Investigative Study on an Energy Theorem for Developing Testing Functions for Numerical Simulations of Multibody Dynamic Systems

Kulasegaram Gugaratshan

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INVESTIGATIVE STUDY ON AN ENERGY THEOREM FOR DEVELOPING
TESTING FUNCTIONS FOR NUMERICAL SIMULATIONS OF
MULTIBODY DYNAMIC SYSTEMS

by

Kulasegaram Gugaratshan

A Thesis
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Master of Science in Engineering (Mechanical)
Department of Mechanical and Aeronautical Engineering

Western Michigan University
Kalamazoo, Michigan
April 2008
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2008
ACKNOWLEDGEMENTS

I like to thank Dr. James Kamman, advisor, for his priceless help throughout my class work and thesis work. His support and patience helped me overcome my life changing events to fulfill my degree requirements.

Also, I like to thank my thesis committee members Dr. Koorosh Naghshineh and Dr. Dennis VandenBrink for their support.

Finally, I want to thank my wife, Watzala, for her patience with me when I was working on the thesis nights after nights sitting in front of my computer.

Kulasegaram Gugaratshan
INVESTIGATIVE STUDY ON AN ENERGY THEOREM FOR DEVELOPING TESTING FUNCTIONS FOR NUMERICAL SIMULATIONS OF MULTIBODY DYNAMIC SYSTEMS

Kulasegaram Gugaratshan, M.S.E.
Western Michigan University, 2008

Multibody dynamic systems designers utilize computer software extensively to simulate the motion of mechanical systems. The governing differential equations of motion are formulated and solved numerically. The accuracy of the numerical solutions is always a concern to the analyst. There have been many investigations done around the world to check the accuracy of these simulations, and based on these investigations, many methods have been developed. These methods are usually based on energy balance equations and can be used to test the validity or the correctness of the numerical solutions. This thesis analyzes and compares these methods based on their formulation and their applicability to test the accuracy of the simulations. A MATLAB/Simulink model of a triple pendulum example system is used to demonstrate the utility of each method. The methods are used to identify numerical inaccuracies associated with the differential equation solver, the integration step size, and errors in the equations of motion.
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CHAPTER 1

SOLVING EQUATIONS OF MOTION

1.1. Introduction

Dynamics has been the subject of extensive research over the past three decades. The advent of computers contributed significant improvement in the field of dynamics. Scientists and engineers realized the need for understanding the dynamics of interconnected bodies by computer simulation before spending money on physical experiments. Dynamics of the bodies are highly nonlinear in nature and their analysis requires the use of matrix, numerical, and computer methods [1].

The non-linear equations of motion of large-scale mechanical systems are too complicated to be developed in closed-form. Consequently, they are usually formulated and solved numerically. Given the complexity of this computational procedure, the correctness of the equations and the accuracy of their solution is always in question. Generally, simulation errors may be a result of inaccuracies associated with the differential equation solver, the integration step size, and errors in the equations of motion [2].

Recently, there has been increasing focus by analysts in finding ways to test the accuracy of numerical solutions [2,3,5,6,8,9]. Accuracy of the solution can be checked by intuitive or physical reasoning or using a conservation principle regulating the solution [3]. Testing functions have been developed based on the conservation principles.
However, these testing functions are generally dependent upon the mechanical systems specified. Analysts like T.R Kane, D.A Levinson, R. L. Huston etc. [2,3,4], have worked on many forms of testing functions for the mechanical systems. The development of testing functions has generally been dependent upon the mechanical system under consideration, and often also upon the insight of the analyst [3]. There seems to be no fundamental starting point for testing function development. System constraints, dissipation forces, and nonlinearities seem to control the applicability of the developed testing functions.

Keeping this in mind, it will be very useful to understand the applicability of testing functions on various forms of differential equations that were developed for mechanical systems. Also, it is useful to see the differences and similarities between the testing functions in an effort to pave the way to develop a universal testing function for dynamic systems. This can help scientists to formulate a method, which corrects the computer simulations using the testing functions.

1.2. Mechanical System and Coordinates

A multibody system consists of number of parts, subject to interconnections and constraints of various kinds. In a constrained mechanical system, the total work performed by the effective forces due to infinitesimal virtual displacement is zero and it became a foundation to the development of Lagrangian dynamics [12]. There are several ways of representing a rigid body in space. Depending on the number of generalized coordinates selected to define the configuration of a mechanical system, different equation structures can be obtained and different solution procedures can be adopted. Some of the formulations lead to equations that are expressed in terms of the constraint
forces, while in other formulations, the constraint forces are eliminated automatically. Equations of motion of simple systems can be formulated using a minimum number of independent coordinates or using a redundant set of coordinates. It is always possible to obtain a set of dynamic equations which do not include constraint forces when the independent sets of coordinates are used. Use of a dependent set of coordinates can have computational advantages and can also increase the generality and flexibility of the formulation used. When the equations of motion are formulated using an independent set of coordinates (equal in number to the number of degree of freedom of the system), one obtains differential equations that can be solved using a simpler numerical strategy. When the equations of motion are formulated in terms of dependent set of coordinates, a more elaborate numerical method must be used and that may increase the numerical solution error.
CHAPTER 2

DEVELOPING TESTING FUNCTIONS

Scientist developed many testing functions to check the simulation of a dynamic system. All the testing functions are based on some form of energy balance equation. Of all the testing functions developed, three testing functions stand out as unique. They were developed by Kane and Levinson [2,4], Liu and Huston [3], and Wang [8]. This chapter will discuss these testing functions and compare them analytically using Wang’s mathematical notations [8].

2.1. Kane and Levinson’s Method, 1988

For discrete holonomic dynamic systems having “n” degrees of freedom, Kane and Levinson [2] proposed the following procedure for checking the accuracy of numerical simulations of the system dynamics.

1. Define a set of generalized speeds \( u_r \) \( (r = 1, \ldots, n) \) from the derivatives of the generalized coordinates as

\[
U_r = \sum_{s=1}^{n} (Y_{rs} \dot{q}_s) + Z_r \quad (r = 1, \ldots, n) \tag{2.1.1}
\]

where \( Y_{rs} \) and \( Z_r \) are in general functions of the generalized coordinates \( q_i \) \( (i = 1, \ldots, n) \) and time \( t \). Then, invert this relationship to find

\[
\dot{q}_s = \sum_{r=1}^{n} (W_{rs} u_r) + X_s \quad (s = 1, \ldots, n) \tag{2.1.2}
\]
Here again, $W_{sr}$ and $X_s$ are functions of the generalized coordinates and time.

2. Find $\vec{F}_r$ $(r = 1, \ldots, n)$ the generalized active forces (section 2.5.1.e) associated with the generalized coordinates.

3. Find the functions $V(q_1, \ldots, q_n, t)$ and $G_r(q_1, \ldots, q_n, t)$ that satisfy the following equations

$$\vec{F}_r = -\sum_{s=1}^{n} \left( \frac{\partial V}{\partial q_s} \right) W_{sr} + G_r \quad (r = 1, \ldots, n) \quad (2.1.3)$$

and

$$\frac{\partial V}{\partial t} + \sum_{s=1}^{n} \left( \frac{\partial V}{\partial q_s} X_s \right) = 0 \quad (2.1.4)$$

Note that the functions $V$ and $G_r$ are found by inspection of the generalized forces. A good candidate for the function $V(q_1, \ldots, q_n, t)$ is the potential energy function for the system.

4. Find the kinetic energy function for the system expressed in the form $K = K_2 + K_1 + K_0$. Here, $K_2$ is quadratic in the $\dot{q}_s$, $K_1$ is linear in the $\dot{q}_s$, and $K_0$ depends only on the generalized coordinates and time.

5. Integrate the equations of motion of the system along with the following differential equation for the variable $z$

$$\dot{z} = -\sum_{r=1}^{n} G_r u_r + \frac{\partial K}{\partial t} + \sum_{s=1}^{n} \left( \frac{\partial K}{\partial q_s} X_s \right) \quad (2.1.5)$$

where $K$ is the kinetic energy of the system.

6. Calculate the following equation.

$$C = V + z + K_2 - K_0 \quad (2.1.6)$$
If C is not constant, then not all integration results are correct.

The difficulty with this method is identifying the functions \( V(q_1, \ldots, q_n, t) \) and \( G_r(q_1, \ldots, q_n, t) \) \( (r = 1, \ldots, n) \) in step 3. The forms of these functions depend on the mechanical system being analyzed. As mentioned earlier, however, the potential energy function for the system is a good candidate for the function \( V(q_1, \ldots, q_n, t) \).

### 2.2. Kane and Levinson’s Method Extended, 1990

The method described above was later found not to apply to all dynamic systems, so Kane and Levinson [4] modified their procedure. The new method was essentially the same as the old method, except the differential equation for the variable \( z \) was changed to be

\[
\dot{z} = -\sum_{r=1}^{n}(G_r u_r) + \sum_{\text{bodies}} \left[ (m v_G \cdot \dot{v}_t) - \left( \omega_B \cdot \dot{L} \cdot \dot{q}_t \right) \right]
\]  

Here, the second summation is over all the bodies in the system. The vectors \( \dot{v}_t \) and \( \dot{q}_t \) are the time derivatives of \( v_t \) and \( q_t \) which are found by inspection of the following equations of the velocities of the mass centers and the angular velocities of the bodies.

\[
v_G = \sum_{r=1}^{n} \left( \frac{\partial v_G}{\partial u_r} \right) u_r + v_t
\]

\[
\omega_B = \sum_{r=1}^{n} \left( \frac{\partial \omega_B}{\partial u_r} \right) u_r + \omega_t
\]

The line of reasoning that leads to this new equation (2.2.1) begins with the observation from Kane and Levinson [4,7] shown in the equation (2.2.4)

\[
-\sum_{r=1}^{p} \vec{F}_r u_r = \dot{K}_2 - \dot{K}_9 + \sum_{\text{bodies}} \left[ (m v_G \cdot \dot{v}_t) + \left( \omega_B \cdot \dot{L} \cdot \dot{q}_t \right) \right]
\]
Here, $\tilde{F}_r$ is the generalized inertia forces (section 2.5.1.f) associated with $u_r$ and $I$ is the inertia dyadic. The difficulty of finding the functions $V$ and $G_r$ ($r = 1, \ldots, n$) is still present in this method.

### 2.3. Liu and Huston’s Method, 1995

Liu and Huston’s Method [3], proposed a testing function that uses Kane’s equations for general mechanical systems. The testing function is given by equation (2.3.1) for a simple nonholonomic system. It says that the derivative of the kinetic energy $K$ of a system may be expressed as a linear combination of products of generalized active forces $\tilde{F}_r$ (see section 2.5.1.e) and generalized speeds $u_r$ ($r = 1, \ldots, n$).

$$\frac{dK}{dt} = \sum_{r=0}^{n} \tilde{F}_r u_r \quad (2.3.1)$$

When expanding equation (2.3.1), $\frac{dK}{dt} = \tilde{F}_0 + \tilde{F}_1 u_1 + \ldots + \tilde{F}_n u_n$, $\tilde{F}_0$ is the acatastatic term (see Appendix A). Equation (2.3.1) must be satisfied in addition to the governing dynamical equations [6].

Integrating the equation (2.3.1) over the time interval $t_0 \to t$ gives

$$\int_{t_0}^{t} \left( \frac{dK}{dt} \right) dt - \int_{t_0}^{t} \left( \sum_{r=0}^{n} \tilde{F}_r u_r \right) dt = 0 \quad (2.3.2)$$

$$K - K_0 - \int_{t_0}^{t} \left( \sum_{r=0}^{n} \tilde{F}_r u_r \right) dt = 0$$

or

$$K - K_0 = \int_{t_0}^{t} \left( \sum_{r=0}^{n} \tilde{F}_r u_r \right) dt \quad (2.3.3)$$
If the initial kinetic energy \( K_0 = 0 \)

\[
K = \int_{t_0}^{t} \left( \sum_{r=0}^{n} \tilde{F}_r u_r \right) dt \tag{2.3.4}
\]

If the initial kinetic energy \( K_0 \) is zero, then the instantaneous value of the kinetic energy can be compared directly to the integral as stated in Equation(2.3.4). However, if the \( K_0 \neq 0 \), then the integral gives the kinetic energy difference as suggested by Equation(2.3.3).

A testing function that should remain constant through any simulation could be defined to be

\[
\Delta C = \left[ K - \int_{t_0}^{t} \left( \sum_{r=0}^{n} \tilde{F}_r u_r \right) dt \right] - K_0 \tag{2.3.5}
\]

2.4. Wang’s Method, 1993

Wang [8] formulated a more general form of testing function which is based on Kane’s multibody formulations of equation of motion[7]. The testing function can be given using kinetic energy, potential energy and net energy flow into the system.

\[
\dot{E} = \dot{K} + \dot{V} - \dot{Q} = 0 \tag{2.4.1}
\]

where \( \dot{E}, \dot{K} \) and \( \dot{V} \) are the time derivatives of the total mechanical energy, kinetic energy, and potential energy of the system respectively. \( \dot{Q} \) is the net power flowing into the system or work being done on the system. A positive \( \dot{Q} \) signifies that the net power is flowing into the system and, conversely, a negative \( \dot{Q} \) indicates that the net power is
flowing out of the system. The integral of the equation (2.4.1) is a constant.

\[ K + V - Q = \text{constant} \]  

(2.4.2)

Equation (2.4.2) is the adjusted total energy of the system as discussed by Wang[8]. Both equations (2.4.1) and (2.4.2) can be used to test the simulations to identify any simulation errors.

2.5. Testing Function Development

Wang’s energy conservation methodology[8] was analyzed in depth in order to understand the similarities between testing functions developed by Kane and Levinson[2,4], Liu and Huston[3], and Wang [8]. Matrix mathematical notations are adopted similar to Wang’s paper[8].

2.5.1. Unconstrained System of \( N \) Bodies

Consider a set of \( N \) rigid bodies whose motion is unconstrained. The position and velocity of the system at any time can be described by a \( 6N \times 1 \) vector of translational and angular coordinates and speeds.

a) Generalized Coordinates

Define a vector of generalized coordinates. The first coordinates are the Cartesian coordinates of the mass centers of the bodies relative to an inertial frame, and the last are absolute orientation angles of each of the bodies relative to an inertial frame.

\[ \bar{x} = (x_1, x_2, \ldots, x_{6N})^T = (\bar{x}_G^T | \bar{\theta}^T)^T \]  

(2.5.1)
b) Generalized Speeds

Define a vector of generalized speeds. The first coordinates are the inertial components of the velocities of the mass centers of the bodies relative to an inertial frame, and the last are the body-fixed components of the angular velocities of the bodies relative to an inertial frame. Moreover, the body-fixed directions are assumed to be principal directions of the bodies.

\[ \mathbf{v} = (v_1, v_2, \ldots, v_{6N})^T = \left( \begin{array}{cc} \mathbf{v}^T \end{array} \mathbf{\bar{\omega}}^T \right)^T = \left( \begin{array}{cc} \dot{\mathbf{x}}^T \mathbf{\dot{\theta}}^T \end{array} \right)^T \]  

(2.5.2)

c) Partial Velocities

From the definitions of \( \mathbf{\bar{x}} \) and \( \mathbf{v} \), we can write

\[ \mathbf{v} = \begin{bmatrix} \mathbf{v} \\ \mathbf{\bar{\omega}} \end{bmatrix} = \begin{bmatrix} I_D & 0 \\ 0 & \hat{P}_I \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\theta} \end{bmatrix} = \hat{P}_I \dot{\mathbf{x}} \]  

(2.5.3)

where \( P_I \) is a 6N x 6N matrix, \( I_D \) is a 3N x 3N identity matrix of partial velocity components, and \( \hat{P}_I \) is a 3N x 3N matrix of partial angular velocity components.

d) Kane’s Equations of Motion of the Unconstrained System

Using Kane’s equations, the equations of motion of the unconstrained system may be written as

\[ F + F^* = 0 \]  

(2.5.4)
where $F$ and $F^*$ are the generalized active and generalized inertia force vectors for the system.

