The Physics of Electron Degenerate Matter in White Dwarf Stars

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THE PHYSICS OF ELECTRON DEGENERATE MATTER
IN WHITE DWARF STARS

by

Subramanian Vilayur Ganapathy

A Thesis
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Master of Arts
Department of Physics

Western Michigan University
Kalamazoo, Michigan
June 2008
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I wish to express my infinite gratitude and sincere appreciation to my advisor, Professor Kirk Korista, for suggesting the topic of the thesis, and for all of his direction, encouragement and great help throughout the course of this research. I would also like to express my gratitude to the other members of my committee, Professor Dean Halderson and Professor Clement Burns for their valuable advice and help.

Subramanian Vilayur Ganapathy
White dwarfs are the remnant cores of medium and low mass stars with initial mass less than 8 times the mass of our sun. As the aging giant star expels its surface layers as planetary nebulae, the core is exposed as a white dwarf progenitor. The density of matter in white dwarfs is so high that thermal or radiation pressure no longer supports the star against the relentless pull of gravity. The white dwarf is supported by a new kind of pressure known as the degeneracy pressure, which is forced on the electrons by the laws of quantum mechanics. The matter in the white dwarf can be explained by using the Fermi gas distribution function for degenerate electrons. Using this we have found the pressure due to electron degeneracy in the non-relativistic, relativistic and ultra-relativistic regimes. Polytropic equations of state were used to calculate the mass-radius relation for white dwarfs and also to find their limiting mass, which is known as the Chandrasekhar limit.
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CHAPTER I

OVERVIEW OF STARS AND THEIR LIFE CYCLE

Introduction

It would be very difficult to find a more beautiful sight than looking at the stars on a clear night. However the subject of this work is not about the stars in the prime of their life but what happens when they die. Depending on their initial mass, stars reach their end stages of evolution in three different ways. One of the ways is to end up as a white dwarf, which is the topic of this work. We start off by explaining how a star forms upon the collapse of an interstellar cloud, its life on the main sequence, the post main sequence where a star having the mass of the sun collapses to form a white dwarf. A pressure called the electron degeneracy pressure, which is the predominant pressure at these densities, supports the white dwarf. We calculate the Fermi energy and Fermi momentum of the electron gas and estimate the pressure due to electron degeneracy for the relativistic and non-relativistic case. Using the method of polytropes and an equation known as the hybrid Equation (Equation 101) we study the mass radius relationship for white dwarfs and compare it to a representative sample of existing white dwarfs (Figure 8). We also calculate the Chandrasekhar limit, which gives the upper limit for the mass of an ideal white dwarf. The physics of white dwarfs is studied for a range of densities from $10^3$ g cm$^{-3}$ to $10^9$ g cm$^{-3}$.
Stellar Evolution

Stars are formed when an interstellar cloud collapses under its own gravity. As the molecular cloud collapses its density increases by many orders of magnitude. This releases gravitational potential energy, which is radiated away from the cloud. Still the cloud is not dense enough to be opaque to its own radiation and hence gravitational potential energy is effectively released into space. The cloud is said to be in free fall and the temperature remains approximately constant throughout; in other words the process is isothermal. Due to inhomogeneities in the density of the molecular cloud, segments of the cloud begin to collapse locally forming fragments and this process is known as fragmentation. Obviously the collapsing cannot go on forever or we would not have any star formation. Something changes after a period of free fall to stop the collapse of the cloud fragment. After a point of time the process loses its isothermal nature and the temperature starts to change. This is because the collapsing cloud has gained sufficient density such that it begins to become opaque to radiation. The gravitational potential energy is not radiated away but is trapped inside the cloud. This means that the collapse begins to slow down because an increase in temperature leads to a pressure gradient and this pressure counteracts the gravitational pull more effectively. At a certain stage the core of the cloud, which is at a slightly higher density than the surrounding gas is nearly in hydrostatic equilibrium and the rate of collapse slows down. However material from the periphery of the cloud is still falling in on the hydrostatic core causing an increase in temperature. The temperature reaches a stage where it is high enough to cause molecular hydrogen to disassociate
into individual atoms. This process absorbs some energy as a result of which the pressure gradient decreases and the core becomes unstable and begins to collapse for the second time and finally settles down in a newly established hydrostatic equilibrium. The rate of evolution of the protostar thus formed is governed by the rate at which the star can thermally adjust to collapse. The temperature of the star increases due to its contraction and the central temperature become high enough to initiate energy production via the pp chain (converting 4 hydrogen nuclei into helium nucleus). This makes the contribution towards energy from gravitational term insignificant and the star enters the main sequence stage where it will stay for most of its lifetime as a result of which we are more likely to find a star in the main sequence stage. In the main sequence the star converts four hydrogen nuclei into a helium nucleus via the pp chain (for low mass stars like our Sun) as a result of which the mean molecular weight of the core increases slowly over time. The density and temperature of the core must, therefore, also increase to provide sufficient gas pressure to support the outer lying layers of the star. The higher mass stars are short lived compared to the low mass stars because they convert hydrogen into helium (via the carbon-nitrogen-oxygen cycle) faster because of the higher temperatures required to generate the pressure needed to support the massive stars against gravity.

Estimation of Required Central Pressure

The physics of stellar structure is very well known and is governed by four basic laws namely, hydrostatic equilibrium, mass conservation, energy transport and energy conservation. The law of hydrostatic equilibrium states that the pressure
decreases from the inner central region to the outer regions of the star and in doing so offsets the weight of the star above each layer. The negative sign on the equation below shows that the pressure gradient is negative which means that the pressure is a maximum at the center of the star and is known as the central pressure \((P_c)\),

\[
\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}
\]

(1)

The equation for mass conservation is

\[
\frac{dM}{dr} = 4\pi r^2 \rho(r).
\]

(2)

Substituting \(\rho(r)dr\) from Equation 2 into Equation 1 we have,

\[
dP(r) = -\left(\frac{G}{4\pi}\right) \frac{M(r)}{r^4} dM(r).
\]

(3)

Integrating the above equation on both sides,

\[
\int_0^r dP(r) = -\left(\frac{G}{4\pi}\right) \int_0^r \frac{M(r)}{r^4} dM(r).
\]

(4)

Let us introduce dimensionless variables

\(x = r/R\) and \(m(x) = M(x)/M\), hence \(Mdmd(x) = dM(r)\),

where \(R\) is the radius of the star and \(M\) is its mass. As \(r\) tends to \(R\), then \(x\) and \(m(x)\) both tend to 1. To isolate the central pressure, the limits of integration in \(r\) and \(x\) should then run from 0 to \(R\) and 0 to 1, respectively. Equation 3 then becomes

\[
P(R) - P(0) = -\frac{G}{4\pi} \frac{M^2}{R^4} \int_0^1 \frac{m(x)}{x^4} dm(x),
\]

(5)
where $P(R)$ is vanishingly small in comparison to the central pressure $P(0)$. Designating $P(0)$ as $P_c$, the central pressure, we obtain

$$P_c = \frac{G M^2}{4\pi R^4} \int_0^1 m(x) \frac{dm(x)}{x^4}.$$  \hfill (6)

Designating the above integral divided by $4\pi$ to be a numerical constant $\alpha$,

$$\alpha = \frac{1}{4\pi} \int_0^1 \frac{m(x)}{x^4} \, dm(x),$$  \hfill (7)

then Equation 6 reduces to,

$$P_c = \alpha \frac{GM^2}{R^4}.$$  \hfill (8)

As a special case let us now determine the required central pressure $P_c$ for

$$\rho(r) = \mathcal{<\rho>} = M/(4\pi R^3/3),$$  the simple case of constant density. At constant density

$$m(x) = x^3$$ which means,

$$m(x) = M \left( \frac{r}{R} \right)^3 \Rightarrow dm(x) = 3x^2 \, dx.$$  \hfill (9)

Inserting the above values in Equation 7, we find

$$\alpha = \frac{1}{4\pi} \int_0^1 \frac{x^3}{x^4} 3x^2 \, dx = \frac{3}{8\pi}.$$  \hfill (10)

Substituting the value of $\alpha$ we just found in Equation 8, we obtain

$$P_c = \frac{3}{8\pi} \frac{GM^2}{R^4}.$$  \hfill (11)

This is the pressure required at the center of a (unphysical) constant density star.
Using Equation 8 the pressure required by hydrostatic equilibrium at the center of a star is

\[ P_c = 1.125 \times 10^{16} \left[ \alpha \left( \frac{M/M_0}{R/R_0} \right)^2 \right] \text{dyne cm}^{-2}, \quad (12) \]

where \( M_0 \) and \( R_0 \) are the mass and radius of the Sun and \( \alpha \) can be determined for a realistic density distribution using Equation 7. Note dyne cm\(^{-2}\) is equivalent to ergs cm\(^{-3}\) and from here onwards we will use these latter units for pressure. In normal stars like our sun this pressure is supplied by the thermal energy of the particles constituting the matter, as well as smaller contributions from radiation pressure.

