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Tilt-Rotor Aircraft Flight Control Design with PID, $H_2$, $H_\infty$ and $\mu$ Synthesis Approach

Yinyang Zhai
Western Michigan University

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TILT-ROTOR AIRCRAFT FLIGHT CONTROL DESIGN WITH PID, $H_2$, $H_{\infty}$ AND $\mu$ SYNTHESIS APPROACH

by

Yinyang Zhai

A Thesis
Submitted to the
Faculty of The Graduate College
in partial fulfillment of the
requirements for the
Degree of Master of Science in Engineering (Electrical)
Department of Electrical and Computer Engineering

Western Michigan University
Kalamazoo, Michigan
April 2005
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Finally, I want to thank my wife Shunhua for her patience and continuous support to overcome the encountered difficulties.

Yinyang Zhai
The main objective of this project is to design optimal controllers for Hover and Cruise Dynamic Systems of XV-15 Tilt Rotor Aircraft. The design criterion is to make system Robust in sense of both Stability and Performance. \( \mu \) synthesis is applied to system uncertainty; this method is less conservative as compared to the worst-case \( H_\infty \) synthesis. The power of \( \Delta K \) architecture is to represent uncertainties and system interconnections in \( \mu \) synthesis. For comparison, the classical control techniques, such as Root Locus Algorithm, \( PID \) controller design with ITAE Performance Index, and Pole Placement Approach are also investigated. For application of \( H_\infty \) and \( \mu \) synthesis, the modern control techniques, such as Linear Fractional Transformation, Mixed Sensitivity and Loop Shaping, are explored, too.
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Research for Tilt Rotor Aircraft (TRA) was initiated in the 1940s. A commercial TRA would be a direct descendant of the XV-3, XV-15, and V-22. The XV-3 type was built in 1953, and this experimental aircraft flew until 1966. This development established the fundamental soundness of the TRA concept and gathered data about technical improvements needed for future designs. The development of XV-15 TRA started in 1973. The first XV-15 TRA flight for NASA/Dryden occurred in October 1980 at the Army contingent of Edwards Air Force Base, Edwards of California. Bell Helicopter Textron and Boeing Helicopters began developing the V-22 "Osprey" in 1981, the latest version of TRA. In general, there are three-operation modes for TRA: Hover Dynamics, Transition Dynamics, and Cruise Dynamics. TRA has brilliant advantages: it can fly as a helicopter in Hover mode. Also, it can transition rapidly from the helicopter mode to the airplane/ Cruise mode. It takes about fifteen seconds in conversion period, and then the aircraft speed
increases and lift is transferred from the rotors to the wing. It would be optimistically predicted that TRA could be used as an unique Civil Vehicle with a large market potential, particularly in high-density areas. To achieve desired performance and stability, a robust automatic flight control system (AFCS) is strongly required.

1.1 Transfer Function (TF) Model Identification

In this paper, XV-15 TRA plant models are studied as controller objectives. The classical control technique and modern control theorem will be used to design proper controllers for different dynamic systems. Normally, the cruise dynamic systems are internally stable and classical controller approaches, such as PID controller, can be applied. However, in Hover Mode the longitudinal dynamic systems and lateral dynamic systems are usually unstable. Because most of the hover longitudinal and lateral plant models have to be mathematically expressed as 4\textsuperscript{th}-order or higher transfer functions, it is difficult to design a practical compensator to meet desired performance using classical control method. Fortunately, modern state space control techniques, especially the robust control theorems are developed and demonstrated in wide applications in the complicated control environment.
The TF models of most of the study cases in this paper are taken from the plant models in open literature [08]. The frequency-domain models were constructed using the following Frequency-Response Identification Method.

- Collecting Flight Data
- Making Spectral Analysis
- Bode Plotting
- Deriving Transfer Function Models
- Model Comparison and Verification

The TF models do not perfectly fit the Flight Data. The differences between TF models and Flight Data can be treated as un-modeled dynamics and/or parameter uncertainty, and the uncertainty models can be constructed based on the boundary curves. Basically, uncertainty can be classified into parametric uncertainty and unmodeled dynamic uncertainty, and the unmodeled dynamics can be treated as additive or multiplicative types at the plant input or output point.

1.2 Dynamic Plant Model Analysis

In general, most of Hover Dynamic Systems are unstable, while all Cruise Dynamic Systems are internally stable, or easily made stable using feedback architecture with proper gain factor.
Plant 01: Pitch Rate over Elevator Surface Deflection

In Hover Longitudinal Dynamics

For Pitch Rate over Elevator Surface Deflection in hover Longitudinal Dynamics, the Open Loop TF has one pair of complex eigenvalues, so this plant is unstable. The model TF and pole locations are listed in Table 1.1. To analyze this plant, different solutions are obtained using MatLab.

Figure 1.1 shows the Poles and Zeros mapping and Root Locus of the plant. Using Root Locus technique, we can observe that the Closed Loop TF changes the system running points while the gain is tuned. Obviously, this is unstable plant, and we cannot simply make it stable by adjusting the gain, so as to shift all the Closed Loop Poles to the Left Hand of s-plane (LHP). This is 4th order system. If we arbitrarily place poles and/or zeros on the LHP to attract the loci to the LHP, this job is tedious and the objective is hard to achieve yet. In Chapter Three, modern control technique is introduced to solve this kind of problem.

Bode Plot is another useful tool to evaluate the Closed Loop System stability using Open Loop System properties. In the Bode Plot, stability margins can be determined. Figure 1.2 gives Bode Plot for this plant.
Table 1.1 Hover Longitudinal Dynamics - Plant Model and Data for Pitch Rate over Elevator Surface Deflection

<table>
<thead>
<tr>
<th>Hover Longitudinal Dynamics: $q/\delta_e$ ((deg/s)/deg)</th>
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<tr>
<td>Model TF</td>
</tr>
<tr>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>$-2.66s(s + 0.508)(s - 0.271)$</td>
</tr>
<tr>
<td>$(s + 1.32)(s + 0.105)(s^2 - 0.5362s + 0.3352)$</td>
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Figure 1.1 Hover Longitudinal Dynamics - Open Loop Poles/Zeros Distribution and Root Locus for Pitch Rate over Elevator Surface Deflection
Figure 1.2 Hover Longitudinal Dynamics – Open Loop Bode Plotting for Pitch Rate over Elevator Surface Deflection

Since it is justified from the Root Locus of this plant that its feedback system is still unstable, there is not much meaning to evaluate the Gain Margin and Phase Margin from the Bode Plot. However, in a stable system, it is a good idea to evaluate its Marginal stability using Gain Margin and Phase Margin obtained from Open Loop TF Bode Plot. It is shown at frequency 0.638 rad/sec, which is near the unstable pole frequency 0.579 rad/sec, the magnitude reaches its peak point 9.23 dB. The minimum phase 21.7° happens at frequency 0.270 rad/sec, which is near the plant RHP zero.
The simple way to evaluate system performance and command tracking is to test system step response. Figure 1.3 shows this feature for Open Loop and Closed Loop cases. The curves coincide with above analysis using Root Locus method. Part A of Figure 1.3 indicates that the output of the Open Loop plant oscillates and the amplitude becomes larger with time increasing. Part B of Figure 1.3 shows that the output of the negative feedback plant magnifies all the time and loses control accordingly.

Figure 1.3 Hover Longitudinal Dynamics - Step Response in Open Loop and Closed Loop for Pitch Rate over Elevator Surface Deflection
Plant 02: Vertical Acceleration over Power Level Deflection In Hover Longitudinal Dynamics

This plant information is given in Table 1.2. This is stable plant itself, and from the Root Locus, Part B of Figure 1.4, it is easily identified that the negative

Table 1.2 Hover Longitudinal Dynamics - Plant Model and Data for Vertical Acceleration over Power Level Deflection

<table>
<thead>
<tr>
<th>Hover Longitudinal Dynamics: (a_z/\delta_c) (g/inch)</th>
</tr>
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<tr>
<td>Model TF</td>
</tr>
<tr>
<td>(-0.0098s)</td>
</tr>
<tr>
<td>(s+0.105)</td>
</tr>
<tr>
<td>Poles</td>
</tr>
<tr>
<td>-------</td>
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<td>-0.105</td>
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Figure 1.4 Hover Longitudinal Dynamics - Open Loop Poles/Zeros Distribution and Root Locus for Vertical Acceleration over Power Level Deflection
feedback system is stable as long as the gain is not too high. It is calculated using Matlab, as long as the Gain Factor is less than 102, the feedback system pole is constrained on the LHP, and the system remains stable.

From the Bode Plot of Figure 1.5, the Gain Margin can be obtained to be 40.2 dB, and the Phase Margin is to be infinite.

The Open Loop and Closed Loop system step responses are shown in Figure 1.6. The performance is not satisfactory and should be improved by proper compensation.

Figure 1.5 Hover Longitudinal Dynamics - Open Loop Bode Plotting for Vertical Acceleration over Power Level Deflection
since there is serious overshooting. Furthermore, the output does not follow input command. In fact, the steady state value is always zero for this plant model, because the only zero of the plant transfer function is located on the origin point. To drive the steady state error to zero for step input, the system should be at least type one [07]. This issue is related to reference command tracking, and is discussed in detail in Chapter Two.

Figure 1.6 Hover Longitudinal Dynamics - Step Response in Open Loop and Closed Loop for Vertical Acceleration over Power Level Deflection
Plant 03: Roll Rate over Aileron Surface Deflection In Hover Lateral Dynamics

This plant is similar to the one of Pitch Rate over Elevator Surface Deflection in Hover Longitudinal Dynamics. The plant model and data are given in Table 1.3. Figure 1.7

Table 1.3 Hover Lateral Dynamics - Plant Model and Data for Roll Rate over Aileron Surface Deflection

<table>
<thead>
<tr>
<th>Hover Longitudinal Dynamics: ( p/ \delta_a \ (\text{deg/s})/\text{deg} )</th>
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<tbody>
<tr>
<td>Model TF</td>
</tr>
<tr>
<td>----------</td>
</tr>
<tr>
<td>(-3.71s(s+0.412)(s-0.107)/(s+1.23)(s+0.102)(s^2-0.3737s+0.1998))</td>
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Figure 1.7 Hover Lateral Dynamics - Open Loop Poles/Zeros Distribution and Root Locus for Roll Rate over Aileron Surface Deflection
shows the Poles/Zeros Mapping and Root Locus. From the Root Locus, we know it is hard to make the system stable using classical controller design technique, but this issue can be perfectly solved using modern state space control theorem - Pole Placement Technique (Refer to Chapter Three for detail). Figure 1.8 indicates the bode plot. At frequency 0.465 rad/sec, the magnitude reaches its maximum point 13.4 dB. The minimum phase is 6.98°, which corresponds to frequency 0.205 rad/sec. Figure 1.9 gives system step response in Open and/or Closed Structure. To clearly observe step response, the time is extended to 30 seconds.

Figure 1.8 Hover Lateral Dynamics - Open Loop Bode Plotting for Roll Rate over Aileron Surface Deflection
It is well known that the plant RHP poles cause system to be unstable. However, the plant RHP zeros also play a significant role in system stabilization, since they correspond to the closed loop poles when the closed loop gain tends to be infinite.

Plant 04: Yaw Rate over Rudder Surface Deflection In Hover Lateral Dynamics

This plant is the only one in Hover Dynamics, which is internally stable and is able to perform excellent command tracking by only calibrating the gain. The plant model is the 1st order, and its information is given in Table 1.4.
Table 1.4 Hover Lateral Dynamics - Plant Model and Data for Yaw Rate over Rudder Surface Deflection

<table>
<thead>
<tr>
<th>Hover Longitudinal Dynamics: $\frac{r}{\delta_r}$ ((deg/s)/deg)</th>
<th>Model TF</th>
<th>Poles</th>
<th>Damping</th>
<th>Freq. (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{0.619}{s+0.102}$</td>
<td>-0.102</td>
<td>1.00</td>
<td>0.102</td>
<td></td>
</tr>
</tbody>
</table>

In fact, the yaw rate in Hover Lateral Dynamics not only responds to rudder surface deflection, but also to aileron surface deflection (Refer to plant 05 for detail). To more precisely analyze this coupled lateral dynamics, MIMO state space theorem should be applied.

![Part A - Poles/Zeros Map: $r/\delta_r$ in Hover Dynamics](image)

![Part B - Root Locus: $r/\delta_r$ in Hover Dynamics](image)

Figure 1.10 Hover Lateral Dynamics - Open Loop Poles/Zeros Distribution and Root Locus for Yaw Rate over Rudder Surface Deflection
Figure 1.11 Hover Lateral Dynamics – Open Loop Bode Plotting for Yaw Rate over Rudder Surface Deflection

Part A of Figure 1.10 shows there is only one pole located on the LHP, and Part B of Figure 1.10 points out that the trajectory of the Closed Loop system pole is always limited in the LHP to whatever the gain value is. The Bode Plot of Figure 1.11 gives that the Gain Margin is infinite and the Phase Margin is 99.5°. It also shows that the maximum magnitude is 15.6 dB in frequencies smaller than 0.02 rad/sec. Normally for classical controller design specification, only 50° Phase Margin is required, so this feedback system is robust in sense of Gain Margin and
Phase margin. Part B of Figure 1.12 gives the Closed Loop System step response. Both steady state error and setting time could be improved by increasing the Gain Factor. In this case, the simplest way to make the system meet design specifications is to first directly close the loop, and then employ a pre-filter to shape the time response and completely eliminate the steady-state error. A pre-filter is an effective component to improve system performance.

Figure 1.12 Hover Lateral Dynamics - Step Response in Open Loop and Closed Loop for Yaw Rate over Rudder Surface Deflection
Plant 05: Yaw Rate over Aileron Surface Deflection In Hover Lateral Dynamics

This is the coupled TF in Hover Lateral Dynamics. That is, the Aileron Surface Deflection interferes to Yaw Rate. It can be treated as a disturbance to Yaw Rate, or it can be analyzed using Multiple Input Multiple Output (MIMO) method in State Space sense.

Table 1.5 gives information for the TF. There exist one pair of complex poles in the Right half of the s-plane (RHP), so this TF is inherently unstable.

<table>
<thead>
<tr>
<th>Hover Longitudinal Dynamics: r/δa ((deg/s)/deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model TF</td>
</tr>
<tr>
<td>0.344s(s-0.345)(s^2 + 0.8454s + 0.2372)</td>
</tr>
<tr>
<td>(s+1.23)(s+0.102)(s^2 - 0.3737s + 0.1998)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

It is impossible to shift the 'running points' of Closed Loop Poles to the LHP by only adjusting the Gain Factor. As demonstrated on Part B of Figure 1.13, when the Gain Factor increases, the Closed Loop poles on Loci A will cross over the image-axis and move to the LHP; however, the pole on Locus B conversely crosses over the image-axis and
move to the RHP, which will make the system unstable. This is a good case to demonstrate that the plant RHP zeros significantly influence feedback system stability.

Figure 1.14 gives the Bode Plot of the TF, and Figure 1.15 shows the step responses of Open Loop and Closed Loop cases.

As we mentioned above, this TF is a model, indicating Aileron to Yaw Rate. In essence, the pilot would like to make attempts to eliminate the Yaw Rate response to Aileron deflection [08].

Figure 1.13 Hover Lateral Dynamics - Open Loop Poles/Zeros Distribution and Root Locus for Yaw Rate over Aileron Surface Deflection
Figure 1.14 Hover Lateral Dynamics - Open Loop Bode Plotting for Yaw Rate over Aileron Surface Deflection

Figure 1.15 Hover Lateral Dynamics - Step Response in Open Loop and Closed Loop for Yaw Rate over Aileron Surface Deflection
We would like to either design a controller to stabilize it and squelch the output to Yaw Rate, or to make the plant, Rudder Surface Deflection to Yaw Rate, robustness enough to withstand the effect of Aileron Deflection. To more precisely analyze this coupling problem, the modern state space multiple input single output (MISO) technique can be applied in this case to realize optimal performance in the sense of the whole hover lateral dynamic system.

**Plant 06: Roll Rate over Aileron Surface Deflection In Cruise Lateral Dynamics**

Although this is 4\textsuperscript{th} order plant, it is internally stable, which can be derived from Table 1.6 plant information and Figure 1.16, which gives Poles/Zeros Map and Root Locus. The unique feature of this plant is that there is a zero in the origin. In the sense of parametric uncertainty, it can spoil system robust stability. For instance, let's assume it deviates 10\% from the origin. Clearly, there is 50\% probability in which this zero is located on the right-hand of s-plane (RHP). As we know, the RHP zeros always cause trouble for system performance. In other words, the command tracking will be poor.
Table 1.6 Cruise Lateral Dynamics - Plant Model and Data for Roll Rate over Aileron Surface Deflection

<table>
<thead>
<tr>
<th>Model TF</th>
<th>Poles</th>
<th>Damping</th>
<th>Freq. (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.063</td>
<td>1.00</td>
<td>0.063</td>
</tr>
<tr>
<td></td>
<td>-1.09</td>
<td>1.00</td>
<td>1.09</td>
</tr>
<tr>
<td>-4.49s(s^2 + 1.183s + 3.572)</td>
<td>-0.392</td>
<td>0.248</td>
<td>1.58</td>
</tr>
<tr>
<td></td>
<td>(s+1.09)(s+0.063)(s^2 + 0.7837s + 2.496)</td>
<td>+1.53i</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.392</td>
<td>0.248</td>
<td>1.58</td>
</tr>
<tr>
<td></td>
<td>-1.53i</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Part A - Poles/Zeros Map: $p/\delta_a$ in Cruise Dynamics

Part B - Root Locus: $p/\delta_a$ in Cruise Dynamics

Figure 1.16 Cruise Lateral Dynamics - Open Loop Poles/Zeros Distribution and Root Locus for Roll Rate over Aileron Surface Deflection
Clearly, if the Gain Factor is inhibited in some proper range, the Root Loci can be kept on the LHP to achieve stable feedback system. Using 'sisotool' toolbox in MatLab, we figure out as long as the Gain Factor is restrained in 0 to 0.177, the feedback system is stable. Figure 1.17 shows the Bode Plot of this plant, and it gives the Gain Margin -15.0dB, Phase Margin 79.7°. The peak magnitude 15 dB happens at 0.291 rad/sec. The Gain Margin is negative, so the feedback plant is unstable if remaining the Gain Factor to be 1, which is bigger than 0.177.

Figure 1.17 Cruise Lateral Dynamics - Open Loop Bode Plotting for Roll Rate over Aileron Surface Deflection
The step responses for Open Loop and Closed Loop Plant are given in Figure 1.18. It is indicated that the performance is poor, although this plant is internally stable. Normally, the system time response can be described using Setting Time, Rising Time, Overshoot Ratio, and Steady State Error [03]. The Step Response steady-state value would be always zero, since there is a zero at the origin in the plant model, and this zero cannot be eliminated using the feedback configuration.