**(e) Generalized Active Forces**

Vector $F_i$ is a $3\times1$ vector of inertial components of the resultant force acting at the center of mass of body “$i$” and vector $M_i$ is a $3\times1$ vector of inertial components of the corresponding resultant moment/torque acting on body “$i$”. Given these definitions, force and moment vectors can be formed for the multibody system as follows.

\[
E_f = \left( E_{1i}^T | E_{2i}^T | \cdots | E_{Ni}^T \right)^T \quad \text{and} \quad E_M = \left( M_{1i}^T | M_{2i}^T | \cdots | M_{Ni}^T \right)^T
\]  

Using the velocity and angular velocity components as the generalized speeds for Kane’s equations, the generalized active forces may be written as

\[
F = \left( \frac{\partial v}{\partial \dot{v}} \right)^T F_f + \left( \frac{\partial v}{\partial \dot{\theta}} \right)^T F_M = \begin{bmatrix} I_D & 0 \\ 0 & I_D \end{bmatrix} \begin{bmatrix} E_f \\ E_m \end{bmatrix} = \begin{bmatrix} E_f \\ E_m \end{bmatrix}
\]  

Also, we can state the active force vector into two terms such as potential force vector $F_v$ and non potential force vector $F_D$.

\[
F = F_v + F_D
\]
f) Generalized Inertia Forces

Vector \( \mathbf{E}_i \) is a 3x1 vector of inertial components of the inertia force of body “i” and \( \mathbf{M}_i \) is a 3x1 vector of inertial components of the corresponding inertia moment/torque of body “i”. Given these definitions, inertia force and inertia moment vectors can be formed for the multibody system as follows.

\[
\mathbf{E}_f = \left( (\mathbf{E}_1)^T \right) \cdots \left( (\mathbf{E}_N)^T \right)^T \quad \text{and} \quad \mathbf{E}_M = \left( (\mathbf{M}_1)^T \right) \cdots \left( (\mathbf{M}_N)^T \right)^T
\]

Using the velocity and angular velocity components as the generalized speeds for Kane’s equations, the generalized inertia forces may be written as

\[
\frac{\partial \mathbf{v}}{\partial \mathbf{q}} \mathbf{v} + \frac{\partial \mathbf{v}}{\partial \mathbf{\omega}} \mathbf{\omega} + \mathbf{E}_f = \left[ \begin{array}{c} \mathbf{I}_D \mathbf{0} \\ \mathbf{0} \mathbf{I}_D \end{array} \right] \mathbf{E}_f = \left( \begin{array}{c} \mathbf{E}_f^i \\ \mathbf{E}_M^i \end{array} \right) = \left\{ \begin{array}{c} -\mathbf{m}_B \mathbf{\ddot{v}} \\ -(\mathbf{I} \mathbf{\dot{\omega}} + \mathbf{\Omega} \mathbf{\dot{\omega}}) \end{array} \right\}
\]

Here the mass matrix \( \mathbf{M}_{m,i} = \left[ \begin{array}{cc} \mathbf{m}_B & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{array} \right] \) and \( B = 1,2,3,...,N \)

where \( \mathbf{m}_B \) is a 3N \times 3N system mass matrix, \( \mathbf{I} \) is a 3N \times 3N system inertia matrix, \( \mathbf{\ddot{v}} \) is a 3N \times 1 vector of inertial components of mass center velocities, \( \mathbf{\dot{\omega}} \) is a 3N \times 1 vector of body-fixed angular velocity components, and \( \mathbf{\Omega} \) is the 3N \times 3N dual matrix of \( \mathbf{\dot{\omega}} \) (see equation (2.5.13)). As defined by equation (2.5.9), \( \mathbf{M}_{m,i} \) is a 6N \times 6N generalized mass matrix. More detailed definitions of the system mass matrix \( \mathbf{m}_B \), the system inertia matrix \( \mathbf{I} \), and the system angular velocity dual matrix \( \mathbf{\Omega} \) are given below.
g) System Mass Matrix

\[
(E_f^*)_i = \begin{bmatrix}
m_i & 0 & 0 \\
0 & m_i & 0 \\
0 & 0 & m_i \\
\end{bmatrix}
\]  
(2.5.10)

The inertia force vector for the system may be then written as

\[
E_f^* = \begin{bmatrix}
(E_f^*)_1 \\
(E_f^*)_2 \\
\vdots \\
(E_f^*)_N \\
\end{bmatrix} = \begin{bmatrix}
m_1 & 0 & \ldots & 0 \\
0 & m_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & m_N \\
\end{bmatrix} \begin{bmatrix}
\dot{v}_1 \\
\dot{v}_2 \\
\vdots \\
\dot{v}_N \\
\end{bmatrix} \pm m_b \ddot{v}
\]  
(2.5.11)

h) System Inertia Matrix and Angular Velocity Dual Matrix

The inertia moment (torque) for a single body “i” of the multibody system may be written as

\[
(E_M^*)_i = \begin{bmatrix}
I_{ii} & 0 & 0 \\
0 & I_{12} & 0 \\
0 & 0 & I_{13} \\
\end{bmatrix} - \begin{bmatrix}
0 & -\bar{\omega}_{13} & \bar{\omega}_{12} \\
\bar{\omega}_{13} & 0 & -\bar{\omega}_{11} \\
-\bar{\omega}_{12} & \bar{\omega}_{11} & 0 \\
\end{bmatrix} \begin{bmatrix}
I_{ii} & 0 & 0 \\
0 & I_{12} & 0 \\
0 & 0 & I_{13} \\
\end{bmatrix}
\]  
(2.5.12)

Dual matrix \( \Omega_i \) is in fact the skew symmetric matrix of \( \bar{\omega} \) (see appendix for the skew symmetric matrix). Here it is assumed that the chosen body-fixed directions are aligned with the body principal directions. The system inertia moment vector may then be written as

\[
E_M^* = \begin{bmatrix}
(E_M^*)_1 \\
(E_M^*)_2 \\
\vdots \\
(E_M^*)_N \\
\end{bmatrix} = \begin{bmatrix}
I_1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix}
\bar{\omega}_1 \\
\bar{\omega}_2 \\
\vdots \\
\bar{\omega}_N \\
\end{bmatrix} \begin{bmatrix}
\Omega_1 & 0 & \ldots & 0 \\
0 & \Omega_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \Omega_N \\
\end{bmatrix} \begin{bmatrix}
I_1 & 0 & \ldots & 0 \\
0 & I_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & I_N \\
\end{bmatrix} \begin{bmatrix}
\bar{\omega}_1 \\
\bar{\omega}_2 \\
\vdots \\
\bar{\omega}_N \\
\end{bmatrix}
\]  
(2.5.13)
or

\[
E_N^+ \triangleq (I\ddot{\omega} + \Omega I\ddot{\omega})
\]  

(2.5.14)

where \( I = \begin{bmatrix} I_1 & 0 & \cdots & 0 \\ 0 & I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_N \end{bmatrix} \) and \( \Omega = \begin{bmatrix} \Omega_1 & 0 & \cdots & 0 \\ 0 & \Omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Omega_N \end{bmatrix} \)

2.5.2. System of \( N \) Bodies with Configuration Constraints

a) Configuration Constraints: Non-Time Varying

If the system is now subjected to \( m_c \) configuration constraints that are not time varying

\[
f_i(\bar{x}) = 0 \quad (i = 1, \ldots, m_c)
\]

(2.5.15)

then not all of the \( x_i \) are independent. These equations can also be presented as a set of linear equations in \( \dot{x} \) by simply differentiating them to give

\[
\left[ \begin{array}{c} \frac{\partial f_i}{\partial x} \end{array} \right] \ddot{x} = P_2 \ddot{x} = 0
\]

(2.5.16)

where \( P_2 \) is an \( m_c \times 6N \) matrix that is a function of \( \bar{x} \).

For the constrained system, define a new set of \( n = 6N - m_c \) independent coordinates

\[
\bar{q} = (q_1, q_2, \ldots, q_n)^T
\]

such that

\[
\ddot{x} = A_t \ddot{\bar{q}}
\]

(2.5.17)
Here, $A_i$ is a function of $q$ (or $x$). The original generalized speeds can be written in terms of $\dot{q}$ substituting from Equation (2.5.17) into (2.5.3)

$$v = P_i \dot{x} = P_i A_i \dot{q} \triangleq R \dot{q}$$

(2.5.18)

Here $R \triangleq P_i A_i$ is a $6N \times n$ matrix of partial velocity components associated with $\dot{q}$ and is a function of $q$ (or $x$).

If we can find a matrix $J$ such that $JR = 0$, then Equation (2.5.18) can be rewritten solely in terms of $v$ as follows

$$Jv = JR \dot{q} = 0$$

(2.5.19)

Here, $J$ is a $m \times 6N$ matrix and is an orthogonal complement of matrix $R$ [11].

b) Kane’s Equations of Motion of the System with Configuration Constraints: Non-Time Varying

Using Kane’s equations, the equations of motion of the constrained system may be written as[11]

$$F + F^s + F^c = F + F^s + J^T \dot{\lambda}_c = 0$$

(2.5.20)

As before, $F$ and $F^s$ represent the generalized active and generalized inertia forces for the system. $F^c$ represents the generalized active forces associated with the constraint forces and torques required to maintain the constraints, and $\dot{\lambda}_c$ represents a vector of undetermined multipliers related to the constraint forces and torques.

These equations can be transformed into a simpler form by pre-multiplying by $R^T$.

$$R^T (F + F^s + J^T \dot{\lambda}_c) = R^T F + R^T F^s + R^T J^T \dot{\lambda}_c \triangleq \hat{F} + \hat{F}^s = 0$$

(2.5.21)
Here, \( \mathbf{\hat{F}} \triangleq R^T \mathbf{F} \) and \( \mathbf{\hat{F}^*} \triangleq R^T \mathbf{F}^* \) are the generalized active and generalized inertia forces for the constrained system. The vectors \( \mathbf{F} \) and \( \mathbf{F}^* \) are defined by equations (2.5.6) and (2.5.9). Note that the term associated with the constraint forces and torques is zero, because \( R^T J^T = (JR)^T = 0 \) [8].

### 2.5.3. Configuration Constraints and Motion Constraints

This section discusses the system of bodies with configuration constraints and motion constraints.

**a) Motion Constraint**

As discussed by J.T. Wang [8], now, consider constraining the system further with a set of \( m_m \) motion constraints

\[
A\mathbf{\dot{v}} + b = AR\mathbf{\dot{q}} + b = \mathbf{\hat{A}\dot{q} + b = 0 (2.5.22)}
\]

where \( A \) is an \( m_m \times 6N \) matrix, \( \mathbf{\hat{A}} \triangleq AR \) is an \( m_m \times n \) matrix, \( b \) is an \( m_m \times 1 \) vector, and all are functions of the generalized coordinates \( \mathbf{q} \) and time \( t \). Given these constraints, only \( n-m_m \) of the \( q_i \) are independent. Now, define a new set of independent generalized speeds \( u \) such that

\[
\mathbf{\dot{q}} = \mathbf{\hat{R}\dot{u} + b (2.5.23)}
\]

where \( \mathbf{\hat{R}} \) is an \( n \times (n-m_m) \) matrix \( u \) is \( (n-m_m) \times 1 \) matrix and \( \mathbf{b} \) is an \( n \times 1 \) vector. If \( \mathbf{\hat{R}} \) and \( \mathbf{b} \) are chosen such that

\[
\mathbf{\hat{A}\hat{R} = 0 \ and \ \hat{A}\mathbf{\dot{b}} = -b (2.5.24)}
\]
then, multiplying equation (2.5.23) by \( \hat{A} \) gives

\[
\hat{A} \dot{q} = \hat{A} \hat{R} u + \hat{A} \hat{b} = -b \\
\text{or} \\
\hat{A} \dot{q} + b = 0
\]  

(2.5.25)

This last result is the constraint equation (2.5.22). Note that the matrix \( \hat{R} \) is an orthogonal complement of \( \hat{A} \).

The velocity vector \( v \) can now be found in terms of \( u \) the new vector of generalized speeds by substituting equation (2.5.23) into equation (2.5.18) to give

\[
v = R \dot{q} = R \left( \hat{R} u + \hat{b} \right) = \tilde{R} u + v_i
\]  

(2.5.26)

Here, \( \tilde{R} \triangleq R \hat{R} \) is an \( 6N \times (n - m_m) \) matrix, and \( v_i = R \hat{b} \) is a \( 6N \times 1 \) vector. Both \( \tilde{R} \) and \( v_i \) can be functions of generalized coordinates \( q \) and time \( t \). For the nonholonomic system, \( v_i = R \hat{b} \neq 0 \), therefore \( v_i \) is the catastatic term.

b) Partial Velocities for the Fully Constrained System

Using equations (2.5.2) and (2.5.26), we can use partitioned matrix multiplication to write

\[
v = \begin{pmatrix} \dot{v} \\ \dot{\omega} \end{pmatrix} = \begin{bmatrix} \tilde{R} & 0 \\ \tilde{R} & 0 \end{bmatrix} u + \begin{bmatrix} \dot{v}_i \\ \dot{\omega}_i \end{bmatrix} 
\]  

(2.5.27)
where \( \tilde{R}_1 \) and \( \tilde{R}_2 \) are \( 3N \times (n - m_w) \) partial velocity and partial angular velocity matrices, and \( \mathbf{\bar{v}}_i \) and \( \mathbf{\bar{\omega}}_i \) are \( 3N \times 1 \) vectors.

c) Kane’s Equations of Motion for the Fully Constrained System

Using Kane’s equations, the equations of motion of the constrained system may be written as [11],

\[
F + F^* + F' = F + F^* + \begin{bmatrix} J^T \mid A^T \end{bmatrix} \begin{bmatrix} \lambda_c \\lambda_m \end{bmatrix} = 0
\]  

(2.5.28)

As before, \( F \) and \( F^* \) represent the generalized active and generalized inertia forces for the system. \( F' \) represents the generalized active forces associated with the constraint forces and torques required to maintain the constraints, and \( \lambda_c \) and \( \lambda_m \) represent vectors of undetermined multipliers related to the constraint forces and torques associated with the configuration and motion constraints.

\[
F' = \begin{bmatrix} J^T \mid A^T \end{bmatrix} \begin{bmatrix} \lambda_c \\lambda_m \end{bmatrix}
\]  

(2.5.29)

These equations can be transformed into a simpler form by pre-multiplying by \( \tilde{R}^T \) [8].

\[
\tilde{R}^T \left( F + F^* + F' \right) = \tilde{R}^T F + \tilde{R}^T F^* + \tilde{R}^T \begin{bmatrix} J^T \mid A^T \end{bmatrix} \begin{bmatrix} \lambda_c \\lambda_m \end{bmatrix} = 0
\]  

(2.5.30)

Here, \( \tilde{R}^T F = F \) and \( \tilde{R}^T F^* = F^* \) are the generalized active force and generalized inertia force vectors for the constrained system, and
\[
\hat{R}^T \left[ J^T \mid A^T \right] \begin{bmatrix} \frac{\dot{\omega}}{\omega_m} \\ \frac{\dot{\omega}}{\omega_m^2} \end{bmatrix} = \hat{R}^T \hat{R}^T \left( J^T \right) \begin{bmatrix} \frac{\dot{\omega}}{\omega_m} \\ \frac{\dot{\omega}}{\omega_m^2} \end{bmatrix} \\
= \left( \hat{R}^T \hat{R}^T J^T \right) \begin{bmatrix} \frac{\dot{\omega}}{\omega_m} \\ \frac{\dot{\omega}}{\omega_m^2} \end{bmatrix} \\
= \left( \hat{R}^T \hat{R}^T J^T \right) \begin{bmatrix} \frac{\dot{\omega}}{\omega_m} \\ \frac{\dot{\omega}}{\omega_m^2} \end{bmatrix} \\
= 0
\]

The result of equation (2.5.31) is zero, because \( R^T J^T = (JR)^T = 0 \) and \( \hat{R}^T \hat{A}^T = (\hat{A}^T)^T = 0 \). So, the equations of the fully constrained system are

\[ \hat{F} + \hat{F}^s = 0 \] (2.5.32)