Evolution of a Sun-like Star

The evolution of a Sun-like star is shown in Figure 3. A star spends most of its lifetime on the main sequence where it supports itself from gravity by fusing hydrogen into helium in its core. Once all the hydrogen in the star’s core is converted to helium, fusion stops in the core and the core can no longer support the overlying layers of the star. As a result the star’s core compresses increasing the temperature in the core. This increase in temperature ignites nuclear fusion in a surrounding thick shell of hydrogen. This is called as the hydrogen shell burning stage. The temperature and density of the hydrogen burning shell increases and the rate at which energy is generated by the shell also increases rapidly forcing the envelope of the star to expand. At the same time the core continues to contract and the star enters the red giant phase of evolution. The contraction of the helium core results in a temperature high enough for helium fusion \((T > 10^8 \text{ K}, \rho = 10^4 \text{ g cm}^{-3})\) resulting in the production
of carbon via the triple alpha process and some oxygen via the capture of another alpha particle (helium nucleus). As the intermediate mass star, i.e. stars with mass less than eight solar masses, continues to evolve the hydrogen burning shell converts more and more helium into carbon and then oxygen, forming a carbon-oxygen core. Eventually, the star has a non-burning carbon-oxygen core surrounded by a helium burning shell, which in turn is surrounded by a hydrogen burning shell. As the helium in the core becomes completely exhausted the carbon-oxygen core begins to shrink, causing an increase in the burning rates of the hydrogen and helium shells. The star's envelope (non-fusing outer layers) expands and the star again becomes a red giant. In this phase the inner core of the star continues shrinking and heating up while the outer envelope continues to expand and cool. Eventually the envelope becomes unstable and is ejected into space forming a cooling shell of matter. The expanding shell of gas around the newly appearing white dwarf progenitor absorbs ultraviolet radiation from the newly formed hot central star causing the atoms to become ionized. When the electrons in excited states of the ionized gas return to lower energy levels, they emit photons in the visible region of the electromagnetic spectrum. This phase is called as “planetary nebula”. The carbon-oxygen core, initially at a temperature in excess of $10^8$ K [9], with a thin layer of hydrogen and helium gas that is now devoid of a surrounding envelope is hot, with initial surface temperatures of 100,000 K to 200,000 K[1], and is known as a white dwarf (see Figure 3). Further shrinking of the white dwarf is prevented by a new kind of pressure which is due to the degenerate electrons whose pressure is independent of temperature. The white dwarf cools down
at a nearly constant radius, as light and, during early phases, when the interior
temperature is still high neutrinos [9]) are radiated away. This can be seen in Figure 3
where the luminosity $L$ decreases at a rate proportional to the cooling white dwarf
surface temperature: $L \propto T_{\text{eff}}^4$. A white dwarf cooling towards absolute zero is the fate
of a solar-type star, provided it does not have a close binary companion.

White dwarfs in close binary systems can steadily accrete material from a
companion star thereby increasing its mass. When the mass of a carbon-oxygen white
dwarf nears the Chandrasekhar limiting mass carbon burning begins in the center. The
initiation of fusion increases the temperature of the star’s interior without an increase
in the pressure, which is dominated by the degenerate electrons. Hence the white
dwarf does not expand or cool. The increased temperature increases the rate of fusion
and hence leads to runaway thermonuclear explosion called a Type 1A supernova.

Final Stages of Evolution in Massive Stars

The post main sequence stages of stellar evolution are a set of stages that end
in the death of the star and the end fate of the star depends on the star’s initial mass.
Depending on whether the initial mass of the star is less than eight solar masses
(intermediate mass stars) or greater than ten solar masses (massive stars) they reach
their end by different means. The previous section discusses the final stages of
evolution of intermediate mass stars while this section is devoted to more massive
stars whose centers have iron cores which are supported against gravitational collapse
until a certain point by electron degeneracy pressure.
Stars with initial masses greater than ten times the mass of the sun will reach the end of their life in a spectacular astronomical event called a supernova (Type II), which is the result of the collapse of a massive star’s iron core. During the later stages of stellar evolution the helium burning shell of a massive star continues to add mass to the carbon-oxygen core, as a result of which the core contracts and the temperature becomes high enough to initiate carbon burning and the process goes on producing heavier and heavier elements until it ends up with an iron core in its center. Iron has the largest binding energy per nucleon, thus no more energy can be obtained by fusing iron. The growing iron core is initially supported by electron degeneracy pressure. However, as the mass of this iron core approaches the critical Chandrasekhar mass limit, several things occur which result in the core’s collapse, as gravity overwhelms the available pressure (largely dominated by the degenerate electrons). We will briefly explain how this happens. At the very high temperatures \( T \approx 8 \times 10^9 \text{ K} \) now present in the iron core, some of the photons possess enough energy to strip the iron nuclei into individual protons and neutrons in a process known as photodisintegration. Under the really high densities \( \rho_c \approx 10^{10} \text{ g cm}^{-3} \) for a 15 solar mass star[1] that is now present in the core it becomes energetically favorable for the free electrons to be captured by the heavy nuclei or protons that were formed through photodisintegration. Due to the photodisintegration of iron, combined with electron capture, most of the pressure the core had in the form of electron degeneracy pressure is gone and the core collapses catastrophically. The collapse of the inner core continues to densities approaching that in an atomic nucleus. At these enormous densities the neutrons are squeezed into a
smaller and smaller region and they start repelling each other in accordance with Pauli’s exclusion principle, and neutron degeneracy pressure halts the collapse. The net result is that the inner core recoils producing shock waves. If the initial mass of the star is not too large the remnant in the inner core will stabilize and become a neutron star (with a radius of approximately 10 km), supported by degenerate neutron pressure. However if the initial mass is much larger even the pressure due to neutron degeneracy cannot support the remnant against gravity and the final collapse will be complete, producing a black hole. Meanwhile the shock waves cause the overlying matter to be ejected in an explosion called a Type II supernova. A tremendous amount of energy is released into space during this time and the envelope is ejected at thousands of kilometers per second. The tremendous amount of energy has its origin from the stored gravitational potential energy, an estimate of which can be obtained from the equation for potential energy difference under the condition that the final radius is much smaller than the initial radius.

\[ E_{\text{gravity}} = \frac{GM^2}{R} = 2.64 \times 10^{53} \left( \frac{10 \text{km}}{R} \right) \left( \frac{M}{M_0} \right)^2 \text{ergs}. \]  

(13)

Most of this energy is carried away by neutrinos \((\approx 10^{53} \text{ergs})\). The total kinetic energy in the expanding material is of the order of \(10^{51} \text{ergs}\) which is about one percent of the energy carried away by neutrinos. Finally, when the material becomes optically thin at a radius of \(10^{15} \text{cm}\) a tremendous optical display result which releases approximately \(10^{49} \text{ergs}\) in the form of photons, the peak luminosity output of which rivals that of an entire galaxy. The development of this whole chapter is based on [1].
CHAPTER II

ELECTRON DEGENERACY PRESSURE

Origin of Electron Degeneracy Pressure

In the previous chapter we mentioned that a Sun like star would reach the end point of its life as a white dwarf which is supported by electron degeneracy pressure. In this chapter we take a look at the origin of the degeneracy pressure and justify our assumption that the degeneracy pressure is the dominant form of support which holds a white dwarf from gravitational collapse. When Sirius B was first discovered its physical parameters were astounding. It had about the mass of the Sun confined in a volume similar to the earth. This means that the density of matter in Sirius B was much greater than ever encountered before. Obviously Sirius B is not a normal star. As we will see thermal and radiation pressure that supports a normal star from gravity is no longer sufficient to counteract the enormous inward pull of gravity caused by the enormous densities present in the white dwarfs. White dwarfs are supported from collapse by a pressure arising from electron degeneracy.

Electron degeneracy pressure is forced on the electrons by the laws of quantum mechanics. Electrons belong to a class of particles known as fermions. They obey the Pauli’s exclusion principle, which states, “No two electrons can occupy the same quantum state”. The degeneracy pressure arises because only one electron can
occupy a single quantum state and hence as the temperature starts falling the electrons start occupying the lower energy levels. At temperature $T = 0$ K all the lower energy levels up to a particular level are completely filled and the higher energy levels are completely empty. Such a fermion gas is said to be completely degenerate. The pressure due to electron degeneracy can be understood in terms of wave/particle duality of electrons. Since matter is so much denser in the interior of white dwarfs the volume available for an electron becomes that much smaller. Now if we think of the electron as a wave, the reduction in volume of the space surrounding the electrons means that the wavelength of the electron becomes smaller to confine it to the smaller volume, making it more energetic. It flies about at greater speeds in its cell and by bumping with other particles gives rise to the degeneracy pressure. This pressure is an unavoidable consequence of the laws of quantum mechanics. The degeneracy pressure can also be explained from Heisenberg’s uncertainty principle, which can be written in the form of an equation as

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$  \hspace{1cm} (14)

Let us now rewrite the uncertainty principle in a form which will help us better understand the origin of degeneracy pressure. Considering $\Delta x \Delta p \approx \hbar = (\hbar/2\pi)$ we infer that the minimum value for the electron momentum is $\Delta p$. Hence as the value of $\Delta x$ becomes smaller, in other words we are confining the electron to a smaller and smaller volume, the momentum of the electron correspondingly increases and this contributes to the pressure.
Calculation of Fermi Energy and Fermi Momentum

In this section we derive the Fermi momentum for electrons starting with the density integral. We then obtain the Fermi energies for electrons traveling at non-relativistic and relativistic speeds by substituting the Fermi momentum in the energy equation. We also compute the numerical values for the Fermi energy in a typical white dwarf and compare it with the energy due to thermal motions, electron-electron coulomb interaction and the electron-ion coulomb interaction.

For an ideal fully degenerate electron gas \((T = 0 \text{ K})\) all the energy levels below a particular energy level known as the Fermi energy level are completely filled and all the energy levels above the Fermi energy level are completely empty. The momentum associated with the Fermi energy is known as the Fermi momentum and it can be calculated from the density integral. In a white dwarf the temperature is never zero and hence the electron gas is never completely degenerate. There will be some electrons with enough energy to stay above the Fermi level as a result of which thermal or other effects might become important. However, the assumption of complete degeneracy is an excellent approximation in white dwarfs and will be justified at the end of this section.

Calculation of Fermi Momentum

The number density of electrons is

\[
  n_e = \int_0^\infty n(p)dp;
\]

where

\[(15)\]
\[ n(p)dp = g_s \frac{4\pi p^2}{h^3} \frac{1}{e^{(E-\mu)/kT} + 1} \] \hspace{1cm} (16)

is the Fermi-Dirac distribution function for fermions, where \( p \) is the momentum of the electrons, \( E \) is the kinetic energy of the electrons, \( \mu \) is the chemical potential and the quantity in square brackets is the occupation number. Electrons have spin \( s = 1/2 \), and hence the statistical weight for electrons, \( g_s = 2s + 1 = 2 \).

