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On Cup-Products of Cofibers of Maps Between Moore Spaces, Hopf Invariant, and Lusternik-Schnirelmann Category

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ON CUP-PRODUCTS OF COFIBERS OF MAPS BETWEEN MOORE SPACES, HOPF INVARIANT, AND LUSTERNIK-SCHNIRELMANN CATEGORY

by

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In this thesis we make a detailed investigation of the cohomology rings of the cofibers C_β of Moore spaces of dimension 2 by computing the cup products in cofibers and to do so we prove that the Hopf invariant in case of Moore spaces in the zero and nonzero homomorphism case is a homomorphism. We have shown when is $x_{r,k}$ a Co-H-Map. We calculated the homologies and cohomologies of Moore spaces of dimension 2 and of the cofibers C_β where $\beta = x_{r,k}$. We used Lusternik-Schnirelmann category to determine the complexity of C_β.

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I dedicate this thesis to my father Assem Soliman Mosallam who strongly believed in me, who gave me a tremendous support starting from my childhood until he died in November 2011, Who loved me by all his heart and whom I loved with all my heart and finally who taught me perseverance by practicing it everyday in front of my eyes.

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Marwa A.S. Mosallam

Contents

CHAPTER 1

INTRODUCTION

1.1 Motivation

Because of the wonderful and spectacular information on Moore spaces of type $(G, 2)$ that my advisor, Dr. Strom, has found in the paper written by Martin Arkowitz and Marek Golasinski named Co-H-Structures on Moore spaces of type $(G, 2)$, we decided continuing on proving some more details about the co-fibers of maps between Moore spaces of type $(G, 2)$.

During the Topology seminar at Western Michigan University in 2019/2020, my advisor has said that in the previous mentioned paper, in section 5, in proposition 11, Arkowitz and Golasinski had completely laid out maps between Moore spaces $M = M(\mathbb{Z}_m, 2)$ and $N = M(\mathbb{Z}_n, 2)$.At the beginning of this research, my advisor asked me to come to an understanding of what these maps are, by working up to and through section 5 in this paper [AG94] in details. For easy reference, I am writing the formula of these maps below

$$x = x_{r,k} = r\lambda + k(i'\phi q) \tag{1.1}$$

Where $x \in \pi_2(\mathbb{Z}_m; N) = [M, N]$ and they can be uniquely written as above and where r and k are integers such that $0 \leq r < d$ and $0 \leq k < \delta$ where $d = (m, n)$ and $\delta = (m, (2n, n^2))$.

For the sake of calculation simplicity, we will focus on the case where $m = p^a$ and $n = p^b$, where p is a prime number and $a, b \in \mathbb{N}$, this will make some of the stuff done in the paper simpler, such as easily writing down greatest common divisors.

Our first goal was to understand what can we say about the cohomology rings of the cofibers $C_{x_{r,k}}$ where we will rename $x_{r,k}$ as β later. Then our second goal was to estimate how complicated is our cofiber $C_{x_{r,k}}$ and to do so we will use something called Lusternik-Schnirelmann category.

1.2 Preliminaries

Definition 1.1.

Let G be an abelian group and n an integer ≥ 2. A *Moore space* of type (G, n) is a 1-connected, CW-complex X such that

$$\widetilde{H_i}(X) = \begin{cases} 0 & if i \neq n \\ G & if i = n, \end{cases}$$

Where $\widetilde{H_i}(X)$ denotes the i-th reduced homology group. We denote a Moore space of type (G, n) by $M(G, n)$.

It is well known that Moore spaces exist, for a precise proof of this look at [Ark11], Lemma 2.5.2, actually you will see how they are constructed. It is also well known that any two Moore spaces of type (G, n) have the same homotopy type or we can say uniqueness up to homotopy of Moore spaces, for a detailed proof of this fact you can read the proof of Lemma 6.4.16 in [Ark11] also.

From the definition of the Moore space given above and since the suspension of any path-connected space is simply connected and because the suspension raises the dimension of homology by 1, then the suspension $\Sigma M(G, n-1)$ of a Moore space of type $(G, n-1)$ will have the same homology of a Moore space of type (G, n) for $n \geq 3$.

In fact, $M(G, 2)$ is also a suspension, namely the suspension of any space whose only nontrivial reduced homology group is G in degree 1. This was proved in [AG94] at the beginning of section 3.

Even though we did not use the definition of the cup-product to calculate it, we will give its definition here for the sake of completeness of our exposition. Also, if you want to read more about it you can check [Str11].

Definition 1.2.

Let $u \in \widetilde{H^n}(X; G)$ and $v \in \widetilde{H^m}(Y; H)$ be cohomology classes. If the reduced diagonal map is the composite map $\bar{\Delta}$ in the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\bar{\Delta}\quad} & X \wedge X \\ & \searrow^{\Delta} \quad\quad \nearrow^{\wedge} & \\ & X \times X & \end{array}$$

where the symbol $\wedge : X \times X \to X \wedge X$ denotes the standard quotient map, then the cup product of u and v is the composition in the following diagram

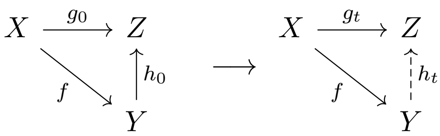
$$\begin{array}{ccc} X & \xrightarrow{u.v} & K(G \otimes H, n+m) \\ {\scriptstyle\bar{\Delta}}\downarrow & & \uparrow{\scriptstyle c} \\ X \wedge X & \xrightarrow{u \wedge v} & K(G, n) \wedge K(H, m) \end{array}$$

Definition 1.3.

The cohomology ring of a topological space $X$is a ring formed from the cohomology groups of X together with the cup-product serving as the ring multiplication. Here cohomology is usually understood as singular cohomology.

Definition 1.4.

A map $f : X \to Y$ is called a cofiber map if for every space Z, maps $g_0 : X \to Z$ and $h_0 : Y \to Z$ and homotopy $g_t : X \to Z$ of g_0 such that $h_0 f = g_0$, there exists a homotopy $h_t : Y \to Z$ of h_0 such that $h_t f = g_t$,



If f is a cofiber map, then $Q = Y/f(X)$ is called the cofiber of f. Below is a figure of the cofiber sequence of f.

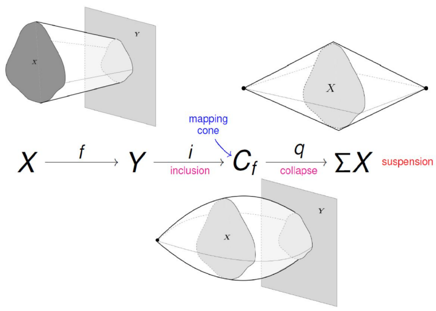


Figure 1.1: The cofiber sequence of f

Example 1. For any spaces A and B, the injections $i_1 : A \to A \vee B$ and $i_2 : B \to A \vee B$ are cofiber maps.

Definition 1.5.

Let S^n denote an oriented n-sphere, where $n \geq 2$. Let there be given a map $f : S^{2n-1} \to S^n$. Denote by σ and τ the generators determined by the given orientations of the cohomology groups in dimensions n and $2n$, respectively. Then the cup-product square σ^2 is some integral multiple of τ. Define the Hopf invariant of f as the integer $H(f)$ such that $\sigma^2 = H(f).\tau$.

1.3 Categorical Context

The context that we have in mind that suggests this line of investigation is that we want to know the complexity of the cofibers C_β and we will do that using L-S category and how it is linked to a space being a co-H-space. A very helpful reference in category theory is [Awo10] and for ample reading on L-S category you can check [Cor+03].

CHAPTER 2

HOMOLOGY OF COFIBERS OF MAPS BETWEEN MOORE SPACES

2.1 Algebraic Preliminaries

We want to study homomorphisms $f : \mathbb{Z}_m \to \mathbb{Z}_n$ for $m, n \in \mathbb{N}$. One such map is of particular importance, namely the unique homomorphism ℓ that makes the following diagram, in which the rows are exact, and $d = (m, n)$, the greatest common divisor of m and n.

$$
\begin{array}{ccccccc}
\mathbb{Z} & \xrightarrow{m} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/m & \longrightarrow & 0 \\
\downarrow{\scriptstyle \frac{m}{d}} & & \downarrow{\scriptstyle \frac{n}{d}} & & \vdots\,\textcircled{\scriptsize ℓ} & & \| \\
\mathbb{Z} & \xrightarrow{n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n & \longrightarrow & 0,
\end{array}
$$

We know that every homomorphism $f : \mathbb{Z}_m \to \mathbb{Z}_n$ is equal to $r.\ell$ for some $r \in \mathbb{Z}$. In fact r can be chosen to be less than or equal to $d = (m, n)$. We know that the kernel of $r.\ell$ is a cyclic subgroup of $\mathbb{Z}_m$ and we are interested in knowing it is isomorphic to what. We also know that the cokernel of $r.\ell$ is a cyclic quotient group of $\mathbb{Z}_n$ and we are interested in knowing it is isomorphic to what.

In particular, we are interested in maps between Moore spaces for cyclic groups in dimension 2. Among these maps are some special ones, namely

$$
\begin{array}{ccccc}
S^2 & \xrightarrow{m} & S^2 & \longrightarrow & M(\mathbb{Z}/m, 2) \\
\downarrow{\scriptstyle \frac{m}{d}} & & \downarrow{\scriptstyle \frac{n}{d}} & & \vdots\,\textcircled{\scriptsize λ} \\
S^2 & \xrightarrow{n} & S^2 & \longrightarrow & M(\mathbb{Z}/n, 2)
\end{array}
$$

in which the rows are cofiber sequences. We are interested in knowing what are the induced homomorphisms $r.\lambda_* : H_*(M(\mathbb{Z}/m, 2); \mathbb{Z}) \to H_*(M(\mathbb{Z}/n, 2); \mathbb{Z})$ and $k.(i' \circ \phi \circ q)_*$ and also what are the groups $\widetilde{H}_2(C)$ and $\widetilde{H}_3(C)$ and this is what we will focus on in the next two sections.

2.2 Homology of C_β in the Zero/Nonzero Homomorphism Cases

We will not be able to figure out the homology groups of the cofibers of $x_{r,k}$ without knowing the induced homomorphisms $x_* : H_2(M) \to H_2(N)$. Consider the following commutative diagram (where we deviated from the notation of [AG94] by assigning the induced map an asterisk below it)

$$\begin{array}{ccccccccc} S^2 & \xrightarrow{m} & S^2 & \xrightarrow{i} & M & \xrightarrow{q} & S^3 & \xrightarrow{\Sigma m} & S^3 \\ {\scriptstyle \frac{m}{d}}\downarrow & & {\scriptstyle \frac{n}{d}}\downarrow & & \downarrow{\scriptstyle \lambda} & & {\scriptstyle \frac{m}{d}}\downarrow & & {\scriptstyle \frac{n}{d}}\downarrow \\ S^2 & \xrightarrow{n} & S^2 & \xrightarrow{i'} & N & \xrightarrow{q'} & S^3 & \xrightarrow{\Sigma n} & S^3 \end{array}$$

In the previous figure λ is the map that is induced by the cofibers. Also, in the previous figure, the rows are cofiber sequences which means that if we apply the homology functor H_* to them we will get exact sequences. We know that S^2 has only nonzero stuff in dimension 2 and also S^3 has only nonzero stuff in dimension 3, so the Moore space $M = M(\mathbb{Z}_m, 2)$ maybe has nonzero stuff in dimension 2 and 3. So, we are interested at least in applying $\widetilde{H_2}$ to the previous diagram. We will also, have homology groups for M and $N = M(\mathbb{Z}_n, 2)$ and we will be interested to know what they are. Before applying $\widetilde{H_2}$ to the previous diagram, notice that, by the definition of Moore space, we have that $\widetilde{H_2}(M) = \mathbb{Z}_m$ and $\widetilde{H_2}(N) = \mathbb{Z}_n$ and also recall that, the homology groups of the sphere with coefficients in $\mathbb{Z}$ are defined as follows

$$\widetilde{H_i}(S^n) \cong \begin{cases} 0 & if i \notin \{0, n\} \\ \mathbb{Z} & if i \in \{0, n\}, \end{cases}$$

Then applying $\widetilde{H_2}$ to the previous commutative diagram we will get the following commutative diagram

$$\begin{array}{ccccccccc} \mathbb{Z} & \xrightarrow{m_*} & \mathbb{Z} & \xrightarrow{i_*} & \mathbb{Z}_m & \xrightarrow{q_*} & 0 & \xrightarrow{(\Sigma m)_*} & 0 \\ {\scriptstyle (\frac{m}{d})_*}\downarrow & & {\scriptstyle (\frac{n}{d})_*}\downarrow & & {\scriptstyle \lambda_*}\downarrow & & {\scriptstyle (\frac{m}{d})_*}\downarrow & & {\scriptstyle (\frac{n}{d})_*}\downarrow \\ \mathbb{Z} & \xrightarrow{n_*} & \mathbb{Z} & \xrightarrow{i'_*} & \mathbb{Z}_n & \xrightarrow{q'_*} & 0 & \xrightarrow{(\Sigma n)_*} & 0 \end{array}$$

Then we decided to take $r = 1, k = 0$ in $x_{r,k}$ to figure out what is the induced homomorphism of $r.\lambda_*$ and then we took m, n as different numbers to understand the induced homomorphism of $r.\lambda_*$. For example, taking $m = 12, n = 16$ and then $d = 4$, we will get the following diagram

$$\begin{array}{ccccccccc} \mathbb{Z} & \xrightarrow{.12} & \mathbb{Z} & \xrightarrow{i^*} & \mathbb{Z}_{12} & \xrightarrow{q^*} & 0 & \xrightarrow{(\Sigma m)^*} & 0 \\ {\scriptstyle .3}\downarrow & & {\scriptstyle .4}\downarrow & & {\scriptstyle \lambda_*}\downarrow{\scriptstyle ?} & & {\scriptstyle (\frac{m}{d})^*}\downarrow & & \downarrow{\scriptstyle (\frac{n}{d})^*} \\ \mathbb{Z} & \xrightarrow{.16} & \mathbb{Z} & \xrightarrow{(i')^*} & \mathbb{Z}_{16} & \xrightarrow{(q')^*} & 0 & \xrightarrow{(\Sigma n)^*} & 0 \end{array}$$

We wanted to figure out what is $\lambda^*([1])$, so we used the second rectangle in the previous figure to get the following commutative diagram:

$$\begin{array}{ccc} 1 & \longmapsto & \bar{1} \\ {\scriptstyle .4}\downarrow & & \downarrow{\scriptstyle \lambda_*} \\ 4 & \longmapsto & \bar{4} \end{array}$$

Therefore, $\lambda_* : \mathbb{Z}_{12} \to \mathbb{Z}_{16}$ is defined as $\bar{1} \mapsto \bar{4}$ and so in total λ_* do the following in dimension 2

$$\bar{1} \mapsto \bar{4}$$
$$\bar{2} \mapsto \bar{8}$$
$$\bar{3} \mapsto \bar{12}$$
$$\bar{4} \mapsto \bar{0}$$

Then we repeated this step for about 20 different values of m and n and then we decided that it will be easier if we only consider m and n to be of prime powers. Finally, we reached that $(r.\lambda)_* = r\frac{n}{d}$ where $0 \le r < d$. Actually, we found it after that clearly stated in [AG94] on pg. 681.

Now, we want to figure out what does $x_{0,1}$ induces in homology. From pg.677 of [AG94], if i' denote the inclusion $S^2 \to N$ and q' denote the projection $N \to S^3$, consider the sequence of the composition $i'\phi q$,

$$M \xrightarrow{q} S^3 \xrightarrow{\phi} S^2 \xrightarrow{i'} N.$$

Now, applying $\widetilde{H_2}$ to the previous sequence we will get

$$\widetilde{H_2}(M) \xleftarrow{q_*} \widetilde{H_2}(S^3) \xleftarrow{\phi_*} \widetilde{H_2}(S^2) \xleftarrow{i'_*} \widetilde{H_2}(N).$$

Which gives us the following sequence

$$\mathbb{Z}_m \xleftarrow{q_*} 0 \xleftarrow{\phi_*} \mathbb{Z} \xleftarrow{i'_*} \mathbb{Z}_n.$$

And since $\widetilde{H_2}(S^3) = 0$, then the whole composite composite induces zero in homology in dimension 2. If we repeat the previous steps for $\widetilde{H_3}$, we will find that the composition $i'\phi q$ also induces zero in homology in dimension 3.

Now, to figure out $\widetilde{H_2}(C)$ and $\widetilde{H_3}(C)$ consider the long exact sequence in homology of the following cofiber sequence $M \xrightarrow{x_{r,k}} N \to C_{x_{r,k}}$, which is

$$\cdots \to 0 \to \widetilde{H_3}(M) = 0 \xrightarrow{(x_{r,k})_*} \widetilde{H_3}(N) = 0 \to \widetilde{H_3}(C_{x_{r,k}}) \to \widetilde{H_2}(M) \xrightarrow{(x_{r,k})_*} \widetilde{H_2}(N) \to \widetilde{H_2}(C_{x_{r,k}}) \to 0$$

And so the kernel of β_* is $\widetilde{H_3}(C_\beta)$ and the cokernel of β_* is $\widetilde{H_2}(C_\beta)$. the other terms in the previous sequence is zero because of the calculations we wrote in chapter 4. Note that the previous homology is with integer coefficients.

From the numeric example of λ_* given above, we know that $\ker(\lambda_*) = \{\bar{0}, \bar{4}, \bar{8}\}$ and $\ker(\lambda_*) \cong \mathbb{Z}_4$. Then doing many other numerical examples with different values for m and n, we were able to conclude when are the kernel and cokernel of λ_* equals 0 and so we concluded the following theorem about λ_*

Theorem 1.

If we have the following homomorphism $\lambda_* : \mathbb{Z}_{p^a} \to \mathbb{Z}_{p^b}$ *defined by multiplication by* n/d *where* $n = p^b$ *and* $d = \gcd(p^a, p^b)$. *We will prove the following:*

1. *if $a \geq b$ then λ_* is onto.*
2. *if $a \leq b$ then λ_* is 1-1.*

Proof.

1. The map
$$\lambda_* : \mathbb{Z}/p^a\mathbb{Z} \longrightarrow \mathbb{Z}/p^b\mathbb{Z}$$
is given explicitly by $\lambda_*(x) = \frac{n}{d}x$. In the first case, this simply sends a number mod p^a to itself mod p^b. To see that this is surjective note that
$$\forall y \in Z_{p^b}, \exists y \in Z_{p^a} \text{ such that} \lambda_*(y \pmod{p^a}) = y \pmod{p^p}.$$
Therefore λ_* is surjective as required.

2. **First: showing that λ_* is well-defined:**

   we have $d = p^a$ and hence $n/d = p^{b-a}$first we will check why f is well-defined. here is our check. Assume that $x \pmod{p^a} = y \pmod{p^a}$, then that means that $p^a \mid (x-y)$, which means that $p^{b-(b-a)}k = x - y$ for some k. Multiplying by p^{b-a} on both sides we get $p^b k = p^{b-a}(x-y)$, therefore $p^{b-a}x \pmod{p^b} = p^{b-a}y \pmod{p^b}$ and hence the required $\lambda_*(x \pmod{p^a}) = \lambda_*(y \pmod{p^a})$ is proved. Therefore, λ_* is well defined as required.