The generalized active and inertia force vectors for the fully constrained system (associated with the generalized speeds \( \dot{\mathbf{u}} \)) may now be written as

\[ \hat{F} = \hat{R}\hat{F} = \hat{R}\begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{bmatrix} = \left( \begin{array}{c} \hat{F}_1 \\ \hat{F}_2 \end{array} \right) = \hat{R}\hat{F}_1 + \hat{R}\hat{F}_2 \] (2.5.33)

\[ \hat{F}^s = \hat{R}\hat{F}_1^s + \hat{R}\hat{F}_2^s = -\hat{R}\hat{M}_{\hat{v}} - \hat{R}\hat{F}_2^s \left( I\hat{\omega} + \hat{\Omega}\hat{\omega} \right) \]

\[ = -\hat{R}\begin{bmatrix} \hat{M} \\ 0 \end{bmatrix} \begin{bmatrix} \hat{\dot{\mathbf{v}}} \\ \hat{\dot{\omega}} \end{bmatrix} - \hat{R} \begin{bmatrix} \hat{\dot{\mathbf{v}}} \\ \hat{\dot{\omega}} \end{bmatrix} \begin{bmatrix} 0 \\ \hat{\Omega}\hat{\omega} \end{bmatrix} \] (2.5.34)

\[ = -\hat{R}\begin{bmatrix} M_{\hat{m}} \hat{\dot{\mathbf{v}}} + \frac{\hat{\dot{\omega}}}{\hat{\omega}} \end{bmatrix} \]

\[ \Delta = \hat{R}\begin{bmatrix} M_{\hat{m}} \hat{\dot{\mathbf{v}}} + \frac{\hat{\dot{\omega}}}{\hat{\omega}} \end{bmatrix} \]

\[ d) \text{ Kinetic Energy} \]

The kinetic energy of the multibody system may be written as the sum of the kinetic energies of the individual bodies

\[ K = \sum_i (K)_i = \sum_i \left( \frac{1}{2} \hat{v}^T \hat{m}_i \hat{v} + \frac{1}{2} \hat{\omega}_i^T \hat{I}_i \hat{\omega}_i \right) \] (2.5.35)
This can be written in a more compact matrix form as

\[
K = \frac{1}{2} \begin{bmatrix} 0 & m_b \\ m_b & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\theta} \end{bmatrix} = \frac{1}{2} \gamma^T M_{m,l} \gamma
\]  

(2.5.36)

This result can be expressed in terms of the generalized speeds by using equation (2.5.26) as follows.

\[
K = \frac{1}{2} \gamma^T M_{m,l} \gamma = \frac{1}{2} \left( \tilde{R} \dot{u} + \gamma \right)^T M_{m,l} \left( \tilde{R} \dot{u} + \gamma \right) \\
= \frac{1}{2} \gamma^T \tilde{R}^T M_{m,l} \tilde{R} \dot{u} + \gamma^T M_{m,l} \tilde{R} \dot{u} + \frac{1}{2} \gamma^T M_{m,l} \gamma \\
\triangleq K_2 + K_1 + K_0
\]  

(2.5.37)

Here \( K_2 = \frac{1}{2} \gamma^T \tilde{R}^T M_{m,l} \tilde{R} \dot{u} \) which is quadratic in \( \gamma \), \( K_1 = \gamma^T M_{m,l} \tilde{R} \dot{u} \) which is linear in \( \gamma \), and \( K_0 = \frac{1}{2} \gamma^T M_{m,l} \gamma \) which is independent of \( \gamma \).

The time derivative of the kinetic energy can be found by differentiating these results as follows.

\[
\frac{dK}{dt} = \dot{K}_2 + \dot{K}_1 + \dot{K}_0
\]  

(2.5.38)

where

\[
\dot{K}_2 = \frac{1}{2} \frac{d}{dt} \left( \left( \tilde{R} \dot{u} \right)^T M_{m,l} \left( \tilde{R} \dot{u} \right) \right) = \frac{1}{2} \left( \frac{d}{dt} \left( \tilde{R} \dot{u} \right)^T M_{m,l} \left( \tilde{R} \dot{u} \right) + \left( \tilde{R} \dot{u} \right)^T \frac{d}{dt} M_{m,l} \left( \tilde{R} \dot{u} \right) \right) \\
= \left( \tilde{R} \dot{u} \right)^T M_{m,l} \frac{d}{dt} \left( \tilde{R} \dot{u} \right) \\
= \dot{u}^T \tilde{R}^T M_{m,l} \tilde{R} \dot{u} + \dot{\gamma}^T \tilde{R}^T M_{m,l} \tilde{R} \dot{u}
\]  

(2.5.39)

\[
\dot{K}_1 = \gamma^T M_{m,l} \tilde{R} \dot{u} + \gamma^T M_{m,l} \tilde{R} \dot{u} + \gamma^T M_{m,l} \tilde{R} \dot{u}
\]  

(2.5.40)

\[
\dot{K}_0 = \frac{1}{2} \gamma^T M_{m,l} \gamma + \frac{1}{2} \gamma^T M_{m,l} \gamma = \gamma^T M_{m,l} \gamma
\]  

(2.5.41)
e) Potential Energy

From the definition of the potential energy \([7]\), we can say

\[
\dot{V} = -y^T E_v
\]  

(2.5.42)

Recall that \(E_v\) is the potential force vector defined by equation (2.5.7).

2.5.4. Derivation of Energy Conservation Theorem

This section discusses the derivation of the energy conservation theorem which is also known as power equation. The equations of motion as expressed in equation (2.5.32) can be converted into a single power equation by pre-multiplying the equation by \(-y^T\) as follows.

\[
-y^T (\tilde{F} + \tilde{F}^*) = 0
\]  

(2.5.43)

From equations (2.5.33) and (2.5.34)

\[
-u^T \tilde{R}^T F + u^T \tilde{R}^T M_{mi} \dot{y} + u^T \tilde{R}^T \left\{ \frac{0}{\Omega^T \omega} \right\} = 0
\]  

(2.5.44)

Based on the above equation (2.5.44), the power equation will be formed.
a) First Term $-\mathbf{u}^T \tilde{\mathbf{R}}^T \mathbf{F}$

The first term involves the generalized force vector $\mathbf{F}$. In the development of equation (2.5.45) below, the force vector is separated into contributions from conservative forces and moments ($\mathbf{F}_v$) and non-conservative forces and moments ($\mathbf{F}_d$).

\[
-\mathbf{u}^T \tilde{\mathbf{R}}^T \mathbf{F} = -\left( \tilde{\mathbf{R}}\mathbf{u} \right)^T \mathbf{F} \\
= - (\mathbf{y} - \mathbf{y}_r)^T (\mathbf{F}_v + \mathbf{F}_d) \\
= -\mathbf{y}^T (\mathbf{F}_v + \mathbf{F}_d) + \mathbf{y}_r^T \mathbf{F} \\
= -\mathbf{y}^T \mathbf{F}_v - \mathbf{y}^T \mathbf{F}_d + \mathbf{y}_r^T \mathbf{F} \\
= \mathbf{\dot{V}} - \mathbf{\dot{y}}^T \mathbf{F}_d + \mathbf{\dot{y}}_r^T \mathbf{F}
\]  

(2.5.45)

b) Second Term $\mathbf{u}^T \tilde{\mathbf{R}}^T \mathbf{M}_{m,l} \mathbf{\ddot{\mathbf{y}}}$

The second term can be related to changes in the kinetic energy function as follows.

\[
\mathbf{u}^T \tilde{\mathbf{R}}^T \mathbf{M}_{m,l} \mathbf{\ddot{\mathbf{y}}} = \mathbf{u}^T \tilde{\mathbf{R}}^T \mathbf{M}_{m,l} \frac{d}{dt} (\tilde{\mathbf{R}}\mathbf{u} + \mathbf{y}_r) \\
= \mathbf{u}^T \tilde{\mathbf{R}}^T \mathbf{M}_{m,l} \tilde{\mathbf{R}}\mathbf{u} + \mathbf{u}^T \tilde{\mathbf{R}}^T \mathbf{M}_{m,l} \tilde{\mathbf{R}}\mathbf{\dot{u}} + \mathbf{u}^T \tilde{\mathbf{R}}^T \mathbf{M}_{m,l} \mathbf{\ddot{y}}_l \\
= \dot{\mathbf{K}}_2 + \left( \tilde{\mathbf{R}}\mathbf{u} \right)^T \mathbf{M}_{m,l} \mathbf{\ddot{y}}_l \\
= \dot{\mathbf{K}}_2 + (\mathbf{y} - \mathbf{y}_r)^T \mathbf{M}_{m,l} \mathbf{\ddot{y}}_l \\
= \dot{\mathbf{K}}_2 + \mathbf{y}^T \mathbf{M}_{m,l} \mathbf{\ddot{y}}_l - \mathbf{y}_r^T \mathbf{M}_{m,l} \mathbf{\ddot{y}}_l \\
= \dot{\mathbf{K}}_2 - \mathbf{K}_0 + \mathbf{y}^T \mathbf{M}_{m,l} \mathbf{\ddot{y}}_l \\
= \dot{\mathbf{K}}_2 - \mathbf{K}_0 + \mathbf{\dot{y}}^T \mathbf{M}_{m,l} \mathbf{\ddot{y}}_l
\]  

(2.5.46)

c) Third Term $\mathbf{u}^T \tilde{\mathbf{R}}^T \left\{ \frac{0}{\Omega^T \mathbf{\bar{\omega}}} \right\}$

In equation (2.5.47), $\Omega^T \mathbf{\bar{\omega}}$ is perpendicular to $\mathbf{\bar{\omega}}$. So, the scalar product of $\mathbf{\bar{\omega}}$ and $\Omega^T \mathbf{\bar{\omega}}$ is zero

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\[
\begin{align*}
\mathbf{u}^T \mathbf{R}^T \begin{bmatrix}
0 \\
\Omega \mathbf{I} \bar{\omega}
\end{bmatrix}
&= \mathbf{u}^T \left[ \begin{bmatrix} \mathbf{R}^T \mathbf{I} \mathbf{R}^T \end{bmatrix} \begin{bmatrix} 0 \\
\Omega \mathbf{I} \bar{\omega}
\end{bmatrix} \right] \\
&= \mathbf{u}^T \mathbf{R}^T (0) + \mathbf{u}^T \mathbf{R}^T \Omega \mathbf{I} \bar{\omega} \\
&= (\mathbf{R}_2 \mathbf{u})^T \Omega \mathbf{I} \bar{\omega} \\
&= (\bar{\omega} - \bar{\omega}_0)^T \Omega \mathbf{I} \bar{\omega} \\
&= \bar{\omega}^T \Omega \mathbf{I} \bar{\omega} - \bar{\omega}_0^T \Omega \mathbf{I} \bar{\omega} \\
&= \bar{\omega}^T \Omega \mathbf{I} \bar{\omega} \\
&= \mathbf{u}_t^T \begin{bmatrix} 0 \\
\Omega \mathbf{I} \bar{\omega}
\end{bmatrix}
\end{align*}
\]

(2.5.47)

The above matrix manipulation can be explained by vector notation and the following example will elaborate. Here assume the elements of the equation (2.5.48) are vectors for this elaboration only.

\[
(\bar{\omega} - \bar{\omega}_0)^T (\bar{\omega} \times \mathbf{I}_B \bar{\omega}) = (\bar{\omega} - \bar{\omega}_0) \cdot (\bar{\omega} \times \mathbf{I}_B \bar{\omega}) \\
= \bar{\omega} \cdot (\bar{\omega} \times \mathbf{I}_B \bar{\omega}) - \bar{\omega}_0 \cdot (\bar{\omega} \times \mathbf{I}_B \bar{\omega}) \\
= \bar{\omega} \times \bar{\omega} \cdot (\mathbf{I}_B \bar{\omega}) - \bar{\omega}_0 \cdot (\bar{\omega} \times \mathbf{I}_B \bar{\omega}) \\
= 0 - \bar{\omega}_0 \cdot (\bar{\omega} \times \mathbf{I}_B \bar{\omega}) \\
= -\bar{\omega}_0 \cdot (\bar{\omega} \times \mathbf{I}_B \bar{\omega})
\]

(2.5.48)

Substituting the results from equations (2.5.45), (2.5.46), and (2.5.47) into equation (2.5.44) gives

\[
\dot{V} - \mathbf{y}^T \mathbf{F}_D + \mathbf{y}_t^T \mathbf{F} + \dot{K}_2 - \dot{K}_0 + \mathbf{y}^T \mathbf{M}_{m,t} \dot{v}_t - \mathbf{y}_t^T \begin{bmatrix} 0 \\
\Omega \mathbf{I} \bar{\omega}
\end{bmatrix} = 0
\]

(2.5.49)

or

\[
\dot{K}_2 - \dot{K}_0 + \dot{V} + \dot{Z} = 0
\]

(2.5.50)
Equation (2.5.50) is similar to the equation presented by Kane and Levinson as shown in equation (2.1.6). As written, it contains the system’s potential energy, but only part of the system’s kinetic energy. To make use of the complete kinetic energy function, it can be written as in equation (2.5.51).

\[
\dot{K} - \dot{K}_0 = \left( \dot{K}_2 + \dot{K}_1 + \dot{K}_0 \right) - \dot{K}_1 - 2\dot{K}_0
\]

(2.5.51)

Substituting this result into equation (2.5.50) gives

\[
\dot{K} + \dot{V} = 0
\]

(2.5.52)

\[
\left( \dot{Z} - \dot{K}_1 - 2\dot{K}_0 \right)
\]

can be manipulated as below.

\[
\dot{Z} - \dot{K}_1 - 2\dot{K}_0 = -\dot{y}^T F_D + x_i^T F + y^T M_{m,i} \ddot{y}_i - y_i^T \left\{ \frac{0}{\Omega^T \Omega} \right\} - \dot{y}_i^T M_{m,i} \ddot{R}_u - \frac{d}{dt} \left( \ddot{R}_u \right) - 2y_i^T M_{m,i} \ddot{y}_i
\]

\[
= -\dot{y}^T F_D + x_i^T F - y^T M_{m,i} \ddot{y}_i - y_i^T \left\{ \frac{0}{\Omega^T \Omega} \right\} - \dot{y}_i^T M_{m,i} \ddot{y}_i - y_i^T M_{m,i} \left( \ddot{y} - \ddot{y}_i \right) - 2y_i^T M_{m,i} \ddot{y}_i
\]

\[
= -\dot{y}^T F_D + x_i^T F + y^T M_{m,i} \ddot{y}_i - y_i^T M_{m,i} \ddot{y}_i - y_i^T \left\{ \frac{0}{\Omega^T \Omega} \right\} - y_i^T M_{m,i} \ddot{y}_i
\]

\[
+ y_i^T M_{m,i} \ddot{y}_i + y_i^T M_{m,i} \ddot{y}_i - 2y_i^T M_{m,i} \ddot{y}_i
\]

\[
= -\dot{y}^T F_D + x_i^T F - y_i^T \left\{ \frac{0}{\Omega^T \Omega} \right\} - y_i^T M_{m,i} \ddot{y}_i
\]

\[
= y_i^T \left( F - M_{m,i} \ddot{y} - \left\{ \frac{0}{\Omega^T \Omega} \right\} \right) - y_i^T F_D
\]

\[
= y_i^T \left( F + F^* \right) - y_i^T F_D
\]

\[
\Delta = -\dot{Q}
\]

(2.5.53)

Substituting from equation (2.5.53) into equation (2.5.52) gives the final form of Wang’s
power equation.

\[ \dot{K} + \dot{V} - \dot{Q} = 0 \]  \hspace{1cm} (2.5.54)

Where

\[ \dot{Q} \triangleq \nu^T E_D - \nu_i^T (E + F') \]  \hspace{1cm} (2.5.55)

Substitute equation (2.5.28),

\[ \dot{Q} \triangleq \nu^T E_D + \nu'_i F' \]  \hspace{1cm} (2.5.56)

Wang's interpretation of \( \dot{Q} \) is that it is the net power flowing into the system.

The total energy of the system is

\[ K + V - Q = \text{constant} \]  \hspace{1cm} (2.5.57)

where \( K \) is the kinetic energy of the system, \( V \) is the potential energy of the system, and \( Q \) is the net energy flowing into the system.

