The Calculation of the Cross-Section of the Interaction between Two Yang-Mills Fields

Richard A. Starr

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THE CALCULATION OF THE CROSS-SECTION
OF THE INTERACTION BETWEEN TWO YANG-MILLS FIELDS

by

Richard A. Starr

A Thesis submitted to the
Faculty of the School of Graduate
Studies in partial fulfillment
of the
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Richard A. Starr
THE PROBLEM AND ITS BACKGROUND

This thesis presents a study of the interaction between two arbitrary fields, named Yang-Mills fields\(^1\) for their originators. The basic equations for the calculations of the Yang-Mills interaction are due to modern quantum field theory and its specific application to quantum electrodynamics\(^2\).

Much useful information can be obtained from the formulation of the dynamical equations for a classical system: for example, the classical description of the electromagnetic field, based on Maxwell's equations, leads to wave concepts. Then using this formalism as a starting point, one can proceed to the quantum theory of free fields, and finally to the quantum theory of interacting fields.

Treatment of a field as a mechanical system with an infinite number of degrees of freedom leads to formulating a theory of the field by analogy with the classical mechanics of a system of particles. In this procedure, the displacements \(\eta_\alpha\) of the particles are replaced by a field function \(\phi(x)\) which corresponds to an infinite number of degrees of freedom. It is usually possible to obtain in a formal way the dynamical equations for the field functions (in

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the form of the Euler-Lagrange equations) from a properly chosen Lagrangian function of the system. Often it is convenient to consider these equations in turn as equivalent to a principle of stationary action, in the manner outlined below.

From classical mechanics the action is defined as \( S = \int L(x) \, d^4x \), where \( L(x) = L(x,t) \); that is, dependence on all four space-time variables is assumed. The Lagrangian \( L \), which does not explicitly depend on the coordinates \( x^\mu \), is generally taken to be a real function of the field variables \( \phi^i(x) \) and of their first derivatives \( \partial \phi^i / \partial x^\mu \). So we have \( L(x) = L(\phi(x), \partial \phi / \partial x^\mu) \).

The variational principle of stationary action states that the value of \( S \) is unchanged when the \( \phi^i \) are varied arbitrarily over a finite volume of space-time. Hence it is required that \( \delta S = \int \delta L(\phi; \partial \phi / \partial x^\mu) \, d^4x = 0 \). Now \( \delta S = \int \delta L + \int \left( \frac{\partial L}{\partial \phi^i} \delta \phi^i \right) \, d^4x \); using this expansion and integrating the second term by parts leads directly to the Euler-Lagrange equations.

The principle is more easily expressed in terms of the functional derivative, \( \frac{\delta}{\delta \phi^i} \). (Let \( \delta S \) be the change in \( S \) produced when the field variables \( \phi^i \) are altered by amounts \( \delta \phi^i \) over some finite volume of space-time. Then the functional derivative \( \left( \frac{\delta S}{\delta \phi^i} \right) \) is the coefficient of \( \delta \phi^i d^4x \) in the integral, over that volume, which produces \( \delta S \).) Thus if \( \delta S = \int \delta L(\phi; \partial \phi / \partial x^\mu) \, d^4x = 0 \), the only non-trivial solution is that \( \frac{\delta S}{\delta \phi^i} = 0 \). In the formal theory underlying the calculation of the interaction of Yang-Mills fields, it is assumed that the dynamical equations can always be derived from the proper choice of an action functional.

\[^{31}\text{loc. cit., p. 14.}\]
The theory of Yang-Mills groups and fields is based on a generalization of the well-known gauge group of electrodynamics. The two-dimensional rotation group from which the gauge group is obtained is replaced by a three-dimensional rotation group, then later extended to include any Lie group. The Yang-Mills group is an infinite-dimensional group since the "angle of rotation" is allowed to vary from point to point in space-time. The group is formed by taking the continuous direct product of any arbitrary Lie group with itself over space-time; thus the Lie group will be called the generating group for the Yang-Mills group.

Discussing the various transformations provided by any group leads to two possibilities: homogeneous transformations and inhomogeneous transformations. In the homogeneous case, the representation is provided by a matrix transformation written in the Einstein convention as (I) \[ \phi'^i = D^i_j (g(x)) \phi^j . \]

Expanding the matrix \( D^i_j \) in a Taylor series about the unit element yields the following:

\[
\phi'^i = \left\{ \delta^i_j + \int \frac{\delta D^i_j}{\delta g(x')} g \left[ g^x(x') - e^x(x') \right] d^4x' \right\} \phi^j
\]

for the case of a space-time dependent matrix. The \( g^x(x') \)'s are the group elements providing the transformation. Defining \( \left[ g^x(x') - e^x(x') \right] \) as \( \delta g^x(x') \), this becomes \( \phi'^i = \phi^i + \int \frac{\delta D^i_j}{\delta g(x')} \delta g^x(x') d^4x' \phi^j \). The change in the field variable due to the homogeneous transformation can now be written \( \phi'^i - \phi^i = \int \left[ G^i_j \phi^j \delta g^x(x') \right] d^4x' \), where \( G^i_j \) is the generator of the representation and is equal to \( \frac{\delta D^i_j}{\delta g^x(x')} g \).
this is true for the homogeneous case, the change in any field variable due to any infinitesimal group transformation may be written

\[ \delta \phi = \int R^i_i(x, x') \delta x' \]

where the \( R^i_i \) are linear combinations of the delta function and its derivatives and the \( \delta x' \) are defined as mentioned above.

