted: configurations outside the cellular automaton's image are referred to as Garden of Eden configurations. These configurations, discernible only at time 0 in the sequence of consecutive automaton iterations, underscore the theorem's essence. Thus, the surjectivity of a cellular automaton aligns with the absence of Garden of Eden configurations. In this chapter we prove a near-linear analog to the Garden of Eden theorem for amenable groups.

## 5.1 Reversibility of Near-Linear Cellular Automata

When $G$ is an arbitrary group and $V = D^{n\times 1}$ is a finite-dimensional near space, we show that every bijective near-linear cellular automaton $\tau : V^G \to V^G$ is reversible (invertible) i.e there exist an inverese mapping which also a near-linear cellular automata. We call upon the following lemma from Mittag and Leffler [12] in order to proceed.

**Lemma 5.1.1.** [Mittag-Leffler Lemma]. Let $(X_n, f_{nm})$ be a projective sequence of non-empty sets such that for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$, with $m \geq n$, such that $f_{nh}(X_h) = f_{nm}(X_m)$ for all $h \geq m$. Then its projective limit $X = \varprojlim X_n$ is non-empty.

**Theorem 5.1.2.** Let $G$ be a group and let $V = D^{n\times 1}$ be a finite-dimensional vector space over a near-field or A.A.A division algebra $D$. Then every bijective near-linear cellular automaton $\tau : V^G \to V^G$ is reversible.

*Proof.* Let $\tau : V^G \to V^G$ be a bijective near-linear cellular automaton. We aim to demonstrate the reversibility of $\tau$ by splitting the proof into two steps.

Firstly, suppose the group $G$ is countable. Since $\tau$ is near-linear and $G$-equivariant, its inverse map $\tau^{-1} : V^G \to V^G$ is also near-linear and $G$-equivariant. We shall establish the following local property for $\tau^{-1}$: there exists a finite subset $N \subset G$ such that

$$(*) \text{ for } y \in V^G, \text{ the element } \tau^{-1}(y)(1_G) \text{ depends solely on the restriction of } y \text{ to } N.$$

This verifies the reversibility of $\tau$. Specifically, if $(*)$ holds for a finite subset $N \subset G$, then there exists a unique map $\nu : V^N \to V$ satisfying $\tau^{-1}(y)(1_G) = \nu(\pi_N(y))$. Utilizing the $G$-equivariance of $\tau^{-1}$, we then derive

$$\tau^{-1}(y)(g) = g^{-1}\tau^{-1}(y)(1_G) = \tau^{-1}(g^{-1}y)(1_G) = \nu(\pi_N(g^{-1}y)).$$

for all $y \in V^G$, implying that $\tau^{-1}$ is the cellular automaton with memory set $N$ and local defining map $\nu$.

Assuming by contradiction that no finite subset $N \subset G$ satisfies condition $(*)$, let $M$ be a memory set for $\tau$ with $1_G \in M$. As $G$ is countable, we can find a sequence $(A_n)_{n\in N}$ of finite subsets of $G$ such that $G = \bigcup_{n\in N} A_n$, $M \subset A_0$, and $A_n \subset A_{n+1}$ for all $n \in N$. Define $B_n = \{g \in G : gM \subset A_n\}$. Note that $G = \bigcup_{n\in N} B_n$, $1_G \in B_0$, and $B_n \subset B_{n+1}$ for all $n \in N$.

Given the absence of a finite subset $N \subset G$ satisfying $(*)$, for each $n \in N$, we can find two configurations $y'_n$ and $y''_n$ in $V^G$ such that $y'_n|_{B_n} = y''_n|_{B_n}$ and $\tau^{-1}(y'_n)(1_G) \neq \tau^{-1}(y''_n)(1_G)$. By the near-linearity of $\tau^{-1}$, the configuration $y_n = y'_n - y''_n$ in $V^G$ satisfies

$$y_n|_{B_n} = 0 \quad \text{and} \quad \tau^{-1}(y_n)(1_G) \neq 0.$$

This implies that if $x$ and $x'$ are elements in $V^G$ such that they coincide on $A_n$, then the configurations $\tau(x)$ and $\tau(x')$ coincide on $B_n$. Consequently, given $x_n \in V^{A_n}$ and denoting by $\widetilde{x_n} \in V^G$ a configuration extending $x_n$, the element

$$u_n = \tau(\widetilde{x_n})|_{B_n} \in V^{B_n}$$

does not depend on the particular choice of the extension $\widetilde{x_n}$ of $x_n$. Thus, we can define a map $\tau_n : V^{A_n} \to V^{B_n}$ by setting $\tau_n(x_n) = u_n$. We note that $\tau_n$ is $D$-linear.

Consider, for each $n \in \mathbb{N}$, the subset $X_n \subseteq V^{A_n}$ consisting of all $x_n \in V^{A_n}$ such that $x_n \in \mathrm{Ker}(\tau_n)$ and $x_n(1_G) \neq 0$. Note that $X_n$ is not empty since $(\tau^{-1}(y_n))|A_n \in X_n$ by (4.1). Now observe that, for $m \geq n$, the restriction map $\rho_{nm} : V^{A_m} \to V^{A_n}$ is $D$-linear and induces a map $f_{nm} : X_m \to X_n$. Indeed, if $u \in X_m$, then $u|A_n \in X_n$ since $\tau_n(u|A_n) = (\tau_m(u))|B_n = 0$ and $(u|A_n)(1_G) = u(1_G) \neq 0$. Note that $(X_n, f_{nm})$ is a projective sequence of nonempty sets. Let us show that $(X_n, f_{nm})$ also satisfies the Mittag-Leffler condition.

For all $m \geq n$, the set $f_{nm}(X_m) \subseteq X_n \subseteq V^{A_n}$. By definition, we have that $f_{nm}(X_m) = \rho_{nm}(\mathrm{Ker}(\tau_m)) \cap X_n$. Now, if $n \leq m \leq m'$, then $\rho_{nm'}(\mathrm{Ker}(\tau_{m'})) \subseteq \rho_{nm}(\mathrm{Ker}(\tau_m))$. Therefore, if we fix $n$, the sequence $\rho_{nm}(\mathrm{Ker}(\tau_m))$, where $m = n, n+1, \ldots$, is a non-increasing sequence of near space subspaces of $V^{A_n}$. As the near space $V^{A_n}$ is finite-dimensional, this sequence stabilizes. Thus, for each $n \in \mathbb{N}$, there exists an integer $m \geq n$ such that $f_{nk}(X_k) = f_{nm}(X_m)$ if $k \geq m$. This demonstrates that the Mittag-Leffler condition is satisfied. Consequently, we deduce from Lemma 4.1.1 that the projective limit $X = \varprojlim X_n$ is non-empty. Let $(z_n) \in X$. Then, there exists a unique $z \in V^G$ such that $\pi_{A_n}(z) = z_n$ for all $n \in \mathbb{N}$. However, $\tau(z) = 0$ since $\pi_{B_n}(\tau(z)) = \tau_n(z_n) = 0$ for all $n$, and $z(1_G) = z_0(1_G) \neq 0$. This contradicts the injectivity of $\tau$. Thus, there must exist a finite subset $N \subseteq G$ satisfying $(*)$, and consequently, $\tau$ is reversible.