For a fully degenerate gas occupation number is 1 since all the energy levels up to the Fermi energy level are completely filled and hence we obtain,

\[ n(p)dp = \frac{8\pi p^2}{h^3} dp \] \hspace{1cm} (17)

Under the assumption that the electron gas is fully degenerate there are no electrons above the energy level corresponding to the Fermi momentum. So we can change the limits of integration in Equation 15 from 0 to \( \infty \) to 0 to \( p_f \).

\[ n_e = \int_0^{p_f} n(p)dp = \int_0^{p_f} \frac{8\pi p^2}{h^3} dp = \frac{8\pi p_f^3}{3h^3} \] \hspace{1cm} (18)

Rearranging the above equation we can obtain the Fermi momentum in terms of the number density of particles,

\[ p_f = \left( \frac{3h^3 n_e}{8\pi} \right)^{1/3} \] \hspace{1cm} (19)

Calculation of Fermi Energy of Electrons in the Small Momentum Limit

The total energy of an electron is given by
Since the electron is traveling at non-relativistic speeds (the speed of the electron is small compared to the speed of light), we can expand the above equation for small $p$.

$$E = \sqrt{(m_e c^2)^2 + (pc)^2} = m_e c^2 \left(1 + \frac{p^2}{m_e^2 c^2}\right)^{1/2}. \quad (20)$$

$$E = \sqrt{(m_e c^2)^2 + (pc)^2} = m_e c^2 \left(1 + \frac{p^2}{m_e^2 c^2}\right)^{1/2} \approx m_e c^2 \left(1 + \frac{p^2}{2m_e^2 c^2}\right). \quad (21)$$

From the above we obtain the kinetic energy of electrons ($E(KE) = E - m_e c^2$) in the small momentum (i.e., classical) limit,

$$E(KE) = \frac{p^2 m_e c^2}{2m_e c^2} = \frac{p^2}{2m_e}. \quad (22)$$

To find the corresponding kinetic energy at the Fermi momentum we use Equation 19

$$E_{f, nr}(KE) = \frac{\frac{p_f^2}{2m_e}}{2m_e} = \frac{1}{2m_e} \left(\frac{3\hbar^2 n_e}{8\pi}\right)^{2/3}. \quad (23)$$

The above equation gives the Fermi energy of a degenerate electron gas in the non-relativistic limit.

We now introduce a parameter known as the Fermi parameter which compares the electron’s $pc$ with its rest mass energy,

$$x_f = \frac{p_f}{m_e c} = \frac{p_f c}{m_e c^2}, \quad (24)$$

into Equation 22 we obtain the kinetic energy of the electrons in terms of new parameter $x_f$, valid in the (non-relativistic) limit $x_f << 1$,

$$E_{f, nr}(KE) = \frac{\frac{p_f^2}{2m_e}}{2m_e} = \frac{x_f^2 m_e c^2}{2} \Rightarrow \frac{E_{f, nr}(KE)}{m_e c^2} = \frac{x_f^2}{2}. \quad (25)$$
Let us take a moment to derive the relation between the density and \( x_f \). The relation between electron number density and total matter density is given by the following expression

\[
n_e = \frac{\rho}{m_H \mu_e},
\]

where \( m_H \) is the mass of hydrogen and \( \mu_e \) is the mean molecular weight per electron. Substituting for \( n_e \) from Equation 19 into Equation 26 and using the Fermi parameter we obtain,

\[
\rho = \frac{8\pi (m_e c)^3 m_H}{h^3} \mu_e x_f^3 = \rho_0 \mu_e x_f^3,
\]

where \( \rho_0 = \frac{8\pi (m_e c)^3 m_H}{h^3} = 9.815535 \times 10^5 \text{ g cm}^{-3} \).

Substituting the value for \( \rho_0 \) in Equation 27 and after rearranging we obtain,

\[
x_f = \left( \frac{1}{\rho_0} \right)^{1/3} \left( \frac{\rho}{\mu_e} \right)^{1/3} = 1.006226 \times 10^{-2} \left( \frac{\rho}{\mu_e} \right)^{1/3}.
\]

**Calculation of Fermi Energy of Electrons for all Momenta**

As will be shown, for densities greater than \( \rho \approx 10^6 \text{ g cm}^{-3} \) the electrons start traveling at appreciable percentages of the speed of light and the previous equation (Equation 19) for calculating the Fermi energy is not adequate because the momentum of electrons is not small anymore. Now we have entered the realm of relativity and hence to account for relativistic effects we have to use relativistic corrections while calculating the Fermi energy. The Fermi kinetic energy is now given as
\[ E_{f,r}(KE) = \sqrt{\left( m_e c^2 \right)^2 + \left( p_{rf} c \right)^2} - m_e c^2. \] (29)

In the above equation we are subtracting the rest mass energy from the total energy to obtain the kinetic energy of the electron. Substituting Equation 24 in the above we find,

\[ E_{f,r}(KE) = \left[ \left( 1 + x_f^2 \right)^{1/2} - 1 \right] m_e c^2 \implies \frac{E_{f,r}(KE)}{m_e c^2} = \left[ \left( 1 + x_f^2 \right)^{1/2} - 1 \right]. \] (30)

This is the equation for Fermi energies of electrons traveling at all momenta in terms of \( x_f \). It is noteworthy that in the small \( x_f \) limit the above equation becomes the same as the equation for the Fermi energy in the small momentum limit. In that limit we can expand the above expression as a binomial series, which gives,

\[ \frac{E_{f,r}(KE)}{m_e c^2} = \left[ \left( 1 + x_f^2 \right)^{1/2} - 1 \right] \frac{E_{f,r}(KE)}{m_e c^2} \approx \left[ \left( 1 + \frac{x_f^2}{2} - \cdots \right) - 1 \right] \approx \frac{x_f^2}{2}, \] (31)

for \( x_f \ll 1 \), keeping the first two terms only, which thus reduces to the equation for the classical kinetic energy of an electron (Equation 25).

Let us now evaluate the Fermi (kinetic) energy for a typical mass density within white dwarf stars, \( \rho/\mu_e \approx 10^6 \text{ g cm}^{-3} \). From Equation 28 we find that this corresponds to \( x_r \approx 1 \), which by definition (Equation 24) indicates that relativistic effects must be important to the electron kinematics. Substituting \( x_f = 1 \) into Equation 30 we obtain the Fermi energy,

\[ \frac{E_{f,r}(KE)}{m_e c^2} = \left[ \left( 1 + x_f^2 \right)^{1/2} - 1 \right] \approx 0.414 \] (32)

which is a fair fraction of the electron’s rest mass energy (0.511 MeV).
Figure 4 plots the relation between Fermi energies of electrons with \( x_f \). The solid curve represents the Fermi energy obtained using Equation 30, which is valid at all speeds. The dashed line represents the case where we approximate Equation 30 by assuming \( x_f \ll 1 \). Both curves agree very well at low values of \( x_f \) but begin deviating significantly about \( x_f \approx 1 \) corresponding to \( (\rho/\mu_e) = 10^6 \text{g cm}^{-3} \) illustrating that the electrons are becoming relativistic.

Potential Deviations to an Ideal Degenerate Electron Gas Equation of State

In our analysis of white dwarfs we have made the assumption that the electron gas is fully degenerate, but in real situations the gas is never precisely fully degenerate. Moreover, since the gas contains electrons and ions, electrostatic and thermal ion corrections to the pressure equation of state might become important. Hence, we now attempt to justify our assumption of complete electron degeneracy by showing that the above corrections are much smaller than the Fermi energy of the electrons for conditions found within typical white dwarf stars.

First, let us compare the Fermi energy equation with the energy equation due to thermal motions of electrons. The energy per electron due to thermal motions is given by \( E_{th} = (3/2)kT \) where \( k \) is the Boltzmann constant and \( T \) is the temperature in the interior of the white dwarf where the energy due to thermal excitations is at its highest. The ratio between the thermal energy and the Fermi energy is given by,
\[
\frac{E_{th}}{E_f} = \frac{(3/2)kT}{\left[1 + x_f^2\right]^{1/2} - 1}m_e c^2
\]

\[
= 2.53 \times 10^{-3} \left( \frac{1}{\left[1 + 1.0125 \times 10^{-4}(\rho/\mu_e)^{2/3} \right]^{1/2} - 1} \right) \left( \frac{T}{10^7 K} \right),
\]

(33)

with \( T = 10^7 K \) an appropriate temperature of the interior after approximately \( 10^9 \) years of cooling [9]. The above ratio yields \( 6.04 \times 10^{-3} \) for \( (\rho/\mu_e) = 10^6 \text{g cm}^{-3} \) corresponding to \( x_f \approx 1 \) (typical conditions found in a white dwarf), which is a small number. A glance at Table 3 shows that the thermal energy contribution to the total pressure decreases as the density increases and we conclude that the contribution of the pressure due to the thermal energy of the electrons (or ions) in the bulk of the white dwarf is negligible and the total pressure is dominated by the Fermi energy of electrons.

Let us now introduce a parameter known as the Coulomb coupling parameter (\( \Gamma \)) which gives the strength of the Coulomb interaction between ions relative to the thermal kinetic energy of ions, \( kT \)

\[
\Gamma = \left( \frac{Ze}{kT} \right)^2 \left( \frac{4\pi \cdot n_{am}}{3} \right)^{1/3} = \left( \frac{Ze}{kT} \right)^2 \left( \frac{4\pi}{3} \right)^{1/3} \left( \frac{\rho}{\mu_0 m_i} \right)^{1/3}
\]

\[
= 35.68 \left( \frac{Z}{6} \right)^2 \left( \frac{12}{\mu_0} \right)^{1/3} \left( \frac{\rho}{10^6 \text{g cm}^{-3}} \right)^{1/3} \left( \frac{10^7 K}{T} \right),
\]

(34)

where \( \mu_0 \) is the mean mass per ion. For \( \Gamma \) of the order of unity the ions begin experiencing short range correlations and the assumption of non-interacting gas is no longer valid for the ions, but it is only at values of \( \Gamma = 150-200 \) [9] that the ions start arranging themselves into a crystalline lattice.
The degenerate electrons in a white dwarf's interior can travel long distances without losing energy because virtually all the lower electron energy levels are completely filled. Hence the interior is highly conductive and the temperature is nearly isothermal. The surface layers however have a temperature gradient because electrons at the surface are only partially degenerate or even largely non-degenerate. This results in an inefficient transfer of thermal energy via radiation (and sometimes convection) resulting in energy loss at the surface. The surface is therefore much cooler than the interior, as can be seen in the H-R diagram (Figure 3) which shows the surface temperature (1-2×10^5 K initially) of the white dwarf to be much lower than the interior temperature (greater than ~10^8 K, initially). As the white dwarf cools the ions within the interior are initially in an ideal gas equation of state, but as the cooling continues the ions eventually crystallize into a lattice (Equation 34). As the crystallization continues the ions undergo a phase change and release their latent heat thereby increasing the cooling time. Crystallization starts at the center (where the density is highest) and the temperature at which this happens is known as the melting temperature that can be calculated from Equation 34 for \( \Gamma \approx 175 \),

\[
T_m \approx \left( \frac{4\pi}{3m_H} \right)^{1/3} \frac{(Ze)^2}{k} \left( \frac{\rho}{\mu_0} \right)^{1/3} \frac{1}{\Gamma}
\]

\[
= 2.04 \times 10^6 K \left( \frac{Z}{6} \right)^{1/3} \left( \frac{12}{\mu_0} \right)^{1/3} \left( \frac{\rho}{10^6 \text{ g.cm}^{-3}} \right)^{1/3} \frac{1}{(175/\Gamma)}
\]  

(35)

or about 2 million K for typical values. As the temperature drops below this critical value, the ions crystallize and form a body-centered cubic lattice structure (like that of
metallic sodium). However, the Fermi energy of the electrons still dominates the Coulomb energy of the ions as we show below.