As shown in equation (2.5.54) the fundamental equation of Wang's method is

\[ \dot{E} = \dot{K} + \dot{V} - \dot{Q} = 0. \] This equation can be integrated over time interval \( t_0 \rightarrow t \) to give

\[ \int_{t_0}^{t} \left( \frac{dK}{dt} \right) dt + \int_{t_0}^{t} \left( \frac{dV}{dt} \right) dt - \int_{t_0}^{t} \left( \frac{dQ}{dt} \right) dt = 0 \]  \hspace{1cm} (2.5.58)

or

\[ (K - K_0) + (V - V_0) - (Q - Q_0) = 0 \]  \hspace{1cm} (2.5.59)

Wang's testing function can be stated as

\[ \Delta C = (K + V - Q) - (K_0 + V_0 - Q_0) \]  \hspace{1cm} (2.5.60)
The testing function $\Delta C$ should remain zero throughout any simulation.

d) Interpretation of the Power Equation

From equation (2.5.28), the equations of motion of the constrained system may be written as

$$F + F^* + F'' = 0 \quad (2.5.61)$$

where $F'$ represents the contribution of the constraint forces to the generalized active forces. So, $\dot{Q}$ as defined in equation (2.5.55) can be rewritten as

$$\dot{Q} = y^T E_D - y^T \left( F + F^* \right)$$

$$= y^T E_D + y^T F'$$

$$= y^T E_D + (v - \tilde{R}u)^T F'$$

$$= y^T E_D + y^T F' - y^T \tilde{R}^T F'$$

$$= y^T E_D + y^T F' + y^T \tilde{R}^T \left( F + F^* \right)$$

$$= y^T E_D + y^T F' + y^T \left( \tilde{F} + \tilde{F}^* \right)$$

$$= y^T E_D + y^T F'$$

Here $\left( \tilde{F} + \tilde{F}^* \right) = 0$ as shown in equation (2.5.32).

The second term in the last of equation (2.5.62) can be further reduced as follows

$$y^T F' = y^T \left[ J^T \right] \left[ A^T \right] \left\{ \frac{\lambda_c}{\lambda_m} \right\}$$

$$= \left[ y^T J^T \right] \left[ y^T A^T \right] \left\{ \frac{\lambda_c}{\lambda_m} \right\}$$

$$= \left[ (yv)^T \right] \left[ (Av)^T \right] \left\{ \frac{\lambda_c}{\lambda_m} \right\}$$

$$= \left[ 0 \right] \left[ b^T \right] \left\{ \frac{\lambda_c}{\lambda_m} \right\}$$

$$= -b^T \lambda_m$$

26
Substituting this last result back into equation (2.5.62) gives

$$\dot{Q} = \mathbf{v}^T \mathbf{F}_D - \mathbf{b}^T \mathbf{\lambda}_m$$  \hspace{1cm} (2.5.64)

Equation (2.5.64) needs an unknown multiplier $\mathbf{\lambda}_m$ to calculate $\dot{Q}$. $\mathbf{\lambda}_m$ is a set of unknown multipliers related to the motion constraint forces and moments. It provides an interpretation for $\dot{Q}$. The first term ($\mathbf{v}^T \mathbf{F}_D$) represents power flowing into the system associated with the non-conservative forces and moments, and the second term ($-\mathbf{b}^T \mathbf{\lambda}_m$) represents power flowing into the system associated with the motion constraint forces and moments.

e) A Second Form of the Power Equation

The power term $\dot{Q}$ as expressed in equation (2.5.62) is calculated from the generalized forces for the unconstrained system. A second form of the power equation using generalized forces for the constrained system may be developed as follows. From equations (2.5.45), (2.5.54) and (2.5.55), we have

$$\mathbf{K} = \dot{\mathbf{Q}} - \mathbf{\dot{v}}$$

$$\mathbf{K} = \mathbf{v}^T \mathbf{F}_D - \mathbf{v}^T \left( \mathbf{F} + \mathbf{F}^* \right) + \mathbf{v}^T \mathbf{F}_v$$

$$\mathbf{K} = \mathbf{v}^T \left( \mathbf{F}_D + \mathbf{F}_v \right) - \mathbf{v}^T \left( \mathbf{F} + \mathbf{F}^* \right)$$

$$\mathbf{K} = \mathbf{v}^T \mathbf{F} - \mathbf{v}^T \mathbf{F}^*$$

$$\mathbf{K} = \left( \mathbf{v}^T - \mathbf{v}_r^T \right) \mathbf{F} - \mathbf{v}_r^T \mathbf{F}^*$$

(2.5.65)

Now, using equations (2.5.26) and (2.5.21)

$$\mathbf{K} = \left( \mathbf{\tilde{R}} \mathbf{u} \right)^T \mathbf{F} - \left( \mathbf{\tilde{R}} \mathbf{\dot{b}} \right)^T \mathbf{F}^*$$

$$\mathbf{K} = \mathbf{u}^T \mathbf{\tilde{R}}^T \mathbf{F} - \mathbf{\dot{b}}^T \mathbf{R}^T \mathbf{F}^*$$

(2.5.66)
or

\[ \dot{K} = u^T \ddot{\mathbf{F}} - \dot{\mathbf{b}}^T \dot{\mathbf{F}}^* \tag{2.5.67} \]

Equation (2.5.67) is equivalent to the energy theorem published by Liu and Huston.

Using matrix notation, their energy theorem can be stated as

\[ \dot{K} = u^T \ddot{\mathbf{F}} + F_0 \tag{2.5.68} \]

where

\[ F_0 \triangleq -F_0^* = -v_0^T \mathbf{F}^* \tag{2.5.69} \]

and

\[ v_0 \triangleq \mathbf{y} - \mathbf{R} \mathbf{u} = \mathbf{v}_i = R\mathbf{b} \tag{2.5.70} \]

So, Liu and Huston’s energy theorem can be stated as

\[ \dot{K} = u^T \ddot{\mathbf{F}} + F_0 \]
\[ = u^T \ddot{\mathbf{F}} - v_0^T F^* \]
\[ = u^T \ddot{\mathbf{F}} - (R\mathbf{b})^T \mathbf{F}^* \tag{2.5.71} \]
\[ = u^T \ddot{\mathbf{F}} - \dot{\mathbf{b}}^T \dot{\mathbf{F}}^* \]

Equations (2.5.67) and (2.5.71) prove the equivalence of the Wang and Liu and Huston results.
CHAPTER 3

TRIPLE PENDULUM SYSTEM

In an effort to apply the testing functions discussed in the previous chapters, an example system, a triple pendulum, was selected. Motion of the pendulum is a very well understood subject as it helped develop many scientific understandings which may include mass, gravitation, acceleration, and chaos theory. The triple pendulum example is a *holonomic* system where the equations of motions can be given in the form of algebraic and differential equations. A simple *nonholonomic* system example was not selected here as it is beyond the scope of this thesis, nevertheless the formulation of the testing functions encompasses both systems. One of the goals of the analysis is to show the simulation errors due to the numerical integration error. Simulations of the equations of motion of the triple pendulum can be performed with and without numerical integration in Matlab and Simulink and they can be compared for errors. Also, triple pendulum is suitable candidate to demonstrate multiple scenarios that show free motion, specified motion, and damper effect with a prior understanding of their dynamic behavior. Familiar dynamic behavior helps analyze and understand the results without running experiments, yet the formulation of the equations of motion is complicated enough to check the validity of the testing functions. Robotic arms and GEO 600 interferometric gravitational wave detector [14] are few examples that use the triple pendulum. Following sections discusses the development of the equations of motions and testing function formulations.
3.1. Equations of Motion

The Figure 3.1 shows a triple pendulum of three bodies \((N = 3)\) with driving moments and rotational dampers at the connecting joints. The pendulum is hanging in the vertical plane with the \(y\)-axis pointing up. The mass centers of the bodies are assumed to lie along the joint connecting lines. This triple pendulum is moving in vertical plane with frictionless pins.

Here \(\mathbf{e}_{B_1}, \mathbf{e}_{B_2}, \mathbf{e}_{B_3} (k = 1, 2, 3)\) are unit vectors rotating with the bodies \(B_1, B_2,\) and \(B_3\) respectively. These unit vectors coincide with the principle moment of inertia of the bodies. Here the reference frame \(S\) has the unit vectors represented by \(\mathbf{i}, \mathbf{j},\) and \(\mathbf{k}.\)
\( \mathbf{v}_{\text{cg}1}, \mathbf{v}_{\text{cg}2}, \text{ and } \mathbf{v}_{\text{cg}3} \) are the velocity vectors of the center of gravity \( \text{cg}_1, \text{cg}_2, \text{cg}_3 \) respectively.

\( l_1, l_2, \text{ and } l_3 \) are the length of the bodies. The length between the connecting joints at \( O \) and \( A \) is \( l_1 \), and the length between the connecting joints at \( A \) and \( B \) is \( l_2 \).

\( m_1, m_2, \text{ and } m_3 \) are the uniform mass of the bodies.

\( r_1, r_2, \text{ and } r_3 \) are the distance of the center of gravity of the bodies from \( O, A, B \) respectively.

\( I_1, I_2, \text{ and } I_3 \) are the mass moment of inertia of the bodies.

\( \theta_1, \theta_2, \text{ and } \theta_3 \) are the swing angles related to the vertical axis. One of the angles may be a specified motion. Example: \( \theta_1(t) = A \sin \omega t \) where \( A \) is amplitude and \( \omega \) is frequency.

\( M_1, M_2, \text{ and } M_3 \) are the torques at joints \( O, A, \text{ and } B \).

\( c_1, c_2, \text{ and } c_3 \) are damping coefficients.

Following steps shows the development of the equation of motion of triple pendulum shown in Figure 3.1.1.

Trigonometric notations are abbreviated as follows.

\[
S_i = \sin \theta_i, \quad C_i = \cos \theta_i, \quad C_{2-1} = \cos(\theta_2 - \theta_1), \quad S_{2-1} = \sin(\theta_2 - \theta_1)
\]

### 3.1.1. Developing Equations of Motions using Lagrange's Method

Assuming that \( \theta_1, \theta_2, \text{ and } \theta_3 \) for a set of independent generalized coordinates, the equation of motion of the three link pendulum may be written as shown in equation (3.1.1).
Here \( q^T = [\theta_1 \quad \theta_2 \quad \theta_3] \). \( L = K - V \) is the Lagrangian of the system. \( K \) is the kinetic energy and \( V \) is the potential energy function for the conservative forces and torques.

Velocity of the center of gravity of the bodies are

\[
\begin{align*}
\nu_{cg_1} &= l_1 \dot{\theta}_1 e_{\theta_1} \\
\nu_{cg_2} &= l_1 \dot{\theta}_1 e_{\theta_1} + l_2 \dot{\theta}_2 e_{\theta_2} \\
\nu_{cg_3} &= l_1 \dot{\theta}_1 e_{\theta_1} + l_2 \dot{\theta}_2 e_{\theta_2} + l_3 \dot{\theta}_3 e_{\theta_3}
\end{align*}
\]

Kinetic energy of the system from fundamentals in dynamics is given

\[
K = K_{B_1} + K_{B_2} + K_{B_3} = \frac{1}{2} \sum_{i=1}^{3} (I_i \dot{\theta}_i^2 + m_i v_{cg_i}^2)
\]

Where \( K \) is the kinetic energy of the system, \( v_{cg_i} \) is the velocity of the center of the body \( N \). Here, \( N = 1, 2, 3 \).

\[
K_{B_1} = \frac{1}{2} \left[ I_1 \dot{\theta}_1^2 + m_1 r_1^2 \dot{\theta}_1^2 \right]
\]

\[
K_{B_2} = \frac{1}{2} \left[ I_2 \dot{\theta}_2^2 + m_2 r_2^2 \dot{\theta}_2^2 + 2m_1 l_1 r_1 \dot{\theta}_1 \dot{\theta}_2 C_{2-1} \right]
\]

\[
K_{B_3} = \frac{1}{2} \left[ I_3 \dot{\theta}_3^2 + m_3 l_3^2 \dot{\theta}_3^2 + m_2 l_2 r_2 \dot{\theta}_2 \dot{\theta}_3 C_{2-1} + 2m_1 l_1 r_1 \dot{\theta}_1 \dot{\theta}_2 C_{2-1} \right]
\]
Potential energy of the system is given below.

\[ V = V_{b_1} + V_{b_2} + V_{b_3} \]  \hspace{1cm} (3.1.9)

\[ V_{b_1} = -r_1 C_1 m_1 g \]  \hspace{1cm} (3.1.10)

\[ V_{b_2} = -(l_2 C_1 + r_2 C_2) m_2 g \]  \hspace{1cm} (3.1.11)

\[ V_{b_3} = -(l_1 C_1 + l_2 C_2 + r_3 C_3) m_3 g \]  \hspace{1cm} (3.1.12)

\[ \dot{V} = (m_1 r_1 + m_2 l_1 + m_3 l_1) g S_i \dot{\theta}_i + (m_2 r_2 + m_3 l_2) g S_j \dot{\theta}_j + (m_3 r_3) g S_k \dot{\theta}_k \]  \hspace{1cm} (3.1.13)

Non-conservative forces \((F_{q})_{\text{non-conservative}}\) are shown below.

\[ (F_{\theta_1})_{\text{non-conservative}} = M_1 - M_2 + \left[ -c_1 \dot{\theta}_1 - c_2 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) \right] \]  \hspace{1cm} (3.1.14)

\[ (F_{\theta_2})_{\text{non-conservative}} = M_2 - M_3 + \left[ -c_2 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) + c_3 \left( \dot{\theta}_3 - \dot{\theta}_2 \right) \right] \]  \hspace{1cm} (3.1.15)

\[ (F_{\theta_3})_{\text{non-conservative}} = M_3 - c_3 \left( \dot{\theta}_3 - \dot{\theta}_2 \right) \]  \hspace{1cm} (3.1.16)

Using \( L = K - V \) and equation (3.1.1), following equations of motions can be derived.

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = \left[ l_1 + m_1 r_1^2 + (m_2 + m_3) l_1^2 \right] \ddot{\theta}_1 + (m_1 r_1 + m_2 r_2 + m_3 l_2) l_1 C_{2-1} \ddot{\theta}_2 
+ \left[ m_1 l_1 C_{3-1} \right] \ddot{\theta}_3 - (m_2 r_2 + m_3 l_1) l_1 S_{2-1} \dot{\theta}_2^2 -(m_1 l_1 r_3) S_{3-1} \dot{\theta}_3^2 
+ \left[ m_2 r_2 + (m_2 + m_3) l_2 \right] g S_i 
= M_1 - M_2 + \left[ -c_1 \dot{\theta}_1 + c_2 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) \right] \]  \hspace{1cm} (3.1.17)

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = \left( m_2 r_2 + m_3 l_2 \right) l_1 C_{2-1} \ddot{\theta}_1 + \left[ l_2 + m_2 r_2^2 + m_3 l_2^2 \right] \ddot{\theta}_2 + \left[ m_2 l_2 r_3 C_{3-2} \right] \ddot{\theta}_3 
+ (m_2 r_2 + m_3 l_2) l_1 S_{2-1} \dot{\theta}_2^2 -(m_1 l_2 r_3) S_{3-2} \dot{\theta}_3^2 + \left[ m_2 r_2 + m_3 l_2 \right] g S_2 
= M_2 - M_3 + \left[ -c_2 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) + c_3 \left( \dot{\theta}_3 - \dot{\theta}_2 \right) \right] \]  \hspace{1cm} (3.1.18)
\[
\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i}
= \left[ m_l l_i r_i C_{i-j} \right] \ddot{\theta}_i + \left[ m_l l_i r_i C_{j-i} \right] \ddot{\theta}_i
+ \left[ I_j + m_l r_j \right] \ddot{\theta}_j + \left[ m_l l_j r_j \right] S_{j-i} \ddot{\theta}_j + \left[ m_l l_j r_j \right] S_{i-j} \ddot{\theta}_j
+ m_l r_j g S_j
= M_j - c_j \left( \dot{\theta}_j - \dot{\theta}_i \right)
\]

By adding all equations (3.1.17), (3.1.18), and (3.1.19), the following equation (3.1.20)
can be derived.

\[
\begin{align*}
\left[ I_i \ddot{\theta}_i + I_j \ddot{\theta}_j + I_k \ddot{\theta}_k \right] + (m_l r_i^2 + m_l l_i^2 + m_l l_i^2) \ddot{\theta}_i + (m_l r_j^2 + m_l l_j^2) \ddot{\theta}_j + (m_l r_k^2 + m_l l_k^2) \ddot{\theta}_k
+ (m_j r_j + m_j l_j + m_j l_j) C_{j-i} \ddot{\theta}_i + m_j l_j r_j C_{j-i} \ddot{\theta}_j - (m_j l_j + m_j l_j) \ddot{\theta}_j - m_j r_j S_{j-i} \ddot{\theta}_j
+ (m_j r_j + m_j l_j) C_{i-j} \ddot{\theta}_i + m_j l_j r_j C_{i-j} \ddot{\theta}_j + (m_j l_j + m_j l_j) S_{i-j} \ddot{\theta}_j - m_j r_j S_{i-j} \ddot{\theta}_j
+ m_j r_j C_{i-j} \ddot{\theta}_i + m_j l_j r_j C_{j-i} \ddot{\theta}_j - m_j r_j S_{j-i} \ddot{\theta}_j
\end{align*}
\]

\[
= \left\{ M_1 - M_2 + \left[ -c_1 \dot{\theta}_1 - c_2 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) \right] \right\} + \left\{ M_2 - M_3 + \left[ -c_2 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) + c_3 \left( \dot{\theta}_3 - \dot{\theta}_1 \right) \right] \right\}
+ \left\{ M_3 - c_3 \left( \dot{\theta}_3 - \dot{\theta}_2 \right) \right\} - \left\{ \left[ m_r l_j + (m_j l_j + m_l l_j) g S_j \right] - \left[ m_j l_j + m_j r_j g S_j \right] - \left[ m_l r_j g S_j \right] \right\}
\]