Now, we drop the countability assumption on $G$ and prove the theorem in the general case. We choose a memory set $M \subset G$ for $\tau$ and denote by $H$ the subgroup of $G$ generated by $M$. Note that $H$ is countable since $M$ is finite. Observing that the restriction cellular automaton $\tau_H : V^H \to V^H$ is near-linear and bijective, we can apply the previous step to conclude that $\tau_H$ is reversible, implying that the inverse map $(\tau_H)^{-1} : V^H \to V^H$ is a cellular automaton. Thus, we establish the reversibility of $\tau$ in the general case as well. □

**Remark 5.1.3.** For a non-periodic group $G$ and a near-field or A.A.A. division algebra $D$, consider an infinite-dimensional near space $V$ over $D$. Then, it is guaranteed that there exists a bijective near-linear cellular automaton $\tau : V^G \to V^G$ that lacks reversibility.

Furthermore, when dealing with an infinite set $A$, it is always feasible to find a near space with a cardinality equivalent to that of $A$. One approach is to construct the $D$ near space utilizing $A$ as its basis.

## 5.2 The Closed Image Property for Near-Linear Cellular Automata

When $G$ is an arbitrary group and $V = D^{n\times 1}$ is a finite-dimensional near space, we show that $\tau : V^G \to V^G$ is closed in $V^G$ for the prodiscrete topology, where $\tau \in \mathrm{NLCA}(G, V)$. We call upon the Mittag and Leffler [12] lemma in order to proceed.

When considering a set $X$ and a topological space $Y$, a map $f : X \to Y$ is said to possess the closed image property if the set $f(X)$ forms a closed subset of $Y$. This property proves instrumental in establishing surjectivity results. Specifically, to demonstrate that a map $f : X \to Y$ with the closed image property is surjective, it suffices to establish that $f(X)$ densely covers $Y$.

In scenarios where the alphabet $A$ is finite, every cellular automaton $\tau : A^G \to A^G$ inherently exhibits the closed image property (considering the prodiscrete topology), owing to the compactness of $A^G$. In the near-linear context, the following holds:

**Theorem 5.2.1.** Let $G$ be a group and let $V = D^{n\times 1}$ be a finite-dimensional near space over a near-field or A.A.A division algebra $D$. Then every near-linear cellular automaton $\tau : V^G \to V^G$ has the closed image property with respect to the prodiscrete topology on $V^G$.

*Proof.* Let $\tau : V^G \to V^G$ be a near-linear cellular automaton. We aim to show that $\tau$ has the closed image property with respect to the prodiscrete topology on $V^G$. We split the proof into two steps as in the proof of Theorem 4.1.2.

Suppose first that the group $G$ is countable. Choose, as in the first step of the preceding proof of Theorem 4.1.2, a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of $G$ such that $G = \bigcup_{n\in\mathbb{N}} A_n$, $M \subset A_0$, and $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$. Consider, for each $n \in \mathbb{N}$, the $D$-linear map $\tau_n : V^{A_n} \to V^{B_n}$, where $B_n = \{g \in G : gM \subset A_n\}$, and $\tau_n$ is defined by $\tau_n(x_n) = (\tau(\widetilde{x_n}))|_{B_n}$ for all $x_n \in V^{A_n}$ and $\widetilde{x_n} \in V^G$ extending $x_n$.

Let $y \in V^G$ and suppose that $y$ is in the closure of $\tau(V^G)$. Then, for all $n \in \mathbb{N}$, there exists $z_n \in V^G$ such that

$$\pi_{B_n}(y) = \pi_{B_n}(\tau(z_n)) \tag{5.1}$$

Consider, for each $n \in \mathbb{N}$, the subspace $X_n \subset V^{A_n}$ defined by

$$X_n = \tau_n^{-1}(\pi_{B_n}(y)).$$

We have $X_n \neq \emptyset$ for all $n$ by (4.1). For $m \geq n$, the restriction map $V^{A_m} \to V^{A_n}$ induces a map $f_{nm} : X_m \to X_n$. Note that $(X_n, f_{nm})$ is a projective sequence.

We claim that $(X_n, f_{nm})$ also satisfies the Mittag-Leffler condition. Indeed, consider, for all $m \geq n$, the subspace $f_{nm}(X_m) \subset X_n$. We have $f_{nm'}(X'_m) \subset f_{nm}(X_m)$ for all $n \leq m \leq m'$ since $f_{nm'} = f_{nm} \circ f_{mm'}$. As the sequence $f_{nm}(X_m)$ $(m = n, n+1, \ldots)$ is a non-increasing sequence of finite-dimensional affine subspaces, it stabilizes, i.e., for each $n \in \mathbb{N}$ there exists an integer $m \geq n$ such that $f_{nk}(X_k) = f_{nm}(X_m)$ if $k \geq m$. Thus, the condition is satisfied. It follows from the Mittag-Leffler lemma that the projective limit $\varprojlim X_n$ is nonempty.

We choose an element $(x_n)_{n\in\mathbb{N}} \in \varprojlim X_n$. We have that $x_{n+1}$ coincides with $x_n$ on $A_n$ and that $x_n \in V^{A_n}$ for all $n \in \mathbb{N}$. As $G = \cup_{n\in\mathbb{N}} A_n$, we deduce that there exists a unique configuration $x \in V^G$ such that $x|A_n = x_n$ for all $n$. We have $\tau(x)|B_n = \tau_n(x_n) = y_n = y|B_n$ for all $n$. Since $G = \cup_{n\in\mathbb{N}} B_n$, this shows that $\tau(x) = y$. This completes the proof in the case that $G$ is countable.

Let us treat now the case of an arbitrary (possibly uncountable) group $G$. As in the second step of the preceding proof of Theorem 4.1.2, choose a memory set $M \subset G$ for $\tau$ and consider the countable subgroup $H$ of $G$ generated by $M$. By the previous step, we have that the restriction cellular automaton $\tau_H : V^H \to V^H$ has the closed image property, that is, $\tau_H(V^H)$ is closed in $V^H$ for the prodiscrete topology. We can then deduce that $\tau(V^G)$ is also closed in $V^G$ for the prodiscrete topology. Thus $\tau$ satisfies the closed image property. $\square$

**Remark 5.2.2.** Let $G$ be a non-periodic group and let $D$ be a near-field or A.A.A division algebra. Let $V$ be an infinite-dimensional near space over $D$. Then there exists a near-linear cellular automaton $\tau' : V^G \to V^G$ such that $\tau'(V^G)$ is not closed in $V^G$ with respect to the prodiscrete topology.

## 5.3 A Garden of Eden Theorem for Near-Linear Cellular Automata

In this section, we give a near-linear analogue to the Garden of Eden Theorem [3]. That is to say, we provide a suffiecnt and nessasry condtion for a near-linear cellular automata to be surjective over an amenable group.