   Second: showing that λ_* is one - one:

   Now, assume that $\lambda_*(x \pmod{p^a}) = \lambda_*(y \pmod{p^a})$ we want to show $x = y$ in $\mathbb{Z}/p^a\mathbb{Z}$. Assume that $\lambda_*(x \pmod{p^a}) = \lambda_*(y \pmod{p^a})$ then $p^{b-a}x \pmod{p^b} = p^{b-a}y \pmod{p^b}$, then p^b divides $p^{b-a}x - p^{b-a}y = p^{b-a}(x-y)$. Therefore, there exists k such that $p^b k = p^{b-a}(x-y)$. Cancelling, we get $p^{b-(b-a)}k = x - y$. And that means that $p^a \mid (x-y)$ which means that $x \pmod{p^a} = y \pmod{p^a}$ as required.

□

But actually our map was not λ_*, it was $(r.\lambda)_*$ and since we know the following theorem:

Theorem 2.
If $f : X \to Y$, and $r \in \mathbb{Z}$, then $(r.f)_* = r.f_*$.

Then we will prove the following theorem:

Theorem 3.
Let $(r\lambda)_* : \mathbb{Z}_{p^a} \to \mathbb{Z}_{p^b}$ be the map defined by multiplication by rn/d where $n = p^b$ and $d = \gcd(p^a, p^b)$ and $r = p^t$ and $0 \leq r < d$ and we know that if $a \geq b$ and we are not in the case of the zero homomorphism i.e. we took $t = b - 1$, we will have:

1. *ker $(r\lambda)_* = Z_{p^{a+t-\min\{a,b\}}}$.*
2. *coker $(r\lambda)_* = Z_{p^{b+t-\min\{a,b\}}}$.*

Proof.

So $(r\lambda)_*$ takes $\overline{1} \in \mathbb{Z}/p^a$ to $\overline{p^{t+b}/d} \in \mathbb{Z}/p^b$, where $d = \gcd(p^a, p^b)$. But since the only positive divisors of p^a and p^b are powers of p, so in fact $d = p^{\min\{a,b\}}$, and we can write

$(r\lambda)_*(\overline{1}) = \overline{p^{t+b-\min\{a,b\}}}$. To see that $(r\lambda)_*$ is a well-defined map, we need to show that $\overline{p^a}(r\lambda)_*(\overline{1}) = \overline{0} \in \mathbb{Z}/p^b$. This amounts to showing that $p^a p^{t+b-\min\{a,b\}}$ is divisible by p^b in $\mathbb{Z}$, i.e. that $a + t + b - \min\{a, b\} \geq b$. Since $t \geq 0$, this is clear.

Now, $\overline{n} \in \ker(r\lambda)_*$ if and only if $\overline{0} = (r\lambda)_*(\overline{n}) = \overline{np^{t+b-\min\{a,b\}}} in \mathbb{Z}/p^b$, which is in turn true if and only if $np^{t+b-\min\{a,b\}}$ is divisible by p^b in $\mathbb{Z}$. If p^s is the highest power of p dividing n, then this is true if and only if $s + t + b - \min\{a, b\} \geq b$, i.e. if and only if $s + t \geq \min\{a, b\}$. but we are assuming that $(r\lambda)_*$ is not the zero map, so we know $t < \min\{a, b\}$ so this number makes sense. Thus a generator for $\ker(r\lambda)_*$ is given by $\overline{p^{\min\{a,b\}-t}}$. The order of $\overline{p^{\min\{a,b\}-t}}$ is $p^{a-(\min\{a,b\}-t)} = p^{a+t-\min\{a,b\}}$, and subgroups of cyclic groups are cyclic, so

$$\boxed{\ker(r\lambda)_* \cong \mathbb{Z}/p^{a+t-\min\{a,b\}}} \tag{2.1}$$

In particular, in the case mentioned in the question, where $b \leq a$ and $t = b - 1$, we have

$$\boxed{\ker(r\lambda)_* \cong \mathbb{Z}/p^{a-1}}$$

Now, the cokernel of $(r\lambda)_*$ is the quotient of $\mathbb{Z}/p^b$ by the image of $(r\lambda)_*$. Quotients of cyclic groups are cyclic, and cyclic groups are uniquely determined by their cardinalities, so the cokernel of $(r\lambda)_*$ will be determined by its size. Now, by the first isomorphism theorem, the image of $(r\lambda)_*$ is isomorphic to the quotient $\frac{\mathbb{Z}/p^a}{\ker(r\lambda)_*}$. Since $|\mathbb{Z}/p^a| = p^a$, and $|\ker(r\lambda)_*| = p^{a+t-\min\{a,b\}}$ by the computation above, we have

$$|\operatorname{im}(r\lambda)_*| = \frac{p^a}{p^{a+t-\min\{a,b\}}} = p^{\min\{a,b\}-t}.$$

(Again, we are assuming $t < \min\{a, b\}$, so this number makes sense.) Since $|\mathbb{Z}/p^b| = p^b$, we can hence compute

$$|\operatorname{coker}((r\lambda)_*)| = \frac{|\mathbb{Z}/p^b|}{|\operatorname{im}(r\lambda)_*|} = \frac{p^b}{p^{\min\{a,b\}-t}} = p^{b+t-\min\{a,b\}},$$

and so

$$\boxed{\operatorname{coker}(r\lambda)_* \cong \mathbb{Z}/p^{b+t-\min\{a,b\}}} \tag{2.2}$$

In particular, in the case we mentioned above, where $b \leq a$ and $t = b - 1$, we have

$$\boxed{\operatorname{coker}(r\lambda)_* \cong \mathbb{Z}/p^{b-1}}.$$

Now, if $a \leq b$ and also we are not in the case of the zero homomorphism i.e. we took $t = a - 1$.

We will have also, $\ker(r\lambda)_* \cong \mathbb{Z}/p^{a-1}$, and $\operatorname{coker}(r\lambda)_* \cong \mathbb{Z}/p^{b-1}$

□

Now, calculating $\ker(r\lambda)_*$ and $\operatorname{coker}(r\lambda)_*$ if we are in the zero homomorphism case:

Consider the following sequence

$$0 \to \cdots \to \mathbb{Z}_{p^a} \xrightarrow{\lambda_*} \mathbb{Z}_{p^b} \to \cdots \to 0$$

Then,

$$d = \begin{cases} p^b & if \quad a \geq b \\ p^a & if \quad a < b, \end{cases}$$

So, for $\lambda_* = \frac{n}{d}$, where $n = p^b$, we have

$$\lambda_* : \bar{1} \mapsto \begin{cases} \bar{1} & if \quad a \geq b \\ \overline{p^{b-a}} & if \quad a < b, \end{cases}$$

And so,

$$r.\lambda_* : \bar{1} \mapsto \begin{cases} \overline{p^t} & if \quad a \geq b \\ \overline{p^{b-a+t}} & if \quad a < b, \end{cases}$$

Where we assumed that $r = xp^t$ and we took $x = 1$ for simplicity. Under what circumstances is $r.\lambda_* = p^t.\lambda_*$ is the zero homomorphism? i.e., $(p^t.\lambda_*)[1] = [0]$. The answer is the zero homomorphism occurs if

$$\begin{cases} p^b | p^t & if \quad a \geq b \\ p^b | p^{b-a+t} & if \quad a < b, \end{cases}.$$

And so, $r.\lambda_*$ is the zero homomorphism when the following happens

$$\begin{cases} b \leq t & if \quad a \geq b \\ a \leq t & if \quad a < b, \end{cases}.$$

Now, in the following sequence

$$0 \to \ker(r\lambda)_* \to \mathbb{Z}_{p^a} \xrightarrow{(r.\lambda)_*} \mathbb{Z}_{p^b} \to \operatorname{coker}(r\lambda)_* \to 0$$

If $\ker(r\lambda)_* = 0$, then we will get that

$$\boxed{\ker(r\lambda)_* \cong \mathbb{Z}_{p^a} \text{ and } \operatorname{coker}(r\lambda)_* \cong \mathbb{Z}_{p^b}.} \tag{2.3}$$

We will complete calculating all other homologies of the cofiber C_β later in chapter 4.

CHAPTER 3

CO-H-STRUCTURE ON MOORE SPACE COFIBERS

3.1 Main Goal and Main Results

We want to study the cofiber of $x_{r,k} : M(\mathbb{Z}_{p^a}, 2) \to M(\mathbb{Z}_{p^b}, 2)$, specifically, we want to know for what values of r and k is C_β a co-H-space. The main results of this chapter are:

Case(I) - When $l = 0$ (i.e., we are using the suspension operator), we will give the values of j for which the following 2 congruences are solvable

First:

$$\boxed{0 \equiv xp^{a+s-b}j + yp^t \pmod{p^b}} \tag{3.1}$$

where p is an odd prime and x, y are not divisible by p. And s, t are any nonnegative integers, b is a positive integer, and $a \geq b$.

Second:

$$\boxed{0 \equiv xp^s j + yp^t \pmod{p^a}} \tag{3.2}$$

where where p is an odd prime and x, y are not divisible by p. and s, t are any nonnegative integers, a is a positive integer and $a \leq b$.

Case(II) - For what values of l and j simultaneously are the following 2 congruences solvable

First:

$$\boxed{l(xp^s)^2 \equiv xp^s jp^{a-b} + yp^t \pmod{p^b}} \tag{3.3}$$

where x, y are not divisible by p. and s, a, t are any nonnegative integers, b is a positive integer and $a \geq b$.

Second:

$$\boxed{p^{b-a}l(xp^s)^2 \equiv xp^s j + yp^t \pmod{p^a}} \tag{3.4}$$

where x, y are not divisible by p. and s, t are any nonnegative integers, a is a positive integer and $a \leq b$.

Why exactly the above 4 congruences? that is exactly what we are going to explain in the following section.

3.2 Preliminaries

Recall the following theorems

Theorem 4.
$M(G, 2)$ is a suspension, and hence has at least one co-H-structure.

Theorem 5.
If $f : X \to Y$, and there is at least one pair of co-H structures φ(for X) and ψ(for Y) such that f is a co-H-map from the co-H-space (X, φ) to the co-H-space (Y, ψ), then the cofiber C_f has at least one co-H structure.

The previous theorem can be concluded from the first line of Theorem 3.4 in [BH60] where quasiprimitive for spaces of category 1 is the same as being a co-H-map.

Also, Arkowitz and Golasinski have

1. Enumerated all of the co-H-structures for $M(\mathbb{Z}_n, 2)$ in Proposition 13 in [AG94] ; these are maps φ_l for various l. When $l = 0, \varphi_l$ is the suspension co-H-structure.
2. Given,in section 6 of [AG94] , which was speaking about the Co-H-Maps and comultiplications of $M(\mathbb{Z}_m, 2)$, an explicit congruence involving r, k, j, l that is true exactly when $x_{r,k} : (M, \varphi_l) \to (N, \psi_j)$ is a co-H-map. Here is the statement of theorem 14 there

Theorem 6.
The map $x_{r,k} : (M, \varphi_l) \to (N, \psi_j)$ is a co-H-map if and only if

$$\left(\frac{n}{d}\right)^2 lr^2 \equiv rj\frac{m}{d} + k - \frac{r(r-1)}{2}\frac{m(m-1)}{2}\left(\frac{n}{d}\right)^2 \pmod d \tag{3.5}$$

Now, we got the congruence in (3.1) in the previous section by substituting $m = p^a, n = p^b, r = xp^s$ and $k = yp^t$ and noticing that in this case $d = p^b$ as $a \geq b$ and also noticing that $\frac{r(r-1)}{2}\frac{m(m-1)}{2} \equiv 0 \pmod{p^b}$ because $\frac{r(r-1)}{2}\frac{m(m-1)}{2} \equiv 0 \pmod{p^a}$ because $m = p^a$ and because $a \geq b$ and similarly for the congruence in (3.3).

And also, we got the congruence in (3.2) in the previous section by substituting $m = p^a, n = p^b, r = xp^s$ and $k = yp^t$ and noticing that in this case $d = p^a$ as $a \leq b$ and also noticing that $\frac{r(r-1)}{2}\frac{m(m-1)}{2}\left(\frac{n}{d}\right)^2 \equiv 0 \pmod{p^a}$ because $m = p^a$ and similarly for the congruence in (3.4).

Definition 3.1.
A *comultiplication* or *co-H-structure* on a space X is a map $\varphi : X \to X \vee X$ such that $j\varphi \simeq \Delta$, where $j : X \vee X \to X \times X$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. Equivalently, $\varphi : X \to X \vee X$ is a comultiplication if and only if $q_1\varphi = 1 = q_2\varphi$ where $q_1, q_2 : X \vee X \to X$ are the two projections.

Definition 3.2.
A space X together with a comultiplication φ is called a *co-H-space*.

Definition 3.3.

If (X', φ') and (X, φ) are co-H-spaces and $h\colon X' \to X$ is a map, we say that $h\colon (X', \varphi') \to (X, \varphi)$ is a *co-H-map* if $\varphi h = (h \vee h)\varphi'$. You can also look at definition 2.2.8 in [Ark11] to see this definition using a commutative diagram.

3.2 When is $x_{r,k}$ a Co-H-Map?

We want to complete the statement of the following proposition

Proposition 1.

Fix r, k.

(1) *There is a j such that $x_{r,k}$ is a co-H-map from the co-H-space $(M(\mathbb{Z}_m, 2), \varphi_0)$ to the co-H-space $(M(\mathbb{Z}_n, 2), \psi_j)$ if and only if*

what?

(2) *There is a j and an l such that $x_{r,k}$ is a co-H-map from the co-H-space $(M(\mathbb{Z}_m, 2), \varphi_l)$ to the co-H-space $(M(\mathbb{Z}_n, 2), \psi_j)$ if and only if*

what?

Starting to find an answer for (1) in the previous proposition,

First: assuming that $l = 0$ in theorem 14 formula in [AG94] i.e. in (5) and $a \geq b$.

We will get the following congruence:

$$0 \equiv xp^{a+s-b}j + yp^t \pmod{p^b}$$

where p is an odd prime and x, y are not divisible by p. And s, t are any nonnegative integers, b is a positive integer, and $a \geq b$.

Now, For what values of j is this congruence solvable?

Since the given congruence above is equivalent to

$$xp^{a+s-b}j + yp^t = k'p^b$$

for some k'; and then we have to divide both sides by $p^{m'}$ where $m' = \min\{a + s - b, t, b\}$, then to know the values of j that makes this congruence solvable:

Since we do not know which of the quantities is minimal we can split the problem up into 3 different cases. since we have that $a + s - b \geq 0$.

If $m' = b$. Then the equation is satisfied for all j because irrespective of the value of j both terms are divisible by $p^{m'}$ and so the congruence is automatically satisfied.

If $m^{'} \neq b, m^{'} = a+s-b$. Then $xj + yp^{t-m^{'}} = k'p^{b-m^{'}}$ and the equation can be solved if and only if j is a multiple of $p^{n^{'}}$, where $n^{'}$ is the minimum of $t-m^{'}$ and $b-m^{'}$.

If $m^{'} \neq b, m^{'} \neq a+s-b$. Then $xjp^{a+s-b-m^{'}} + y = k'p^{b-m^{'}}$ and since p is not a factor of y there are no solutions. Note that there is no need for p to be odd in the previous case i.e. when $a \geq b$.

Second: assuming that $l = 0$ in theorem 14 formula in [AG94] and $a \leq b$.

We will get the following congruence

$$0 \equiv xp^s j + yp^t \pmod{p^a}$$

where x, y are not divisible by p. and s, t are any nonnegative integers, a is a positive integer and $a \leq b$.

Since the given congruence above is equivalent to

$$xp^s j + yp^t = up^a \text{ for some } u;$$

and then dividing both sides by $p^{m^{'}}$ where $m^{'} = \min\{s, t, a\}$, we will get the following cases for the values of j that makes this congruence solvable:

If $m^{'} = a$**.** Then the equation is satisfied for all j.
If $m^{'} \neq a, m^{'} = t$ **and** $t \neq s$. Then

$$xp^{s-t}j + y = up^{a-t}$$

and since p is not a factor of y then there are no solutions.
If $m^{'} \neq a, m^{'} = t$ **and** $t = s$.Then

$$xj + y = up^{a-m^{'}}.$$

For any j not divisible by p we can therefore choose $y = up^{a-m^{'}} - xj$. But if j is divisible by p we will have no solution.

If $m^{'} \neq a, m^{'} \neq t$. Then

$$xj + yp^{t-s} = up^{a-s};$$

Now, distinguish between 2 cases:

Case(I): j is not divisible by p. Since p is not a factor of x there are no solutions.
Case(II): j is divisible by p. There may be solutions when j is divisible by p. For example, when $a = t$ and $j = Jp^{a-s}$, the equation becomes $xJ + y = u$ and so there are infinitely many solutions for x and y.

Note also: there is no need for p to be odd in the previous case i.e. when $a \leq b$.

Now, starting to find an answer for (2) in proposition 1

For what values of l and j simultaneously is the congruence of theorem 14 in [AG94] solvable?

1. **CASE A.** $a \geq b$

Here is the congruence given in theorem 14 after substituting the appropriate values for m, n, r and k:

$$l(xp^s)^2 \equiv xp^s jp^{a-b} + yp^t \pmod{p^b}$$

where x, y are not divisible by p. and s, a, t are any nonnegative integers, b is a positive integer and $a \geq b$.

Roughly (there is some overlapping) we could say $4! = 24$ for different orders of the four variables($2s, s+a-b, t, b$) multiplied by $2^3 = 8$ possibilities for the signs($>, =$).

A mechanical way of proceeding would be to analyse separately all the different possibilities (dreadful!) such as the following one.

$$2s = s + a - b > t = b.$$

CASE(I): b is the minimal i.e. the right-hand number.

Then p^b divides all the terms and so the congruence is satisfied for all pairs (l, j).

CASE(II): If $t \geq b > 2s = s + a - b$.

Then the equation simplifies to

$$lx^2 \equiv xj \pmod{p^{b-2s}}.$$

So the equation is solvable if and only if either $l \equiv j \equiv 0 \pmod{p^{b-2s}}$ or l and j are divisible by the same power (possibly zero) of p.

Even a more efficient way to know when is the congruence in theorem 14 is solvable is to apply the following general result about the solution of equations of this type.

Theorem 7.

Let $M = \min\{A, B, C, D\}$. Then the equation

$$Xp^A + Yp^B + Zp^C \equiv 0 \pmod{p^D}$$

can be solved with $XYZ \not\equiv 0 \pmod p$ *if and only if either*

(a) $M = D$

(b) M *is equal to precisely two of* $\{A, B, C\}$

(c) M *is equal to all three of* $\{A, B, C\}$ *and* $p \neq 2$.

Proof.

(a) CASE I. If $M = D$.Then each of Xp^A, Yp^B and Zp^C is divisible by p^D whatever the values of X, Y or Z.

(b) CASE II. If $M = A = B, M < C$.Then the equation is solved by, for example, $Y = Z = 1, X = -1 - p^{C-M}$. By symmetry, we have similar solutions when $M = A = C$ and $M = B = C$.

(c) CASE III. If $M = A = B = C, M < D$.If $p \neq 2$ then the equation is solved by, for example, $Y = Z = 1, X = p^{D-M} - 2$. However, if $p = 2$ then $X + Y + Z$ has to be even and so at least one of X, Y, Z is even.

(d) CASE IV. If B, C and D are all greater than $M = A$.Then dividing the equation through by p^M gives

$$X + Yp^{B-M} + Zp^{C-M} \equiv 0 \pmod{p^{D-M}}$$

and then we have the contradiction that p divides X.By symmetry, we have similar contradictions when $M = B$ and $M = C$.

□

TRANSFORMATIONS TO BE CONSIDERED.

Let $l = Lp^U$ and $j = Jp^V$ where $LJ \not\equiv 0 \pmod p$. Then let

$$A = a - b + s + V, B = t, C = 2s + U, D = b.$$

Theorem 8.