Here \[
\begin{align*}
\begin{bmatrix}
(F_{\theta_1})_{\text{non-conservative}} \\
(F_{\theta_2})_{\text{non-conservative}} \\
(F_{\theta_3})_{\text{non-conservative}}
\end{bmatrix}
&= \begin{bmatrix}
M_1 - M_2 + \left[ -c_1 \dot{\theta}_1 - c_2 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) \right] \\
M_2 - M_3 + \left[ -c_2 \left( \dot{\theta}_2 - \dot{\theta}_1 \right) + c_3 \left( \dot{\theta}_3 - \dot{\theta}_1 \right) \right] \\
M_3 - c_3 \left( \dot{\theta}_3 - \dot{\theta}_2 \right)
\end{bmatrix}
\end{align*}
\]
3.1.2. Derivation of Kinetic Energy Rate \( \dot{K} \)

From equations (3.1.6), (3.1.7), and (3.1.8), \( \dot{K} \) can be calculated. Here the equation (3.1.22) is shown in matrix form with three terms \( K_{\text{term}1}, K_{\text{term}2}, \) and \( K_{\text{term}3} \). 

\[
\dot{K} = \frac{1}{2} \begin{bmatrix} I_1 & I_2 & I_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \\ \dot{\theta}_3^2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} m_1r_1^2 + m_2l_1^2 + m_3l_1^2 \\ m_2r_2^2 + m_3l_2^2 \\ m_3r_3^2 \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
\]

\[+ \begin{bmatrix} (m_2l_1r_2 + m_3l_1l_2)C_{2-1} \\ m_2l_2r_2C_{2-2} \\ m_3l_1r_3C_{3-1} \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \]

\[= \begin{bmatrix} \dot{K}_{\text{term}1} \\ \dot{K}_{\text{term}2} \\ \dot{K}_{\text{term}3} \end{bmatrix} \]  

(3.1.22)

\[
\frac{dK}{dt} = \frac{dK_{\text{term}1}}{dt} + \frac{dK_{\text{term}2}}{dt} + \frac{dK_{\text{term}3}}{dt} 
\]

(3.1.23)

\[
\frac{dK_{\text{term}1}}{dt} = \begin{bmatrix} I_1 \dot{\theta}_1 & I_2 \dot{\theta}_2 & I_3 \dot{\theta}_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} 
\]

(3.1.24)

\[
\frac{dK_{\text{term}2}}{dt} = \begin{bmatrix} (m_1r_1^2 + m_2l_1^2 + m_3l_1^2) \dot{\theta}_1 \\ (m_2r_2^2 + m_3l_2^2) \dot{\theta}_2 \\ (m_3r_3^2) \dot{\theta}_3 \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} 
\]

(3.1.25)
\[
\begin{align*}
\frac{dK_{term}}{dt} &= \left[ (m_1l_1 + m_2l_2)(-S_{2,1}(\dot{\theta}_2 - \dot{\theta}_1)\dot{\theta}_2 + C_{2,1}\ddot{\theta}_2 + C_{2,2}\ddot{\theta}_2) \right]^T \\
&\quad + m_1l_2 \left( -S_{3,2}(\dot{\theta}_3 - \dot{\theta}_2)\dot{\theta}_3 + C_{3,2}\ddot{\theta}_3 + C_{3,3}\ddot{\theta}_3 \right) \\
&\quad + m_2l_3 \left( -S_{4,3}(\dot{\theta}_4 - \dot{\theta}_3)\dot{\theta}_4 + C_{4,3}\ddot{\theta}_4 + C_{4,4}\ddot{\theta}_4 \right) \\
&= \left[ (m_1l_1 + m_2l_2)C_{2,2}\ddot{\theta}_2 + m_1l_2r_2C_{2,3}\ddot{\theta}_3 - (m_1l_2 + m_2l_3)\dot{\theta}_2^2 - m_1l_2r_3S_{2,2}\ddot{\theta}_3^2 \\
&\quad + m_2l_3r_3C_{3,3}\ddot{\theta}_3 + m_2l_3r_3S_{3,3}\ddot{\theta}_3^2 + m_2l_3r_3S_{3,3}\ddot{\theta}_3^2 \\
&= \left[ \begin{array}{c} \dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 
\end{array} \right] \left[ \begin{array}{c} \dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 
\end{array} \right]^T
\end{align*}
\]

(3.1.26)

By adding equations (3.1.24), (3.1.25), and (3.1.26)

\[
\dot{K} = \left[ I_1\ddot{\theta}_1 \quad I_2\ddot{\theta}_2 \quad I_3\ddot{\theta}_3 \right] \left[ \begin{array}{c} \dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 
\end{array} \right] + \left[ \begin{array}{c} (m_1r_1^2 + m_2r_2^2 + m_3r_3^2)\ddot{\theta}_1 \\
(m_2r_2^2 + m_3r_3^2)\ddot{\theta}_2 \\
(m_3r_3^2)\ddot{\theta}_3 
\end{array} \right] \left[ \begin{array}{c} \dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 
\end{array} \right]
\]

\[
= \left[ \begin{array}{c} \dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 
\end{array} \right] \left[ \begin{array}{c} \dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 
\end{array} \right]^T
\]

(3.1.27)

From equations (3.1.20) and (3.1.27) we can show that

\[
\dot{K} = \left[ M_1 - M_2 - c_1(\dot{\theta}_1 - \dot{\theta}_2) - (m_1r_1 + m_1l_1 + m_1g \frac{s_1}{2}) \right] \dot{\theta}_1 + \\
\left[ M_2 - M_3 - c_2(\dot{\theta}_2 - \dot{\theta}_3) + c_3(\dot{\theta}_3 - \dot{\theta}_2) - (m_2r_2 + m_2l_2 + m_2g \frac{s_2}{2}) \right] \dot{\theta}_2 + \\
\left[ M_3 - c_3(\dot{\theta}_3 - \dot{\theta}_2) - m_3r_3g \frac{s_3}{2} \right] \dot{\theta}_3
\]

(3.1.28)

Here the \( \dot{K} \) was derived with the help of basic dynamics fundamentals and Lagrange’s method.

### 3.2. Application of Energy Theorems

This section will demonstrate the application of the testing functions using the
3.2.1. Coordinates

Define the vector $\mathbf{x}$ to contain the $XY$ coordinates of the mass centers and the angles that each body makes with the vertical, and define the vector $\mathbf{v}$ to contain the velocity and angular velocity components.

\[ \mathbf{x} = \begin{pmatrix} x_1, y_1, x_2, y_2, x_3, y_3, \theta_1, \theta_2, \theta_3 \end{pmatrix}^T \quad \mathbf{v} = \begin{pmatrix} \dot{x}_G \mathbf{\theta}^T \end{pmatrix}^T \]

\[ \mathbf{v} = \begin{pmatrix} \mathbf{v}^T \mathbf{\omega}^T \end{pmatrix} \triangleq \dot{\mathbf{x}} = \mathbf{P} \ddot{\mathbf{x}} \]

Here, $\mathbf{P}$ is a $9 \times 9$ identity matrix of partial velocity and partial angular velocity components.

3.2.2. Generalized Forces

The generalized active force vector is defined to be

\[ \mathbf{F} = \mathbf{P} \left[ \begin{array}{c} \mathbf{F}_f \\ \mathbf{F}_M \end{array} \right] \]

where $\mathbf{F}_f$ is a $6 \times 1$ vector of resultant force components acting on the mass centers of the bodies, and $\mathbf{F}_M$ is a $3 \times 1$ of resultant moments acting on the bodies.

The generalized inertia force vector is defined to be

\[ \mathbf{F}' = \mathbf{P} \left[ \begin{array}{c} \mathbf{F}'_f \\ \mathbf{F}'_M \end{array} \right] \]
Where

\[
\begin{bmatrix}
    m_1 & 0 & 0 & 0 & 0 \\
    0 & m_1 & 0 & 0 & 0 \\
    0 & 0 & m_2 & 0 & 0 \\
    0 & 0 & 0 & m_3 & 0 \\
    0 & 0 & 0 & 0 & m_3 \\
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{y}_1 \\
    \dot{x}_2 \\
    \dot{y}_2 \\
    \dot{x}_3 \\
\end{bmatrix}
\]

(3.2.5)

\[
\begin{bmatrix}
    I_1 & 0 & 0 \\
    0 & I_2 & 0 \\
    0 & 0 & I_3 \\
\end{bmatrix}
\begin{bmatrix}
    \dot{\theta}_1 \\
    \dot{\theta}_2 \\
    \dot{\theta}_3 \\
\end{bmatrix}
\]

(3.2.6)

or

\[
E_r^* = \begin{bmatrix}
E_r^m \\
0
\end{bmatrix}
= -m_0 \ddot{y} = -M_{m, r} \ddot{y}
\]

(3.2.7)

Note that \( I_i \) \((i = 1, 2, 3)\) are the moments of inertia of the bodies about their mass centers.

### 3.2.3. Configuration Constraints

The elements of \( \bar{x} \) can be related through a set of configuration constraints

\[
x_1 - r_1 S_1 = 0 \\
y_1 + r_1 C_1 = 0
\]

(3.2.8)

\[
x_2 - r_1 S_1 - r_2 S_2 = 0 \\
y_2 + r_1 C_1 + r_2 C_2 = 0
\]

(3.2.9)

\[
x_3 - l_1 S_1 - l_2 S_2 - r_3 S_3 = 0 \\
y_3 + l_1 C_1 + l_2 C_2 + r_3 C_3 = 0
\]

(3.2.10)
These equations can be differentiated and arranged into the following matrix form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
\dot{x}_2 \\
\dot{y}_2 \\
\dot{x}_3 \\
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 \\
r_1C_1 & 0 & 0 \\
r_1S_1 & 0 & 0 \\
l_1C_1 & r_2C_2 & 0 \\
l_1S_1 & r_2S_2 & 0 \\
l_1C_1 & l_2C_2 & r_3C_3 \\
l_1S_1 & l_2S_2 & r_3S_3 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix} = R\dot{q}
\] (3.2.11)

3.2.4. Wang’s Energy Theorem: No Motion Constraints

\[
\dot{K} = \dot{\mathcal{Q}} - \dot{\mathcal{V}}
\] (3.2.12)

From equation (2.5.56), \( y_i^TF^i = 0 \) for no motion constraint.

\[
\dot{\mathcal{Q}} \triangleq y_i^TF_D + \begin{cases} \frac{v_i^TF}{v_i} & \text{if no motion constraints} \\ 0 & \text{if no motion constraints} \end{cases}
\] (3.2.13)

\[
= (R\dot{q})^T F_D
\]

\[
= \dot{q}^T (RF_D)
\]

\[
= \dot{q}^T \hat{F}_D
\]

Here, \( \hat{F}_D = \left( (F_{\theta_1})_D, (F_{\theta_2})_D, (F_{\theta_3})_D \right)^T \) represents the generalized forces associated with \( \dot{q} \) and the non-potential forces and moments and \( \dot{q}^T = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3) \). The vector of generalized forces may be calculated by noting that

\[
(F_{\theta_i})_D = (M_{\theta_i}^Rk) \cdot \frac{\partial \omega_i}{\partial \theta_i} + (M_{\theta_i}^Rk) \cdot \frac{\partial \omega_i}{\partial \theta_i} + (M_{\theta_i}^Rk) \cdot \frac{\partial \omega_i}{\partial \theta_i}
\] (3.2.14)
where $M_i^R$ represents the resultant moment on body $i$ in the reference frame $(i, j, k)$. So

\[
\dot{\mathbf{E}}_D = \begin{bmatrix}
(F_{\theta_i})_{\text{non-conservative}} \\
(F_{\theta_2})_{\text{non-conservative}} \\
(F_{\theta_3})_{\text{non-conservative}}
\end{bmatrix} = \begin{bmatrix}
M_1 - M_2 - c_1\dot{\theta}_1 + c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) \\
M_2 - M_3 - c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) + c_3\left(\dot{\theta}_3 - \dot{\theta}_2\right) \\
M_3 - c_3\left(\dot{\theta}_3 - \dot{\theta}_2\right)
\end{bmatrix}
\]  
(3.2.15)

The time derivative of the potential energy may be calculated as follows

\[
-\dot{V} = v^T F_\nu = (R\dot{q})^T F_\nu = \dot{q}^T (R^T F_\nu) = \dot{q}^T \dot{E}_\nu
\]  
(3.2.16)

where $\dot{E}_\nu = \left( (F_{\theta_1})_\nu, (F_{\theta_2})_\nu, (F_{\theta_3})_\nu \right)^T$. The elements of this vector can be calculated by noting that

\[
(F_{\theta_i})_\nu = (-m_i g j) \frac{\partial v_i}{\partial \theta_i} + (-m_2 g j) \frac{\partial v_2}{\partial \theta_i} + (-m_3 g j) \frac{\partial v_3}{\partial \theta_i}
\]  
(3.2.17)

Where ($i = 1, 2, 3$) Using this relationship, we find

\[
\dot{E}_\nu = \begin{bmatrix}
-(m_1 r_i + m_2 l_i + m_3 l_i) g S_1 \\
-(m_2 r_i + m_3 l_i) g S_2 \\
m_3 r_3 g S_3
\end{bmatrix}
\]  
(3.2.18)

\[
\dot{V} = (m_1 r_i + m_2 l_i + m_3 l_i) g S_1 \dot{\theta}_i + (m_2 r_i + m_3 l_i) g S_2 \dot{\theta}_2 + (m_3 r_3) g S_3 \dot{\theta}_3
\]  
(3.2.19)

Equation (3.2.19) was the exact same solution for $\dot{V}$ as shown in equation (3.1.13)

Substituting from equations(3.2.13), (3.2.15), (3.2.16), and (3.2.19) into equation (3.2.12) gives the following energy statement which is shown in equation (3.1.28).
3.2.5. Liu and Huston’s Energy Theorem: No Motion Constraints

\[ \dot{K} = u^T \dot{F} - \dot{\bar{b}}^T \bar{F}' = u^T \bar{F} \quad (3.2.21) \]

where

\[ u^T = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3) \quad (3.2.22) \]

\[ \bar{F} = \begin{bmatrix} F_{\theta_1} \\ F_{\theta_2} \\ F_{\theta_3} \end{bmatrix} = \begin{bmatrix} (F_{\theta_1})_D + (F_{\theta_1})_V \\ (F_{\theta_2})_D + (F_{\theta_2})_V \\ (F_{\theta_3})_D + (F_{\theta_3})_V \end{bmatrix} = \begin{bmatrix} M_1 - M_2 - c_1 \dot{\theta}_1 + c_2 (\dot{\theta}_2 - \dot{\theta}_1) - (m_1 r_1 + m_2 l_1 + m_3 l_1) g S_1 \\ M_2 - M_3 - c_2 (\dot{\theta}_2 - \dot{\theta}_1) + c_3 (\dot{\theta}_3 - \dot{\theta}_2) - (m_2 r_2 + m_3 l_2) g S_2 \\ M_3 - c_3 (\dot{\theta}_3 - \dot{\theta}_2) - m_3 r_3 g S_3 \end{bmatrix} \quad (3.2.23) \]

Substituting from equations (3.2.22) and (3.2.23) into equation (3.2.21) gives the same result as with Wang’s theorem as presented in equation (3.2.20).

3.2.6. Wang’s Energy Theorem: With Motion Constraint

\[ \dot{K} = \hat{Q} - \dot{V} \quad (3.2.24) \]

where

\[ \hat{Q} \triangleq \dot{v}^T \hat{F}_D - \dot{\bar{b}}^T (\hat{F} + \hat{F}') \]

\[ = \left( R \hat{q} \right)^T \hat{F}_D - \left( R \hat{b} \right)^T (\hat{F} + \hat{F}') \]

\[ = \hat{q}^T \left( R^T \hat{F}_D \right) - \hat{b}^T (R^T \hat{F} + R^T \hat{F}') \]

\[ = \hat{q}^T \hat{F}_D - \hat{b}^T (\hat{F} + \hat{F}') \quad (3.2.25) \]
The term $q^T \hat{E}_D$ is the same as that presented in the previous section. The other terms are associated with the motion constraints.