![Part A - Open-Loop Step Response: $p/a_\alpha$ in Cruise Dynamics](image1)

**Part A** - Open-Loop Step Response: $p/a_\alpha$ in Cruise Dynamics

- **From:** Aileron, Deg
- **Setting Time:** 64.3 Sec
- **Overshoot:** Inf
- **DC Gain:** 0

![Part B - Feedback Step Response: $p/a_\alpha$ in Cruise Dynamics](image2)

**Part B** - Feedback Step Response: $p/a_\alpha$ in Cruise Dynamics

- **From:** Aileron, Deg

Figure 1.18 Cruise Lateral Dynamics - Step Response in Open Loop and Closed Loop for Roll Rate over Aileron Surface Deflection
Plant 07: Sideslip over Rudder Surface Deflection In Cruise Lateral Dynamics

Sideslip is used to measure the Yaw movement with respect to the center of gravity. The pilot endeavors to control the Yaw movement by maneuvering the Rudder Surface Deflection. The plant model for Sideslip over Rudder Surface Deflection in Cruise Lateral Dynamics is shown in Table 1.7. It is internally stable system, too. Figure 1.19 gives the Poles/Zeros map and Root Locus. The pole at -0.063 may potentially cause trouble in stability due to parametric uncertainty since it is too near the origin and could potentially move to the right-hand side of the s-plane. Also, in a parametric uncertainty sense, the zero at 0.086 will cause trouble for system to follow command tracking.

Table 1.7 Cruise Lateral Dynamics - Plant Model and Data for Sideslip over Rudder Surface Deflection

<table>
<thead>
<tr>
<th>Cruise Lateral Dynamics: $\beta_{cg} / \delta_r$ (deg/deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model TF</td>
</tr>
<tr>
<td>$-0.051(s + 48)(s + 0.818)(s + 0.086)$</td>
</tr>
<tr>
<td>$\quad / (s + 1.09)(s + 0.063)(s^2 + 0.7837s + 2.496)$</td>
</tr>
<tr>
<td>Poles</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>-0.063</td>
</tr>
<tr>
<td>-1.09</td>
</tr>
<tr>
<td>-0.392</td>
</tr>
<tr>
<td>+1.53i</td>
</tr>
<tr>
<td>-0.392</td>
</tr>
<tr>
<td>-1.53i</td>
</tr>
</tbody>
</table>
Figure 1.19 Cruise Lateral Dynamics - Open Loop Poles/Zeros Distribution and Root Locus for Sideslip over Rudder Surface Deflection

Bode Plot of this plant is shown in Figure 1.20. The peak magnitude is 5.48 dB at the frequency 1.48 rad/sec, and minimum phase is 17.1° at frequency 7.86 rad/sec. The Gain Margin and Phase Margin are measured as 0.0 dB and -2.1°, respectively. The negative Phase margin indicates that the feedback plant is unstable. To stabilize it, the Gain Factor should be constrained at 0 to 0.995, as calculated using MatLab. Figure 1.21 shows the step response.
Figure 1.20 Cruise Lateral Dynamics - Open Loop Bode Plotting for Sideslip over Rudder Surface Deflection

Figure 1.21 Cruise Lateral Dynamics - Step Response in Open Loop and Closed Loop for Sideslip over Rudder Surface Deflection
Plant 08: Pitch Rate over Elevator Surface Deflection

In Cruise Longitudinal Dynamics

This is 2nd order plant, and it is internally stable. The plant model TF and data is listed on Table 1.8.

Table 1.8 Cruise Longitudinal Dynamics - Plant Model and Data for Pitch Rate over Elevator Surface Deflection

<table>
<thead>
<tr>
<th>Cruise Longitudinal Dynamics: $q/\delta_e$ ((deg/s)/deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model TF</td>
</tr>
<tr>
<td>---------------------------------------------------------</td>
</tr>
<tr>
<td>$-7.38(s + 0.89)$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

From the Root Locus plotting in part B of Figure 1.22, the Poles and Zero are all located on the LHP. However, when the Gain Factor is large enough, the Root Loci will cross over the image line and travel to the RHP. As a result, the feedback system is unstable if its eigenvalues are located in the RHP, as demonstrated on Figure 1.22. It is calculate that the feedback system stays stable if the Gain Factor is forced in the range of 0 to 0.293. In terms of system robust stability, a compensator should be designed so that the system remains stable even if the Gain Factor goes to infinite.
Figure 1.22 Cruise Longitudinal Dynamics - Open Loop Poles/Zeros Distribution and Root Locus for Pitch Rate over Elevator Surface Deflection

Figure 1.23 Cruise Longitudinal Dynamics - Open Loop Bode Plotting for Pitch Rate over Elevator Surface Deflection
Figure 1.24 Cruise Longitudinal Dynamics - Step Response in Open Loop and Closed Loop for Pitch Rate over Elevator Surface Deflection

The Bode Plot in Figure 1.23 gives the Gain Margin -4.0 dB, and Phase Margin -79.7°, which indicates that the feedback system is unstable when the Gain Factor remains as 1. It is also shown that the maximum magnitude is 11.5 dB, corresponding to frequency 1.91 rad/sec, which is near the plant natural frequency 2.02 rad/sec. Figure 1.24 shows the step responses for Open and Closed Loop Plant. Because the Gain Factor is out of the stable region, feedback system is unstable in a direct close loop.
Plant 09: Vertical Acceleration over Elevator Surface Deflection In Cruise Longitudinal Dynamics

This is the other coupled TF case for XV-15 Tilt Rotor Aircraft. In Cruise Longitudinal Dynamics, the Pitch Rate and Vertical Acceleration are dependent on the Elevator Surface Deflection, and they form a Single Input Multiple Output system (SIMO). Table 1.19, Figure 1.25, 1.26 and 1.27 record the plant dynamics. The feedback is stable under the gain range $0 \sim 43.4$. It is noticed that this plant has one zero in the RHP, too. As mentioned previously, it will degrade the system robust performance. However, this RHP zero location is far away the origin. This property is beneficial when the Loop-shaping technique is applied to design a controller using $H_{\infty}$ synthesis, as demonstrated in chapter 06.

Table 1.9 Cruise Longitudinal Dynamics - Plant Model and Data for Vertical Acceleration over Elevator Surface Deflection

<table>
<thead>
<tr>
<th>Cruise Longitudinal Dynamics: $a_z / \delta_e$ (g/deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Model TF</strong></td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>$-0.023(s+7.59)(s-6.7)$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$(s^2+2.167s+4.084)$</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Figure 1.25 Cruise Longitudinal Dynamics - Open Loop Poles/Zeros Distribution and Root Locus for Vertical Acceleration over Elevator Surface Deflection

Figure 1.26 Cruise Longitudinal Dynamics - Open Loop Bode Plotting for Vertical Acceleration over Elevator Surface Deflection
Figure 1.27 Cruise Longitudinal Dynamics - Step Response in Open Loop and Closed Loop for Vertical Acceleration over Elevator Surface Deflection

1.3 **MIMO State Space Denotation**

Whether in Hover or in Cruise operation mode, the system can be further divided into Longitudinal and Lateral Dynamic subsystems, and these two subsystems are completely decoupled. If using the state space theorem, the coupling functions can easily be expressed in each complete independent subsystem. In the view of MIMO analysis, we can manipulate them and classify the study in the following four subsystems:
• Hover Longitudinal Dynamics
• Hover Lateral Dynamics
• Cruise Longitudinal Dynamics
• Cruise Lateral Dynamics

The four subsystem configurations are shown in Figure 1.28 to Figure 1.31, respectively.

---

**Figure 1.28** Block Diagram for Hover Longitudinal Dynamics

---

**Figure 1.29** Block Diagram for Hover Lateral Dynamic System
In general, any LTI system can be depicted using state space form as the following:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]  

(EQ-1.1)

where \( x \), \( y \) and \( u \) denote state vector, output vector and input vector, respectively. \( A \), \( B \), \( C \) and \( D \) are constant matrices.
Assuming system initial state values to be zeros, then any LTI system can be expressed as:

\[ Y(s) = [C(SI - A)^{-1} B + D]U(s) \]  
(EQ-1.2)

In other words, any LTI system information can be specified using four matrices: \( A, B, C \) and \( D \). In the Robust Control Section of Matlab, three types of matrices are commonly used: Constant Matrix, System Matrix and Varying Matrix, each of which is derived from the matrix below:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

The four subsystems depicting XV-15 Tilt Rotor Aircraft are denoted in matrices form as the following:

1) Hover Longitudinal Dynamics in State Space Form

\[
\begin{bmatrix}
-0.8888 & 0.2903 & -0.4033 & -0.1858 & 0 & 2.0000 & 0 \\
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.2500 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.1050 & 0 & 0.0313 \\
-1.3300 & -0.3152 & 0.1831 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0329 & 0 & -0.0098
\end{bmatrix}
\]

2) Hover Lateral Dynamics in State Space Form
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
-0.1020 & 0 & 0 & 0 & 0 & 1.0000 & 0 \\
0 & -0.9583 & 0.1725 & -0.2192 & -0.1003 & 0 & 2.0000 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.2500 & 0 & 0 & 0 \\
0.6190 & 0.1720 & 0.0861 & -0.0094 & -0.0563 & 0 & 0 \\
0 & -1.8550 & -0.5658 & 0.0818 & 0 & 0 & 0
\end{bmatrix}
\]

3) Cruise Longitudinal Dynamics in State Space Form

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
-1.937 & -0.867 & -0.366 & -0.171 & 0 & 0 & 0 & 0 & 0 & 2.000 & 0 \\
4.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1.937 & -0.867 & -0.366 & -0.086 & 0 & 1.000 & 0 \\
0 & 0 & 0 & 0 & 4.000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2.000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.250 & 0 & 0 & 0 & 0 \\
-2.245 & -0.664 & -1.002 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.051 & -0.624 & -0.277 & -0.086 & 0 & 0
\end{bmatrix}
\]

4) Cruise Lateral Dynamics in State Space Form

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
-2.1665 & -2.0422 & 4.0000 \\
2.0000 & 0 & 0 \\
0 & 0 & 0 \\
-1.8450 & -0.8210 & 0 \\
0.0073 & 0.1579 & -0.0230
\end{bmatrix}
\]

**Remarks:** This subsystem is one input two output system, so the D is not square matrix.
2.1 **PID Controller Design**

In classical controller design, the controller types can be classified as below:

- Phase Lag
- Phase Lead
- Phase Lag/Lead
- PI
- PD
- PID

In general, the PI controller plays a similar role to the Phase Lag Filter for the purpose of reducing steady state error, and the PD controller is analogous to Phase Lead filter, which compensates Open Loop TF with additional positive phase angles to increase the bandwidth and the stability margin [03].

PID holds the Phase Lag/Lead properties, i.e., in the low spectrum of frequencies it enhances the low frequency gains, and in the high spectrum of frequencies, it provides positive phase angles, thus increasing the stability
margin. It consists of three terms: proportional term, integral term, and differential term. The proportional term is normally used to shorten the system transient response, but higher proportional gain often causes serious overshoot. The integral term is employed to reduce the steady state error. In aircraft control command tracking is always required, so a simple way to follow reference input is to add one integrator at each input channel [01]. To overcome the overshooting problem for light damped system and improve system speed response, the differential term can be applied.

The PID is usually expressed as:

\[ G_c = K_p + \frac{K_i}{s} + K_ds \]  \hspace{1cm} (EQ-2.1)

By simple conversion, (EQ-1.1) presents another form:

\[ G_c = \frac{K_p(s-z_1)(s-z_2)}{s} \]  \hspace{1cm} (EQ-2.2)

From (EQ-1.2) by observation, it is simply implemented in Root Locus by place one pole as integrator, and one pair of complex or two real zeros in the LHP. Matlab provides a good platform for controller design with Root Locus method using 'sisitool' command.
In the design of PID controller, normally pre-filter is requested to assist shaping system time response to meet desired performance. The standard block diagram for PID controller design is shown in Figure 2.1. The $R(s)$, $E(s)$, $D(s)$, and $Y(s)$ represent reference input, steady state error, disturbance, and measured output, respectively.

![Figure 2.1 Standard Block Diagram for PID Controller](image)

One widely used performance index is the Integral of Time multiplied by Absolute Error (ITAE) [11]. It is mathematically expressed as the following:

$$\text{ITAE} = \int_0^t t |e(t)| dt \quad \text{(EQ-2.3)}$$

Based on the ITAE performance criterion, the Closed Loop TF of any 2\textsuperscript{nd} order plant will take the special form as below [11]:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^3}{s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3} \quad \text{(EQ-2.4)}$$
The natural frequency $\omega_n$ is determined by the desired setting time $T_s$, and the damping ratio $\zeta$. The relationship is as below:

$$\omega_n = \frac{4}{T_s \zeta} \quad (EQ-2.5)$$

Assuming the plant is strictly proper 2\textsuperscript{nd} order system, the general form is below:

$$G(s) = \frac{c_i s + c_0}{s^2 + a s + b} \quad (EQ-2.6)$$

Applying Mason’s Law [03] to Block Diagram Figure 2.1, we obtain the following Closed Loop TF:

$$\frac{Y(s)}{R(s)} = \left( \frac{K_D c_1}{K_D c_1 + 1} \right) s^3 + \left( \frac{K_D c_0 + K_P c_1}{K_D c_1 + 1} \right) s^2 + \left( \frac{K_P c_1 + K_P c_0}{K_D c_1 + 1} \right) s + \left( \frac{K_I c_0}{K_D c_1 + 1} \right) G_p$$

$$\quad (EQ-2.7)$$

Comparing (EQ-2.6) with (EQ-2.4), we get the optimal PID controller coefficients by solving the following linear algebraic matrix equation:

$$\begin{bmatrix} c_1 & 0 & -1.75\omega_n c_1 \\ c_0 & c_1 & -2.15\omega_n^2 c_1 \\ 0 & c_0 & -\omega_n^3 c_1 \end{bmatrix} \begin{bmatrix} K_p \\ K_i \\ K_D \end{bmatrix} = \begin{bmatrix} 1.75\omega_n - a \\ 2.15\omega_n^2 - b \\ \omega_n^3 \end{bmatrix} \quad (EQ-2.8)$$
The pre-filter TF is also derived:

\[
G_p = \frac{\omega_n^3 \left( \frac{K_D c_1 + 1}{K_D c_1} \right)}{s^3 + \left( \frac{K_D c_0 + K_p c_1}{K_D c_1} \right) s^2 + \left( \frac{K_I c_1 + K_p c_0}{K_D c_1} \right) s + \left( \frac{K_I c_0}{K_D c_1} \right)}
\]

(EQ-2.9)

The above formula is in the condition of \( c_1 \neq 0 \). If \( c_1 = 0 \), the pre-filter can be simplified as below:

\[
G_p = \frac{\omega_n^3}{s^3 + \left( \frac{K_p}{K_D} \right) s + \frac{K_I}{K_D}}
\]

(EQ-2.10)

Case Study 01: G08 Pitch Rate over Elevator Surface Deflection in Cruise Longitudinal Dynamics

Method One: Applying ITAE Performance Index Criterion

In this case, the plant TF is predicted as the following:

\[
G(s) = \frac{-7.38s - 6.568}{s^2 + 2.167s + 4.084}
\]

(EQ-2.11)

The PID controller design procedure is listed as the following:

**Step One:** Determine \( \omega_n \)
We desire to have the setting time to be less than 0.5 seconds, and assume the nominal damping ratio $\zeta$ is 0.8. Substituting in (EQ-2.5), we obtain $\omega_n = 10(\text{rad/sec})$.

Conservatively, $\omega_n$ is selected to be 15.

Step Two: Find the optimal coefficients of PID controller according to the performance index ITAE.

The rest of the parameters are known by the comparison of (EQ-2.11) with (EQ-2.6). All the data needed to compute the PID coefficients is listed in table 2.1.

Table 2.1 Known Data to Compute PID

<table>
<thead>
<tr>
<th>Data Prepared to Compute PID Coefficients</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>2.167</td>
</tr>
<tr>
<td>$b$</td>
<td>4.084</td>
</tr>
<tr>
<td>$c_0$</td>
<td>-6.568</td>
</tr>
<tr>
<td>$c_1$</td>
<td>-7.380</td>
</tr>
<tr>
<td>$T_s$</td>
<td>0.500</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.800</td>
</tr>
<tr>
<td>$\omega_n$</td>
<td>15.00</td>
</tr>
</tbody>
</table>

Plugging the above data in the (EQ-2.8), we get PID controller:

$$G_c = 0.1760 + 0.4548/s + 0.1356s$$ (EQ-2.12)

The Pre-filter TF is also found by using (EQ-2.9) as the following:
The complete output over input TF is easily obtained as the following:

\[ \frac{Y(s)}{R(s)} = \frac{3375}{(s+10.62)(s^2 + 15.63s + 317.8)} \]  \hspace{1cm} (EQ-2.14)

In contrast with the Part A of Figure 1.24, Figure 2.2 shows quite obvious improvement for step response in this system.

**Figure 2.2 Step Response Using PID Controller for Pitch Rate over Elevator Surface Deflection in Cruise Longitudinal Dynamics**

- Setting Time: 0.50 Sec
- Overshoot: 1.92%
- DC Gain: 1.00
Method Two: Using Root Locus Algorithm and User Interface in the Matlab "sisotool" Tool Box.

Typing "sisotool(-G)" in Matlab command line, and placing one pole at the origin, one pair of complex zeros in the LHP in the sense of "trial and error" basis, we obtain the Root Locus as shown in Figure 2.3.

![Root Locus Placing PID Controller: \( \dot{q}/\dot{s} \) in Cruise Dynamics](image)

Figure 2.3 Root Locus Placing PID Controller for Pitch Rate over Elevator Surface Deflection in Cruise Longitudinal Dynamics

During the placement of one pair of complex zeros of the PID controller, the Closed Loop Step Response can be dynamically observed. However, compared with the method one, method two is tedious and less effective.
The optimal PID controller is finally found out as the following:

\[ G_e = 0.7289 + \frac{18.14}{s} + 113.9s \]  

(EQ-2.15)

The system Step Response is plotted in Figure 2.4, which is marked as 'Curve A'. There exists a deficiency in this curve: the initial output value is not zero. This means that there is a 'shock' at the instant \( t_{0+} \). To overcome this drawback, it is necessary to employ a Pre-filter to guarantee the smooth response.

Figure 2.4  Step Response With PID Controller, With/ Without Pre-filter for Pitch Rate Over Elevator Surface Deflection in Cruise Longitudinal Dynamics
The Pre-filter TF is tentatively chosen as the following:

\[ G_p = \frac{1}{0.1s+1} \]  
(EQ-2.16)

The complete output over input TF is also obtained as the following:

\[ \frac{Y(s)}{R(s)} = \frac{8.4324(s + 0.89)(s^2 + 24.89s + 156.3)}{(s + 10)(s + 0.8864)(s^2 + 21.19s + 132.3)} \]  
(EQ-2.17)

The Step Response with Pre-filter is also drawn in Figure 2.4 and marked as 'Curve B'. It is almost perfect!

Table 2.2 gives the comparison of the results from Method One and Method Two. It shows that the Interactive Root Locus Method is a bit better than ITAE Performance Index Technique. However, the ITAE Performance Index Technique is straight forward, and can be performed with computer programming, so it is a more effective method to design PID controller.