For non-negative integers a, b, s *and* t *such that* $a \geq b$, *the equation*

$$l(xp^s)^2 \equiv xp^s jp^{a-b} + yp^t \pmod{p^b} \tag{3.6}$$

can be solved with $xy \not\equiv 0 \pmod p$ *if and only if the equation*

$$Xp^A + Yp^B + Zp^C \equiv 0 \pmod{p^D} \tag{3.7}$$

can be solved with $XYZ \not\equiv 0 \pmod p$.

Proof.

If the x, y equation can be solved then simply take

$$X = xJ, Y = y, Z = -Lx^2.$$

Conversely, suppose we have a solution of the X, Y, Z equation. Since X and L are coprime to p we can get:

$$x \equiv -\frac{JZ}{LX} \pmod{p^D}, y \equiv -\frac{J^2YZ}{LX^2} \pmod{p^D}.$$

Details of solution for $x \equiv -\frac{JZ}{LX} \pmod{p^D}$

LX and p^D are coprime. Therefore there are integers α and β such that $\alpha LX + \beta p^D = 1$. Then $\alpha LX \equiv 1 \pmod{p^D}$ and so $-\alpha JZLX \equiv -JZ \pmod{p^D}$. Then take $x = -\alpha JZ$.

Details of solution for $y \equiv -\frac{J^2YZ}{LX^2} \pmod{p^D}$.

LX^2 and p^D are coprime. Therefore there are integers α' and β' such that $\alpha' LX^2 + \beta' p^D = 1$.

Then $\alpha' LX^2 \equiv 1 \pmod{p^D}$ and so multiplying by $-J^2YZ$ we will get $-\alpha' LX^2 J^2YZ \equiv -J^2YZ \pmod{p^D}$.Then take $y = -\alpha' J^2YZ$.

Now, returning back to our original proof, from the congruence (3.6) we then have

$$xp^s jp^{a-b} + yp^t - l(xp^s)^2 = xJp^A + yp^B - Lx^2p^C$$

and by the subtituting the previous congruences of x and y, we get

$$-\frac{J^2Z}{LX^2}(Xp^A + Yp^B + Zp^C) \equiv 0 \pmod{p^D}.$$

because we know that $Xp^A + Yp^B + Zp^C \equiv 0 \pmod{p}^D$.

So,we can now see that the original equation can be solved if and only if the quantities

$$A = a - b + s + V, B = t, C = 2s + U, D = b$$

satisfy the conditions of Theorem 7.

□

CONCLUSION.

Let $v_p(n)$ represent the highest power of p which divides the number n. You will notice that the precise values of j and l do not affect the existence of solutions; all that matters is $v_p(j)$ and $v_p(l)$.

CASE 1. $t \geq b$, $v_p(l) \geq b - 2s$ and $v_p(j) \geq 2b - a - s$.

CASE 2. There are three possibilities.

$v_p(l) > t - 2s$ and $v_p(j) = b - a - s + t$.

$v_p(l) < t - 2s$ and $v_p(j) = b - a + s + v_p(l)$.

$v_p(l) = t - 2s$ and $v_p(j) > b - a + t - s$.

CASE 3.

-If $p \neq 2$. We will have that $t < b$, $v_p(l) = t - 2s$ and $v_p(j) = b - a + t - s$.

-If $p = 2$ then, for a solution, we require at least one of $X = xJ, Y = y, Z = -Lx^2$ to be even. However, these quantities cannot be multiples of p and so are all odd. So there is no solution for case 3 when $p = 2$.

2. **CASE B.** $a \leq b$

Here is the congruence given in theorem 14 of [AG94] after substituting the appropriate values for m, n, r and k:

$$p^{b-a}l(xp^s)^2 \equiv xp^s j + yp^t \pmod{p^a}$$

where x, y are not divisible by p. and s, b, t are any nonnegative integers, a is a positive integer and $a \leq b$.

We will also use theorem 7 of the previous case as a reference of the cases at which the solution exists.

TRANSFORMATIONS TO BE CONSIDERED.

Let $l = Lp^U$ and $j = Jp^V$ where $LJ \not\equiv 0 \pmod{p}$. Then let

$$A = s + V, B = t, C = 2s + b - a + U, D = a.$$

Theorem 9.

For non-negative integers b, s and t and for a positive integer a with $a \leq b$, the equation

$$p^{b-a}l(xp^s)^2 \equiv xp^s j + yp^t \pmod{p^a} \tag{3.8}$$

can be solved with $xy \not\equiv 0 \pmod{p}$ if and only if the equation

$$Xp^A + Yp^B + Zp^C \equiv 0 \pmod{p^D} \tag{3.9}$$

can be solved with $XYZ \not\equiv 0 \pmod{p}$.

Proof. If the x, y equation can be solved then simply take

$$X = xJ, Y = y, Z = -Lx^2.$$

Conversely, suppose we have a solution of the X, Y, Z equation. Since X and L are coprime to p, then by the same calculations given in the previous case, we get the following congruence equations:

$$x \equiv -\frac{JZ}{LX} \pmod{p^D}, y \equiv -\frac{J^2YZ}{LX^2} \pmod{p^D}.$$

Now, from (3.8), we have

$$xp^sj + yp^t - p^{b-a}l(xp^s)^2 = xJp^A + yp^B - Lx^2p^C$$

and by substituting the previous congruences of x and y, we get

$$-\frac{J^2Z}{LX^2}(Xp^A + Yp^B + Zp^C) \equiv 0 \pmod{p^D}.$$

Because we know that $Xp^A + Yp^B + Zp^C \equiv 0 \pmod{p^D}$.

We can now see that the original equation can be solved if and only if the quantities

$$A = s + V, B = t, C = 2s + b - a + U, D = a$$

satisfy the conditions of Theorem 7.

□

CONCLUSION.

Let $v_p(n)$ represent the highest power of p which divides the number n. You will notice that the precise values of j and l do not affect the existence of solutions; all that matters is $v_p(j)$ and $v_p(l)$.

CASE 1. $t \geq a$, $v_p(l) \geq 2a - 2s - b$ and $v_p(j) \geq a - s$.

Where the first inequality because $B \geq D$, the second inequality because $C \geq D$ and the third inequality because $A \geq D$.

CASE 2. There are three possibilities.

-$v_p(l) > t - 2s - b + a$ and $v_p(j) = t - s$.

Where the inequality because $B < C$ and the equality because $A = B$.

-$v_p(j) < t - s$ and $v_p(l) = -b + a - s + v_p(j)$.

Where the inequality because $A \leq B$ and the equality because $A = C$.

-$v_p(l) = t - 2s - b + a$ and $v_p(j) > t - s$.

Where the equality because $B = C$ and the inequality because $B < A$.

CASE 3. We have two possibilities.

-If $p \neq 2$, then $t < a$, $v_p(j) = t - s$ and $v_p(l) = a - b + t - 2s$. Where the inequality because $B < D$, the first equality because $B = A$ and the second equality because $B = C$.

-If $p = 2$ then there is no solution for the same reason as that in the conclusion after theorem 8.

Finally,from all the previous, I will leave you to complete proposition 1 statement.

3.3 What is Next?

Now we have the conditions under which C_β has a co-H-structure. But it can happen that the map f in theorem 3.3.1 is not a co-H-map, and yet its cofiber is a co-H-space. Since our goal is to completely nail this down, we will have the next 3 chapters.

CHAPTER 4

HOMOLOGY AND COHOMOLOGY

4.1 Main Goal and Main Results

Our main goal is calculating homologies and cohomologies of $M(G,2)$ (and consequently of $N(G,2)$) and of C_β in case of either integral or non-integral coefficients to be used in the proof of that $x_{r,k}$ is an additive function in the next chapter.

Our main results that we will get in this chapter are Calculating

1. $H_*(M(\mathbb{Z}_{p^a},2);G)$.
2. $H^*(M(\mathbb{Z}_{p^a},2);G)$.
3. $H_*(C_\beta;\mathbb{G})$.
4. $H^*(C_\beta;G)$ where $G=\mathbb{Z}$ or $G=\mathbb{Z}_{p^k}$.

Also,Using commutative diagrams, we will prove that induced map of addition is the same as addition of induced maps either for homology or cohomology and we will explain what does addition mean in our case.

4.2 Preliminaries

In our calculations, we use the statements of the Universal Coefficient Theorems in [Ark11], specifically Theorems C.4, C.5 on pg. 311. We begin with the universal coefficient theorem for cohomology. This theorem expresses cohomology with coefficients in an abelian group in terms of integral homology. Here is its statement

Universal Coefficient Theorem for cohomology

Let X be a space and let G be an abelian group. Then there exists a short exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(X),G) \to H^n(X;G) \to \mathrm{Hom}(H_n(X),G) \to 0.$$

that splits.

Universal Coefficient Theorem for homology

Let X be a space and let G be an abelian group. Then there exists a short exact sequence

$$0 \to H_n(X) \otimes G \to H_n(X;G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0$$

that splits.

For any abelian group G, the tensor product $\mathbb{Z}/m \otimes G$ and the Tor group $\mathrm{Tor}(\mathbb{Z}/m, G)$ can be defined as the kernel and the cokernel in the following exact sequence

$$0 \to \mathrm{Tor}(\mathbb{Z}/m, G) \to G \xrightarrow{m} G \to \mathbb{Z}/m \otimes G \to 0.$$

Dually, the Hom and the Ext functors can be defined as the kernel and the cokernel in the following exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/m, G) \to G \xrightarrow{m} G \to \mathrm{Ext}(\mathbb{Z}/m, G) \to 0.$$

Recall the following propositions:

Proposition 2.

The homomorphism groups satisfy the following isomorphisms where $d = (m, n)$

$$1 - \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$
$$2 - \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n) \cong \mathbb{Z}_n$$
$$3 - \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0$$
$$4 - \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d.$$

Proposition 3.

The Ext *functor is a derived functor, the name comes from the fact that the first* Ext *group* Ext^1 *classifies extensions of one module by another.The* Ext *functor satisfies the following isomorphisms where* $d = (m, n)$

$$1 - \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$$
$$2 - \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}_n) = 0$$
$$3 - \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \cong \mathbb{Z}_m$$
$$4 - \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$$

Proposition 4.

The tensor product symbolized by $\otimes$ *satisfies the following isomorphisms where* $d = (m, n)$

$$1 - \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$$
$$2 - \mathbb{Z} \otimes \mathbb{Z}_n = \mathbb{Z}_n$$
$$3 - \mathbb{Z}_m \otimes \mathbb{Z} = \mathbb{Z}_m$$
$$4 - \mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_d.$$

Proposition 5.

The Tor *functor satisfies the following isomorphisms where* $d = (m, n)$

$$1 - \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}) = 0$$
$$2 - \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}_n) = 0$$
$$3 - \mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}) = 0$$
$$4 - \mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d.$$

For ample details about Hom, Ext, Tor and tensor product see [Fuc03; Rot88; Rot79].

4.3 Homologies and Cohomologies of $M(\mathbb{Z}_{p^a}, 2)$

Now we will make use of the previous formulas to understand the homology and cohomology of maps between Moore spaces.

The Moore space for $\mathbb{Z}/m$ in dimension k can be defined as any space homotopy equivalent to the cofiber in the following cofiber sequence

$$S^k \xrightarrow{m} S^k \to M(\mathbb{Z}/m, k),$$

where m denotes the degree m map between spheres.

Now, we want to complete the following proposition

Proposition 6.

1. $H_*(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) =?$
2. $H^*(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) =?$
3. $H_*(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) =?$
4. $H^*(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) =?$

Here is what we did:

1**- Finding** $H_*(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z})$.

Using the following definition of Moore space:

Let G be an abelian group and n an integer ≥ 2. A Moore space of type (G, n) is $1-$connected, $CW-$complex X such that:

$$\widetilde{H}_i(X) = \begin{cases} 0 & if\, i \neq n, \\ G & if i = n. \end{cases}$$

We can say that $H_0 = \mathbb{Z} \oplus \widetilde{H_0}$ and $H_i = \widetilde{H_i}$ for $i \geq 1$.

Or we can say more explicitly that:

$$H_i(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = \begin{cases} \mathbb{Z}_{p^a} & \text{if } i = 2, \\ \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

2- **Finding** $H^*(M(\mathbb{Z}_{p^a},2);\mathbb{Z})$.

Here we want to calculate the cohomology of Moore spaces with integral coefficients and we will do that below:

$a - H^1(M(\mathbb{Z}_{p^a},2);\mathbb{Z})$.

$$0 \to \mathrm{Ext}(H_0(M(\mathbb{Z}_{p^a},2),\mathbb{Z}) \to H^1(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) \to \mathrm{Hom}(H_1(M(\mathbb{Z}_{p^a},2),\mathbb{Z}) \to 0$$

But $H_1(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = 0$ and $H_0(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = \mathbb{Z}$ then

$$H^1(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = 0.$$

$b - H^2(M(\mathbb{Z}_{p^a},2);\mathbb{Z})$.

$$0 \to \mathrm{Ext}(H_1(M(\mathbb{Z}_{p^a},2),\mathbb{Z}) \to H^2(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) \to \mathrm{Hom}(H_2(M(\mathbb{Z}_{p^a},2),\mathbb{Z}) \to 0$$

But $H_1(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = 0$ and $H_2(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = \mathbb{Z}_{p^a}$ then

$$H^2(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}_{p^a},\mathbb{Z}) = 0.$$

$c - H^3(M(\mathbb{Z}_{p^a},2);\mathbb{Z})$.

$$0 \to \mathrm{Ext}(H_2(M(\mathbb{Z}_{p^a},2),\mathbb{Z}) \to H^3(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) \to \mathrm{Hom}(H_3(M(\mathbb{Z}_{p^a},2),\mathbb{Z}) \to 0$$

But $H_3(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = 0$ because our Moore space has dimension 2, so

$$H^3(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) \cong \mathrm{Ext}(H_2(M(\mathbb{Z}_{p^a},2),\mathbb{Z}) = \mathbb{Z}_{p^a}$$

$d - H^4(M(\mathbb{Z}_{p^a},2);\mathbb{Z})$.

$$0 \to \mathrm{Ext}(H_3(M(\mathbb{Z}_{p^a},2),\mathbb{Z}) \to H^4(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) \to \mathrm{Hom}(H_4(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) \to 0$$

But $H_4(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = H_3(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = 0$ because our Moore space has dimension 2, so

$$H^4(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = 0.$$

3 - **Finding** $H_*(M(\mathbb{Z}_{p^a},2);\mathbb{Z}_{p^k})$.

Here we want to calculate the homology of Moore spaces with non-integral coefficients and we will do that below:

$a - H_1(M(\mathbb{Z}_{p^a},2);\mathbb{Z}_{p^k})$.

$$0 \to H_1(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) \otimes \mathbb{Z}_{p^k} \to H_1(M(\mathbb{Z}_{p^a},2);\mathbb{Z}_{p^k}) \to \mathrm{Tor}(H_0(M(\mathbb{Z}_{p^a},2),\mathbb{Z}_{p^k}) \to 0$$

But $H_1(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = 0$ and $H_0(M(\mathbb{Z}_{p^a},2);\mathbb{Z}) = \mathbb{Z}$ then

$$H_1(M(\mathbb{Z}_{p^a},2);\mathbb{Z}_{p^k}) \cong \mathrm{Tor}(\mathbb{Z},\mathbb{Z}_{p^k}) = 0.$$

$b - H_2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k})$.

$$0 \to H_2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) \otimes \mathbb{Z}_{p^k} \to H_2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \to \text{Tor}(H_1(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to 0$$

But $H_1(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = 0$ and $H_2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = \mathbb{Z}_{p^a}$ then

$$H_2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^a} \otimes \mathbb{Z}_{p^k} = \mathbb{Z}_d$$

Where $d = \gcd(p^a, p^k)$.

Now, distinguish between 2 cases:

Case I. $a \geq k$.

$$H_2(M; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k}$$

Case II. $a \leq k$

$$H_2(M; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^a}.$$

$c - H_3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k})$.

$$0 \to H_3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) \otimes \mathbb{Z}_{p^k} \to H_3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \to \text{Tor}(H_2(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to 0$$

But $H_3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = 0$ and $H_2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = \mathbb{Z}_{p^a}$ then

$$H_3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \cong \text{Tor}(\mathbb{Z}_{p^a}, \mathbb{Z}_{p^k}) = \mathbb{Z}_d$$

Where $d = \gcd(p^a, p^k)$.

Now, distinguish between 2 cases:

Case I. $a \geq k$.

$$H_3(M; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k}$$

Case II. $a \leq k$

$$H_3(M; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^a}.$$

$d - H_4(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k})$.

$$0 \to H_4(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) \otimes \mathbb{Z}_{p^k} \to H_4(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \to \text{Tor}(H_3(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to 0$$

But $H_4(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = H_3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = 0$ then

$$H_4(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) = 0.$$

4- **Finding** $H^*(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k})$.

Here we want to calculate the cohomology of Moore spaces with non-integral coefficients and we will do that below:

$a - H^1(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k})$.

$$0 \to \mathrm{Ext}(H_0(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to H^1(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \to \mathrm{Hom}(H_1(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to 0$$

But $H_1(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = 0$ and $H_0(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = \mathbb{Z}$ then

$$H^1(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \cong \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}_{p^k}) = 0.$$

$b - H^2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k})$.

$$0 \to \mathrm{Ext}(H_1(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \to H^2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \to \mathrm{Hom}(H_2(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to 0$$

But $H_1(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = 0$ and $H_2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = \mathbb{Z}_{p^a}$ then

$$H^2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \cong \mathrm{Hom}(\mathbb{Z}_{p^a}, \mathbb{Z}_{p^k}) = \mathbb{Z}_d$$

Where $d = \gcd(p^a, p^k)$.

Now, distinguish between 2 cases:

Case I. $a \geq k$.

$$H^2(M; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k}$$

Case II. $a \leq k$

$$H^2(M; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^a}.$$

$c - H^3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k})$.

$$0 \to \mathrm{Ext}(H_2(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to H^3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \to \mathrm{Hom}(H_3(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to 0$$

But $H_3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = 0$ because our Moore space has dimension 2 and $H_2(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = \mathbb{Z}_{p^a}$ so

$$H^3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \cong \mathrm{Ext}(\mathbb{Z}_{p^a}, \mathbb{Z}_{p^k}) = \mathbb{Z}_d$$

Where $d = \gcd(p^a, p^k)$

Now, distinguish between 2 cases:

Case I. $a \geq k$.

$$H^3(M; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k}$$

Case II. $a \leq k$

$$H^3(M; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^a}.$$

$d - H^4(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k})$.

$$0 \to \text{Ext}(H_3(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to H^4(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) \to \text{Hom}(H_4(M(\mathbb{Z}_{p^a}, 2), \mathbb{Z}_{p^k}) \to 0$$

But $H_4(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = H_3(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}) = 0$ because our Moore space has dimension 2, so

$$H^4(M(\mathbb{Z}_{p^a}, 2); \mathbb{Z}_{p^k}) = 0.$$

Now, in sections $4.5 - 4.8$, we are going to complete the following proposition

Proposition 7.

1. $H_*(C_\beta; \mathbb{Z}) =?$
2. $H_*(C_\beta; \mathbb{Z}_{p^k}) =?$
3. $H^*(C_\beta; \mathbb{Z}) =?$
4. $H^*(C_\beta; \mathbb{Z}_{p^k}) =?$

4.4 Homology With Integral Coefficients of the Cofiber C_β

To calculate this, we will use the long exact sequence in homology. Consider the map $M \xrightarrow{r\lambda} N \to C$

The general form of the long exact sequence in homology is

$$\dots \to H_4(M) \xrightarrow{(r\lambda)_*} H_4(N) \to H_4(C) \to H_3(M) \xrightarrow{(r\lambda)_*} H_3(N) \to H_3(C) \to H_2(M) \xrightarrow{(r\lambda)_*} H_2(N) \to H_2(C) \to H_1(M) \xrightarrow{(r\lambda)_*} H_1(N) \to H_1(C) \to H_0(M) \xrightarrow{(r\lambda)_*} H_0(N) \to H_0(C) \to 0$$

$a - H_0(C; \mathbb{Z})$.