The Wigner-Seitz model is employed to calculate the electrostatic energy in which the electron degenerate gas (with $\Gamma > 175$) is imagined to be divided into neutral spheres of radius $r_0$ about each nucleus, enclosing the $Z$ electrons closest to the nucleus. Since the cells are considered as neutral spheres, the interaction between the electrons and nuclei of different cells are ignored. The total electrostatic energy is the sum of energies due to electron-electron interaction and electron-ion interaction. The total Coulomb energy of a cell is given by $-E_c = 0.9(Ze)^2/r_0$ [2]. Using the relation between the volume of the cell and the number density of electrons, which is given by $(4\pi/3)r_0^3/Z = 1/n_e$, we can write the total electrostatic energy per electron as

$$\frac{E_c}{Z} = -\frac{9}{10} \left( \frac{4\pi}{3} \right)^{1/3} Z^{2/3} e^2 n_e^{1/3}. \quad (36)$$

This electrostatic correction arises from the fact that the mean distance between nuclei and electrons is smaller than the mean distance between electrons, which are approximately uniformly distributed. Hence repulsion is weaker than attraction and the energy and pressure of the electrons decreases. The effect of this electrostatic correction to the ideal degenerate electron equation of state is that it reduces the total pressure that would otherwise be available to support the white dwarf from gravitational collapse. The ratio between the Fermi energy and the electrostatic energy per electron is given by
\[
\frac{E_C}{E_f} = \frac{-9}{10} \left( \frac{4\pi}{3} \right)^{1/3} Z^2 e^2 \left[ \left( 1 + x_f^2 \right)^{1/2} - 1 \right] m_e c^2 n_e^3 =
\]
\[
-1.137 \times 10^{-2} \left[ \frac{\left( \frac{\rho/\mu_e}{10^6 \text{gcm}^{-3}} \right)^{1/3} \left( \frac{Z}{6} \right)^{2/3}}{1 + \left( 1.0125 \left( \frac{\rho/\mu_e}{10^6 \text{gcm}^{-3}} \right)^{2/3} \right)^{1/2}} \right] - 1
\]

(37)

where we have again used \( n_e = \rho/\mu_e m_H \) (Equation 26).

The above ratio shows that the Coulomb energy is small compared to the Fermi energy of the electrons for typical densities. A look at Table 3 shows that the Coulomb corrections (assuming \( Z = 6 \)) becomes less important at higher densities such as those found in the interiors of white dwarfs whereas these corrections would be significant near the surface layers where the densities are lower. In summary, the above calculations show that the Fermi energy of the electrons is far greater than the thermal energy of the electrons or ions, and also greater than the total electrostatic energy for typical conditions found in the interiors of white dwarf stars. While in any complete analysis of white dwarfs we would have to include these corrections, in the remainder of the thesis we neglect these corrections and assume that the electron gas is completely degenerate (and non-interacting) from the time a white dwarf star is formed to the time it cools down to a cold dark sphere of crystallized carbon supported largely by electron degeneracy pressure. A more complete treatment of the corrections to the ideal gas equation of state is given by Salpeter (1961) [12] and
Salpeter & Zapolsky (1967) [13]. For densities greater than $10^4 \text{ g cm}^{-3}$ the Coulomb corrections obtained using the Wigner-Seitz approximation (see Equation 36) are sufficient, but at lower densities one should use the results of Feynman, Metropolis & Teller [14] for the Thomas-Fermi-Dirac model.

We have introduced a plot (Figure 5) which is a log-log plot of temperature vs. density and summarizes the results obtained in the present subsection. Figure 5 shows the approximate regimes for various equations of state: ideal gas pressure, radiation pressure, fully degenerate electron pressure, as well as two values of the Coulomb coupling parameter $\Gamma$. The straight solid line in the upper left hand corner with logarithmic slope $= 1/3$ is obtained by setting the gas pressure equal to the radiation pressure for a $\mu = 0.6$ which is appropriate for a Hydrogen-Helium mix in normal stars. The radiation pressure dominates conditions within any star that falls in the region above that line, such as might occur in very massive stars. At the high densities for a fixed temperature, the electron degeneracy pressure becomes important. This boundary is shown by the slope $= 2/3$ line in the graph, which is obtained by setting the ideal electron gas pressure equal to the non-relativistic electron degenerate gas pressure. The degenerate equation of state transitions from the non-relativistic electron gas to the relativistic electron gas as the electrons become relativistic. Here we have defined that to be $P_{ur} = P_{nr}$ which corresponds to a density of $3.83 \times 10^6 \text{ g cm}^{-3}$ for a $\mu_e = 2$ ($x_f = 1.25$). For illustrative purposes we show this transition as a sudden change in slope. Note too, that at a temperature of $10^7 \text{ K}$ and density of $10^6 \text{ g cm}^{-3}$, the pressure is dominated by the electron degeneracy pressure,
which is the result we arrived at when we compared the Fermi energies of electrons with the thermal energies and Coulomb energies (see Equations 33 and 37). We have plotted the temperature as a function of density for a carbon white dwarf (i.e., Z=6, $\mu_0 = 12$) for two different values of the Coulomb coupling parameter, $\Gamma = 1$ and 175. The line corresponding to $\Gamma = 175$ lies near the bottom of the chart and points below this boundary indicate the ions are in a crystallized state. The present central conditions of our sun, which is a main sequence star, lies within the classical ideal gas regime.

Thus we conclude from this chapter that the electron degeneracy pressure is the dominant pressure which supports the white dwarf. The next step is to calculate this degeneracy pressure for various cases of a fully degenerate, non-interacting, electron gas which we do in Chapter III.
CHAPTER III

CALCULATION OF PRESSURE

As we saw in the previous chapter the pressure in the interior of white dwarfs is dominated by the electron degeneracy pressure. In this chapter we derive this degeneracy pressure for the following three cases assuming that the electron gas is fully degenerate.

- Fully degenerate relativistic gas
- Fully degenerate ultra-relativistic gas
- Fully degenerate non-relativistic gas

The fully degenerate relativistic case gives the exact solution to the degeneracy pressure and the solution is valid over the whole range of densities encountered in the white dwarf. The other two cases are asymptotic limits to the exact equation and are valid only when the speeds of electrons are small compared to the speed of light (non-relativistic) or when the speeds approach the speed of light (ultra-relativistic). We start off by substituting the relativistic electron momentum in the pressure integral to obtain the pressure of a fully degenerate relativistic gas. We then approximate the exact solution and obtain two solutions that are valid at lower densities and higher densities. We also obtain asymptotic solutions to the fully degenerate relativistic gas by using approximations to the momentum in the pressure integral. We then calculate
the kinetic energy density and derive its relation with pressure. Finally, we plot pressure as functions of density and $x_f$.

Fully Degenerate Relativistic Gas

The pressure due to electron degeneracy can be calculated by solving the pressure integral for an isotropic gas, which is given by

$$ P = \frac{1}{3} \int_0^\infty p v_p n(p) dp , $$

where $p$ is the momentum of the electrons, $v_p$ is the velocity of electrons. The factor of one third in front of the integral is present because for a sufficiently large collection of particles in random motion, the likelihood of motion in each of the three directions is the same and the magnitude of the velocity vector is averaged out. The number density of particles is denoted by $n(p)dp$ which is again given by Equation 16,

$$ n(p)dp = \frac{2}{h^3} 4\pi p^2 \eta(p) dp , $$

where $\eta(p)$ is the occupation number. For a fully degenerate gas $\eta(p) = 1$. The pressure integral becomes,

$$ P = \frac{1}{3} \int_0^\infty p v_p \frac{8\pi p^2}{h^3} dp . $$

For electrons traveling at relativistic speeds the momentum is given by

$$ p = m_e v \Rightarrow v = \frac{p}{m_e} \left[ 1 - \left( \frac{v}{c} \right)^2 \right]^{\frac{1}{2}} \left[ 1 + \left( \frac{p}{m_e c} \right)^2 \right]^{\frac{1}{2}} . $$

Substituting for velocity in the pressure integral we obtain,
and then, 

\[ P_e = \frac{8\pi}{3h^3} \int_0^{p_f} \frac{p^4}{m_e} \left[ 1 + \left( \frac{p}{m_e c} \right)^2 \right]^{1/2} dp. \]  

Substituting the Fermi parameters \( x_f = \frac{p_f}{m_e c} \) (Equation 24), and \( x = \frac{p}{m_e c} \), which compares the electron’s \( pc \) with its rest mass energy, we obtain 

\[ P_e = \frac{8\pi(m_e c)^5 x_f}{3h^3 m_e} \int_0^{\infty} \frac{x^4}{\sqrt{1 + x^2}} dx. \]  

Solving the above integral using the integral tables[15] we find,

\[ P_{e,r} = u_0 \left[ \frac{x_f^3(1 + x_f^2)^{3/2}}{4} - \frac{3}{8} x_f (1 + x_f^2)^{1/2} + \frac{3}{8} \ln \left[ x_f + (1 + x_f^2)^{1/2} \right] \right]. \]

where \( u_0 = \frac{8\pi(m_e c)^5}{3h^3 m_e} = 4.801867 \times 10^{23} \text{ ergs cm}^{-3}. \) 

\[ P_{e,r} = 4.801867 \times 10^{23} \text{ ergs cm}^{-3} \left[ \frac{x_f^3(1 + x_f^2)^{3/2}}{4} - \frac{3}{8} x_f (1 + x_f^2)^{1/2} + \frac{3}{8} \ln \left[ x_f + (1 + x_f^2)^{1/2} \right] \right] \]