To proceed we need the concept of mean dimension, which takes on the role that entropy played in the scenario of finite alphabets. We proceed by outlining some fundamental properties of mean dimension. In this definition, the dimension of finite-dimensional near spaces takes the place of the cardinality of finite sets.

But first, let's recall some basic facts about amenable groups. There are gonna be the groups in question that serve as our universe.

**Definition 5.3.1.** Let $G$ be a countable group (e.g., a finitely generated group) and let $P(G)$ denote the set of all subsets of $G$. The group $G$ is said to be amenable if there exists a right-invariant mean, that is, a map $\mu : P(G) \to [0,1]$ satisfying the following conditions:

1. $\mu(G) = 1$ (normalization);

2. $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in P(G)$ such that $A \cap B = \emptyset$ (finite additivity);

3. $\mu(Ag) = \mu(A)$ for all $g \in G$ and $A \in P(G)$ (right-invariance).

**Remark 5.3.2.** It's noteworthy that if $G$ is amenable, meaning it possesses a right-invariant mean, then it also harbors left-invariant means, and notably, bi-invariant means as well. The category of amenable groups encompasses finite groups, solvable groups, and finitely generated groups exhibiting subexponential growth. This category remains invariant under several operations, including subgroup formation, factorization, extension, and directed unions. Notably, the free group $F_2$ generated by two elements, and consequently, any groups housing non-abelian free subgroups, fall outside the amenable classification.

Now, in [6], Følner showed that a countable group $G$ is amenable if and only if it admits a Følner sequence, i.e. a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of non-empty finite subsets of $G$ such that

$$\lim_{n\to\infty} \frac{|\partial_E(\Omega_n)|}{|\Omega_n|} = 0 \quad \text{for all finite subsets } E \subset G. \tag{5.2}$$

With $E = \{1, g^{-1}\}$, where $g \in G$, one has $\Omega^{-E} = \Omega \cap g\Omega$ and $\Omega^{+E} = \Omega \cup g\Omega$, so that $\partial_E(\Omega) = \Omega \bigtriangleup g\Omega$, where $\bigtriangleup$ denotes the symmetric difference and (4.2) above gives

$$\lim_{n\to\infty} \frac{|\Omega_n \bigtriangleup g\Omega_n|}{|\Omega_n|} = 0 \quad \text{for all elements } g \in G,$$

which shows that the Følner sequence is asymptotically left-invariant.

The following statement is proved in [14], Theorem 6.1, and [8] Theorem 1.3.A. It can be deduced from a result of Ornstein and Weiss [17].

**Lemma 5.3.3** (Ornstein–Weiss lemma)**.** Let $G$ be a countable amenable group and let $F(G)$ denote the set of all finite subsets of $G$. Let $\phi : F(G) \to [0, \infty)$ be a function satisfying the following properties:

(a) $\phi(\Omega) \le \phi(\Omega')$ for all $\Omega, \Omega' \in F(G)$ such that $\Omega \subset \Omega'$ (monotonicity);

(b) $\phi(\Omega \cup \Omega') \le \phi(\Omega) + \phi(\Omega')$ for all $\Omega, \Omega' \in F(G)$ such that $\Omega \cap \Omega' = \emptyset$ (subadditivity);

(c) $\phi(g\Omega) = \phi(\Omega)$ for all $g \in G$ and $\Omega \in F(G)$ (left-invariance).

Then there is a real number $\lambda = \lambda(G, \phi) \geq 0$ depending only on $G$ and $\phi$ such that

$$\lim_{n\to\infty} \frac{\phi(\Omega_n)}{|\Omega_n|} = \lambda$$

for any Følner sequence $(\Omega_n)_{n\in\mathbb{N}}$ of $G$.

**Definition 5.3.4.** Let $G$ be a group. Let $E$ and $F$ be subsets of $G$. A subset $N \subset G$ is called an $(E, F)$-net if it satisfies the following conditions:

(i) the subsets $(gE)_{g\in N}$ are pairwise disjoint, i.e. $gE \cap g'E = \emptyset$ for all $g, g' \in N$ such that $g \neq g'$;

(ii) $G = \bigcup_{g\in N} gF$.

Note that if $N$ is an $(E, F)$-net then it is also an $(E', F')$-net for all $E', F'$ such that $E' \subset E$ and $F \subset F' \subset G$.

**Lemma 5.3.5.** Let $G$ be a group. Let $E$ be a non-empty subset of $G$ and let $F = EE^{-1} = xy^{-1} : x, y \in E$. Then $G$ contains an $(E, F)$-net.

*Proof.* The set of $S \subset G$ such that the subsets $(gE)_{g\in S}$ are pairwise disjoint is clearly inductive. Hence, by the Zorn lemma, it admits a maximal element. This element is an $(E, F)$-net. □

For the remainder of this section let $G$ denotes a countable amenable group, while $V = D^{n\times 1}$ represents a finite-dimensional near space over a near-field or $A.A.A$ division algebra $D$.

It is assumed that a Følner sequence $(\Omega_n)_{n\in\mathbb{N}}$ for $G$ has been chosen once and for all. Given a near subspace $X$ of $V^G$ and a subset $\Omega \subset G$, we denote by $X_\Omega$ the projection of $X$ on $V^\Omega$, that is, the subspace of $V^\Omega$ defined by

$$X_\Omega = \{x|_\Omega : x \in X\},$$

where, as above, $x|_\Omega$ denotes the restriction of $x$ to $\Omega$.

**Definition 5.3.6.** Let $X$ be a near subspace of $V^G$. The *mean dimension* of $X$ (with respect to the Følner sequence $(\Omega_n)_{n\in\mathbb{N}}$) is the non-negative number

$$(X) = \liminf_{n\to\infty} \frac{\dim(X_{\Omega_n})}{|\Omega_n|}. \tag{5.3}$$

Note that it immediately follows from this definition that $(V^G) = \dim(V)$ and that $(X) \leq \dim(V)$ for every subspace $X$ of $V^G$. More generally, one has $(X) \leq (Y)$ if $X$ and $Y$ are subspaces of $V^G$ such that $X \subset Y$.

**Proposition 5.3.7.** Suppose that $X$ is a $G$-invariant subspace of $V^G$ (e.g. $X = \tau(V^G)$ where $\tau : V^G \to V^G$ is a near-linear cellular automaton). Then one has

$$(X) = \lim_{n\to\infty} \frac{\dim(X_{\Omega_n})}{|\Omega_n|}.$$

Moreover, $(X)$ does not depend on the choice of the Følner sequence $(\Omega_n)_{n\in\mathbb{N}}$.