Table 2.2 Results for Comparison of Different PID Controller Design Method for Pitch Rate over Elevator Surface Deflection in Cruise Longitudinal Dynamics

<table>
<thead>
<tr>
<th></th>
<th>Original Plant</th>
<th>ITAE Index</th>
<th>Interactive Root Locus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Setting Time (Sec)</td>
<td>3.76</td>
<td>0.50</td>
<td>0.362</td>
</tr>
<tr>
<td>Overshoot (%)</td>
<td>76.4</td>
<td>1.91</td>
<td>0</td>
</tr>
<tr>
<td>Steady State Error (%)</td>
<td>261</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
2.2 **Command Tracking**

In aircraft controller design, the output must always explicitly follow the reference command signal. This is so called Command Tracking.

The TF of the continuous time control system shown in Figure 2.1 is given by Mason's Law as the following:

\[
\frac{Y(s)}{R(s)} = \frac{G_p G_c G}{1 + G_c G}
\]

(EQ-2.18)

In general, the above formula has the standard fractional polynomial form below:

\[
\frac{Y(s)}{R(s)} = \frac{b_m S^m + b_{m-1} S^{m-1} + \cdots + b_1 S + b_0}{S^n + a_{n-1} S^{n-1} + \cdots + a_1 S + a_0}
\]

(EQ-2.19)

To force the output to explicitly follow the reference command, \(b_0\) should be equal to \(a_0\). Generally, the condition:

\[
b_0 = a_0
\]

(EQ-2.20)

can be accomplished by selecting an appropriate Pre-filter.

In the case if \(G_p(s) = 1\), the Block Diagram in Figure 2.1 is revised to Figure 2.5.
The Open Loop TF is $L(s) = G(s)G_c(s)$, or in general, it takes the following form:

$$L(s) = \frac{K(\tau_0 s + 1)(\tau_1 s + 1) \cdots (\tau_m s + 1)}{s^N(\tau_1 s + 1)(\tau_2 s + 1) \cdots (\tau_n s + 1)}$$

(EQ-2.21)

The error signal is given by:

$$E(s) = \frac{1}{1 + L(s)} R(s)$$

(EQ-2.22)

To drive the system steady state error to zero, this system should be of type one ($N=1$), or higher [07]. In that way, of course, the tracking command requirement is achieved.

**Case Study 02: G02 Vertical Acceleration over Power Level Inching in Hover Longitudinal Dynamics**

The plant model is rewritten here:
This model depicts a 1st order system. It is a simple plant, but its time response is very poor (Refer to Figure 1.6 for detail). In fact, it is not a strictly proper system, and its steady state value would be always zero. Now we’ll design a controller to make the system robust and to realize good quality in performance.

**Step One:** Using Model Following Technique to get Compensator TF

For the 1st order system, it is known that the ideal model will take the form as following, according to ITAE performance index:

\[
\frac{Y(s)}{R(s)} = \frac{\omega_n}{s + \omega_n}
\]

(EQ-2.24)

The \(\omega_n\) is determined by (EQ-2.5). Here we let: \(\omega_n = 10\). In this way, the compensator is determined to be:

\[
G_c = \frac{s + 0.105}{s + 10}
\]

(EQ-2.25)

This is a Phase Lead Filter. As we know, a Phase Lead Filter can augment stable margin, and further more, it will improve system speed response. This would be helpful for
the original plant, since its Setting Time is 37.3 sec according to Figure 1.6.

**Step Two: Closing the Loop to form Robust Feedback Architecture**

As we know, feedback structure can reduce the sensitivity to disturbance. The Closed Loop TF is given as the following:

\[
T_1 = \frac{GG_c}{1 + GG_c} \frac{-0.009897s}{s+10.1} \tag{EQ-2.26}
\]

**Step Three: Employing Pre-filter For Command Tracking**

The Step Response for \( T_1 \) is plotted in Part A of Figure 2.6. Now the Setting Time is decreased to 0.39 Sec, but there is no improvement for steady state error, so a Pre-filter is required to reach this objective.

Let:

\[
G_pT_1 = \frac{10.1}{s+10.1} \tag{EQ-2.27}
\]

It should be noticed that the right hand of (EQ-2.27) is an explicitly ideal model. From (EQ-2.27), the \( G_p \) is derived as:

\[
G_p = \frac{-1020}{s} \tag{EQ-2.28}
\]
Step Four: Simulating Step Response for the system with Compensator and Pre-filter

The complete system TF with Compensator and Pre-filter is given as:

$$\frac{Y(s)}{R(s)} = \frac{10.1}{s + 10.1}$$  \hspace{1cm}  (EQ-2.29)

The system performance is now almost perfect, and its Step Response is shown in Part B of Figure 2.6.

---

**Figure 2.6** Step Response with Compensator for Vertical Acceleration over Power Level in Hover Longitudinal Dynamics
3.1 Multivariable System Stability

When system is higher than 2\textsuperscript{nd} order, and it is internally unstable, for instance the 4\textsuperscript{th} order plants in the Hover Dynamic System of Tilt Rotor Aircraft, it would be difficult to search a controller using classical controller design technique, such as 'Root Locus Design Technique'.

Normally, there are two concerns regarding a control system. First, system should be stable; secondly, system should meet desired performance. This chapter will deal with how to make the system stable.

Any Linear Time Invariant system (LTI) could be mathematically expressed in state space form:

\[ \dot{x} = Ax + Bu \]  
\[ y = Cx + Du \]

where \( x, \ y, \) and \( u \) denotes state, output, and input variable vectors, respectively; and \( x \in \mathbb{R}^n, \ y \in \mathbb{R}^p, \ u \in \mathbb{R}^m. \ A, \ B, \ C, \) and \( D \) are constant matrices for LTI system; and
A \in R^{nxn}, B \in R^{nxm}, C \in R^{pxn}, D \in R^{pxm}. In most practical cases, it is known that \( D = 0 \) (This assumption is set through out this paper). Such a system can be depicted as Block Diagram of Figure 3.1.

\[ x(t) = Me^h M^{-1} x(0) + \int_0^t Me^{j(t-\tau)} M^{-1} Bu(\tau) d\tau \]  

\[ y(t) = CM e^h M^{-1} x(0) + C \int_0^t Me^{j(t-\tau)} M^{-1} Bu(\tau) d\tau + Du(t) \]  

where \( M \) is the Modal Matrix of \( A \), and \( J \) is the Jordan Block of Matrix \( A \).

**Asymptotic Lyapunov Stability**

A LTI system is said to be asymptotically Lyapunov stable if all the eigenvalues of Matrix \( A \) are in the closed
left-half plant. In other words, this system is internally stable and boundary input boundary output stable if all the real parts of the eigenvalues of Matrix $A$ are negative [06]. The eigenvalues are computed using the formula below:

$$|\lambda I - A| = 0$$

(EQ-3.5)

**Controllability and Observability**

To design a controller using the state space technique, the prerequisite conditions in this system should be controllable and observable. Consider the LTI system described by (Eq-3.1) and (EQ-3.2), and define:

$$P = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

(EQ-3.6)

$$Q = \begin{bmatrix} C & CA & \cdots & CA^{n-1} \end{bmatrix}^T$$

(EQ-3.7)

For these matrices, if $\text{Rank}(P) = n$, the system is controllable; if $\text{Rank}(Q) = n$, the system is observable.

**State Feedback System**

The ideal control law is to feedback all the state variables. If this is permissible, system can be guaranteed stable, because we are allowed to place poles to any desired positions, and loops are closed simultaneously. The simplified architecture is exhibited in Figure 3.2.
The Matrix $K$ denotes the dynamic loop gain. From Figure 3.2, the input to the plant is given as:

$$u = r - Kx$$  \hspace{1cm} (EQ-3.8)

where $r$ is the reference command, and $x$ represents state variable vector. Substitute (EQ-3.8) into (EQ-3.1), we obtain:

$$\dot{x} = (A - BK)x + Br$$  \hspace{1cm} (EQ-3.9)

It is clear, due to the introduction of Loop Gain Matrix $K$, that the system eigenvalues are now determined by the matrix $(A - BK)$, not by matrix $A$. In conclusion, if some eigenvalues of matrix $A$ have positive real parts, which cause system unstable, this system is guaranteed stable through the proper selection of Loop Gain Matrix $K$, as long as this system is observable.
Linear Quadratic Regulator (LQR) with State Feedback

LQR is one solution to find the optimal Loop Gain Matrix $K$.

First, the Linear Quadratic Performance Index is set up as the following:

$$ J = \frac{1}{2} \int_0^\infty (x^T Q x + u^T R u) dt, $$  

(EQ-3.10)

where $J$ is the cost function based on Linear Quadratic Performance; and $Q$, $R$ are symmetric positive semidefinite weighting matrices, which determine the Closed Loop dynamics [01].

Secondly, solve the Algebraic Riccati Equation as the following to get a positive semidefinite matrix $P$.

$$ A^T P + PA + Q - PB R^{-1} B^T P = 0 $$  

(EQ-3.11)

Finally, the optimal Loop Gain Matrix $K$ is found as below, which is also called the Kalman Gain Matrix.

$$ K = R^{-1} B^T P $$  

(EQ-3.12)

If $(A, B)$ is controllable and $(\sqrt{Q}, A)$ is observable, the Closed Loop system depicted in Figure 3.2 is guaranteed to be stable [01].
3.2 **Pole Placement Technique**

The Root Locus Algorithm in classical controller design is under the category of Pole Placement Technique, because in the feedback system, we are allowed to choose the desired operation points, or the Closed Loop Poles' positions on any locus of the determined Root Loci. However, its contribution to stabilize an internally unstable system is constricted, because it can only close one loop at a time, and the poles cannot be arbitrarily placed, i.e., they should be on the Root Loci, which are solely determined by the locations of the Open Loop TF Zeros and Poles.

While the State feedback approach is a perfect solution for both system stability and system robustness, in this approach, the poles' locations of this state feedback system are allowed to be selected arbitrarily, and if this system is controllable, it is guaranteed to be stable. It is also known that each loop structure will reduce the sensitivity for disturbance, hence the Multiple Loop Structure is more robust than Single Loop One. However, not all state variables are measurable in real world. Fortunately, we can build an estimator to obtain the approximation of all state variables in dynamical sense.
Construction of Estimator Using Separation Principle

The block diagram in Figure 3.3 is utilized to perform the construction of state variable estimator, where the \( \hat{x} \) and \( \hat{y} \) stand for the estimation of state variable vector \( x \) and measurement output vector \( y \), respectively. The plant state space model is rewritten as:

\[
\dot{x} = Ax + Bu \quad \text{(EQ-3.13)}
\]
\[
y = Cx \quad \text{(EQ-3.14)}
\]

![Block Diagram for the Implementation of Estimator to fulfill the Pole Placement Technique](image)

Figure 3.3 Block Diagram for the Implementation of Estimator to fulfill the Pole Placement Technique

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The control law now is given as:

\[ u = r - K\hat{x} \quad (EQ-3.15) \]

Substitute (EQ-3.15) into (EQ-3.13), we get:

\[ \dot{x} = Ax - BK\hat{x} + Br \quad (EQ-3.16) \]

It is implied in Figure 3.3 that:

\[ \dot{x} = Ax + Bu + Ly \quad (EQ-3.17) \]

\[ \tilde{y} = y - \hat{y} \quad (EQ-3.18) \]

\[ \hat{y} = C\hat{x} \quad (EQ-3.19) \]

Now we manipulate for \( \hat{x} \) to make it more sensible:

\[ \dot{\hat{x}} = A\hat{x} + Bu + Ly \]
\[ = A\hat{x} + B(r - K\hat{x}) + L(\tilde{y} - \hat{y}) \quad (EQ-3.20) \]
\[ = (A - BK)\hat{x} + Br + L(Cx - C\hat{x}) \]
\[ = LCx + (A - BK - LC)\hat{x} + Br \]

Combine (EQ-3.16) with (EQ-3.20):

\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{bmatrix} =
\begin{bmatrix}
A & -BK \\
LC & A - BK - LC
\end{bmatrix}
\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix} + 
\begin{bmatrix}
B \\
B
\end{bmatrix} r
\quad (EQ-3.21)
\]

To evaluate the error between the state variable vector and its estimator, we denote:
Thus, we have:

\[
\ddot{x} = \dot{x} - \dot{\hat{x}}
\]

\[
= (Ax - BK\dot{x} + Br) - [LCx + (A - BK - LC)\dot{\hat{x}} + Br]
\]

\[
= (A - LC)(x - \dot{x})
\]

\[
= (A - LC)\ddot{x}
\]

(EQ-3.23)

It is also noticed that (EQ-3.16) can be expressed as:

\[
\dot{x} = Ax - BK\dot{x} + Br
\]

\[
= (A - BK)(x - \dot{x}) + Br
\]

\[
= (A - BK)\ddot{x} + Br
\]

(EQ-3.24)

Combining (EQ-3.23) with (EQ-3.24), we derive an interesting equation:

\[
\begin{bmatrix}
\ddot{x} \\
\dot{\hat{x}}
\end{bmatrix} =
\begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x}
\end{bmatrix}
+ 
\begin{bmatrix}
B \\
0
\end{bmatrix} r
\]

(EQ-3.25)

As we know, eigenvalues of a block triangular matrix are equal to the eigenvalues of the matrices along the diagonal blocks [10]. From (EQ-3.25), it is clear that the poles of state variable vector \(x\) are the eigenvalues of matrix \((A - BK)\), and the poles of error vector \(\dot{x}\) are the eigenvalues of matrix \((A - LC)\), which are also the poles of the state estimator. If the minimum absolute value of the
negative real parts of eigenvalues of matrix \((A - LC)\) is much bigger than the maximum absolute value of the negative real parts of the eigenvalues of matrix \((A - BK)\), in Dynamical sense, the error vector \(\dot{x}\) will elapse fast enough in the time domain to ensure that the estimative vector \(\hat{x}\) could be accurate enough to approximate the state variable vector \(x\).

In another point of view, the (EQ-3.25) implies that we can design estimator Gain Matrix \(L\) and Loop Gain Matrix \(K\), separately, as long as the \((A, B)\) is controllable, and \((C, A)\) is observable. Thus the Closed Loop Poles are the union of the independently placed poles of state Estimator and Loop Gain. This property is well known as the Separation Principle.

One effective method to compute the Loop Gain Matrix \(K\) and the estimator Gain Matrix \(L\) is Ackerman's Formula.

When desired poles are assigned for Loop Gain Matrix \(K\), the corresponding characteristic equation is also determined as the following:

\[
\Delta(s) = s^n + \alpha_s s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n
\]  
(EQ-3.26)

Substitute \(s\) with \(A\), we get:

\[
\Delta(A) = A^n + \alpha_s A^{n-1} + \cdots + \alpha_{n-1}A + \alpha_nI
\]  
(EQ-3.27)
The Loop Gain Matrix $K$ is then determined by [10]:

$$K = [0 \cdots 0 I]P^{-1}\Delta(A) \quad (EQ-3.28)$$

where $P$ and $\Delta(A)$ are determined by (EQ-3.6) and (EQ-3.27), respectively.

Estimator Gain Matrix $L$ can be determined using the same method as above.

**Case Study 03: G03 Roll Rate over Aileron Surface Deflection in Hover Longitudinal Dynamics**

This is a 4th order internally unstable system, and its plant is modeled as the following:

$$G = \frac{-3.71s(s+0.412)(s-0.107)}{(s+1.23)(s+0.102)(s^2-0.3737s+0.1998)} \quad (EQ-3.29)$$

Now we'll apply Pole Placement Technique to design a Regulator to make system stable.

We'll use the simplified Block Diagram as shown in Figure 3.4 for the convenience of setting up the Regulator design procedure by applying the Pole Placement Technique. To make the design easier, we'll not consider the reference command during the Regulator design. When the desired Regulator is found, the reference command can be put back.
Figure 3.4 Simplified Block Diagram for Case Study 03

Step One: Transfer Plant Model from Laplace form to State Space form using Matlab Command: `ss(sys)`

\[
\dot{x} = \begin{bmatrix}
-0.9583 & 0.1725 & -0.4385 & -0.2005 \\
1 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 \\
0 & 0 & 0.25 & 0
\end{bmatrix} x + \begin{bmatrix} 2 \end{bmatrix} u \tag{EQ-3.30}
\]

\[
y = \begin{bmatrix}
-1.855 & -0.5658 & 0.1636 & 0
\end{bmatrix} x \tag{EQ-3.31}
\]

Step Two: Verify whether the System is Controllable and/or Observable

Use (EQ-3.6) and (EQ-3.7) to compute matrix \( P \) and \( Q \) as the following:

\[
P = \begin{bmatrix}
2 & -1.9166 & 2.1817 & -2.8598 \\
0 & 2 & -1.9166 & 2.1817 \\
0 & 0 & 1 & -0.9583 \\
0 & 0 & 0 & 0.25
\end{bmatrix}
\]
It is found that \( \text{rank}(P) = 4 \), \( \text{rank}(Q) = 4 \). We know that this is a 4\(^{th}\) order system, so this system is controllable and observable, and Pole Placement Technique is qualified to apply.

**Step Three:** Assign Desired Poles separately to Design Loop Gain Matrix \( K \) and Estimator Gain Matrix \( L \)

Let Poles for Estimator Gain Matrix \( L \) be 3 times faster than that for Loop Gain Matrix \( K \), as shown:

\[
p = [-0.4000 \ -0.4080 \ -0.3464 + 0.2000i \ -0.3464 - 0.2000i]
\]

\[
q = [-1.2000 \ -1.2080 \ -0.9672 + 0.6000i \ -0.9672 - 0.6000i]
\]

Now let's compute Loop Gain Matrix \( K \) using Ackerman's Formula. According to the assigned poles \( p \) as above, the corresponding characteristic equation is given as:

\[
\Delta(s) = s^4 + 1.5008s^3 + 0.8830s^2 + 0.2423s + 0.0261
\]

Substitute \( s \) with matrix \( A \), which is given in (EQ-3.30), and then compute it to get:
\[ \Delta(A) = \begin{bmatrix}
0.3546 & -0.1994 & 0.1471 & 0.0795 \\
-0.3966 & -0.0255 & -0.2621 & -0.1074 \\
0.2678 & 0.0583 & -0.1179 & -0.0544 \\
0.0678 & 0.1319 & 0.0058 & 0.0010
\end{bmatrix} \]

Hence, the Loop Gain Matrix \( K \) is found as the following:

\[ K = [0 \ 0 \ 0 \ 1] P^{-1} \Delta(A) \]

\[ = [0 \ 0 \ 0 \ 1] \begin{bmatrix}
2 & -1.9166 & 2.1817 & -2.8598 \\
0 & 2 & -1.9166 & 2.1817 \\
0 & 0 & 1 & -0.9583 \\
0 & 0 & 0 & 0.25
\end{bmatrix}^{-1} \begin{bmatrix}
0.3546 & -0.1994 & 0.1471 & 0.0795 \\
-0.3966 & -0.0255 & -0.2621 & -0.1074 \\
0.2678 & 0.0583 & -0.1179 & -0.0544 \\
0.0678 & 0.1319 & 0.0058 & 0.0010
\end{bmatrix} \]

\[ = [0.2712 \ 0.5277 \ 0.0231 \ 0.0042] \]

The Estimator Gain Matrix \( L \) could be acquired the same way:

\[ L = [-0.4566 \ 4.1524 \ 29.8767 \ -49.5358]^T \]

**Step Four:** Construct Estimator and Regulator

The Estimator is built using the structure shown in Figure 3.3, and the Regulator's formation is given in Figure 3.4. The Estimator and Regulator are easily obtained using Matlab Command: \texttt{estim(sys, L)} and \texttt{reg(sys, K, L)}. The Regulator TF Model is returned from the Matlab Code computation, and shown as the following:
\[ TF(\text{Regulator}) = \frac{-2.5505 (s+1.23)(s-0.8363)(s+0.287)}{(s+0.412)(s+0.2839)(s^2+4.189s+16.72)} \quad (\text{EQ-3.32}) \]

The Regulator, or Controller is a 4\textsuperscript{th} order system, the same order as the original plant model, and it is an internally stable Controller.