This is the exact solution for the pressure due to a non-interacting fully degenerate electron gas.
Large x Limit

We can approximate the above exact equation to be valid for particular range of densities by considering the momentum of electrons to be either small or large compared to $m_e c$, i.e. for large or small values of the Fermi parameter. For the case of large $x_f$, we can neglect the logarithmic term in Equation 44 and hence

$$P_e(x_f \text{ large}) = u_0 \left[ \frac{x_f^3 (1 + x_f^2)^{1/2}}{4} - \frac{3}{8} x_f (1 + x_f^2)^{1/2} \right] = u_0 \left[ (1 + x_f^2)^{1/2} \left( \frac{x_f^3}{4} - \frac{3}{8} x_f \right) \right]$$

(47)

Similarly in this limit we can approximate the term under the radical as $(1 + x^2 \approx x^2)$, we obtain

$$P_e(x_f \text{ large}) = u_0 \left( \frac{x_f^4}{4} - \frac{3x_f^2}{8} \right).$$

(48)

Small x Limit

In the limit of $x$ being small which means the electrons are traveling at momentum that are small compared to $m_e c$ we can expand the denominator of Equation 43 in a binomial series to obtain the pressure in that limit. Rewriting Equation 43, we obtain,

$$P_e = u_0 \int_0^{x_f} x^4 (1 + x^2)^{-1/2} \, dx.$$ 

(49)

Expanding $(1 + x^2)^{-1/2}$ in a binomial series we obtain,

$$P_e = u_0 \int_0^{x_f} \left( x^4 - \frac{3x^6}{2} + \frac{3x^8}{8} \cdots \right) \, dx.$$ 

(50)
Keeping the first two terms of the expansion we obtain after integrating,

\[ P_e(x \text{ small}) = u_0 \left( \frac{x_f^5}{5} - \frac{x_f^7}{14} \right). \]  

\[ (51) \]

Two Asymptotic Limits

Asymptotic Limit of Ultra-relativistic Electrons

When in general \( x = p/m_c \) is large as is the case with electrons traveling at highly relativistic speeds we have \((1 + x^2) \approx x^2\) and the integral in Equation 43 simplifies to,

\[ P_{e,ur} = u_0 \int_0^{x_f} \frac{x^4}{\sqrt{x^2}} \, dx = u_0 \int_0^{x_f} x^3 \, dx \]

\[ (52) \]

\[ P_{e,ur} = u_0 \frac{x_f^4}{4}. \]  

\[ (53) \]

Note that Equation 48 reduces to the above equation if we neglect the second term. To obtain the pressure due to the electron gas in terms of the Fermi momentum we make the substitution \( p_f/m_c \) (Equation 24) in Equation 53 to obtain,

\[ P_{e,ur} = u_0 \frac{p_f^4}{4(m_c)^4} \cdot \]  

\[ (54) \]

Substituting Equation 19 in the above expression we obtain the pressure due to a fully degenerate ultra-relativistic electron gas in terms of the number density of particles.

\[ P_{e,ur} = \frac{u_0}{4(m_c)^4} \left( \frac{3h^3}{8\pi} \right)^{4/3} n_e^{4/3} = \left( \frac{2\pi c}{3h^3} \right) \left( \frac{3h^3}{8\pi} \right)^{4/3} n_e^{4/3}. \]  

\[ (55) \]

Using Equation 26 we obtain \( P_{e,ur} \) in terms of the mass density \( \rho \),
\[ P_{e,nr} = \frac{u_0}{4(m_e c)^4} \left( \frac{3h^3}{8\pi n_H} \right)^{4/3} \left( \frac{\rho}{\mu_c} \right)^{4/3} = \left( \frac{2\pi c}{3h^3} \right) \left( \frac{3h^3}{8\pi n_H} \right)^{4/3} \left( \frac{\rho}{\mu_c} \right)^{4/3} \] (56)

Asymptotic limit of Non-relativistic electrons

When in general \( x = p/m_e c \) is very small, as is the case with electrons traveling at non-relativistic speeds, we have \( (1+x^2) \approx 1 \), and the integral in Equation 43 reduces to

\[ P_e = u_0 \int_0^{x_f} \frac{x^4}{1} dx. \] (57)

Hence we obtain the pressure due to a fully degenerate non-relativistic electron gas

\[ P_{e,nr} = u_0 \frac{x_f^5}{5}. \] (58)

Again note that if we neglect the second term in Equation 51 we arrive at the above equation. Following the same procedure as we did for the ultra-relativistic case we obtain the pressure in terms of the Fermi momentum,

\[ P_{e,nr} = u_0 \frac{p_f^5}{5(m_e c)^5}. \] (59)

Again using Equation 19 and Equation 26 we find the pressure in terms of the number density of electrons and in terms of the density respectively.

\[ P_{e,nr} = \left( \frac{u_0}{5(m_e c)^5} \right) \left( \frac{3h^3}{8\pi} \right)^{5/3} n_e^{5/3} = \frac{8\pi}{15h^3 m_e} \left( \frac{3h^3}{8\pi} \right)^{5/3} n_e^{5/3}. \] (60)

In terms of mass density, we have,
We will apply these important results of the asymptotic limits in Chapter IV.

Calculation of Kinetic Energy Density

Let us now proceed to calculate the kinetic energy density for the two asymptotic cases that we just found out. The kinetic energy density is given by

\[ u_e = \int E(KE)(p)n(p)dp \; , \quad (62) \]

where \( E(KE)(p) \) gives the Fermi momentum of electrons for a given momentum \( p \).

\[ u_e = \left[ c^2 p^2 + m_e^2 c^4 \right]^{1/2} - m_e c^2 \frac{8\pi e^2}{h^3} dp \; , \quad (63) \]

where we have made use of Equations 29 and 16 for a general momentum \( p \).

Again making the substitution \( x_f = p_f/ m_e c \) and \( x = p/ m_e c \), we find

\[ u_e = \int_0^{x_f} \left[ c^2 x^2 m_e^2 c^2 + m_e^2 c^4 \right]^{1/2} - m_e c^2 \frac{8\pi e^2}{h^3} x^2 dx \; . \quad (64) \]

Rearranging terms and taking constants out of the integral we find,

\[ u_e = 3u_0 \int_0^{x_f} \left( x^2 + 1 \right)^{1/2} - 1 \right] x^2 dx \; . \quad (65) \]

Now if we apply the limit of \( x \) being very small the above expression does not provide any meaningful insight so let us rewrite the above equation in a more illuminating form, which will enable us to apply both the asymptotic limiting cases.

Multiply and divide the above integrand by \( (1+x^2)^{1/2} + 1 \). We obtain
$u_e = 3u_0 \int_{0}^{x_f} \frac{x^4}{\sqrt{1 + x^2} + 1} \, dx. \quad (66)$

Now we can apply the two asymptotic limits to Equation 67.

**Kinetic Energy Density for the Asymptotic Cases**

For electrons traveling at non-relativistic speeds we can say $x_f$ is much smaller than unity and hence $(1 + x^2 \approx 1)$, we find

$$u_{e, nr} = 3u_0 \int_{0}^{x_f} \frac{x^4}{2} \, dx, \quad (67)$$

$$u_{e, nr} = 3u_0 \frac{x_f^5}{10} \Rightarrow u_{e, nr} = \frac{3}{10} u_0 x_f^5. \quad (68)$$

For electrons traveling at ultra-relativistic speeds $x_f$ is much greater than unity and we can make the approximation $(1 + x^2 \approx x^2)$ in Equation 66

$$u_{e, ur} = 3u_0 \int_{0}^{x_f} \frac{x^4}{x} \, dx, \quad (69)$$

$$u_{e, ur} = 3u_0 \int_{0}^{x_f} x^3 \, dx = \frac{3u_0}{4} x_f^4 \Rightarrow u_{e, ur} = \frac{3u_0}{4} x_f^4. \quad (70)$$

The above two results (Equations 68 and 70) give the kinetic energy density of the electrons which are traveling at non-relativistic and ultra-relativistic speeds (asymptotic cases). We will use these two results to find the relation between kinetic energy density and pressure of a non-relativistic and ultra-relativistic electron gas. Let us now rewrite Equation 53 and 58 in the form of a polytrope to enable us to use these two equations in the next chapter to calculate the mass-radius relationship and the
Chandrasekhar's limiting mass for an ideal white dwarf. Substituting for $x_f$ (Equation 27) in the equation for pressure due to a fully degenerate non-relativistic gas (Equation 58), we find

$$P_{e, nr} = \frac{u_0}{5} \left( \frac{\rho}{\rho_0 \mu_e} \right)^{5/3},$$

(71)

$$P_{e, nr} = K_{nr} \left( \frac{\rho}{\mu_e} \right)^{5/3},$$

(72)

where $K_{nr} = \frac{u_0}{5 \rho_0^{5/3}} = 9.906422 \times 10^{12} \text{ ergs cm}^2 \text{ g}^{-5/3}$

(73)

and $\rho_0$ is given by Equation 27.

Using Equation 28 we obtain,

$$\left( \frac{\rho}{\rho_0 \mu_e} \right)^{4/3} = x_f^4.$$

(74)

Substituting the above equation in Equation 53 we find,

$$P_{e, nr} = \frac{u_0}{4} \left( \frac{\rho}{\rho_0 \mu_e} \right)^{4/3},$$

(75)

$$P_{e, nr} = K_{nr} \left( \frac{\rho}{\mu_e} \right)^{4/3},$$

(76)

where $K_{nr} = 1.230641 \times 10^{15} \text{ ergs cm}^2 \text{ g}^{-4/3}$

(77)

Also from the equations for the pressure and kinetic energy density (Equations 58 and 68) for a fully degenerate non-relativistic electron gas we can infer that the kinetic energy density is one and a half times the value of pressure which also holds true for a classical ideal gas.
\[ u_{e,ur} = \frac{3}{2} P_{e,ur}. \] (78)

Eliminating \( x_f \) between Equation 71 and Equation 54 we find,
\[ u_{e,ur} = 3 P_{e,ur}. \] (79)

The kinetic energy density is three times the pressure for ultra relativistic particles which also holds true for photons. This is to be contrasted with three halves the pressure when electrons are traveling at non-relativistic speeds.