*Proof.* Let $F(G)$ denote the set of finite subsets of $G$. Let us verify that the function $\phi : F(G) \to \mathbb{N}$ defined by $\phi(\Omega) = \dim(X_\Omega)$ satisfies the hypotheses of the Ornstein–Weiss lemma. Let $\Omega, \Omega' \in F(G)$. Property (a) of the Ornstein–Weiss lemma follows from the fact that, if $\Omega \subset \Omega'$, then there is a surjective linear map (hence near-linear) $X_{\Omega'} \to X_\Omega$ given by restriction. If $\Omega \cap \Omega' = \emptyset$, then there is a natural embedding $X_{\Omega\cup\Omega'} \subset X_\Omega \times X_{\Omega'}$. This gives property (b). Finally, property (c) follows from the $G$-invariance of $X$, which implies that the map $x \mapsto x^g$ induces an isomorphism of near spaces $X_{g\Omega} \to X_\Omega$. □

**Lemma 5.3.8.** Let $E$ and $F$ be finite subsets of $G$ and let $N \subset G$ be an $(E, F)$-net. Let $N_n^- \subset N$ denote the set of $g \in N$ such that $gE \subset \Omega_n$. Then there exist $\alpha \in (0, 1]$ and $n_0 \in \mathbb{N}$ such that $|N_n^-| \geq \alpha|\Omega_n|$ for all $n \geq n_0$.

*Proof.* We can assume $E \subset F$. Let $N_n^+$ denote the set of $g \in N$ such that $gF$ meets $\Omega_n$. Since $\Omega_n$ is covered by the sets $gF$, $g \in N_n^+$, we have $|\Omega_n| \leq |F| \cdot |N_n^+|$. Now observe that $N_n^+ \setminus N_n^- \subset \partial_F(\Omega_n)$. Thus, setting $\alpha = |F|^{-1}$ we get

$$|N_n^-| \geq \alpha|\Omega_n| - |\partial_F(\Omega_n)|$$

for all $n \geq 0$. This shows the lemma, by the Ornstein–Weiss lemma. □

**Proposition 5.3.9.** Let $X$ be a subspace of $V^G$. Suppose that there exist finite subsets $E$ and $F$ of $G$, and an $(E, F)$-net $N \subset G$ such that $X_{gE} \subsetneq V^{gE}$ for all $g \in N$. Then $\mathrm{mdim}(X) < \dim(V)$.

*Proof.* Let us set as above $N_n^- = \{g \in N : gE \subset \Omega_n\}$ and let $\Omega_n^* = \Omega_n \setminus \bigsqcup_{g\in N_n^-} gE$. By hypothesis, we have $\dim(X_{gE}) \leq \dim(V^{gE}) - 1$ for all $g \in N$ so that

$$\begin{aligned}
\dim(X_{\Omega_n}) &\leq \dim(X_{\Omega_n^*}) + \sum_{g\in N_n^-} \dim(X_{gE}) \\
&\leq \dim(V^{\Omega_n^*}) + \sum_{g\in N_n^-} (\dim(V^{gE}) - 1) \\
&= \dim(V^{\Omega_n}) - |N_n^-| \\
&= |\Omega_n| \dim(V) - |N_n^-|.
\end{aligned}$$

Since, by Lemma 4.3.8, we can find $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that $|N_n^-| \geq \alpha|\Omega_n|$ for all $n \geq n_0$, this gives us $\mathrm{mdim}(X) \leq \dim(V) - \alpha < \dim(V)$. □

**Corollary 5.3.10.** Let $X$ be a $G$-invariant subspace of $V^G$. Suppose that there exists a finite subset $\Omega \subset G$ such that $X_\Omega \subsetneq V^\Omega$. Then $\mathrm{mdim}(X) < \dim(V)$.

*Proof.* Let $E = \Omega$ and $F = EE^{-1}$. By Lemma 4.3.5, there exists an $(E, F)$-net $N \subset G$. Since $X$ is $G$-invariant and $X_\Omega \subsetneq V^\Omega$, we have $X_{gE} \subsetneq V^{gE}$ for all $g \in N$. This implies $\mathrm{mdim}(X) < \dim(V)$ by Proposition 4.3.9. □

**Lemma 5.3.11.** Let $\tau : V^G \to V^G$ be a near-linear cellular automaton. Suppose that $\tau$ is not surjective. Then $(\tau(V^G)) < \dim(V)$.

*Proof.* Let $X = \tau(V^G)$ and consider an element $y \in V^G \backslash X$. By the closed image property for near-linear cellular automata, we can find a finite subset $\Omega \subset G$ such that $y|_\Omega \notin X_\Omega$. Therefore we have $X_\Omega \subsetneq V^\Omega$. This implies $\mathrm{mdim}(X) < \dim(V)$ by Corollary 4.3.10 □

**Proposition 5.3.12.** Let $\tau : V^G \to V^G$ be a near-linear cellular automaton and let $X$ be a subspace of $V^G$. Then $(\tau(X)) \leq \mathrm{mdim}(X)$.

*Proof.* Let us set $Y = \tau(X)$. Let $M \subset G$ be a memory set for $\tau$. We can assume $1_G \in M$. For each subset $\Omega \subset G$, the automaton $\tau$ induces a surjective linear map from $X_\Omega$ onto $Y_{\Omega^{-M}}$. As $Y_\Omega$ is a subspace of $Y_{\Omega^{-M}} \times Y_{\Omega\backslash\Omega^{-M}} \subset Y_{\Omega^{-M}} \times V^{\Omega\backslash\Omega^{-M}}$, this yields

$$\begin{aligned}\dim(Y_\Omega) &\leq \dim(Y_{\Omega^{-M}}) + |\Omega\backslash\Omega^{-M}| \dim(V) \\ &\leq \dim(X_\Omega) + |\partial_M(\Omega)| \dim(V).\end{aligned}$$

Therefore, we have

$$\frac{\dim(Y_{\Omega_n})}{|\Omega_n|} \leq \frac{\dim(X_{\Omega_n})}{|\Omega_n|} + \frac{|\partial_M(\Omega_n)|}{|\Omega_n|} \dim(V)$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and invoking equation (4.2), we get $\mathrm{mdim}(Y) \leq \mathrm{mdim}(X)$. □

**Lemma 5.3.13.** Let $\tau : V^G \to V^G$ be a near-linear cellular automaton. If $\tau$ is pre-injective then $(\tau(V^G)) = \dim(V)$.

*Proof.* Let us set $Y = \tau(V^G)$. Let $M$ be a memory set for $\tau$ such that $1_G \in M$. Suppose, by contradiction, that $\mathrm{mdim}(Y) < \dim(V)$. As $Y_{\Omega_n^{+M}}$ is a subspace of $Y_{\Omega_n} \times V^{\Omega_n^{+M}\backslash\Omega_n}$, we have

$$\begin{aligned}\dim(Y_{\Omega_n^{+M}}) &\leq \dim(Y_{\Omega_n}) + |\Omega_n^{+M}\backslash\Omega_n| \dim(V) \\ &\leq \dim(Y_{\Omega_n}) + |\partial_M(\Omega_n)| \dim(V).\end{aligned}$$

It follows from eqautaion (4.2) that we can find $n_0 \in \mathbb{N}$ such that $\dim(Y_{\Omega_{n_0}^{+M}}) < |\Omega_{n_0}| \dim(V)$. Let $Z$ denote the (finite-dimensional) near subspace of $V[G]$ consisting of the elements of $V^G$ whose support is contained in $\Omega_{n_0}$. Observe that $\tau(x)$ vanishes outside $\Omega_{n_0}^{+M}$ for every $x \in Z$. Thus we have

$$\dim(\tau(Z)) = \dim(\tau(Z)\Omega_{n_0}^{+M}) \leq \dim(Y_{\Omega_{n_0}^{+M}}) < |\Omega_{n_0}| \dim(V) = \dim(Z).$$

This shows that the restriction of $\tau$ to $Z$ is not injective. Therefore $\tau$ is not pre-injective. □

**Lemma 5.3.14.** Let $\tau : V^G \to V^G$ be a near-linear cellular automaton. Suppose that $\tau$ is not pre-injective. Then $\text{mdim}(\tau(V^G)) < \dim(V)$.