The four-vector potential, \( A_\mu \), of electromagnetism, which has the regular three-vector and the scalar potentials as components, serves as an example of a field variable providing an inhomogeneous representation of an invariance group. The \( E \) and \( B \) fields are not changed if (II) \( A'_\mu = A_\mu + \frac{\partial F}{\partial x^\mu} \) replaces \( A_\mu \), where \( F \) is an arbitrary function of space-time and can be associated with an element in the electromagnetic gauge group. Since form (I) does not apply here, it is stated that \( A_\mu \) transforms inhomogeneously under the gauge group.

If in the Yang-Mills group a field variable \( \phi^i \) transforms homogeneously, its derivatives will not obey a simple transformation law since the matrix of the transformation depends on space-time. (This last is analogous to the electromagnetic case above (II), in which \( F \) is dependent on the same space-time point as \( A_\mu \).) That is, the first derivative of the field variable is \( \phi'^i_{,\mu} = D^i_j \phi^j_{,\mu} + D^i_{j\mu} \phi^j \), where the comma denotes the ordinary partial derivative. Thus, if an attempt is made to construct invariant action functions, a more manageable type of derivative must be sought. The introduction of the Yang-Mills field can be based on the need for this "invariant derivative" to replace the ordinary derivative in the original action functional. This derivative\(^5\) is defined by \( \phi'^i_{,\mu} = \phi^i_{,\mu} + G^i_{,\mu} A^\mu \phi^j \).

\(^{5}\)loc. cit., p. 437.
It has the same transformation properties under the Yang-Mills group as $\phi'$ has, provided the $A_\mu^\alpha$, components of the Yang-Mills field, transform inhomogeneously — much like the components of the electromagnetic field — and $G_\alpha$ is the generator matrix of $\phi'$'s representation.

The Lagrangian for the Yang-Mills field itself is obtained from the analogy with the electromagnetic field. The Lagrangian for the electromagnetic field is $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, where the $F$s are the components of the field tensor given by $F_{\mu\nu} = A_\nu^\alpha - A_\mu^\alpha$ . In the Yang-Mills case, similarly, $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, with $F_{\mu\nu} = A_\nu^\alpha - A_\mu^\alpha + C_\beta^\gamma_{\alpha} A_\mu^\beta A_\nu^\gamma$.

The $C_\beta^\gamma_{\alpha}$ are known as structure constants of the generating group and are defined in terms of the group elements ($g$ and $g'$) as $C_\beta^\gamma_{\alpha} = \left( \frac{\partial (g g')^\beta}{\partial g'^\gamma} - \frac{\partial (g g')^\gamma}{\partial g'^\beta} \right) g = g' = e$. (In this section, "e" is the group identity.) The $F$s provide a matrix representation of the Yang-Mills group generated by the structure constants; hence, $F_{\mu\nu}^\alpha = F_{\mu\nu}^\alpha + C_\beta^\gamma_{\alpha} A_\mu^\beta F_{\nu}^\gamma$.

The action functional for the Yang-Mills field is then $S_A = -\frac{1}{4} \int q^{\frac{1}{2}} F_{\mu\nu}^\alpha C_\gamma^\mu_{\alpha} \delta A_\mu^\nu$, where $q^{\frac{1}{2}}$ is a scalar density of unit weight included to make $S_A$ invariant under general coordinate transformations. The field equations corresponding to $S_A$ can now be obtained if desired. They have the form $\frac{\delta S_A}{\delta A_\mu^\alpha} = -q^{\frac{1}{2}} F_{\mu\nu}^\alpha \delta J$, using the invariant derivative. This result is analogous to the electromagnetic equation $F_{\mu\nu}^\alpha = 0$, which states Maxwell's equations, $\nabla \cdot E = \rho/\epsilon_0$, and $\nabla \times B = \mu_0 (\sigma_0 + \epsilon_0 \frac{\partial E}{\partial t})$, in field tensor form for free space. Hereafter, commas followed by Latin indices will sometimes be used to denote functional derivatives: thus, $\frac{\delta^2 S}{\delta \phi^i \delta \phi^j}$ will be abbreviated by the symbol $S_{ij}$. 
THE SCATTERING AMPLITUDE

The primary problem of any scattering theory is that of computing the scattering amplitudes $\langle A'_1, \ldots, A'_n | A_{i_1}, \ldots, A_{i_m} \rangle$, where the $A_i$'s correspond to particular quanta separated at large distances from one another before and after the scattering process. The information derived is the subsequent configuration of these quanta after they approach closely enough to interact with one another and are again separated at large distances. A linearized version of scattering theory (in which $\phi \rightarrow 0$ when no disturbances are present) applies and, in order to emphasize this so-called linearization of the theory, explicit labels are affixed whenever appropriate, writing $|A_1, \ldots, A_m, -\infty \rangle$ for the state vector of the collection of free quanta in the infinite past, $(S_{ij})$ for $(S_{ij})$ evaluated at $\phi = 0$, etc. Once the scattering amplitudes have been calculated the transition probabilities and scattering cross-sections can easily be computed.