**Motion Constraint** $\theta_i = A \sin(\omega t)$:

To identify the vector $\hat{\theta}$, we write the motion constraint equation

$$\dot{q} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} A \omega \cos(\omega t) \\ 0 \\ 0 \end{bmatrix} \triangleq \hat{R} \dot{u} + \hat{b} \quad (3.2.26)$$

Using the definition of $\hat{b}$ given by equation (3.2.26), we can write

$$\hat{b}^T (\hat{E} + \hat{E}^*) = A \omega \cos(\omega t) \left( F_{\theta_i} + F_{\theta_i}^* \right) = \hat{\theta}_1 \left( F_{\theta_i} + F_{\theta_i}^* \right) \quad (3.2.27)$$

Here, $F_{\theta_i}$ is given by the first term in equation (3.2.23), and $F_{\theta_i}^*$ may be calculated as follows.

$$F_{\theta_i}^* \triangleq (-m_{a_1} \cdot \frac{\partial v_i}{\partial \theta_i}) + (-m_{a_2} \cdot \frac{\partial v_i}{\partial \theta_i}) + (-m_{a_3} \cdot \frac{\partial v_i}{\partial \theta_i})$$

$$+ \left(-I_{k\theta_i} \frac{\partial \omega_{k\theta_i}}{\partial \theta_i} \right) + \left(-I_{k\theta_i} \frac{\partial \omega_{k\theta_i}}{\partial \theta_i} \right) + \left(-I_{k\theta_i} \frac{\partial \omega_{k\theta_i}}{\partial \theta_i} \right) \quad (3.2.28)$$

where, by direct differentiation, it can be shown that

$$a_1 = n^1_i \left( C_i \ddot{\theta}_1 - S_i \ddot{\theta}_1^2 \right) i + \dot{n}_i \left( S_i \ddot{\theta}_1 + C_i \ddot{\theta}_1 \right) j \quad (3.2.29)$$

$$a_2 = \left( l_i C_i \ddot{\theta}_1 - l_i S_i \ddot{\theta}_1^2 + r_2 C_i \ddot{\theta}_2 - r_2 S_i \ddot{\theta}_2^2 \right) i + \left( l_i S_i \ddot{\theta}_1 + l_i C_i \ddot{\theta}_1^2 + r_2 S_i \ddot{\theta}_2 + r_2 C_i \ddot{\theta}_2 \right) j \quad (3.2.30)$$

$$a_3 = \left( l_i C_i \ddot{\theta}_1 - l_i S_i \ddot{\theta}_1^2 + l_i C_i \ddot{\theta}_1 \right) i + \left( l_i S_i \ddot{\theta}_1 + l_i C_i \ddot{\theta}_1^2 + l_i S_i \ddot{\theta}_2 + l_i C_i \ddot{\theta}_2 \right) j \quad (3.2.31)$$

Substituting from equations (3.2.29), (3.2.30), and (3.2.31) into equation (3.2.28), and...
simplifying using some trigonometric identities, it can be shown that

\[
F_{\theta_1}^* = -(l_1 + m_1r_1^2 + m_2l_2^2 + m_3l_3^2)\ddot{\theta}_1 - (m_1r_1 + m_2l_2 + m_3l_3)C_{2-3}\ddot{\theta}_2 - (m_3l_3C_{3-1})\ddot{\theta}_3 \\
+ (m_2l_2r_2 + m_3l_3l_3)S_{2-3}\ddot{\theta}_2 + (m_3l_3r_3S_{3-1})\ddot{\theta}_3^2
\]  \(3.2.32\)

Substituting equations (3.1.17) and (3.2.23)

\[
F_{\theta_1}^* = -\left[ M_1 - M_2 - c_1\dot{\theta}_1 + c_2(\dot{\theta}_2 - \dot{\theta}_1) - (m_1r_1 + m_2l_2 + m_3l_3)gS_1 \right] \\
F_{\theta_1}^* = -F_{\theta_1} \\
F_{\theta_1}^* + F_{\theta_1} = 0
\]  \(3.2.33\)

Using equation (3.2.33)

\[
\dot{\mathbb{E}}^T(\mathbb{E} + \dot{\mathbb{E}}^*) = A\omega \cos(\omega t)(F_{\theta_1} + F_{\theta_1}^*) = \dot{\theta}_1(F_{\theta_1} + F_{\theta_1}^*) = 0
\]  \(3.2.35\)

The results shown in equation (3.2.35) can also be derived using equation (2.5.21) where \((\mathbb{E} + \dot{\mathbb{E}}^*) = 0\). Therefore, for both situations with the motion constraint and without the motion constraint,

\[
\dot{Q} = \dot{q}^T\dot{E}_0
\]  \(3.2.36\)

The energy theorem can also be found by substituting from equations (3.2.25), (3.2.25), (3.2.27), (3.2.23), and (3.2.32) into equation (3.2.24) and canceling terms.

\[
\dot{K} = \dot{Q} - \dot{V} = \left[ M_1 - M_2 - c_1\dot{\theta}_1 + c_2(\dot{\theta}_2 - \dot{\theta}_1) \right]\dot{\theta}_1 + \left[ M_2 - M_3 - c_2(\dot{\theta}_2 - \dot{\theta}_1) + c_3(\dot{\theta}_3 - \dot{\theta}_2) \right]\dot{\theta}_2 + \\
+ \left[ M_3 - c_3(\dot{\theta}_3 - \dot{\theta}_2) \right]\dot{\theta}_3 - \left[ (m_1r_1 + m_2l_2 + m_3l_3)gS_1 \right]\dot{\theta}_1 - \left[ (m_1r_1 + m_2l_2 + m_3l_3)gS_2 \right]\dot{\theta}_2 - \left[ m_3r_3gS_3 \right]\dot{\theta}_3 \\
+ \left[ (l_1 + m_1l_1^2 + m_2l_2^2 + m_3l_3^2)\dot{\theta}_1 + (m_2l_2r_2 + m_3l_3l_3)C_{2-3}\dot{\theta}_2 + (m_3l_3r_3C_{3-1})\dot{\theta}_3 \right] \dot{\theta}_1 - \left[ (m_2l_2r_2 + m_3l_3l_3)S_{2-3}\dot{\theta}_2 + (m_3l_3r_3S_{3-1})\dot{\theta}_3^2 \right] \dot{\theta}_1
\]
\[
\dot{K} = \left[ M_2 - M_3 - c_2 (\dot{\theta}_2 - \dot{\theta}_1) + c_3 (\dot{\theta}_3 - \dot{\theta}_1) - (m_z r_2 + m_z l_z) g S_2 \right] \dot{\theta}_2 + \left[ M_3 - c_3 (\dot{\theta}_3 - \dot{\theta}_2) - m_z r_2 g S_3 \right] \dot{\theta}_3 \\
+ \left[ (I_1 + m_i r_i^2 + m_i l_i^2) \ddot{\theta}_1 + (m_i r_i + m_i l_i) C_{2-1} \ddot{\theta}_2 + (m_i r_i C_{2-1}) \ddot{\theta}_3 \right] \dot{\theta}_1 \\
- (m_z l_z + m_z l_z) S_{3-1} (\dot{\theta}_2 - \dot{\theta}_3)
\]

(3.2.37)

Substitute equation (3.1.17) into (3.2.37), the \( \dot{K} \) will be same result as shown in equation (3.2.20).

\[
\dot{K} = \left[ M_1 - M_2 - c_1 \dot{\theta}_1 + c_2 (\dot{\theta}_2 - \dot{\theta}_1) - (m_1 r_1 + m_2 l_1 + m_3 l_1) g S_1 \right] \dot{\theta}_1 + \\
\left[ M_2 - M_3 - c_2 (\dot{\theta}_2 - \dot{\theta}_1) + c_3 (\dot{\theta}_3 - \dot{\theta}_2) - (m_z r_2 + m_z l_z) g S_2 \right] \dot{\theta}_2 + \\
\left[ M_3 - c_3 (\dot{\theta}_3 - \dot{\theta}_2) - m_z r_3 g S_3 \right] \dot{\theta}_3
\]

(3.2.38)

3.2.7. Liu and Huston’s Energy Theorem: With Motion Constraint

\[
\dot{K} = u^T \ddot{F} - \dot{b}^T \dot{F}^*
\]

(3.2.39)

where

\[
u^T = (\dot{\theta}_2, \dot{\theta}_3)
\]

(3.2.40)

\[
\hat{F} = \hat{R}^T F = \hat{R}^T (R F) = \hat{R}^T \hat{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{\theta_1} \\ F_{\theta_2} \\ F_{\theta_3} \end{bmatrix} = \begin{bmatrix} F_{\theta_1} \\ \hat{F}_{\theta_2} \\ \hat{F}_{\theta_3} \end{bmatrix}
\]

(3.2.41)

or

\[
\hat{F} = \begin{bmatrix} F_{\theta_1} \\ F_{\theta_2} \end{bmatrix} = \begin{bmatrix} M_2 - M_3 - c_2 (\dot{\theta}_2 - \dot{\theta}_1) + c_3 (\dot{\theta}_3 - \dot{\theta}_2) - (m_z r_2 + m_z l_z) g S_2 \\ M_3 - c_3 (\dot{\theta}_3 - \dot{\theta}_2) - m_z r_3 g S_3 \end{bmatrix}
\]

(3.2.42)

The term associated with the motion constraint is

\[
-\dot{b}^T \dot{F}^* = -\dot{\theta}_1 F_{\theta_1}^*
\]

(3.2.43)
where $F_q^*$ is given by equation (3.2.32) or equation (3.2.33).

Substituting from equations (3.2.40), (3.2.42), (3.2.43), and (3.2.32) into equation (3.2.39) gives the same result as with Wang’s theorem as presented in equation (3.2.20) which is shown below.

\[
\dot{\theta} = \left[ M_1 - M_2 - c_1 \dot{\theta}_1 + c_2 (\dot{\theta}_2 - \dot{\theta}_1) - (m_1r_1 + m_2l_1 + m_3l_1) gS_1 \right] \dot{\theta}_1 + \\
\left[ M_2 - M_3 - c_2 (\dot{\theta}_2 - \dot{\theta}_1) + c_3 (\dot{\theta}_3 - \dot{\theta}_2) - (m_2r_2 + m_3l_2) gS_2 \right] \dot{\theta}_2 + \\
\left[ M_3 - c_3 (\dot{\theta}_3 - \dot{\theta}_2) - m_3r_3 gS_3 \right] \dot{\theta}_3
\] (3.2.44)
CHAPTER 4

MATLAB AND SIMULINK SIMULATION RESULTS

This chapter will discuss example simulations to understand Liu and Huston’s testing function and Wang’s testing function to evaluate the accuracy of the simulation by changing the time step, integration method, making error in equations of motion formulation, and using a damper on the pendulum. Kane and Levinson’s testing function is not used here because the testing function based on the total energy of the system encompasses their approach. In multibody dynamics system simulation, selecting the method of integration and the time step or the step size are important. Any of the fixed-step continuous solvers in Simulink can simulate a model to any desired level of accuracy as long as it is given an adequate time and a small enough time step. However, without much experience, it is difficult to determine what type of a solver or time step would provide an accurate simulation. A good simulation practice is to optimize the time step. Smaller time step would increases the computer processing time and bigger time step would increase the simulation error. During the process of developing equations of motions or building a computer model, we may make careless mistakes. Unless there is an experimental result or an intuitive reasoning, these types of mistakes may never be detected. Also, some mistakes may not be very obvious for a set of conditions. The triple pendulum example simulation will demonstrate how these testing functions will help overcome the concerns discussed here.
4.1. Modeling in Matlab and Simulink

The three pendulums were selected as uniform slender bars having lengths of \( l_1 = 0.1 \, \text{m}, l_2 = 0.25 \, \text{m}, \) and \( l_3 = 0.5 \, \text{m}. \) The weights of the bars are \( m_1g = 10 \, \text{N}, \) \( m_2g = 30 \, \text{N}, \) and \( m_3g = 20 \, \text{N}. \) The gravitational acceleration is \( g = 9.8 \, \text{m/s}^2. \) Standard simulation time is 10 seconds.

For the uniform slender bar mass moments of inertia are

\[
I_1 = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{12} & 0 \\ 0 & 0 & I_{13} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} ml_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} ml_2^2 \end{bmatrix}, \quad I_2 = \begin{bmatrix} I_{21} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} ml_2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} ml_2^2 \end{bmatrix}, \quad \text{and} \quad I_3 = \begin{bmatrix} I_{31} & 0 & 0 \\ 0 & I_{32} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} ml_3^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} ml_3^2 \end{bmatrix},
\]

The differential equations of motions were solved numerically using Matlab and Simulink. All the example simulations results are shown in three plots that contain time history of pendulums’ swing angle, Liu and Huston’s testing function solution, and Wang’s testing function solution.

4.1.1. Liu and Huston’s Testing Function in Matlab and Simulink

For the triple pendulum example, the kinetic energy \( K \) was calculated by using the fundamental principles of dynamics (section 3.1) without using the numerical integration. Therefore, we can assume the numerical simulation error is insignificant. Let’s call this kinetic energy as \( K_a. \) \( \dot{K} \) was calculated by using the Liu and Huston’s
method and numerically integrated to get the kinetic energy $K$ as shown in equations (2.3.1) and (2.3.4)

Let’s call this kinetic energy as $K_b$.

$$K_b = \int_{t_0}^{t} \left( \sum_{r=0}^{n} \tilde{F}_r \tilde{u}_r \right) dt \quad (4.1.1)$$

For this numerical integration, Simulink numerical “integrator” was used. Equation (2.3.5) was used to calculate the kinetic energy differences by both methods.

$$\Delta C = [K_a - K_b] - K_0 \quad (4.1.2)$$

Assumption was made that there was no simulation error generated when calculating the kinetic energy $K_a$ using the fundamentals of dynamics as there was no integration or differentiation used. In Liu and Huston’s method, numerical integration was used to calculate the $K_b$ from $\dot{K}$. During this calculation, generated error can be detected by the $\Delta C$. Ideally, $\Delta C$ should be zero.

Also, when modeling the specified motion, $\theta_i(t) = A \sin \omega t$ was not differentiated to get the $\dot{\theta}_i$ and $\ddot{\theta}_i$ as it will introduce errors when calculating the kinetic energy $K_a$ using the fundamentals of dynamics.

$\dot{\theta}_i$ and $\ddot{\theta}_i$ were modeled using the equations (4.1.3) and (4.1.4).

$$\dot{\theta}_i = A \omega \cos \omega t \quad (4.1.3)$$

$$\ddot{\theta}_i = -A \omega^2 \sin \omega t \quad (4.1.4)$$

**4.1.2. Wang’s Testing Function in Matlab and Simulink**

In Wang’s testing function, $\dot{K}$, $\dot{V}$, and $\dot{Q}$ were calculated and integrated to get
Using the testing function $\Delta C = (K + V - Q) - (K_0 + V_0 - Q_0)$, simulation errors were checked. Ideally, $\Delta C$ should be zero.

4.2. Free Motion

4.2.1. Initial Kinetic Energy of the System $= 0$

Free motion of the triple pendulum can be achieved by a simple push on one of the pendulums. The pendulum can freely swing due to gravity influence. The free motion of the triple pendulum was simulated for three integration time steps such as 0.01 second, 0.001 second, and 0.0001 second. Dormand-Prince fixed-step continuous solver formula, also known as ode5 in Matlab was used for the integration. The initial conditions for the pendulums were $\theta_1 = 0^\circ$, $\theta_2 = 60^\circ$, and $\theta_3 = 15^\circ$. Also, initial conditions for $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ were zero and all the damping coefficients were zero. This initial kinetic energy of the system is zero.
From the three plots shown in Figure 4.1, the testing function developed by Liu and Huston and Wang methods showed similar plots. The testing function \( \Delta C \) gets closer and closer to zero as time step decreases. Further time step reduction from 0.001 seconds to 0.0001 seconds does not remarkably affect \( \Delta C \) in both methods. Therefore, the time step 0.001 second can be chosen as the suitable value for this simulation.