Because C is path connected, we have that

$$H_0(C; \mathbb{Z}) = \mathbb{Z}.$$

$b - H_1(C; \mathbb{Z})$.

Using the following part of the long exact sequence in homology

$$\dots \to H_1(N) \to H_1(C) \to H_0(M) \to \dots$$

And for easier understanding, consider the reduced homology version of the previous sequence i.e.,

$$\cdots \to \widetilde{H}_1(N) \to \widetilde{H}_1(C) \to \widetilde{H}_0(M) \to \dots$$

And knowing that $H_1(N) = \widetilde{H}_1(N) = 0 = \widetilde{H}_0(M)$, because we have a Moore space of dimension 2, then $H_1(C) = \widetilde{H}_1(C) = 0$.

$c - H_2(C;\mathbb{Z})$.

Using the following part of the long exact sequence in homology

$$\cdots \to H_3(M) \xrightarrow{(r\lambda)_*} H_3(N) \to H_3(C) \to H_2(M) \xrightarrow{(r\lambda)_*} H_2(N) \to H_2(C) \to \dots$$

This tells us that $H_2(C)$ is the cokernel of $(r\lambda)_*$, which we have calculated before, and it turns out to be the following:

CASE(I): In case of zero homomorphism.

We have:

$$\widetilde{H}_2(C) = \text{coker}(p^t\lambda)_* = \mathbb{Z}_{p^b}.$$

CASE(II): In case of nonzero homomorphism.

And we have $a \leq b$ i.e. we took $t = a - 1$ or we have $a \geq b$ i.e. we took $t = b - 1$, we have

$$\widetilde{H}_2(C) = \text{coker}(p^t\lambda)_* = \mathbb{Z}_{p^{b-1}}.$$

$d - H_3(C;\mathbb{Z})$.

Using the following part of the long exact sequence in homology

$$\cdots \to H_3(M) \xrightarrow{(r\lambda)_*} H_3(N) \to H_3(C) \to H_2(M) \xrightarrow{(r\lambda)_*} H_2(N) \to H_2(C) \to \dots$$

This tells us that $H_3(C)$ is the kernel of $(r\lambda)_*$, which we have calculated before, and it turns out to be the following:

CASE(I): In case of the zero homomorphism.

We have:

$$H_3(C) = \widetilde{H}_3(C) = \text{coker}(p^t\lambda)_* = \mathbb{Z}_{p^a}.$$

CASE(II): In case of a nonzero homomorphism.

And we have $a \leq b$ i.e. we took $t = a - 1$ or we have $a \geq b$ i.e. we took $t = b - 1$, we have

$$H_3(C) = \widetilde{H}_3(C) = \text{coker}(p^t\lambda)_* = \mathbb{Z}_{p^{a-1}}.$$

$e - H_4(C; \mathbb{Z})$.

Using the following part of the long exact sequence in homology

$$\cdots \to H_4(M) \xrightarrow{(r\lambda)_*} H_4(N) \to H_4(C) \to H_3(M) \xrightarrow{(r\lambda)_*} H_3(N) \to H_3(C) \to \dots$$

And knowing that $H_4(M; \mathbb{Z}) = H_4(N; \mathbb{Z}) = H_3(M; \mathbb{Z}) = H_3(N; \mathbb{Z}) = 0$ because our Moore space has dimension 2, so

$$H_4(C; \mathbb{Z}) = 0.$$

4.5 Homology With Nonintegral Coefficients of the Cofiber C_β.

Knowing that ,from chapter 2 (I will add it later to this document), in case of zero homomorphism, we have:

$$\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^a},$$

And,

$$\widetilde{H}_2(C) = \text{coker}(p^t\lambda)_* = \mathbb{Z}_{p^b}.$$

And knowing that, from chapter 2 also, in case of the nonzero homomorphism we have $a \leq b$ i.e. we took $t = a - 1$ or we have $a \geq b$ i.e. we took $t = b - 1$, we have

$$\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^{a-1}},$$

And,

$$\widetilde{H}_2(C) = \text{coker}(p^t\lambda)_* = \mathbb{Z}_{p^{b-1}}.$$

And using the following definition of Moore space:

Let G be an abelian group and n an integer ≥ 2. A Moore space of type (G, n) is $1-$connected, $CW-$complex X such that:

$$\widetilde{H}_i(X) = \begin{cases} 0 & if\, i \neq n, \\ G & ifi = n. \end{cases}$$

We will calculate the following homology groups with coefficients

$$H_4(C; \mathbb{Z}_{p^k}), H_3(C; \mathbb{Z}_{p^k}), H_2(C; \mathbb{Z}_{p^k}), H_1(C; \mathbb{Z}_{p^k})$$

and the reason why those specifically was mentioned in chapter 2 (I will add it there)

1- $H_4(C;G) = H_4(C;\mathbb{Z}_{p^k})$.

$$H_4(C;G) \cong H_4(C) \otimes \mathbb{Z}_{p^k} \oplus \operatorname{Tor}(H_3(C), \mathbb{Z}_{p^k})$$

But since $H_4(C) = 0$, by construction of Moore space, we will have

$$H_4(C;G) \cong \operatorname{Tor}(H_3(C), \mathbb{Z}_{p^k})$$

Now distinguish between 2 cases:

(a) In case of the zero homomorphism i.e. $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^a}$,:

We have

$$H_4(C;G) \cong \operatorname{Tor}(\mathbb{Z}_{p^a}, \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{\gcd(p^a, p^k)}.$$

Now distinguish between 2 cases:

Case I. $a \geq k$.

$$H_4(C;G) \cong \mathbb{Z}_{p^k}.$$

Case II. $a \leq k$.

$$H_4(C;G) \cong \mathbb{Z}_{p^a}.$$

(b) In case of a non-zero homomorphism i.e. $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^{a-1}}$,:

We have

$$H_4(C;G) \cong \operatorname{Tor}(\mathbb{Z}_{p^{a-1}}, \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{\gcd(p^{a-1}, p^k)}.$$

Now distinguish between 2 cases:

Case I. $a-1 \geq k$.

$$H_4(C;G) \cong \mathbb{Z}_{p^k}.$$

Case II. $a-1 \leq k$.

$$H_4(C;G) \cong \mathbb{Z}_{p^{a-1}}.$$

2- $H_3(C;G) = H_3(C;\mathbb{Z}_{p^k})$.

$$H_3(C;G) \cong H_3(C) \otimes \mathbb{Z}_{p^k} \oplus \operatorname{Tor}(H_2(C), \mathbb{Z}_{p^k})$$

Now distinguish between 2 cases:

(a) In case of the zero homomorphism i.e. $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^a}$, **and** $\widetilde{H}_2(C) = \operatorname{coker}(p^t\lambda)_* = \mathbb{Z}_{p^b}$. **:**

We have

$$H_3(C;G) \cong \mathbb{Z}_{p^a} \otimes \mathbb{Z}_{p^k} \oplus \operatorname{Tor}(\mathbb{Z}_{p^b}, \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{\gcd(p^a,p^k)} \oplus \mathbb{Z}_{\gcd(p^b,p^k)}$$

Now distinguish between 4 cases:

Case I. $a \geq k$. **and** $b \geq k$.

$$H_3(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}.$$

Case II. $a \leq k$ **and** $b \leq k$.

$$H_3(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^b}.$$

Case III. $a \geq k$ **and** $b \leq k$.

$$H_3(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^b}.$$

Case IV. $a \leq k$ **and** $b \geq k$.

$$H_3(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^k}.$$

(b) In case of a non-zero homomorphism i.e. $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^{a-1}}$ **and** $\widetilde{H}_2(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^{b-1}}$**:**

We have

$$H_3(C;G) \cong \mathbb{Z}_{p^{a-1}} \otimes \mathbb{Z}_{p^k} \oplus \operatorname{Tor}(\mathbb{Z}_{p^{b-1}}, \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{\gcd(p^{a-1},p^k)} \oplus \mathbb{Z}_{\gcd(p^{b-1},p^k)}$$

Now distinguish between 4 cases:

Case I. $a-1 \geq k$. **and** $b-1 \geq k$.

$$H_3(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$$

Case II. $a-1 \leq k$ **and** $b-1 \leq k$.

$$H_3(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^{a-1}} \oplus \mathbb{Z}_{p^{b-1}}.$$

Case III. $a-1 \geq k$ **and** $b-1 \leq k$.

$$H_3(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^{b-1}}.$$

Case IV. $a-1 \leq k$ **and** $b-1 \geq k$.

$$H_3(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^{a-1}} \oplus \mathbb{Z}_{p^k}.$$

3- $H_2(C;G) = H_2(C;\mathbb{Z}_{p^k})$.

$$H_2(C;G) \cong H_2(C) \otimes \mathbb{Z}_{p^k} \oplus \mathrm{Tor}(H_1(C),\mathbb{Z}_{p^k})$$

But since $H_1(C) = 0$, by the definition of the Moore space we are using, we then have

$$H_2(C;G) \cong H_2(C) \otimes \mathbb{Z}_{p^k}$$

Now distinguish between 2 cases:

(a) In case of the zero homomorphism i.e. $\widetilde{H}_2(C) = \mathrm{coker}(p^t\lambda)_* = \mathbb{Z}_{p^b}$,**:**

We have

$$H_2(C;G) \cong H_2(C) \otimes \mathbb{Z}_{p^k} \cong \mathbb{Z}_{p^b} \otimes \mathbb{Z}_{p^k} \cong \mathbb{Z}_{\mathrm{gcd}(p^b,p^k)}$$

Now distinguish between 2 cases:

Case I. $b \geq k$.

$$H_2(C;G) \cong \mathbb{Z}_{p^k}.$$

Case II. $b \leq k$.

$$H_2(C;G) \cong \mathbb{Z}_{p^b}.$$

(b) In case of a non-zero homomorphism i.e. $\widetilde{H}_2(C) = \mathrm{coker}(p^t\lambda)_* = \mathbb{Z}_{p^{b-1}}$,**:**

We have

$$H_2(C;G) \cong H_2(C) \otimes \mathbb{Z}_{p^k} \cong \mathbb{Z}_{p^{b-1}} \otimes \mathbb{Z}_{p^k} \cong \mathbb{Z}_{\mathrm{gcd}(p^{b-1},p^k)}$$

Now distinguish between 2 cases:

Case I. $b-1 \geq k$.

$$H_2(C;G) \cong \mathbb{Z}_{p^k}.$$

Case II. $b-1 \leq k$.

$$H_2(C;G) \cong \mathbb{Z}_{p^{b-1}}.$$

4 - $H_1(C;G) = H_1(C;\mathbb{Z}_{p^k})$.

$$H_1(C;G) \cong H_1(C) \otimes \mathbb{Z}_{p^k} \oplus \operatorname{Tor}(H_0(C), \mathbb{Z}_{p^k})$$

But since $H_1(C) = H_0(C) = 0$, by the definition of the Moore space we are using, and because we are considering Moore space of dimension 2. And so we then have

$$H_1(C;G) = 0.$$

4.6 Cohomology With Integral Coefficients of the Cofiber C_β

1- $H^1(C;\mathbb{Z})$.

By the UCT, the following SES exists:

$$0 \to \operatorname{Ext}(H_0(C,\mathbb{Z});\mathbb{Z}) \to H^1(C;\mathbb{Z}) \to \operatorname{Hom}(H_1(C,\mathbb{Z});\mathbb{Z}) \to 0$$

But $H_0(C,\mathbb{Z}) = H_1(C,\mathbb{Z}) = 0$, therefore

$$H^1(C;\mathbb{Z}) = 0.$$

2- $H^2(C;\mathbb{Z})$.

By the UCT, the following SES exists:

$$0 \to \operatorname{Ext}(H_1(C,\mathbb{Z});\mathbb{Z}) \to H^2(C;\mathbb{Z}) \to \operatorname{Hom}(H_2(C,\mathbb{Z});\mathbb{Z}) \to 0$$

But $H_1(C,\mathbb{Z}) = 0$, therefore by the properties of the SES, we have

$$H^2(C;\mathbb{Z}) \cong \operatorname{Hom}(H_2(C,\mathbb{Z});\mathbb{Z}).$$

Now distinguish between 2 cases:

(a) In case of the zero homomorphism i.e. $\widetilde{H}_2(C) = \operatorname{coker}(p^t\lambda)_* = \mathbb{Z}_{p^b}$**:**

We have:

$$H^2(C;\mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}_{p^b};\mathbb{Z}) = 0,$$

Therefore,

$$H^2(C;\mathbb{Z}) = 0.$$

(b) In case of a non-zero homomorphism i.e. $\widetilde{H}_2(C) = \operatorname{coker}(p^t\lambda)_* = \mathbb{Z}_{p^{b-1}}$**:**

We have:

$$H^2(C;\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}_{p^{b-1}};\mathbb{Z}) = 0,$$

Therefore,

$$H^2(C;\mathbb{Z}) = 0.$$

3- $H^3(C;\mathbb{Z})$.

By the UCT, the following SES exists:

$$0 \to \mathrm{Ext}(H_2(C,\mathbb{Z});\mathbb{Z}) \to H^3(C;\mathbb{Z}) \to \mathrm{Hom}(H_3(C,\mathbb{Z});\mathbb{Z}) \to 0$$

Therefore, by the properties of the SES, we have

$$H^3(C;\mathbb{Z}) \cong \mathrm{Ext}(H_2(C,\mathbb{Z});\mathbb{Z}) \oplus \mathrm{Hom}(H_3(C,\mathbb{Z});\mathbb{Z}).$$

Now distinguish between 2 cases:

(a) In case of the zero homomorphism i.e. $\widetilde{H}_2(C) = \mathrm{coker}(p^t\lambda)_* = \mathbb{Z}_{p^b}$ and $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^a}$:

We have:

$$H^3(C;\mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}_{p^b};\mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}_{p^a};\mathbb{Z}).$$

Therefore,

$$H^3(C;\mathbb{Z}) = \mathbb{Z}_{p^b} \oplus 0 = \mathbb{Z}_{p^b}.$$

(b) In case of a non-zero homomorphism i.e. $\widetilde{H}_2(C) = \mathrm{coker}(p^t\lambda)_* = \mathbb{Z}_{p^{b-1}}$ and $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^{a-1}}$:

We have:

$$H^3(C;\mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}_{p^{b-1}};\mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}_{p^{a-1}};\mathbb{Z})$$

Therefore,

$$H^3(C;\mathbb{Z}) = \mathbb{Z}_{p^{b-1}} \oplus 0 = \mathbb{Z}_{p^{b-1}}.$$

4- $H^4(C;\mathbb{Z})$.

By the UCT, the following SES exists:

$$0 \to \mathrm{Ext}(H_3(C,\mathbb{Z});\mathbb{Z}) \to H^4(C;\mathbb{Z}) \to \mathrm{Hom}(H_4(C,\mathbb{Z});\mathbb{Z}) \to 0$$

But $H_4(C,\mathbb{Z}) = 0$, using the long exact sequence in homology for the following map $M \to N \to C$.
therefore by the properties of the SES, we have

$$H^4(C;\mathbb{Z}) \cong \mathrm{Ext}(H_3(C,\mathbb{Z});\mathbb{Z}).$$

Now distinguish between 2 cases:

(a) In case of the zero homomorphism i.e. $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^a}$**:**

We have:

$$H^4(C;\mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}_{p^a};\mathbb{Z})$$

Therefore,

$$H^4(C;\mathbb{Z}) \cong \mathbb{Z}_{p^a}.$$

(b) In case of a nonzero homomorphism i.e. $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^{a-1}}$**:**

We have:

$$H^4(C;\mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}_{p^{a-1}};\mathbb{Z}) = \mathbb{Z}_{p^{b-1}}.$$

Therefore,

$$H^4(C;\mathbb{Z}) \cong \mathbb{Z}_{p^{a-1}}.$$

4.7 Cohomology With Nonintegral Coefficients of the Cofiber C_β.

We will calculate the following cohomology groups with non-integral coefficients

$$H^4(C;\mathbb{Z}_{p^k}), H^3(C;\mathbb{Z}_{p^k}), H^2(C;\mathbb{Z}_{p^k}), H^1(C;\mathbb{Z}_{p^k}).$$

1- $H^1(C;\mathbb{Z}_{p^k})$.

By the UCT, the following SES exists:

$$0 \to \mathrm{Ext}(H_0(C;\mathbb{Z}),\mathbb{Z}_{p^k}) \to H^1(C;\mathbb{Z}_{p^k}) \to \mathrm{Hom}(H_1(C;\mathbb{Z}),\mathbb{Z}_{p^k}) \to 0$$

But $H_0(C,\mathbb{Z}) = H_1(C,\mathbb{Z}) = 0$, therefore

$$H^1(C;\mathbb{Z}_{p^k}) = 0.$$

2- $H^2(C;\mathbb{Z}_{p^k})$.

By the UCT, the following SES exists:

$$0 \to \mathrm{Ext}(H_1(C;\mathbb{Z}),\mathbb{Z}_{p^k}) \to H^2(C;\mathbb{Z}_{p^k}) \to \mathrm{Hom}(H_2(C;\mathbb{Z}),\mathbb{Z}_{p^k}) \to 0$$

But $H_1(C,\mathbb{Z}) = 0$, therefore by the properties of the SES, we have

$$H^2(C;\mathbb{Z}_{p^k}) \cong \mathrm{Hom}(H_2(C;\mathbb{Z}),\mathbb{Z}_{p^k}).$$

Now distinguish between 2 cases:

(a) In case of the zero homomorphism i.e., $\widetilde{H}_2(C) = \mathrm{coker}(p^t\lambda)_* = \mathbb{Z}_{p^b}$**:**

We have:

$$H^2(C;\mathbb{Z}_{p^k}) \cong \mathrm{Hom}(\mathbb{Z}_{p^b};\mathbb{Z}_{p^k}) = \mathbb{Z}_d,$$

Where $d = \gcd(p^b, p^k)$

Therefore,

$$H^2(C; \mathbb{Z}_{p^k}) = \mathbb{Z}_d.$$

Now, distinguish between 2 cases:

Case I. $b \geq k$.

$$H^2(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k}.$$

Case II. $b \leq k$.

$$H^2(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^b}.$$

(b) In case of a non-zero homomorphism i.e. $\widetilde{H}_2(C) = \text{coker}(p^t\lambda)_* = \mathbb{Z}_{p^{b-1}}$**:**

We have:

$$H^2(C; \mathbb{Z}_{p^k}) \cong \text{Hom}(\mathbb{Z}_{p^{b-1}}; \mathbb{Z}_{p^k}) = \mathbb{Z}_d,$$

Where $d = \gcd(p^{b-1}, p^k)$.

Therefore,

$$H^2(C; \mathbb{Z}_{p^k}) = \mathbb{Z}_d.$$

Now, distinguish between 2 cases:

Case I. $b - 1 \geq k$.

$$H^2(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k}.$$

Case II. $b - 1 \leq k$.

$$H^2(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^{b-1}}.$$

3- $H^3(C; \mathbb{Z}_{p^k})$.