In quantum theory it is generally necessary to replace the field variable by operators satisfying definite commutation relations, assuming that this change does not alter the functional dependence of the action ($S$) on these variables.

The variational principle due to Schwinger\textsuperscript{6} enables the rewriting of the amplitude $\langle A, B, \infty | C, D, -\infty \rangle$ as a sum of terms each of which

\footnote{\textit{loc. cit.}, p. 760.}
contains the vacuum-to-vacuum amplitude of a product of \( \phi \)'s arranged from right to left in chronological order, that is, of a \( T \) - product. It is possible to rewrite these as sums of products containing correlation functions \( G_{ij}^{j''} \ldots \) such that
\[
\langle 0, \infty | \phi^i | 0, -\infty \rangle = \langle 0, \infty | o, -\infty \rangle \phi^i, \\
\langle 0, \infty | T(\phi^i \phi^j) | o, -\infty \rangle = \langle 0, \infty | o, -\infty \rangle [\phi^i \phi^j - i G^{ij}], \\
\langle 0, \infty | T(\phi^i \phi^j \phi^{k''}) | o, -\infty \rangle = \langle 0, \infty | o, -\infty \rangle \left[ \phi^i \phi^j \phi^{k''} - i \mathcal{P}(\phi^i G^{j''}) + (-i)^2 G^{ij'} \right],
\]
where \( \mathcal{P} \) indicates the sum of all permutations.

The higher-order correlation functions can be expressed in terms containing only \( G_{ij}^{j''} \) and derivatives of the action. The important point to be made here is that these two-point correlation functions can be approximated, in the linearized theory of scattering, by the \( ij \)th component of the Feynman Green's function of the classical \( S_{ij}^{j''} \) matrix (modified slightly to become non-singular).

The final form of the scattering amplitude\(^7\) is now a matter of straightforward computation, since all \( \phi \)'s can now be dropped due to linearization. All that remains is a combination of Green functions and derivatives of the action \( S(\phi) \). In this first-order theory
\[
\langle 0, \infty | o, -\infty \rangle \text{ is unity and } \langle A, B, \infty | C, D, -\infty \rangle \text{ is}
\]
\[
(I) \; i \mathcal{U}_A^i \mathcal{U}_B^{j''} \left[ S_{ij'k''} \mathcal{U}_C^{j'} + S_{ij'm''} G^{m''} n'' S_{n''k''} \mathcal{U}_D^{k''} + S_{ik''} m'' G^{m''} m'' S_{i'k''} \mathcal{U}_C^{k''} + S_{ik''} m'' G^{m''} n'' S_{n''k''} \mathcal{U}_D^{k''} \right] \mathcal{U}_C^{j'} \mathcal{U}_D^{k''},
\]
where the primes denote integration over a particular space-time variable, and where \( \mathcal{U}_A \) is the free wave function appropriate to particle \( A \). For the Yang-Mills quanta, the wave functions and Green's function are:

\(^7\text{loc. cit., p. 783.}\)
where $G(x,x')$ is the Feynman Green's function for a scalar field, $\epsilon_\pm$ is a helicity four-vector, $\delta^{\alpha\beta}$ is a Kronecker delta and $\delta^{\mu\nu}$ is a component of the Minkowski metric.

Feynman diagrams can be used to outline the results of the calculations. The first term in the scattering amplitude arises from the direct interaction of the two particles:

\[ u_{\mu z}^{\alpha}(p;x) = \frac{e^{ipx}}{\sqrt{2p^0}} \epsilon_{\pm \mu}(p)(2\pi)^{-\frac{3}{2}} \]

(III) \[ G_{\mu \nu} = \delta^{\alpha\beta} \eta_{\mu \nu} G(x,x') \]

The second term arises from the possibility that the initial particles may be annihilated and then re-created in their final states after passing through a one-particle virtual intermediate state:

8 loc. cit., p. 615-619.
The third and fourth terms, which are obtainable from one another through interchange of the initial or final states, result from the exchange of a virtual particle:

\[ \text{(3)} \]

\[ \begin{array}{c}
A \\
\text{C}
\end{array} \quad \begin{array}{c}
B \\
\text{D}
\end{array} \]

\[ \text{(4)} \]

\[ \begin{array}{c}
A \\
\text{C}
\end{array} \quad \begin{array}{c}
B \\
\text{D}
\end{array} \]

In each case there are sixteen different helicity arrangements\(^9\), but a simplification is possible. In terms of transverse polarization states, the helicities for the Yang-Mills quanta are:

\[ \epsilon_{\mu}(P) = \frac{1}{\sqrt{2}} \left( \epsilon_1(P) \pm i \epsilon_2(P) \right) \]

and the customary relation holds that \((\epsilon_2(P))^* = (\epsilon_1(P))\).