We will summarize our findings in plots of \( \log(P) \) vs. \( \log(x_f) \) and \( \log(p/\mu_e) \) in Figures 6 and 7 for the following cases:

- the two asymptotic limits, \( x_f >> 1 \) and \( x_f << 1 \)
- the limit of small \( x_f \) and large \( x_f \)
- the exact equation
- the hybrid equation of state [11]

The motivation behind doing these plots was to see firsthand the range of values of density and Fermi parameter over which the above approximations to the equation of state are valid. Once we know that, we can apply these approximations to real white dwarf stars making the calculations simpler without much loss in accuracy because we do not have to use the exact equation say in the asymptotic limits. Obviously the plot made using the exact equation is valid over all values of density and the Fermi parameter encountered in white dwarfs. It is clear from the plot that the approximate equations of state mimic the exact equation very well over their regimes of validity. For example in Figure 6 the non-relativistic asymptote which is equal to the exact
equation only asymptotically deviates appreciably from the exact equation at around \( \log x_f = 0.3 \) and \( \log (\rho/\mu_e) = 5.2 \). This can also be seen in Figure 4 where the line and the curve representing the kinetic energy of the relativistic and non-relativistic electron start deviating from each other at the same value of \( x_f \). Also the ultra-relativistic asymptote deviates appreciably from the exact equation until \( \log x_f = 0.5 \) and \( \log (\rho/\mu_e) = 7.6 \), and then nearly matches it for larger values. It is clear from the plot that the approximate equations of state for large and small \( x_f \) agree well with the exact equation at high and low values of the Fermi parameter respectively. We noted earlier that the equation for small \( x \) reduces to the non-relativistic asymptotic equation if we neglect the second term in Equation 51. This behavior can also be seen from the graph because the two curves match very well at small \( x_f \). The same holds true for large \( x_f \) and the ultra-relativistic asymptote.

We have also introduced a hybrid equation of state (as developed in the next chapter; see Equation 101) which is a combination of the two asymptotic cases. The purpose of introducing this equation in the plot is to find out how well this equation behaves over the range of densities found in white dwarfs. It is clear from the graph (Figures 6 and 7) that the hybrid equation of state mimics the exact solution very well with the maximum deviation from the exact equation of state occurring at \( \log x_f = 0.167 \) and a corresponding \( \log (\rho/\mu_e) = 6.50 \). The success of the hybrid equation of state lies in the fact that it weighs out the lesser of the asymptotic pressures. At higher densities the non-relativistic asymptote overshoots the exact equation and so the hybrid equation chooses the ultra-relativistic asymptote's contribution to the pressure.
and vice versa for the other asymptote. This enables us to use a solution that is easier than the exact solution to calculate the mass-radius relationship.
CHAPTER IV

MASS-RADIUS RELATIONSHIP

The Method of Polytropes

In the previous chapter we calculated the pressure due to a fully degenerate electron gas and in Chapter I we calculated the required central pressure to keep a star in equilibrium. In this chapter we equate the two asymptotic equations to the required central pressure to find the mass-radius relationship (using the non-relativistic equation of state) and the Chandrasekhar's limiting mass (using the ultra-relativistic equation of state) for white dwarfs. We achieve that by using the method of polytropes. A polytrope refers to a solution to the Lane-Emden equation, and as such is useful if simple gas equation of state. The two asymptotic equations of state (Equations 72 and 76) are in the form of a polytropic equation of state, \( P = K \rho^n \).

Consider the equation for hydrostatic equilibrium (Equation 1) and the equation for mass conservation (Equation 2). Multiplying Equation 1 by \( r^2/\rho(r) \) and differentiating with respect to \( r \) we obtain,

\[
\frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -G \frac{dm}{dr} .
\]

(80)

Substituting Equation 2 in the above equation we obtain,

\[
\frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho .
\]

(81)
We assume that the pressure equation of state can be written in the form

\[ P = K \rho^\gamma, \]  

(82)

where \( k \) and \( \gamma \) are constants and \( \gamma \) is given by,

\[ \gamma = 1 + \frac{1}{n}, \]  

(83)

where \( n \) is known as the polytropic index.

This is simply a relationship that expresses an assumption regarding the run of \( P \) with radius in terms of the run of \( \rho \) with radius, and this assumption yields a solution to the Lane-Emden equation. Different values of \( n \) correspond to equation of state for different gases. For example \( n = 1.5 \) corresponds to a \( \gamma = 5/3 \) polytrope (non-relativistic gas) and \( n = 3 \) corresponds to a \( \gamma = 4/3 \) polytrope (ultra-relativistic gas).

When we substitute the above values for \( \gamma \) in Equation 82 we arrive at the general forms of the two asymptotic limits (non-relativistic and ultra-relativistic respectively) to the ideal electron degenerate equation of state.

Substituting the above assumptions in Equation 81 we get a second order differential equation:

\[ \frac{(n+1)K}{4\pi G n} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\rho}{\rho^{n+1}} \right) = -\rho. \]  

(84)

The solution \( \rho(r) \) for \( 0 \leq r \leq R \) is called a polytrope subject to the boundary conditions \( \rho(R) \) at the surface and \( d\rho/dr = 0 \) at the center. Hence a polytrope is uniquely defined by three parameters \( K, n \) and \( R \) and it enables the calculation of additional quantities.
as functions of radius through the structure, such as the density. Let us define a dimensionless variable in the range $0 \leq \theta \leq 1$,

$$\theta^n = \frac{\rho}{\rho_c}. \quad (85)$$

Substituting into Equation 84 we obtain,

$$\alpha^2 \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\theta}{dr} \right) = -\theta^n, \quad (86)$$

where

$$\alpha^2 = \left[ \frac{(n+1)K}{4\pi G n \rho_c} \right]. \quad (87)$$

is a constant having the dimensions of length squared. This can be used to replace $r$ by a dimensionless variable $\xi$,

$$r = \alpha \xi. \quad (88)$$

Substituting this in (Equation 86) we get the Lane-Emden equation of index $n$,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n, \quad (89)$$

subject to the boundary conditions $\theta = 1$ and $d\theta/d\xi = 0$ at $\xi = 1$.

We can now use the method of polytropes to calculate the mass-radius relationship which approximates that of white dwarf stars. Using Equation 4 in Chapter I, we can compute the pressure for any region of interest within the star by choosing the appropriate limits of integration. For example, Equation 8 in Chapter I determines the central pressure in hydrostatic equilibrium. Setting the available
pressure for a non-relativistic ideal degenerate electron gas to the pressure required by hydrostatic equilibrium \( P_{e, nr} = P_{req} \) at the center of the star we obtain,

\[
K_{nr} \left( \frac{\rho_c}{\mu_e} \right)^{5/3} = \frac{\alpha G M^2}{R^4}. 
\]

where \( A = \rho_c/\langle \rho \rangle \) (scaling factor), and we have substituted for the central density. Canceling out like terms and grouping the constants we obtain,

\[
R = K' M^{-1/3}. 
\]

where

\[
K' = K_{nr} \left( \frac{3A}{4\pi \mu_e} \right)^{5/3}, 
\]

and \( K_{nr} \) is given in Equation 73 (Chapter III). It can be seen from Equation 92 that more massive white dwarfs are smaller in size. This interesting behavior results from a structure which increases its pressure by an increase in density alone. The pressure equation for a non-relativistic fully degenerate electron gas is in the form \( P = K \rho'' \) and hence we can apply the law of polytropes to calculate the value of \( K' \). We know from Equation 83 that

\[
\gamma - 1 = \frac{1}{n} \Rightarrow \gamma = 1 + \frac{1}{n}. 
\]

In our case \( \gamma = 5/3 \) (non-relativistic fully degenerate electron gas) and hence the polytropic index is \( n = 1.5 \). From the polytropic tables [7] we find for \( n = 1.5 \), \( A = 5.99071 \) and \( \alpha = 0.770140 \). Substituting for \( A \) and \( \alpha \) in the expression for \( K' \) we
obtain the mass-radius relationship for an ideal white dwarf whose electrons remain non-relativistic

\[(1.10 \times 10^{20} \text{ cm.g}^{-1/3}) M^{-1/3} = R \Rightarrow R = 0.0126 R_0 \left( \frac{2}{\mu_e} \right)^{5/3} \left( \frac{M}{M_\odot} \right)^{-1/3}. \] (94)

Chandrasekhar's Limiting Mass

Let us now consider the pressure equation for a fully degenerate ultra-relativistic gas (Equation 76) and set it equal to the required central pressure (Equation 8)

\[ K_{\mu_r} \left( \frac{\rho_c}{\mu_e} \right)^{4/3} = \frac{\alpha G M^2}{R^4}. \] (95)

As before, with \( \rho_c = A <\rho> = 3MA/4\pi R^3 \), we can rewrite the above equation as

\[ \frac{K_{\mu_r}}{\alpha G} \left( \frac{3AM}{4\pi R^3 \mu_e} \right)^{4/3} = M^2 \Rightarrow M^2 = \left[ \frac{K_{\mu_r}}{\alpha G} \left( \frac{3A}{4\pi \mu_e} \right)^{4/3} \right]^{3/2}. \] (96)