*Proof.* Since $\tau$ is not pre-injective, we can find an element $x_0 \in V^G$ with non-empty finite support $\Omega \subset G$ such that $\tau(x_0) = 0$. Let $M$ be a memory set for $\tau$ such that $1_G \in M$ and $M = M^{-1}$. Let $E = \Omega^{+M^2}$. By Lemma 4.3.5, we can find a finite subset $F \subset G$ such that $G$ contains an $(E, F)$-net $N$. Note that for each $g \in G$, the support of $x_0^g$ is $g^{-1}\Omega \subset g^{-1}E$. Let us choose, for each $g \in N$, a hyperplane $H_g \subset V^{g^{-1}\Omega}$ which does not contain the restriction to $g^{-1}\Omega$ of $x_0^g$. Consider the subspace $X \subset V^G$ consisting of all $x \in V^G$ such that the restriction of $x$ to $g^{-1}\Omega$ belongs to $H_g$ for each $g \in N$. We claim that $\tau(V^G) = \tau(X)$. Indeed, let $z \in V^G$. Then, for each $g \in N$, there exists a scalar $\lambda_g \in D$ such that the restriction to $g^{-1}\Omega$ of $z + \lambda_g x_0^g$ belongs to $H_g$. Let $z' \in V^G$ be such that $z'|_{g^{-1}\Omega} = (z + \lambda_g x_0^g)|_{g^{-1}\Omega}$ for each $g \in N$ and $z' = z$ outside $\bigsqcup_{g \in N} g^{-1}\Omega$. We have $z' \in X$ by construction. On the other hand, since $z'$ and $z$ coincide outside $\bigsqcup_{g \in N} g^{-1}\Omega$, we have $\tau(z') = \tau(z)$ outside $\bigsqcup_{g \in N} g^{-1}\Omega^{+M}$. Now, if $h \in g^{-1}\Omega^{+M}$ for some $g \in N$, then $hM \subset g^{-1}\Omega^{+M^2} = g^{-1}E$ and therefore $\tau(z')(h) = \tau(z + \lambda_g x_0^g)(h) = \tau(z)(h)$ since $x_0^g$ lies in the kernel of $\tau$. Thus $\tau(z) = \tau(z')$ and the claim follows. Thus we have

$$\text{mdim}(\tau(V^G)) = \text{mdim}(\tau(X)) \leq \text{mdim}(X) < \dim(V),$$

where the first inequality follows from Proposition 4.3.12and the second one from Proposition 4.3.9. □

We are now ready to give and prove the near-linear analog of the Garden of Eden Theorem.

**Theorem 5.3.15.** Let $V = D^{n \times 1}$ be a finite-dimensional near space over a near-field or A.A.A division algebra $D$ and let $G$ be a countable amenable group. Let $\tau : V^G \to V^G$ be a near-linear cellular automaton. Then the following conditions are equivalent:

(a) $\tau$ is surjective;

(b) $\text{mdim}(\tau(V^G)) = \dim(V)$;

(c)$\tau$ is pre-injective.

*Proof.* Condition (a) implies (b) since $\mathrm{mdim}(V^G) = \dim(V)$. The converse implication follows from Lemma 4.3.11 On the other hand, condition (c) implies (b) by Lemma 4.3.13. and, conversely, (b) implies (c) by Lemma 4.3.14. □

We wrap up this section by giving an example that the theorem above fails to hold when the near space $V$ is infinite-dimensional, in particular we will take $V$ to be an infinite dimensional vector space.

**Example 5.3.16.** Let $V$ denote an infinite-dimensional vector space over a field $K$, and let $G$ be any group. Suppose we select a basis $B$ for $V$. Any mapping $\alpha : B \to B$ uniquely extends to a linear map $\tilde{\alpha} : V \to V$. Consider the product map $\tau = \tilde{\alpha}^G : V^G \to V^G$, which constitutes a linear cellular automaton with a memory set $M = \{1_G\}$ and a local defining map $\tilde{\alpha}$. Since $B$ is infinite, it is possible to find a mapping $\alpha : B \to B$ that is surjective but not injective (or injective but not surjective). Clearly, the associated linear cellular automaton $\tau$ is surjective but not pre-injective (or injective but not surjective).

# Chapter 6

# Sofic Groups and Near-Linear Surjunctivity

Gromov (1999) [7] introduced the concept of *sofic groups*, initially referred to as initially subamenable groups, to offer a unified framework encompassing both residually finite and amenable groups. The term "sofic," derived from Hebrew meaning "finite," was coined by Weiss [20]. Sofic groups attract significant attention due to their role in confirming several fundamental conjectures in group theory. One notable example is Gromov's proof (1999) of Gottschalk's surjunctivity conjecture. A major unresolved question in this area pertains to whether all countable groups are sofic. These groups can be characterized in three equivalent manners: through their local resemblance to finite symmetric groups using the Hamming distance, by their local resemblance to finite labeled graphs in their Cayley graphs, and by their embeddability into ultraproducts of finite symmetric groups (this characterization is credited to Gábor Elek and Endre Szabó). Sofic groups represent the most extensive known category of surjunctive groups.

In this chapter we introduce the notion of a group being near-linearly surjunctive. We use this definition and the characterizations of sofic groups to show that all sofif groups are near-linearly surjunctive.

## 6.1 Sofic Groups

**Definition 6.1.1** ([2])**.** Let $G$ and $C$ be two groups. Given a finite subset $K \subset G$, a map $\phi : G \to C$ is called a $K$-almost-homomorphism of $G$ into $C$ if it satisfies the following conditions:

(K-AH-1)]$\phi(k_1k_2) = \phi(k_1)\phi(k_2)$ for all $k_1, k_2 \in K$;

[(K-AH-2)] the restriction of $\phi$ to $K$ is injective.

**Remark 6.1.2.** Note that in the preceding definition, the map $\varphi$ is not required to be a homomorphism nor to be globally injective.