Therefore, taking the complex conjugate of any amplitude, say, for example, the process given by \(+x-\) yields that of the process \(-x+\), where the signs refer to the helicities. This procedure then will enable the handling of the sixteen arrangements by calculating eight cases and taking their complex conjugates. It will also be convenient to split these eight cases into a pair of two-by-two matrices, one matrix for the four incoming possibilities when the two outgoing particles have like helicities, and the other matrix to handle the other four cases.

---

The expression for the amplitude has been reduced to form (I). The calculation is similar for all four terms of the expression, so that only the procedure for one term will be outlined.

For the diagram \( A \rightarrow B \), the corresponding expression in the amplitude expansion is

\[
U_A^* U_B^m \left[ S_{ij} G^{m'n} S_{j}^{n'} \right] U_C \cdot U_D^v.
\]

In addition to equations (II) and (III), only the third functional derivative of the classical action is needed: it has the form

\[
\frac{\delta^3 S_A}{\delta A_{\mu} \delta A_{\rho} \delta A_{\gamma} \delta A_{\tau}} = \frac{\eta^{1/2}}{2} C_{\mu \rho \gamma \tau} \left[ 2 \delta_{\mu \rho} \delta_{\gamma \tau} \right] - 2 \delta_{\mu \rho} \delta_{\gamma \tau} \left[ \delta_{\mu \rho} \delta_{\gamma \tau} \right]
\]

where \( \delta_{\mu \rho} \delta_{\gamma \tau} \) is defined as \( \delta_{\mu \rho} \delta(x, x') \).

In calculating the final form, the helicity terms have dependence only on the momenta and, since the integrals are carried out over space-time, these terms will be neglected in the proceedings.

Thus the second term becomes

\[
e^{-i \int A(x) - i \int A(x')} \left[ S_{\mu \rho} \delta^{\gamma \tau} G^{\mu \rho} S_{\gamma}^{\tau'} \right] \delta(x, x') \]

where the field variable \( \phi^i \) is now \( A_{\mu} \).

Upon substituting for the various quantities and performing the integrations, one reduces the amplitude to

\[
\frac{C_{\mu \rho \gamma \tau}}{(P_a + P_b)^2} \left[ \eta^{\mu \rho} \left( (P_a - P_b) (2 P_c + P_D) - (P_a - P_b) (2 P_D + P_c) \right) + \eta^{\mu \rho} \left( (2 P_a + P_b)^2 (P_a - P_b) - (P_a + 2 P_b)^2 (P_a - P_b) \right) + \eta^{\mu \rho} (P_a + 2 P_b)^2 - \eta^{\mu \rho} (P_a + 2 P_b)^2 \right].
\]
For the first term, the amplitude becomes
\[ C_{\alpha \beta} \delta \left( (2 \eta^\sigma \eta^\mu - \eta^\mu \eta^\sigma)(\eta \eta^\rho) + C_{\alpha \beta} \delta \left( (2 \eta^\sigma \eta^\mu - \eta^\mu \eta^\sigma)(\eta \eta^\rho) \right) \right] (2\pi)^4 \delta(E). \]

The amplitude for the third term is
\[
\frac{C_{\alpha \beta} \delta (2\pi)^4 \delta(E)}{(P_a - P_c)^2} \left[ \eta_{\mu \rho} \left\{ (P_a + P_c)^3 (P_a - 2P_c)^\phi + (P_c + P_a)^{\phi^*} (P_b - 2P_b)^3 \right\} \right.
\]
\[ + \eta_{\mu \rho} \eta_{\sigma \tau} (P_a - P_b)(P_a - P_b) + \eta_{\mu \rho} (P_a - P_b)^3 (P_b - 2P_b)^\phi + \eta_{\mu \rho} (P_b - 2P_b)^3 (P_b - 2P_b) \]
\[ + \eta_{\mu \rho} \left\{ (P_a - 2P_a)^3 (P_b - P_b) + (P_a - 2P_a)^3 (P_b + P_b) \right\} + \eta_{\mu \rho} (P_a - 2P_a)^3 (P_b - 2P_b)^\phi + \eta_{\mu \rho} (P_b - 2P_b)^3 (P_b - 2P_b)^\phi \right].
\]