Note that the radius drops out of the above equation! Polytropes have specific values for \( A \) and \( \alpha \). In this case \( \gamma = 4/3 \), (obtained by looking at the form of the equation for ultra-relativistic fully degenerate gas which mimics a polytrope) from which we obtain \( n = 3 \). From the polytropic tables [6] we can find that corresponding to the \( n = 3 \) polytrope, \( A = 54.1825 \) and \( \alpha = 11.05066 \), and so we find

\[ M = 2.90 \times 10^{33} \left( \frac{2}{\mu_e} \right)^2 \text{ g}. \] (98)
Converting into solar mass units $M_o$ we find,

$$M = 1.43 M_o \left( \frac{2}{\mu_e} \right)^2. \quad (99)$$

The above is the maximum mass that an ideal white dwarf can have because the available pressure can only approach the ultra-relativistic case and cannot exceed it. This is because the electrons cannot travel at or faster than the speed of light. As the density of the white dwarf increases the electrons start becoming more energetic and they begin traveling at speeds close to the speed of light and reaches a point where it cannot travel any faster and that point determines the maximum pressure that the electrons can provide against gravity resulting in an upper limit on the mass of a white dwarf. This is known as the Chandrasekhar's limit. Hence a white dwarf of mass $M$ and mean molecular weight $\mu_e$ should obey

$$M < M_{ch} = 1.43 M_o \left( \frac{2}{\mu_e} \right)^2. \quad (100)$$

Mass-Radius Relationship using a Hybrid Polytropic Equation of State

The mass-radius relation we derived in Equation 94 is valid only so long as the electrons remain non-relativistic. Therefore in this section we derive the mass-radius relationship for white dwarfs using a hybrid polytropic equation of state which is applicable in both limits and well approximates the exact equation. Of course the exact equation would be the first choice, but use of it requires numerical methods to produce a solution. Hence, we appeal to a simplified equation of state that accurately mimics the exact equation. The equations of state for the high electron momentum
limit (ultra-relativistic case) and the small electron momentum limit (non-relativistic case) work well only in their regimes, as shown in Figures 6 and 7. Between densities of $10^3$-10$^9$ g cm$^{-3}$, we saw from the discussion of Figures 6 and 7 that the hybrid equation of state satisfies all these requirements: the maximum deviation of $P$ vs. $\rho/\mu_e$ from exact ideal equation of state is 1.8%, occurring at a density $\rho/\mu_e \approx 3.1 \times 10^6$ g cm$^{-3}$ corresponding to $x_f \approx 1.47$. The hybrid equation of state is represented by

$$P_{e,d} = \left[ P_{e,dr}^{-2} + P_{e,nr}^{-2} \right]^{-1/2},$$

where $P_{e,d}$ stands for pressure due to fully degenerate electron gas and $P_{e,nr}$ and $P_{e,ur}$ (Equations 61 and 56) stands for the pressure due to a fully degenerate non-relativistic electron gas and a fully degenerate ultra-relativistic electron gas respectively. The success of the hybrid equation of state is that it is weighted toward the weaker of the two pressures, and thus it is able to accurately follow the exact equation over the above mentioned range of densities. The meaning of the above statement will be clearer if we rewrite Equation 101 in the following form

$$P_{e,d} = P_{e,nr} \left[ 1 + \left( \frac{P_{e,nr}}{P_{e,ur}} \right)^2 \right]^{-1/2}.$$  (102)

The non-relativistic equation of state overestimates the pressure obtained using the exact equation for $x_f >> 1.25$ ($\rho/\mu_e \approx 1.9 \times 10^6$ g cm$^{-3}$), while the ultra-relativistic equation overestimates the pressure obtained using the exact equation for $x_f << 1.25$. The form of the above equation suggests that the hybrid equation of state adds the two asymptotic equations of state in such a way that the lesser of the two pressures at a
given value of $x_f$ or equivalently $\rho/\mu_e$ is given more weight in the solution, thus ensuring that the pressure is not overestimated, resulting in a very good approximation to the exact equation of state.

Substituting Equations 72 and 76 into Equation 101 we obtain

$$P_{c,fr} = \left[ \left( K_{nr} \left( \frac{\rho}{\mu_e} \right)^{5/3} \right)^{-2} + \left( K_{nr} \left( \frac{\rho}{\mu_e} \right)^{4/3} \right)^{-2} \right]^{-1/2} \tag{103}$$

We can write down a simple relation for the density as $\rho \approx \frac{M}{4\pi/3 R^3}$ and substitute in the above equation. Similarly we can write a simplified relation for the pressure required for hydrostatic equilibrium as $(P \approx \frac{GM^2}{R^4})$. If we substitute the above simplifications in Equation 103, which sets them equal to each other and after some algebra we obtain,

$$R \approx \frac{K_{nr} / \mu_e^{5/3}}{GM^{1/3}} \left[ 1 - \frac{G^2 M^{4/3}}{(K_{nr} / \mu_e^{4/3})^2} \right]^{1/2} \tag{104}$$

Each polytrope follows a specific mass-radius relationship. By a rigorous analysis of polytropes we obtain the mass-radius relationship for polytropes. Using the general form of the polytropic equation of state (Equation 82) and substituting $P = \alpha GM^2/R^4$ and $\rho_c = A <\rho> = A \times 3M/4\pi R^3$ we obtain,

$$\frac{\alpha GM^2}{R^4} = K \left( \frac{3MA}{4\pi R^3} \right)^{1+n} \Rightarrow R^{n} M^{n-1} = \frac{K}{\alpha G A^{1-n}} \left( \frac{3}{4\pi} \right)^{1+n} \tag{105}$$

Introducing a constant $N_n$, we can write the above equation as
\[
R^n M^n = \frac{K}{GN_n}.
\]  

(106)

where

\[
N_n = \left( \frac{\alpha_n}{A_n^{1 + \frac{1}{n}}} \right) \left( \frac{4\pi}{3} \right)^{1 + \frac{1}{n}}
\]

is a numerical factor, \(1 + 1/n = \gamma\), \(P = K\rho^n\) is the polytropic equation of state of index \(n\), and \(K\) is a constant. If we consider the two asymptotic cases of non-relativistic and ultra-relativistic ideal electron degenerate equations of state, and substitute Equation 107 into Equation 106, we will obtain the following two mass-radius relationships:

For \(n = 1.5\) (non-relativistic polytrope), we find from numerical tables \([3]\) and Equation 108 that \(N_n = 0.4242158\) and so,

\[
RM^{1/3} = \frac{K_{\text{nr}}/\mu_e^{5/3}}{0.4242158G},
\]

the mass-radius relation already seen in Equation 94. For \(n = 3\) (ultra-relativistic polytrope) we find from numerical tables and Equation 107 that \(N_n = 0.3639382\) and so,

\[
M^{2/3} = \frac{K_{\text{nr}}/(\mu_e^{4/3})}{0.3639382G}.
\]

(109)

Note that the radius drops out in the above expression, as we found out in the previous section – this is the Chandrasekhar limiting mass. Combining Equations 109 and 108, Equation 104 can be written more accurately as
Rearranging the above we obtain,

\[
R \approx \frac{K_{nr} / \mu_e^{5/3}}{0.4242158GM^{1/3}} \left[ 1 - \left( \frac{0.3639382G}{(K_{nr} / \mu_e^{4/3})^2} \right)^{1/2} \right],
\]

or

\[
R \approx \frac{K_{nr} / \mu_e^{5/3}}{0.4242158GM^{1/3}} \left[ 1 - \left( \frac{M_{ch}}{M_{ch}} \right)^{4/3} \right]^{1/2},
\]

or

\[
R \approx 0.0126R_0 \left( \frac{2}{\mu_e} \right)^{5/3} \left( \frac{M}{M_{ch}} \right)^{-1/3} \left[ 1 - \left( \frac{M}{M_{ch}} \right)^{4/3} \right]^{1/2},
\]

where \( M_{ch} = 1.43M_0 (2/\mu_e)^2 \) is Chandrasekhar’s limiting mass (Equation 100). This expression gives a very accurate mass-radius relationship for white dwarfs which is proved by the fact that all the observed white dwarfs fall along the curve obtained using the above equation (see Figure 8).

Figure 8 plots the mass-radius relationships obtained using the hybrid equation of state for various compositions of white dwarfs. Also included in the plot are observed values for the masses and radii for a representative sample of white dwarfs. The three solid curves are plotted using Equation 113, while the dashed line is plotted using the mass-radius relation obtained by setting the required pressure equal to the
non-relativistic asymptote (Equation 94). The two solid curves represent a carbon white dwarf (top curve) and an iron white dwarf (bottom curve). It can be noticed from the plot that the dashed curve deviates from the solid one and does not fall toward zero radius. This is a consequence of not incorporating relativistic effects to the equation of state at higher densities. The solid curve drops to zero radius at a certain limiting mass \((M_{ch})\) depending on its composition. This is because the equation of state softens from a \(\gamma = 5/3\) to a \(\gamma = 4/3\) polytrope. The pressure required by hydrostatic equilibrium scales as \(\langle \rho \rangle^{4/3} M^{2/3}\). So for a fixed mass, if the run of available pressure scales as density to the 4/3 power (or less), then it cannot keep up with the required pressure and dynamical instability results with the slightest perturbation from equilibrium, resulting in a collapse. Of course in real white dwarfs many processes occur which prevent a white dwarf from falling collapsing to zero radius- the equation of state might change or the white dwarf could undergo runaway thermonuclear fusion and explode as a Type 1A supernova.

Figure 9 shows the relation between mean density \(\langle \rho \rangle\) and mass in solar mass units for carbon white dwarfs using the mass-radius relation in Equation 114. For masses \(M < 0.2\) solar mass, the curve behaves in a way expected of the non-relativistic equation of state \((P \propto \rho^{5/3})\), in which the mean density increases rapidly with the mass by the relation \(\langle \rho \rangle \propto M^2\). This is obtained by setting the required pressure \(P \propto M^2/R^4 \propto \langle \rho \rangle^{4/3} M^{2/3}\) equal to the available pressure \(P \propto \rho^{5/3}\). For masses exceeding about 0.2 solar mass the mean density-mass relation begins deviating significantly from the above relation. This can also be seen in Figure 8, where for a
masses beyond about 0.2 $M_\odot$ the more accurate hybrid mass-radius relation starts deviating from the mass-radius relation for the $\gamma = 5/3$ polytrope, since the electrons start traveling at speeds that are an appreciable fraction of the speed of light, and thus the electron kinetic energy can no longer be approximated by the non-relativistic kinetic energy equation $E(KE) = p^2/2m_e$. This means that the equation of state is changing from the non-relativistic asymptote to a somewhat softer equation of state, as evidenced by the increasing value in slope in Figure 9.