Let $F$ be a nonempty finite set and consider the symmetric group $\mathrm{Sym}(F)$. For $\alpha \in \mathrm{Sym}(F)$, we denote by $\mathrm{Fix}(\alpha)$ the set $\{x \in F : \alpha(x) = x\}$ of fixed points of $\alpha$. The support of $\alpha$ is the set $\{x \in F : \alpha(x) \neq x\} = F \setminus \mathrm{Fix}(\alpha)$, so that we have

$$|\{x \in F : \alpha(x) \neq x\}| = |F| - |\mathrm{Fix}(\alpha)|.$$

Consider the map $d_F : \mathrm{Sym}(F) \times \mathrm{Sym}(F) \to \mathbb{R}$ defined by

$$d_F(\alpha_1, \alpha_2) = \frac{|\{x \in F : \alpha_1(x) \neq \alpha_2(x)\}|}{|F|}$$

for all $\alpha_1, \alpha_2 \in \mathrm{Sym}(F)$. Observe that the set $\{x \in F : \alpha_1(x) \neq \alpha_2(x)\} = \{x \in F : x \neq \alpha_1^{-1}\alpha_2(x)\}$ is the support of $\alpha_1^{-1}\alpha_2$, so that the equation above gives us

$$d_F(\alpha_1, \alpha_2) = 1 - \frac{|\mathrm{Fix}(\alpha_1^{-1}\alpha_2)|}{|F|}.$$

**Remark 6.1.3.** Let $F$ be a nonempty finite set. Then $d_F$ is a bi-invariant metric on $\mathrm{Sym}(F)$.

**Definition 6.1.4** ([2])**.** Let $F$ be a nonempty finite set. The bi-invariant metric $d_F$ is called the (normalized) Hamming metric on $\mathrm{Sym}(F)$.

**Definition 6.1.5** ([2])**.** Let $G$ be a group, $K \subset G$ a finite subset, and $\varepsilon > 0$. Let $F$ be a nonempty finite set. A map $\phi : G \to \mathrm{Sym}(F)$ is called a $(K, \varepsilon)$-almost-homomorphism if it satisfies the following conditions:

($(K, \varepsilon)$-AH-1) for all $k_1, k_2 \in K$, one has $d_F(\phi(k_1k_2), \phi(k_1)\phi(k_2)) \leq \varepsilon$;

($(K, \varepsilon)$-AH-2) for all $k_1, k_2 \in K$, $k_1 \neq k_2$, one has $d_F(\phi(k_1), \phi(k_2)) \geq 1 - \varepsilon$,

where $d_F$ denotes the normalized Hamming metric on Sym.

**Definition 6.1.6** ([2])**.** A group $G$ is called sofic if it satisfies the following condition: for every finite subset $K \subset G$ and every $\varepsilon > 0$, there exist a nonempty finite set $F$ and a $(K, \varepsilon)$-almost-homomorphism $\phi : G \to \mathrm{Sym}(F)$.

**Theorem 6.1.7** ([2])**.** Every finite group is sofic.

**Theorem 6.1.8** ([2])**.** Every subgroup of a sofic group is sofic.

**Theorem 6.1.9** ([2])**.** Every locally sofic group is sofic..

**Theorem 6.1.10** ([2])**.** Every amenable group is sofic.

**Theorem 6.1.11** ([2])**.** Let $(G_i)_{i\in I}$ be a family of sofic groups. Then, their direct product $G = \prod_{i\in I} G_i$ is sofic.

**Theorem 6.1.12** ([2])**.** Let $(G_i)_{i\in I}$ be a family of sofic groups. Then their direct sum $G = \bigoplus_{i\in I} G_i$ is sofic.

**Theorem 6.1.13** ([2])**.** The limit of a projective system of sofic groups is sofic..

**Theorem 6.1.14** ([2])**.** Every group which is locally embeddable into the class of sofic groups is sofic.

**Theorem 6.1.15** ([2])**.** Let $G$ be a group. Suppose that $G$ contains a normal subgroup $N$ such that N is sofic and $G/N$ is amenable. Then $G$ is sofic.

We know describe an alternate characterization of sofic groups.

**Definition 6.1.16** ([20])**.** Let $G$ be a finitely generated group, and $S \subseteq G$ a fixed finite, symmetric (i.e., $S = S^{-1}$) generating set. The Cayley graph of $G$ is a directed graph $\mathrm{Cay}(G, S)$, whose edges are labeled by the elements of $S$: the set of vertices equals $G$, and the edges with label $s \in S$ are the pairs $(g, sg)$ for all $g \in G$. Let $B_r(1)$ denote the $r$-ball around $1 \in \mathrm{Cay}(G, S)$ (it is an edge-colored graph, and also a finite subset in $G$).

**Theorem 6.1.17.** Let $G$ be a finitely generated group and let $S$ be a finite symmetric generating subset of $G$. The following conditions are equivalent:
(a) The group $G$ is sofic;
(b) For all $\epsilon > 0$ and $r \in \mathbb{N}$, there exists a finite $S$-labeled graph $Q = (Q, E)$ such that

$$|Q(r)| \geq (1 - \epsilon)|Q|,$$

where $Q(r) \subset Q$ denotes the set consisting of all vertices $q \in Q$ for which there exists an $S$-labeled graph isomorphism $\psi_{q,r} : B_S(r) \to B(q, r)$ from the ball $B_S(r)$ in the Cayley graph $\mathcal{C}_S(G)$ of $G$ with respect to $S$ onto the ball $B(q, r)$ in $Q$ satisfying $\psi_{q,r}(1_G) = q$.

## 6.2 Near-Linear Surjunctivity

**Definition 6.2.1.** A group $G$ is said to be *near-linearly surjunctive*, NL-surjunctive if, for any near-field or A.A.A division algebra $D$ and any finite-dimensional near space $V = D^{n\times 1}$ over $D$, every injective near-linear cellular automaton $\tau : V^G \to V^G$ is surjective.

**Proposition 6.2.2.** Every subgroup of an NL-surjunctive group is NL-surjunctive.

*Proof.* Suppose that $H$ is a subgroup of an NL-surjunctive group $G$. Let $V = D^{n\times 1}$ be a finite-dimensional near space over a near-field or A.A.A division algebra $D$, and let $\tau : V^H \to V^H$ be an injective near-linear cellular automaton over $H$. Consider the cellular automaton $\tau^G : V^G \to V^G$ over $G$ obtained from $\tau$ by induction. The fact that $\tau$ is injective implies that $\tau^G$ is injective. Also, $\tau^G$ is near-linear. Since $G$ is NL-surjunctive, it follows that $\tau^G$ is surjective. Hence we may deduce that $\tau$ is surjective. This shows that $H$ is NL-surjunctive. □

**Proposition 6.2.3.** Let $G$ be a group. Then the following conditions are equivalent:

(a) $G$ is NL-surjunctive;

(b) every finitely generated subgroup of $G$ is L-surjunctive.

*Proof.* ($a \implies b$) Suppose that $G$ is NL-surjunctive, then it follows from proposition 5.2.2 that every finitely generated subgroup of $G$ is NL-surjunctive.