And for the fourth term, the amplitude is
\[
\frac{C_{\alpha \beta} \delta (2\pi)^4 \delta(E)}{(P_a - P_b)^2} \left[ \eta_{\mu \rho} \left\{ (P_a + P_b)^3 (P_a - 2P_b)^\phi + (P_a + P_b)^{\phi^*} (P_a - 2P_a)^3 \right\} \right.
\]
\[ + \eta_{\mu \rho} \eta_{\sigma \tau} (P_a + P_b)(P_a + P_b) + \eta_{\mu \rho} (P_a - 2P_a)^3 (P_a - 2P_a)^\phi + \eta_{\mu \rho} (P_a - 2P_a)^3 (P_a - 2P_a)^\phi \]
\[ + \eta_{\mu \rho} \left\{ (P_a - 2P_a)^3 (P_a + P_a)^\phi + (P_a - 2P_a)^3 (P_a + P_a)^{\phi^*} \right\} + \eta_{\mu \rho} \]
\[ (P_a - 2P_a)^3 (P_a - 2P_a)^\phi + \eta_{\mu \rho} (P_a - 2P_a)^3 (P_a - 2P_a)^\phi \right].
\]

In these results, \( \delta(E) \) signifies the four-dimensional delta function,
\[ \delta(P_a + P_b - P_c - P_d). \]
GAUGE INVARIANCE

As the scattering amplitudes obtained are somewhat lengthy, it would be advisable to have some way of verifying these results. A convenient means of performing such a check on the scattering amplitudes is effected by using the fact, mentioned in Chapter I, that the action is left essentially unchanged by transformations of the type

\[ \phi^i \rightarrow \phi^i + \int R^i_{\alpha}(x,x') \delta_\alpha \xi^\alpha(x') d^4x' \]

In the quantized theory this invariance leads to the scattering amplitudes being left unchanged when

\[ U^i_A \rightarrow U^i_A + \int R^i_{\alpha}(x,x') \delta_\alpha \xi^\alpha(x') d^4x' \]

since the \( U^i_A \) are merely disturbances in the field \( \phi^i \) and can be varied independently.

From this result it can be shown that, if the amplitudes are correct, they must be observed to vanish when the Yang-Mills wave functions \( U^\alpha_{\mu} \) are replaced by \( \delta \xi^\alpha_{,\mu} \). It is more convenient to work with the Fourier transforms of the above functions, since the calculation involves the exponential space-time dependence of the wave functions. Thus, if in any given amplitude \( U^\alpha_{\mu}(P,x) \) is replaced by

\[ \frac{\partial}{\partial \alpha} \int \delta \xi^\alpha(p) e^{ipx} d^4p = \int i \delta \xi^\alpha(p) P_\mu e^{ipx} d^4p, \]

the result must vanish when integrated over \( p \).

If in the amplitude the wave function for particle \( D, U^\alpha_{\mu}(P,x) \), is replaced by \( \int \delta \xi^\alpha(p) P_\mu e^{ipx} d^4p \), the four elements of the amplitude become

\[ \text{loc. cit., pp. 17-18.} \]
where, because of the delta function, the variable $P$ has become $(P_a + P_b - P_c)$, which has been called $P_0'$. Collecting terms and rearranging yields
\[
\begin{align*}
\eta^{\mu\nu} & \left[ C_{\alpha\beta\varepsilon} \epsilon_{\rho\theta} \left( 2 P^\sigma_0 - 2 P^\sigma_3 \right) + C_{\alpha\beta\varepsilon} \epsilon_{\rho \theta} \left( -P^\sigma_0 + P^\sigma_3 - P^\sigma_4 \right) \\
& + C_{\alpha\beta\kappa} \epsilon_{\rho \varepsilon} \left( -2 P^\sigma_0 \right) \right] + \eta^{\mu\rho} \left[ C_{\alpha\beta\varepsilon} \epsilon_{\theta \rho} \left( -P^\sigma_0 + P^\sigma_3 + P^\sigma_3 \right) + C_{\alpha\beta\varepsilon} \epsilon_{\theta \varepsilon} \left( 2 P^\sigma_0 - 2 P^\sigma_3 \right) + C_{\alpha\beta\kappa} \epsilon_{\theta \varepsilon} \left( -2 P^\sigma_3 \right) \right] + \eta^{\rho \sigma} \left[ C_{\alpha\beta\varepsilon} \epsilon_{\theta \sigma} \left( -P^\mu_0 - P^\mu_3 \right) + C_{\alpha\beta\varepsilon} \epsilon_{\theta \sigma} \left( -P^\mu_0 + 2 P^\mu_3 \right) + C_{\alpha\beta\kappa} \epsilon_{\theta \sigma} \left( P^\mu_0 + P^\mu_3 \right) \right].
\end{align*}
\]