The observed range in white dwarf masses is between 0.3 and 1.3 $M_\odot$, while most lie between 0.4-0.8 $M_\odot$. This can be seen in Figure 8 where we have added observed masses and radii for a representative sample of white dwarfs. It is interesting to note that not many white dwarfs are found in the purely non-relativistic regime ($M < 0.2 M_\odot$) where the electrons travel with small momenta and are governed by the non-relativistic asymptote (Equation 72), or in the highly-relativistic regime (as the mass approaches Chandrasekhar's limit).

For masses exceeding about 1.1 $M_\odot$ for which $<\rho>$ exceeds $10^6$ g cm$^{-3}$ and the corresponding value of $x_f$ is approximately unity (for $\mu_e = 2$), the behavior of the mean density with respect to mass in Figure 9 begins to change very rapidly, as we have now entered the regime where the electron equation of state begins taking on the characteristics of an ultra-relativistic gas ($P \propto \rho^{4/3}$). The asymptotic rise in the mean density near $M_{ch}$ in Fig. 9 corresponds to the rapid fall in radius in Fig. 8. A carbon-oxygen white dwarf may reach masses very near to the Chandrasekhar limit by accreting mass from a binary companion. The resulting explosive ignition of carbon
fusion under highly degenerate conditions lead to a runaway thermonuclear explosion (Type 1A supernova) as mentioned in Chapter I.
Summary

In this thesis work we investigated some of the physics of electron degenerate matter in white dwarf stars. We started Chapter I by briefly explaining the life cycle of stars focusing on the possible end stages of evolution and introduced white dwarfs as the stellar corpses of intermediate mass stars \((M < 8M_0)\). We also obtained an estimate of the required central pressure of a star which we used later in Chapter IV when we calculated the mass-radius relationship and the Chandrasekhar's limiting mass. We concluded Chapter I with a discussion on the fate of massive stars \((M > 8M_0)\) and estimated the energy released from a Type II supernova explosion.

In Chapter II we explained the origin of electron degeneracy pressure which supports the white dwarf against gravity induced collapse. We calculated the Fermi energy and Fermi momentum of electrons using the density integral under the assumption of complete electron degeneracy. We introduced a parameter known as the Fermi parameter which compares the electrons \(pc\) with its rest mass and calculated the Fermi energy of electrons using both the non-relativistic and relativistic kinetic energy equations and showed that they both begin to deviate from each other around \(x_f \approx 1\) (see Figure 4). We shifted our attention to electrostatic corrections to the ideal degenerate electron equation of state and justified our assumption of complete electron degeneracy by showing that the corrections to the equations of state due to thermal and Coulomb interactions are small compared to the Fermi energy of electrons. We introduced a Coulomb coupling parameter which gives the relative
strength of the Coulomb interaction between ions relative to the thermal energy \((kT)\) of ions and we showed how the equation of state changes as the white dwarf cools. We included a plot which shows the approximate regimes for the various equations of state namely ideal gas pressure, radiation pressure and fully degenerate electron pressure. Our present sun which is a main sequence star falls in the ideal gas regime in this plot as it should.

We devoted Chapter III to calculating the degeneracy pressure of the electron degenerate gas for the following three cases: fully degenerate non-relativistic, fully degenerate relativistic, fully degenerate ultra-relativistic. The asymptotic solutions obtained were in the form of polytropes and were used in Chapter IV to calculate the mass-radius relationship and Chandrasekhar’s limiting mass. We summarized all the results obtained in Chapter III in the form of two plots which show the approximate regimes of validity for the equation of state obtained using the various assumptions. We also introduced a hybrid polytropic equation of state and showed that it mimics the exact equation of state very closely. The hybrid polytropic equation of state was used later in Chapter IV to obtain an accurate mass-radius relationship for white dwarf stars.

In Chapter IV we introduced the method of polytropes and used it to calculate the mass-radius relationship and the Chandrasekhar limiting mass. We set the required central pressure equal to the ideal degenerate non-relativistic equation of state and ideal ultra-relativistic equation of state respectively (which we calculated in Chapter III—see Equations 72 and 76) to obtain the mass-radius relationship and the
Chandrasekhar’s limiting mass. To obtain an accurate mass-radius relationship we would have to use the exact equation of state which is quite complicated. Moreover the Coulomb corrections (which we calculated in Chapter II - see Equation 37 and Table 3) to the equation of state are important especially for low mass white dwarfs and significantly affect their mass-radius relationship. To overcome the complexity of the exact equation we used the hybrid polytropic equation of state which we have already shown mimics the exact equation of state very closely and obtain an accurate mass-radius relationship. The accuracy of this mass-radius relationship is validated by the fact that all white dwarfs from a representative sample lie along the curve or very near to it (see Figure 8).
Table 1. Observed Mass and Radii of Selected White Dwarfs [7]

<table>
<thead>
<tr>
<th>White Dwarf</th>
<th>Mass (in units of $M_\odot$)</th>
<th>Radius (in units of $R_\odot$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sirius B</td>
<td>1.000±0.016</td>
<td>0.0084±0.0002</td>
</tr>
<tr>
<td>Procyon B</td>
<td>0.604±0.018</td>
<td>0.0096±0.0004</td>
</tr>
<tr>
<td>40 Eri B</td>
<td>0.501±0.011</td>
<td>0.0136±0.0002</td>
</tr>
<tr>
<td>EG 50</td>
<td>0.50±0.06</td>
<td>0.0104±0.0006</td>
</tr>
<tr>
<td>GD 140</td>
<td>0.79±0.09</td>
<td>0.0085±0.0005</td>
</tr>
<tr>
<td>CD-38 10980</td>
<td>0.74±0.04</td>
<td>0.01245±0.0004</td>
</tr>
<tr>
<td>W485A</td>
<td>0.59±0.04</td>
<td>0.0150±0.0001</td>
</tr>
<tr>
<td>G226-29</td>
<td>0.750±0.030</td>
<td>0.01040±0.0003</td>
</tr>
<tr>
<td>G93-48</td>
<td>0.750±0.060</td>
<td>0.01410±0.0020</td>
</tr>
<tr>
<td>L268-92</td>
<td>0.700±0.120</td>
<td>0.01490±0.0010</td>
</tr>
<tr>
<td>Stein 2051B</td>
<td>0.660±0.040</td>
<td>0.0110±0.0010</td>
</tr>
<tr>
<td>L711-10</td>
<td>0.540±0.040</td>
<td>0.01320±0.0010</td>
</tr>
<tr>
<td>L481-60</td>
<td>0.530±0.050</td>
<td>0.01200±0.0040</td>
</tr>
<tr>
<td>G151-B5B</td>
<td>0.460±0.080</td>
<td>0.01300±0.0020</td>
</tr>
<tr>
<td>Wolf 1346</td>
<td>0.440±0.010</td>
<td>0.01342±0.0006</td>
</tr>
</tbody>
</table>

Table 2. $\mu_e$ for Selected Elements

<table>
<thead>
<tr>
<th>$\mu_e$</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^4$He</td>
<td>2.001302</td>
</tr>
<tr>
<td>$^{12}$C</td>
<td>2.000000</td>
</tr>
<tr>
<td>$^{16}$O</td>
<td>1.999364</td>
</tr>
<tr>
<td>$^{20}$Ne</td>
<td>1.999244</td>
</tr>
<tr>
<td>$^{28}$Si</td>
<td>1.998352</td>
</tr>
<tr>
<td>$^{56}$Fe</td>
<td>2.151344</td>
</tr>
<tr>
<td>( (\rho/\mu_e) \text{ g cm}^{-3} )</td>
<td>( E_{th}/E_{f} )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>( 10^3 )</td>
<td>0.501</td>
</tr>
<tr>
<td>( 10^4 )</td>
<td>0.109</td>
</tr>
<tr>
<td>( 10^5 )</td>
<td>0.0244</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>( 6.04 \times 10^{-3} )</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>( 1.82 \times 10^{-3} )</td>
</tr>
<tr>
<td>( 10^8 )</td>
<td>( 6.70 \times 10^{-4} )</td>
</tr>
<tr>
<td>( 10^9 )</td>
<td>( 2.78 \times 10^{-4} )</td>
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Figure 1. Hubble space telescope image of Sirius A&B located in the constellation Canis Major. The brighter star Sirius A is a main sequence star and its dim companion Sirius B (lower left) is a white dwarf star.

Source: http://hubblesite.org/newscenter/newsdesk/archive/releases/2005/36/
Figure 2. Chandra X-Ray Observatory image of Sirius A&B. The central bright star is Sirius B, a dense hot white dwarf with a surface temperature of about 25,200 Kelvin, and the dim source is its companion Sirius A.

Source: http://chandra.harvard.edu/photo/2000/0065/0065_hand.html
Figure 3. Hertzsprung-Russell diagram (luminosity vs. surface temperature) showing the evolutionary phases of a one solar mass star starting with the contraction phase on the pre-main sequence through to the final stages of evolution where it becomes a white dwarf and cools down at nearly constant radius. The sun is currently a main sequence star.

Figure 4. Fermi energies of electrons over a range of Fermi parameter $x_f$. Also included is the small momentum limit extrapolation (dashed line).
Figure 5. Log-Log plot of temperature (K) vs. density (g cm\(^{-3}\)) denoting the regions where various equations of state predominate.
Figure 6. Log-Log plot of pressure (ergs cm$^{-3}$) of the electron gas vs. $x_f$ for the two asymptotic cases, the exact equation, the equations for large and small $x_f$ and the hybrid equation of state. The range of values of $x_f$ corresponds to densities of $10^3$ to $10^9$ g cm$^{-3}$. Note that the exact solution lies underneath the hybrid polytropic equation of state solution.
Figure 7. Same as Figure 6, but with pressure (ergs cm$^{-3}$) plotted against density (g cm$^{-3}$).
Figure 8. Relationship between mass and radius for an ideal fully degenerate electron gas using the $\gamma = 5/3$ and hybrid polytropic ($\gamma$) equations of state. The observed masses and radii of a selection of white dwarf stars are plotted for comparison. The black curve is appropriate for a carbon composition while the red curve is appropriate for an iron composition. The curve for the oxygen composition lies beneath the curve for carbon composition.
Figure 9. Relation between mass in solar mass units and the white dwarf mean density for the hybrid polytropic ($\gamma$) equation of state assuming ideal electron degeneracy, and $\mu_e = 2$. 
BIBLIOGRAPHY