By the UCT, the following SES exists:

$$0 \to \text{Ext}(H_2(C; \mathbb{Z}), \mathbb{Z}_{p^k}) \to H^3(C; \mathbb{Z}_{p^k}) \to \text{Hom}(H_3(C; \mathbb{Z}), \mathbb{Z}_{p^k}) \to 0$$

But $H_2(C, \mathbb{Z})$ and $H_3(C, \mathbb{Z})$ each has 2 values depending on pg.5, therefore we have

$$H^3(C; \mathbb{Z}_{p^k}) \cong \text{Ext}(H_2(C; \mathbb{Z}), \mathbb{Z}_{p^k}) \oplus \text{Hom}(H_3(C; \mathbb{Z}), \mathbb{Z}_{p^k})$$

Now distinguish between 2 cases:

(a) In case of the zero homomorphism :

Since we have $\widetilde{H}_2(C) = \mathrm{coker}(p^t\lambda)_* = \mathbb{Z}_{p^b}$ and $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^a}$, then

$$H^3(C; \mathbb{Z}_{p^k}) \cong \mathrm{Ext}(\mathbb{Z}_{p^b}, \mathbb{Z}_{p^k}) \oplus \mathrm{Hom}(\mathbb{Z}_{p^a}, \mathbb{Z}_{p^k})$$

Distinguish between 4 cases:

Case I. $a \geq k$. **and** $b \geq k$.

$$H^3(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}.$$

Case II. $a \leq k$ **and** $b \leq k$.

$$H^3(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^b} \oplus \mathbb{Z}_{p^a}.$$

Case III. $a \geq k$ **and** $b \leq k$.

$$H^3(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^b} \oplus \mathbb{Z}_{p^k}.$$

Case IV. $a \leq k$ **and** $b \geq k$.

$$H^3(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^a}.$$

(b) In case of a nonzero homomorphism:

Since we have $\widetilde{H}_2(C) = \mathrm{coker}(p^t\lambda)_* = \mathbb{Z}_{p^{b-1}}$ and $\widetilde{H}_3(C) = \ker(p^t\lambda)_* = \mathbb{Z}_{p^{a-1}}$, then

$$H^3(C; \mathbb{Z}_{p^k}) \cong \mathrm{Ext}(\mathbb{Z}_{p^{b-1}}, \mathbb{Z}_{p^k}) \oplus \mathrm{Hom}(\mathbb{Z}_{p^{a-1}}, \mathbb{Z}_{p^k})$$

Distinguish between 4 cases:

Case I. $a-1 \geq k$. **and** $b-1 \geq k$.

$$H^3(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}.$$

Case II. $a-1 \leq k$ **and** $b-1 \leq k$.

$$H^3(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^{b-1}} \oplus \mathbb{Z}_{p^{a-1}}.$$

Case III. $a-1 \geq k$ **and** $b-1 \leq k$.

$$H^3(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^{b-1}} \oplus \mathbb{Z}_{p^k}.$$

Case IV. $a-1 \leq k$ **and** $b-1 \geq k$.

$$H^3(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^{a-1}}.$$

4- $H^4(C;\mathbb{Z}_{p^k})$.

By the UCT, the following SES exists:

$$0 \to \mathrm{Ext}(H_3(C;\mathbb{Z}),\mathbb{Z}_{p^k}) \to H^4(C;\mathbb{Z}_{p^k}) \to \mathrm{Hom}(H_4(C;\mathbb{Z}),\mathbb{Z}_{p^k}) \to 0$$

But $H_4(C,\mathbb{Z}) = 0$ and $H_3(C,\mathbb{Z})$ has 2 values depending on pg.4, therefore we have

$$H^4(C;\mathbb{Z}_{p^k}) \cong \mathrm{Ext}(H_3(C;\mathbb{Z}),\mathbb{Z}_{p^k})$$

Now distinguish between 2 cases:

(a) In case of the zero homomorphism :

Since we have $H_3(C) = \mathbb{Z}_{p^a}$, then

$$H^4(C;\mathbb{Z}_{p^k}) \cong \mathrm{Ext}(\mathbb{Z}_{p^a},\mathbb{Z}_{p^k})$$

Distinguish between 2 cases:

Case I. $a \geq k$.

$$H^4(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k}.$$

Case II. $a \leq k$.

$$H^4(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^a}.$$

(b) In case of a nonzero homomorphism:

Since we have $H_3(C) = \mathbb{Z}_{p^{a-1}}$, then

$$H^4(C;\mathbb{Z}_{p^k}) \cong \mathrm{Ext}(\mathbb{Z}_{p^{a-1}},\mathbb{Z}_{p^k})$$

Distinguish between 2 cases:

Case I. $a-1 \geq k$.

$$H^4(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^k}.$$

Case II. $a-1 \leq k$.

$$H^4(C;\mathbb{Z}_{p^k}) \cong \mathbb{Z}_{p^{a-1}}.$$

4.8 Proving That $(f+g)_* = f_* + g_*$ and $(f+g)^* = (f^* + g^*)$

Even though the proof of the following proposition is given in [[Hat02], Lemma 4.60], we will present the proof here using commutative diagrams :

Proposition 4.9.1

For any maps $f, g : \Sigma X \to Y$

1. The homomorphisms $(f+g)_* : H_*(\Sigma X; G) \to H_*(Y; G)$ and $(f_* + g_*) : H_*(\Sigma X; G) \to H_*(Y; G)$ are equal, for any coefficient group G.

2. The homomorphisms $(f+g)^* : H^*(Y; G) \to H_*(\Sigma X; G)$ and $(f^* + g^*) : H^*(Y; G) \to H_*(\Sigma X; G)$ are equal, for any coefficient group G.

Proof of 1.**:**

First, addition in this context is defined as follows:

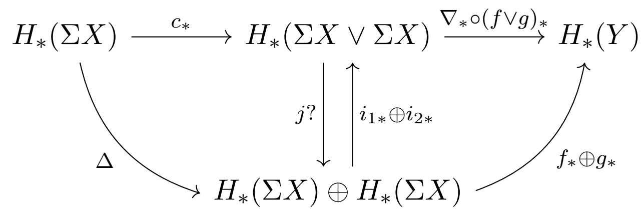
$$f + g = \nabla \circ (f \vee g) \circ c : \Sigma X \xrightarrow{c} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y \qquad (1)$$

i.e., addition is induced by what is called the pinch map $c : \Sigma X \to \Sigma X \vee \Sigma X$, which is the map that quotients out an equatorial copy of X in the suspension. Now, applying the homology functor to $f+g$, we will get $(f+g)_*$.

Note that we will be using both $\times$ and $\oplus$ in places where $\oplus$ should be used only. Even though the direct sum and direct product of two summands/factors are the same. This is to differentiate between maps into vs. out of the direct product/sum, and to make it obvious what changes should we make when we dualize.

The main idea of this proof is to think of $f_* + g_*$ as the composition of the diagonal homomorphism $\Delta : H_*(\Sigma X) \to H_*(\Sigma X) \times H_*(\Sigma X)$ given by $x \mapsto (x, x)$ with the direct sum $f_* \oplus g_*$. Now recall that $H_i(\Sigma X \vee \Sigma X) \cong H_i(\Sigma X) \oplus H_i(\Sigma X)$. Thus if we can show that the following diagram commutes, and the two vertical arrows cancel (i.e. j is the inverse of $(i_{1*} \oplus i_{2*})$), then we will have obtained the equality between the composition of the top row, which by definition is just $(f+g)_*$, and the composition of the curved arrows at the bottom, which is $f_* + g_*$, since

$$f_* + g_* = (f_* \oplus g_*) \circ \Delta = \nabla_* \circ (f \vee g)_* \circ (i_{1*} \oplus i_{2*}) \circ j \circ c_* = \nabla_* \circ (f \vee g)_* \circ c_* = (f+g)_* \ .$$

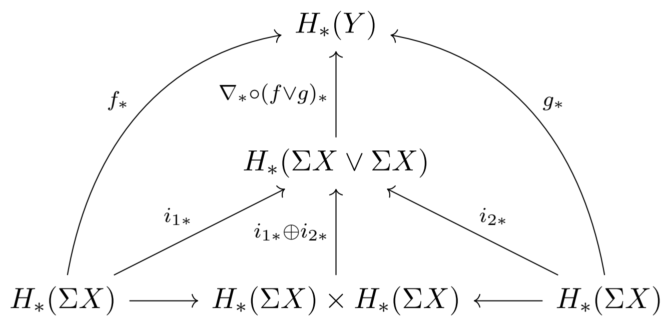


Now we tackle this proof in three steps.

First we show that the triangle on the right commutes.

The isomorphism $i_{1*} \oplus i_{2*} : H_*(\Sigma X) \oplus H_*(\Sigma X) \to H_*(\Sigma X \vee \Sigma X)$ is induced by the inclusion maps $i_1 : \Sigma X \hookrightarrow \Sigma X \vee \Sigma X,\ a \mapsto (a, *)$,$i_2 : \Sigma X \hookrightarrow \Sigma X \vee \Sigma X,\ a \mapsto (*, a)$ (we are viewing $\Sigma X \vee \Sigma X \subseteq \Sigma X \times \Sigma X$).

Now, since $(f \vee g) \circ i_1 = f \vee *$ and $(f \vee g) \circ i_2 = g \vee *$, using the universal property of direct sums, we can show that the composition $\nabla_* \circ (f \vee g)_* \circ (i_{1*} \oplus i_{2*})$ is the direct sum map $f_* \oplus g_*$. (See below for the relevant diagram. The horizontal arrows at the bottom are the canonical injections into the direct sum.)

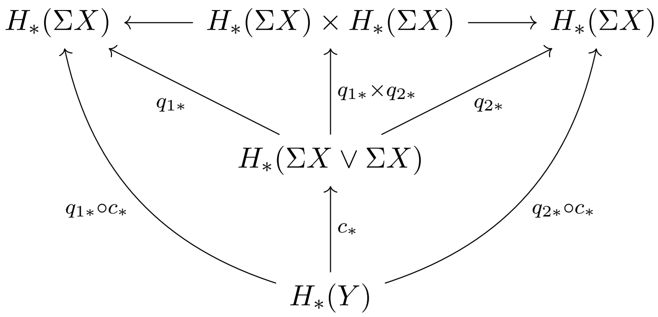


Second, we want to construct the inverse of $i_{1*} \oplus i_{2*}$.

Consider the quotient maps $q_i : \Sigma X \vee \Sigma X \to \Sigma X, i = 1, 2$ that are identity on the i-th component, and collapsing to a point the other component. It's clear that $q_1 \circ i_1$ and $q_2 \circ i_2$ are the identity maps, and $q_1 \circ i_2$ and $q_2 \circ i_1$ are constant maps (collapses the other component to a point), hence an easy calculation shows that $q_{1*} \times q_{2*}$ is the inverse of $i_{1*} \oplus i_{2*}$.

Finally to show that the left triangle commutes.

Note that $q_1 \circ c$ and $q_2 \circ c$ are both homotopic to the identity. This means that the induced maps $q_{1*} \circ c_*$ and $q_{2*} \circ c_*$ are equal to the identity, so $(q_{1*} \times q_{2*}) \circ c_*$ is the diagonal map. Here is the diagram of this part



This completes the proof.

Proof of 2.**:**

For the cohomology version of the statement, we will dualize all diagrams (reverse all arrows, direct sums become direct products and vice versa), and argue that in diagram 1 the top composition, which becomes $(f+g)^*$, is equal to the bottom composition, which becomes $f^* + g^*$. The central idea is similar, which is to break up $f^* + g^*$ into the composition of $f^* \times g^*$ with the addition $(x, y) \mapsto x + y$. The former is the curved arrow on the right, now pointing from $H^*(Y)$ to $H^*(\Sigma X) \times H^*(\Sigma(X)$, and the latter is the curved arrow on the left, from $H^*(\Sigma X) \times H^*(\Sigma X)$ to $H^*(\Sigma X)$. Commutativity on the left and right follows from the dualization of diagrams 2 and 3 above, respectively.

Corollary 4.9.2

If $f : \Sigma X \to Y$, and $r \in \mathbb{Z}$, then $(r.f)_* = r.f_*$ and $(r.f)^* = r.f^*$.

CHAPTER 5

HOPF INVARIANT AND CUP PRODUCT FORMULA

5.1 Main Goal and Main Results

In [Jam57], theorem 4.1 that I.M. James proved is a generalization of the Hopf invariant. If we take $p = q = n$, $X = S^n$, and $a = b \in H^n(S^n)$ the fundamental class, we will recover the definition of the usual Hopf invariant. In his paper, James managed to prove that Hopf invariant is a homomorphism in case of spheres as a domain and any space X as a target.

Our main goal is to compute the cup-product in cofibers and a step towards doing that is to adapt I.M. James' proof to our case which is Moore spaces of dimension 2.

For more reading on Hopf invariant see chapter 4 in [MT68] and section 4.B in [Hat02].

The main results that we achieved in this chapter is proving that the Hopf invariant is a homomorphism in case of Moore spaces of dimension 2 in both the zero and non-zero homomorphism case.

5.2 Preliminaries

In algebraic topology, the cup product is a method of adjoining two cocycles of degree p and q to form a composite cocycle of degree $p + q$. This defines an associative (and distributive) graded commutative product operation in cohomology, turning the cohomology of a space X into a graded ring, $H^*(X)$, called the cohomology ring.

Here is the story of the general idea of what we will do, We will look at the cofiber sequences

$$M \xrightarrow{x_{r,k}} N \to C_{x_{r,k}}$$

Where M and N are Moore spaces for cyclic groups in dimension 2. For simplicity, and without loss of generality, we assumed that $M = M(\mathbb{Z}_{p^a}, 2)$ and $N = M(\mathbb{Z}_{p^b}, 2)$, where p is a prime number. Therefore C_β will have nontrivial cells only in dimensions $2, 3$, and 4,(if time permits I will explain why in chapter 1 or 2) which means that the only nontrivial cup products in $H^*(C_\beta; G)$ are those of the form

$$\text{CUP} : H^2(C; \mathbb{Z}_{p^k}) \otimes H^2(C; \mathbb{Z}_{p^k}) \to H^4(C; \mathbb{Z}_{p^k})$$

And because of the work we have done in the previous chapter, we know (given a, b, r

and k) exactly what these groups are; they are cyclic of order p to some power that comes out of our formulas. Let's say $H^2(C_\beta;\mathbb{Z}_{p^k})$ is generated by u and $H^4(C_\beta;\mathbb{Z}_{p^k})$ is generated by v. Then $H^2(C_\beta;\mathbb{Z}_{p^k})\otimes H^2(C_\beta;\mathbb{Z}_{p^k})$ is also cyclic, and generated by $u\otimes u$. It follows that the cup product map CUP is totally determined by its value on $u\otimes u$. Furthermore, since $H^4(C_\beta;\mathbb{Z}_{p^k})$ is cyclic and generated by v, we know that

$$\mathrm{CUP}(u\otimes u)=m.v$$

for some $m\in\mathbb{Z}$(well-defined module the order of $H^4(C_\beta;\mathbb{Z}_{p^k})$). Therefore, determining the cup product structure in $H^*(C_\beta;\mathbb{Z}_{p^k})$ comes down to determining the number m. Since m is a function of the map $x_{r,k}$, we will write for the moment $m(r,k)$ for this coefficient(later in my formal prove I will change this notation). The idea is to prove the following formulas

$$m(r_1+r_2,k)=m(r_1,k)+m(r_2,k)$$
$$m(r,k_1+k_2)=m(r,k_1)+m(r,k_2)$$

Which all comes down to determining $m(1,0)$ and $m(0,1)$, which should be easy.

The paper of I.M. James proves something very much like these formulas. So, we first worked through section 4 of it which is pretty much self-contained and then we will try below to adapt the proof to our situation.

5.3 A Formula for Cup-Product in the Zero Homomorphism Case

The cohomology theory in what follows has non-integral coefficients i.e., $G=\mathbb{Z}_{p^k}$ as in case of coefficients in the ring of integers we have $H^2(N)=0$ and if C is the cofiber of the map $\beta\in[M,N]$ then $H^2(C)=0$ as well by section 4.7. Consider a Moore space $N=N(\mathbb{Z}_{p^b},2)$, and an element $s\in H^2(N;\mathbb{Z}_{p^k})$, such that $s^2=0$ because in our case we have that $H^4(N;\mathbb{Z}_{p^k})=0$. Since $H^3(M;\mathbb{Z}_{p^k})$ is finite from section 4.4, then we can define in case of the zero homomorphism a $\mathbb{Z}_d$-valued function where $d=\gcd(p^a,p^k),h$, on $[M,N]$ as follows. Let $\beta=x_{r,k}\in[M,N]$. Remember that C, the cofiber of $\beta:M\to N$, is formed by attaching a cone on M to N by a map of homotopy class $\beta=x_{r,k}$. Let c the generator of $\mathbb{Z}_d$, denote the cohomology class which is carried by the cone. Then by Mayer Vietoris sequence applied to $C=S^2\cup_{S^3,\beta}D^3\cup_{S^3,\beta}D^3\cup_{S^4,\beta}D^4$, there is the following exact sequence:

$$\cdots\to\mathbb{Z}_d\cong H^3(M;\mathbb{Z}_{p^k})\xrightarrow{q}H^4(C;\mathbb{Z}_{p^k})\cong\mathbb{Z}_d\to H^4(N;\mathbb{Z}_{p^k})=0\to\dots \qquad (1)$$

It is clear from the sequence above that q is an isomorphism.

Also, since part of our main sequence was

$$M\to N\xrightarrow{\alpha}C_\beta\to\Sigma M$$

then applying the cohomology functor H^2 to it, we will get the following sequence

$$H^2(\Sigma M) \to H^2(C_\beta) \xrightarrow{\alpha^*} H^2(N) \to H^2(M)$$

But, by section 4.4, we know that $H^2(\Sigma M) = H^1(M) = 0$, then α^* is injective and hence there is a unique $s' \in H^2(C_\beta; \mathbb{Z}_{p^k})$ that is sent to $s \in H^2(N; \mathbb{Z}_{p^k})$ under the injection α^* .

So, there is $(s')^2 \in H^4(C; \mathbb{Z}_{p^k})$ that is sent to $s^2 \in H^4(N; \mathbb{Z}_{p^k})$which is zero by the properties of a $2-$dimensional Moore space, so by exactness $(s')^2$ can be identified with an element coming from $\mathbb{Z}_d \cong H^3(M; \mathbb{Z}_{p^k})$ which has the form mc for some $m \in \mathbb{Z}_d$ *i.e.*, $(s')^2 = mc$. Now, I define $h(\beta) = m$. We prove:

Theorem (A.1). The function h constitutes a homomorphism of $[M, N]$ into the group $\mathbb{Z}_d$.