($b \implies a$) Conversely, let $G$ be a group all of whose finitely generated subgroups are NL-surjunctive. Let $V = D^{n\times 1}$ be a finite-dimensional near space over a near-field or A.A.A division algebra $A$, and let $\tau : V^G \to V^G$ be an injective near-linear cellular automaton with memory set $S$. Let $H$ denote the subgroup of $G$ generated by $S$, and consider the near-linear cellular automaton $\tau_H : V^H \to V^H$ obtained by restriction of $\tau$. The fact that $\tau$ is injective implies that $\tau_H$ is injective. As $H$ is finitely generated, it is NL-surjunctive by our hypothesis on $G$. It follows that $\tau_H$ is surjective. We have that $\tau$ is also surjective. Hence (b) implies (a). □

**Remark 6.2.4.** The proposition mentioned above can be restated as follows: a group is NL-surjunctive if and only if it exhibits local NL-surjunctivity.

**Remark 6.2.5.** Every finite group is NL-surjunctive. Indeed, if $G$ is a finite group and $V$ is a finite-dimensional near space, then the near space $V^G$ is also finite-dimensional and therefore every injective endomorphism of $V^G$ is surjective. As such, we have the following result, which is a near-linear analogue to the fact that all sofic groups are surjunctive.

**Theorem 6.2.6.** Every sofic group is NL-surjunctive.

*Proof.* Let $G$ be a sofic group. Let $V = D^{n\times 1}$ be a finite-dimensional near space over a near-field of A.A.A division algebra $D$ of dimension $\dim_D(V) = d \geq 1$, and let $\tau : V^G \to V^G$ be an injective near-linear cellular automaton. We wish to show that $\tau$ is surjective.

We first note that by proposition 5.1.8, every subgroup of a sofic group is sofic. On the other hand, it follows from Proposition 5.2.3 that a group is NL-surjunctive if all its finitely generated subgroups are NL-surjunctive. Thus, we can assume that $G$ is finitely generated.

Let then $S \subset G$ be a finite symmetric generating subset of $G$. In other words for any $g$ in $G$, there exist $s_1, s_2, \ldots, s_k$ in $S$ and their inverses such that $g = s_1^{e_1} \cdot s_2^{e_2} \cdot \ldots \cdot s_k^{e_k}$, where $e_i = \pm 1$ for all $i$. Now, for $r \in \mathbb{N}$, we denote by $B_S(r) \subset G$ the ball of radius $r$ centered at $1_G$ in the Cayley graph associated with $(G, S)$. We set $Y = \tau(V^G)$. Note that $Y$ is a $G$-invariant subspace of $V^G$. On the other hand, it follows from the closed image property for near-linear cellular automata that $Y$ is closed in $V^G$ with respect to the prodiscrete topology. Moreover, since $\tau$ is an injective near-linear cellular automaton, there exists a near-linear cellular automaton $\sigma : V^G \to V^G$ such that

$$\sigma \circ \tau = \mathrm{Id}_{V^G}.$$

We next choose an $r_0$ large enough so that the ball $B_S(r_0)$ is a memory set for both $\tau$ and $\sigma$. Let $\mu : V^{B_S(r_0)} \to V$ and $\nu : V^{B_S(r_0)} \to V$ denote the corresponding local defining maps for $\tau$ and $\sigma$ respectively.

By way of contradiction, suppose that $\tau$ is not surjective, that is, $Y \subsetneq V^G$. Then, since $Y$ is closed in $V^G$, there exists a finite subset $\Omega \subset G$ such that $Y|_\Omega \subsetneq V^\Omega$. It is not restrictive, up to taking a larger $r_0$, again if necessary, to suppose that $\Omega \subset B_S(r_0)$. Thus, $Y|_{B_S(r_0)} \subsetneq V^{B_S(r_0)}$.

Let $\epsilon > 0$ be such that

$$\epsilon < \frac{1}{d|B(2r_0)| + 1}. \tag{6.1}$$

Note that from (5.1) we have

$$1 - \epsilon > 1 - \frac{1}{d|B(2r_0)| + 1},$$

which gives us

$$(1 - \epsilon)^{-1} < 1 + \frac{1}{d|B(2r_0)|}. \tag{6.2}$$

Since $G$ is sofic, we can find a finite $S$-labeled graph $(Q, E, \lambda)$ such that $|Q(3r_0)| \geq (1 - \epsilon)|Q|$, where we recall that $Q(r)$, $r \in \mathbb{N}$, denotes the set of all $q \in Q$ such that there exists an $S$-labeled graph isomorphism $\psi_{q,r} : B_S(r) \to B(q, r)$ satisfying $\psi_{q,r}(1_G) = q$ by theorem 5.1.17. We note the following inclusions

$$Q(r_0) \supset Q(2r_0) \supset \cdots \supset Q(ir_0) \supset Q((i+1)r_0) \supset \cdots. \tag{6.3}$$

Also we note that $B(q, r_0) \subset Q(ir_0)$ for all $Q((i+1)r_0)$ and $i \geq 0$. For each integer $i \geq 1$, we define the map $\mu_i : V^{Q(ir_0)} \to V_{Q((i+1)r_0)}$ by setting, for all $u \in V^{Q(ir_0)}$ and $q \in Q((i+1)r_0)$,

$$\mu_i(u)(q) = \mu(u|_{B(q,r_0)} \circ \psi_{q,r_0}(1_G)), \tag{6.4}$$

where $\psi_{q,2r_0}$ is the unique isomorphism of $S$-labeled graphs from $B_S(r_0) \subset G$ to $B(q, r_0) \subset Q$ sending $1_G$ to $q$. Similarly, we define the map $\nu_i : V^{Q(ir_0)} \to V^{Q((i+1)r_0)}$ by setting, for all $u \in V^{Q(ir_0)}$ and $q \in Q((i+1)r_0)$,

$$\nu_i(u)(q) = \nu(u|_{B(q,r_0)} \circ \psi_{q,r_0}(1_G)). \tag{6.5}$$

From the fact that $\tau^{-1} \circ \tau$ is the identity map on $V^G$, we deduce that the composite $\nu_{i+1} \circ \mu_i : V^{Q(ir_0)} \to V^{Q((i+2)r_0)}$ is the identity on $V^{Q((i+2)r_0)}$. More precisely, denoting by $\rho_i : V^{Q(ir_0)} \to V^{Q((i+2)r_0)}$ the restriction map, we have that $\nu_{i+1} \circ \mu_i = \rho_i$ for all $i \geq 1$. In particular, we have $\nu_2 \circ \mu_1 = \rho_1$. Thus, setting $Z = \mu_1(V^{Q(r_0)}) \subset V^{Q(2r_0)}$, we deduce that $\nu_2(Z) = \rho_1(V^{Q(r_0)}) = V^{Q(3r_0)}$. It follows that

$$\dim(Z) \geq d|Q(3r_0)|$$

Let $Q' \subset Q(3r_0)$ such that

$$|Q'| \geq \frac{|Q(3r_0)|}{|B(2r_0)|}$$

and set $\hat{Q}' = \bigcup_{q \in Q} B(q, r_0)$. Note that $\hat{Q}' \subseteq Q(2r_0)$ so that