Since the helicity vector of a particle is orthogonal to the particle's momentum, \( P^\mu_0, P^\mu_3 \), and \( P^\sigma \) vanish when multiplied by \( \epsilon_{\tau \mu} \left( P^\rho_0 \right) \), \( \epsilon_{\tau \rho} \left( P^\rho_3 \right) \) and \( \epsilon_{\tau \sigma} \left( P^\rho_3 \right) \), respectively; using these facts in the above expression one obtains

\[
\begin{align*}
\eta^{\mu\nu} & \left[ C_{\alpha\beta\varepsilon} \epsilon_{\rho\theta} \left( 2 P^\sigma_0 \right) + C_{\alpha\beta\varepsilon} \epsilon_{\rho \theta} \left( -2 P^\sigma_0 \right) + C_{\alpha\beta\kappa} \epsilon_{\rho \varepsilon} \left( -2 P^\sigma_0 \right) \right] + \eta^{\mu\rho} \left[ C_{\alpha\beta\varepsilon} \epsilon_{\theta \rho} \left( 2 P^\sigma_3 \right) + C_{\alpha\beta\varepsilon} \epsilon_{\theta \varepsilon} \left( -2 P^\sigma_3 \right) + C_{\alpha\beta\kappa} \epsilon_{\theta \varepsilon} \left( -2 P^\sigma_3 \right) \right] + \eta^{\rho \sigma} \left[ C_{\alpha\beta\varepsilon} \epsilon_{\theta \sigma} \left( -P^\mu_0 - P^\mu_3 \right) + C_{\alpha\beta\varepsilon} \epsilon_{\theta \sigma} \left( P^\mu_0 + P^\mu_3 \right) + C_{\alpha\beta\kappa} \epsilon_{\theta \sigma} \left( P^\mu_0 + P^\mu_3 \right) \right].
\end{align*}
\]

Use must now be made of the cyclic identity which is obeyed by any set of structure constants: \( C^a_{2a} C^a_{34} + C^a_{4a} C^a_{23} + C^a_{a3} C^a_{42} = 0 \). Also note that \( C^x_{2x} C^x_{34} = -C^x_{2x} C^x_{43} \).

When these properties are utilized, the amplitude reduces to

\[
\begin{align*}
\eta^{\mu
u} & \left( -2 P^\sigma_0 \right) \left[ C_{\alpha\beta\varepsilon} \epsilon_{\rho\theta} \left( P^\sigma_0 \right) + C_{\alpha\beta\varepsilon} \epsilon_{\theta\rho} \left( P^\sigma_0 \right) + C_{\alpha\beta\kappa} \epsilon_{\rho \varepsilon} \left( P^\sigma_0 \right) \right] + \eta^{\mu\rho} \left( -2 P^\sigma_3 \right) \left[ C_{\alpha\beta\varepsilon} \epsilon_{\theta \rho} \left( P^\sigma_3 \right) + C_{\alpha\beta\varepsilon} \epsilon_{\theta \rho} \left( P^\sigma_3 \right) + C_{\alpha\beta\kappa} \epsilon_{\theta \varepsilon} \left( P^\sigma_3 \right) \right] + \eta^{\sigma \rho} \left( P^\mu_0 + P^\mu_3 \right) \left[ C_{\alpha\beta\varepsilon} \epsilon_{\theta \sigma} \left( P^\mu_0 \right) + C_{\alpha\beta\kappa} \epsilon_{\theta \sigma} \left( P^\mu_0 \right) \right] = 0.
\end{align*}
\]
SPECIALIZATION TO CENTER-OF-MOMENTUM COORDINATES

The amplitudes calculated in Chapter II are expressed for an arbitrary reference frame. It is convenient to specialize these amplitudes to the center-of-momentum system, defined by \( \mathbf{R} + \mathbf{B} = \mathbf{0} \). Every amplitude contains the four-dimensional delta function, \( \delta(\mathbf{R} + \mathbf{R}_0 - \mathbf{P} - \mathbf{P}_0) \), which ensures the conservation of four-momentum. This in turn can be rewritten as \( \delta^3(\mathbf{R} + \mathbf{R}_0 - \mathbf{P} - \mathbf{P}_0) \delta(\mathbf{P}_0^0 + \mathbf{P}_0^0 - \mathbf{P}_0^0 - \mathbf{P}_0^0) \), where \( \mathbf{R} \) refers to the vector momentum. Now the properties of the delta function state that \( \mathbf{R} + \mathbf{R}_0 - \mathbf{P} - \mathbf{P}_0 = \mathbf{0} \) exactly, which, by the choice of the reference system, gives \( \mathbf{R} + \mathbf{B} = \mathbf{0} \). In terms of magnitudes this expression becomes \( |\mathbf{R}_0| = |\mathbf{P}_0| \).

If the relativistic expression for the energy of a particle, 
\[ E^2 = (mc^2)^2 + |\mathbf{P}|^2 c^2, \]
is now used, with \( C \) as usual set numerically equal to one, the result is that \( E^2 - |\mathbf{P}|^2 = m^2 \). Furthermore, all Yang-Mills quanta are massless, which leads to the fact that \( E^2 = |\mathbf{P}|^2 \), i.e., that \( \mathbf{P}_0^2 = |\mathbf{R}_0|^2 \), \( \mathbf{P}_0^2 = |\mathbf{P}_0|^2 \), and since \( |\mathbf{R}^0| = |\mathbf{P}_0|^2 \), \( \mathbf{P}_0 = \mathbf{P}_0^0 \). Similarly, \( \mathbf{P}_o = \mathbf{P}_o \), and now, using \( \delta(\mathbf{P}_o^0 + \mathbf{P}_o^0 - \mathbf{P}_0^0 - \mathbf{P}_0^0) \), i.e., that \( \mathbf{P}_o^0 + \mathbf{R}_0^0 = \mathbf{P}_o^0 + \mathbf{P}_o^0 \), one finds that \( \mathbf{R}_0^0 = \mathbf{P}_o^0 \).