We prove **Theorem (A.1)** in the following form:

Lemma (A.2). Let $\beta_1, \beta_2, \beta_3 \in [M, N]$ be elements such that $\beta_1 + \beta_2 + \beta_3 = 0$. Then

$$h(\beta_1) + h(\beta_2) + h(\beta_3) = 0_{\mathbb{Z}_d}.$$

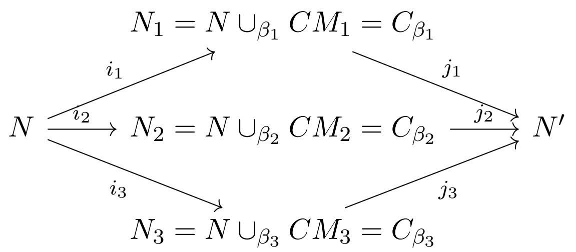
In what follows, let t be an indexing integer which takes values $1, 2, 3$. Let N_t denote the complex which is obtained by attaching a cone on M, call it $e_t = CM_t$, to N by a map of homotopy class β_t, so that

$$N_1 \cap N_3 = N_2 \cap N_3 = N_1 \cap N_2 = N.$$

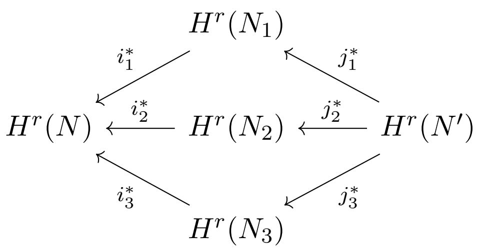
Let $N' = N_1 \cup N_2 \cup N_3 = N \cup_{\beta_1} e_1 \cup_{\beta_2} e_2 \cup_{\beta_3} e_3$. Consider the induced injections:

$$H^r(N') \xrightarrow{j_t^*} H^r(N_t) \xrightarrow{i_t^*} H^r(N).$$

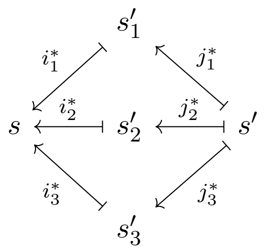
The situation, before applying H^r, can be described by the following diagram:



The situation, after applying H^r, can be described by the following diagram:



Let s' denote the cohomology class of N' such that $i_t^* j_t^*(s') = s$.Then, from the previous figure, we can get the following figure:



let c_t denote the class which is carried by e_t. Since $s^2 = 0$ in N,and by a reasoning similar to the one given in the paragraph just before Theorem(A.1), there are integers modulo d, called m_t such that

$$(s')^2 = m_1 c_1 + m_2 c_2 + m_3 c_3 \qquad (1)$$

However, by the naturality of the cup-product, the first equality in the following is correct,

$$\begin{aligned} j_t^*(s') \cup j_t^*(s') &= j_t^*(s' \cup s') \\ &= j_t^*((s')^2) \\ &= j_t^*(m_1 c_1 + m_2 c_2 + m_3 c_3) \\ &= m_1 j_t^*(c_1) + m_2 j_t^*(c_2) + m_3 j_t^*(c_3) \\ &= m_t j_t^*(c_t) + 0 + 0 \\ &= m_t c_t. \quad \text{(I)} \end{aligned}$$

In the above, the second equality is correct by the definition of the cup product, the third equality is correct from (1), the fourth equality is correct because, in general, the total cohomology $H^*(X)$ for any space X is a ring whose multiplication is the cup product and because j_t^* is the map of cohomology induced by the inclusion $j_t : N_t \hookrightarrow N'$ as can be seen from figure 1 and so j_t^* is a ring homomorphism. And the fifth equality is by definition of the induced inclusion function j_t^* and because of the following:

$$j_t^*(c_s) = \begin{cases} c_t & if s = t, \\ 0 & if s \neq t. \end{cases} \qquad (2)$$

Finally, the sixth equality is also because of (2).

On the other hand, we have that

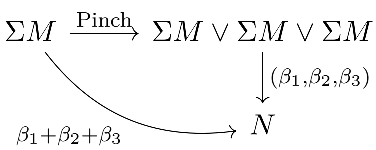
$$\begin{aligned} j_t^*(s') \cup j_t^*(s') &= j_t^*(s' \cup s') \\ &= j_t^*((s')^2) \\ &= (j_t^*(s'))^2 \\ &= (s_t^{'})^2 \\ &= h(\beta_t).c_t \quad \text{(II)} \end{aligned}$$

Where the third equality because j_t^* is a ring homomorphism, the fourth equality by figure 3 and the fifth equality by our definition for the function h.

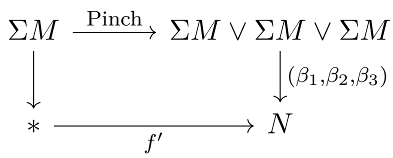
So, by comparing (I) and (II), we get that $m_t = h(\beta_t)$.

Note that in the above, j_t^* acts on 2 different things,the first is c_t which is the cohomology class attached with the cone e_t, while the second is s' which is the cohomology class of N' such that $i_t^* j_t^*(s') = s$.

Now, since by page 33 and 34, the addition $\beta_1+\beta_2+\beta_3$ means that we have the following diagram:



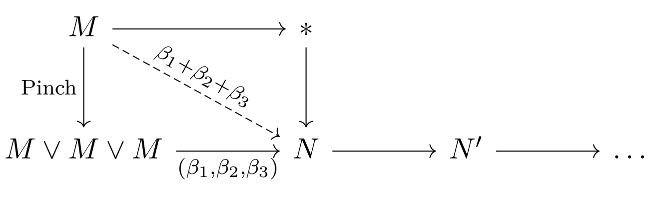
And since $\beta_1 + \beta_2 + \beta_3 = 0$, by hypothesis, then this means that we have the following commutative diagram (up to homotopy):



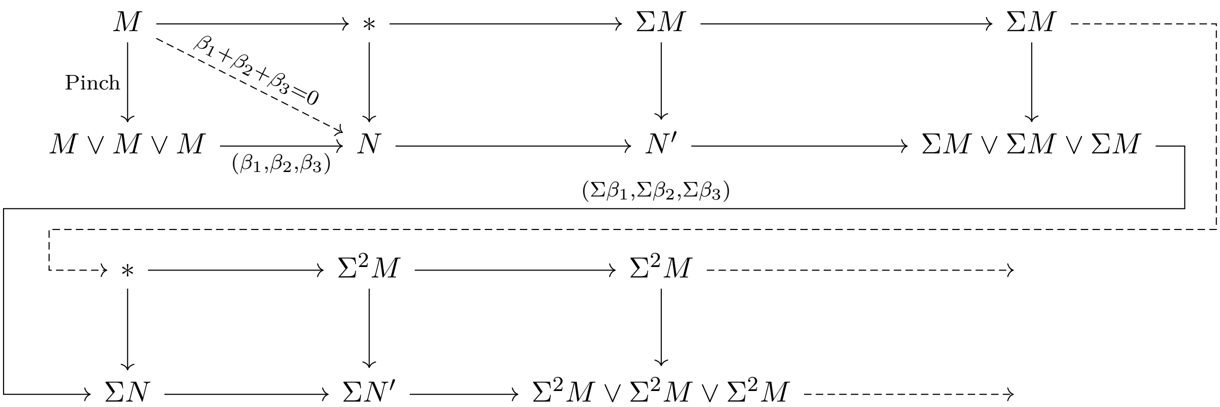
i.e., $\beta_1 + \beta_2 + \beta_3 = 0$ means that, up to homotopy, it factors through a single point which means that it is homotopic to a constant map.

Also, since $\beta_1 + \beta_2 + \beta_3 = 0$, by hypothesis, then we have the following diagram (our hypothesis square):

If $M = \Sigma A$,



Now, after forming the cofibres multiple times we got the following diagram:



If time permits I will continue drawing the above commutative diagram until having 16 rectangles, I already have it drawn in my hand written notes.

So, we can see that as a result of the existence of the first commutative square in this figure, there is a map $f : \Sigma M \to N'$ which maps ΣM onto each of the three cells e_t.

Let $f^* : H^r(N') \to H^r(\Sigma M) = H^{r-1}(M)$ denote the homomorphism induced by f, which, by page section 4.4, is trivial unless $r = 3, 4$. Then, by our construction of f which maps ΣM(whose $4^{th}-$cohomology represents c) onto e_t (which represents c_t), we have $f^*(c_t) = c$, the cohomology class of $H^4(\Sigma M)$. Hence

$$
\begin{aligned}
(m_1 + m_2 + m_3)c &= (m_1 c + m_2 c + m_3 c) \\
&= (m_1 f^*(c_1) + m_2 f^*(c_2) + m_3 f^*(c_3)) \\
&= f^*(m_1 c_1 + m_2 c_2 + m_3 c_3) \\
&= f^*(s' \cup s') \\
&= f^*((s')^2) \\
&= (f^*(s'))^2 \\
&= f^*(s') \cup f^*(s'),
\end{aligned}
$$

which is zero because the cup product of suspension is trivial on reduced cohomology. Therefore $m_1 + m_2 + m_3 = 0$ because $H^4(\Sigma M)$ is freely generated by c. Since $m_t = h(\beta_t)$, this proves $(A.2)$. Therefore

$$h(\beta_1) + h(\beta_2) + h(\beta_3) = 0_{\mathbb{Z}_d} = h(0) = h(\beta_1 + \beta_2 + \beta_3)$$

and this proves $(A.1)$ as required, Note that $h(0) = 0_{\mathbb{Z}_d}$ because in the zero-homomorphism case M is sent to a single point on N by the zero homomophism β and then we glue the cone on M to a single point so it becomes a suspension of M *i.e.*, the cofiber in case of the zero-homomorphism is a suspension and then the cup product is zero i.e.,

$$(s')^2 = 0$$

If time permits I will draw a figure of this here.

5.4 A Formula for Cup-Product in the Nonzero Homomorphism case

The proof we will give below is applied for Moore spaces, but actually this proof can be made very general to any spaces ΣA and X which satisfy the hypotheses used below.

The cohomology theory in what follows has non-integral coefficients i.e., $G = \mathbb{Z}_{p^k}$ as in case of coefficients in the ring of integers we have $H^2(M) = 0$ (as can be seen in section 4.4) and if C_β is the cofiber of the map $\beta = x_{r,k}$ then $H^2(C_\beta) = 0$ as well. Consider a Moore space $N = N(\mathbb{Z}_{p^b}, 2)$, and an element $u \in H^2(N; \mathbb{Z}_{p^k})$, such that $u^2 = 0$ because in our case we have that $H^4(N; \mathbb{Z}_{p^k}) = 0$. Since $H^3(M; \mathbb{Z}_{p^k})$ is finite from section 4.4, then we can define in case of the non-zero homomorphism a $\mathbb{Z}_{d'}$-valued function where $d' = \gcd(p^{a-1}, p^k), \mathcal{H}$, on $[M, N]$ as follows. Let $\beta \in [M, N]$. Remember that C_β, the cofiber of $\beta : M \to N$, is formed by attaching a cone on M to N by a map of homotopy class $x_{r,k}$. Let c the generator of $\mathbb{Z}_{d'}$ denote the cohomology class which is carried by the cone. Then by Mayer Vietoris sequence applied to $C_\beta = N \cup_\beta CM = S^2 \cup_{S^3,\lambda} D^3 \cup_{S^3,\lambda} D^3 \cup_{S^4,\lambda} D^4$, there is the following exact sequence:

$$\cdots \to \mathbb{Z}_d \cong H^3(M; \mathbb{Z}_{p^k}) \xrightarrow{q} H^4(C; \mathbb{Z}_{p^k}) \cong \mathbb{Z}_{d'} \to H^4(N; \mathbb{Z}_{p^k}) = 0 \to \dots \qquad (1)$$

It is clear from the sequence above that q- note that we left the name of the function here q for the sake of comparison with the previous proof only and we will change it later- is surjective only and not an isomorphism as in the zero-homomorphism case.

Also, since part of our main sequence was

$$M \xrightarrow{\beta} N \xrightarrow{i_\beta} C_\beta \xrightarrow{\partial_\beta} \Sigma M \xrightarrow{\Sigma\beta} \Sigma N \qquad (2)$$

then applying the cohomology functor H^2 to it, we will get the following sequence

$$H^2(\Sigma N) \xrightarrow{(\Sigma\beta)^*} H^2(\Sigma M) \xrightarrow{\partial_\beta^*} H^2(C_\beta) \xrightarrow{i_\beta^*} H^2(N) \xrightarrow{\beta^*} H^2(M) \qquad (3)$$

But, by section 4.4, we know that $H^2(\Sigma M) = H^1(M) = 0$, then i_β^* is injective and hence there is a unique $v_\beta \in H^2(C_\beta; \mathbb{Z}_{p^k})$ that is sent to $u \in H^2(N; \mathbb{Z}_{p^k})$ under the injection i_β^* .

Now, applying the cohomology functor H^4 to (2), we will get the following sequence

$$H^4(\Sigma N) \xrightarrow{(\Sigma\beta)^*} H^4(\Sigma M) \xrightarrow{\partial_\beta^*} H^4(C_\beta) \xrightarrow{i_\beta^*} H^4(N) \xrightarrow{\beta^*} H^4(M) \qquad (4)$$

So, there is ${v_\beta}^2 \in H^4(C_\beta; \mathbb{Z}_{p^k})$ that is sent to $u^2 \in H^4(N; \mathbb{Z}_{p^k})$which is zero by the properties of a 2−dimensional Moore space, so by exactness ${v_\beta}^2$ can be identified with an element, say h_1, coming from $\mathbb{Z}_d \cong H^3(M; \mathbb{Z}_{p^k})$ which has the form $m.c$ for some $m \in \mathbb{Z}_d \quad i.e., {v_\beta}^2 = mc.$ But since the map q in (1) is only surjective then there maybe many elements $h_i, i \geq 1$ that goes to ${v_\beta}^2$. This imposes a little difficulty on us in defining $\mathcal{H}$ to keep it well-defined function, but to overcome this, we will mode out the group $H^4(\Sigma M; G)$ by a subgroup of it which we will name K i.e., collapsing K to a point. Now, here is how our adjusted construction will look like:

Hopf-Type Functions

First, remember that Moore space of dimension 2 is also a suspension by section 3 in [2], so we can assume that $M = \Sigma A$, for some space A. Then we will define a function $\mathcal{H}$ that takes certain maps $\beta : \Sigma A \to N$ and produces some kind of information about the cup products in $H^*(C_\beta; G)$. Here is what we will take as givens:

- A Moore space N and a cohomology class $u \in H^2(N; G)$ such that $u^2 = 0$.
- A space A with $H^2(\Sigma^2 A; G) = 0$.
- A subgroup $K \subseteq H^4(\Sigma^2 A; G) \neq 0$.

The Construction.

1. Let

$$\mathcal{A} = \{\beta \in [\Sigma A, N] | \beta^*(u) = 0 \text{ and } \mathrm{Im}(H^{2n}(\Sigma N; G) \xrightarrow{(\Sigma\beta)^*} H^{2n}(\Sigma^2 A; G)) \subseteq K\}$$

2. Define a function

$$\mathcal{H} : \mathcal{A} \to H^{2n}(\Sigma^2 A; G)/K$$

as follows:

a. Write $C_\beta = N \cup_\beta C\Sigma A$, so that there is a long cofiber sequence

$$\Sigma A \xrightarrow{\beta} N \xrightarrow{i_\beta} C_\beta \xrightarrow{\partial_\beta} \Sigma^2 A \xrightarrow{\Sigma\beta} \Sigma N$$

b. Since $\beta^*(u) = 0$ and $H^2(\Sigma^2 A; G) = 0$, there is a unique $v_\beta \in H^2(C_\beta; G)$ such that $i^*_\beta(v_\beta) = u$.

c. Since $u^2 = 0$, then $i^*_\beta(v^2_\beta) = 0$, so there is an $h_1 \in H^4(\Sigma^2 A; G)$ such that $\partial^*_\beta(h_1) = v^2_\beta$.

d. Note that h_1 is not uniquely defined in general because ∂^*_β is onto only i.e., we can choose h_1 or h_2 or h_i. So that means that

$$h_1 \equiv h_2 \equiv \cdots \equiv h_i \pmod{\mathrm{Im}((\Sigma\beta)^*)}$$

But because of our construction of $\mathcal{A}$, we have $\mathrm{Im}((\Sigma\beta)^*) \subseteq K$, therefore

$$h_1 \equiv h_2 \equiv \cdots \equiv h_i \pmod{K}$$

So, there could be bunch of h_i but only one equivalence class congruent modulo K.So, Define

$$\mathcal{H}(\beta) = [h]$$

Which is a well-defined function from the group $\mathcal{A}$(to be proved later) to the group $H^4(\Sigma^2 A; G)/K$. So, we can ask if $\mathcal{H}$ is a group homomorphism or no. Note that $H^4(\Sigma^2 A; G)/K \cong H^4(C_\beta)$ by the first isomorphism theorem.

Proving that $\mathcal{A} \subseteq [\Sigma A, X]$ is a subgroup.

It is well known result that the homotopy set $[\Sigma A, X]$ is a group because the domain is a suspension. Now, $\mathcal{A}$ is a subgroup because if $\alpha, \beta \in \mathcal{A}$, then

$$(\alpha + \beta)^*(u) = (\alpha^* + \beta^*)(u) = \alpha^*(u) + \beta^*(u) = 0 + 0 = 0$$

And

$$\mathrm{Im}((\Sigma(\alpha+\beta))^*) = \mathrm{Im}((\Sigma\alpha+\Sigma\beta)^*) = \mathrm{Im}((\Sigma\alpha)^*+(\Sigma\beta)^*) \subseteq \mathrm{Im}((\Sigma\alpha)^*)+\mathrm{Im}((\Sigma\beta)^*) \subseteq K+K = K.$$

Theorem (A.1). The function $\mathcal{H} : \mathcal{A} \to H^{2n}(\Sigma^2 A; G)/K$ constitutes a homomorphism of $\mathcal{A}$ into the group $\mathbb{Z}_{d'}$.

We prove **Theorem (A.1)** in the following form:

Lemma (A.2). Let $\beta_1, \beta_2, \beta_3 \in \mathcal{A}$ be elements such that $\beta_1 + \beta_2 + \beta_3 = 0$. Then

$$\mathcal{H}(\beta_1) + \mathcal{H}(\beta_2) + \mathcal{H}(\beta_3) = 0_{\mathbb{Z}_{d'}}.$$

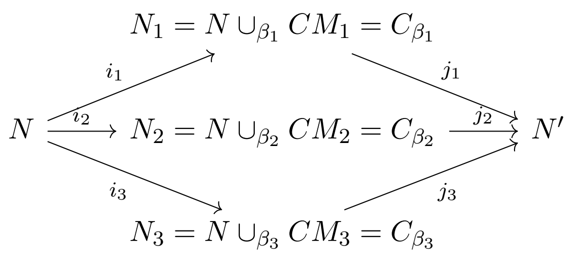
In what follows, let t be an indexing integer which takes values $1, 2, 3$. Let N_t denote the complex which is obtained by attaching a cone on M, call it $e_t = CM_t$, to N by a map of homotopy class β_t, so that

$$N_1 \cap N_3 = N_2 \cap N_3 = N_1 \cap N_2 = N.$$

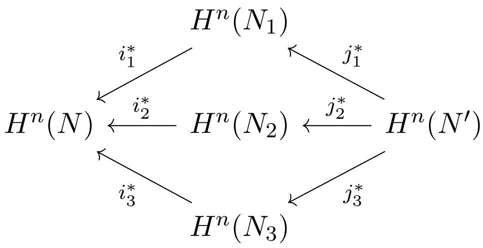
Let $N' = N_1 \cup N_2 \cup N_3 = N \cup_{\beta_1} e_1 \cup_{\beta_2} e_2 \cup_{\beta_3} e_3$. Consider the induced injections:

$$H^n(N') \xrightarrow{j_t^*} H^n(N_t) \xrightarrow{i_t^*} H^n(N).$$

The situation, before applying H^n, can be described by the following diagram:



The situation, after applying H^n, can be described by the following diagram:



Let's focus on $H^n(N_1) \xrightarrow{i_1^*} H^n(N)$ for the moment. We know from our construction of $\mathcal{A}$ that $\beta_1^*(u) = 0$, we also know that $H^n(\Sigma M; G) = 0$ then i_1^* is injective and hence there is a unique $v_{\beta_1} \in H^n(N_1)$ such that $i_1^*(v_{\beta_1}) = u$. Similarly, there is a unique $v_{\beta_2} \in H^n(N_2)$ such that

$$i_2^*(v_{\beta_2}) = u.$$

and there is a unique $v_{\beta_3} \in H^n(N_3)$ such that

$$i_3^*(v_{\beta_3}) = u.$$

Now, in the case of N', since we are attaching 3 cones simultaneously to N, we have the following sequence

$$\Sigma A \vee \Sigma A \vee \Sigma A \to N \to N' \to \Sigma^2 A \vee \Sigma^2 A \vee \Sigma^2 A \to \Sigma N \qquad (5)$$

And after applying H^n, we will get the following sequence

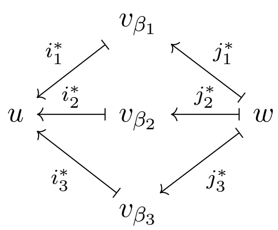
$$H^n(\Sigma^2 A \vee \Sigma^2 A \vee \Sigma^2 A) \to H^n(N') \to H^n(N) \to H^n(\Sigma A \vee \Sigma A \vee \Sigma A)$$

But since $H^n(\Sigma^2 A \vee \Sigma^2 A \vee \Sigma^2 A) \cong H^n(\Sigma^2 A) \oplus H^n(\Sigma^2 A) \oplus H^n(\Sigma^2 A)$, and since by our construction we also have that $H^n(\Sigma^2 A) = 0$, then $H^n(\Sigma^2 A \vee \Sigma^2 A \vee \Sigma^2 A) = 0$.