**Step Five:** Form the Complete System and Evaluate its Performance in Time Domain

The Conventional Feedback Structure, as shown in Figure 3.5, is our favorite. To restore this formation, let:

\[ G_c = -TF(\text{Regulator}) \]
\[ = \frac{-2.5505 (s+1.23)(s-0.8363)(s+0.287)}{(s+0.412)(s+0.2839)(s^2+4.189s+16.72)} \quad (\text{EQ-3.33}) \]

Thus, the Closed Loop System TF Model is acquired as the following:

![Figure 3.5 Conventional Feedback Architecture with Controller and Pre-filter for SISO System](image)
It is observed that the poles of this Closed Loop System are explicitly the union of the assigned poles, by which we separately design the Loop Gain Matrix $K$, and the Estimator Gain Matrix $L$.

The step response for this Closed Loop System is shown in Figure 3.6 as Curve A. Clearly, this system is stable, although its performance is not the best. However, since this system is stable, it is possible for us to make use of any classical controller design technique to further improve its performance. For instance, we can utilize a Pre-filter to force it to follow the reference command. The Pre-filter is just an integrator, and its TF takes the following form:

$$G_p = \frac{-0.4}{s}$$  \hspace{1cm} (EQ-3.35)

The whole system TF model is given by:

$$T_f = \frac{3.785 \, s \, (s+1.23) \, (s+0.412) \, (s+0.287) \, (s-0.8363) \, (s-0.107)}{s \, (s+1.2) \, (s+1.208) \, (s+0.4) \, (s+0.408) \, (s^2 + 0.6928s + 0.16) \, (s^2 + 1.934s + 1.295)}$$  \hspace{1cm} (EQ-3.36)
Figure 3.6 Step Response with Controller for Roll Rate over Aileron Surface Deflection in Hover Lateral Dynamics

Curve B in Figure 3.6 is the step response of the original Plant with Controller and Pre-filter. Now it exactly meets the Command Tracking, although the setting time and the overshoot need to be further improved.

The advantage of Pole Placement Technique is that it can stabilize any internally unstable system, as long as this system is controllable and observable. The drawback of this controller design technique is it is hard to give good quality in performance. Basically, the design is a 'trial and error' procedure.
Chapter Four

Dynamic Uncertainty Modeling

It is known that any system and the environment outside it are not perfect in real world. When a system is mathematically depicted with a model, it is under certain conditions. When conditions change, whether they are internal or external, the nominal model will no more precisely describe the system. To design a controller working in these conditions, it is necessary to augment the model of this system to express the possible deviations from the nominal model. This is called plant or model uncertainty. Generally, the objective to design a controller is to make sure that when the system work with the designed controller, it should be robust in performance to meet the desired requirement, and be robust in stability to withstand external disturbance and/or internal parametric deviation. To analyze system robustness in performance and stability, it is necessary to mathematically effectively deal with uncertainty. The prevailing method is to reconstruct system into PΔK structure using Linear Fractional Transformation (LFT)
approach, where uncertainty is structured at the system level.

### 4.1 Classification of Uncertainty

In general, uncertainty sources take the following forms:

- Sensor Measurement Noise and Environment Disturbance
- Plant Mathematically Modeling Error
- Unmodeled Nonlinear Distortion

From the structural point of view, uncertainty can be classified into two types: structured uncertainty and unstructured uncertainty.

**Unstructured Uncertainty:**

Unstructured Uncertainty, also referred to model uncertainty, usually represents frequency dependent elements. The following is a list of unstructured uncertainty cases:

- Physical actuator uncertainty
- Unmodeled modes in the high frequency dynamics
- Environment disturbance to plant in the low frequency range, for instance, gust disturbance in aircraft control
Mathematically, unstructured uncertainty can be merged into the nominal plant with additive and/or multiplicative models. Figure 4.1 gives the corresponding Block Diagram.

Part A: Additive Uncertainty  
Part B: Multiplicative Uncertainty

Figure 4.1 Block Diagram for Modeling Unstructured Uncertainty

Structured Uncertainty:

Structured Uncertainty, also named Parametric Uncertainty, represents parametric variations in the plant dynamics. It includes the following:

- Uncertainties in specific poles and/or zeros of the plant TF
- Uncertainties in specific loop gains and time constants
- Uncertain time delay

4.2 Structured Uncertainty and LFT Approach

To efficiently analyze system with uncertainty, it is a good idea to make uncertainty structured at the system
LFT approach provides a professional method to represent uncertainty. With LFT method, each uncertainty source in control system is separated within its own perturbation, and each perturbation corresponds to a delta block in the Block Diagram of the complete system model.

**LFT Approach in Uncertainty System**

The structure in Figure 4.2 is used to analyze system robustness with the perturbation of model uncertainty and/or parametric uncertainty, where $\Delta$ represents uncertainty, and $M_{11}$, $M_{12}$, $M_{21}$, $M_{22}$ are sub-matrices.

![Diagram](image)

**Part A: Upper Loop Structure**

**Part B: Lower Loop Structure**

Figure 4.2 Uncertainty Depicted in $\Delta M$ Module Using LFT Approach

In the case of part A of Figure 4.2, the following equations is straightforwardly established:

\[ w = \Delta z \quad \text{(EQ-4.1)} \]

\[
\begin{bmatrix}
  z \\
  y
\end{bmatrix} =
\begin{bmatrix}
  M_{11} & M_{12} \\
  M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
  w \\
  u
\end{bmatrix} \quad \text{(EQ-4.2)}
\]
We'll derive the TF with $y$ over $u$. From (EQ-4.2) and with (EQ-4.1), the $z$ is shown:

$$z = M_{11}w + M_{12}u$$
$$= M_{11}\Delta z + M_{12}u$$  \hspace{1cm} (EQ-4.3)

Solve $z$ from (EQ-4.3), we get:

$$z = (I - M_{11}\Delta)^{-1}M_{12}u$$  \hspace{1cm} (EQ-4.4)

where $I$ is identification matrix. Once again use (EQ-4.2), and utilize (EQ-4.1), (EQ-4.4): 

$$y = M_{21}w + M_{22}u$$
$$= M_{21}\Delta z + M_{22}u$$
$$= M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}u + M_{22}u$$  \hspace{1cm} (EQ-4.5)

$$= \left\{ M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \right\} u$$

Here we define:

$$F_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$$  \hspace{1cm} (EQ-4.6)

where $F_u(M, \Delta)$ represents transfer function in the condition that $M$ is upper looped with uncertainty $\Delta$, and $M$ is a matrix, which represents the augmented plant model. It is defined as:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$  \hspace{1cm} (EQ-4.7)
Thus, we have:

\[ y = F_u(M, \Delta)u \]  (EQ-4.8)

If the structure of part B in Figure 4.2 is applied, the above results need just a bit of modification to fit this structure. The counterpart results for it are given as the following:

\[ F_L(M, \Delta) = M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21} \]  (EQ-4.9)

\[ y = F_L(M, \Delta)u \]  (EQ-4.10)

**Parametric Uncertainty Expression Using LFT**

Now we can describe parametric uncertainty in sense of LFT. Assume that a parameter \( c \) is variable, but it is constrained in some range. This can be mathematically expressed as:

\[ c = c_1 + c_2\Delta_c \]  (EQ-4.11)

where \( \Delta_c \in [-1, +1] \), and \( c_1, c_2 \) are constant scalars with \( c_1, c_2 \in \mathbb{R} \). Let’s say, we want express it as the structure of Part A in Figure 4.2. Rewrite (EQ-4.11) as the following:

\[ c = c_1 + c_2\Delta_c(1 - 0\Delta_c)I \]  (EQ-4.12)
Compare (EQ-4.12) with (EQ-4.6), we get: $M_{11} = 0$, $M_{12} = 1$, $M_{21} = c_2$, and $M_{22} = c_1$, which is shown in Figure 4.3.

![Figure 4.3](image)

**Figure 4.3 Equivalent Block Diagram Representation of Uncertain Parameter**

Sometimes, the parameter is presented in the reciprocal form in System Block Diagram. We can make some manipulation to obtain the corresponding $M$ matrix, as shown the following:

\[
\frac{1}{c} = \frac{1}{c_1 + c_2 \Delta_c} = \frac{1}{c_1} \left( 1 + \frac{c_2 \Delta_c}{c_1} \right) \left( 1 + \frac{c_2 \Delta_c}{c_1} \right)^{-1} \left( \frac{1}{c_1} \right).
\]  

(EQ-4.13)
Compare (EQ-4.13) with (EQ-4.6), we have:

\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} -\frac{c_2}{c_1} & 1 \\ -\frac{c_2}{c_1} & 1 \end{bmatrix}
\]  

(EQ-4.14)

The results are shown in Figure 4.4, where \( \Delta_c \in [-1, +1] \).

Figure 4.4 Equivalent Block Diagram Representation of the Inversive Form of Uncertain Parameter

Now we use the above outcomes to analyze a general first order system, then form corresponding system level uncertainty structure.

Let’s assume:

\[
\frac{y(s)}{u(s)} = \frac{K}{\tau s + 1}
\]  

(EQ-4.15)

To form Block Diagram using Mason’s Low, change (EQ-4.15) to the following form:
\[
\frac{y(s)}{u(s)} = \frac{K \left( \frac{1}{\tau} \right) s^{-1}}{1 + \left( \frac{1}{\tau} \right) s^{-1}}
\]  

(EQ-4.16)

The system depicted by (EQ-4.16) is easily illustrated in Block Diagram according to Mason's Gain Formula, which is shown in Figure 4.5.

![Block Diagram for General First Order System](image)

Figure 4.5 Block Diagram for General First Order System

To simulate a more realistic plant, we assume the DC Gain \( K \) and the Time Constant \( \tau \) to be functions of various constraints. We can express these variations as the following:

\[
K = b_1 + b_2 \Delta_1
\]  

(EQ-4.17)

\[
\tau = a_1 + a_2 \Delta_2
\]  

(EQ-4.18)

where \( \Delta_1, \Delta_2 \in [-1, +1] \). Now we combine Figure (4.3), Figure (4.4), and Figure (4.5), to represent the perturbation of parameter DC Gain \( K \) and Time Constant \( \tau \). The results are shown in Figure 4.6.
Figure 4.6 Block Diagram for Separating Perturbation Structure with Plant Structure

From Figure 4.6, four equations are obtained as the following, two of which are for the two two-port blocks:

\[
\begin{bmatrix}
  z_1 \\
r_1
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  b_2 & b_1
\end{bmatrix}
\begin{bmatrix}
w_1 \\
u
\end{bmatrix} \quad \text{(EQ-4.19)}
\]

\[
\begin{bmatrix}
  z_2 \\
r_2
\end{bmatrix} =
\begin{bmatrix}
  \frac{-a_2}{a_1} & \frac{1}{a_1} \\
  \frac{-a_2}{a_1} & \frac{1}{a_1}
\end{bmatrix}
\begin{bmatrix}
w_2 \\
e
\end{bmatrix} \quad \text{(EQ-4.20)}
\]

\[e = r_1 - y \quad \text{(EQ-4.21)}\]

\[y = r_2 s^{-1} \quad \text{(EQ-4.22)}\]

Separate (EQ-4.19) into two equations:

\[z_1 = u \quad \text{(EQ-4.23)}\]

\[r_1 = b_2 w_1 + b_1 u \quad \text{(EQ-4.24)}\]
From (EQ-4.20), with (EQ-4.21), (EQ-4.22) and (EQ-4.24), the equation related to $r_2$ is obtained:

$$r_2 = -\frac{a_2}{a_1}w_2 + \frac{1}{a_1}e$$

$$= -\frac{a_2}{a_1}w_2 + \frac{1}{a_1}(r_1 - y)$$

$$= -\frac{a_2}{a_1}w_2 + \frac{1}{a_1}(b_2w_1 + b_1u - r_s^{-1})$$

$$= \frac{b_2}{a_1}w_1 - \frac{a_2}{a_1}w_2 + \frac{b_1}{a_1}u - \frac{r_s}{a_1}s \quad \text{(EQ-4.25)}$$

Solving $r_2$ from (EQ-4.25):

$$r_2 = \frac{b_2s}{a_1s+1}w_1 - \frac{a_2s}{a_1s+1}w_2 + \frac{b_1s}{a_1s+1}u \quad \text{(EQ-4.26)}$$

From (EQ-4.20), the output $z_2$ is found:

$$z_2 = r_2$$

$$= \frac{b_2s}{a_1s+1}w_1 - \frac{a_2s}{a_1s+1}w_2 + \frac{b_1s}{a_1s+1}u \quad \text{(EQ-4.27)}$$

Substitute (EQ-4.25) into (EQ-4.22), the $y$ is also acquired as the following:

$$y = \frac{b_2}{a_1s+1}w_1 - \frac{a_2}{a_1s+1}w_2 + \frac{b_1}{a_1s+1}u \quad \text{(EQ-4.28)}$$

Combine (EQ-4.23), (EQ-4.27) and (EQ-4.28) to form matrix equation:
\[
\begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & 1 \\
  \frac{b_2 s}{a_1 s + 1} & \frac{a_1 s}{a_2} & \frac{b_2 s}{a_1 s + 1} \\
  \frac{b_2}{a_1 s + 1} & \frac{a_2}{a_1 s + 1} & \frac{b_2 s}{a_1 s + 1}
\end{bmatrix}
\begin{bmatrix}
  w_1 \\
  w_2 \\
  u
\end{bmatrix}
\]  
(EQ-4.29)

To form the uncertainty diagonal block, which is an efficient method to process matrix operation when there are many zero elements in it, let's define:

\[
z = \begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix}
\]  
(EQ-4.30)

\[
w = \begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}
\]  
(EQ-4.31)

From Figure 4.6, list out two uncertainty blocks' equations:

\[
w_i = \Delta_1 z_1
\]  
(EQ-4.32)

\[
w_2 = \Delta_2 z_2
\]  
(EQ-4.33)

Express (EQ-4.32) and (EQ-4.33) as a diagonal matrix equation:

\[
\begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix} = \begin{bmatrix}
  \Delta_1 & 0 \\
  0 & \Delta_2
\end{bmatrix}
\begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix}
\]  
(EQ-4.34)
The two equations (EQ-4.29) and (EQ-4.34) give a full depiction of system model with completely separated and system level structured uncertainty. To simplify these results, let's define:

\[
\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \quad (EQ-4.35)
\]

Thus, the general first order system with two uncertainty parameters can be mathematically express in equations as the following:

\[
\begin{align*}
\begin{cases}
w = \Delta z \\
z = M \begin{bmatrix} w \\ u \end{bmatrix}
\end{cases}
\end{align*} \quad (EQ-4.36)
\]

where \( w, z, \) and \( \Delta \) are defined by (EQ-4.30), (EQ-4.31) and (EQ-4.35), respectively; and \( M \) is given by:

\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \\
\begin{bmatrix}
0 & 0 & 1 \\
\frac{b_2 s}{a_i s + 1} & \frac{a_i s}{a_i s + 1} & \frac{b_2 s}{a_i s + 1} \\
\frac{b_1}{a_i s + 1} & \frac{a_i s + 1}{a_i s + 1} & \frac{b_1}{a_i s + 1}
\end{bmatrix} \quad (EQ-4.37)
\]

The simplified Block Diagram is shown in Figure 4.7.
Case Study 04: G04 Uncertainty Analysis for Yaw Rate over Rudder Surface Deflection in Hover Lateral Dynamics

The Plant nominal model is given as:

\[
G(s) = \frac{0.619}{s+0.102} \quad (EQ-4.38)
\]

To make the above TF meaningful in physical sense, we convert it to the following:

\[
G(s) = \frac{6.069}{9.804s+1} \quad (EQ-4.39)
\]

This implies that the nominal DC Gain and Time Constant of this system are 6.069 and 9.08, respectively. Now we assume these two parameters have 10% variation around their nominal values. The deviations of these two parameters can be expressed as:
\[
K = 6.069 + 0.61\Delta_1,
\]
\[
\tau = 9.804 + 0.98\Delta_2.
\]

where \(\Delta_1, \Delta_2 \in [-1, +1]\).

Thus, the coefficients in (EQ-4.17) and (EQ-4.18) are determined:

\[
b_1 = 6.069, \ b_2 = 0.61
\]
\[
a_1 = 9.804, \ a_2 = 0.98
\]

Substituting these data into (EQ-4.37), we can determine the matrix \(M\):

\[
M = \begin{bmatrix}
0 & 0 & 1 \\
0.61s & 0.98s & 6.069s \\
\frac{9.804s + 1}{9.804s + 1} & \frac{9.804s + 1}{9.804s + 1} & 6.069 \\
\frac{0.61}{9.804s + 1} & \frac{0.98}{9.804s + 1} & \frac{6.069}{9.804s + 1}
\end{bmatrix}
\]

The data for \(M\Delta\) structure is ready, and this is necessary preparation for robust controller design.
Chapter Five

Robust Performance And Stability

When designing a controller, we need to set up certain objectives. Above all, desired performance and robust stability in the sense of withstand model and parametric uncertainty are benchmarks in control systems.

5.1 Norms

To analyze a system quantitatively, the concept norm is used to depict the 'size' of the system. In this section, the norms for vectors, signals, matrices, and systems, will be discussed. How to compute the system norms $H_2$ and $H_\infty$, which are fundamental to measure robust system, are also introduced.

**Vector Norm:**

The 2-norm, or Euclidean norm of vector $x$, is defined as:

$$
\|x\|_2 = \sqrt{\langle x^*, x \rangle} \\
= \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}
$$

(EQ-5.1)
where \( x' \) denotes the conjugate transpose of vector \( x \), and \( x \in \mathbb{C}^n \).

**Matrix Norm:**

The 2-norm, or spectral norm of matrix \( A, A \in \mathbb{C}^{m \times n} \), is defined as its maximum singular value. That is:

\[
\|A\|_2 = \sigma_{\text{max}}(A) = \sqrt{\lambda_{\text{max}}(A^*A)}
\]

(EQ-5.2)

where \( A^* \), \( A_{\text{max}} \) denote the conjugate transpose of matrix \( A \) and the maximum eigenvalue of matrix \( (A^*A) \), respectively.