$$|Q(2r_0)| = |Q'| \cdot |B_S(r_0)| + |Q(2r_0) \setminus \hat{Q}'| \tag{6.6}$$

Now observe that, for all $q \in Q(2r_0)$, we have a natural isomorphism of near spaces $Z|_{B(q,r_0)} \to Y|_{B_S(r_0)}$ given by $u \mapsto u \circ \psi_{q,r_0}$, where $\psi_{q,r_0}$ denotes as above the unique isomorphism of $S$-labeled graphs from $B_S(r_0)$ to $B(q, r_0)$ such that $\psi_{q,r_0}(1_G) = q$. Since $Y|_{B_S(r_0)} \subsetneq V^{B_S(r_0)}$, this implies that

$$\dim(Z|_{B(q,r_0)}) = \dim(Y|_{B_S(r_0)}) \leq d \cdot |B_S(r_0)| - 1, \tag{6.7}$$

for all $q \in Q'$. Thus we have

$$\begin{aligned}
\dim(Z) &\leq \dim(Z|_{\hat{Q}'}) + \dim(Z|_{Q(2r_0) \setminus \hat{Q}'}) \\
&\leq |Q'| \cdot (d \cdot |B_S(r_0)| - 1) + d \cdot |Q(2r_0) \setminus \hat{Q}'| \\
&= d\left(|Q(2r_0)| - \frac{|Q'|}{d}\right),
\end{aligned}$$

where the last equality follows from (5.6). Comparing this with (5.7), we obtain

$$|Q(3r_0)| \leq |Q(2r_0)| - \frac{|Q'|}{d}.$$

Thus,

$$\begin{aligned}|Q| \geq |Q(2r_0)| \geq |Q(3r_0)| + \frac{|Q'|}{d} \\ \geq |Q(3r_0)| + \frac{|Q(3r_0)|}{d|B(2r_0)|} \\ = |Q(3r_0)| \left(1 + \frac{1}{d|B(2r_0)|}\right) \\ > |Q(3r_0)|(1-\epsilon)^{-1}\end{aligned}$$

where the last inequality follows from (5.2). This yields

$$|Q(3r_0)| < (1-\epsilon)|Q|,$$

which contradicts the fact that $|Q(3r_0)| \geq (1-\epsilon)|Q|$. This shows that $\tau(V^G) = Y = V^G$, that is, $\tau$ is surjective. It follows that the group $G$ is NL-surjunctive. $\square$

# Chapter 7

# Conclusion and Future Directions

In this dissertation, we have successfully extended the theory of linear cellular automata into the realm of near-linear cellular automata over near spaces. By building upon the foundational work of Ceccherini-Silberstein and Coornaert, we have introduced the concept of $G$-equivariant $D$-endomorphisms and explored their algebraic properties.

Key achievements include:

1. Establishing a near-linear analog of the Curtis-Hedlund Theorem.
2. Proving a Garden of Eden Theorem for near-linear cellular automata.
3. Demonstrating that all sofic groups are near-linearly surjunctive.

These results not only deepen our understanding of cellular automata in abstract algebraic settings but also pave the way for future investigations into more complex structures.

## 7.1 Future Work

Recent efforts have been made to further generalize the notion of a cellular automata over a group. In particular, the existence and structure of automata from a configuration space $A^G$ to a configuration space $A^H$ where $A$ is a finite alphabet and $G$ and $H$ are arbitrary groups.

The following are results obtained by the efforts from A. Castillo-Ramirez, M. Sanchez-Alvarez, A. Vazquez-Aceves, and A. Zaldivar-Corichi in [1].

**Definition 7.1.1.** Let $A$ be a finite set, and let $G$ and $H$ be groups. Denote by $\mathrm{Hom}(H, G)$ the set of all group homomorphisms from $H$ to $G$. For any $\phi \in \mathrm{Hom}(H, G)$, a $\phi$-*cellular automaton* from $A^G$ to $A^H$ is a function $\tau : A^G \to A^H$ such that there is a finite subset $S \subseteq G$, called a *memory set* of $\tau$, and a *local function* $\mu : A^S \to A$ satisfying

$$\tau(x)(h) = \mu((\phi(h^{-1}) \cdot x)|_S), \quad \forall x \in A^G, h \in H.$$

This definition is line with the classical definition of cellular automata over the same group. It still gives the fundamental characteristic that cellular automata over groups must adhere to a local defining principle.

The following are examples of a $\phi$-*cellular automaton*.

**Example 7.1.2.** Every cellular automaton from $A^G$ to $A^G$ is an $\mathrm{id}_G$-cellular automaton, where $\mathrm{id}_G$ is the identity function on $G$. However, note that we may define $\phi$-cellular automata from $A^G$ to $A^G$, where $\phi$ is a nontrivial element of $\mathrm{End}(G) := \mathrm{Hom}(G, G)$.

**Example 7.1.3.** Let $G = \mathbb{Z}$, $H = \mathbb{Z}^2$ and $S = \{-1, 0, 1\} \subseteq \mathbb{Z}$. Recall that a configuration $x \in A^{\mathbb{Z}}$ may be seen as a bi-infinite sequence $x = \dots x_{-1}, x_0, x_1, \dots$.

Consider the homomorphism $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ given by $\phi(a, b) = a + b$, for all $(a, b) \in \mathbb{Z}^2$. Then, for any function $\mu : A^S \to A$, the $\phi$-cellular automaton $\tau : A^{\mathbb{Z}} \to A^{\mathbb{Z}^2}$ with memory set $S$ and local function $\mu$ is given by

$$\tau(x)(a, b) = \mu(x_{a+b-1}, x_{a+b}, x_{a+b+1}).$$

for all $x \in A^{\mathbb{Z}}$ and $(a, b) \in \mathbb{Z}^2$.

Classical cellular automata possess the crucial trait of being $G$-equivariant, meaning that for every $g \in G$ and $x \in A^G$, the transformation $\tau(g \cdot x)$ equals $g \cdot \tau(x)$. On the other hand, $\phi$-cellular automata exhibit a broader form of this property.

**Definition 7.1.4.** Let $\phi \in \mathrm{Hom}(H, G)$. A function $\tau : A^G \to A^H$ is called $\phi$-*equivariant* if

$$h \cdot \tau(x) = \tau(\phi(h) \cdot x), \quad \forall x \in A, h \in H.$$

**Theorem 7.1.5.** Every $\phi$-cellular automaton is $\phi$-equivariant.

**Theorem 7.1.6.** Every $\phi$-cellular automaton is continuous.

As a result, the following is then generalized version of Curtis-Hedlund Theorem.

**Theorem 7.1.7.** Let $\phi \in \mathrm{Hom}(H, G)$. A function $\tau : A^G \to A^H$ is a $\phi$-cellular automaton if and only if $\tau$ is continuous and $\phi$-equivariant.

The current inquiry pertains to the scenario wherein set $A$ does not possess finiteness. Specifically, we endeavor to discern the feasibility of extending the definition and implications of $\phi$-cellular automata to instances where the alphabet comprises elements of a finite-dimensional vector space. Furthermore, an exploration into the possibility of extending the definition and implications of $\phi$-cellular automata to situations involving a nearspace as the alphabet is warranted.

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