This can all be summarized in the statement that the interaction can be described in terms of two three-vectors \( \mathbf{R} = \mathbf{P} \) and \( \mathbf{R} = \mathbf{P}' \), where \( |\mathbf{P}| = |\mathbf{P}'| = E \). Therefore a process might be diagrammed as follows:
In this system the four-vector inner products of momenta become $P_a \cdot P_b = P_a \cdot P_b = -2E^2$; $P_a \cdot P_c = P_b \cdot P_c = -E^2(1 - \cos \theta)$; and $P_a \cdot P_d = P_b \cdot P_c = -E^2(1 + \cos \theta)$.

The helicity four-vectors $\epsilon_x$ will form inner products with each other, and with the momenta appearing in the amplitudes, because of the components of the Minkowski tensor. In general, with $\theta$ being the angle between vectors $P$ and $P'$, the products are those listed below with $\xi$ equal to $-2\omega t$, $\omega$ being the common rotational frequency.\(^{11}\)

\[
\frac{\epsilon_x(P) \cdot \epsilon_x(P')}{2} = \begin{pmatrix} \epsilon_i^i \xi \cos \theta - 1 \\ \epsilon_i^i \xi \cos \theta + 1 \\ \epsilon_i^i \xi \cos \theta + 1 \\ \epsilon_i^i \xi \cos \theta - 1 \\ \epsilon_i^i \xi \cos \theta - 1 \\ \epsilon_i^i \xi \cos \theta + 1 \end{pmatrix}
\]

\[
\frac{\epsilon_x(P') \cdot P'}{\sqrt{2}} = \begin{pmatrix} \epsilon^i \xi E \sin \theta \\ \epsilon^i \xi E \sin \theta \\ \epsilon^i \xi E \sin \theta \\ \epsilon^i \xi E \sin \theta \\ \epsilon^i \xi E \sin \theta \\ \epsilon^i \xi E \sin \theta \end{pmatrix}
\]

Special cases of these results are also needed in the computations, such as $\epsilon_+(P) \cdot \epsilon_+(-P)$ and $\epsilon_-(P) \cdot \epsilon_-(P)$. These expressions

\(^{11}\) loc. cit., p. 19.
can be obtained from the above by setting $\theta = \pi$ in the appropriate case; thus they are $(-e^{i\xi})$ and $(-e^{-i\xi})$, respectively. Substitution of these results will lead to the center-of-momentum amplitudes.

In their present form the scattering amplitudes are not easily interpreted. They should be easier to interpret if they are converted to some particular and convenient reference frame. Such a coordinate system is that of the center-of-momentum. As is the case in calculating the original amplitudes, the procedure is similar for all components. Therefore only one specialization to the center-of-momentum frame will be outlined. As stated in Chapter II, the handling of the sixteen arrangements is simplified by calculating eight cases and taking their complex conjugates. These eight cases can be handled by considering two matrices in which are included all the helicity possibilities.

For the configuration $A \times B$ having the helicities $A_-, B_-, C_\pm, D_\pm$, the amplitude becomes

$$\frac{i \delta(E)}{16 \pi^2 p^2} \epsilon_{\mu} \epsilon_{+\nu} \left( \begin{array}{cc} \epsilon_{+\sigma} & \epsilon_{+\phi} \\ \epsilon_{-\sigma} & \epsilon_{-\phi} \end{array} \right) \left[ C_{\nu \rho \sigma} C_{\sigma \phi} \left( 2 \eta^{\sigma \phi} \eta^{\mu \nu} - \eta^{\nu \sigma} \eta^{\phi \mu} \right) \right] .$$

Performing the inner products indicated by the Minkowski tensors, one reduces the above to

$$\frac{i \delta(E)}{16 \pi^2 p^2} C_{\nu \rho \sigma} C_{\sigma \phi} \left\{ \begin{array}{c} 2 (\epsilon_{+} \cdot \epsilon_{-} ) \\ (\epsilon_{+} \cdot \epsilon_{-} ) \end{array} \right\} \left( \begin{array}{cc} (\epsilon_{+} \cdot \epsilon_{+} ) & (\epsilon_{-} \cdot \epsilon_{-} ) \\ (\epsilon_{-} \cdot \epsilon_{-} ) & (\epsilon_{+} \cdot \epsilon_{+} ) \end{array} \right) .$$
After substituting for the various products from the table on page 11, one obtains the amplitude for this configuration in the form

\[
\left( \epsilon' \cdot \epsilon \right) \left( \epsilon \cdot \epsilon' \right) - \left( \epsilon' \cdot \epsilon \right) \left( \epsilon \cdot \epsilon' \right) + C_{\alpha \beta} C^\epsilon_{\alpha \beta} \left\{ 2 \left( \left( \epsilon' \cdot \epsilon \right) \left( \epsilon' \cdot \epsilon' \right) - \left( \epsilon' \cdot \epsilon \right) \left( \epsilon' \cdot \epsilon' \right) \right) - \left( \epsilon' \cdot \epsilon \right) \left( \epsilon' \cdot \epsilon' \right) - \left( \epsilon' \cdot \epsilon \right) \left( \epsilon' \cdot \epsilon' \right) \right\}.
\]