Assume \( x, x \in \mathbb{R}^n \) is an \( n \)-dimensional real vector, then the following property exist for matrix norm [06]:

\[
\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|Ax\|_2
\]

(EQ-5.3)

The physical meaning of (EQ-5.3) is that the 2-norm of matrix represents the maximum gain when it operates the input vector \( x \) to the output vector \( Ax \) in a 2-norm sense.

**Finite Energy Signal Norm:**

The 2-norm of a signal \( x(t) \) is the square root of its total energy over \(-\infty < t < +\infty\) and is defined as:
\[ \|x(t)\|_2 = \left( \int_{-\infty}^{\infty} |x(t)|^2 \, dt \right)^{\frac{1}{2}} \]  
\hspace{1cm} (EQ-5.4)\

Sometimes it is inconvenient to compute the signal 2-norm in the time domain. In this situation, Parseval's theorem can be applied and the 2-norm computation in the time domain is converted into frequency domain:

**Theorem 5.1  (Parseval's Theorem)**

\[ \|x(t)\|_2 = \left( \int_{-\infty}^{\infty} |x(t)|^2 \, dt \right)^{\frac{1}{2}} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{x}(j\omega)|^2 \, d\omega \right)^{\frac{1}{2}} \]
\hspace{1cm} (EQ-5.5)

where \( \hat{x}(j\omega) \) denotes the Fourier Transform of the signal \( x(t) \).

**System Norms:**

The 2-norm and the infinite-norm of systems are often used to measure the system performance. They are symbolized as \( H_2 \) and \( H_\infty \), respectively.

\( H_2 \) Norm of a system with TF \( G(s) \) is defined as the following:

\[ \|G\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left[ G(j\omega)^* G(j\omega) \right] d\omega \right)^{\frac{1}{2}} \]
\hspace{1cm} (EQ-5.6)

where \( G(j\omega)^* \) is the conjugate transpose of \( G(j\omega) \).
Let’s assume that a system is depicted as \((A, B, C, D)\) in the state space format, and it is assumed stable and strictly proper. A practical method to compute \(H_2\) Norm is given as follows:

- Define the controllability grammian matrix \(L\)

\[
L = \int_{0}^{\infty} e^{At}BB^t e^{A^t} dt \tag{EQ-5.7}
\]

- Solve \(L\) from the Lyapunov equation

\[
AL + LA^T + BB^T = 0 \tag{EQ-5.8}
\]

- Compute \(H_2\)

\[
\|G\|_{2} = \left[ \text{trace}(CLC^T) \right]^{\frac{1}{2}} \tag{EQ-5.9}
\]

The \(H_{\infty}\) Norm of a system with TF \(G(s)\) is defined as the following:

\[
\|G\|_{\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}} \left[ G(j\omega) \right] \tag{EQ-5.10}
\]

where \(\sigma_{\text{max}}\) is the maximum singular value of matrix \(G(j\omega)\). In words, the \(H_{\infty}\) Norm is the maximum gain of the frequency response of the system.
To compute $H_\infty$ Norm, we introduce the following theorem [05]:

**Theorem 5.2:**

For a system depicted as $(A, B, C, D)$ in the state space, which is assumed stable and proper, define a $2n \times 2n$ Hamiltonian matrix as:

$$J = \begin{bmatrix} A & \gamma^{-2}BB^r \\ -C^TC & -A^r \end{bmatrix}$$ (EQ-5.11)

$\|H\|_\infty < \gamma$ iff $J$ has no eigenvalues on the image-axis.

Unlike the computation of $H_2$ Norm, which can be found one time straightforwardly, the $H_\infty$ Norm of a system can only be found using iterative procedure. The following shows the $H_\infty$ Norm computation procedure:

**Step One:** given a large positive value $\gamma_0$ (let’s say: $\gamma_0=100$), test if $\|H\|_\infty < \gamma_0$, or equivalently, $\|\gamma_0^{-1}H\|_\infty < 1$. This is done by checking whether $J(\gamma_0)$ has no eigenvalues on the image-axis, which is given as:

$$J(\gamma_0) = \begin{bmatrix} A & \gamma_0^{-2}BB^r \\ -C^TC & -A^r \end{bmatrix}$$ (EQ-5.12)
Step Two: if $J(\gamma_0)$ has no imaginary eigenvalues, select $\gamma_1 = (1/2)\gamma_0$, check condition again.

Step Three: if $J(\gamma_1)$ has imaginary eigenvalues, then it is supposed to select $\gamma_2 = (1/2)(\gamma_0 + \gamma_1)$; otherwise, we should select $\gamma_2 = (1/2)\gamma_1$. The iterations are continued according to the same criterion for gamma value selection, until two consecutive values of gamma representing lower and upper bounds on $\|H\|_\infty$ are found to be close enough. The tolerance can be any desired small value.

Case Study 05: G07 Computing $H_2$ and $H_\infty$ norms for the Plant, Sideslip over Rudder Surface Deflection in Cruise Lateral Dynamics

The plant TF is the following:

$$G(s) = \frac{-0.051(s+48)(s+0.818)(s+0.086)}{(s+1.09)(s+0.063)(s^2 + 0.7837s + 2.496)}$$

(EQ-5.13)

Its state space form is given as:

$$\begin{bmatrix}
-1.9367 & -0.8672 & -0.3665 & -0.0857 \\
4.0000 & 0 & 0 & 0 \\
0 & 2.0000 & 0 & 0 \\
0 & 0 & 0.2500 & 0
\end{bmatrix} \begin{bmatrix}
x \\
x \\
x \\
y
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} u$$

(EQ-5.14)
From (EQ-5.14), the matrices \((A, B, C, D)\) are obtained.

Substitute \((A, B)\) into (EQ-5.8), and solve this matrix equation. We get:

\[
L = \begin{bmatrix}
0.4704 & 0 & -1.1216 & 0 \\
0 & 2.2432 & 0 & -0.7408 \\
-1.1216 & 0 & 5.9266 & 0 \\
0 & -0.7408 & 0 & 4.2235
\end{bmatrix}
\]  

Substitute matrices \((C, L)\) into (EQ-5.9), and solve it.

The 2-norm of this plant is: \(\|H\|_2 = 1.1173\)

To compute \(\|H\|_{\infty}\), first, let's assume \(\gamma_0 = 2\), using (EQ-5.12) to compute matrix \(J(\gamma_0)\), and its eigenvalues, which is shown in Table 5.1. There are no imaginary eigenvalues for \(\gamma_0 = 2\), so this gamma value can be reduced. Select \(\gamma_0 = 1\), and the imaginary eigenvalues come out, so this gamma is too small. Continue the iteration until the desired tolerance is achieved. When \(\gamma = 1.881\), the \(J(\gamma)\) is given as:

\[
J(1.881) = \begin{bmatrix}
-1.9367 & -0.8672 & -0.3665 & -0.0857 & 0.2826 & 0 & 0 & 0 \\
4.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\
-0.0026 & -0.0318 & -0.0141 & -0.0044 & 1.9367 & -4.0000 & 0 & 0 \\
-0.0318 & -0.3888 & -0.1728 & -0.0537 & 0.8672 & 0 & -2.0000 & 0 \\
-0.0141 & -0.1728 & -0.0768 & -0.0239 & 0.3665 & 0 & 0 & -0.2500 \\
-0.0044 & -0.0537 & -0.0239 & -0.0074 & 0.0857 & 0 & 0 & 0
\end{bmatrix}
\]
Table 5.1 $H_\infty$ Norm Iterative Computation for Plant Sideslip over Rudder Surface Deflection in Cruise Lateral Dynamics

<table>
<thead>
<tr>
<th>$\gamma$ Values</th>
<th>2</th>
<th>1</th>
<th>1.5</th>
<th>1.880</th>
<th>1.881</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.120</td>
<td>2.046i</td>
<td>1.773i</td>
<td>-1.124</td>
<td>-1.124</td>
<td></td>
</tr>
<tr>
<td>-0.135+1.497i</td>
<td>-2.046i</td>
<td>-1.773i</td>
<td>1.124</td>
<td>1.124</td>
<td></td>
</tr>
<tr>
<td>-0.135-1.497i</td>
<td>-1.234</td>
<td>-1.146</td>
<td>-1.503i</td>
<td>0.007+1.492i</td>
<td></td>
</tr>
<tr>
<td>0.135+1.497i</td>
<td>1.234</td>
<td>1.146</td>
<td>1.503i</td>
<td>0.007-1.492i</td>
<td></td>
</tr>
<tr>
<td>0.135-1.497i</td>
<td>0.725i</td>
<td>1.166i</td>
<td>1.481i</td>
<td>-0.007+1.492i</td>
<td></td>
</tr>
<tr>
<td>1.120</td>
<td>-0.725i</td>
<td>-1.166i</td>
<td>-1.481i</td>
<td>-0.007-1.492i</td>
<td></td>
</tr>
<tr>
<td>-0.0586</td>
<td>0.009i</td>
<td>-0.054</td>
<td>-0.058</td>
<td>-0.058</td>
<td></td>
</tr>
<tr>
<td>0.0586</td>
<td>-0.009i</td>
<td>0.054</td>
<td>0.058</td>
<td>0.058</td>
<td></td>
</tr>
</tbody>
</table>

The iterative results are shown in Table 5.1. The solution for the $H_\infty$ Norm of this plant is 1.881, and it happens at the frequency 1.492 rad/sec. Recall that this norm is, in fact, the maximum gain of the frequency response of this system. Refer to the open loop plant Bode Plot in Figure 1.20: the maximum gain is 5.48dB, which represents 1.88, and the corresponding frequency is 1.48 rad/sec. Comparing these two results, we find they are very close each other. In fact, in the case of SISO system, they are the same.

5.2 $\mu$ Analysis and Robust Performance

System robust performance and robust stability are only meaningful in sense of uncertainty consideration. Structured Singular Value is a mathematical object, and is developed to analyze the effect of unmodeled dynamics and
parametric uncertainty on the stability and performance of multiloop feedback systems that are depicted in linear algebra equations. $\mu$ analysis is less conservative in controller design in comparison to $H_\infty$ norm concept.

**Definition of Structured Singular Value**

In Chapter Four, it is illustrated that any system with uncertainty can be transformed into $M\Delta$ structure, for $M, \Delta \in \mathbb{C}^{n \times n}$. Under this architecture the structured singular value (SSV) is defined as the inverse of the size of the smallest structured perturbation that can cause or make matrix $I - M\Delta$ singular. That is:

$$\mu_\Delta(M) = \frac{1}{\min\{\bar{\sigma}(\Delta) | \Delta \in \Delta, \det(I - M\Delta) = 0\}}$$  \hspace{1cm} (EQ-5.15)

where $\bar{\sigma}(\Delta)$ denotes the maximum singular value of perturbation matrix $\Delta$, and is computed using the following formula:

$$\bar{\sigma}(\Delta) = \|\Delta\|_2$$

$$= \left[\rho(\Delta^T \Delta)\right]^{\frac{1}{2}}$$  \hspace{1cm} (EQ-5.16)

$$= \left\{\max_{\lambda_i} |\lambda_i|^{\frac{1}{2}} : \det(\lambda I - \Delta^T \Delta) = 0\right\}$$
The denotation $\mu_{\Delta}(M)$ implies that the structured singular value not only depends on generalized plant $M$, but also the perturbation $\Delta$.

An alternative expression for $\mu_{\Delta}(M)$ is as the following:

$$\mu_{\Delta}(M) = \max_{\Delta \in B_\Delta} \rho(M\Delta)$$  \hspace{1cm} (EQ-5.17)

where $\rho(M\Delta)$ represents the spectral radius of matrix $M\Delta$, which is computed from:

$$\rho(M\Delta) = \max \{|\lambda_i|: \det(\lambda I - M\Delta) = 0\}$$  \hspace{1cm} (EQ-5.18)

and $B_\Delta$ denotes limited uncertainty range:

$$B_\Delta = \{\Delta \in \Delta | \bar{\sigma}(\Delta) \leq 1\}$$  \hspace{1cm} (EQ-5.19)

In general, it is known that it is difficult to compute $\mu_{\Delta}(M)$ by solving (EQ-5.15), because this is not a convex problem [15]. Fortunately, it is achievable to compute its upper and lower bound, and the bounds can be refined by making transformations on matrix $M$ that do not affect $\mu_{\Delta}(M)$, which makes $\mu$ synthesis much practical. The bounded $\mu$ is expressed as:
\[
\max_{Q \in Q_\Delta} \rho(QM) \leq \mu_\Delta(M) \leq \inf_{D \in D_\Delta} \bar{\sigma}(DMD^{-1}) \tag{EQ-5.20}
\]

where \(\rho(QM)\) represents the spectral radius of matrix \(QM\), and \(Q_\Delta, D_\Delta\) denote the ranges of scaling matrices:

\[
Q_\Delta = \{Q \in \Delta | Q^*Q = I\} \tag{EQ-5.21}
\]

\[
D_\Delta = \left\{ \text{diag}[D_1, \ldots, D_s, d_iI_m, \ldots, d_{f-1}I_{m_{f-1}}, I_{m_f}] \right\} \tag{EQ-5.22}
\]

The scaling matrices \(Q\) and \(D\) are employed for the purpose to narrow the gap between the upper bound and lower bound of \(\mu_\Delta(M)\) and it is always true that \(\Delta = D\Delta D^{-1}\).

We’ve mentioned that there are two types of Uncertainty: Parametric Uncertainty and Unmodeled Uncertainty. The elements of uncertainty matrix \(\Delta\) are also classified into two categories: repeated scalar blocks and full blocks, and can be expressed as the following:

\[
\Delta = \{\text{diag}[\delta_iI_{r_i}, \ldots, \delta_iI_{r_s}, \Delta_1, \ldots, \Delta_F] | \delta_i \in \mathbb{R}, \text{ or } C; \Delta_j \in \mathbb{C}^{m_j \times m_j} \} \tag{EQ-5.23}
\]

where \(I_{r_i}\) represents identity matrix, and the dimensions must meet the following:

\[
\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n = \dim(\Delta) \tag{EQ-5.24}
\]
Theoretically, the function $\rho(QM)$ can have multiple local maxima, so the local search can only yield a lower bound, not be guaranteed to obtain $\mu_\alpha(M)$. However, the upper bound can be reformulated as a convex optimization problem [15], so the global minimum can be found. Unfortunately, the upper bound is not always equal to $\mu_\alpha(M)$. The fitness condition is related to the number of repeated scalar and full blocks, and the results are shown in Table 5.2.

Table 5.2 The Uncertainty Conditions in Which the Upper Bound Equals the SSV

<table>
<thead>
<tr>
<th>Number of Full block (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

Using (EQ-5.20) to compute the upper bound of SSV, we need to scale the generalized plant $M$. In general, the $M$ represent two port plant $P$, which is lower looped with desired controller $K$. Now we derive how to augment plant $P$ with Diagonal Scaling Matrix $D$. Referring to the Figure 5.1, we list the related equations below:

$$
\begin{bmatrix}
\hat{y}_1 \\
\hat{y}_2
\end{bmatrix} =
\begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{u}_1 \\
u_2
\end{bmatrix}
$$

(EQ-5.25)
Solving above equations with respect to $y_1$ and $y_2$, we get:

\[
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix} = \begin{bmatrix}
  DP_{11}D^{-1} & DP_{12} \\
  P_{21}D^{-1} & P_{22}
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

(EQ-5.28)

But we know:

\[
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix} = DPD^{-1} \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

(EQ-5.29)

Compare (EQ-5.28) with (EQ-5.29), we have the augmented two-port plant after scaling, which simplifies system structure.
System Robust Stability and Performance in Sense of Structured Singular Value

Using \( \mu \) to analyze control system, we often transform system structure into so called \( P\Delta K \) formation, where the \( P, \Delta \) and \( K \) represent the generalized plant, structured perturbation and desired controller, respectively. This architecture is shown in Part A of Figure 5.2, and Part B of Figure gives the equivalent \( M-\Delta \) structure interconnection, which highlights the relationship between the perturbation and the plant LFT transfer function.

\[
DPD^{-1} = \begin{bmatrix} DP_{11}D^{-1} & DP_{12} \\ P_{21}D^{-1} & P_{22} \end{bmatrix}
\]

(EQ-5.30)

Figure 5.2 System Structures Employed in Sense of \( \mu \)
In Figure 5.2, we have:

\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\]  

(EQ-5.31)

\[
M = F_k[P, K] = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}
\]  

(EQ-5.32)

Employing the above structure, we have so called small gain theorem [12] to describe system robust stability in sense of \( \mu \).

**Theorem 5.3: (Small Gain Theorem)**

Assume controller \( K \) is stabilizing for the nominal plant \( P \). The closed loop system in the above figure is well-posed and internally stable iff:

\[
\sup_{\omega \in \mathbb{R}} \mu_\alpha(M) \leq 1
\]  

(EQ-5.33)

where the structured perturbation is supposed to be normalized.

Not only does exogenous disturbance (for instance: wind gusts, sensor noise) perturb system stability, but also degrades system performance. It is possible to use \( \mu \) concept to depict system robust performance specification.

As we know, any system with disturbance and controller can be expressed as \( P\Delta K \) structure. To evaluate system performance, we add controlled output \( z \) and exogenous input
w to this structure, as shown in Part A of Figure 5.3. To find out the relationship between the controlled output and exogenous input, by using LFT technique, we first convert Part A of Figure 5.3 to Part B of Figure 5.3; then convert Part B of Figure 5.3 to Part C of Figure 5.3. The flow of the structure transformation is all shown above.

![Diagram of system structure](image)

**Figure 5.3** System $P\Delta K$ structure with Exogenous Input and Controlled Output

Under the expression of Figure 5.3, the system performance specification can be depicted as the following:

$$\|F_L[F_u(P,\Delta),K]\|_\infty < 1$$  \hspace{1cm} (EQ-5.34)
under the condition:

\[ \Delta \in B_\alpha, \|\Delta\|_\infty < 1 \]  

(EQ-5.35)

where \( B_\alpha \) is defined in (EQ-5.19).

The performance criterion given in (EQ-5.34) specifies the worst-case level of performance associated with a given structured uncertainty set.

If we introduce a fictitious uncertainty \( \Delta_{fi} \), which links the controlled output \( z \) to the exogenous input \( w \), then the robust performance problem will be transformed into a robust stability problem with respect to the augmented structured perturbation \( \Delta_p \) [14]. This idea is shown in Figure 5.4.

![Figure 5.4](image)

Figure 5.4 The Expression of Performance into Stability
Clearly, Part A of Figure 5.4 depicts a robust performance problem, but in Part B or Part C of Figure 5.4, it becomes a robust stability problem. In other words, $\mu$ can be employed to analyze system robust performance.

**Theorem 5.4  (Robust Performance With $\mu$)**

Assume controller $K$ is stabilizing for the nominal plant $P$. Then the closed loop system shown in Part A of Figure 5.3 is well-posed, internally stable and the performance specification $\| F_L [F_u (P, \Delta), K] \|_\infty < 1$, iff:

$$\sup_{\omega \in \mathbb{R}} \mu_{\Delta, \{ F_L [P, K] \}} \leq 1$$

(EQ-5.36)

where the perturbation structure has been normalized and augmented by a fictitious full complex performance block $\Delta_{fi}$. That is:

$$\Delta_p = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_{fi} \end{bmatrix} | \Delta \in B_\Delta, \Delta_{fi} \in \mathbb{C} \right\}$$

(EQ-5.36)

where $B_\Delta$ is defined in (EQ-5.19).