Next example is to compare the simulation accuracy between the fixed-step solvers. Euler's fixed-step continuous solver known as ode1 and fourth-order Runge-Kutta fixed step continuous solver formula, also known as ode4 in Matlab, were compared with the Dormand-Prince fixed-step continuous solver formula. The time step
selected for this integration is 0.001 second.

From Figure 4.2, testing functions from both methods clearly show that the Euler’s method cannot be used to simulate this dynamic system due to the fact that at the end of the simulation testing function $\Delta C = 7.7 \text{J}$. Dormand-Prince integration method yielded $\Delta C \approx 10^{-10} \text{J}$ and Runge-Kutta integration method yielded $\Delta C \approx 10^{-7} \text{J}$ for the testing function. Therefore, based on these observations, Dormand-Prince (ode5) integration method and Runge-Kutta (ode4) integration method can be used for the triple pendulum simulation. Dormand-Prince integration method will yield the lowest
4.2.2. Initial Kinetic Energy of the System ≠ 0

When the initial kinetic energy is not zero, it will be interesting to see if the testing functions can detect the simulation error. Initial conditions for the pendulums were \( \theta_1 = 0 \) degree, \( \theta_2 = 60 \) degree, and \( \theta_3 = 15 \) degree. Also, initial conditions for \( \dot{\theta}_1 = 10 \text{rad/sec} \), \( \dot{\theta}_2 = 0 \), and \( \dot{\theta}_3 = 0 \) and all the damping coefficients were zero. The initial kinetic energy of the system is 2.721 J.

Figure 4.3. Time steps of 0.01, 0.001, and 0.0001 seconds with an initial kinetic energy
Figure 4.3 shows that the testing functions detect the simulation error with an initial kinetic energy in the system. Again, both Liu and Huston’s testing functions and Wang’s testing functions show that the 0.001 second and 0.0001 second time steps are better than the 0.01 second time step for accurate simulation.

4.2. Specified Motion

In this example, the triple pendulum motion was obtained creating a specified motion for the link OA. Specified motion was achieved by applying a torque $M_j$ (see Figure 4.2.2) at the joint O in such a way that $\theta_j(t) = A \sin \omega t$. Select $A = 1$ radian and $\omega = 0.75$ radian/sec. $M_j$ can be found using equation (3.1.17). The initial conditions were $\theta_2 = 60$ degree and $\theta_3 = 15$ degree. Also, initial conditions for $\dot{\theta}_2$ and $\dot{\theta}_3$ were zero and all of the damping coefficients were set to zero.
Figure 4.4. Specified motion for $\theta_1(t) = A \sin(\omega t)$

As shown in Figure 4.4, smaller time step show smaller simulation error as expected. Time step of 0.001 second and 0.0001 second showed very similar $\Delta C$ and therefore, reducing the time step smaller than 0.001 second would not add much value to the simulation.
Figure 4.5. Driving torque $M_1$ to maintain Body 1 pendulum specified motion

Applied torque to maintain the specified motion of the link OA is shown in Figure 4.5. Euler, Runge-Kutta, and Dormand-Prince integration methods were compared for the time step of 0.001 second for this specified motion example.
Figure 4.6. Euler, and Runge-Kutta, and Dormand-Prince integration method for the time step=0.001 second

Figure 4.6 shows that the Euler’s integration method produced significant simulation error as compared to the Runge-Kutta and Dormand-Prince testing functions. Runge-Kutta’s method and Dormand-Prince’s method yielded similar results and therefore, both of these methods can be utilized to simulate the specified motion example.

4.3. Make a Mistake in Formulation of Equations of Motions

Developing differential equations of motions and programming them are very tedious and time consuming process. During this process, even a small mistake such as incorrect signs (positive and negative signs) or missing out a term can cause errors in the
simulation results. Some errors are very hard to detect in a complex multibody system simulations, but it is extremely important to identify if there is an error in the simulations. Using the testing functions, it is relatively easy to detect if there was a mistake in the formulations of the equations of motion or programming. Free motion of the triple pendulum discussed in the example problem section 4.1 was selected to demonstrate a mistake in simulation.

Let’s take one of equations of motions developed using the Lagrange’s method in equation (3.1.18) and make a sign mistake as shown in equation (4.3.1) from

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = (m_2r_2 + m_3l_2)l_1C_{2,1}\ddot{\theta}_1 + [I_2 + m_2r_2^2 + m_3l_2^2]\ddot{\theta}_2 + [m_3l_2C_{2,1}]\ddot{\theta}_3 
+ (m_2r_2 + m_3l_2)l_1S_{2,1}\ddot{\theta}_1 - (m_3l_2r_3)S_{3,2}\ddot{\theta}_3 \times - [m_2r_2 + m_3l_2]gS_2 
= M_2 - M_3 + [-c_2(\ddot{\theta}_2 - \dot{\theta}_1) + c_3(\ddot{\theta}_3 - \dot{\theta}_2)] \quad (4.3.1) \]

The initial conditions for the pendulums were \( \theta_1 = 0 \) degree, \( \theta_2 = 60 \) degree, and \( \theta_3 = 15 \) degree. Also, initial conditions for \( \dot{\theta}_1, \dot{\theta}_2, \) and \( \dot{\theta}_3 \) were zero and all the damping coefficients were zero. Time step used for this simulation is 0.001 second established as the adequate time step for an accurate simulation. The integration method selected is Dormand-Prince fixed-step continuous solver formula.

In Figure 4.7, both testing functions detected the incorrect equation where the \( \Delta C \)'s are more than 10° or 1. For correct equations, the testing functions are about \( \Delta C \approx 10^{10} \) J.
Figure 4.7. Mistake in the equations of motion that affect the kinetic energy calculation

Based on this observation, both of the testing functions can be incorporated into the dynamic simulations softwares to check the accuracy of the models or formulations of the equations.

4.4. Contribution of a Small Damper in Simulation Accuracy

Systems that have no damping can exhibit erratic behavior, and with the damper the erratic behaviors can vanish. Damping can be used to help the energy dissipation which is dependent of displacement and velocity or combination of both.

The purpose of this example is to see the simulation error with the damper in the system.
The example system selected was the triple pendulum with an initial kinetic energy. The following parameters were selected to simulate the system. \( l_1 = 0.1 \text{ m}, l_2 = 0.25 \text{ m}, \) and \( l_3 = 0.5 \text{ m}. \) The weights of the bars are \( m_1 g = 10 \text{ N}, \) \( m_2 g = 40 \text{ N}, \) and \( m_3 g = 10 \text{ N}. \) The gravitational acceleration is \( g = 9.8 \text{ m/s}^2. \) The initial conditions for the pendulums were \( \theta_1 = 30 \text{ degree}, \) \( \theta_2 = 60 \text{ degree}, \) and \( \theta_3 = 15 \text{ degree}. \) Also, initial angular speeds of the bars are \( \dot{\theta}_1 = 10 \text{ rad/sec}, \) \( \dot{\theta}_2 = 0, \) and \( \dot{\theta}_3 = 0. \) Time step of 0.01 second was selected for the simulation. Iterations were performed for the following conditions. Without Dampers \( c_1 = c_2 = c_3 = 0 \) and with dampers \( c_1 = c_2 = c_3 = 0.001 \text{ Nms/rad}. \) The simulation was run for 50 seconds to capture all the details. First plot in Figure 4.8 shows the erratic behavior of the triple pendulum motion when the system does not have a damper, but when the system has a very small damper (see Figure 4.8 second plot), the erratic behavior was eliminated.
In Figure 4.9, the system with the damper shows much smaller $\Delta C$ than the
system without the damping. From 0-2 seconds, magnitude of the initial error was much more than the previous simulation examples due to the fact that there was an initial speed \( \dot{\theta}_r = 10 \text{rad/sec} \) applied to the arm OA. Without this initial speed, simulation of damping effect could not be achieved. Application of the damper in the system helped eliminate the erratic behavior of the system. Further analysis was performed to understand whether the damper interferes with the ability of the testing functions to detect the mistakes in simulations.

Example simulation from section 4.3 was rerun here. A mistake was incorporated into Lagrange’s equations of motions with the damper and without the damper. Simulations were run at the time step of 0.001 second in the following order.

Incorrect Equation with No Damper - IE+ND
Incorrect Equation with Damper- IE+D
Correct Equation with No Damper- CE+ND
Based on the Figure 4.10, damper does not interfere with the ability of the testing functions to detect the mistakes in the equations of motions. However, it helps to stabilize the dynamic simulations. Thus understanding the effectiveness of the testing functions in the presence of a damper will be beneficial when we incorporate the testing functions in simulation programs.
CHAPTER 5

CONCLUSION

The testing functions developed by Kane and Levinson, Liu and Huston, and Wang have similar approaches based on Kane's equation even though they are in different forms. This thesis demonstrated the application of Liu and Huston's testing function and Wang's testing function using a triple pendulum example. Even though the forms of the testing functions are different, they were able to demonstrate that they can be used to identify the correct time step, to identify the careless mistakes made in formulations of the equations of motions or computer programming, to identify the numerical integration method suitable for the dynamic system, and to detect the dampers effect in the dynamic simulations. Also, the testing functions were able to suggest the best time step for accurate numerical integration. The testing functions discussed in this thesis are applicable to any holonomic and simple nonholonomic systems. By applying these methods on multiple example systems, a robust procedure can be developed to be incorporated into the dynamic simulation softwares to check the accuracy of the simulation.
BIBLIOGRAPHY

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<http://www.mae.wmich.edu/faculty/kamman/Me459GenForcesPVW.pdf>
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APPENDICES

A. Definition of Terms

Holonomic Constraints and Nonholonomic Constraint

If the constraints can be given in an algebraic form of equations, they are called holonomic constraints. If they cannot be given in an algebraic form of equations, they are called nonholonomic constraints. In a nonholonomic system, the constraint relationships are given only in differential form and the coordinates of the system cannot be expressed by the rest of coordinates in an algebraic form. Time-independent holonomic constraint is called scleronomic constraint. Time-dependent holonomic constraint is called rheonomic constraint. Table 1 shows the classification of the holonomic and nonholonomic constraints [13].

Table 1. Classification of Constraints [13]

<table>
<thead>
<tr>
<th>Holonomic</th>
<th>Scleronomic</th>
<th>Rheonomic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_j(q_1, q_2, ..., q_n) = 0$</td>
<td>$f_j(q_1, q_2, ..., q_n, t) = 0$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nonholonomic</th>
<th>Catastatic</th>
<th>Acatastatic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{k=1}^{n} a_{jk} dq_k = 0$</td>
<td>$\sum_{k=1}^{n} a_{jk} dq_k + a_{j\bullet} dt = 0$</td>
<td></td>
</tr>
</tbody>
</table>
Here \( a_{jk} = a_{jk}(q_1, q_2, \ldots, q_n, t) \) and \( a_{j0} = a_{j0}(q_1, q_2, \ldots, q_n, t) \). \( q_1, q_2, \ldots, q_n \) are generalized coordinates and \( t \) is time. When \( a_{j0} \text{d}t = 0 \) the system is said to be catastatic ("orderly" or "homogenous") system and the constraint equations become homogeneous. When \( a_{j0} \text{d}t \neq 0 \), the system is called an acatastatic system. An acatastatic system is a non-conservative system where constraint forces do work on the system.

**Skew Symmetric Matrix**

Skew symmetric matrix is a square matrix. When it is transposed, it becomes its negative.

Let's take two vectors \( \vec{a} \) and \( \vec{b} \)

\[
\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3 \tag{A1}
\]

\[
\vec{b} = b_1 \vec{e}_1 + b_2 \vec{e}_2 + b_3 \vec{e}_3 \tag{A2}
\]

Cross or vector product of the two vectors can be given as follows.

\[
\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \vec{e}_1 + (a_3 b_1 - a_1 b_3) \vec{e}_2 + (a_1 b_2 - a_2 b_1) \vec{e}_3 \tag{A3}
\]

\[
\vec{a} \times \vec{b} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{A4}
\]

\[
\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}
\]

is the skew symmetric matrix [10].
B. Matlab/ Simulink Programming for Free Motion of the Pendulum

Combination of Matlab codes and Simulink was used to simulate example problems.

Following program was used in all the examples by changing some of the parameters as shown in the examples in section 4. Examples are shown for the free motion and specified motion of the pendulums.

%This Program is to simulate Trippe pendulum free motion

%Define Variables

Specify_Time_Step=[0.01 0.001 0.0001]

for j=1:3

Time_Step=Specify_Time_Step(1,j)

End_Time=10

L1=0.1; L2=0.25; L3=0.5;

m1=10/9.8; m2=30/9.8; m3=20/9.8;

r1=L1/2; r2=L2/2; r3=L3/2;

I1=1/12*m1*L1^2;

I2=1/12*m2*L2^2;

I3=1/12*m3*L3^2;

cl=0;

c2=0;

c3=0;

g=9.8 % m/s^2 Gravity

M1=0;
M2=0;
M3=0;
Theta1=0;  % in degrees
Theta2=60;  % in degrees
Theta3=15;  % in degrees
Theta1dot=0;
Theta2dot=0;
Theta3dot=0;

%Calculate Initial Kinetic Energy
InitialKE=1/2*(11+m1*r1^2)*Theta1dot^2+1/2*...
(l2*Theta2dot^2+m2*L1^2*Theta1dot^2+m2*r2^2*Theta2dot^2+2*m2*L1*r2*Theta1d
ot*Theta2dot*cos(Theta2-Theta1))...  
1/2*(l3*Theta3dot^2+m3*L1^2*Theta1dot^2+m3*L2^2*Theta2dot^2+2*m3*L1*r2*Theta3
dot^2+2*m3*L1*L2*Theta1dot*Theta2dot*cos(Theta2-Theta1)+...  
2*m3*L2*r3*Theta2dot*Theta3dot*cos(Theta3-
Theta2)+2*m3*L1*r3*Theta1dot*Theta3dot*cos(Theta3-Theta1));

%To Run the simulation in Simulink
sim('TPSC')

%===To Plot the pendulum motion
subplot(3,1,1)
plot(tout,Theta1,tout,Theta2,tout,Theta3)
title('bf Triple Pendulum Motion','FontName','Times')
ylabel('Swing Angle \it\theta \rm (degree)','FontName','Times')
% To process the data to draw plots for KE difference and total energy of the system versus time

tstep=(tout(2)-tout(1));
a=who;
if (ismember('data',a) & ismember('time',a))
    c=size(data,2)+1;
    if length(tout)<size(time,1)
        time(length(tout)+1:end,c)=NaN;
        data(length(tout)+1:end,c)=NaN;
    elseif length(tout)>size(time,1)
        time(size(time,1)+1:length(tout,:),)=NaN;
        data(size(data,1)+1:length(tout,:),)=NaN;
    end
else
    c=1;
end

if size(KEdiff,2)>size(KEdiff,1)
KEdiff = KEdiff;
end

if size(tout,2)>size(tout,1)
    tout = tout';
end

data(1:length(tout),c) = KEdiff;
time(1:length(tout),c) = tout;

if ismember('leg', a)
    d = length(leg) + 1;
else
    d = 1;
end

leg{d} = ['Time step: ', num2str(tstep), ' sec'];

% To plot Kinetic energy difference versus time
subplot(3,1,2)
semilogy(time, data)
ylim([10^-20 1])
ylabel('\(\Delta C\) (J)', 'FontName', 'Times')
%xlabel('Time (sec)', 'FontName', 'Times')
title('\textbf{Testing Function by Liu and Huston: Kinetic Energy Difference} \(\Delta C\)', 'FontName', 'Times')
Grid on
legend(leg)

% to plot K+V-Q versus time
dataKVQ(1:length(tout),c)=KVQ;
delta=abs(dataKVQ-dataKVQ(1,1));
subplot(3,1,3)
semilogy(time,delta)
ylim([10^-20 1])
ylabel('\Delta C (J)','FontName','Times')
xlabel('Time (sec)','FontName','Times')
title('\bf Testing Function by Wang: \it \Delta C = (K+V-Q)-(K_0+V_0-Q_0)\rm ','FontName','Times')
Grid on
legend(leg)

%-----------------------------
clear tout KEdiff a c d tstep KVQ
end

Simulink program was used to solve the differential equations of motion. Matlab was used to run the overall solutions procedure. Following pages show the Simulink programming diagrams.
Free Motion of Triple Pendulum Using Dependent Set of Coordinates

Equation 1

\[ \theta_1 \]

Equation 2

\[ \theta_2 \]

Equation 3

\[ \theta_3 \]