After substituting for the various products from the table on page 11, one obtains the amplitude for this configuration in the form

\[
\frac{i \delta(E)}{16\pi^2 P^2} \begin{pmatrix}
C_{\alpha \beta} C^\epsilon_{\alpha \beta} & \frac{e^{2i\delta}}{2} (3 - \cos^2 \theta) & \frac{e^{i\delta}}{2} \sin^2 \theta \\
\frac{e^{i\delta}}{2} \sin^2 \theta & \frac{1}{2} (3 - \cos^2 \theta) & \\
\end{pmatrix}
\]

\[
+ C_{\alpha \beta} C^\epsilon_{\alpha \beta} \begin{pmatrix}
\frac{e^{2i\delta}}{4} (\cos^2 \theta - 6 \cos \theta - 3) & -\frac{e^{i\delta}}{4} \sin^2 \theta \\
-\frac{e^{i\delta}}{4} \sin^2 \theta & \frac{1}{4} (\cos^2 \theta + 6 \cos \theta - 3) \\
\end{pmatrix}
\]
The amplitudes for the other possible choices are

\[-\times^+ : \frac{i \delta(E)}{16\pi^2 P^2} \begin{bmatrix} C_{\alpha\beta} C_{\alpha\beta}^e \left( \frac{e^{i\delta}}{2} \sin^2 \theta \right) & - \frac{(1+\cos \theta)^2}{2} \\ \frac{1}{2} \sin^2 \theta & \frac{e^{i\delta}}{2} \sin^2 \theta \end{bmatrix} \]

\[+ C_{\alpha\beta} C_{\alpha\beta} e^{i\delta} \begin{bmatrix} -\frac{e^{i\delta}}{4} \sin^2 \theta & \frac{(\cos \theta + 1)^2}{4} \\ \frac{(\cos \theta - 1)^2}{4} & -\frac{e^{i\delta}}{4} \sin^2 \theta \end{bmatrix} \]

\[-\times^- : \frac{i \delta(E)}{16\pi^2 P^2} \begin{bmatrix} C_{\alpha\beta} C_{\alpha\beta}^e \left( e^{i\delta} \cos \theta \right) & 0 \\ 0 & \cos \theta \end{bmatrix} \]

\[-\times^+ : \frac{i \delta(E)}{16\pi^2 P^2} \begin{bmatrix} C_{\alpha\beta} C_{\alpha\beta}^e \left( \frac{e^{i\delta}}{2} \cos \theta \right) & 0 \\ 0 & 0 \end{bmatrix} \]

\[-\times^- : \frac{i \delta(E)}{16\pi^2 P^2} \begin{bmatrix} C_{\alpha\beta} C_{\alpha\beta}^e \left( \frac{e^{i\delta}}{2} \sin^2 \theta \right) (\cos \theta - 1) \\ -\frac{e^{i\delta}}{2} \left[ (1-\cos \theta)^2 \left( \frac{3}{2} + \frac{\cos \theta}{2} \right) \right] \end{bmatrix} \]

\[-\frac{e^{i\delta}}{2} \sin^2 \theta (\cos \theta - 1) \]

\[-\left[ (1+\cos \theta)^2 \left( \frac{3}{2} + \frac{\cos \theta}{2} \right) + 8 \sin^2 \theta \right] \]
If one combines these results into a general interaction term, the two possibilities are found to be
It will be noted that the sum of the helicities is conserved in the total interaction.

These last two results give the total amplitudes for the self-scattering of Yang-Mills quanta by means of virtual quanta of the same type. They are identical to the results obtained by Edward A. Remler, who treated this problem by use of an effective Hamiltonian in the interaction representation. Therefore, the underlying formalism for the calculations of this paper is, in this instance, seen to be compatible with the Hamiltonian approach, which is perhaps more familiar.

\[ \frac{G(E)}{16\pi^2p^2} \left( \frac{1+\cos\theta}{\sin\theta} \right)^2 \left[ C_{\mu\nu}^e C_{\alpha\beta}^e \cos\theta + C_{\alpha\beta}^e C_{\mu\nu}^e - C_{\mu\nu}^e C_{\alpha\beta}^e \right] \]

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\(^{12}\text{Remler, Edward A., Cross Sections of Yang-Mills Quanta. Institute of Field Physics, Department of Physics, University of North Carolina, Chapel Hill, North Carolina, (1963), 8-9.}\)
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