If the uncertainty is bounded by the largest singular value, it is possible using $\mu$ to check for robust stability and robust performance in a non-conservative way, so $\mu$ is very useful to analyze robust stability and performance.
5.3 Mixed Sensitivity and Loop Shaping

It is known that system 'size' can be measured using $H_2$ and/or $H_\infty$ norms. $H_2$ synthesis and $H_\infty$ synthesis (Refer to Chapter Six for detail) are effective methods, which produce controllers meeting design specification and achieving system robust stability and performance. Mixed Sensitivity and Loop Shaping Techniques play important roles in the implementation of $H_2$ synthesis and $H_\infty$ synthesis.

We consider the general feedback system configuration in Figure 5.5, and assume $d(s)$, $n(s)$ to represent plant output disturbance signal and measurement noise, respectively. Figure 5.5 gives the three equations below:

\[
\begin{align*}
\hat{e}(s) &= r(s) - n(s) - y(s) \\
u(s) &= K[r(s) - n(s) - y(s)] \\
y(s) &= d(s) + G(s)u(s)
\end{align*}
\]

(EQ-5.37)

![Figure 5.5 General Feedback System Configuration](image_url)
To evaluate command tracking, we introduce the tracking error:

\[ e(s) = r(s) - y(s) \]  \hspace{1cm} (EQ-5.38)

Solving (EQ-5.37) and (EQ-5.38), we obtain:

\[ y(s) = T(s)[r(s) - n(s)] + S(s)d(s) \]  \hspace{1cm} (EQ-5.39)

\[ u(s) = R(s)[r(s) - n(s) - d(s)] \]  \hspace{1cm} (EQ-5.40)

\[ \hat{e}(s) = S(s)[r(s) - n(s) - d(s)] \]  \hspace{1cm} (EQ-5.41)

\[ e(s) = S(s)[r(s) - d(s)] + T(s)n(s) \]  \hspace{1cm} (EQ-5.42)

where \( S(s) \), \( T(s) \) and \( R(s) \) are called Sensitivity, Complementary Sensitivity and Control Sensitivity, respectively, and defined as the following:

\[ S(s) = \frac{1}{1 + G(s)K(s)} \]  \hspace{1cm} (EQ-5.43)

\[ T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} \]  \hspace{1cm} (EQ-5.44)

\[ R(s) = \frac{K(s)}{1 + G(s)K(s)} \]  \hspace{1cm} (EQ-5.45)

Obviously, they have the relationship below:

\[ S(s) + T(s) = 1 \]  \hspace{1cm} (EQ-5.46)
Based on (EQ-5.40) and (EQ-5.42), the following important observations can be made:

- In order for disturbance and measurement noise not to deteriorate the input from the controller to the plant, the control sensitivity $R(s)$ has to be small over the frequency domain.

- To minimize the command tracking error, good disturbance attenuation and satisfying suppression of measurement noise are required. Normally, disturbance happens at low frequencies, and measurement noise at high frequencies. To achieve this objective, the Sensitivity $S(s)$ has to be small in the low frequency domain, and the Complementary Sensitivity $T(s)$ has to be small in the high frequency domain.

To achieve design specification, the weighting matrices $W_1$, $W_2$, and $W_3$ are introduced to penalize the tracking error, control input, and measurement output, respectively. This is shown in Figure 5.6, which is modified from Figure 5.5.

Now we’ll study this system configuration in sense of desired performance.
From Figure 5.6, it is easily shown:

\[
\begin{align*}
\begin{cases}
  e &= u_1 - y \\
  u &= Fe \\
  y &= Gu
\end{cases} \\
\text{(EQ-5.48)}
\end{align*}
\]

Solving (EQ-5.48), with respect to input \( u_1 \), it is given:

\[
\begin{bmatrix}
y \\ u \\ e
\end{bmatrix} =
\begin{bmatrix}
  T \\
  R \\
  S
\end{bmatrix}
\begin{bmatrix}
  u_1
\end{bmatrix}
\]

\[
\text{(EQ-5.49)}
\]

where \( T \), \( R \), and \( S \) are defined in (EQ-5.43) to (EQ-5.45), respectively.
Consequently, we obtain the penalized output with respect to input $u_1$:

\[
\begin{bmatrix}
y_{11} \\
y_{12} \\
y_{13}
\end{bmatrix} = \begin{bmatrix} W_1S \\ W_2R \\ W_3T \end{bmatrix} u_1 
\]  
(EQ-5.50)

Figure 5.7 gives the simplified architecture of Figure 5.6. In system Norm and LFT sense, the robust performance of this system can be expressed as:

\[
\|T_{y_1u_1}(j\omega)\|_\infty = \|F_L(P,K)\|_\infty \leq 1 
\]  
(EQ-5.51)

where $y_1$ represents:

\[
y_1 = \begin{bmatrix} y_{11} & y_{12} & y_{13} \end{bmatrix}^T 
\]  
(EQ-5.52)

![Figure 5.7 Simplified General Feedback Block Diagram with Augmented Plant](image)

According to (EQ-5.50), this will equivalently require:
\[ \overline{\sigma}(S^{-1}(j\omega)) \geq |W_1(j\omega)| \quad (\text{EQ-5.53}) \]
\[ \overline{\sigma}(R(j\omega)) \leq |W_2^{-1}(j\omega)| \quad (\text{EQ-5.54}) \]
\[ \overline{\sigma}(T(j\omega)) \leq |W_3^{-1}(j\omega)| \quad (\text{EQ-5.55}) \]

where \( \overline{\sigma}(\bullet) \) denotes the upper bound singular value of corresponding matrix. In fact, the effects of Control Sensitivity \( R(j\omega) \) can be merged into Sensitivity \( S(j\omega) \), and Complementary Sensitivity \( T(j\omega) \). Normally it is enough to use (EQ-5.53) and (EQ-5.55) to depict Robust Design Specification. The robust design criterion is vividly shown in Figure 5.8.

![Figure 5.8 Demonstration of Loop Shaping Technique](image)

**Remarks:** (Guide Lines for the Selection of \( W_1 \) and \( W_3 \))

- It is important that when choosing design specification \( W_1 \) and \( W_3 \), the 0dB crossover frequency
\( \omega_1 \) in the bode plot \( W_1 \) must be sufficiently below the 0dB crossover frequency \( \omega_3 \) in the bode plot \( W_3 \), otherwise, the performance requirements depicted in (EQ-5.53) and (EQ-5.55) will not be achievable.

- The penalty weight matrix \( W_1 \) is supposed to be chosen large at low frequencies and small at high frequencies to achieve good command tracking and disturbance attenuation. However, it should be small at frequencies corresponding to plant zeros in the RHP.

- The penalty weight matrix \( W_3 \) is supposed to be intentionally selected small at low frequencies and large at high frequencies. However, it should be small at frequencies corresponding to plant poles in the RHP.

The Criterion depicted in (EQ-5.51) is called Mixed Sensitivity Approach since in this cost function it penalizes both Sensitivity \( S(j\omega) \) and Complementary Sensitivity \( T(j\omega) \). The guidelines and procedures choosing weights \( W_1 \) and \( W_3 \) are so-called Loop Shaping Technique, which will be used in the \( H_2 \) and \( H_\infty \) controller design synthesis.
Chapter Six

Modern Robust Control Technique
Application

6.1 $H_2$ Synthesis Control

To design controller using $H_2$ and/or $H_\infty$ norm concepts, the system configuration depicted in Figure 6.1 is normally

![Figure 6.1 General Configuration for $H_2$ and $H_\infty$ Synthesis]
engaged, where the augmented plant takes the following matrix form:

\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\]  
\(\text{EQ-6.1}\)

According to Part A of Figure 6.1, it is easy to derive that:

\[
\begin{bmatrix} y_{11} \\ y_{13} \\ y_{2} \end{bmatrix} = \begin{bmatrix} W_1 & -W_1G \\ 0 & W_3G \\ I & -G \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]  
\(\text{EQ-6.2}\)

Then, \(P\) matrix is determined:

\[
\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} W_1 & -W_1G \\ 0 & W_3G \\ I & -G \end{bmatrix}
\]  
\(\text{EQ-6.3}\)

The \(H_2\) and \(H_\infty\) Synthesis are generally manipulated in state space. The State Space Form of (EQ-6.1) is given by:

\[
\begin{bmatrix} \dot{x} \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}
\]  
\(\text{EQ-6.4}\)

For the convenience of studying the problem, the following assumptions are made:

- \((A, B_1, C_1)\) is stabilizable and detectable.
• \((A, B_2, C_2)\) is stabilizable and detectable.

• \(D_{12}^T [C_1 \quad D_{12}] = [0 \quad I]\)

• \(
\begin{bmatrix}
B_1 \\
D_{21}
\end{bmatrix} D_{21}^T = [0 \\
I]
\)

Let's assume the State Space Form of the original plant \(G(s)\), weighting matrix \(W_1(s), W_2(s)\) and \(W_3(s)\) as the following:

\[
G(s) = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix}
\]

\[
W_1(s) = \begin{bmatrix} A_{W_1} & B_{W_1} \\ C_{W_1} & D_{W_1} \end{bmatrix}; \quad W_2(s) = \begin{bmatrix} A_{W_2} & B_{W_2} \\ C_{W_2} & D_{W_2} \end{bmatrix}
\]

\[
W_3(s) = \begin{bmatrix} A_{W_3} & B_{W_3} \\ C_{W_3} & D_{W_3} \end{bmatrix} + p_n S^n + \cdots + p_1 S + p_0
\]

where the \(W_3(s)\) can be improper and can be expressed as the sum of the strictly proper part and the polynomial form part. However, \(W_3(s)G(s)\) should be proper system, otherwise, the criterion depicted in (EQ-5.55) in Loop Shaping Technique could not hold.

The State Space Form of the augmented plant can be computed using the following formula:
\[ \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} A_G & 0 & 0 & 0 & 0 & B_G \\ -B_{w_i}C_G & A_{w_i} & 0 & 0 & B_{w_1} & -B_{w_i}D_G \\ 0 & 0 & A_{w_2} & 0 & B_{w_2} \\ B_{w_i}C_G & 0 & 0 & A_{w_i} & 0 & B_{w_i}D_G \\ -D_{w_i}C_G & C_{w_i} & 0 & 0 & D_{w_i} & -D_{w_i}D_G \\ 0 & 0 & C_{w_2} & 0 & 0 & D_{w_2} \\ \tilde{C}_G + D_{w_i}C_G & 0 & 0 & C_{w_2} & 0 & \tilde{D}_G + D_{w_i}D_G \\ \tilde{C}_G & 0 & 0 & 0 & I & -D_G \end{bmatrix} \]  

(EQ-6.5)

where:

\[ \tilde{C}_G = p_0C_G + p_1C_GA_G + \ldots + p_nC_GA_G^n \]

\[ \tilde{D}_G = p_0D_G + p_1C_GB_G + \ldots + p_nC_GA_G^{n-1}B_G \]

The \( p_0, p_1, \ldots, p_n \) are the polynomial coefficients of weighting matrix \( W_3(s) \).

It is admissible to assume that \( D_{11} = 0 \), and \( D_{22} = 0 \) [05], so the augmented plant can be rationally depicted by:

\[
\begin{bmatrix}
\dot{x} \\
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}
\]

Remarks: The condition \( D_{22} = 0 \) need not strictly hold when \( H_\infty \) Synthesis is implemented to design a controller, but a small non-zero value is still required.

Comparing Figure 6.1 to Figure 5.6, it is found that \( W_2 \) is no longer in use. In General, if \( D_{12} \) matrix is not of
full column rank, the matrix $W_2$ is employed by assigning a small value, i.e., $W_2 = \varepsilon I$. However, the function of $W_2$ can be mostly represented by $W_3$, as long as $W_3$ is chosen such that $W_3(s)G(s)$ is a proper system, which ensures the $D_{12}$ matrix is full column rank as required by $H_2$ and $H_\infty$ Synthesis.

Part B of Figure 6.1 is a more general architecture to deal with $H_2$ and $H_\infty$ Synthesis problems. The signal $u_1$ could represent all external inputs, including plant disturbance, measurement noise, and commands.

To find $H_2$ optimal controller, the cost function of $H_2$ Synthesis is defined as the following in the sense of LFT:

$$J = \min \|T_{y,u}\|_2 = \min \|F_L(P(s), K(s))\|_2 \quad (EQ-6.6)$$

In fact, the $H_2$ optimal controller is realized using usual Linear Quadratic Gaussian (LQG) technique, combined with Loop Shaping Algorithm. The core concept of LQG is similar to Pole Placement. For instance, the observer, which is called Kalman Filter in LQG, is constructed to recover full-state feedback. Thus the obvious advantage of $H_2$ Synthesis is that the system is guaranteed to be stable using $H_2$ controller. The Separation Principle is also suitable and is applied to compute the full-state feedback $K_c$ and Kalman Filter with residual gain matrix $K_f$. 
Furthermore, in LQG controller design, the plant disturbance and measurement noise are considered, and treated as white (Gaussian) noises. The LQG approach is quite popular and has been extensively by Honeywell and others to design multivariable aircraft flight control system [01].

The LQG system configuration is shown in Figure 6.2 (in next page), which shares a similar structure as that of the Pole Placement Technique.

The full-state approximation is reconstructed using Kalman Filter, and is given by:

\[ \dot{x} = A\hat{x} + B_f u_2 + K_f (y_2 - C_2 \hat{x}) \]  \hspace{1cm} (EQ-6.7)

The full-state estimations are feedback. Thus, the LQG control law is as follows:

\[ u_2 = K_c \hat{x} \]  \hspace{1cm} (EQ-6.8)

The cost function of LQG is a quadratic type, and is depicted as:

\[ J_{LQG} = \lim_{T \to \infty} E \left\{ \frac{1}{T} \int_0^T \left[ C_1^T C_1 \begin{bmatrix} x^T & u_2^T \end{bmatrix} \begin{bmatrix} C_1^T D_{12} & C_1^T D_{12} \end{bmatrix} \begin{bmatrix} x \\ u_2 \end{bmatrix} \right] dt \right\} \]  \hspace{1cm} (EQ-6.9)
The plant disturbance \( \xi(t) \) and sensor noise \( \theta(t) \) are included in \( u_1 \), so they perturb the system through the channels \( B_1 \) and \( D_{21} \). These two noises are treated as white noises, and are correlated as:

\[
E\left\{ \begin{bmatrix} \xi(t) \\ \theta(t) \end{bmatrix} \begin{bmatrix} \xi(\tau) \\ \theta(\tau) \end{bmatrix} \right\} = \begin{bmatrix} B_1B_1^T & B_1D_{21}^T \\ D_{21}B_1^T & D_{21}D_{21}^T \end{bmatrix} \]

(EQ-6.10)
Now we are ready to give the \( H_2 \) synthesis design procedure. Refer to [05] for the derivation of formulae.

**Step One:** Construct Plant Model in State-Space Form

\[
G = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{EQ-6.11}
\]

**Step Two:** Express the Design Specification using matrices \( W_1 \) and \( W_3 \), and make the augmented plant in State-Space Form, as shown in (EQ-6.6).

**Step Three:** Build a Kalman Filter using the following formula:

\[
K_f = (QC_2^T + B_1D_{21}^T)(D_{21}D_{21}^T)^{-1} \tag{EQ-6.12}
\]

where the \( Q \) is symmetric matrix, and is obtained by solving the following Algebraic Ricatti Equation (ARE):

\[
QA^T + AQ - (QC_2^T + B_1D_{21}^T)(D_{21}D_{21}^T)^{-1}(C_2Q + D_{21}B_1^T) + B_1B_1^T = 0 \tag{EQ-6.13}
\]

**Step Four:** Develop full-state estimation feedback by finding the feedback gain matrix \( K_c \):

\[
K_c = (D_{12}D_{12})^{-1}(B_2^TP + D_{12}^TC_1) \tag{EQ-6.14}
\]

where \( P \) is a symmetric matrix, and satisfies the following Algebraic Ricatti Equation (ARE):
\[ A^T P + PA - (PB_2 + C_1^T D_{12})(D_{12}^T D_{12})^{-1}(B_2^T P + D_{12}^T C_1) + C_1^T C_1 = 0 \] (EQ-6.15)

**Step Five:** Compute positive feedback $H_2$ optimal controller in State-Space Form using the following formula:

\[
K(s) = \begin{bmatrix}
A - K_f C_2 - B_2 K_c & K_f \\
- K_c & 0
\end{bmatrix}
\] (EQ-6.16)

**Step Six:** Construct the feedback system using the original plant and designed controller. Evaluate the system performance in both the time domains and frequency domains.

**Case Study 06: G06 Roll Rate Over Aileron Surface Deflection in Cruise Lateral Dynamics**

**Step One:** Construct Original Plant State-Space

\[
G(s) = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
-1.9367 & -0.8672 & -0.3665 & -0.1714 & 2.0000 \\
4.0000 & 0 & 0 & 0 & 0 \\
0 & 2.0000 & 0 & 0 & 0 \\
0 & 0 & 0.1250 & 0 & 0 \\
-2.2450 & -0.6640 & -1.0024 & 0 & 0
\end{bmatrix}
\]

**Step Two:** Construct $W_1$ and $W_3$ to meet design spec., and form the two-port type augmented plant using (EQ-6.6). The $W_1$ should be proper, but $W_3$ could be improper, as long as $W_3(s)G(s)$ is proper.

Design Specification:
• Step Response Setting Time is less than 0.5 Seconds, and Overshoot Ratio is less than 3%.

• The Sensitivity Function should be as small as possible in low frequencies, such that the system is robust enough to withstand plant disturbance.