Also, we assumed in $\mathcal{A}$ that $\beta^*(u) = 0$, then in our case here we have $H^n(\Sigma A \vee \Sigma A \vee \Sigma A) \cong H^n(\Sigma A) \oplus H^n(\Sigma A) \oplus H^n(\Sigma A)$, and so $\beta^*(u) = (0, 0, 0) = 0$ as required. So we will get the following sequence

$$0 \to H^n(N') \to H^n(N) \to H^n(\Sigma A \vee \Sigma A \vee \Sigma A)$$

and hence there is a unique $w \in N'$ that goes to something in N_1 that goes to u. But since there is only a unique $v_{\beta_1} \in N_1$ that goes to u, and similarly for v_{β_2} and v_{β_3}. So, we can say that w denote the cohomology class of N' such that $i_t^* j_t^*(w) = u$.Then, from the previous information, we will get the following diagrame:



Now, applying H^{2n} to (5), we will get the following sequence

$$H^{2n}(\Sigma N) \to H^{2n}(\Sigma^2 A \vee \Sigma^2 A \vee \Sigma^2 A) \to H^{2n}(N') \to H^{2n}(N) \to H^{2n}(\Sigma A \vee \Sigma A \vee \Sigma A)$$

And since $H^{2n}(\Sigma^2 A \vee \Sigma^2 A \vee \Sigma^2 A) \cong H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A)$. Specifically, in our case $n = 2$ and $M = \Sigma A$, then $H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A) = H^3(M) \oplus H^3(M) \oplus H^3(M)$

And if we assume that c_t denote the cohomology class which is carried by e_t, then $H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A) = H^3(M) \oplus H^3(M) \oplus H^3(M) = (c_1, c_2, c_3)$.

Now, by all what is said before, we can have the following sequence

$$(c_1, c_2, c_3) \mapsto w^2 \mapsto u^2 = 0.$$

Now, we can do the following

$$H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A) \to \Big(H^{2n}(\Sigma^2 A)/K\Big) \oplus \Big(H^{2n}(\Sigma^2 A)/K\Big) \oplus \Big(H^{2n}(\Sigma^2 A)/K\Big)$$

and so

$$(c_1, c_2, c_3) \mapsto ([c_1], [c_2], [c_3])$$

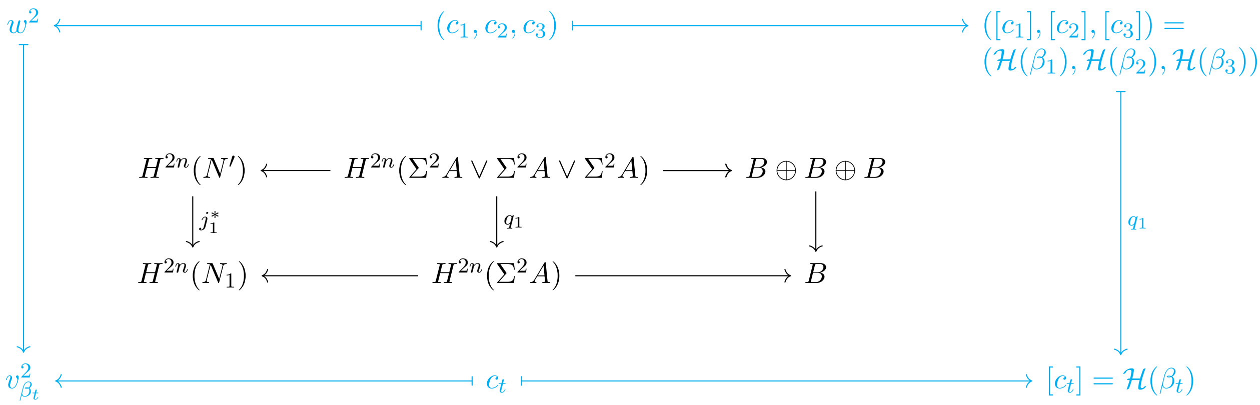
Now, define

$$\mathcal{H}(\beta) = [c]$$

However, by the naturality of the cup-product, the first equality in the following is correct,

$$\begin{aligned} j_t^*(w) \cup j_t^*(w) &= j_t^*(w \cup w) \\ &= j_t^*(w^2) \\ &= v_{\beta_t}^2 \end{aligned}$$

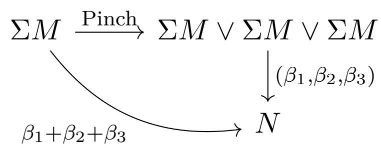
Next, from the diagram below with $t = 1, 2, 3$ and $B = H^{2n}(\Sigma^2 A)/K$



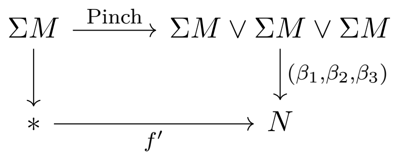
we can deduce that

$$[c_t] = \mathcal{H}(\beta_t).$$

Now, since by section 4.9, the addition $\beta_1 + \beta_2 + \beta_3$ means that we have the following diagram:



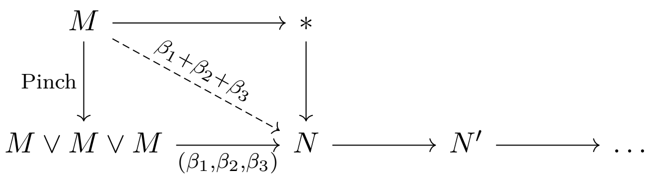
And since $\beta_1 + \beta_2 + \beta_3 = 0$, by hypothesis, then this means that we have the following commutative diagram (up to homotopy):



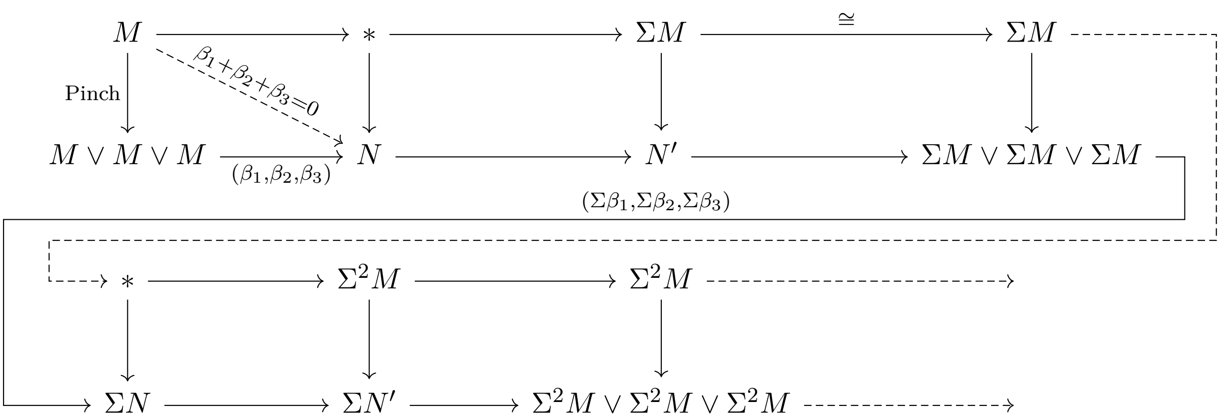
i.e., $\beta_1 + \beta_2 + \beta_3 = 0$ means that, up to homotopy, it factors through a single point which means that it is homotopic to a constant map.

Also, since $\beta_1 + \beta_2 + \beta_3 = 0$, by hypothesis, then we have the following diagram (our hypothesis square):

If $M = \Sigma A$,



Now, after forming the cofibres multiple times we got the following diagram:



If time permits I will continue drawing the above commutative diagram until having 16 rectangles, I already have it drawn in my hand written notes.

So, we can see that as a result of the existence of the first commutative square in this figure, there is a map $f : \Sigma M \to N'$ which maps ΣM onto each of the three cells e_t.

Let $f^* : H^{2n}(N') \to H^{2n}(\Sigma M)$ denote the homomorphism induced by f. Then we will apply H^{2n} to the third square in the previous figure - noting that the map Pinch* is just adding - and try to link the new square we get with the upper row in the inner commutative diagram of the figure containing two rectangles inside each other above (figure $3'$). Here is the figure that we will get if $B = H^{2n}(\Sigma^2 A)/K$

$$\begin{array}{ccccc}
H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A) \oplus H^{2n}(\Sigma^2 A) & \longrightarrow & B \oplus B \oplus B & \xrightarrow{add} & B \\
\downarrow & & & \nearrow & \\
H^{2n}(N') & \xrightarrow{f^*} & H^{2n}(\Sigma^2 A) & &
\end{array}$$

Then from the previous figure, in terms of the elements of the groups above, we will get the following figure:

$$\begin{array}{ccccc}
(c_1, c_2, c_3) & \longmapsto & (\mathcal{H}(\beta_1), \mathcal{H}(\beta_2), \mathcal{H}(\beta_3)) & \longmapsto & \mathcal{H}(\beta_1) + \mathcal{H}(\beta_2) + \mathcal{H}(\beta_3) \\
\downarrow & & & \nearrow & \\
w^2 & \longmapsto & (f^*(w))^2 & &
\end{array}$$

Where $(f^*(w))^2$ is because $f^*(w^2) = f^*(w \cup w) = f^*(w) \cup f^*(w) = (f^*(w))^2$ and this is zero because the cup product of suspension is trivial on reduced cohomology. Therefore

$$\mathcal{H}(\beta_1) + \mathcal{H}(\beta_2) + \mathcal{H}(\beta_3) = 0 \pmod K$$

as required.

5.5 Understanding the Cup-Product Structure in $H^*(C_\beta)$ Using Hopf Invariant

How will what we did in the previous section help us in determining - in terms of the parameters r and k - the cup products structure

$$H^2(C_\alpha; \pi_2(C_\beta)) \otimes H^2(C_\beta; \pi_2(C_\beta)) \to H^4(C_{\alpha+\beta}; \pi_2(C_\alpha) \otimes \pi_2(C_\beta)).$$

Specifically, we want to know whether this function is zero or non-zero. And then we will use the following theorem, which can be concluded from Proposition (5.3) in [Jam78] and example 1.49 in‘[Cor+03].

Theorem 10. *Let G be an abelian group and X be an $(n-1)$-connected space that is at most $2n$-dimensional with $\pi_n(X) \cong G$. Then X is a co-H-space if and only if the cup product map*

$$H^n(X;G) \otimes H^n(X;G) \to H^{2n}(X;G \otimes G)$$

is zero.

Relationship between the meaning of a co-H-map and Hopf invariant.

What we will explain below is based on the definition of Hopf invariant used in [9]. If $(\Sigma X, \varphi_i)$ and (Y, ψ_j) are co-H-spaces and $h : \Sigma X \to Y$ is a map, we say that $h : (\Sigma X, \varphi_i) \to (Y, \psi_j)$ is a co-H-map if the following diagram commutes up to homotopy

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\quad h \quad} & Y \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \psi_j} \\ \Sigma X \vee \Sigma X & \xrightarrow{h \vee h} & Y \vee Y \end{array}$$

which means that $(h \vee h) \circ \varphi_i \simeq \psi_j \circ h$. We hope that this happens, but in any case we can take their difference and call it $H_{ij}(h)$ i.e., $(h \vee h) \circ \varphi_i - \psi_j \circ h = H_{ij}(h)$. And so, if there are i, j such that the diagram commutes i.e., $H_{ij}(h)$ then C_β will has at least one co-H-structure by Theorem3.3.1. In[9], Berstein and Hilton are always taking $i = 0$, which means, according to them, that they are using the suspension structure on X and this is their definition for Hopf invariant i.e., they are defining the Hopf invariant as follows

$$\mathcal{H}_j(h) = H_{0,j}(h).$$

Also, recall that we have

$$x_{r,k} = r.\lambda + k.(i \circ \phi \circ q)$$

and we know that $r.\lambda$ is a suspension map from [2, pg. 681], so, if we form the cofiber and take the cup-product this term will be zero. i.e., $\mathcal{H}(r.\lambda) = 0$. Hence,

$$\mathcal{H}(x_{r,k}) = k.\mathcal{H}(i \circ \phi \circ q)$$

Also, recall that $\mathcal{H}$ was defined using:

•$u \in H^2(N)$.
•$K \supseteq \mathrm{Im}((\Sigma\alpha)^*) \subseteq H^4(\Sigma M)$.

So, we have 2 things to do:

• Find the right K and u.
• Figure out what is $\mathcal{H}(i \circ \phi \circ q)$.

Recall that our assumptions on u were:

1. $u \in H^2(N)$
2. $\beta^*(u) = 0$
3. $u^2 = 0$

First: Figuring out K and u.

1. For choosing u, consider the following sequence in dimension 2:

$$H^2(M) \xleftarrow{\beta^*} H^2(N) \xleftarrow{i_\beta^*} H^2(C_\beta) \xleftarrow{\partial_\beta^*} H^2(\Sigma M) \leftarrow \dots$$

   Note that $H^2(\Sigma M) = 0$ by section 4.4, so that means that i_β^* is injective. Now, since $\ker(\beta^*) \hookrightarrow H^2(N)$ and $H^2(N) \cong \mathbb{Z}_d$ is a cyclic group and a subset of a cyclic group is cyclic, then $\ker(\beta^*)$ is cyclic and so we can say that

$$\ker(\beta^*) \cong Z_{p^{w'}}. \text{ a generator}$$

   and we will take the u to be the generator of $\ker(\beta^*)$.

2. For choosing K, consider the following sequence in dimension 4:

$$H^4(M) \xleftarrow{\beta^*} H^4(N) \xleftarrow{i_\beta^*} H^4(C_\beta) \xleftarrow{\partial_\beta^*} H^4(\Sigma M) \xleftarrow{(\Sigma\beta)^*} H^4(\Sigma N)$$

   In the above sequence $\ker(\partial_\beta^*) = \mathrm{Im}((\Sigma\beta)^*)$
   Choose $K = \mathrm{Im}((\Sigma\beta)^*) = \ker(\partial_\beta^*)$.

If $\beta = x = x_{r,k} = r.\lambda + k.(i \circ \phi \circ q) = x_1 + x_2$, then we have the following claim:

CLAIM: Same u and K will work for x_1 and x_2 i.e., $x_1, x_2 \in \mathcal{A}$.

Proof:

(a) checking that the claim is correct for u.

1. $u \in H^2(N)$ is automatically true for all x_1 and x_2.
2. $x_2^*(u) = 0$ as we proved in chapter 2 due to the Hopf map. Now $x_1^*(u) = 0$ because of the following:

$$x^* = x_1^* + x_2^* = x_1^*$$

   and then

$$x^*(u) = 0 = (x_1^* + x_2^*)(u) = x_1^*(u)$$

   Therefore $x_1^*(u) = 0$ as required.

3. $u^2 = 0$ is automatically true for all x_1 and x_2.

(b) checking that the claim is correct for K.

Since

$$K \supseteq \mathrm{Im}((\Sigma\beta)^*) = \mathrm{Im}((\Sigma x_1)^*) \supseteq 0 = \mathrm{Im}((\Sigma x_2)^*)$$

where the first containment is by our construction, the first equality is by 2 in the proof of (a) and the second equality is by 2 in the proof of (a) also. Then, the claim is correct for K also.

Second: Figuring out what is $\mathcal{H}(i \circ \phi \circ q)$ by comparing with $\mathbb{C}P^2$.

We know that $H^2(C_\beta)$ is a cyclic group by section 4.8 and since we know that there is a unique v_β corresponding to u by the argument given in section 5.5 after the sequence given in (3),then this v_β is a generator for $H^2(C_\beta)$. Therefore, the following are equivalent:

1. All cup-products in $H^*(C_\beta)$ are zero.
2. $v_\beta^2 = 0 \in H^4(C_\beta)$

This is because for $v, w \in H^2(C_\beta)$, we have $v = a_1 v_\beta, w = a_2 v_\beta$, and then $v.w = a_1 a_2 v_\beta^2$.

The way we compute our $\mathcal{H}$ is as follows, we have the following diagram:

$$\begin{array}{ccccccccc}
 & & & & H^4(\Sigma M)/K & & & & \\
 & & & & \text{surjection}\uparrow\pi & \searrow\cong & & & \\
H^4(\Sigma C_\beta) & \longrightarrow & H^4(\Sigma N) & \xrightarrow{(\Sigma\beta)^*} & H^4(\Sigma M) & \xrightarrow[\text{Onto}]{\partial_\beta^*} & H^4(C_\beta) & \longrightarrow & H^4(N) = 0
\end{array}$$

Where by the first isomorphism theorem $H^4(\Sigma M)/K \cong H^4(C_\beta)$ as π is a surjection. And the corresponding diagram of elements is given below

$$\begin{array}{ccccccccc}
 & & & & \mathcal{H}(\beta) = [w] & & & & \\
 & & & & \text{surjection}\uparrow & \searrow\cong & & & \\
\dots & \longmapsto & \dots & \xmapsto{(\Sigma\beta)^*} & w^2 & \xmapsto[\text{Onto}]{\partial_\beta^*} & v_\beta^2 & \longmapsto & 0
\end{array}$$

Therefore $v_\beta^2 = 0$ if and only if $\mathcal{H}(\beta) = 0$ because of the isomorphism in the figure. But we already know that

$$\mathcal{H}(\beta) = k.\mathcal{H}(i \circ \phi \circ q).$$

Now, we know that $H^4(\Sigma M)/K \cong H^4(C_\beta)$ and we also know that $H^4(C_\beta) \cong \mathbb{Z}_d$ in the zero homomorphism case and $H^4(C_\beta) \cong \mathbb{Z}_{d'}$ in the nonzero homomorphism case. So, the question is:

Is $\mathcal{H}(i\circ\phi\circ q)$ divisible by $\mathbb{Z}_d$ or $\mathbb{Z}_{d'}$? or is k divisible by $\mathbb{Z}_d$ or $\mathbb{Z}_{d'}$? and this is what we will try to answer by comparing the cofiber C_β with the complex projective plane $\mathbb{C}P^2$.

Note that since section 5.4 in this chapter can be derived from section 5.5 in the same chapter also by just setting the subgroup K to be the zero group *i.e.*, $\mathrm{Im}((\Sigma\beta)^*) = 0$, so what we are doing here in this section is valid for both the zero and the non-zero homomorphism case of the previous 2 sections.

