THE k-CORES OF A GRAPH

Allan Bickle, Ph.D.

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The k-core of a graph is the maximal subgraph with minimum degree at least k. It is easily shown that this subgraph is unique, the cores of a graph are nested, and that it can be found by iteratively deleting vertices with degree less than k. The maximum k such that G has a k-core is the maximum core number of G, $\widehat{C}(G)$, and if $\widehat{C}(G) = \delta(G)$, we say G is k-monocore. Many common graph classes including trees and regular graphs are monocore. A deletion sequence is formed by iteratively deleting a vertex of smallest degree, and a construction sequence reverses a deletion sequence.

Following these basic results, chapter one defines the k-shell of a graph as the subgraph induced by edges in the k-core and not in the $k + 1$-core. The 1-shell is a forest with no trivial components. The structure of 2-cores and 3-cores is analyzed and an operation characterization of 2-monocore graphs is presented.

Chapter two examines the extremal classes of k-cores. Maximal k-degenerate graphs are the upper extremal graphs. Results on their size, degree sequence, diameter, and more are presented. Labeled maximal k-degenerate graphs are shown to correspond bijectively to a certain type of sequences. The k-trees, a special type of maximal k-degenerate graph, are characterized.

The degree sequences of k-monocore graphs are characterized. Collapsible and core-critical graphs, classes of lower extremal graphs, are defined and analyzed. How graphs collapse is analyzed.

In Chapter three, the structure of the k-core of a line graph or Cartesian product or join of graphs is characterized. Ramsey core numbers, a new variation of Ramsey numbers, are defined and an exact formula is proven.

Chapter four considers applications of cores to problems in graph theory. The core number bound for chromatic number, $\chi(G) \leq 1 + \widehat{C}(G)$, is proved using construction sequences. It leads to short proofs of Brooks' Theorem and the Nordhaus-Gaddum Theorem. Extremal decompositions attaining Plesnik's Conjecture for $k = 2$ and 3 are characterized. Similar coloring techniques are discussed for edge coloring, list coloring, $L(2,1)$ coloring, arboricity, vertex arboricity, and point partition numbers. Applications of cores to problems in planarity, integer embeddings, domination, total domination, and the Reconstruction Conjecture are discussed.