The design Specification is interpreted as the following in sense of weighting matrices:

\[ W_1(s) = \frac{12.5 s^2 + 273.9 s + 1500}{15 s^2 + 104.6 s + 9} \]

\[ W_3(s) = \frac{S}{100} \]

Thus, the augmented plant is obtained using (EQ-6.6):

\[
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} = \begin{bmatrix}
-1.9367 & -0.8672 & -0.3665 & -0.1714 & 0 & 0 & 0 & 2.0000 \\
4.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.1250 & 0 & 0 & 0 & 0 & 0 \\
2.2450 & 0.6640 & 1.0024 & 0 & -6.9714 & -0.6000 & 1.0000 & 0 \\
0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \\
1.8708 & 0.5533 & 0.8354 & 0 & 12.4479 & 99.5000 & 0.8333 & 0 \\
0.0169 & -0.0006 & 0.0082 & 0.0038 & 0 & 0 & 0 & -0.0449 \\
2.2450 & 0.6640 & 1.0024 & 0 & 0 & 0 & 1.0000 & 0
\end{bmatrix}
\]
Step Three: Compute full-state feedback $K_c$ and Kalman Filter with residual gain matrix $K_f$ by solving two ARE:

$$K_f = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

$$K_c = \begin{bmatrix} -48.3 & -14.4 & -21.5 & 0.1 & -280.2 & -2214.2 \end{bmatrix}$$

Step Four: Compute $H_2$ Synthesis Controller $K(s)$:

$$K(s) = \frac{-280.2179(s+1.09)(s+7.902)(s+0.063)(s^2+0.7837s+2.496)}{s(s+97.27)(s+6.884)(s+0.08716)(s^2+1.183s+3.572)}$$

Step Five: Make in series with original plant $G(s)$, and close the negative loop to form feedback structure:

$$C_{sys}(s) = \frac{1258.1784(s+7.902)}{(s+82.14)(s+11.63)(s+10.47)}$$

Step Six: Evaluate System Performance:

The simulation results are shown from Figure 6.3 to Figure 6.6. System Performance can be evaluated using step response, which is shown in Figure 6.3. The setting Time is 0.21 Seconds, the Overshoot Ratio is 1.1%, and the Steady State Error is 0.6%, which all meet the design specifications. This confirms that closed loop system with designed controller performs good command tracking. The effectiveness of Loop Shaping implementation is given in Figure 6.4. Figure 6.5 demonstrates the cost function $T_y u u_1$. 
Figure 6.3  Step Response for Roll Rate Over Aileron Surface Deflection in Cruise Lateral Dynamics Using $H_2$ Synthesis Controller

Figure 6.4  Loop Shaping for Roll Rate Over Aileron Surface Deflection in Cruise Lateral Dynamics Using $H_2$ Synthesis Controller
Figure 6.5 Cost Function $T_{y1u1}$ for Roll Rate Over Aileron Surface Deflection in Cruise Lateral Dynamics Using $H_2$ Synthesis Controller

Figure 6.6 Open Loop Bode Plot for Roll Rate Over Aileron Surface Deflection in Cruise Lateral Dynamics Using $H_2$ Synthesis Controller
It is interesting to draw the open loop bode plot for the original plant with $H_2$ Synthesis Controller to investigate the robustness stability in sense of Gain Margin and Phase Margin. From Figure 6.6, it is measured that the Gain Margin is infinite, and the Phase margin is 79.3°. Thus it verified that controller designed using linear quadratic theorem will guarantee system robustness with the Gain Margin infinite and Phase margin at least 60° [01].

Obviously, in this case, the $H_2$ Synthesis is just showing a model following technique, because the controller obtained first cancel the low quality poles/zeros of the original plant, then shape it to optimal model. Applying this Technique to internal unstable system, we found system could be stabilized effectively. However, the performance is poor. In conclusion, $H_2$ Synthesis is a good method to make system stable, which performs the same role as Pole Placement Technique. It perfectly provides robust stability and performance for internal stable system, in which it shows up the same properties as optimal Model Following Algorithm. However, it generally loses power when dealing with internal unstable system, in which the Loop Shaping Technique is lost, so the performance, such as command tracking requirement, is out of control.
6.2 $H_\infty$ Synthesis Control

The general system configuration in $H_\infty$ Synthesis Control is the same as in $H_2$ Synthesis Control, which is redrawn in Figure 6.7, and the state space form of the augmented plant is the same as depicted in (EQ-6.6).

The optimal controllers are more difficult to characterize than suboptimal ones [05], so in $H_\infty$ Synthesis, suboptimal control problems are considered first, and then through iterations, the optimal controller is taken as the limiting case. The objective of suboptimal $H_\infty$ control is to find an admissible controller $K(s)$ such that $\|T_{y_2u_1}(s)\|_\infty < \gamma$, while the optimal control problem is to iteratively produce a controller, such that $\|T_{y_2u_1}(s)\|_\infty$ is minimized.

The $H_\infty$ control solution involves two Hamiltonian Matrices as the following:
Theorem 6.1: There exists an admissible controller such that $\|T_{\gamma_{w1}}(s)\|_m < \gamma$ iff the following three conditions hold:

- $H \in \text{dom}(\text{Ric})$ and $X = \text{Ric}(H) \succeq 0$
- $J \in \text{dom}(\text{Ric})$ and $Y = \text{Ric}(J) \succeq 0$
- $\rho(XY) < \gamma^2$

When above three conditions hold, one such suboptimal controller is determined as follows:

$$K(s) = \begin{bmatrix}
A + \gamma^2 B_1 B_1^T X - B_2 B_2^T X - (I - \gamma^2 Y X)^{-1} Y C_2^T C_2 \gamma (I - \gamma^2 Y X)^{-1} Y C_2^T X \\
-B_2^T X
\end{bmatrix}$$

(EQ-6.18)

If it is necessary, such suboptimal controller can be parameterized by a fixed LFT with a free parameter matrix $Q$, which is called Youla Parameterization Algorithm [05].

Case Study 07: G09 Vertical Acceleration Over Elevator Surface Deflection in Cruise Longitudinal Dynamics

The plant TF is given:

$$a_z/\delta_e = \frac{-0.023(s + 7.59)(s - 6.7)}{s^2 + 2.167s + 4.084}$$
There are some difficulties to design controllers for this system. First it is not strictly proper system, which will make trouble when weighting matrix $W_3(s)$ is constructed to meet design specification. It is difficult to meet $W_3(s)G(s)$ as proper system, and if this condition does not hold, the Complementary Sensitivity $T(s)$ will not meet the criterion required by Loop Shaping Technique. Secondly, there is a RHP zero, which will constrain the selection of weighting matrix $W_i(s)$, and furthermore, its performance is seriously deteriorated.

Step One: Form original plant State Space Form as the following:

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -2.1665 & -1.0211 & 0.5000 \\ 4.0000 & 0 & 0 \\ 0.0587 & 0.6318 & -0.0230 \end{bmatrix}$$

Step Two: Construct Matrices $W_1(s)$ and $W_3(s)$ according to design specification, and produce augmented plant in State Space Form. (Design Spec.: Setting Time < 1 Sec; Overshoot Ratio < 5%; Sensitivity $S(s)$ should be as small as possible.)

$$W_i(s) = \frac{1}{s + 0.001}$$

$$W_3(s) = \frac{s^2(10^{-5}S+1)}{1000} \approx \frac{s^2}{1000}$$
Step Three: Search suboptimal controller using iterations by changing the gamma value, such that 
\[ \| T_{\gamma u} (s) \|_\infty < \gamma \]. During this procedure, two AREs are solved, which are associated with two Hamiltonian Matrices, (EQ-6.17), and (EQ-6.18), respectively. One such controller is found as the following:

\[
K(s) = \frac{883.2071 (s^2 + 2.166s + 4.084)}{(s+44.52) (s+7.706) (s+0.001)}
\]

Step Four: Build new system using the original plant and designed controller, which is in series with the original plant, and close the loop. Evaluate the performance of this new system.

\[
C_{sys}(s) = \frac{-20.3138 (s-6.7) (s+7.59)}{(s+6.354) (s+11.93) (s+13.63)}
\]

System step response is plotted as Curve A in Figure 6.8. The amplitude first drops to -0.43, then rises. This performance can be improved using Prefilter.
Figure 6.8 Step Response for Vertical Acceleration Over Elevator Surface Deflection in Cruise Longitudinal Dynamics Using $H_\infty$ Synthesis Controller

Such a pre-filter is found to be:

$$G_p(s) = \frac{1}{0.01 s^2 + 0.2 s + 1}$$

Curve B in Figure 6.8 is plotted after the employment of a pre-filter. The negative amplitude is reduced to 0.14 with the sacrifice of setting time, which increases from 0.63 seconds to 0.98 seconds. To more practically simulate the input signal, the ramp input is considered, and the system with pre-filter response is plotted in Figure 6.9.
Figure 6.9 Ramp Response for Vertical Acceleration Over Elevator Surface Deflection in Cruise Longitudinal Dynamics Using $H_{\infty}$ Synthesis Controller and Prefilter

The $H_{\infty}$ Synthesis cost function is shown in Figure 6.10. As required, the gamma value is equal or less than one. To evaluate the effectiveness of the implementation of the Loop Shaping Technique, Figure 6.11 demonstrates the weighting matrices $W_1(s), W_3(s)$, and the Loop TF $L(s)$, which is defined as: $L(s)=K(s)G(s)$. In fact, in low frequencies, $L(s)$ represents the $S(s)$; while in high frequencies, it illustrates the $T(s)$. Also notice that, due to RHP zero, $L(s)$ deviates the design specification in high frequencies.
Figure 6.10 Cost Function $T_yu_1$ for Vertical Acceleration Over Elevator Surface Deflection in Cruise Longitudinal Dynamics Using $H_\infty$ Synthesis Controller

Figure 6.11 Loop Shaping Plot for Vertical Acceleration Over Elevator Surface Deflection in Cruise Longitudinal Dynamics Using $H_\infty$ Synthesis Technique
Case Study 08: G01 Pitch Rate Over Elevator Surface

Deflection in Hover Longitudinal Dynamics

The plant TF is given as below:

\[ \frac{q}{\delta_e} = \frac{-2.66s(s + 0.508)(s - 0.271)}{(s + 1.32)(s + 0.105)(s^2 - 0.5362s + 0.3352)} \]

This plant has one RHP zero and one pair of complex RHP poles. In this case, it is difficult to design a controller such that the system is stabilized and provides good command tracking, since there exists too small a RHP zero. One strategy is to design two controllers: one controller makes system stable, and the other ensures command tracking. These two controllers can interact so as to insure both system stability and performance.

One such stabilizing controller is found below:

\[ K_n(s) = \frac{0.019661 (s+1000) (s+1.32) (s-0.5673) (s-0.2569) (s+0.105)}{(s+5.487) (s+0.508) (s+0.001)s (s^2 + 0.4534s + 15.18)} \]

The TF of the closed loop system with the above controller is given as:

\[ Cl_{sys,n} = \frac{-0.052297 (s+1000) (s-0.5673) (s-0.271) (s-0.2569) (s+0.508)}{(s+2.108) (s+0.6366) (s+0.508) (s^2 + 0.5361s + 0.3352)(s^2 + 2.125s + 4.654)} \]

Figure 6.12 and 6.13 plot the step response and loop shaping. The step response is obviously unsatisfactory.
Figure 6.12 Step Response for Pitch Rate Over Elevator Surface Deflection in Hover Longitudinal Dynamics Using $H_\infty$ Synthesis Stabilizing Controller

Figure 6.13 Loop Shaping Plot for Pitch Rate Over Elevator Surface Deflection in Hover Longitudinal Dynamics Using $H_\infty$ Synthesis Stabilizing Controller
One such controller, which helps system improve performance, and the corresponding closed loop system are found as the following:

\[ K_{pc}(s) = \frac{-135.2474(s+19)(s+1.32)(s+0.105)(s^2 + 0.5443s + 0.3195)}{s(s+29)(s+18.39)(s+0.508)(s+0.3915)(s+0.09988)} \]

\[ Cl_{sys}(s) = \frac{359.7582(s-0.271)(s+19)(s^2 + 0.5443s + 0.3195)}{(s+19.65)(s-0.2618)(s^2 + 0.5362s + 0.3352)(s^2 + 27.42s + 339.2)} \]

The step response of this closed loop system with its performance controller is plotted in Figure 6.14.

![Step Response for q/s in Hover Dynamics](image)

**Figure 6.14** Step Response for Pitch Rate Over Elevator Surface Deflection in Hover Longitudinal Dynamics Using $H_\infty$ Synthesis Performance Controller
To evaluate the performance of this system in sense of tolerating perturbation, the Loop Shaping plot is shown in Figure 6.15. The corresponding cost function $T_y u_1$ is given in Figure 6.16.

Figure 6.15 Loop Shaping Plot for Pitch Rate Over Elevator Surface Deflection in Hover Longitudinal Dynamics Using $H_\infty$ Synthesis Performance Controller

Figure 6.16 Cost Function $T_y u_1$ Plotting for Pitch Rate Over Elevator Surface Deflection in Hover Longitudinal Dynamics Using $H_\infty$ Synthesis Performance Controller
6.3 $\mu$ Synthesis Control

Uncertainty is a prevailing problem in control system. System with uncertainty can be modeled with $\Delta P$ Structure, where $\Delta$ and $P$ denote uncertainty and augmented plant, respectively. Consider one controller $K$ need to stabilize a system with uncertainty, thus a structure called $P\Delta K$ is formed to implement $\mu$ Synthesis Control, which is shown in Figure 6.17.

![Figure 6.17 $P\Delta K$ Block Diagram For $\mu$ Synthesis Control](image)

The objective of $\mu$ Synthesis is to search a stabilizing controller $K$, and a diagonal scaling matrix $D$, such that:

$$\sup_{\omega \in \mathbb{R}} \mu_\Delta \{ F_\omega [P, K] \} = \frac{1}{\min \{ \sigma(\Delta) | \Delta \in \Delta, \det(I - F_\omega [P, K] \Delta) = 0 \}} \leq 1 \quad (EQ-6.19)$$

The $\mu$ Synthesis controller design involves a $D-K$ iterative procedure (Step 4, 5, and 6), as shown below:
Step One: Modeling Uncertainty with weighting matrix $W_u$. Assuming the uncertainty is modeled as multiplicative type at the plant input, the nominal plant is denoted as $G$.

Step Two: The performance specification is interpreted as weighting matrix $W_p$. The configuration of the analyzed system is shown in Part A of Figure 6.18, where $\|\Delta_G\|_2 \leq 1$.

Figure 6.18 Block Diagram Flow for $\mu$ Synthesis Design Procedure
denotes the uncertainty range, and $K$ is the desired stabilizing controller.

**Step Three:** Construct augmented plant $P$, or so-called system interconnection, and form $P\Delta K$ module. The performance from input $d$ to output $e$ is represented by a fictitious uncertainty $\Delta_F$ to form a complete $P\Delta K$ module. Thus, the robust stability and performance problems are converted to a purely stability problem. This conversion is shown in Part B and C of Figure 6.18.

**Step Four:** With $D$ fixed, apply $H_\infty$ optimal controller design for the open loop interconnection (i.e., break the controller loop) to obtain $K$. In this step, the scaling matrix $D$ is absorbed in augmented plant $P$, and the method is shown in Chapter Five. Usually at the first iteration, the identity matrix is set. If $P_D$ is denoted as the plant $P$ with absorbed rational $D$ scaling, this step design objective can be expressed the following cost function:

$$J = \min_{k} \left\| F_{\infty}(P_D, K) \right\|_{\infty}$$

(EQ-6.20)

The solutions of $H_\infty$ optimal control problem are well known and involve solving ARE in terms of a state space model for $P_D$. 

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Step Five: With K held fixed, analyze the robust performance properties of the resulting closed loop system using \( \mu \) concept. If \( M \) is used to denote the closed loop system, then its LFT form is the following:

\[
M = F_i (P, K)
\]  \hspace{1cm} (EQ-6.20)

The scaling matrix \( D \) and the allowable maximum size of perturbation are figured out in this step. To obtain the optimal frequency-dependent scaling matrix \( D \), the fitting algorithm is applied, which is based on Fast Fourier Transformation and least square errors. The detail computation algorithm of scaling matrices is given in [16].

Step Six: Use frequency dependent similarity scaling, obtained in the \( \mu \) analysis step, to scale the open loop interconnection and redesign the \( H_\infty \) optimal controller.

Case Study 09: G04 Yaw Rate Over Rudder Surface Deflection in Hover Lateral Dynamics

The nominal plant model is given as the following:

\[
G(s) = \frac{0.619}{s + 0.102}
\]

The detail Flight Data record in the frequency domain and nominal frequency response is given in Figure 6.19 [08].
Figure 6.19 Frequency Identification for Yaw Rate Over Rudder Surface Deflection in Hover Lateral Dynamics

Figure 6.20 Uncertainty Modeling for Yaw Rate Over Rudder Surface Deflection in Hover Lateral Dynamics
A more approximate model can be found to simulate the flight data, which is given below:

\[ \hat{G}(s) = \frac{0.6(s+1)}{(s^2 + 0.75s + 0.3)} \]

It's frequency response, and that of nominal plant are plotted in Figure 6.20 (This Figure is put on last page for Comparison with Figure 6.19). Consider that the unmodeled dynamics are multiplicative type at the plant input. Thus, the uncertainty weighting matrix \( W_u \) is easily computed to be the follows:

\[ W_u(s) = \frac{-0.0307(s-9.69)(s-0.6763)}{(s^2 + 0.75s + 0.3)} \]

The performance objective is chosen to be a stable, closed loop system, and the output disturbance rejection up to 0.6 rad/sec, with at least 100:1 disturbance rejection at DC to meet command tracking. This design specification can be interpreted as performance weighting matrix \( W_p \):

\[ W_p(s) = \frac{0.25s+0.6}{s+0.006} \]

\( \mu \) Synthesis Control carries out in state space, so every TF will be translated into its state space matrix form. Below are the state space matrices of above TFs:
The state space matrix of interconnection plant $P$ is computed as below (using the same method introduced in $H_2$ synthesis):

$$G = \begin{bmatrix} -0.1020 & 0.7868 \\ 0.7868 & 0 \end{bmatrix}; \quad W_p = \begin{bmatrix} -0.0060 & 0.7736 \\ 0.7736 & 0.2500 \end{bmatrix}$$

$$W_u = \begin{bmatrix} -0.0539 & -0.5124 & -0.2568 \\ 0.5124 & -0.6961 & 0.6381 \\ 0.2568 & 0.6381 & -0.0307 \end{bmatrix}$$

Now we can use $D-K$ iteration to design a controller. It involves the following four steps iteration in detail:

- Use the $H_\infty$ optimal controller design algorithm to search for a suitable controller. In the first iteration, the scaling matrix $D$ is assigned as identity matrix. It is absorbed to plant $P$ to form new plant interconnection $P_D$ (Refer to Figure 5.1).

- The plant $P$ is looped back by computed controller $K$. Thus, the closed loop matrix $M$ is formed.
• Compute the frequency response of $M$, $M_{11}(j\omega)$ and $M_{22}(j\omega)$ represent system stability and performance, respectively.

• Find the optimal frequency-dependent scaling matrix $D$ at a large, but finite set of frequencies to compute the upper bound of $\mu$. Fit this optimal frequency-dependent scaling matrix with a stable, minimum-phase, real-rational transfer function. In most applications, $\mu$ could be represented by its upper bound. The plotting of $\mu$ will depict whether the system robust performance is achieved.

It is found after the first $D-K$ iteration, the computed controller $K$ is 4th order, and in the second iteration, controller increases to 6th order. To implement the controller for real applications, it is necessary to reduce the controller’s order using the algorithm introduced in [02]. Finally, a second order controller is derived. (The corresponding peak $\mu$ is 0.97, less than one; thus, the stability is guaranteed even for worst case perturbation):

$$K = \begin{bmatrix}
-12.6591 & 3.4772 & -7.6099 \\
2.7845 & -0.8237 & 0.4446 \\
6.0522 & -0.4415 & 0
\end{bmatrix}$$
Its TF form is easily to be converted as below:

\[ K(s) = \frac{-46.2528(s+0.4693)}{(s+13.43)(s+0.05544)} \]

The matrix \( M \) represents the closed loop system of plant \( P \) looping with controller \( K \), and its state space form is shown below:

\[
M = \begin{bmatrix}
-0.1020 & 0 & 0 & 0 & 4.7617 & -0.3474 & 0.7868 & 0 \\
0 & -0.0539 & -0.5124 & 0 & -1.5542 & 0.1134 & 0 & 0 \\
0 & 0.5124 & -0.6961 & 0 & 3.8619 & -0.2817 & 0 & 0 \\
0.6087 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7736 \\
-5.9872 & 0 & 0 & 0 & -12.6591 & 3.4772 & 0 & -7.6099 \\
0.3498 & 0 & 0 & 0 & 2.7845 & -0.8237 & 0 & 0.4446 \\
0 & 0.2568 & 0.6381 & 0 & -0.1857 & 0.0136 & 0 & 0 \\
0.1967 & 0 & 0 & 0.7736 & 0 & 0 & 0 & 0.2500
\end{bmatrix}
\]

System configuration in terms of matrix \( M \) is shown in Figure 6.21, which is developed from Part C of Figure 6.18.