Convert \( \theta_1 \) to Degree

Convert \( \theta_2 \) to Degree

Convert \( \theta_3 \) to Degree

KE Difference

Kinetic Energy Comparison

To Workspace 4

To Workspace 5

Theta KE Difference

Convert \( \theta_1 \) to Degree

\[ -K \]

\[ 180\pi \]

\[ 360\pi \]

Mux

To Workspace 2

To Workspace 3

Theta 1

Theta 2

Theta 3
Integrations to Calculate Theta Vectors

1. \( f(z) \) \( f(z) = 0 \) solving Algebraic Constraint 1
   - \( \theta_1 \) DDot
   - \( \theta_1 \) dot
   - \( \theta_1 \) DDot
   - \( \theta_1 \) dot
   - \( \theta_1 \) DDot
   - \( \theta_1 \)

2. \( f(z) \) \( f(z) = 0 \) solving Algebraic Constraint 2
   - \( \theta_2 \) DDot
   - \( \theta_2 \) dot
   - \( \theta_2 \) DDot
   - \( \theta_2 \) dot
   - \( \theta_2 \) DDot
   - \( \theta_2 \)

3. \( f(z) \) \( f(z) = 0 \) solving Algebraic Constraint 3
   - \( \theta_3 \) DDot
   - \( \theta_3 \) dot
   - \( \theta_3 \) DDot
   - \( \theta_3 \) dot
   - \( \theta_3 \) DDot
   - \( \theta_3 \)
Calculate Equation 1
Dependent Set of Coordinates

\[ \text{Eqn1} = (1 + m_1 r_1^2 + (m_2 + m_3) L_1^2) u[3] \]
\[ \text{Eqn2} = (m_2 r_2 + m_3 L_2) L_1 \cos(u[4] - u[1]) u[6] \]
\[ \text{Eqn3} = m_3 L_1 r_3 \cos(u[7] - u[1]) u[9] \]
\[ \text{Eqn4} = -(m_2 r_2 + m_3 L_2) L_1 \sin(u[4] - u[1]) (u[5])^2 \]
\[ \text{Eqn5} = -(m_3 L_1 r_3) \sin(u[7] - u[1]) (u[8])^2 \]
\[ \text{Eqn6} = (m_1 r_1 + (m_2 + m_3) L_1) g \sin(u[1]) \]
\[ \text{Eqn7} = -(M_1 - M_2 - c_1 u[2] + c_2 (u[5] - u[2])) \]

Signal Guide:
Theta 1 = u[1]
Theta 1 dot = u[2]
Theta 1 dd = u[3]
Theta 2 = u[4]
Theta 2 dot = u[5]
Theta 2 dd = u[6]
Theta 3 = u[7]
Theta 3 dot = u[8]
Theta 3 dd = u[9]
dd = double dot
Signal Guide
Theta 1 = u[1]
Theta 1dot= u[2]
Theta 1dd= u[3]
Theta 2 = u[4]
Theta 2dot= u[5]
Theta 2dd= u[6]
Theta 3 = u[7]
Theta 3dot= u[8]
Theta 3dd= u[9]

Calculate Equation 2
Dependent Set of Coordinates

\[ Fcn1 = \frac{(m_2 r_2 + m_3 L_2) L_1 \cos(u[4]-u[1]) u[3]}{Fcn2} \]

\[ Fcn2 = (l^2 + m_2 r_2^2 + m_3 L_2^2) u[6] \]

\[ Fcn3 = m_3 L_2 r_3 \cos(u[7]-u[4]) u[9] \]

\[ Fcn4 = \frac{(m_2 r_2 + m_3 L_2) L_1 \sin(u[4]-u[1])(u[2])^2}{Fcn5} \]

\[ Fcn5 = -(m_3 L_2 r_3) \sin(u[7]-u[4]) (u[8])^2 \]

\[ Fcn6 = (m_2 r_2 + m_3 L_2) g \sin(u[4]) \]

\[ Fcn7 = -(M_2 - M_3 - c_2(u[5]-u[2]) + c_3(u[8]-u[5])) \]
Signal Guide
Theta 1 = u[1]
Theta 1dot = u[2]
Theta 1dd = u[3]
Theta 2 = u[4]
Theta 2dot = u[5]
Theta 2dd = u[6]
Theta 3 = u[7]
Theta 3dot = u[8]
Theta 3dd = u[9]
dd = double dot

Calculate Equation 3
Dependent Set of Coordinates

m3*L1*r3*cos(u[7]-u[1])*u[3]
Fcn1

m3*L2*r3*cos(u[7]-u[4])*u[6]
Fcn2

(I3+m3*r3^2)*u[9]
Fcn3

m3*L1*r3*sin(u[7]-u[1])*(u[2])^2
Fcn4

m3*L2*r3*sin(u[7]-u[4])*(u[5])^2
Fcn5

m3*r3*g*sin(u[7])
Fcn6

-(M3-c3*(u[8]-u[5]))
Fcn7

Equation 3
Kinetic Energy Comparison

KE Fundamentals

Calculate KE Using Dynamic Fundamentals

KE Houston's Method

Calculate KE from Huston's Method

Energy Input Q

V

Out1

Potential Energy V

Energy Input Q

K + V - Q

To Workspace 4

KE Difference 1

KE Houston

To Workspace 1

KE Fundamentals

Theta

KE Houston

To Workspace 2

abs(u(1))

Fcn 1

KE Difference

V1

Kdot + Vdot - Odot

Build Derivative

To Workspace 4

Q
Calculate Kinetic Energy from Fundamentals in Dynamics

\[ \frac{1}{2} \left( I_1 + m_1 r_1^2 + m_2 L_1^2 + m_3 L_1^2 \right) (u(2))^2 \]

\[ \frac{1}{2} \left( I_2 + m_2 r_2^2 + m_3 L_2^2 \right) (u(5))^2 \]

\[ \frac{1}{2} \left( I_3 + m_3 r_3^2 \right) (u(8))^2 \]

\[ \frac{1}{2} \left( m_2 L_1 r_2 + m_3 L_1 L_2 \right) \cos(u(4) - u(1)) (u(2)) (u(5)) \]

\[ m_3 L_2 r_3 \cos(u(7) - u(4)) (u(5))^2 (u(8)) \]

\[ m_3 L_1 r_3 \cos(u(7) - u(1)) (u(2)) (u(8)) \]
Calculate Kinetic Energy from Liu and Huston's Method

\[ (-m_1 r_1 - m_2 L_1 - m_3 L_1) g \sin(\theta_1) \dot{\theta}_1 + (-m_2 r_2 - m_3 L_2) g \sin(\theta_4) \dot{\theta}_4 + (-m_3 r_3) g \sin(\theta_7) \dot{\theta}_7 + \ldots \]
**Energy Input Q**

\[ Q = \frac{1}{s} \]


\[ Fcn2: \quad (M2 - M3 - c2 \cdot (u[5] - u[2]) + c3 \cdot (u[8] - u[5])) \cdot u[5] \]

\[ Fcn4: \quad (M3 - c3 \cdot (u[8] - u[5])) \cdot u[8] \]

**Signal Guide**
- \( \Theta_1 = u[1] \)
- \( \Theta_1 \dot{=} u[2] \)
- \( \Theta_1 \ddot{=} u[3] \)
- \( \Theta_2 = u[4] \)
- \( \Theta_2 \dot{=} u[5] \)
- \( \Theta_2 \ddot{=} u[6] \)
- \( \Theta_3 = u[7] \)
- \( \Theta_3 \dot{=} u[8] \)
- \( \Theta_3 \ddot{=} u[9] \)
- \( dd = \) double dot
Potential Energy \( V \)

\[ -\frac{1}{2} \cos(u[1]) m_1 g \]

\[ -(L_1 \cos(u[1]) + L_2/2 \cos(u[4])) m_2 g \]

\[ -(L_1 \cos(u[1]) + L_2 \cos(u[4]) + L_3/2 \cos(u[7])) m_3 g \]
Specified Motion of Triple Pendulum Using Dependent Set of Coordinates

Equation 1
\[ \Theta_1 + \Theta_2 + \Theta_3 = \text{Driving Torque} \]

Equation 2
\[ \Theta_1 \text{ Vector} \]

Equation 3
\[ \Theta_2 \text{ Vector} \]

\[ \Theta_3 \text{ Vector} \]

Calculate Theta Vectors

To Workspace 1

Convert Theta 1 to Degree

To Workspace 2

Convert Theta 2 to Degree

To Workspace 3

Convert Theta 3 to Degree

To Workspace 4

Mux

To Workspace 5

Kinetic Energy Difference

To Workspace 4

Theta 1

Theta 2

Theta 3

Theta 1

Theta 2

Theta 3
Integrations to Calculate Theta Vectors

If \( f(z) = 0 \)

\[ \frac{d}{dz} \theta_1 = \text{signal} \]

\[ \frac{d^2}{dz^2} \theta_1 = \text{DDot} \]

\[ \frac{1}{s} \theta_2 = \text{Dot} \]

\[ \frac{1}{s} \theta_2 = \text{DDot} \]

\[ \frac{1}{s} \theta_3 = \text{Dot} \]

\[ \frac{1}{s} \theta_3 = \text{DDot} \]
Calculate Equation 1
Dependent Set of Coordinates

Signal Guide
Theta 1 = u[1]
Theta 1dot= u[2]
Theta 1dd= u[3]
Theta 2 = u[4]
Theta 2dot= u[5]
Theta 2dd= u[6]
Theta 3 = u[7]
Theta 3dot= u[8]
Theta 3dd= u[9]
dd= double dot

Fcn1
\((11+m1*r1^2+(m2+m3)*L1^2)*u[3]\)

Fcn2
\((m2*r2+m3*L2)^2*L1*\cos(u[4]-u[1])*u[6]\)

Fcn3
\(m3*L1*r3*\cos(u[7]-u[1])*u[9]\)

Fcn4
\(-m2*r2+m3*L2)^2*L1*\sin(u[4]-u[1])^2(u[5])^2\)

Fcn5
\(-m3*L1*r3*\sin(u[7]-u[1])^2(u[8])^2\)

Fcn6
\(m1*r1+(m2+m3)*L1*g*\sin(u[1])\)

Fcn7
\(-M2-c1*u[2]+c2*(u[5]-u[2])\)

Solve
\(f(x,z) = 0\)

Algebraic Constraint4
Calculate Equation 2
Dependent Set of Coordinates

\[ \text{Signal Guide} \]
\[ \theta_1 = u[1] \]
\[ \dot{\theta}_1 = u[2] \]
\[ \ddot{\theta}_1 = u[3] \]
\[ \theta_2 = u[4] \]
\[ \dot{\theta}_2 = u[5] \]
\[ \ddot{\theta}_2 = u[6] \]
\[ \theta_3 = u[7] \]
\[ \dot{\theta}_3 = u[8] \]
\[ \ddot{\theta}_3 = u[9] \]

\[ \text{Fcn1: } (m_2 r_2 + m_3 L_2) L_1 \cos(u[4] - u[1]) \dot{u}[3] \]
\[ \text{Fcn2: } (I_2 + m_2 r_2 + m_3 L_2^2) u[6] \]
\[ \text{Fcn3: } m_3 L_2 r_3 \cos(u[7] - u[4]) \dot{u}[9] \]
\[ \text{Fcn4: } (m_2 r_2 + m_3 L_2) L_1 \sin(u[4] - u[1]) \dot{u}[2]^2 \]
\[ \text{Fcn5: } -(m_3 L_2^2 r_3) \sin(u[7] - u[4]) \dot{u}[8]^2 \]
\[ \text{Fcn6: } (m_2 r_2 + m_3 L_2) g \sin(u[4]) \]
\[ \text{Fcn7: } -(M_2 - M_3 - c_2 (u[5] - u[2]) + c_3 (u[8] - u[5])) \]
Signal Guide
Theta 1 = u[1]
Theta 1dot= u[2]
Theta 1dd= u[3]
Theta 2 = u[4]
Theta 2dot= u[5]
Theta 2dd= u[6]
Theta 3 = u[7]
Theta 3dot= u[8]
Theta 3dd= u[9]
dd= double dot

Calculate Equation 3
Dependent Set of Coordinates

\[ Fcn1 = m_3 L_1 r_3 \cos(u_7-u_1) u_3 \]

\[ Fcn2 = m_3 L_2 r_3 \cos(u_7-u_4) u_6 \]

\[ Fcn3 = (I_3+m_3 r_3^2) u_9 \]

\[ Fcn4 = m_3 L_1 r_3 \sin(u_7-u_1) (u_2)^2 \]

\[ Fcn5 = m_3 L_2 r_3 \sin(u_7-u_4) (u_5)^2 \]

\[ Fcn6 = m_3 r_3 g \sin(u_7) \]

\[ Fcn7 = -(M_3-c_3 (u_8-u_5)) \]
Kinetic Energy Comparison

Calculate KE Using Dynamic Fundamentals

Calculate KE from Huston's Method

Potential Energy V

Energy Input Q

KVQ

Derivative Kdot+Vdot-Qdot

abs(u[1])

KE Difference

KE Difference1

To Workspace1

To Workspace2

To Workspace4
Signal Guide

\[ \Theta_1 = u(1) \]
\[ \Theta_1' = u(2) \]
\[ \Theta_1'' = u(3) \]
\[ \Theta_2 = u(4) \]
\[ \Theta_2' = u(5) \]
\[ \Theta_2'' = u(6) \]
\[ \Theta_3 = u(7) \]
\[ \Theta_3' = u(8) \]
\[ \Theta_3'' = u(9) \]

\[ \text{dd} = \text{double dot} \]

I Calculate Kinetic Energy from Fundamentals in Dynamics.

\[ \frac{1}{2} (I_1 + m_1 r_1^2 + m_2 L_1^2 + m_3 L_2^2) (u_1')^2 \]

\[ \text{Fcn1} \]

\[ \frac{1}{2} (I_2 + m_2 r_2^2) (u_2')^2 \]

\[ \text{Fcn2} \]

\[ \frac{1}{2} (I_3 + m_3 r_3^2) (u_3')^2 \]

\[ \text{Fcn3} \]

\[ (m_2 L_1 r_2 + m_3 L_1 r_1) \cos(u_1 - u_3) \]

\[ \text{Fcn4} \]

\[ m_3 L_2 r_3 \cos(u_7 - u_4) \]

\[ \text{Fcn5} \]

\[ m_3 L_1 r_3 \cos(u_7 - u_1) \]

\[ \text{Fcn6} \]
Calculate Kinetic Energy from Liu and Huston's Method

\[ \text{KE Huston's Method} \]

\[ \text{Signal Guide} \]
\[ \text{Theta 1} = \theta_1 \]
\[ \text{Theta 1 dot} = \dot{\theta}_1 \]
\[ \text{Theta 1 double dot} = \ddot{\theta}_1 \]
\[ \text{Theta 2} = \theta_2 \]
\[ \text{Theta 2 dot} = \dot{\theta}_2 \]
\[ \text{Theta 2 double dot} = \ddot{\theta}_2 \]
\[ \text{Theta 3} = \theta_3 \]
\[ \text{Theta 3 dot} = \dot{\theta}_3 \]
\[ \text{Theta 3 double dot} = \ddot{\theta}_3 \]

dd = double dot
Method 3

\[ Fcn15: \frac{(-M2 - c1 \cdot u[2] + c2 \cdot (u[5] - u[2]) - (m1 \cdot r1 + m2 \cdot L1 + m3 \cdot L1) \cdot g \cdot \sin(u[1])) \cdot u[2]}{u[2]} \]

\[ Fcn16: \frac{(M2 - M3 - c2 \cdot (u[5] - u[2]) + c3 \cdot (u[8] - u[5]) - (m2 \cdot r2 + m3 \cdot L2) \cdot g \cdot \sin(u[4])) \cdot u[5]}{u[5]} \]

\[ Fcn17: \frac{(M3 - c3 \cdot (u[8] - u[5]) - m3 \cdot r3 \cdot g \cdot \sin(u[7])) \cdot u[8]}{u[8]} \]

\[ Fcn18: \frac{1}{s} \]

\[ [\text{Theta}] \]

\[ \text{Out1} \]
Potential Energy $V$

\[ V = -\frac{1}{2} \cos(u_1) m_1 g \]

\[ + \frac{1}{2} \cos(u_4) m_2 g \]

\[ - \cos(u_1) + \frac{1}{2} \cos(u_4) \]

\[ + \frac{1}{2} \cos(u_7) m_3 g \]