Now, let's see what is going on in this $\mathcal{H}(i\circ\phi\circ q)$. Recall that Hopf map is a map from S^3 to S^2 i.e., $\phi : S^3 \to S^2$ and its cofiber is the complex projective plane $\mathbb{C}P^2$ i.e., $C_\phi = \mathbb{C}P^2$.Using [4,6], we know that for any ring R with unit, the cohomology of $\mathbb{C}P^2$ is

$$H^*(C_\phi, R) = \begin{cases} R.1 & if \quad *=0, \\ R.t & if \quad *=2, \\ R.t^2 & if \quad *=4, \\ 0 & \quad \text{otherwise} . \end{cases}$$

Also, we know that the map $i\circ\phi\circ q$ is given by the following diagram

$$\begin{array}{ccc} S^2\cup_m D^3 & \xrightarrow{i\circ\phi\circ q} & S^2\cup_n D^3 \\ \big\downarrow q & & \big\uparrow i \\ S^3 & \xrightarrow[\phi]{\qquad\qquad} & S^2 \end{array}$$

So what we want to do is to create a relation between the cofiber of $i\circ\phi\circ q$ and the cofiber of ϕ. But the diagram we had is a little bit weird as the map q arrow is pointing downward while the map i arrow is pointing upward. we will try to fix this problem, consider the following commutative diagram

$$\begin{array}{ccccccc} M = S^2\cup_m D^3 & \xrightarrow{i\circ\phi\circ q} & N = S^2\cup_n D^3 & \longrightarrow & C_{v_{x_2}} & \xrightarrow{\partial} & \Sigma M \\ \| & & \big\uparrow i & & \big\uparrow g & & \| \\ S^2\cup_m D^3 & \xrightarrow{\phi\circ q} & S^2 & \longrightarrow & D & \xrightarrow{\delta} & \Sigma M \\ \big\downarrow q & & \| & & \big\downarrow f & & \big\downarrow \Sigma q \\ S^3 & \xrightarrow{\phi} & S^2 & \longrightarrow & \mathbb{C}P^2 & \xrightarrow{d} & \Sigma S^3 \end{array}$$

Where each row in the previous diagram is a cofiber sequence.We will work on the lower part of the diagram first and then on the upper part. On applying H^2 to the lower part of the previous diagram, we will get the following diagram:

$$\begin{array}{ccccccc} H^2(M) & \xleftarrow{(\phi\circ q)^*=0} & H^2(S^2) & \xleftarrow{\cong} & H^2(D) & \xleftarrow{\delta^*} & H^2(\Sigma M) = 0 \\ \big\uparrow q^* & & \| & \square & \cong\big\uparrow f^* & & \big\uparrow (\Sigma q)^* \\ 0 = H^2(S^3) & \xleftarrow{\phi^*} & H^2(S^2) & \xleftarrow{\cong} & H^2(\mathbb{C}P^2) & \xleftarrow{d^*} & H^2(\Sigma S^3) = 0 \end{array}$$

Taking into account, by a way similar to what we proved in chapter 2 for homology, that ϕ induces the zero map 0 in cohomology and so $(\phi \circ q)^* = 0$. Also, in the rectangle that contains the mini-square, the 2 isomorphisms and the equal sign makes f^* an isomorphism. So, we get the following diagram from the middle rectangle in the previous figure :

$$
\begin{array}{ccc}
\sigma & \longleftarrow\!\shortmid & \tau \\
\Vert & & \uparrow \\
\sigma & \longleftarrow\!\shortmid & t
\end{array}
$$

Where, because of the isomorphism, the generator t of $H^2(\mathbb{C}P^2)$ will go to the generator τ of $H^2(D)$. And where σ is the generator of $H^2(S^2)$. Now, we will work on the upper part of the big diagram above. Apply H^2 to it, we will get the following diagram

$$
\begin{array}{ccccccc}
H^2(M) & \xleftarrow{(i\circ\phi\circ q)^*=0} & H^2(N) & \xleftarrow{\cong} & H^2(C_{v_{x_2}}) & \xleftarrow{\partial^*} & H^2(\Sigma M) = 0 \\
\Vert & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle g^*} & & \Vert \\
H^2(M) & \xleftarrow{(\phi\circ q)^*=0} & H^2(S^2) & \xleftarrow{\cong} & H^2(D) & \xleftarrow{\delta^*} & H^2(\Sigma M) = 0
\end{array}
$$

Recall that the ring of coefficients that we have is $G = \mathbb{Z}_{p^k}$ where this k is different from the k in $x_{r,k}$. Now, the key idea is that we want $H^2(C) \cong H^2(D)$. Note that i^* is an isomorphism because, in the rectangle that contains it, we have two zero maps and an equal sign. Therefore g^* is an isomorphism as required because the rectangle in which it lies contains 3 isomorphisms. So τ will come from $v_{x_2} \in H^2(C)$ i.e., we have

$$t \mapsto \tau \leftarrow\!\shortmid v_{x_2} \mapsto u \mapsto 0.$$

Now, if we squared t and v_{x_2}, this will come to squaring of τ. So, we will learn about $v_{x_2}^2$ via τ^2. Since we know that $H^4(C_{v_{x_2}})$ is cyclic, then our guess is that $v_{x_2}^2$ is a generator of it. To see this, we will apply H^4 to the big diagram on the previous page after increasing some groups in the cofiber sequences and here is the diagram that we will get

$$
\begin{array}{ccccccccc}
H^4(M) & \xleftarrow{(i\circ\phi\circ q)^*=0} & H^4(N) = 0 & \longleftarrow & H^4(C_{v_{x_2}}) & \underset{\cong}{\xleftarrow{\partial^*}} & H^4(\Sigma M) & \xleftarrow{0} & H^4(\Sigma N) \\
\Vert & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle g^*} & & \Vert & & \downarrow \\
H^4(M) & \xleftarrow{(\phi\circ q)^*=0} & H^4(S^2) = 0 & \longleftarrow & H^4(D) & \underset{\cong}{\xleftarrow{\delta^*}} & H^4(\Sigma M) & \longleftarrow & H^4(S^3) = 0 \\
\uparrow{\scriptstyle q} & & \Vert & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle (\Sigma q)^*} & & \Vert \\
0 = H^4(S^3) & \xleftarrow{\phi^*} & H^4(S^2) = 0 & \longleftarrow & H^4(\mathbb{C}P^2) & \underset{\cong}{\xleftarrow{d^*}} & H^4(\Sigma S^3) & \longleftarrow & H^4(S^3) = 0
\end{array}
$$

taking into account that we are in the zero homomorphism case as the map $i \circ \phi \circ q$ induces the zero map in cohomology. And now we have the following diagram:

$$t^2 \mapsto \tau^2 \leftarrow\!\shortmid v_{x_2}^2 \mapsto u^2 \mapsto 0.$$

Claim: $(\Sigma q)^* : H^4(S^4) \to H^4(\Sigma M)$ **is onto.**

Before proving this, we will clarify why this will help us in proving that $v_{x_2}^2$ is the generator of $H^4(C_{v_{x_2}})$. In the previous diagram that we obtained after applying H^4, we know that g^* is an isomorphism because ∂^* and δ^* are isomorphisms and there is an equality sign in this rectangle. So, if we were able to prove that f^* is onto we will be able to prove that $v_{x_2}^2$ is the generator of $H^4(C_{v_{x_2}})$, because if we have a surjection from one cyclic group $H^4(\mathbb{C}P^2)$ to another $H^4(D)$, then the generator has to go to the generator. Now, proving that f^* is onto comes down to proving that $(\Sigma q)^*$ is onto (because the rectangle of f^* contains δ^* and d^* which are isomorphisms) which is what we are going to prove now.

Proof:

Consider the following cofiber sequence:

$$S^2 \xrightarrow{p^a} S^2 \to M \to S^3 \xrightarrow{p^a} S^3 \to \Sigma M \to S^4 \xrightarrow{p^a} S^4 \to \dots$$

Applying H^4 to the previous sequence we will get

$$H^4(S^2) \xleftarrow{(p^a)^*} H^4(S^2) \leftarrow H^4(M) \leftarrow H^4(S^3) \xleftarrow{(p^a)^*} H^4(S^3) \leftarrow H^4(\Sigma M) \xleftarrow{(\Sigma q)^*} H^4(S^4) \xleftarrow{(p^a)^*} H^4(S^4) \leftarrow \dots$$

Since all the groups before $H^4(\Sigma M)$ are zeros, and because of exactness, it can be seen that the map $(\Sigma q)^*$ is onto as required.

So, from all the previous $v_{x_2}^2$ is a generator of $H^4(C_{v_{x_2}})$, and hence is a generator of $H^4(\Sigma M)/K$ because $H^4(\Sigma M)/K \cong H^4(C_{v_{x_2}})$. But also we proved before that $\mathcal{H}(\beta) \cong v_\beta^2$ then $\mathcal{H}(i \circ \phi \circ q)$ is a generator of $H^4(\Sigma M)/K$.

Now, we will take the ring of coefficients to be $\mathbb{Z}_{p^b}$ as we know, from our previous calculations, that in case of the zero homomorphism $\widetilde{H_2}(C;\mathbb{Z}) = \mathbb{Z}_{p^b} \cong H_2(N;\mathbb{Z})$. Then $\mathcal{H}(i \circ \phi \circ q)$ is a generator for $H^4(\Sigma M;\mathbb{Z}_{p^b})/K$. Then we can conclude the following

$$\mathcal{H}(x_{r,k}) = k.\mathcal{H}(i \circ \phi \circ q) = 0 \textbf{ iff } k \textbf{ divides the generator of } H^4(\Sigma M;\mathbb{Z}_{p^b})/K \cong H^4(C_\beta;\mathbb{Z}_{p^b})$$

So we were interested in figuring out the cup-product of $x_{r,k}$, and we get the condition under which it is zero.

CHAPTER 6

LUSTERNIK-SCHNIRELMANN CATEGORY, CO-H-SPACES AND CUP-PRODUCTS

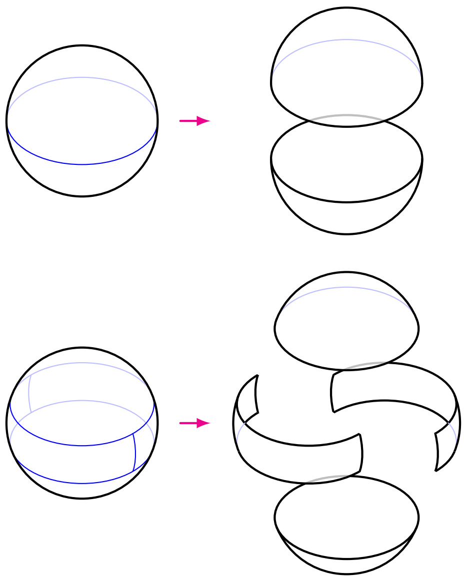
6.1 Preliminaries

The classical reductionist paradigm of mathematics and science has been to decompose an object into simpler pieces and then understand the object by analyzing these pieces and how they fit together to form the object. For a homotopy theorist, the simplest possible pieces of a space are its contractible subsets i.e., subsets that are equivalent to a point.To relate such subsets to the space in which they sit, it is better,however, to look at the subsets which are contractible in the space. So,for a space, we can simply ask, how many such(open)subsets are required to cover the space. This simple numerical invariant provides one measure of the complexity of a space and also provides the starting point in our study. This simple numerical invariant is determined by Lusternik-Schnirelmann category.

Lusternik-Schnirelmann category is like a Picasso painting. Looking at a category from different perspectives produces completely different impressions of category's beauty and applicability.

Definition 6.1. Lusternik-Schnirelmann or LS category of a space X is the least integer n such that there exists an open covering $U_0, ..., U_n$of X with each U_i contractible to a point in the space X. We denote this by $\mathrm{Cat}(X) = n$ and we call such a covering $\{U_i\}$ categorical. If no such integer exists we write $\mathrm{Cat}(X) = \infty$.

Example 2. As can be seen from the diagram below, we can cover the sphere S^2 with 2 or 4 open contractible sets



yet, according to the definition of L-S category, $\mathrm{Cat}(S^2) = 1$.

Example 3. Every suspension has L-S category at most 1. And it is equal to 1 if the suspension is not trivial.

Example 4. $\mathrm{Cat}(S^m \otimes S^n) = 2$. And one way to see that is there is a beautiful theorem that everyone loves in category theory, which is

Theorem 11. *If* $\mathrm{Cat}(X) \leq n$, *then in* $\widetilde{H}^*(X; G)$, *all cup products of length* $> n$ *are zero.*

Corollary 1. In any suspension, all non-trivial cup-products are zero.

We have applied the previous corollary at the end of section 5.4 and 5.5.

6.2 Determining the L-S Category of C_β

Question 1. If we have the following map $f : A \to X$ and if we know that $\mathrm{Cat}(X) = n$ and we looked at the cofiber $X \cup_f CA$, basically when we add a cone upon X, we are increasing the category by at most 1. So, we can conclude that $\mathrm{Cat}(X \cup_f CA) \leq n + 1$. So the question we are asking is, how can you tell if it is actually $n + 1$ or if it is $\leq n + 1$?

We may return to this question later if we have time. In this chapter, we will focus more on spaces X of category 1 and the reason is because of the following theorem

Theorem 12. *The following are equivalent:*

1. $\mathrm{Cat}(X) \leq 1$.

2. *X is a co-H-space.*

Which link the L-S category with co-H-spaces and which can be found in [11, Example 1.49]. Remember that among the co-H-spaces are suspensions and we said previously that every suspension has L-S category at most 1, so this theorem is kind of matching what we already have seen.

So, by the previous theorem, we can restate Theorem 3.3.1 as follows:

Theorem 13. *If* $\mathrm{Cat}(X) \leq 1, \mathrm{Cat}(Y) \leq 1$ *and* $f : X \to Y$ *is a co-H-map, then*

$$\mathrm{Cat}(Y \cup_f CX) \leq 1.$$

Another question we have in mind is

Question 2. Is it possible to find a co-H-structure on X and a co-H-structure on Y that turns f into a co-H-map?

The answer to this question will be based on the congruences we get in chapter 3. If the answer is in affirmative, then the cofiber $Y \cup_f CX$ has L-S category ≤ 1. If the answer is in negation, then $\mathrm{Cat}(Y \cup_f CX) \leq 1$ or $\mathrm{Cat}(Y \cup_f CX) = 2$. We may return to this general question also later if we have enough time.

Let's focus now on maps between Moore spaces $\beta : M \to N$ and we are wondering about $C_\beta = N \cup_\beta CM$. Are there co-H structures for which β is a co-H map and so $\mathrm{Cat}(C_\beta) \leq 1$? If not, is $\mathrm{Cat}(C_\beta) \leq 1$ or $\mathrm{Cat}(C_\beta) = 2$?

There is one last piece to go into this which is, because of that cofiber starting in dimension 2 and going into dimension 4, the answer to what is its category is given entirely by calculating the cup-products. So, this is why we have theorem 9 below.

By what we proved in section 5.6 we know the condition under which the cup-product of C_β is zero. Then using the following theorem which can be concluded from [10]:

Theorem 14. *If X is $(n-1)$-connected and k-times n-dimensional, then* $\mathrm{Cat}(X) = k$ *if and only if there is a class $u \in H^n(X)$ whose k^{th} cup-product is non-zero in $H^{kn}(X)$.*

And knowing that X in our case is $2n$-dimensional i.e., $k = 2$, we can conclude that $\mathrm{Cat}(C_\beta) \neq 2$ and so C_β is a co-H space i.e.$\mathrm{Cat}(C_\beta) \leq 1$. Now we have known the complexity of the cofibers of the maps $x_{r,k}$ between Moore spaces as we claimed that we will do.

CHAPTER 7

CONCLUSION, ONGOING AND FUTURE WORK

7.1 Conclusion

We were able to know the condition that makes the cup-product of of C_β is zero , namely it is zero iff k divides the generator of $H^4(\Sigma M; \mathbb{Z}_{p^b})/K \cong H^4(C_\beta; \mathbb{Z}_{p^b})$.And so, we were able to know the complexity of this cofiber C_β if the condition on k is satisfied.

7.2 Ongoing and Future Work

We are wondering about, how many co-H-structures can the cofiber C_β be given? Among Moore spaces, are there some maps that can be co-H-maps but they can not be co-H-maps for the suspension structure as a domain?

Another question we have in mind is that, if $f : X \to Y$, is it possible to find a co-H-structure on X and a co-H-structure on Y that turns f into a co-H-map? The answer to this question will be based on the congruences we get in chapter 3. If the answer is in affirmative, then the cofiber $Y \cup_f CX$ has L-S category ≤ 1. If the answer is in negation, then $\mathrm{Cat}(Y \cup_f CX) \leq 1$ or $\mathrm{Cat}(Y \cup_f CX) = 2$.

We also did not answer the following question,if we have the following map $f : A \to X$ and if we know that $\mathrm{Cat}(X) = n$ and we looked at the cofiber $X \cup_f CA$, basically when we add a cone upon X, we are increasing the category by at most 1. So, we can conclude that $\mathrm{Cat}(X \cup_f CA) \leq n + 1$. So the question we are asking is, how can you tell if it is actually $n + 1$ or if it is $\leq n + 1$?

Also, we hope in our future work, using the congruences we have calculated for what values of j, l they are solvable and the congruence we calculated in section 5.6, to find a numeric example that gives us a map that is not a co-H-map no matter what co-H structure of X and Y we choose,and yet the co-fiber is a co-H-space.

Bibliography

[Jam57] I. M. James. "Note on cup-products". In: *Proc. Amer. Math. Soc.* 8 (1957), pp. 374–383. ISSN: 0002-9939. DOI: 10.2307/2033748. URL: https://doi.org/10.2307/2033748.

[BH60] I. Berstein and P. J. Hilton. "Category and generalized Hopf invariants". In: *Illinois J. Math.* 4 (1960), pp. 437–451. ISSN: 0019-2082. URL: http://projecteuclid.org/euclid.ijm/1255456060.

[MT68] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory.* Harper & Row, Publishers, New York-London, 1968, pp. x+214.

[Jam78] I. M. James. "On category, in the sense of Lusternik-Schnirelmann". In: *Topology* 17.4 (1978), pp. 331–348. ISSN: 0040-9383. DOI: 10.1016/0040-9383(78)90002-2. URL: https://doi.org/10.1016/0040-9383(78)90002-2.

[Rot79] Joseph J. Rotman. *An introduction to homological algebra.* Vol. 85. Pure and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1979, pp. xi+376. ISBN: 0-12-599250-5.

[Rot88] Joseph J. Rotman. *An introduction to algebraic topology.* Vol. 119. Graduate Texts in Mathematics. Springer-Verlag, New York, 1988, pp. xiv+433. ISBN: 0-387-96678-1. DOI: 10.1007/978-1-4612-4576-6. URL: https://doi-org.libproxy.library.wmich.edu/10.1007/978-1-4612-4576-6.

[AG94] Martin Arkowitz and Marek Golasiński. "Co-H-structures on Moore spaces of type $(G, 2)$". In: *Canad. J. Math.* 46.4 (1994), pp. 673–686. ISSN: 0008-414X. DOI: 10.4153/CJM-1994-037-0. URL: https://doi-org.libproxy.library.wmich.edu/10.4153/CJM-1994-037-0.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0.

[Cor+03] Octav Cornea et al. *Lusternik-Schnirelmann category.* Vol. 103. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xviii+330. ISBN: 0-8218-3403-5. DOI: 10.1090/surv/103. URL: https://doi.org/10.1090/surv/103.

[Awo10] Steve Awodey. *Category theory.* Second. Vol. 52. Oxford Logic Guides. Oxford University Press, Oxford, 2010, pp. xvi+311. ISBN: 978-0-19-923718-0.

[Ark11] Martin Arkowitz. *Introduction to homotopy theory.* Universitext. Springer, New York, 2011, pp. xiv+344. ISBN: 978-1-4419-7328-3. DOI: 10.1007/978-1-4419-7329-0. URL: https://doi-org.libproxy.library.wmich.edu/10.1007/978-1-4419-7329-0.

[Str11] Jeffrey Strom. *Modern classical homotopy theory.* Vol. 127. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxii+835. ISBN: 978-0-8218-5286-6. DOI: 10.1090/gsm/127. URL: https://doi.org/10.1090/gsm/127.

[Fuc03] László Fuchs. "Infinite abelian groups in Hungary". In: *Mat. Lapok (N.S.)* 11.1 (2002/03), 16–26 (2006). ISSN: 0025-519X.