MODULAR MONOCHROMATIC COLORINGS, SPECTRA AND FRAMES IN GRAPHS

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Historically, a number of problems and puzzles introduced over a period of decades initially appeared to have no connection to graph colorings but, upon further analysis, suggested graph coloring concepts and problems. One of these problems is the well-known combinatorial problem called the Lights Out Puzzle, which can be represented by a graph coloring problem which we describe in this work. For a nontrivial connected graph G and an integer $k \geq 2$, let $c : V(G) \to \mathbb{Z}_k$ be a vertex coloring of G where $c(v) \neq 0$ for at least one vertex v of G. Then the coloring c induces a new coloring $\sigma : V(G) \to \mathbb{Z}_k$ of G defined by $\sigma(v) = \sum_{u \in N[v]} c(u)$ where $N[v]$ is the closed neighborhood of v and addition is performed in $\mathbb{Z}_k$. If $\sigma(u) = \sigma(v) = t \in \mathbb{Z}_k$ for every two vertices u and v in G, then the coloring c is called a modular monochromatic (k, t)-coloring of G. Several results dealing with modular monochromatic $(k, 0)$-colorings are presented, particularly the case where $k = 2$. The modular monochromatic $(2, 1)$-coloring and $(2, 0)$- colorings are not only closely related to the Lights Out Puzzle but also related to some well-known studied domination parameters, namely odd and even dominations in graphs.

In a modular monochromatic $(2, 0)$-coloring of a graph G, we have $\sigma(v) = 0 \in \mathbb{Z}_2$ for every vertex v in G. A graph G having a modular monochromatic $(2, 0)$-coloring is a $(2, 0)$-colorable graph. The minimum number of vertices colored 1 in a modular monochromatic $(2, 0)$-coloring of G is the $(2, 0)$-chromatic number $\chi_{(2,0)}(G)$ of G. Thus $2 \leq \chi_{(2,0)}(G) \leq n$ and $\chi_{(2,0)}(G)$ is even for every $(2, 0)$-colorable graph G of order n. A monochromatic $(2, 0)$-colorable graph G of order n is $(2, 0)$-extremal if $\chi_{(2,0)}(G) = n$. It is known that a tree T is $(2, 0)$-extremal if and only if every vertex of T has odd degree. We characterize all trees of order n having $(2, 0)$-chromatic number $n - 1, n - 2$ or $n - 3$ and investigate the structures of connected graphs having the large $(2, 0)$-chromatic numbers in general.

A dominating set S of a graph G is an even dominating set of G if every vertex of G is dominated by an even number of vertices in S and the minimum number of vertices in an even dominating set of G is the even domination number $\gamma_e(G)$ of G. We study the structures of $(2, 0)$-colorable graphs with prescribed order and $(2, 0)$-chromatic number and the relationship between modular monochromatic $(2, 0)$-colorings and even dominating sets in graphs. It is shown that for each pair a, b of even integers with $2 \leq a \leq b$, there is a connected graph G such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$. A triple (a, b, n) of positive integers is realizable if there is a connected graph G of order n such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$. Realizable triples are determined.

For a $(2,0)$-colorable graph G, the monochromatic $(2,0)$-spectrum $S_{(2,0)}(G)$ of G is the set of all positive integers k for which exactly k vertices of G can be colored 1 in a monochromatic $(2,0)$-coloring of G. Monochromatic $(2,0)$-spectra are determined for several well-known classes of graphs. If G is a connected graph of order $n \geq 2$ and $a \in S_{(2,0)}(G)$, then a is even and $1 \leq |S_{(2,0)}(G)| \leq \lfloor n/2 \rfloor$. It is shown that for every pair k, n of integers with $1 \leq k \leq \lfloor n/2 \rfloor$, there is a connected graph G of order n such that $|S_{(2,0)}(G)| = k$. A set S of positive even integers is $(2,0)$-realizable if S is the monochromatic $(2,0)$-spectrum of some connected graph. Although there are infinitely many non-$(2,0)$-realizable sets, it is shown that every set of positive even integers is a subset of some $(2,0)$-realizable set. Other results and questions are also presented on $(2,0)$-realizable sets in graphs.

A graph G is called an odd-degree graph if every vertex of G has odd degree. For a nontrivial odd-degree graph H, a connected graph $G \neq H$ is a monochromatic $(2,0)$-frame of H if G has a minimum monochromatic $(2,0)$-coloring c such that the subgraph induced by the vertices colored 1 by c is H. The monochromatic frame number of H is defined as $fn(H) = \min\{|V(G) - V(H)|\}$ where the minimum is taken over all monochromatic $(2,0)$-frames G of H. Recall that a monochromatic $(2,0)$-colorable graph G of order n is a $(2,0)$-extremal graph if $\chi_{(2,0)}(G) = n$. It is shown that $fn(H) \leq 2$ for every connected $(2,0)$-extremal graph H and the monochromatic frame numbers are determined for several well-known classes of $(2,0)$-extremal graphs. Furthermore, it is shown that if H is the union of $k \geq 2$ connected $(2,0)$-extremal graphs, then $fn(H) = k - 1$. Other results and questions are also presented on monochromatic $(2,0)$-frames in graphs.