Figure 6.21 Block Diagram for System Closed Loop Interconnection Represented by Matrix \( M \)
The two-port input/output relationship is given below:

\[
\begin{bmatrix}
  w \\
  e \\
\end{bmatrix} = \begin{bmatrix}
  M_{11} & M_{12} \\
  M_{21} & M_{22} \\
\end{bmatrix}
\begin{bmatrix}
  z \\
  d \\
\end{bmatrix}
\]

Clearly, if we ignore the coupling influence between performance and perturbation, the $M_{11}$ would be an index showing the system stability, and $M_{22}$ would be an index representing system performance. The state space form and conventional TF form are given as the following:

\[
M_{11} = \begin{bmatrix}
  -0.1020 & 0 & 0 & 0 & 4.7617 & -0.3474 & 0.7868 \\
  0 & -0.0539 & -0.5124 & 0 & -1.5542 & 0.1134 & 0 \\
  0 & 0.5124 & -0.6961 & 0 & 3.8619 & -0.2817 & 0 \\
  0.6087 & 0 & 0 & -0.0060 & 0 & 0 & 0 \\
  -5.9872 & 0 & 0 & 0 & -12.6591 & 3.4772 & 0 \\
  0.3498 & 0 & 0 & 0 & 2.7845 & -0.8237 & 0 \\
  0 & 0.2568 & 0.6381 & 0 & -0.1857 & 0.0136 & 0 \\
\end{bmatrix}
\]

\[
M_{11}(s) = \frac{0.87867(s-9.692)(s-0.6761)(s+0.4693)}{(s+10.87)(s+2.131)(s+0.5833)(s^2 + 0.75s + 0.3)}
\]

\[
M_{22} = \begin{bmatrix}
  -0.1020 & 0 & 0 & 0 & 4.7617 & -0.3474 & 0 \\
  0 & -0.0539 & -0.5124 & 0 & -1.5542 & 0.1134 & 0 \\
  0 & 0.5124 & -0.6961 & 0 & 3.8619 & -0.2817 & 0 \\
  0.6087 & 0 & 0 & -0.0060 & 0 & 0 & 0.7736 \\
  -5.9872 & 0 & 0 & 0 & -12.6591 & 3.4772 & -7.6099 \\
  0.3498 & 0 & 0 & 0 & 2.7845 & -0.8237 & 0.4446 \\
  0.1967 & 0 & 0 & 0.7736 & 0 & 0 & 0.2500 \\
\end{bmatrix}
\]

\[
M_{22}(s) = \frac{0.25(s+13.43)(s+2.4)(s+0.102)(s+0.05544)}{(s+0.006)(s+0.5833)(s+2.131)(s+10.87)}
\]
We can also compute the worse case plant model corresponding to this controller as below:

\[
G_{wc} = \begin{bmatrix}
-0.1020 & 0.0161 & 0.2012 & 0.4998 & 0.7627 \\
0 & -0.0002 & -0.0053 & -0.0131 & 0.0006 \\
0 & 0 & -0.0539 & -0.5124 & -0.2568 \\
0 & 0 & 0.5124 & -0.6961 & 0.6381 \\
0.7868 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
G_{wc}(s) = \frac{0.60009 (s+0.9975) (s+0.1021) (s+0.001071)}{(s+0.102) (s+0.0002114) (s^2 + 0.75s + 0.3)}
\]

The maximum allowable \( \Delta_G \) that corresponds to the worst case is the following (It is found during the \( \mu \) computation)

\[
\Delta_G = \begin{bmatrix}
-0.0002 & -0.0205 \\
0.0205 & 0.9956
\end{bmatrix}
\]

\[
\Delta_G(s) = \frac{0.99563 (s-0.0002114)}{(s+0.0002114)}
\]

The reconfigured system, as shown in Figure 6.22, is employed to evaluate command tracking. The closed loop step response is plotted in Figure 6.23 with initial time 0.5s.

![Figure 6.22 Conventional Command Tracking Block Diagram](image-url)
Figure 6.23 Step Response for Yaw Rate Over Rudder Surface Deflection in Hover Lateral Dynamics using $\mu$ Synthesis Controller

Figure 6.24 shows how the size of perturbation affects the system performance. The $M_{22}$ and $M_{11}$ are plotted in the left part of Figure 6.25, showing the nominal plant performance and stability with the employment of 2nd order controller obtained using $\mu$ Synthesis Technique. The $\mu$ plotting in the right part of Figure 6.25 efficiently reflects this system robust performance. As our desire the $\mu$ is less than one over the frequency range, and its peak value is 0.962. However, the gamma value in sense of $H_\infty$ optimal control is 3.52, so $\mu$ Synthesis is less conservative compared with $H_\infty$ synthesis.
Figure 6.24 Performance Degradation with the Increasing of Perturbation size $\Delta_G$ for Yaw Rate over Rudder Surface Deflection in Hover Lateral Dynamics using $\mu$ Synthesis

Figure 6.25 Robust Performance and Stability Index Plotting for Yaw Rate over Rudder Surface Deflection in Hover Lateral Dynamics using $\mu$ Synthesis Technique
Chapter Seven

Conclusion and Future Work

7.1 Conclusion

State feedback architecture is the core of robust control. It is well known that the system with full state feedback can guarantee stability for any rational system. For the application of real physical world, an observer is needed to recuperate all states, which may be fictitious or measurable. Linear quadratic regulator design criterion is at the foundation of robust control, in which the observer concept is employed, and the system performance index is expressed in linear quadratic form of states and inputs.

In the case of an internal stable system, model following, or model reference technique, is very effective to improve system performance. The pre-filter takes a flexible and efficient role to shape desired command tracking when combined with a controller that is in the loop. The model following concept not only exists in classical control, but also in robust control.

The real world is not perfect, the plant will be always under the influence of low frequency disturbance and
high frequency sensor noise, and these influences are measurable using sensitivity and complementary sensitivity. The loop-shaping design algorithm is extended in the basis of sensitivity and complementary sensitivity, in which design specifications are interpreted into two weighting matrices.

In multivariable systems, the method of weighting matrices is often used since it makes intuitive sense. Moreover, it is powerful when searching for optimal problem solutions. This is reasonable since each variable has its own range of operation, and in the view of system level, it is essential to weight necessary variables to constrain them in the nominal domain. For instance, in the case of infinite norm optimization, the introduction of weights allows the frequency dependent characteristics of signals and systems to be captured as well as their sizes [12]. Especially in $\mu$ synthesis, three times weighting concepts are applied. First, the design specifications are interpreted into relevant weighting matrices. Secondly, $D$ scaling plays an important role to compute $\mu$, thus, to search for the optimal controller. Finally, Youla Parameterization method is utilized to generate stabilizing controllers for all realizable stable closed-loop systems.
In robust control, all controller synthesis problems may be described in the language of LFT. For instance, the system parametric uncertainty can be easily expressed in a LFT form. Also, the system robust stability and performance problems are depicted in simple but explicit LFT forms.

System size is measurable using norm concepts, such as $H_2$ and $H_\infty$ norms. Their solutions involve Hamiltonian Matrices and the corresponding Algebraic Riccati Equations. Mathematically, all system information may be expressed in matrix forms. Thus any controller design is feasible in computerization.

$\mu$ synthesis is the abstract of robust control. First it is powerful enough to deal with system uncertainty. Secondly, the controller design involves iterative optimal $H_\infty$ controller searching. Thirdly, $\mu$ analysis is less conservative in comparison with $H_\infty$ norm. Finally, the controller design objectives in sense of robust stability and performance exactly fit in the $\mu$ framework. One of the drawbacks of $\mu$ synthesis is that the produced controllers are normally high order models. In practice, model reduction, such as truncation of unimportant states from state-space models, is necessary to be implemented.
7.2 **Future Work**

Time delay is ignored in this thesis. Normally, there are two significant time delays in control system. First, the actuators are mechanical systems, so their time-delay response is obvious. Secondly, it needs processing-time for sensors to measure desired outputs. Time-delay problem is more serious in digital-time domain in comparison to continuous-time domain. For instance, a stable system in term of continuous time may be unstable in sense of digital system.

RHP zeros always make trouble for system performance and robust stability, especially when they are near the origin. In command tracking, the setting time and overshooting are always deteriorated due to the existence of RHP zeros. Also, in output feedback system, the RHP zeros are the loci ending points of the closed-loop system poles, so when the loop gain is bigger enough, the feedback system is definitely unstable.

For higher order system with RHP zeros and/or poles, for instance 4th order and above, it is hard to pursue both robust stability and perfect performance. Also, there is a trade off among robust stability, system performance and controller's order.
The application of controller design using robust synthesis control is constrained in some range of uncertainty, out of which the system becomes uncontrollable. The adaptive and reconfigurable controller may be extended in this point of view. Another attempt would be the artificial neural network in sense of adaptation, since the construction of neural network is based on the training sets, and the training sets can be easily augmented to fit the new flight conditions.

The controller hardware implementation would be feasible using a micro-controller. However, some elements should be considered during this phase. First, the controller model should be converted from continuous time domain to digital domain. Secondly, the time-delay should be absorbed in digitized model. Furthermore, the sampling frequency should meet the Shannon’s sampling theorem [03]. Generally, system stability is degenerated in digital domain.

Time delays and discretization effects come from several sources as the follows:

- Remote sampling at the sensor site
- Limited speed between sensors and the micro-processor
• Sampling and multiplexing at the micro-processor
  (Sampling rates can be differ for each signal)
• Processing speed at the micro-processor

In summary, these effects should be incorporated in the practical application and a realistic simulator be implemented to dark performance.
## Appendix A

### Plant Model and Designed Controller

Table AA.1 Original Plant Models for XV-15 Tilt-Rotor Aircraft

<table>
<thead>
<tr>
<th>Mode</th>
<th>Plant</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>G01</td>
<td></td>
<td>( q/\delta_c = \frac{-2.66s(s + 0.508)(s - 0.271)}{(s + 1.32)(s + 0.105)(s^2 - 0.5362s + 0.3352)} )</td>
</tr>
<tr>
<td>G02</td>
<td></td>
<td>( a_z/\delta_c = \frac{-0.0098s}{s + 0.105} )</td>
</tr>
<tr>
<td>G03</td>
<td></td>
<td>( p/\delta_a = \frac{-3.71s(s + 0.412)(s - 0.107)}{(s + 1.23)(s + 0.102)(s^2 - 0.3737s + 0.1998)} )</td>
</tr>
<tr>
<td>G04</td>
<td></td>
<td>( r/\delta_r = \frac{0.619}{s + 0.102} )</td>
</tr>
<tr>
<td>G05</td>
<td></td>
<td>( r/\delta_a = \frac{0.344s(s - 0.345)(s^2 + 0.8454s + 0.2372)}{(s + 1.23)(s + 0.102)(s^2 - 0.3737s + 0.1998)} )</td>
</tr>
<tr>
<td>G06</td>
<td></td>
<td>( p/\delta_a = \frac{-4.49s(s^2 + 1.183s + 3.572)}{(s + 1.09)(s + 0.063)(s^2 + 0.7837s + 2.496)} )</td>
</tr>
<tr>
<td>G07</td>
<td></td>
<td>( \beta_{r\delta}/\delta_r = \frac{-0.051(s + 48)(s + 0.818)(s + 0.086)}{(s + 1.09)(s + 0.063)(s^2 + 0.7837s + 2.496)} )</td>
</tr>
<tr>
<td>G08</td>
<td></td>
<td>( q/\delta_c = \frac{-7.38(s + 0.89)}{(s^2 + 2.167s + 4.084)} )</td>
</tr>
<tr>
<td>G09</td>
<td></td>
<td>( a_z/\delta_c = \frac{-0.023(s + 7.59)(s - 6.7)}{(s^2 + 2.167s + 4.084)} )</td>
</tr>
<tr>
<td>Plant</td>
<td>Technique</td>
<td>Controller</td>
</tr>
<tr>
<td>-------</td>
<td>-----------</td>
<td>------------</td>
</tr>
<tr>
<td>G01</td>
<td>( H_\infty ) Synthesis</td>
<td>( G_c(s) = \frac{s + 0.105}{s + 10} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( G_p(s) = -\frac{1020}{s} )</td>
</tr>
<tr>
<td>G02</td>
<td>Model Following</td>
<td>( G_c(s) = \frac{-2.5505 (s + 1.23)(s - 0.8363)(s + 0.287)}{(s + 0.412)(s + 0.2839)(s^2 + 4.189s + 16.72)} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( G_p(s) = \frac{-0.4}{s} )</td>
</tr>
<tr>
<td>G03</td>
<td>Pole Placement</td>
<td>( G_c(s) = \frac{-46.2528 (s + 0.4693)}{(s + 13.43)(s + 0.05544)} )</td>
</tr>
<tr>
<td>G04</td>
<td>( \mu ) Synthesis</td>
<td>( G_c(s) = \frac{-22.8444 (s + 1.000)(s + 1.23)(s + 0.102)(s^2 - 0.2527s + 0.07296)}{(s + 87.56)(s + 0.001)(s^2 + 0.8454s + 0.2372)(s^2 + 4.282s + 103.6)} )</td>
</tr>
<tr>
<td>G05</td>
<td>( H_\infty ) Synthesis</td>
<td>( G_c(s) = \frac{-280.22 (s + 1.09)(s + 7.902)(s + 0.063)(s^2 + 0.7837s + 2.496)}{s(s + 97.27)(s + 6.884)(s + 0.08716)(s^2 + 1.183s + 3.572)} )</td>
</tr>
<tr>
<td>G06</td>
<td>( H_2 ) Synthesis</td>
<td>( G_c(s) = \frac{-24670 (s + 1.09)(s + 7.902)(s + 0.063)(s^2 + 0.7837s + 2.496)}{(s + 97.27)(s + 48)(s + 6.884)(s + 0.818)(s + 0.08716)(s + 0.086)} )</td>
</tr>
<tr>
<td>G07</td>
<td>( H_2 ) Synthesis</td>
<td>( G_c(s) = \frac{0.1760 + 0.4548/s + 0.1356s}{0.4548/s + 0.1356s} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( G_p(s) = \frac{2.985}{s^3 + 2.188s^2 + 4.509s + 2.985} )</td>
</tr>
<tr>
<td>G08</td>
<td>PID</td>
<td>( G_c(s) = \frac{883.2071 (s^2 + 2.166s + 4.084)}{(s + 44.52)(s + 7.067)(s + 0.001)} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( G_p(s) = \frac{1}{0.01s^2 + 0.2s + 1} )</td>
</tr>
</tbody>
</table>
Appendix B

Matlab Code For PID Controller Using ITAE

% PID controller design for G08: Cruise pitch rate over elevator

% Construct Plant Model

fa=-7.38
fb=[1 .89]
f1=[1 2.1665 4.0844]
um=conv(fa,fb)
den=f1
outputVariable = {'Pitch Rate, Deg/Sec'}
inputVariable = {'Elevator, Deg'}
gCpitchE = tf(num,den,'inputname', inputVariable,'outputname',outputVariable)
gCpitchEzpk = zpk(gCpitchE)
G = gCpitchEzpk

% Using ITAE algorithm to obtain PID controller

[num,den] = tfdata(G,'v')
c1=num(1,2)
c0=num(1,3)
a=den(1,2)
b=den(1,3)
zeta=0.8
Ts=0.5
wn=4/(Ts*zeta)
wn=1.5*wn
Apid=[c1 0 c0-1.75*wn*c1 ;c0 c1 -2.15*wn*wn*c1;0 c0 -wn*wn*wn*wn*c1]
bpid=[1.75*wn-a;2.15*wn*wn-b;wn*wn*wn]
xpid=Apid\bpid
kpm2=xpid(1,1)
kim2=xpid(2,1)
dm2=xpid(3,1)
Gcm2=tf([kdm2 kpm2 kim2],[1 0])

% Form feedback T1 transfer function

Lm2=G*Gcm2
T1m2=feedback(Lm2,1,-1)
% Construct pre-filter to achieve zero steady state error

[num,den] = tfdata(Tlm2,'v')
cn3=num(1,1)
cn2=num(1,2)
cn1=num(1,3)
cn0=num(1,4)
if cn3==0
    if cn2==0
        den=num/cn1
        num=wn*wn*wn/cn1
    else
        den=num/cn2
        num=wn*wn*wn/cn2
    end
else
    den=num/cn3
    num=wn*wn*wn/cn3
end
Gpm2=tf(num,den)

% Get reduced order system
Tm2=minreal(Gpm2*Tlm2)

% Evaluate system
step(Tm2,'b',2)
title ('Step Response With PID Controller: \ltq\delta\lt_e in Cruise Dynamics...'
       'FontSize', 10)
Appendix C

Notations and Acronyms

List of Notations:

$\delta_a$: Aileron Surface Deflection, unit: deg

$\delta_c$: Power Level Deflection, unit: inch

$\delta_e$: Elevator Surface Deflection, unit: deg

$\delta_r$: Rudder Surface Deflection, unit: deg

p: Roll Rate, unit: deg/s

q: Pitch Rate, unit: deg/s

r: Yaw Rate, unit: deg/s

$\alpha_z$: Vertical Acceleration, unit: g

$\beta_{cg}$: Sideslip, unit: deg

$P\Delta K$: It is a module to be used to analyze system robustness, where $P$, $\Delta$ and $K$ represent plant, uncertainty and controller, respectively. This structure is specially conceived to execute $D$-$K$ iteration in the $\mu$ Synthesis Controller Design.

$M^T$: Transpose of any matrix $M$

$\rho(M)$: The spectral radius of any matrix $M$
List of Acronyms:

AFCS: Automatic Flight Control System
ARE: Algebraic Ricatti Equation
BIBO: Boundary Input Boundary Output
c.g.: center of gravity
iff: if and only if
ITAE: Integral of Time multiplied by Absolute Error
LHP: Left Hand s Plane
LQG: Linear Quadratic Gaussian
LQR: Linear Quadratic Regulator
LTI: Linear Time Invariant System
MISO: Multiple Input Single Output
PD: Proportional and Differential
PI: Proportional and Integral
PID: Proportional, Integral and Differential
RHP: Right Hand s Plane
SIMO: Single Input Multiple Output
SISO: Single Input Single Output
SSV: Structured Singular Value
TF: Transfer Function
TRA: Tilt Rotor Aircraft
BIBLIOGRAPHY


